

Zeckendorf's Theorem

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April 18, 2024

Abstract

This work formalizes Zeckendorf's theorem. The theorem states that every positive integer can be uniquely represented as a sum of one or more non-consecutive Fibonacci numbers. More precisely, if N is a positive integer, there exist unique positive integers $c_i \geq 2$ with $c_{i+1} > c_i + 1$, such that

$$N = \sum_{i=0}^k F_{c_i}$$

where F_n is the n -th Fibonacci number. This entry formalizes the proof from Gerrit Lekkerkerker's paper [1].

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1 Zeckendorf's Theorem

theory *Zeckendorf*

imports

Main

HOL-Number-Theory.Number-Theory

begin

1.1 Definitions

Formulate auxiliary definitions. An increasing sequence is a predicate of a function f together with a set I . f is an increasing sequence on I , if $f(x) + 1 < f(x + 1)$ for all $x \in I$. This definition is used to ensure that the Fibonacci numbers in the sum are non-consecutive.

definition *is-fib* :: *nat* \Rightarrow *bool* **where**

$$is-fib\ n = (\exists\ i.\ n = fib\ i)$$

definition *le-fib-idx-set* :: *nat* \Rightarrow *nat set* **where**

$$le-fib-idx-set\ n = \{i.\ fib\ i < n\}$$

definition *inc-seq-on* :: (*nat* \Rightarrow *nat*) \Rightarrow *nat set* \Rightarrow *bool* **where**

$$inc-seq-on\ f\ I = (\forall\ n \in I.\ f(Suc\ n) > Suc(f\ n))$$

definition *fib-idx-set* :: *nat* \Rightarrow *nat set* **where**

$$fib-idx-set\ n = \{i.\ fib\ i = n\}$$

1.2 Auxiliary Lemmas

lemma *fib-values[simp]*:

$$fib\ 3 = 2$$

$$fib\ 4 = 3$$

$$fib\ 5 = 5$$

$$fib\ 6 = 8$$

<proof>

lemma *fib-strict-mono*: $i \geq 2 \implies fib\ i < fib\ (Suc\ i)$

<proof>

lemma *smaller-index-implies-fib-le*: $i < j \implies fib(Suc\ i) \leq fib\ j$

<proof>

lemma *fib-index-strict-mono* : $i \geq 2 \implies j > i \implies fib\ j > fib\ i$

<proof>

lemma *fib-implies-is-fib*: $fib\ i = n \implies is-fib\ n$

<proof>

lemma *zero-fib-unique-idx*: $n = fib\ i \implies n = fib\ 0 \implies i = 0$

<proof>

lemma *zero-fib-equiv*: $fib\ i = 0 \longleftrightarrow i = 0$

<proof>

lemma *one-fib-idxs*: $fib\ i = Suc\ 0 \implies i = Suc\ 0 \vee i = Suc(Suc\ 0)$

<proof>

lemma *ge-two-eq-fib-implies-eq-idx*: $n \geq 2 \implies n = fib\ i \implies n = fib\ j \implies i = j$

<proof>

lemma *ge-two-fib-unique-idx*: $fib\ i \geq 2 \implies fib\ i = fib\ j \implies i = j$

<proof>

lemma *no-fib-lower-bound*: $\neg is-fib\ n \implies n \geq 4$

<proof>

lemma *pos-fib-has-idx-ge-two*: $n > 0 \implies \text{is-fib } n \implies (\exists i. i \geq 2 \wedge \text{fib } i = n)$
<proof>

lemma *finite-fib0-idx*: $\text{finite}(\{i. \text{fib } i = 0\})$
<proof>

lemma *finite-fib1-idx*: $\text{finite}(\{i. \text{fib } i = 1\})$
<proof>

lemma *finite-fib-ge-two-idx*: $n \geq 2 \implies \text{finite}(\{i. \text{fib } i = n\})$
<proof>

lemma *finite-fib-index*: $\text{finite}(\{i. \text{fib } i = n\})$
<proof>

lemma *no-fib-implies-zero-in-le-idx-set*: $\neg \text{is-fib } n \implies 0 \in \{i. \text{fib } i < n\}$
<proof>

lemma *no-fib-implies-le-fib-idx-set*: $\neg \text{is-fib } n \implies \{i. \text{fib } i < n\} \neq \{\}$
<proof>

lemma *finite-smaller-fibs*: $\text{finite}(\{i. \text{fib } i < n\})$
<proof>

lemma *nat-ge-2-fib-idx-bound*: $2 \leq n \implies \text{fib } i \leq n \implies n < \text{fib } (\text{Suc } i) \implies 2 \leq i$
<proof>

lemma *inc-seq-on-aux*: $\text{inc-seq-on } c \{0..k-1\} \implies n - \text{fib } i < \text{fib } (i-1) \implies \text{fib } (c \ k) < \text{fib } i \implies$
 $(n - \text{fib } i) = (\sum_{i=0..k.} \text{fib } (c \ i)) \implies \text{Suc } (c \ k) < i$
<proof>

lemma *inc-seq-zero-at-start*: $\text{inc-seq-on } c \{0..k-1\} \implies c \ k = 0 \implies k = 0$
<proof>

lemma *fib-sum-zero-equiv*: $(\sum_{i=n..m::\text{nat}} \text{fib } (c \ i)) = 0 \iff (\forall i \in \{n..m\}. c \ i = 0)$
<proof>

lemma *fib-idx-ge-two-fib-sum-not-zero*: $n \leq m \implies \forall i \in \{n..m::\text{nat}\}. c \ i \geq 2 \implies$
 $\neg (\sum_{i=n..m.} \text{fib } (c \ i)) = 0$
<proof>

lemma *one-unique-fib-sum*: $\text{inc-seq-on } c \{0..k-1\} \implies \forall i \in \{0..k\}. c \ i \geq 2 \implies$
 $(\sum_{i=0..k.} \text{fib } (c \ i)) = 1 \iff k = 0 \wedge c \ 0 = 2$
<proof>

lemma *no-fib-betw-fibs*:

assumes \neg *is-fib* n

shows $\exists i. \text{fib } i < n \wedge n < \text{fib } (\text{Suc } i)$

<proof>

lemma *betw-fibs*:

shows $\exists i. \text{fib } i \leq n \wedge \text{fib}(\text{Suc } i) > n$

<proof>

Proof that the sum of non-consecutive Fibonacci numbers with largest member F_i is strictly less than F_{i+1} . This lemma is used for the uniqueness proof.

lemma *fib-sum-upper-bound*:

assumes *inc-seq-on* $c \{0..k-1\} \forall i \in \{0..k\}. c \ i \geq 2$

shows $(\sum_{i=0..k} \text{fib } (c \ i)) < \text{fib } (\text{Suc } (c \ k))$

<proof>

lemma *last-fib-sum-index-constraint*:

assumes $n \geq 2 \ n = (\sum_{i=0..k} \text{fib } (c \ i)) \ \text{inc-seq-on } c \ \{0..k-1\}$

assumes $\forall i \in \{0..k\}. c \ i \geq 2 \ \text{fib } i \leq n \ \text{fib}(\text{Suc } i) > n$

shows $c \ k = i$

<proof>

1.3 Theorem

Now, both parts of Zeckendorf's Theorem can be proven. Firstly, the existence of an increasing sequence for a positive integer N such that the corresponding Fibonacci numbers sum up to N is proven. Then, the uniqueness of such an increasing sequence is proven.

lemma *fib-implies-zeckendorf*:

assumes *is-fib* $n \ n > 0$

shows $\exists c \ k. n = (\sum_{i=0..k} \text{fib}(c \ i)) \wedge \text{inc-seq-on } c \ \{0..k-1\} \wedge (\forall i \in \{0..k\}. c \ i \geq 2)$

<proof>

theorem *zeckendorf-existence*:

assumes $n > 0$

shows $\exists c \ k. n = (\sum_{i=0..k} \text{fib } (c \ i)) \wedge \text{inc-seq-on } c \ \{0..k-1\} \wedge (\forall i \in \{0..k\}. c \ i \geq 2)$

<proof>

lemma *fib-unique-fib-sum*:

fixes $k :: \text{nat}$

assumes $n \geq 2 \ \text{inc-seq-on } c \ \{0..k-1\} \forall i \in \{0..k\}. c \ i \geq 2$

assumes $n = \text{fib } i$

shows $n = (\sum_{i=0..k} \text{fib } (c \ i)) \iff k = 0 \wedge c \ 0 = i$

<proof>

theorem *zeckendorf-unique*:

assumes $n > 0$
assumes $n = (\sum_{i=0..k} \text{fib}(c\ i)) \text{ inc-seq-on } c\ \{0..k-1\} \forall i \in \{0..k\}. c\ i \geq 2$
assumes $n = (\sum_{i=0..k'} \text{fib}(c'\ i)) \text{ inc-seq-on } c'\ \{0..k'-1\} \forall i \in \{0..k'\}. c'\ i \geq 2$
shows $k = k' \wedge (\forall i \in \{0..k\}. c\ i = c'\ i)$
 $\langle \text{proof} \rangle$
end

References

- [1] C. G. Lekkerkerker. Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, (ZW 30/51), 1951.