Wetzel's Problem and the Continuum Hypothesis

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Abstract

Let F be a set of analytic functions on the complex plane such that, for each $z \in \mathbb{C}$, the set $\{f(z) \mid f \in F\}$ is countable; must then F itself be countable? The answer is yes if the Continuum Hypothesis is false, i.e., if the cardinality of $\mathbb{R}$ exceeds $\aleph_1$. But if CH is true then such an F, of cardinality $\aleph_1$, can be constructed by transfinite recursion.

The formal proof illustrates reasoning about complex analysis (analytic and homomorphic functions) and set theory (transfinite cardinalities) in a single setting. The mathematical text comes from *Proofs from THE BOOK* [1, pp. 137–8], by Aigner and Ziegler.

Contents

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1 Wetzel's Problem, Solved by Erdös

Martin Aigner and Günter M. Ziegler. Proofs from THE BOOK. (Springer, 2018). Chapter 19: Sets, functions, and the continuum hypothesis Theorem 5 (pages 137–8)

theory *Wetzels-Problem* **imports**
HOL−Complex-Analysis.Complex-Analysis ZFC-in-HOL.General-Cardinals

begin

definition $Wetzel :: (complex \Rightarrow complex)\ set \Rightarrow bool$
where $Wetzel \equiv \lambda F.\ (\forall f{\in}F.\ f\ analytic\text{-}on\ UNIV) \wedge (\forall z.\ countable((\lambda f.\ f\ z)\ `\ F))$

1.0.1 When the continuum hypothesis is false

proposition *Erdos-Wetzel-nonCH*:
assumes W: $Wetzel\ F$ **and** NCH: $C\text{-}continuum > \aleph 1$
shows $countable\ F$
$\langle proof \rangle$

1.0.2 When the continuum hypothesis is true

lemma *Rats-closure-real2*: $closure\ (\mathbb{Q}\times\mathbb{Q}) = (UNIV::real\ set)\times(UNIV::real\ set)$
$\langle proof \rangle$

proposition *Erdos-Wetzel-CH*:
assumes CH: $C\text{-}continuum = \aleph 1$
obtains F **where** $Wetzel\ F$ **and** $uncountable\ F$
$\langle proof \rangle$

theorem *Erdos-Wetzel*: $C\text{-}continuum = \aleph 1 \longleftrightarrow (\exists F.\ Wetzel\ F \wedge uncountable\ F)$
$\langle proof \rangle$

The originally submitted version of this theory included the development of cardinals for general Isabelle/HOL sets (as opposed to ZF sets, elements of type V), as well as other generally useful library material. From March 2022, that material has been moved to the analysis libraries or to *ZFC-in-HOL.General-Cardinals*, as appropriate.

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References

[1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 6th edition, 2018.