A Formalization of Weighted Path Orders and Recursive Path Orders[*]

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April 18, 2024

Abstract

We define the weighted path order (WPO) and formalize several properties such as strong normalization, the subterm property, and closure properties under substitutions and contexts. Our definition of WPO extends the original definition by also permitting multiset comparisons of arguments instead of just lexicographic extensions. Therefore, our WPO not only subsumes lexicographic path orders (LPO), but also recursive path orders (RPO). We formally prove these subsumptions and therefore all of the mentioned properties of WPO are automatically transferable to LPO and RPO as well. Such a transformation is not required for Knuth–Bendix orders (KBO), since they have already been formalized. Nevertheless, we still provide a proof that WPO subsumes KBO and thereby underline the generality of WPO.

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1 Introduction

Path orders are well-founded orders on terms that are useful for automated deduction, e.g., for termination proving of term rewrite systems or for completion-based theorem provers. Well-known path orders are the lexicographic path order (LPO) [3], the recursive path order (RPO) [2], and the Knuth–Bendix order (KBO) [4], and all of these orders are presented in a standard textbook on term rewriting [1, Chapter 5].

Whereas the mentioned path orders date back to the last century, the weighted path order (WPO) has only recently been presented [9, 10]. It has two nice properties. First, the search for suitable parameters is feasible and tools like NaTT and TTT2 implement it. Second, WPO is quite powerful and versatile: in fact, KBO and LPO are just instances of WPO. Moreover, with a slight extension of WPO (adding multiset-comparisons) also RPO is covered.

This AFP-entry provides a full formalization of WPO and also the connection to KBO, LPO, and RPO. Here, for the existing formal version of KBO [5, 6] it is just proven that WPO can simulate it by chosing suitable

parameters, whereas LPO and RPO are defined from scratch and many properties of LPO and RPO—such as strong normalization, closure under contexts and substitutions, transitivity, etc.—are derived from the corresponding WPO properties.

Note that most of the WPO formalization is described in [8]. The formal version deviates from the paper version only by the additional possibility to perform multiset-comparisons instead of lexicographic comparisons within WPO. The formal version of LPO and RPO extend their original definitions as well: the RPO definition is taken from [7], and LPO is defined as this extended RPO where always lexicographic comparisons are performed when comparing lists of terms. The formalization of multiset-comparisons (w.r.t. two orders) is described in more detail in [7].

2 Preliminaries

2.1 Status functions

A status function assigns to each n-ary symbol a list of indices between 0 and n-1. These functions are encapsulated into a separate type, so that recursion on the i-th subterm does not have to perform out-of-bounds checks (e.g., to ensure termination).

theory *Status*
  imports
    First-Order-Terms.Term
begin

typedef $'f$ *status* = { $(\sigma :: 'f \times nat \Rightarrow nat\ list).\ (\forall\ f\ k.\ set\ (\sigma\ (f,\ k)) \subseteq \{0\ ..< k\})$}
  morphisms *status Abs-status*
  $\langle proof \rangle$

setup-lifting *type-definition-status*

lemma *status*: *set* $(status\ \sigma\ (f,\ n)) \subseteq \{0\ ..< n\}$
  $\langle proof \rangle$

lemma *status-aux*[*termination-simp*]: $i \in set\ (status\ \sigma\ (f,\ length\ ss)) \Longrightarrow ss\ !\ i \in set\ ss$
  $\langle proof \rangle$

lemma *status-termination-simps*[*termination-simp*]:
  assumes *i1*: $i < length\ (status\ \sigma\ (f,\ length\ xs))$
  shows *size* $(xs\ !\ (status\ \sigma\ (f,\ length\ xs)\ !\ i)) < Suc\ (size\text{-}list\ size\ xs)$ (**is** *?a < ?c*)
  $\langle proof \rangle$

lemma *status-ne*:
  status $\sigma\ (f,\ n) \neq [] \Longrightarrow \exists i < n.\ i \in set\ (status\ \sigma\ (f,\ n))$

⟨*proof*⟩

lemma *set-status-nth*:
  length xs = n ⟹ i ∈ set (status σ (f, n)) ⟹ i < length xs ∧ xs ! i ∈ set xs
  ⟨*proof*⟩

lift-definition *full-status* :: *′f status* **is** *λ (f, n). [0 ..< n]* ⟨*proof*⟩

lemma *full-status[simp]*: *status full-status (f, n) = [0 ..< n]*
  ⟨*proof*⟩

  An argument position i is simple wrt. some term relation, if the i-th subterm is in relation to the full term.

definition *simple-arg-pos* :: *(′f, ′v) term rel ⇒ ′f × nat ⇒ nat ⇒ bool* **where**
  simple-arg-pos rel f i ≡ ∀ ts. i < snd f ⟶ length ts = snd f ⟶ (Fun (fst f) ts, ts ! i) ∈ rel

lemma *simple-arg-posI*: ⟦⋀ *ts. length ts = n ⟹ i < n ⟹ (Fun f ts, ts ! i) ∈ rel*⟧ ⟹ *simple-arg-pos rel (f, n) i*
  ⟨*proof*⟩

end

2.2 Precedence

A precedence consists of two compatible relations (strict and non-strict) on symbols such that the strict relation is strongly normalizing. In the formalization we model this via a function "prc" (precedence-compare) which returns two Booleans, indicating whether the one symbol is strictly or weakly bigger than the other symbol. Moreover, there also is a function "prl" (precedence-least) which gives quick access to whether a symbol is least in precedence, i.e., without comparing it to all other symbols explicitly.

theory *Precedence*
  imports
    Abstract−Rewriting.Abstract-Rewriting
begin

locale *irrefl-precedence* =
  fixes *prc* :: *′f ⇒ ′f ⇒ bool × bool*
    and *prl* :: *′f ⇒ bool*
  assumes *prc-refl*: *prc f f = (False, True)*
    and *prc-stri-imp-nstri*: *fst (prc f g) ⟹ snd (prc f g)*
    and *prl*: *prl g ⟹ snd (prc f g) = True*
    and *prl3*: *prl f ⟹ snd (prc f g) ⟹ prl g*
    and *prc-compat*: *prc f g = (s1, ns1) ⟹ prc g h = (s2, ns2) ⟹ prc f h = (s, ns) ⟹*
    (ns1 ∧ ns2 ⟶ ns) ∧ (ns1 ∧ s2 ⟶ s) ∧ (s1 ∧ ns2 ⟶ s)
begin

lemma *prl2*:
  assumes *g*: *prl g* **shows** *fst (prc g f) = False*
⟨*proof*⟩

abbreviation *pr ≡ (prc, prl)*

end

locale *precedence = irrefl-precedence +*
  constrains *prc* :: *′f ⇒ ′f ⇒ bool × bool*
    and *prl* :: *′f ⇒ bool*
  assumes *prc-SN*: *SN {(f, g). fst (prc f g)}*

end

2.3 Local versions of relations

theory *Relations*
  imports
    HOL−Library.Multiset
    Abstract−Rewriting.Abstract-Rewriting
begin

    Common predicates on relations

definition *compatible-l* :: *′a rel ⇒ ′a rel ⇒ bool* **where**
  compatible-l R1 R2 ≡ R1 O R2 ⊆ R2

definition *compatible-r* :: *′a rel ⇒ ′a rel ⇒ bool* **where**
  compatible-r R1 R2 ≡ R2 O R1 ⊆ R2

    Local reflexivity

definition *locally-refl* :: *′a rel ⇒ ′a multiset ⇒ bool* **where**
  locally-refl R A ≡ (∀ a. a ∈# A ⟶ (a,a) ∈ R)

definition *locally-irrefl* :: *′a rel ⇒ ′a multiset ⇒ bool* **where**
  locally-irrefl R A ≡ (∀ t. t ∈# A ⟶ (t,t) ∉ R)

    Local symmetry

definition *locally-sym* :: *′a rel ⇒ ′a multiset ⇒ bool* **where**
  locally-sym R A ≡ (∀ t u. t ∈# A ⟶ u ∈# A ⟶
       (t,u) ∈ R ⟶ (u,t) ∈ R)

definition *locally-antisym* :: *′a rel ⇒ ′a multiset ⇒ bool* **where**
  locally-antisym R A ≡ (∀ t u. t ∈# A ⟶ u ∈# A ⟶
       (t,u) ∈ R ⟶ (u,t) ∈ R ⟶ t = u)

    Local transitivity

definition *locally-trans* :: *′a rel ⇒ ′a multiset ⇒ ′a multiset ⇒ ′a multiset ⇒ bool*
where

locally-trans R A B C ≡ (∀ *t u v.*
      t ∈# *A* ⟶ *u* ∈# *B* ⟶ *v* ∈# *C* ⟶
      (t,u) ∈ *R* ⟶ *(u,v)* ∈ *R* ⟶ *(t,v)* ∈ *R*)

Local inclusion

definition *locally-included* :: ′*a rel* ⇒ ′*a rel* ⇒ ′*a multiset* ⇒ ′*a multiset* ⇒ *bool*
where
   locally-included R1 R2 A B ≡ (∀ *t u. t* ∈# *A* ⟶ *u* ∈# *B* ⟶
      (t,u) ∈ *R1* ⟶ *(t,u)* ∈ *R2*)

Local transitivity compatibility

definition *locally-compatible-l* :: ′*a rel* ⇒ ′*a rel* ⇒ ′*a multiset* ⇒ ′*a multiset* ⇒ ′*a*
multiset ⇒ *bool* **where**
   locally-compatible-l R1 R2 A B C ≡ (∀ *t u v. t* ∈# *A* ⟶ *u* ∈# *B* ⟶ *v* ∈# *C*
⟶
      (t,u) ∈ *R1* ⟶ *(u,v)* ∈ *R2* ⟶ *(t,v)* ∈ *R2*)

definition *locally-compatible-r* :: ′*a rel* ⇒ ′*a rel* ⇒ ′*a multiset* ⇒ ′*a multiset* ⇒ ′*a*
multiset ⇒ *bool* **where**
   locally-compatible-r R1 R2 A B C ≡ (∀ *t u v. t* ∈# *A* ⟶ *u* ∈# *B* ⟶ *v* ∈# *C*
⟶
      (t,u) ∈ *R2* ⟶ *(u,v)* ∈ *R1* ⟶ *(t,v)* ∈ *R2*)

included + compatible ⟶ transitive

lemma *in-cl-tr*:
   assumes *R1* ⊆ *R2*
      and *compatible-l R2 R1*
   shows *trans R1*
⟨*proof*⟩

lemma *in-cr-tr*:
   assumes *R1* ⊆ *R2*
      and *compatible-r R2 R1*
   shows *trans R1*
⟨*proof*⟩

If a property holds globally, it also holds locally. Obviously.

lemma *r-lr*:
   assumes *refl R*
   shows *locally-refl R A*
   ⟨*proof*⟩

lemma *tr-ltr*:
   assumes *trans R*
   shows *locally-trans R A B C*
   ⟨*proof*⟩

lemma *in-lin*:
   assumes *R1* ⊆ *R2*

shows *locally-included R1 R2 A B*
⟨*proof*⟩

lemma *cl-lcl*:
  assumes *compatible-l R1 R2*
  shows *locally-compatible-l R1 R2 A B C*
  ⟨*proof*⟩

lemma *cr-lcr*:
  assumes *compatible-r R1 R2*
  shows *locally-compatible-r R1 R2 A B C*
  ⟨*proof*⟩

    If a predicate holds on a set then it holds on all the subsets:

lemma *lr-trans-l*:
  assumes *locally-refl R (A + B)*
  shows *locally-refl R A*
  ⟨*proof*⟩

lemma *li-trans-l*:
  assumes *locally-irrefl R (A + B)*
  shows *locally-irrefl R A*
  ⟨*proof*⟩

lemma *ls-trans-l*:
  assumes *locally-sym R (A + B)*
  shows *locally-sym R A*
  ⟨*proof*⟩

lemma *las-trans-l*:
  assumes *locally-antisym R (A + B)*
  shows *locally-antisym R A*
  ⟨*proof*⟩

lemma *lt-trans-l*:
  assumes *locally-trans R (A + B) (C + D) (E + F)*
  shows *locally-trans R A C E*
  ⟨*proof*⟩

lemma *lin-trans-l*:
  assumes *locally-included R1 R2 (A + B) (C + D)*
  shows *locally-included R1 R2 A C*
  ⟨*proof*⟩

lemma *lcl-trans-l*:
  assumes *locally-compatible-l R1 R2 (A + B) (C + D) (E + F)*
  shows *locally-compatible-l R1 R2 A C E*
  ⟨*proof*⟩

lemma *lcr-trans-l*:
  assumes *locally-compatible-r R1 R2 $(A + B)$ $(C + D)$ $(E + F)$*
  shows *locally-compatible-r R1 R2 A C E*
  ⟨*proof*⟩

lemma *lr-trans-r*:
  assumes *locally-refl R $(A + B)$*
  shows *locally-refl R B*
  ⟨*proof*⟩

lemma *li-trans-r*:
  assumes *locally-irrefl R $(A + B)$*
  shows *locally-irrefl R B*
  ⟨*proof*⟩

lemma *ls-trans-r*:
  assumes *locally-sym R $(A + B)$*
  shows *locally-sym R B*
  ⟨*proof*⟩

lemma *las-trans-r*:
  assumes *locally-antisym R $(A + B)$*
  shows *locally-antisym R B*
  ⟨*proof*⟩

lemma *lt-trans-r*:
  assumes *locally-trans R $(A + B)$ $(C + D)$ $(E + F)$*
  shows *locally-trans R B D F*
  ⟨*proof*⟩

lemma *lin-trans-r*:
  assumes *locally-included R1 R2 $(A + B)$ $(C + D)$*
  shows *locally-included R1 R2 B D*
  ⟨*proof*⟩

lemma *lcl-trans-r*:
  assumes *locally-compatible-l R1 R2 $(A + B)$ $(C + D)$ $(E + F)$*
  shows *locally-compatible-l R1 R2 B D F*
  ⟨*proof*⟩

lemma *lcr-trans-r*:
  assumes *locally-compatible-r R1 R2 $(A + B)$ $(C + D)$ $(E + F)$*
  shows *locally-compatible-r R1 R2 B D F*
  ⟨*proof*⟩

lemma *lr-minus*:
  assumes *locally-refl R A*
  shows *locally-refl R $(A - B)$*
  ⟨*proof*⟩

lemma *li-minus*:
  assumes *locally-irrefl R A*
  shows *locally-irrefl R (A − B)*
  ⟨*proof*⟩

lemma *ls-minus*:
  assumes *locally-sym R A*
  shows *locally-sym R (A − B)*
  ⟨*proof*⟩

lemma *las-minus*:
  assumes *locally-antisym R A*
  shows *locally-antisym R (A − B)*
  ⟨*proof*⟩

lemma *lt-minus*:
  assumes *locally-trans R A C E*
  shows *locally-trans R (A − B) (C − D) (E − F)*
  ⟨*proof*⟩

lemma *lin-minus*:
  assumes *locally-included R1 R2 A C*
  shows *locally-included R1 R2 (A − B) (C − D)*
  ⟨*proof*⟩

lemma *lcl-minus*:
  assumes *locally-compatible-l R1 R2 A C E*
  shows *locally-compatible-l R1 R2 (A − B) (C − D) (E − F)*
  ⟨*proof*⟩

lemma *lcr-minus*:
  assumes *locally-compatible-r R1 R2 A C E*
  shows *locally-compatible-r R1 R2 (A − B) (C − D) (E − F)*
  ⟨*proof*⟩

    Notations

notation *restrict* (**infixl** ↾ *80*)

lemma *mem-restrictI*[*intro!*]: **assumes** $x \in X$ $y \in X$ $(x,y) \in R$ **shows** $(x,y) \in R$ ↾ X
  ⟨*proof*⟩

lemma *mem-restrictD*[*dest*]: **assumes** $(x,y) \in R$ ↾ X **shows** $x \in X$ $y \in X$ $(x,y) \in R$
  ⟨*proof*⟩

end

2.4 Interface for extending an order pair on lists

theory *List-Order*
  imports
    Knuth-Bendix-Order.Order-Pair
begin

type-synonym *'a list-ext = 'a rel ⇒ 'a rel ⇒ 'a list rel*

locale *list-order-extension =*
  fixes *s-list :: 'a list-ext*
    and *ns-list :: 'a list-ext*
  assumes *extension*: *SN-order-pair S NS ⟹ SN-order-pair (s-list S NS) (ns-list S NS)*
    and *s-map*: ⟦$\bigwedge$ *a b. (a,b) ∈ S ⟹ (f a,f b) ∈ S*; $\bigwedge$ *a b. (a,b) ∈ NS ⟹ (f a,f b) ∈ NS*⟧ *⟹ (as,bs) ∈ s-list S NS ⟹ (map f as, map f bs) ∈ s-list S NS*
    and *ns-map*: ⟦$\bigwedge$ *a b. (a,b) ∈ S ⟹ (f a,f b) ∈ S*; $\bigwedge$ *a b. (a,b) ∈ NS ⟹ (f a,f b) ∈ NS*⟧ *⟹ (as,bs) ∈ ns-list S NS ⟹ (map f as, map f bs) ∈ ns-list S NS*
    and *all-ns-imp-ns*: *length as = length bs ⟹* ⟦$\bigwedge$ *i. i < length bs ⟹ (as ! i, bs ! i) ∈ NS*⟧ *⟹ (as,bs) ∈ ns-list S NS*

type-synonym *'a list-ext-impl = ('a ⇒ 'a ⇒ bool × bool) ⇒ 'a list ⇒ 'a list ⇒ bool × bool*

locale *list-order-extension-impl = list-order-extension s-list ns-list* **for**
  s-list ns-list :: 'a list-ext +
  fixes *list-ext :: 'a list-ext-impl*
  assumes *list-ext-s*: $\bigwedge$ *s ns. s-list {(a,b). s a b} {(a,b). ns a b} = {(as,bs). fst (list-ext (λ a b. (s a b, ns a b)) as bs)}*
    and *list-ext-ns*: $\bigwedge$ *s ns. ns-list {(a,b). s a b} {(a,b). ns a b} = {(as,bs). snd (list-ext (λ a b. (s a b, ns a b)) as bs)}*
    and *s-ext-local-mono*: $\bigwedge$ *s ns s' ns' as bs. (set as × set bs) ∩ ns ⊆ ns' ⟹ (set as × set bs) ∩ s ⊆ s' ⟹ (as,bs) ∈ s-list ns s ⟹ (as,bs) ∈ s-list ns' s'*
    and *ns-ext-local-mono*: $\bigwedge$ *s ns s' ns' as bs. (set as × set bs) ∩ ns ⊆ ns' ⟹ (set as × set bs) ∩ s ⊆ s' ⟹ (as,bs) ∈ ns-list ns s ⟹ (as,bs) ∈ ns-list ns' s'*

end

3 Multiset extension of an order pair

Given a well-founded order ≺ and a compatible non-strict order ≾, we define the corresponding multiset-extension of these orders.

theory *Multiset-Extension-Pair*
  imports
    HOL−Library.Multiset

Regular−Sets.Regexp-Method
Abstract−Rewriting.Abstract-Rewriting
Relations
begin

lemma *mult-locally-cancel*:
  assumes *trans s* **and** *locally-irrefl s* $(X + Z)$ **and** *locally-irrefl s* $(Y + Z)$
  shows $(X + Z, Y + Z) \in mult\ s \longleftrightarrow (X, Y) \in mult\ s$ (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *mult-locally-cancelL*:
  assumes *trans s locally-irrefl s* $(X + Z)$ *locally-irrefl s* $(Y + Z)$
  shows $(Z + X, Z + Y) \in mult\ s \longleftrightarrow (X, Y) \in mult\ s$
  ⟨*proof*⟩

lemma *mult-cancelL*:
  assumes *trans s irrefl s* **shows** $(Z + X, Z + Y) \in mult\ s \longleftrightarrow (X, Y) \in mult\ s$
  ⟨*proof*⟩

lemma *wf-trancl-conv*:
  shows $wf\ (r^+) \longleftrightarrow wf\ r$
  ⟨*proof*⟩

3.1 Pointwise multiset order

inductive-set *multpw* :: $'a\ rel \Rightarrow 'a\ multiset\ rel$ **for** *ns* :: $'a\ rel$ **where**
  empty: $(\{\#\}, \{\#\}) \in multpw\ ns$
| *add*: $(x, y) \in ns \Longrightarrow (X, Y) \in multpw\ ns \Longrightarrow (add\text{-}mset\ x\ X, add\text{-}mset\ y\ Y) \in multpw\ ns$

lemma *multpw-emptyL* [*simp*]:
  $(\{\#\}, X) \in multpw\ ns \longleftrightarrow X = \{\#\}$
  ⟨*proof*⟩

lemma *multpw-emptyR* [*simp*]:
  $(X, \{\#\}) \in multpw\ ns \longleftrightarrow X = \{\#\}$
  ⟨*proof*⟩

lemma *refl-multpw*:
  assumes *refl ns* **shows** *refl* (*multpw ns*)
⟨*proof*⟩

lemma *multpw-Id-Id* [*simp*]:
  multpw Id = Id
⟨*proof*⟩

lemma *mono-multpw*:

assumes $ns \subseteq ns'$ **shows** *multpw ns* $\subseteq$ *multpw ns'*
⟨*proof*⟩

lemma *multpw-converse*:
  multpw $(ns^{-1}) = (multpw\ ns)^{-1}$
⟨*proof*⟩

lemma *multpw-local*:
  $(X,\ Y) \in multpw\ ns \implies (X,\ Y) \in multpw\ (ns \cap set\text{-}mset\ X \times set\text{-}mset\ Y)$
⟨*proof*⟩

lemma *multpw-split1R*:
  assumes $(add\text{-}mset\ x\ X,\ Y) \in multpw\ ns$
  obtains $z\ Z$ **where** $Y = add\text{-}mset\ z\ Z$ **and** $(x,\ z) \in ns$ **and** $(X,\ Z) \in multpw\ ns$
  ⟨*proof*⟩

lemma *multpw-splitR*:
  assumes $(X1 + X2,\ Y) \in multpw\ ns$
  obtains $Y1\ Y2$ **where** $Y = Y1 + Y2$ **and** $(X1,\ Y1) \in multpw\ ns$ **and** $(X2,\ Y2) \in multpw\ ns$
  ⟨*proof*⟩

lemma *multpw-split1L*:
  assumes $(X,\ add\text{-}mset\ y\ Y) \in multpw\ ns$
  obtains $z\ Z$ **where** $X = add\text{-}mset\ z\ Z$ **and** $(z,\ y) \in ns$ **and** $(Z,\ Y) \in multpw\ ns$
  ⟨*proof*⟩

lemma *multpw-splitL*:
  assumes $(X,\ Y1 + Y2) \in multpw\ ns$
  obtains $X1\ X2$ **where** $X = X1 + X2$ **and** $(X1,\ Y1) \in multpw\ ns$ **and** $(X2,\ Y2) \in multpw\ ns$
  ⟨*proof*⟩

lemma *locally-trans-multpw*:
  assumes *locally-trans ns S T U*
    and $(S,\ T) \in multpw\ ns$
    and $(T,\ U) \in multpw\ ns$
  shows $(S,\ U) \in multpw\ ns$
  ⟨*proof*⟩

lemma *trans-multpw*:
  assumes *trans ns* **shows** *trans* (*multpw ns*)
  ⟨*proof*⟩

lemma *multpw-add*:
  assumes $(X1,\ Y1) \in multpw\ ns$ $(X2,\ Y2) \in multpw\ ns$ **shows** $(X1 + X2,\ Y1 + Y2) \in multpw\ ns$
  ⟨*proof*⟩

lemma *multpw-single*:
  $(x, y) \in ns \implies (\{\#x\#\}, \{\#y\#\}) \in multpw\ ns$
  $\langle proof \rangle$

lemma *multpw-mult1-commute*:
  assumes *compat*: $s\ O\ ns \subseteq s$ **and** *reflns*: *refl ns*
  shows *mult1 s O multpw ns* $\subseteq$ *multpw ns O mult1 s*
$\langle proof \rangle$

lemma *multpw-mult-commute*:
  assumes $s\ O\ ns \subseteq s$ *refl ns* **shows** *mult s O multpw ns* $\subseteq$ *multpw ns O mult s*
$\langle proof \rangle$

lemma *wf-mult-rel-multpw*:
  assumes *wf s s O ns* $\subseteq$ *s refl ns* **shows** *wf* $((multpw\ ns)^*\ O\ mult\ s\ O\ (multpw\ ns)^*)$
  $\langle proof \rangle$

lemma *multpw-cancel1*:
  assumes *trans ns* $(y, x) \in ns$
  shows $(add\text{-}mset\ x\ X,\ add\text{-}mset\ y\ Y) \in multpw\ ns \implies (X, Y) \in multpw\ ns$ (**is** *?L* $\implies$ *?R*)
$\langle proof \rangle$

lemma *multpw-cancel*:
  assumes *refl ns* *trans ns*
  shows $(X + Z,\ Y + Z) \in multpw\ ns \longleftrightarrow (X, Y) \in multpw\ ns$ (**is** *?L* $\longleftrightarrow$ *?R*)
$\langle proof \rangle$

lemma *multpw-cancelL*:
  assumes *refl ns* *trans ns* **shows** $(Z + X,\ Z + Y) \in multpw\ ns \longleftrightarrow (X, Y) \in multpw\ ns$
  $\langle proof \rangle$

3.2 Multiset extension for order pairs via the pointwise order and *mult*

definition *mult2-s ns s* $\equiv$ *multpw ns O mult s*
definition *mult2-ns ns s* $\equiv$ *multpw ns O* $(mult\ s)^=$

lemma *mult2-ns-conv*:
  shows *mult2-ns ns s* = *mult2-s ns s* $\cup$ *multpw ns*
  $\langle proof \rangle$

lemma *mono-mult2-s*:
  assumes $ns \subseteq ns'\ s \subseteq s'$ **shows** *mult2-s ns s* $\subseteq$ *mult2-s ns' s'*
  $\langle proof \rangle$

lemma *mono-mult2-ns*:

assumes $ns \subseteq ns'\ s \subseteq s'$ **shows** *mult2-ns ns s* $\subseteq$ *mult2-ns ns' s'*
⟨*proof*⟩

lemma *wf-mult2-s*:
  assumes *wf s s O ns* $\subseteq$ *s refl ns*
  shows *wf* (*mult2-s ns s*)
  ⟨*proof*⟩

lemma *refl-mult2-ns*:
  assumes *refl ns* **shows** *refl* (*mult2-ns ns s*)
  ⟨*proof*⟩

lemma *trans-mult2-s*:
  assumes *s O ns* $\subseteq$ *s refl ns trans ns*
  shows *trans* (*mult2-s ns s*)
  ⟨*proof*⟩

lemma *trans-mult2-ns*:
  assumes *s O ns* $\subseteq$ *s refl ns trans ns*
  shows *trans* (*mult2-ns ns s*)
  ⟨*proof*⟩

lemma *compat-mult2*:
  assumes *s O ns* $\subseteq$ *s refl ns trans ns*
  shows *mult2-ns ns s O mult2-s ns s* $\subseteq$ *mult2-s ns s mult2-s ns s O mult2-ns ns s* $\subseteq$ *mult2-s ns s*
  ⟨*proof*⟩

    Trivial inclusions

lemma *mult-implies-mult2-s*:
  assumes *refl ns* $(X,\ Y) \in$ *mult s*
  shows $(X,\ Y) \in$ *mult2-s ns s*
  ⟨*proof*⟩

lemma *mult-implies-mult2-ns*:
  assumes *refl ns* $(X,\ Y) \in$ (*mult s*)$^{=}$
  shows $(X,\ Y) \in$ *mult2-ns ns s*
  ⟨*proof*⟩

lemma *multpw-implies-mult2-ns*:
  assumes $(X,\ Y) \in$ *multpw ns*
  shows $(X,\ Y) \in$ *mult2-ns ns s*
  ⟨*proof*⟩

3.3 One-step versions of the multiset extensions

lemma *mult2-s-one-step*:
  assumes *ns O s* $\subseteq$ *s refl ns trans s*
  shows $(X,\ Y) \in$ *mult2-s ns s* $\longleftrightarrow$ ($\exists$ *X1 X2 Y1 Y2. X = X1 + X2* $\wedge$ *Y = Y1 + Y2* $\wedge$

$(X1, Y1) \in multpw\ ns \land Y2 \neq \{\#\} \land (\forall x.\ x \in \# X2 \longrightarrow (\exists y.\ y \in \# Y2 \land (x, y) \in s)))$ (**is** $?L \longleftrightarrow ?R$)

⟨*proof*⟩

lemma *mult2-ns-one-step*:
  assumes *ns O s ⊆ s refl ns trans s*
  shows $(X, Y) \in mult2\text{-}ns\ ns\ s \longleftrightarrow (\exists X1\ X2\ Y1\ Y2.\ X = X1 + X2 \land Y = Y1 + Y2 \land$
    $(X1, Y1) \in multpw\ ns \land (\forall x.\ x \in \# X2 \longrightarrow (\exists y.\ y \in \# Y2 \land (x, y) \in s)))$ (**is** $?L \longleftrightarrow ?R$)

⟨*proof*⟩

lemma *mult2-s-locally-one-step′*:
  assumes *ns O s ⊆ s refl ns locally-irrefl s X locally-irrefl s Y trans s*
  shows $(X, Y) \in mult2\text{-}s\ ns\ s \longleftrightarrow (\exists X1\ X2\ Y1\ Y2.\ X = X1 + X2 \land Y = Y1 + Y2 \land$
    $(X1, Y1) \in multpw\ ns \land (X2, Y2) \in mult\ s)$ (**is** $?L \longleftrightarrow ?R$)

⟨*proof*⟩

lemma *mult2-s-one-step′*:
  assumes *ns O s ⊆ s refl ns irrefl s trans s*
  shows $(X, Y) \in mult2\text{-}s\ ns\ s \longleftrightarrow (\exists X1\ X2\ Y1\ Y2.\ X = X1 + X2 \land Y = Y1 + Y2 \land$
    $(X1, Y1) \in multpw\ ns \land (X2, Y2) \in mult\ s)$ (**is** $?L \longleftrightarrow ?R$)
  ⟨*proof*⟩

lemma *mult2-ns-one-step′*:
  assumes *ns O s ⊆ s refl ns irrefl s trans s*
  shows $(X, Y) \in mult2\text{-}ns\ ns\ s \longleftrightarrow (\exists X1\ X2\ Y1\ Y2.\ X = X1 + X2 \land Y = Y1 + Y2 \land$
    $(X1, Y1) \in multpw\ ns \land (X2, Y2) \in (mult\ s)^=)$ (**is** $?L \longleftrightarrow ?R$)

⟨*proof*⟩

3.4 Cancellation

lemma *mult2-s-locally-cancel1*:
  assumes *s O ns ⊆ s ns O s ⊆ s refl ns trans ns locally-irrefl s (add-mset z X) locally-irrefl s (add-mset z Y) trans s*
    $(add\text{-}mset\ z\ X, add\text{-}mset\ z\ Y) \in mult2\text{-}s\ ns\ s$
  shows $(X, Y) \in mult2\text{-}s\ ns\ s$

⟨*proof*⟩

lemma *mult2-s-cancel1*:
  assumes *s O ns ⊆ s ns O s ⊆ s refl ns trans ns irrefl s trans s* $(add\text{-}mset\ z\ X, add\text{-}mset\ z\ Y) \in mult2\text{-}s\ ns\ s$
  shows $(X, Y) \in mult2\text{-}s\ ns\ s$
  ⟨*proof*⟩

lemma *mult2-s-locally-cancel*:

assumes *s O ns ⊆ s ns O s ⊆ s refl ns trans ns locally-irrefl s (X + Z) locally-irrefl s (Y + Z) trans s*
  shows $(X + Z, Y + Z) \in mult2\text{-}s\ ns\ s \implies (X, Y) \in mult2\text{-}s\ ns\ s$
  ⟨*proof*⟩

lemma *mult2-s-cancel*:
  assumes *s O ns ⊆ s ns O s ⊆ s refl ns trans ns irrefl s trans s*
  shows $(X + Z, Y + Z) \in mult2\text{-}s\ ns\ s \implies (X, Y) \in mult2\text{-}s\ ns\ s$
  ⟨*proof*⟩

lemma *mult2-ns-cancel*:
  assumes *s O ns ⊆ s ns O s ⊆ s refl ns trans s irrefl s trans ns*
  shows $(X + Z, Y + Z) \in mult2\text{-}s\ ns\ s \implies (X, Y) \in mult2\text{-}ns\ ns\ s$
  ⟨*proof*⟩

3.5 Implementation friendly versions of *mult2-s* and *mult2-ns*

definition *mult2-alt* :: *bool ⇒ 'a rel ⇒ 'a rel ⇒ 'a multiset rel* **where**
  mult2-alt b ns s = {(X, Y). ($\exists X1\ X2\ Y1\ Y2.\ X = X1 + X2 \wedge Y = Y1 + Y2 \wedge$
    $(X1, Y1) \in multpw\ ns \wedge (b \vee Y2 \neq \{\#\}) \wedge (\forall x.\ x \in\# X2 \longrightarrow (\exists y.\ y \in\# Y2$
  $\wedge\ (x, y) \in s)))$}

lemma *mult2-altI*:
  assumes $X = X1 + X2$ $Y = Y1 + Y2$ $(X1, Y1) \in multpw\ ns$
  $b \vee Y2 \neq \{\#\}$ $\forall x.\ x \in\# X2 \longrightarrow (\exists y.\ y \in\# Y2 \wedge (x, y) \in s)$
  shows $(X, Y) \in mult2\text{-}alt\ b\ ns\ s$
  ⟨*proof*⟩

lemma *mult2-altE*:
  assumes $(X, Y) \in mult2\text{-}alt\ b\ ns\ s$
  obtains *X1 X2 Y1 Y2* **where** $X = X1 + X2$ $Y = Y1 + Y2$ $(X1, Y1) \in multpw\ ns$
  $b \vee Y2 \neq \{\#\}$ $\forall x.\ x \in\# X2 \longrightarrow (\exists y.\ y \in\# Y2 \wedge (x, y) \in s)$
  ⟨*proof*⟩

lemma *mono-mult2-alt*:
  assumes $ns \subseteq ns'\ s \subseteq s'$ **shows** $mult2\text{-}alt\ b\ ns\ s \subseteq mult2\text{-}alt\ b\ ns'\ s'$
  ⟨*proof*⟩

abbreviation *mult2-alt-s* ≡ *mult2-alt False*
abbreviation *mult2-alt-ns* ≡ *mult2-alt True*

lemmas *mult2-alt-s-def* = *mult2-alt-def*[**where** *b = False, unfolded simp-thms*]
lemmas *mult2-alt-ns-def* = *mult2-alt-def*[**where** *b = True, unfolded simp-thms*]
lemmas *mult2-alt-sI* = *mult2-altI*[**where** *b = False, unfolded simp-thms*]
lemmas *mult2-alt-nsI* = *mult2-altI*[**where** *b = True, unfolded simp-thms True-implies-equals*]
lemmas *mult2-alt-sE* = *mult2-altE*[**where** *b = False, unfolded simp-thms*]
lemmas *mult2-alt-nsE* = *mult2-altE*[**where** *b = True, unfolded simp-thms True-implies-equals*]

Equivalence to *mult2-s* **and** *mult2-ns* **lemma** *mult2-s-eq-mult2-s-alt*:
  assumes *ns O s ⊆ s refl ns trans s*
  shows *mult2-alt-s ns s = mult2-s ns s*
  ⟨*proof*⟩

lemma *mult2-ns-eq-mult2-ns-alt*:
  assumes *ns O s ⊆ s refl ns trans s*
  shows *mult2-alt-ns ns s = mult2-ns ns s*
  ⟨*proof*⟩

lemma *mult2-alt-local*:
  assumes $(X, Y) \in$ *mult2-alt b ns s*
  shows $(X, Y) \in$ *mult2-alt b (ns ∩ set-mset X × set-mset Y) (s ∩ set-mset X ×*
set-mset Y)
⟨*proof*⟩

3.6   Local well-foundedness: restriction to downward closure of a set

definition *wf-below* :: $'a\ rel \Rightarrow\ 'a\ set \Rightarrow\ bool$ **where**
  wf-below r A = wf (Restr r $((r^*)^{-1}$ *'' A))*

lemma *wf-below-UNIV*[*simp*]:
  shows *wf-below r UNIV* $\longleftrightarrow$ *wf r*
  ⟨*proof*⟩

lemma *wf-below-mono1*:
  assumes $r \subseteq r'$ *wf-below r' A* **shows** *wf-below r A*
  ⟨*proof*⟩

lemma *wf-below-mono2*:
  assumes $A \subseteq A'$ *wf-below r A'* **shows** *wf-below r A*
  ⟨*proof*⟩

lemma *wf-below-pointwise*:
  wf-below r A $\longleftrightarrow$ $(\forall a.\ a \in A \longrightarrow$ *wf-below r {a})* (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *SN-on-Image-rtrancl-conv*:
  SN-on r A $\longleftrightarrow$ *SN-on r (r* '' A)* (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *SN-on-iff-wf-below*:
  SN-on r A $\longleftrightarrow$ *wf-below* (r^{-1}) *A*
⟨*proof*⟩

lemma *restr-trancl-under*:
  shows *Restr* (r^+) $((r^*)^{-1}$ *'' A) = (Restr r* $((r^*)^{-1}$ *'' A))$^+$

⟨*proof*⟩

lemma *wf-below-trancl*:
  shows *wf-below* (r^+) *A* ⟷ *wf-below r A*
  ⟨*proof*⟩

lemma *wf-below-mult-local*:
  assumes *wf-below r* (*set-mset X*) **shows** *wf-below* (*mult r*) {*X*}

⟨*proof*⟩

lemma *qc-wf-below*:
  assumes *s O ns* ⊆ (*s* ∪ *ns*)* *O s wf-below s A*
  shows *wf-below* (*ns** *O s O ns**) *A*
  ⟨*proof*⟩

lemma *wf-below-mult2-s-local*:
  assumes *wf-below s* (*set-mset X*) *s O ns* ⊆ *s refl ns trans ns*
  shows *wf-below* (*mult2-s ns s*) {*X*}
  ⟨*proof*⟩

3.7   Trivial cases

lemma *mult2-alt-emptyL*:
  ({#}, *Y*) ∈ *mult2-alt b ns s* ⟷ *b* ∨ *Y* ≠ {#}
  ⟨*proof*⟩

lemma *mult2-alt-emptyR*:
  (*X*, {#}) ∈ *mult2-alt b ns s* ⟷ *b* ∧ *X* = {#}
  ⟨*proof*⟩

lemma *mult2-alt-s-single*:
  (*a*, *b*) ∈ *s* ⟹ ({#*a*#}, {#*b*#}) ∈ *mult2-alt-s ns s*
  ⟨*proof*⟩

lemma *multpw-implies-mult2-alt-ns*:
  assumes (*X*, *Y*) ∈ *multpw ns*
  shows (*X*, *Y*) ∈ *mult2-alt-ns ns s*
  ⟨*proof*⟩

lemma *mult2-alt-ns-conv*:
  mult2-alt-ns ns s = *mult2-alt-s ns s* ∪ *multpw ns* (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *mult2-alt-s-implies-mult2-alt-ns*:
  assumes (*X*, *Y*) ∈ *mult2-alt-s ns s*
  shows (*X*, *Y*) ∈ *mult2-alt-ns ns s*
  ⟨*proof*⟩

lemma *mult2-alt-add*:
  assumes *(X1, Y1)* ∈ *mult2-alt b1 ns s* **and** *(X2, Y2)* ∈ *mult2-alt b2 ns s*
  shows *(X1 + X2, Y1 + Y2)* ∈ *mult2-alt (b1 ∧ b2) ns s*
⟨*proof*⟩

lemmas *mult2-alt-s-s-add* = *mult2-alt-add*[*of - - False - - - - False, unfolded simp-thms*]
lemmas *mult2-alt-ns-s-add* = *mult2-alt-add*[*of - - True - - - - False, unfolded simp-thms*]
lemmas *mult2-alt-s-ns-add* = *mult2-alt-add*[*of - - False - - - - True, unfolded simp-thms*]
lemmas *mult2-alt-ns-ns-add* = *mult2-alt-add*[*of - - True - - - - True, unfolded simp-thms*]

lemma *multpw-map*:
  assumes ⋀*x y. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *(x, y)* ∈ *ns* ⟹ *(f x, g y)* ∈ *ns′*
    and *(X, Y)* ∈ *multpw ns*
  shows *(image-mset f X, image-mset g Y)* ∈ *multpw ns′*
  ⟨*proof*⟩

lemma *mult2-alt-map*:
  assumes ⋀*x y. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *(x, y)* ∈ *ns* ⟹ *(f x, g y)* ∈ *ns′*
    and ⋀*x y. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *(x, y)* ∈ *s* ⟹ *(f x, g y)* ∈ *s′*
    and *(X, Y)* ∈ *mult2-alt b ns s*
  shows *(image-mset f X, image-mset g Y)* ∈ *mult2-alt b ns′ s′*
⟨*proof*⟩

Local transitivity of *mult2-alt*

lemma *trans-mult2-alt-local*:
  assumes *ss*: ⋀*x y z. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *z* ∈# *Z* ⟹ *(x, y)* ∈ *s* ⟹ *(y, z)* ∈ *s* ⟹ *(x, z)* ∈ *s*
    and *ns*: ⋀*x y z. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *z* ∈# *Z* ⟹ *(x, y)* ∈ *ns* ⟹ *(y, z)* ∈ *s* ⟹ *(x, z)* ∈ *s*
    and *sn*: ⋀*x y z. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *z* ∈# *Z* ⟹ *(x, y)* ∈ *s* ⟹ *(y, z)* ∈ *ns* ⟹ *(x, z)* ∈ *s*
    and *nn*: ⋀*x y z. x* ∈# *X* ⟹ *y* ∈# *Y* ⟹ *z* ∈# *Z* ⟹ *(x, y)* ∈ *ns* ⟹ *(y, z)* ∈ *ns* ⟹ *(x, z)* ∈ *ns*
    and *xyz*: *(X, Y)* ∈ *mult2-alt b1 ns s (Y, Z)* ∈ *mult2-alt b2 ns s*
  shows *(X, Z)* ∈ *mult2-alt (b1 ∧ b2) ns s*
⟨*proof*⟩

lemmas *trans-mult2-alt-s-s-local* = *trans-mult2-alt-local*[*of - - - - - False False, unfolded simp-thms*]
lemmas *trans-mult2-alt-ns-s-local* = *trans-mult2-alt-local*[*of - - - - - True False, unfolded simp-thms*]
lemmas *trans-mult2-alt-s-ns-local* = *trans-mult2-alt-local*[*of - - - - - False True, unfolded simp-thms*]
lemmas *trans-mult2-alt-ns-ns-local* = *trans-mult2-alt-local*[*of - - - - - True True,*

unfolded simp-thms]

end

3.8   Executable version

theory *Multiset-Extension-Pair-Impl*
  imports
    Multiset-Extension-Pair
begin

lemma *subset-mult2-alt*:
  assumes $X \subseteq\# Y$ $(Y, Z) \in$ *mult2-alt b ns s b* $\Longrightarrow b'$
  shows $(X, Z) \in$ *mult2-alt b' ns s*
⟨*proof*⟩

Case distinction for recursion on left argument

lemma *mem-multiset-diff*: $x \in\# A \Longrightarrow x \neq y \Longrightarrow x \in\# (A - \{\#y\#\})$
  ⟨*proof*⟩

lemma *mult2-alt-addL*: $(add\text{-}mset\ x\ X,\ Y) \in$ *mult2-alt b ns s* $\longleftrightarrow$
  $(\exists y.\ y \in\# Y \wedge (x, y) \in s \wedge (\{\#\ x \in\# X.\ (x, y) \notin s\ \#\},\ Y - \{\#y\#\}) \in$
mult2-alt-ns ns s) $\vee$
  $(\exists y.\ y \in\# Y \wedge (x, y) \in ns \wedge (x, y) \notin s \wedge (X,\ Y - \{\#y\#\}) \in$ *mult2-alt b ns s*)
(**is** *?L* $\longleftrightarrow$ *?R1* $\vee$ *?R2*)
⟨*proof*⟩

Auxiliary version with an extra *bool* argument for distinguishing between
the non-strict and the strict orders

context fixes *nss* :: $'a \Rightarrow 'a \Rightarrow bool \Rightarrow bool$
begin

fun *mult2-impl0* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool \Rightarrow bool$
  and *mult2-ex-dom0* :: $'a \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool \Rightarrow bool$
  where
    mult2-impl0 []   [] *b* $\longleftrightarrow b$
  | *mult2-impl0 xs*   [] *b* $\longleftrightarrow$ *False*
  | *mult2-impl0* []   *ys b* $\longleftrightarrow$ *True*
  | *mult2-impl0* $(x \# xs)$ *ys b* $\longleftrightarrow$ *mult2-ex-dom0 x xs ys* [] *b*

  | *mult2-ex-dom0 x xs* []   *ys' b* $\longleftrightarrow$ *False*
  | *mult2-ex-dom0 x xs* $(y \# ys)$ *ys' b* $\longleftrightarrow$
      nss x y False $\wedge$ *mult2-impl0* $(filter\ (\lambda x.\ \neg\ nss\ x\ y\ False)\ xs)\ (ys\ @\ ys')$ *True* $\vee$
      nss x y True $\wedge \neg$ *nss x y False* $\wedge$ *mult2-impl0 xs* $(ys\ @\ ys')$ *b* $\vee$
      mult2-ex-dom0 x xs ys $(y \# ys')$ *b*

end

lemma *mult2-impl0-sound*:

fixes *nss*
defines *ns* ≡ {(*x*, *y*). *nss x y True*} **and** *s* ≡ {(*x*, *y*). *nss x y False*}
shows *mult2-impl0 nss xs ys b* ⟷ (*mset xs*, *mset ys*) ∈ *mult2-alt b ns s*
  mult2-ex-dom0 nss x xs ys ys' b ⟷
    (∃ *y*. *y* ∈# *mset ys* ∧ (*x*, *y*) ∈ *s* ∧ (*mset* (*filter* (λ*x*. (*x*, *y*) ∉ *s*) *xs*), *mset* (*ys* @ *ys'*) − {#*y*#}) ∈ *mult2-alt True ns s*) ∨
    (∃ *y*. *y* ∈# *mset ys* ∧ (*x*, *y*) ∈ *ns* ∧ (*x*, *y*) ∉ *s* ∧ (*mset xs*, *mset* (*ys* @ *ys'*) − {#*y*#}) ∈ *mult2-alt b ns s*)
⟨*proof*⟩

  Now, instead of functions of type *bool* ⇒ *bool*, use pairs of type *bool* × *bool*

definition [*simp*]: *or2 a b* = (*fst a* ∨ *fst b*, *snd a* ∨ *snd b*)

context fixes *sns* :: ′*a* ⇒ ′*a* ⇒ *bool* × *bool*
begin

fun *mult2-impl* :: ′*a list* ⇒ ′*a list* ⇒ *bool* × *bool*
  and *mult2-ex-dom* :: ′*a* ⇒ ′*a list* ⇒ ′*a list* ⇒ ′*a list* ⇒ *bool* × *bool*
  where
    mult2-impl []   [] = (*False*, *True*)
  | *mult2-impl xs*   [] = (*False*, *False*)
  | *mult2-impl* []   *ys* = (*True*, *True*)
  | *mult2-impl* (*x* # *xs*) *ys* = *mult2-ex-dom x xs ys* []

  | *mult2-ex-dom x xs* []   *ys'* = (*False*, *False*)
  | *mult2-ex-dom x xs* (*y* # *ys*) *ys'* =
    (*case sns x y of*
      (*True*, -) ⇒ *if snd* (*mult2-impl* (*filter* (λ*x*. ¬ *fst* (*sns x y*)) *xs*) (*ys* @ *ys'*)) *then* (*True*, *True*)
                else mult2-ex-dom x xs ys (*y* # *ys'*)
    | (*False*, *True*) ⇒ *or2* (*mult2-impl xs* (*ys* @ *ys'*)) (*mult2-ex-dom x xs ys* (*y* # *ys'*))
    | - ⇒ *mult2-ex-dom x xs ys* (*y* # *ys'*))
end

lemma *mult2-impl-sound0*:
  defines *pair* ≡ λ*f*. (*f False*, *f True*) **and** *fun* ≡ λ*p b*. *if b then snd p else fst p*
  shows *mult2-impl sns xs ys* = *pair* (*mult2-impl0* (λ*x y*. *fun* (*sns x y*)) *xs ys*) (**is** *?P*)
    mult2-ex-dom sns x xs ys ys' = *pair* (*mult2-ex-dom0* (λ*x y*. *fun* (*sns x y*)) *x xs ys ys'*) (**is** *?Q*)
⟨*proof*⟩

lemmas *mult2-impl-sound* = *mult2-impl-sound0*(*1*)[*unfolded mult2-impl0-sound if-True if-False*]
end

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4 Multiset extension of order pairs in the other direction

Many term orders are formulated in the other direction, i.e., they use strong normalization of $>$ instead of well-foundedness of $<$. Here, we flip the direction of the multiset extension of two orders, connect it to existing interfaces, and prove some further properties of the multiset extension.

theory *Multiset-Extension2*
  imports
    List-Order
    Multiset-Extension-Pair
begin

4.1 List based characterization of *multpw*

lemma *multpw-listI*:
  assumes *length xs = length ys X = mset xs Y = mset ys*
    $\forall\,i.\ i < length\ ys \longrightarrow (xs\ !\ i,\ ys\ !\ i) \in ns$
  shows $(X,\ Y) \in multpw\ ns$
  ⟨*proof*⟩

lemma *multpw-listE*:
  assumes $(X,\ Y) \in multpw\ ns$
  obtains *xs ys* **where** *length xs = length ys X = mset xs Y = mset ys*
    $\forall\,i.\ i < length\ ys \longrightarrow (xs\ !\ i,\ ys\ !\ i) \in ns$
  ⟨*proof*⟩

4.2 Definition of the multiset extension of $>$-orders

We define here the non-strict extension of the order pair $(\geqslant, >)$ – usually written as (ns, s) in the sources – by just flipping the directions twice.

definition *ns-mul-ext* :: $'a\ rel \Rightarrow {}'a\ rel \Rightarrow {}'a\ multiset\ rel$
  where *ns-mul-ext ns s* $\equiv (mult2\text{-}alt\text{-}ns\ (ns^{-1})\ (s^{-1}))^{-1}$

lemma *ns-mul-extI*:
  assumes $A = A1 + A2$ **and** $B = B1 + B2$
    and $(A1,\ B1) \in multpw\ ns$
    and $\bigwedge b.\ b \in\#\ B2 \Longrightarrow \exists\,a.\ a \in\#\ A2 \wedge (a,\ b) \in s$
  shows $(A,\ B) \in ns\text{-}mul\text{-}ext\ ns\ s$
  ⟨*proof*⟩

lemma *ns-mul-extE*:
  assumes $(A,\ B) \in ns\text{-}mul\text{-}ext\ ns\ s$
  obtains *A1 A2 B1 B2* **where** $A = A1 + A2$ **and** $B = B1 + B2$
    and $(A1,\ B1) \in multpw\ ns$
    and $\bigwedge b.\ b \in\#\ B2 \Longrightarrow \exists\,a.\ a \in\#\ A2 \wedge (a,\ b) \in s$
  ⟨*proof*⟩

lemmas *ns-mul-extI-old = ns-mul-extI[OF - - multpw-listI[OF - refl refl], rule-format]*

Same for the "greater than" order on multisets.

definition *s-mul-ext* :: *'a rel ⇒ 'a rel ⇒ 'a multiset rel*
  where *s-mul-ext ns s ≡ (mult2-alt-s (ns^{-1}) (s^{-1}))$^{-1}$*

lemma *s-mul-extI*:
  assumes *A = A1 + A2* **and** *B = B1 + B2*
    and *(A1, B1) ∈ multpw ns*
    and *A2 ≠ {#}* **and** $\bigwedge$*b. b ∈# B2 ⟹ ∃a. a ∈# A2 ∧ (a, b) ∈ s*
  shows *(A, B) ∈ s-mul-ext ns s*
  ⟨*proof*⟩

lemma *s-mul-extE*:
  assumes *(A, B) ∈ s-mul-ext ns s*
  obtains *A1 A2 B1 B2* **where** *A = A1 + A2* **and** *B = B1 + B2*
    and *(A1, B1) ∈ multpw ns*
    and *A2 ≠ {#}* **and** $\bigwedge$*b. b ∈# B2 ⟹ ∃a. a ∈# A2 ∧ (a, b) ∈ s*
  ⟨*proof*⟩

lemmas *s-mul-extI-old = s-mul-extI[OF - - multpw-listI[OF - refl refl], rule-format]*

4.3 Basic properties

lemma *s-mul-ext-mono*:
  assumes *ns ⊆ ns' s ⊆ s'* **shows** *s-mul-ext ns s ⊆ s-mul-ext ns' s'*
  ⟨*proof*⟩

lemma *ns-mul-ext-mono*:
  assumes *ns ⊆ ns' s ⊆ s'* **shows** *ns-mul-ext ns s ⊆ ns-mul-ext ns' s'*
  ⟨*proof*⟩

lemma *s-mul-ext-local-mono*:
  assumes *sub*: *(set-mset xs × set-mset ys) ∩ ns ⊆ ns' (set-mset xs × set-mset ys)*
∩ *s ⊆ s'*
    and *rel*: *(xs,ys) ∈ s-mul-ext ns s*
  shows *(xs,ys) ∈ s-mul-ext ns' s'*
  ⟨*proof*⟩

lemma *ns-mul-ext-local-mono*:
  assumes *sub*: *(set-mset xs × set-mset ys) ∩ ns ⊆ ns' (set-mset xs × set-mset ys)*
∩ *s ⊆ s'*
    and *rel*: *(xs,ys) ∈ ns-mul-ext ns s*
  shows *(xs,ys) ∈ ns-mul-ext ns' s'*
  ⟨*proof*⟩

lemma *s-mul-ext-ord-s [mono]*:
  assumes $\bigwedge$*s t. ord s t ⟶ ord' s t*
  shows *(s, t) ∈ s-mul-ext ns {(s,t). ord s t} ⟶ (s, t) ∈ s-mul-ext ns {(s,t). ord'
s t}*

⟨*proof*⟩

lemma *ns-mul-ext-ord-s* [*mono*]:
  assumes ⋀*s t. ord s t* ⟶ *ord′ s t*
  shows (*s, t*) ∈ *ns-mul-ext ns* {(*s,t*). *ord s t*} ⟶ (*s, t*) ∈ *ns-mul-ext ns* {(*s,t*).
ord′ s t}
  ⟨*proof*⟩

  The empty multiset is the minimal element for these orders

lemma *ns-mul-ext-bottom*: (*A*,{#}) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-mul-ext-bottom-uniqueness*:
  assumes ({#},*A*) ∈ *ns-mul-ext ns s*
  shows *A* = {#}
  ⟨*proof*⟩

lemma *ns-mul-ext-bottom2*:
  assumes (*A,B*) ∈ *ns-mul-ext ns s*
    and *B* ≠ {#}
  shows *A* ≠ {#}
  ⟨*proof*⟩

lemma *s-mul-ext-bottom*:
  assumes *A* ≠ {#}
  shows (*A*,{#}) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

lemma *s-mul-ext-bottom-strict*:
  ({#},*A*) ∉ *s-mul-ext ns s*
  ⟨*proof*⟩

  Obvious introduction rules.

lemma *all-ns-ns-mul-ext*:
  assumes *length as* = *length bs*
    and ∀ *i. i* < *length bs* ⟶ (*as* ! *i, bs* ! *i*) ∈ *ns*
  shows (*mset as, mset bs*) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *all-s-s-mul-ext*:
  assumes *A* ≠ {#}
    and ∀ *b. b* ∈# *B* ⟶ (∃ *a. a* ∈# *A* ∧ (*a,b*) ∈ *s*)
  shows (*A, B*) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

  Being stricly lesser than implies being lesser than

lemma *s-ns-mul-ext*:
  assumes (*A, B*) ∈ *s-mul-ext ns s*
  shows (*A, B*) ∈ *ns-mul-ext ns s*

⟨*proof*⟩

The non-strict order is reflexive.

lemma *multpw-refl′*:
  assumes *locally-refl ns A*
  shows (*A, A*) ∈ *multpw ns*
⟨*proof*⟩

lemma *ns-mul-ext-refl-local*:
  assumes *locally-refl ns A*
  shows (*A, A*) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-mul-ext-refl*:
  assumes *refl ns*
  shows (*A, A*) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

The orders are union-compatible

lemma *ns-s-mul-ext-union-multiset-l*:
  assumes (*A, B*) ∈ *ns-mul-ext ns s*
    and *C* ≠ {#}
    and ∀ *d. d* ∈# *D* ⟶ (∃ *c. c* ∈# *C* ∧ (*c,d*) ∈ *s*)
  shows (*A + C, B + D*) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

lemma *s-mul-ext-union-compat*:
  assumes (*A, B*) ∈ *s-mul-ext ns s*
    and *locally-refl ns C*
  shows (*A + C, B + C*) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-mul-ext-union-compat*:
  assumes (*A, B*) ∈ *ns-mul-ext ns s*
    and *locally-refl ns C*
  shows (*A + C, B + C*) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

context
  fixes *NS* :: ′*a rel*
  assumes *NS*: *refl NS*
begin

lemma *refl-imp-locally-refl*: *locally-refl NS A* ⟨*proof*⟩

lemma *supseteq-imp-ns-mul-ext*:
  assumes *A* ⊇# *B*
  shows (*A, B*) ∈ *ns-mul-ext NS S*
  ⟨*proof*⟩

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lemma *supset-imp-s-mul-ext*:
  assumes $A \supset\# B$
  shows $(A, B) \in$ *s-mul-ext NS S*
  $\langle proof \rangle$

end

definition *mul-ext* :: $('a \Rightarrow 'a \Rightarrow bool \times bool) \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool \times bool$
  where *mul-ext f xs ys* $\equiv$ *let* $s = \{(x,y).\ fst\ (f\ x\ y)\}$; $ns = \{(x,y).\ snd\ (f\ x\ y)\}$
  in $((mset\ xs, mset\ ys) \in$ *s-mul-ext ns s*, $(mset\ xs,\ mset\ ys) \in$ *ns-mul-ext ns s*$)$

definition *smulextp f m n* $\longleftrightarrow (m, n) \in$ *s-mul-ext* $\{(x, y).\ snd\ (f\ x\ y)\}$ $\{(x, y).$
fst $(f\ x\ y)\}$
definition *nsmulextp f m n* $\longleftrightarrow (m, n) \in$ *ns-mul-ext* $\{(x, y).\ snd\ (f\ x\ y)\}$ $\{(x, y).$
fst $(f\ x\ y)\}$

lemma *smulextp-cong*[*fundef-cong*]:
  assumes *xs1 = ys1*
    and *xs2 = ys2*
    and $\bigwedge x\ x'.\ x \in\# ys1 \Longrightarrow x' \in\# ys2 \Longrightarrow f\ x\ x' = g\ x\ x'$
  shows *smulextp f xs1 xs2 = smulextp g ys1 ys2*
  $\langle proof \rangle$

lemma *nsmulextp-cong*[*fundef-cong*]:
  assumes *xs1 = ys1*
    and *xs2 = ys2*
    and $\bigwedge x\ x'.\ x \in\# ys1 \Longrightarrow x' \in\# ys2 \Longrightarrow f\ x\ x' = g\ x\ x'$
  shows *nsmulextp f xs1 xs2 = nsmulextp g ys1 ys2*
  $\langle proof \rangle$

definition *mulextp f m n = (smulextp f m n, nsmulextp f m n)*

lemma *mulextp-cong*[*fundef-cong*]:
  assumes *xs1 = ys1*
    and *xs2 = ys2*
    and $\bigwedge x\ x'.\ x \in\# ys1 \Longrightarrow x' \in\# ys2 \Longrightarrow f\ x\ x' = g\ x\ x'$
  shows *mulextp f xs1 xs2 = mulextp g ys1 ys2*
  $\langle proof \rangle$

lemma *mset-s-mul-ext*:
  $(mset\ xs,\ mset\ ys) \in$ *s-mul-ext* $\{(x, y).\ snd\ (f\ x\ y)\}$ $\{(x, y).fst\ (f\ x\ y)\} \longleftrightarrow$
    fst (*mul-ext f xs ys*)
  $\langle proof \rangle$

lemma *mset-ns-mul-ext*:
  $(mset\ xs,\ mset\ ys) \in$ *ns-mul-ext* $\{(x, y).\ snd\ (f\ x\ y)\}$ $\{(x, y).fst\ (f\ x\ y)\} \longleftrightarrow$
    snd (*mul-ext f xs ys*)

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⟨*proof*⟩

lemma *smulextp-mset-code*:
  smulextp f (*mset xs*) (*mset ys*) ⟷ *fst* (*mul-ext f xs ys*)
  ⟨*proof*⟩

lemma *nsmulextp-mset-code*:
  nsmulextp f (*mset xs*) (*mset ys*) ⟷ *snd* (*mul-ext f xs ys*)
  ⟨*proof*⟩

lemma *nstri-mul-ext-map*:
  assumes $\bigwedge$*s t. s* ∈ *set ss* ⟹ *t* ∈ *set ts* ⟹ *fst* (*order s t*) ⟹ *fst* (*order′* (*f s*)
(*f t*))
    and $\bigwedge$*s t. s* ∈ *set ss* ⟹ *t* ∈ *set ts* ⟹ *snd* (*order s t*) ⟹ *snd* (*order′* (*f s*) (*f
t*))
    and *snd* (*mul-ext order ss ts*)
  shows *snd* (*mul-ext order′* (*map f ss*) (*map f ts*))
  ⟨*proof*⟩

lemma *stri-mul-ext-map*:
  assumes $\bigwedge$*s t. s* ∈ *set ss* ⟹ *t* ∈ *set ts* ⟹ *fst* (*order s t*) ⟹ *fst* (*order′* (*f s*)
(*f t*))
    and $\bigwedge$*s t. s* ∈ *set ss* ⟹ *t* ∈ *set ts* ⟹ *snd* (*order s t*) ⟹ *snd* (*order′* (*f s*) (*f
t*))
    and *fst* (*mul-ext order ss ts*)
  shows *fst* (*mul-ext order′* (*map f ss*) (*map f ts*))
  ⟨*proof*⟩

lemma *mul-ext-arg-empty*: *snd* (*mul-ext f* [] *xs*) ⟹ *xs* = []
  ⟨*proof*⟩

  The non-strict order is irreflexive

lemma *s-mul-ext-irrefl*: **assumes** *irr*: *irrefl-on* (*set-mset A*) *S*
  and *S-NS*: *S* ⊆ *NS*
  and *compat*: *S O NS* ⊆ *S*
shows (*A,A*) ∉ *s-mul-ext NS S* ⟨*proof*⟩

lemma *mul-ext-irrefl*: **assumes** $\bigwedge$ *x. x* ∈ *set xs* ⟹ ¬ *fst* (*rel x x*)
  and $\bigwedge$ *x y z. fst* (*rel x y*) ⟹ *snd* (*rel y z*) ⟹ *fst* (*rel x z*)
  and $\bigwedge$ *x y. fst* (*rel x y*) ⟹ *snd* (*rel x y*)
shows ¬ *fst* (*mul-ext rel xs xs*)
  ⟨*proof*⟩

  The non-strict order is transitive.

lemma *ns-mul-ext-trans*:
  assumes *trans s trans ns compatible-l ns s compatible-r ns s refl ns*
    and (*A, B*) ∈ *ns-mul-ext ns s*
    and (*B, C*) ∈ *ns-mul-ext ns s*

shows $(A, C) \in$ *ns-mul-ext ns s*
⟨*proof*⟩

The strict order is trans.

lemma *s-mul-ext-trans*:
  assumes *trans s trans ns compatible-l ns s compatible-r ns s refl ns*
    and $(A, B) \in$ *s-mul-ext ns s*
    and $(B, C) \in$ *s-mul-ext ns s*
  shows $(A, C) \in$ *s-mul-ext ns s*
⟨*proof*⟩

The strict order is compatible on the left with the non strict one

lemma *s-ns-mul-ext-trans*:
  assumes *trans s trans ns compatible-l ns s compatible-r ns s refl ns*
    and $(A, B) \in$ *s-mul-ext ns s*
    and $(B, C) \in$ *ns-mul-ext ns s*
  shows $(A, C) \in$ *s-mul-ext ns s*
⟨*proof*⟩

The strict order is compatible on the right with the non-strict one.

lemma *ns-s-mul-ext-trans*:
  assumes *trans s trans ns compatible-l ns s compatible-r ns s refl ns*
    and $(A, B) \in$ *ns-mul-ext ns s*
    and $(B, C) \in$ *s-mul-ext ns s*
  shows $(A, C) \in$ *s-mul-ext ns s*
⟨*proof*⟩

s-mul-ext is strongly normalizing

lemma *SN-s-mul-ext-strong*:
  assumes *order-pair s ns*
    and $\forall y.\ y \in\# M \longrightarrow$ *SN-on s* $\{y\}$
  shows *SN-on* (*s-mul-ext ns s*) $\{M\}$
⟨*proof*⟩

lemma *SN-s-mul-ext*:
  assumes *order-pair s ns SN s*
  shows *SN* (*s-mul-ext ns s*)
⟨*proof*⟩

lemma (**in** *order-pair*) *mul-ext-order-pair*:
  order-pair (*s-mul-ext NS S*) (*ns-mul-ext NS S*) (**is** *order-pair ?S ?NS*)
⟨*proof*⟩

lemma (**in** *SN-order-pair*) *mul-ext-SN-order-pair*: *SN-order-pair* (*s-mul-ext NS S*)
(*ns-mul-ext NS S*)
  (**is** *SN-order-pair ?S ?NS*)
⟨*proof*⟩

lemma *mul-ext-compat*:

assumes *compat*: $\bigwedge$ *s t u.* $\llbracket s \in set\ ss;\ t \in set\ ts;\ u \in set\ us \rrbracket \Longrightarrow$
  (snd (f s t) $\wedge$ fst (f t u) $\longrightarrow$ fst (f s u)) $\wedge$
  (fst (f s t) $\wedge$ snd (f t u) $\longrightarrow$ fst (f s u)) $\wedge$
  (snd (f s t) $\wedge$ snd (f t u) $\longrightarrow$ snd (f s u)) $\wedge$
  (fst (f s t) $\wedge$ fst (f t u) $\longrightarrow$ fst (f s u))
shows
  (snd (mul-ext f ss ts) $\wedge$ fst (mul-ext f ts us) $\longrightarrow$ fst (mul-ext f ss us)) $\wedge$
  (fst (mul-ext f ss ts) $\wedge$ snd (mul-ext f ts us) $\longrightarrow$ fst (mul-ext f ss us)) $\wedge$
  (snd (mul-ext f ss ts) $\wedge$ snd (mul-ext f ts us) $\longrightarrow$ snd (mul-ext f ss us)) $\wedge$
  (fst (mul-ext f ss ts) $\wedge$ fst (mul-ext f ts us) $\longrightarrow$ fst (mul-ext f ss us))
$\langle proof \rangle$

lemma *mul-ext-cong*[*fundef-cong*]:
  assumes *mset xs1 = mset ys1*
    and *mset xs2 = mset ys2*
    and $\bigwedge$ *x x'. x $\in$ set ys1 $\Longrightarrow$ x' $\in$ set ys2 $\Longrightarrow$ f x x' = g x x'*
  shows *mul-ext f xs1 xs2 = mul-ext g ys1 ys2*
  $\langle proof \rangle$

lemma *all-nstri-imp-mul-nstri*:
  assumes $\forall\ i <$ *length ys. snd (f (xs ! i) (ys ! i))*
    and *length xs = length ys*
  shows *snd (mul-ext f xs ys)*
$\langle proof \rangle$

lemma *relation-inter*:
  shows $\{(x,y).\ P\ x\ y\} \cap \{(x,y).\ Q\ x\ y\} = \{(x,y).\ P\ x\ y \wedge Q\ x\ y\}$
  $\langle proof \rangle$

lemma *mul-ext-unfold*:
  $(x,y) \in \{(a,b).\ fst\ (mul\text{-}ext\ g\ a\ b)\} \longleftrightarrow (mset\ x,\ mset\ y) \in (s\text{-}mul\text{-}ext\ \{(a,b).\ snd$
$(g\ a\ b)\}\ \{(a,b).\ fst\ (g\ a\ b)\})$
  $\langle proof \rangle$

The next lemma is a local version of strong-normalization of the multiset extension, where the base-order only has to be strongly normalizing on elements of the multisets. This will be crucial for orders that are defined recursively on terms, such as RPO or WPO.

lemma *mul-ext-SN*:
  assumes $\forall\ x.\ snd\ (g\ x\ x)$
    and $\forall\ x\ y\ z.\ fst\ (g\ x\ y) \longrightarrow snd\ (g\ y\ z) \longrightarrow fst\ (g\ x\ z)$
    and $\forall\ x\ y\ z.\ snd\ (g\ x\ y) \longrightarrow fst\ (g\ y\ z) \longrightarrow fst\ (g\ x\ z)$
    and $\forall\ x\ y\ z.\ snd\ (g\ x\ y) \longrightarrow snd\ (g\ y\ z) \longrightarrow snd\ (g\ x\ z)$
    and $\forall\ x\ y\ z.\ fst\ (g\ x\ y) \longrightarrow fst\ (g\ y\ z) \longrightarrow fst\ (g\ x\ z)$
  shows *SN* $\{(ys,\ xs).$
  $(\forall\ y \in set\ ys.\ SN\text{-}on\ \{(s\ ,t).\ fst\ (g\ s\ t)\}\ \{y\}) \wedge$
  fst (mul-ext g ys xs)$\}$
$\langle proof \rangle$

lemma *mul-ext-stri-imp-nstri*:
  assumes *fst* (*mul-ext f as bs*)
  shows *snd* (*mul-ext f as bs*)
  ⟨*proof*⟩

lemma *ns-ns-mul-ext-union-compat*:
  assumes (*A,B*) ∈ *ns-mul-ext ns s*
    and (*C,D*) ∈ *ns-mul-ext ns s*
  shows (*A + C, B + D*) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *s-ns-mul-ext-union-compat*:
  assumes (*A,B*) ∈ *s-mul-ext ns s*
    and (*C,D*) ∈ *ns-mul-ext ns s*
  shows (*A + C, B + D*) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-ns-mul-ext-union-compat-rtrancl*: **assumes** *refl*: *refl ns*
  and *AB*: (*A, B*) ∈ (*ns-mul-ext ns s*)*
  and *CD*: (*C, D*) ∈ (*ns-mul-ext ns s*)*
shows (*A + C, B + D*) ∈ (*ns-mul-ext ns s*)*
⟨*proof*⟩

4.4 Multisets as order on lists

interpretation *mul-ext-list*: *list-order-extension*
  λ*s ns.* {(*as, bs*). (*mset as, mset bs*) ∈ *s-mul-ext ns s*}
  λ*s ns.* {(*as, bs*). (*mset as, mset bs*) ∈ *ns-mul-ext ns s*}
⟨*proof*⟩

lemma *s-mul-ext-singleton* [*simp, intro*]:
  assumes (*a, b*) ∈ *s*
  shows ({#*a*#}, {#*b*#}) ∈ *s-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-mul-ext-singleton* [*simp, intro*]:
  (*a, b*) ∈ *ns* ⟹ ({#*a*#}, {#*b*#}) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *ns-mul-ext-singleton2*:
  (*a, b*) ∈ *s* ⟹ ({#*a*#}, {#*b*#}) ∈ *ns-mul-ext ns s*
  ⟨*proof*⟩

lemma *s-mul-ext-self-extend-left*:
  assumes *A* ≠ {#} **and** *locally-refl W B*
  shows (*A + B, B*) ∈ *s-mul-ext W S*
⟨*proof*⟩

lemma *s-mul-ext-ne-extend-left*:

assumes $A \neq \{\#\}$ **and** $(B, C) \in$ *ns-mul-ext W S*
shows $(A + B, C) \in$ *s-mul-ext W S*
⟨*proof*⟩

lemma *s-mul-ext-extend-left*:
  assumes $(B, C) \in$ *s-mul-ext W S*
  shows $(A + B, C) \in$ *s-mul-ext W S*
  ⟨*proof*⟩

lemma *mul-ext-mono*:
  assumes $\bigwedge x\ y.$ ⟦$x \in set\ xs;\ y \in set\ ys;\ fst\ (P\ x\ y)$⟧ $\Longrightarrow fst\ (P'\ x\ y)$
    and $\bigwedge x\ y.$ ⟦$x \in set\ xs;\ y \in set\ ys;\ snd\ (P\ x\ y)$⟧ $\Longrightarrow snd\ (P'\ x\ y)$
  shows
    $fst\ (mul\text{-}ext\ P\ xs\ ys) \Longrightarrow fst\ (mul\text{-}ext\ P'\ xs\ ys)$
    $snd\ (mul\text{-}ext\ P\ xs\ ys) \Longrightarrow snd\ (mul\text{-}ext\ P'\ xs\ ys)$
  ⟨*proof*⟩

4.5 Special case: non-strict order is equality

lemma *ns-mul-ext-IdE*:
  assumes $(M, N) \in$ *ns-mul-ext Id R*
  obtains X **and** Y **and** Z **where** $M = X + Z$ **and** $N = Y + Z$
    and $\forall y \in set\text{-}mset\ Y.\ \exists x \in set\text{-}mset\ X.\ (x, y) \in R$
  ⟨*proof*⟩

lemma *ns-mul-ext-IdI*:
  assumes $M = X + Z$ **and** $N = Y + Z$ **and** $\forall y \in set\text{-}mset\ Y.\ \exists x \in set\text{-}mset$
$X.\ (x, y) \in R$
  shows $(M, N) \in$ *ns-mul-ext Id R*
  ⟨*proof*⟩

lemma *s-mul-ext-IdE*:
  assumes $(M, N) \in$ *s-mul-ext Id R*
  obtains X **and** Y **and** Z **where** $X \neq \{\#\}$ **and** $M = X + Z$ **and** $N = Y + Z$
    and $\forall y \in set\text{-}mset\ Y.\ \exists x \in set\text{-}mset\ X.\ (x, y) \in R$
  ⟨*proof*⟩

lemma *s-mul-ext-IdI*:
  assumes $X \neq \{\#\}$ **and** $M = X + Z$ **and** $N = Y + Z$
    and $\forall y \in set\text{-}mset\ Y.\ \exists x \in set\text{-}mset\ X.\ (x, y) \in R$
  shows $(M, N) \in$ *s-mul-ext Id R*
  ⟨*proof*⟩

lemma *mult-s-mul-ext-conv*:
  assumes *trans R*
  shows $(mult\ (R^{-1}))^{-1} =$ *s-mul-ext Id R*
  ⟨*proof*⟩

lemma *ns-mul-ext-Id-eq*:

ns-mul-ext Id R = (s-mul-ext Id R)$^=$
⟨*proof*⟩

lemma *subseteq-mset-imp-ns-mul-ext-Id*:
  assumes *A ⊆# B*
  shows *(B, A) ∈ ns-mul-ext Id R*
⟨*proof*⟩

lemma *subset-mset-imp-s-mul-ext-Id*:
  assumes *A ⊂# B*
  shows *(B, A) ∈ s-mul-ext Id R*
  ⟨*proof*⟩

end

4.6   Executable version

theory *Multiset-Extension2-Impl*
imports
  HOL−Library.DAList-Multiset
  List-Order
  Multiset-Extension2
  Multiset-Extension-Pair-Impl
begin

lemma *mul-ext-list-ext*: ∃ *s ns. list-order-extension-impl s ns mul-ext*
⟨*proof*⟩

context fixes *sns* :: *$'a$ ⇒ $'a$ ⇒ bool × bool*
begin

fun *mul-ext-impl* :: *$'a$ list ⇒ $'a$ list ⇒ bool × bool*
and *mul-ex-dom* :: *$'a$ list ⇒ $'a$ list ⇒ $'a$ ⇒ $'a$ list ⇒ bool × bool*
where
  mul-ext-impl [] []   *= (False, True)*
| *mul-ext-impl [] ys*   *= (False, False)*
| *mul-ext-impl xs []*   *= (True, True)*
| *mul-ext-impl xs (y # ys) = mul-ex-dom xs [] y ys*

| *mul-ex-dom []*   *xs′ y ys = (False, False)*
| *mul-ex-dom (x # xs) xs′ y ys =*
    (case sns x y of
       (True, -) ⇒ if snd (mul-ext-impl (xs @ xs′) (filter (λy. ¬ fst (sns x y)) ys))
then (True, True)
                else mul-ex-dom xs (x # xs′) y ys
    | (False, True) ⇒ or2 (mul-ext-impl (xs @ xs′) ys) (mul-ex-dom xs (x # xs′) y

ys)
  | - ⇒ mul-ex-dom xs (x # xs') y ys)

end

context
begin
lemma *mul-ext-impl-sound0*:
  mul-ext-impl sns xs ys = mult2-impl (λx y. sns y x) ys xs
  mul-ex-dom sns xs xs' y ys = mult2-ex-dom (λx y. sns y x) y ys xs xs'
⟨*proof*⟩ **definition** *cond1* **where**
  cond1 f bs y xs ys ≡
  ((∃ b. b ∈ set bs ∧ fst (f b y) ∧ snd (mul-ext f (remove1 b xs) [y←ys . ¬ fst (f b y)]))
y)]))
  ∨ (∃ b. b ∈ set bs ∧ snd (f b y) ∧ fst (mul-ext f (remove1 b xs) ys)))

private lemma *cond1-propagate*:
  assumes *cond1 f bs y xs ys*
  shows *cond1 f (b # bs) y xs ys*
⟨*proof*⟩ **definition** *cond2* **where**
  cond2 f bs y xs ys ≡ (cond1 f bs y xs ys
  ∨ (∃ b. b ∈ set bs ∧ snd (f b y) ∧ snd (mul-ext f (remove1 b xs) ys)))

private lemma *cond2-propagate*:
  assumes *cond2 f bs y xs ys*
  shows *cond2 f (b # bs) y xs ys*
⟨*proof*⟩ **lemma** *cond1-cond2*:
  assumes *cond1 f bs y xs ys*
  shows *cond2 f bs y xs ys*
⟨*proof*⟩

lemma *mul-ext-impl-sound*:
  shows *mul-ext-impl f xs ys = mul-ext f xs ys*
⟨*proof*⟩

lemma *mul-ext-code* [*code*]: *mul-ext = mul-ext-impl*
  ⟨*proof*⟩

lemma *mul-ext-impl-cong*[*fundef-cong*]:
  assumes ⋀x x'. x ∈ set xs ⟹ x' ∈ set ys ⟹ f x x' = g x x'
  shows *mul-ext-impl f xs ys = mul-ext-impl g xs ys*
⟨*proof*⟩
end

fun *ass-list-to-single-list* :: *('a × nat) list ⇒ 'a list*
  where
    ass-list-to-single-list [] = []
  | ass-list-to-single-list ((x, n) # xs) = replicate n x @ ass-list-to-single-list xs

lemma *set-ass-list-to-single-list* [*simp*]:
  set (*ass-list-to-single-list xs*) = {*x*. ∃ *n*. (*x*, *n*) ∈ *set xs* ∧ *n* > *0*}
  ⟨*proof*⟩

lemma *count-mset-replicate* [*simp*]:
  count (*mset* (*replicate n x*)) *x* = *n*
  ⟨*proof*⟩

lemma *count-mset-lal-ge*:
  (*x*, *n*) ∈ *set xs* ⟹ *count* (*mset* (*ass-list-to-single-list xs*)) *x* ≥ *n*
  ⟨*proof*⟩

lemma *count-of-count-mset-lal* [*simp*]:
  distinct (*map fst y*) ⟹ *count-of y x* = *count* (*mset* (*ass-list-to-single-list y*)) *x*
  ⟨*proof*⟩

lemma *Bag-mset*: *Bag xs* = *mset* (*ass-list-to-single-list* (*DAList.impl-of xs*))
  ⟨*proof*⟩

lemma *Bag-Alist-Cons*:
  x ∉ *fst* ' *set xs* ⟹ *distinct* (*map fst xs*) ⟹
    Bag (*Alist* ((*x*, *n*) # *xs*)) = *mset* (*replicate n x*) + *Bag* (*Alist xs*)
  ⟨*proof*⟩

lemma *mset-lal* [*simp*]:
  distinct (*map fst xs*) ⟹ *mset* (*ass-list-to-single-list xs*) = *Bag* (*Alist xs*)
  ⟨*proof*⟩

lemma *Bag-s-mul-ext*:
  (*Bag xs*, *Bag ys*) ∈ *s-mul-ext* {(*x*, *y*). *snd* (*f x y*)} {(*x*, *y*). *fst* (*f x y*)} ⟷
    fst (*mul-ext f* (*ass-list-to-single-list* (*DAList.impl-of xs*)) (*ass-list-to-single-list*
  (*DAList.impl-of ys*)))
  ⟨*proof*⟩

lemma *Bag-ns-mul-ext*:
  (*Bag xs*, *Bag ys*) ∈ *ns-mul-ext* {(*x*, *y*). *snd* (*f x y*)} {(*x*, *y*). *fst* (*f x y*)} ⟷
    snd (*mul-ext f* (*ass-list-to-single-list* (*DAList.impl-of xs*)) (*ass-list-to-single-list*
  (*DAList.impl-of ys*)))
  ⟨*proof*⟩

lemma *smulextp-code*[*code*]:
  smulextp f (*Bag xs*) (*Bag ys*) ⟷ *fst* (*mul-ext f* (*ass-list-to-single-list* (*DAList.impl-of*
  xs)) (*ass-list-to-single-list* (*DAList.impl-of ys*)))
  ⟨*proof*⟩

lemma *nsmulextp-code*[*code*]:
  nsmulextp f (*Bag xs*) (*Bag ys*) ⟷ *snd* (*mul-ext f* (*ass-list-to-single-list* (*DAList.impl-of*
  xs)) (*ass-list-to-single-list* (*DAList.impl-of ys*)))
  ⟨*proof*⟩

lemma *mulextp-code*[*code*]:
　mulextp f (*Bag xs*) (*Bag ys*) = *mul-ext f* (*ass-list-to-single-list* (*DAList.impl-of xs*)) (*ass-list-to-single-list* (*DAList.impl-of ys*))
　⟨*proof*⟩

end

5　The Weighted Path Order

This is a version of WPO that also permits multiset comparisons of lists of terms. It therefore generalizes RPO.

theory *WPO*
　imports
　　Knuth-Bendix-Order.Lexicographic-Extension
　　First-Order-Terms.Subterm-and-Context
　　Knuth-Bendix-Order.Order-Pair
　　Polynomial-Factorization.Missing-List
　　Status
　　Precedence
　　Multiset-Extension2
　　HOL.Zorn
begin

datatype *order-tag = Lex | Mul*

locale *wpo* =
　fixes *n* :: *nat*
　　and *S NS* :: (′*f*, ′*v*) *term rel*
　　and *prc* :: (′*f* × *nat* ⇒ ′*f* × *nat* ⇒ *bool* × *bool*)
　　and *prl* :: ′*f* × *nat* ⇒ *bool*
　　and *σσ* :: ′*f status*
　　and *c* :: ′*f* × *nat* ⇒ *order-tag*
　　and *ssimple* :: *bool*
　　and *large* :: ′*f* × *nat* ⇒ *bool*
begin

fun *wpo* :: (′*f*, ′*v*) *term* ⇒ (′*f*, ′*v*) *term* ⇒ *bool* × *bool*
　where
　　wpo s t = (*if* (*s,t*) ∈ *S then* (*True*, *True*) *else*
　　　if (*s,t*) ∈ *NS then* (*case s of*
　　Var x ⇒ (*False*,
　　　(*case t of*
　　　　Var y ⇒ *x* = *y*
　　　| *Fun g ts* ⇒ *status σσ* (*g*, *length ts*) = [] ∧ *prl* (*g*, *length ts*)))
　　| *Fun f ss* ⇒
　　　if ∃ *i* ∈ *set* (*status σσ* (*f*, *length ss*)). *snd* (*wpo* (*ss* ! *i*) *t*) *then* (*True*, *True*)
　　　else

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```
          (case t of
             Var - ⇒ (False, ssimple ∧ large (f, length ss))
           | Fun g ts ⇒
             (case prc (f, length ss) (g, length ts) of (prs, prns) ⇒
              if prns ∧ (∀ j ∈ set (status σσ (g, length ts)). fst (wpo s (ts ! j))) then
                if prs then (True, True)
                else let ss' = map (λ i. ss ! i) (status σσ (f, length ss));
                         ts' = map (λ i. ts ! i) (status σσ (g, length ts));
                         cf = c (f,length ss);
                         cg = c (g,length ts)
                     in if cf = Lex ∧ cg = Lex
                        then lex-ext wpo n ss' ts'
                        else if cf = Mul ∧ cg = Mul
                            then mul-ext wpo ss' ts'
                            else (length ss' ≠ 0 ∧ length ts' = 0, length ts' = 0)
              else (False, False))))
     else (False, False))
```

declare *wpo.simps* [*simp del*]

abbreviation *wpo-s* (**infix** ≻ *50*) **where** $s \succ t \equiv$ *fst* (*wpo s t*)
abbreviation *wpo-ns* (**infix** ⪰ *50*) **where** $s \succeq t \equiv$ *snd* (*wpo s t*)

abbreviation *WPO-S* ≡ {(*s,t*). $s \succ t$}
abbreviation *WPO-NS* ≡ {(*s,t*). $s \succeq t$}

lemma *wpo-s-imp-ns*: $s \succ t \implies s \succeq t$
⟨*proof*⟩

lemma *S-imp-wpo-s*: (*s,t*) ∈ *S* $\implies s \succ t$ ⟨*proof*⟩

end

declare *wpo.wpo.simps*[*code*]

definition *strictly-simple-status* :: ′*f status* ⇒ (′*f,*′*v*)*term rel* ⇒ *bool* **where**
  strictly-simple-status σ *rel* =
    (∀ *f ts i. i* ∈ *set* (*status* σ (*f,length ts*)) ⟶ (*Fun f ts, ts ! i*) ∈ *rel*)

definition *trans-precedence* **where** *trans-precedence prc* = (∀ *f g h.*
  (*fst* (*prc f g*) ⟶ *snd* (*prc g h*) ⟶ *fst* (*prc f h*)) ∧
  (*snd* (*prc f g*) ⟶ *fst* (*prc g h*) ⟶ *fst* (*prc f h*)) ∧
  (*snd* (*prc f g*) ⟶ *snd* (*prc g h*) ⟶ *snd* (*prc f h*)))

locale *wpo-with-basic-assms* = *wpo* +
  order-pair + *irrefl-precedence* +

constrains $S :: ('f, 'v)$ *term rel* **and** $NS ::$ -
  and *prc* $:: 'f \times nat \Rightarrow 'f \times nat \Rightarrow bool \times bool$
  and *prl* $:: 'f \times nat \Rightarrow bool$
  and *ssimple* $:: bool$
  and *large* $:: 'f \times nat \Rightarrow bool$
  and $c :: 'f \times nat \Rightarrow$ *order-tag*
  and $n :: nat$
  and $\sigma\sigma :: 'f$ *status*
 assumes *subst-S*: $(s,t) \in S \Longrightarrow (s \cdot \sigma, t \cdot \sigma) \in S$
  and *subst-NS*: $(s,t) \in NS \Longrightarrow (s \cdot \sigma, t \cdot \sigma) \in NS$
  and *irrefl-S*: *irrefl* S
  and *S-imp-NS*: $S \subseteq NS$
  and *ss-status*: *ssimple* $\Longrightarrow i \in set\ (status\ \sigma\sigma\ fn) \Longrightarrow$ *simple-arg-pos* $S\ fn\ i$
  and *large*: *ssimple* $\Longrightarrow$ *large* $fn \Longrightarrow fst\ (prc\ fn\ gm) \lor snd\ (prc\ fn\ gm) \land status$
$\sigma\sigma\ gm = []$
  and *large-trans*: *ssimple* $\Longrightarrow$ *large* $fn \Longrightarrow snd\ (prc\ gm\ fn) \Longrightarrow$ *large* gm
  and *ss-S-non-empty*: *ssimple* $\Longrightarrow S \neq \{\}$
begin
abbreviation $\sigma \equiv status\ \sigma\sigma$

lemma *ss-NS-not-UNIV*: *ssimple* $\Longrightarrow NS \neq UNIV$
$\langle proof \rangle$

lemmas $\sigma = status[of\ \sigma\sigma]$
lemma σE: $i \in set\ (\sigma\ (f, length\ ss)) \Longrightarrow ss\ !\ i \in set\ ss\ \langle proof \rangle$

lemma *wpo-ns-imp-NS*: $s \succeq t \Longrightarrow (s,t) \in NS$
 $\langle proof \rangle$

lemma *wpo-s-imp-NS*: $s \succ t \Longrightarrow (s,t) \in NS$
 $\langle proof \rangle$

lemma *wpo-least-1*: **assumes** *prl* $(f,length\ ss)$
 and $(t, Fun\ f\ ss) \in NS$
 and $\sigma\ (f,length\ ss) = []$
shows $t \succeq Fun\ f\ ss$
$\langle proof \rangle$

lemma *wpo-least-2*: **assumes** *prl* $(f,length\ ss)$ (**is** *prl* ?f)
 and $(Fun\ f\ ss, t) \notin S$
 and $\sigma\ (f,length\ ss) = []$
shows $\neg\ Fun\ f\ ss \succ t$
$\langle proof \rangle$

lemma *wpo-least-3*: **assumes** *prl* $(f,length\ ss)$ (**is** *prl* ?f)
 and *ns*: $Fun\ f\ ss \succeq t$
 and *NS*: $(u, Fun\ f\ ss) \in NS$
 and *ss*: $\sigma\ (f,length\ ss) = []$
 and *S*: $\bigwedge x.\ (Fun\ f\ ss, x) \notin S$

and *u*: *u = Var x*
shows $u \succeq t$
$\langle proof \rangle$

lemma *wpo-compat*: $(s \succeq t \wedge t \succ u \longrightarrow s \succ u) \wedge$
  $(s \succ t \wedge t \succeq u \longrightarrow s \succ u) \wedge$
  $(s \succeq t \wedge t \succeq u \longrightarrow s \succeq u)$ (**is** *?tran s t u*)
$\langle proof \rangle$

context
  assumes *ssimple*: *strictly-simple-status* $\sigma\sigma$ *NS*
begin
lemma *NS-arg'*:
  assumes *i*: $i \in set\ (\sigma\ (f,length\ ts))$
  shows $(Fun\ f\ ts,\ ts\ !\ i) \in NS$
  $\langle proof \rangle$

lemma *wpo-ns-refl'*:
  shows $s \succeq s$
$\langle proof \rangle$

lemma *wpo-stable'*: **fixes** $\delta :: ('f,'v)subst$
  shows $(s \succ t \longrightarrow s \cdot \delta \succ t \cdot \delta) \wedge (s \succeq t \longrightarrow s \cdot \delta \succeq t \cdot \delta)$
    (**is** *?p s t*)
$\langle proof \rangle$

lemma *subterm-wpo-s-arg'*: **assumes** *i*: $i \in set\ (\sigma\ (f,length\ ss))$
  shows $Fun\ f\ ss \succ ss\ !\ i$
$\langle proof \rangle$

context
  fixes *f s t bef aft*
  assumes *ctxt-NS*: $(s,t) \in NS \Longrightarrow (Fun\ f\ (bef\ @\ s\ \#\ aft),\ Fun\ f\ (bef\ @\ t\ \#\ aft))$
$\in NS$
begin

lemma *wpo-ns-pre-mono'*:
  defines $\sigma f \equiv \sigma\ (f,\ Suc\ (length\ bef + length\ aft))$
  assumes *rel*: (*wpo-ns s t*)
  shows $(\forall j \in set\ \sigma f.\ Fun\ f\ (bef\ @\ s\ \#\ aft) \succ (bef\ @\ t\ \#\ aft)\ !\ j)$
    $\wedge (Fun\ f\ (bef\ @\ s\ \#\ aft),\ (Fun\ f\ (bef\ @\ t\ \#\ aft))) \in NS$
    $\wedge (\forall\ i < length\ \sigma f.\ ((map\ ((!)\ (bef\ @\ s\ \#\ aft))\ \sigma f)\ !\ i) \succeq ((map\ ((!)\ (bef\ @\ t$
$\#\ aft))\ \sigma f)\ !\ i))$
    (**is** *- $\wedge$ - $\wedge$?three*)
$\langle proof \rangle$

lemma *wpo-ns-mono'*:

assumes *rel*: $s \succeq t$
  shows *Fun f (bef @ s # aft)* $\succeq$ *Fun f (bef @ t # aft)*
$\langle proof \rangle$

end
end
end

locale *wpo-with-assms = wpo-with-basic-assms + order-pair +*
  constrains S :: $('f, 'v)$ *term rel* **and** *NS* :: -
    and *prc* :: $'f \times nat \Rightarrow 'f \times nat \Rightarrow bool \times bool$
    and *prl* :: $'f \times nat \Rightarrow bool$
    and *ssimple* :: *bool*
    and *large* :: $'f \times nat \Rightarrow bool$
    and *c* :: $'f \times nat \Rightarrow order\text{-}tag$
    and *n* :: *nat*
    and $\sigma\sigma$ :: $'f$ *status*
  assumes *ctxt-NS*: $(s,t) \in NS \Longrightarrow$ (*Fun f (bef @ s # aft)*, *Fun f (bef @ t # aft)*)
$\in NS$
    and *ws-status*: $i \in set$ (*status* $\sigma\sigma$ *fn*) $\Longrightarrow$ *simple-arg-pos NS fn i*
begin

lemma *ssimple*: *strictly-simple-status* $\sigma\sigma$ *NS*
  $\langle proof \rangle$

lemma *trans-prc*: *trans-precedence prc*
  $\langle proof \rangle$

lemma *NS-arg*: **assumes** *i*: $i \in set$ (σ (*f*,*length ts*))
  shows (*Fun f ts*, *ts ! i*) $\in NS$
  $\langle proof \rangle$

lemma *NS-subterm*: **assumes** *all*: $\bigwedge f\ k.\ set$ (σ (*f*,*k*)) = $\{0\ ..<\ k\}$
  shows $s \trianglerighteq t \Longrightarrow (s,t) \in NS$
$\langle proof \rangle$

lemma *wpo-ns-refl*: $s \succeq s$
  $\langle proof \rangle$

lemma *subterm-wpo-s-arg*: **assumes** *i*: $i \in set$ (σ (*f*,*length ss*))
  shows *Fun f ss* $\succ$ *ss ! i*
  $\langle proof \rangle$

lemma *subterm-wpo-ns-arg*: **assumes** *i*: $i \in set$ (σ (*f*,*length ss*))
  shows *Fun f ss* $\succeq$ *ss ! i*
  $\langle proof \rangle$

lemma *wpo-irrefl*: $\neg$ ($s \succ s$)

⟨proof⟩

lemma *wpo-ns-mono*:
  assumes *rel*: $s \succeq t$
  shows *Fun f (bef @ s # aft)* $\succeq$ *Fun f (bef @ t # aft)*
  ⟨proof⟩

lemma *wpo-ns-pre-mono*: **fixes** *f* **and** *bef aft* :: $('f,'v)term\ list$
  defines $\sigma f \equiv \sigma\ (f,\ Suc\ (length\ bef\ +\ length\ aft))$
  assumes *rel*: (*wpo-ns s t*)
  shows ($\forall j \in set\ \sigma f.\ Fun\ f\ (bef\ @\ s\ \#\ aft) \succ (bef\ @\ t\ \#\ aft)\ !\ j$)
    $\land$ (*Fun f (bef @ s # aft), (Fun f (bef @ t # aft))*) $\in$ *NS*
    $\land$ ($\forall\ i < length\ \sigma f.\ ((map\ ((!)\ (bef\ @\ s\ \#\ aft))\ \sigma f)\ !\ i) \succeq ((map\ ((!)\ (bef\ @\ t\ \#\ aft))\ \sigma f)\ !\ i))$
  ⟨proof⟩

lemma *wpo-stable*: **fixes** δ :: $('f,'v)subst$
  shows ($s \succ t \longrightarrow s \cdot \delta \succ t \cdot \delta$) $\land$ ($s \succeq t \longrightarrow s \cdot \delta \succeq t \cdot \delta$)
  ⟨proof⟩

theorem *wpo-order-pair*: *order-pair WPO-S WPO-NS*
⟨proof⟩

theorem *WPO-S-subst*: $(s,t) \in$ *WPO-S* $\implies$ $(s \cdot \sigma,\ t \cdot \sigma) \in$ *WPO-S* **for** σ
  ⟨proof⟩

theorem *WPO-NS-subst*: $(s,t) \in$ *WPO-NS* $\implies$ $(s \cdot \sigma,\ t \cdot \sigma) \in$ *WPO-NS* **for** σ
  ⟨proof⟩

theorem *WPO-NS-ctxt*: $(s,t) \in$ *WPO-NS* $\implies$ (*Fun f (bef @ s # aft), Fun f (bef @ t # aft)*) $\in$ *WPO-NS*
  ⟨proof⟩

theorem *WPO-S-subset-WPO-NS*: *WPO-S* $\subseteq$ *WPO-NS*
  ⟨proof⟩

context
  assumes σ-*full*: $\bigwedge f\ k.\ set\ (\sigma\ (f,k)) = \{0\ ..<\ k\}$
begin

lemma *subterm-wpo-s*: $s \rhd t \implies s \succ t$
⟨proof⟩

lemma *subterm-wpo-ns*: **assumes** *supteq*: $s \unrhd t$ **shows** $s \succeq t$
⟨proof⟩

lemma *wpo-s-mono*: **assumes** *rels*: $s \succ t$

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shows *Fun f (bef @ s # aft) ≻ Fun f (bef @ t # aft)*
⟨*proof*⟩

theorem *WPO-S-ctxt*: *(s,t) ∈ WPO-S ⟹ (Fun f (bef @ s # aft), Fun f (bef @ t # aft)) ∈ WPO-S*
  ⟨*proof*⟩

theorem *supt-subset-WPO-S*: *{▷} ⊆ WPO-S*
  ⟨*proof*⟩

theorem *supteq-subset-WPO-NS*: *{⊵} ⊆ WPO-NS*
  ⟨*proof*⟩

end
end

If we demand strong normalization of the underlying order and the precedence, then also WPO is strongly normalizing.

locale *wpo-with-SN-assms = wpo-with-assms + SN-order-pair + precedence +*
  constrains *S :: ('f, 'v) term rel* **and** *NS :: -*
    and *prc :: 'f × nat ⇒ 'f × nat ⇒ bool × bool*
    and *prl :: 'f × nat ⇒ bool*
    and *ssimple :: bool*
    and *large :: 'f × nat ⇒ bool*
    and *c :: 'f × nat ⇒ order-tag*
    and *n :: nat*
    and *σσ :: 'f status*
begin

lemma *Var-not-S[simp]*: *(Var x, t) ∉ S*
⟨*proof*⟩

lemma *WPO-S-SN*: *SN WPO-S*
⟨*proof*⟩

theorem *wpo-SN-order-pair*: *SN-order-pair WPO-S WPO-NS*
⟨*proof*⟩

end
end

6   The Recursive Path Order as an instance of WPO

This theory defines the recursive path order (RPO) that given two terms provides two Booleans, whether the terms can be strictly or non-strictly oriented. It is proven that RPO is an instance of WPO, and hence, carries over all the nice properties of WPO immediately.

theory *RPO*

imports
  WPO
begin

context
  fixes *pr* :: *$'f \times nat \Rightarrow 'f \times nat \Rightarrow bool \times bool$*
    and *prl* :: *$'f \times nat \Rightarrow bool$*
    and *c* :: *$'f \times nat \Rightarrow order\text{-}tag$*
    and *n* :: *nat*
begin

fun *rpo* :: *$('f, 'v)\ term \Rightarrow ('f, 'v)\ term \Rightarrow bool \times bool$*
  where
    rpo (Var x) (Var y) = (False, x = y) |
    rpo (Var x) (Fun g ts) = (False, ts = [] $\wedge$ prl (g,0)) |
    rpo (Fun f ss) (Var y) = (let con = ($\exists$ s $\in$ set ss. snd (rpo s (Var y))) in (con,con)) |
    rpo (Fun f ss) (Fun g ts) = (
    if ($\exists$ s $\in$ set ss. snd (rpo s (Fun g ts)))
      then (True,True)
      else (let (prs,prns) = pr (f,length ss) (g,length ts) in
       if prns $\wedge$ ($\forall$ t $\in$ set ts. fst (rpo (Fun f ss) t))
       then if prs
         then (True,True)
         else if c (f,length ss) = Lex $\wedge$ c (g,length ts) = Lex
           then lex-ext rpo n ss ts
           else if c (f,length ss) = Mul $\wedge$ c (g,length ts) = Mul
             then mul-ext rpo ss ts
             else (length ss $\neq$ 0 $\wedge$ length ts = 0, length ts = 0)
       else (False,False)))
end

locale *rpo-with-assms = precedence prc prl*
  for *prc* :: *$'f \times nat \Rightarrow 'f \times nat \Rightarrow bool \times bool$*
    and *prl* :: *$'f \times nat \Rightarrow bool$*
    and *c* :: *$'f \times nat \Rightarrow order\text{-}tag$*
    and *n* :: *nat*
begin

sublocale *wpo-with-SN-assms n {} UNIV prc prl full-status c False λ -. False*
  ⟨*proof*⟩

abbreviation *rpo-pr $\equiv$ rpo prc prl c n*
abbreviation *rpo-s $\equiv$ λ s t. fst (rpo-pr s t)*
abbreviation *rpo-ns $\equiv$ λ s t. snd (rpo-pr s t)*

lemma *rpo-eq-wpo*: *rpo-pr s t = wpo s t*
⟨*proof*⟩

abbreviation *RPO-S* ≡ {(s,t). *rpo-s s t*}
abbreviation *RPO-NS* ≡ {(s,t). *rpo-ns s t*}

theorem *RPO-SN-order-pair*: *SN-order-pair RPO-S RPO-NS*
⟨*proof*⟩

theorem *RPO-S-subst*: (s,t) ∈ *RPO-S* ⟹ (s · σ, t · σ) ∈ *RPO-S* **for** σ ::
($'f$,$'a$)*subst*
⟨*proof*⟩

theorem *RPO-NS-subst*: (s,t) ∈ *RPO-NS* ⟹ (s · σ, t · σ) ∈ *RPO-NS* **for** σ ::
($'f$,$'a$)*subst*
⟨*proof*⟩

theorem *RPO-NS-ctxt*: (s,t) ∈ *RPO-NS* ⟹ (*Fun f* (*bef* @ *s* # *aft*), *Fun f* (*bef*
@ *t* # *aft*)) ∈ *RPO-NS*
⟨*proof*⟩

theorem *RPO-S-ctxt*: (s,t) ∈ *RPO-S* ⟹ (*Fun f* (*bef* @ *s* # *aft*), *Fun f* (*bef* @ *t*
aft)) ∈ *RPO-S*
⟨*proof*⟩

theorem *RPO-S-subset-RPO-NS*: *RPO-S* ⊆ *RPO-NS*
⟨*proof*⟩

theorem *supt-subset-RPO-S*: {▷} ⊆ *RPO-S*
⟨*proof*⟩

theorem *supteq-subset-RPO-NS*: {⊵} ⊆ *RPO-NS*
⟨*proof*⟩

end
end

7   The Lexicographic Path Order as an instance of WPO

We first directly define the strict- and non-strict lexicographic path orders
(LPO) w.r.t. some precedence, and then show that it is an instance of WPO.
For this instance we use the trivial reduction pair in WPO (∅, UNIV) and
the status is the full one, i.e., taking parameters [0,..,n-1] for each n-ary
symbol.

theory *LPO*
  imports
    WPO
begin

context
  fixes *pr* :: (*′f* × *nat* ⇒ *′f* × *nat* ⇒ *bool* × *bool*)
    and *prl* :: *′f* × *nat* ⇒ *bool*
    and *n* :: *nat*
begin
fun *lpo* :: (*′f*, *′v*) *term* ⇒ (*′f*, *′v*) *term* ⇒ *bool* × *bool*
  where
    lpo (*Var x*) (*Var y*) = (*False*, *x* = *y*) |
    lpo (*Var x*) (*Fun g ts*) = (*False*, *ts* = [] ∧ *prl* (*g*,0)) |
    lpo (*Fun f ss*) (*Var y*) = (*let con* = (∃ *s* ∈ *set ss*. *snd* (*lpo s* (*Var y*))) *in*
(*con*,*con*)) |
    lpo (*Fun f ss*) (*Fun g ts*) = (
      if (∃ *s* ∈ *set ss*. *snd* (*lpo s* (*Fun g ts*)))
        then (*True*,*True*)
        else (*let* (*prs*,*prns*) = *pr* (*f*,*length ss*) (*g*,*length ts*) *in*
         if prns ∧ (∀ *t* ∈ *set ts*. *fst* (*lpo* (*Fun f ss*) *t*))
         then if prs
           then (*True*,*True*)
           else lex-ext lpo n ss ts
         else (*False*,*False*)))

end

locale *lpo-with-assms* = *precedence prc prl*
  for *prc* :: *′f* × *nat* ⇒ *′f* × *nat* ⇒ *bool* × *bool*
    and *prl* :: *′f* × *nat* ⇒ *bool*
    and *n* :: *nat*
begin

sublocale *wpo-with-SN-assms n* {} *UNIV prc prl full-status* λ -. *Lex False* λ -.
False
  ⟨*proof*⟩

abbreviation *lpo-pr* ≡ *lpo prc prl n*
abbreviation *lpo-s* ≡ λ *s t*. *fst* (*lpo-pr s t*)
abbreviation *lpo-ns* ≡ λ *s t*. *snd* (*lpo-pr s t*)

lemma *lpo-eq-wpo*: *lpo-pr s t* = *wpo s t*
⟨*proof*⟩

abbreviation *LPO-S* ≡ {(*s*,*t*). *lpo-s s t*}
abbreviation *LPO-NS* ≡ {(*s*,*t*). *lpo-ns s t*}

theorem *LPO-SN-order-pair*: *SN-order-pair LPO-S LPO-NS*
  ⟨*proof*⟩

theorem *LPO-S-subst*: (*s*,*t*) ∈ *LPO-S* ⟹ (*s* · σ, *t* · σ) ∈ *LPO-S* **for** σ ::

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$(\,'f,'a)subst$
  $\langle proof\rangle$

theorem *LPO-NS-subst*: $(s,t) \in$ *LPO-NS* $\Longrightarrow (s \cdot \sigma,\ t \cdot \sigma) \in$ *LPO-NS* **for** σ ::
$(\,'f,'a)subst$
  $\langle proof\rangle$

theorem *LPO-NS-ctxt*: $(s,t) \in$ *LPO-NS* $\Longrightarrow$ (*Fun f* (*bef* @ *s* # *aft*), *Fun f* (*bef*
@ *t* # *aft*)) $\in$ *LPO-NS*
  $\langle proof\rangle$

theorem *LPO-S-ctxt*: $(s,t) \in$ *LPO-S* $\Longrightarrow$ (*Fun f* (*bef* @ *s* # *aft*), *Fun f* (*bef* @ *t*
aft)) $\in$ *LPO-S*
  $\langle proof\rangle$

theorem *LPO-S-subset-LPO-NS*: *LPO-S* $\subseteq$ *LPO-NS*
  $\langle proof\rangle$

theorem *supt-subset-LPO-S*: $\{\rhd\} \subseteq$ *LPO-S*
  $\langle proof\rangle$

theorem *supteq-subset-LPO-NS*: $\{\unrhd\} \subseteq$ *LPO-NS*
  $\langle proof\rangle$

end

end

8 The Knuth–Bendix Order as an instance of WPO

Making the Knuth–Bendix an instance of WPO is more complicated than
in the case of RPO and LPO, because of syntactic and semantic differences.
We face the two main challenges in two different theories and sub-sections.

8.1 Aligning least elements

In all of RPO, LPO and WPO there is the concept of a minimal term, e.g.,
a constant term c where c is least in precedence among *all function symbols*.
By contrast, in KBO a constant c is minimal if it has minimal weight and
has least precende *among all constants of minimal weight.*

In this theory we prove that for any KBO one can modify the precedence
in a way that least constants c also have least precendence among *all function
symbols*, without changing the defined order. Hence, afterwards it will be
simpler to relate such a KBO to WPO.

theory *KBO-Transformation*
  imports *WPO Knuth-Bendix-Order.KBO*

begin

context *admissible-kbo*
begin

lemma *weight-w0-unary*:
  assumes ∗: *weight t = w0 t = Fun f ts ts = t1 # ts′*
  shows *ts′ = [] w (f,1) = 0*
⟨*proof*⟩

definition *lConsts* :: (′*f × nat*)*set* **where** *lConsts* = { (*f,0*) | *f. least f*}
definition *pr-strict′* **where** *pr-strict′ f g* = (*f* ∉ *lConsts* ∧ (*pr-strict f g* ∨ *g* ∈ *lConsts*))
definition *pr-weak′* **where** *pr-weak′ f g* = ((*f* ∉ *lConsts* ∧ *pr-weak f g*) ∨ *g* ∈ *lConsts*)

lemma *admissible-kbo′*: *admissible-kbo w w0 pr-strict′ pr-weak′ least scf*
  ⟨*proof*⟩

lemma *least-pr-weak′*: *least f* ⟹ *pr-weak′ g (f,0)* ⟨*proof*⟩

lemma *least-pr-weak′-trans*: *least f* ⟹ *pr-weak′ (f,0) g* ⟹ *least (fst g)* ∧ *snd g = 0*
  ⟨*proof*⟩

context
begin
interpretation *kbo′*: *admissible-kbo w w0 pr-strict′ pr-weak′ least scf*
  ⟨*proof*⟩

lemma *kbo′-eq-kbo*: *kbo′.kbo s t = kbo s t*
⟨*proof*⟩
end
end
end

8.2 A restricted equality between KBO and WPO

The remaining difficulty to make KBO an instance of WPO is the different treatment of lexicographic comparisons, which is unrestricted in KBO, but there is a length-restriction in WPO. Therefore we will only show that KBO is an instance of WPO if we compare terms with bounded arity.

    This restriction does however not prohibit us from lifting properties of WPO to KBO. For instance, for several properties one can choose a large-enough bound restriction of WPO, since there are only finitely many arities occurring in a property.

theory *KBO-as-WPO*
  imports

WPO
KBO-Transformation
begin

definition *bounded-arity* :: *nat* ⇒ (′*f* × *nat*)*set* ⇒ *bool* **where**
  bounded-arity b F = (∀ (*f,n*) ∈ *F. n ≤ b*)

lemma *finite-funas-term*[*simp,intro*]: *finite* (*funas-term t*)
  ⟨*proof*⟩

context *weight-fun* **begin**

definition *weight-le s t* ≡
  (*vars-term-ms* (*SCF s*) ⊆# *vars-term-ms* (*SCF t*) ∧ *weight s ≤ weight t*)

definition *weight-less s t* ≡
  (*vars-term-ms* (*SCF s*) ⊆# *vars-term-ms* (*SCF t*) ∧ *weight s < weight t*)

lemma *weight-le-less-iff*: *weight-le s t* ⟹ *weight-less s t* ⟷ *weight s < weight t*
  ⟨*proof*⟩

lemma *weight-less-iff*: *weight-less s t* ⟹ *weight-le s t* ∧ *weight s < weight t*
  ⟨*proof*⟩

abbreviation *weight-NS* ≡ {(*t,s*). *weight-le s t*}

abbreviation *weight-S* ≡ {(*t,s*). *weight-less s t*}

lemma *weight-le-mono-one*:
  assumes *S*: *weight-le s t*
  shows *weight-le* (*Fun f* (*ss1* @ *s* # *ss2*)) (*Fun f* (*ss1* @ *t* # *ss2*)) (**is** *weight-le ?s ?t*)
⟨*proof*⟩

lemma *weight-le-ctxt*: *weight-le s t* ⟹ *weight-le* (*C*⟨*s*⟩) (*C*⟨*t*⟩)
  ⟨*proof*⟩

lemma *SCF-stable*:
  assumes *vars-term-ms* (*SCF s*) ⊆# *vars-term-ms* (*SCF t*)
  shows *vars-term-ms* (*SCF* (*s · σ*)) ⊆# *vars-term-ms* (*SCF* (*t · σ*))
  ⟨*proof*⟩

lemma *SN-weight-S*: *SN weight-S*
⟨*proof*⟩

lemma *weight-less-imp-le*: *weight-less s t* ⟹ *weight-le s t* ⟨*proof*⟩

lemma *weight-le-Var-Var*: *weight-le* (*Var x*) (*Var y*) ⟷ *x = y*

⟨*proof*⟩
end

context *kbo* **begin**

lemma *kbo-altdef*:
   kbo s t = (if weight-le t s
     then if weight-less t s
       then (True, True)
       else (case s of
         Var y ⇒ (False, (case t of Var x ⇒ x = y | Fun g ts ⇒ ts = [] ∧ least g))
       | Fun f ss ⇒ (case t of
           Var x ⇒ (True, True)
         | Fun g ts ⇒ if pr-strict (f, length ss) (g, length ts)
           then (True, True)
           else if pr-weak (f, length ss) (g, length ts)
           then lex-ext-unbounded kbo ss ts
           else (False, False)))
     else (False, False))
   ⟨*proof*⟩

end

context *admissible-kbo* **begin**

lemma *weight-le-stable*:
  assumes *weight-le s t*
  shows *weight-le (s · σ) (t · σ)*
  ⟨*proof*⟩

lemma *weight-less-stable*:
  assumes *weight-less s t*
  shows *weight-less (s · σ) (t · σ)*
  ⟨*proof*⟩

lemma *simple-arg-pos-weight*: *simple-arg-pos weight-NS (f,n) i*
  ⟨*proof*⟩

lemma *weight-lemmas*:
  shows *refl weight-NS* **and** *trans weight-NS* **and** *trans weight-S*
    and *weight-NS O weight-S ⊆ weight-S* **and** *weight-S O weight-NS ⊆ weight-S*
  ⟨*proof*⟩

interpretation *kbo′*: *admissible-kbo w w0 pr-strict′ pr-weak′ least scf*
  ⟨*proof*⟩

context
  assumes *least-global*: $\bigwedge f\ g.\ least\ f \implies pr\text{-}weak\ g\ (f,0)$
    and *least-trans*: $\bigwedge f\ g.\ least\ f \implies pr\text{-}weak\ (f,0)\ g \implies least\ (fst\ g) \land snd\ g = 0$

fixes *n* :: *nat*
begin

lemma *kbo-instance-of-wpo-with-SN-assms*: *wpo-with-SN-assms*
  weight-S weight-NS (λf *g*. (*pr-strict f g*, *pr-weak f g*))
    ($\lambda(f, n)$. *n = 0* $\land$ *least f*) *full-status False* (λf. *False*)
⟨*proof*⟩

interpretation *wpo*: *wpo-with-SN-assms*
  where *S = weight-S* **and** *NS = weight-NS*
    and *prc* = λf *g*. (*pr-strict f g*, *pr-weak f g*) **and** *prl* = $\lambda(f,n)$. *n = 0* $\land$ *least f*
    and *c* = λ-. *Lex*
    and *ssimple = False* **and** *large* = λf. *False* **and** $\sigma\sigma$ = *full-status*
    and *n = n*
⟨*proof*⟩

lemma *kbo-as-wpo-with-assms*: **assumes** *bounded-arity n* (*funas-term t*)
  shows *kbo s t = wpo.wpo s t*
⟨*proof*⟩
end

    This is the main theorem. It tells us that KBO can be seen as an instance of WPO, under mild preconditions: the parameter *n* for the lexicographic extension has to be chosen high enough to cover the arities of all terms that should be compared.

lemma defines *prec* $\equiv$ ((λf *g*. (*pr-strict′ f g*, *pr-weak′ f g*)))
  and *prl* $\equiv$ ($\lambda(f, n)$. *n = 0* $\land$ *least f*)
  shows
  kbo-encoding-is-valid-wpo: *wpo-with-SN-assms weight-S weight-NS prec prl full-status False* (λf. *False*)
  and
    kbo-as-wpo: *bounded-arity n* (*funas-term t*) $\Longrightarrow$ *kbo s t = wpo.wpo n weight-S weight-NS prec prl full-status* (λ-. *Lex*) *False* (λf. *False*) *s t*
  ⟨*proof*⟩

    As a proof-of-concept we show that now properties of WPO can be used to prove these properties for KBO. Here, as example we consider closure under substitutions and strong normalization, but the following idea can be applied for several more properties: if the property involves only terms where the arities are bounded, then just choose the parameter *n* large enough. This even works for strong normalization, since in an infinite chain of KBO-decreases $t_1 > t_2 > t_3 > ...$ all terms have a weight of at most the weight of t_1, and this weight is also a bound on the arities.

lemma *KBO-stable-via-WPO*: *S s t* $\Longrightarrow$ *S* (*s* $\cdot$ (σ :: ($'f$,$'a$) *subst*)) (*t* $\cdot$ σ)
⟨*proof*⟩

lemma *weight-is-arity-bound*: *weight t* $\leq$ *b* $\Longrightarrow$ *bounded-arity b* (*funas-term t*)
⟨*proof*⟩

lemma *KBO-SN-via-WPO*: *SN* {(*s,t*). *S s t*}
⟨*proof*⟩

end

end

9   Executability of the orders

theory *Executable-Orders*
  imports
    WPO
    RPO
    LPO
    Multiset-Extension2-Impl
begin

If one loads the implementation of multiset orders (in particular for *mul-ext*), then all orders defined in this AFP-entry (WPO, RPO, LPO, multiset extension of order pairs) are executable.

export-code
  lpo
  rpo
  wpo.wpo
  mul-ext
  mult2-impl
  in *Haskell*

end

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