

Pólya's Proof of the Weighted Arithmetic–Geometric Mean Inequality

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Abstract

This article provides a formalisation of the Weighted Arithmetic–Geometric Mean Inequality: given non-negative reals a_1, \dots, a_n and non-negative weights w_1, \dots, w_n such that $w_1 + \dots + w_n = 1$, we have

$$\prod_{i=1}^n a_i^{w_i} \leq \sum_{i=1}^n w_i a_i .$$

If the weights are additionally all non-zero, equality holds if and only if $a_1 = \dots = a_n$.

As a corollary with $w_1 = \dots = w_n = \frac{1}{n}$, the regular arithmetic–geometric mean inequality follows, namely that

$$\sqrt[n]{a_1 \dots a_n} \leq \frac{1}{n}(a_1 + \dots + a_n) .$$

I follow Pólya's elegant proof, which uses the inequality $1 + x \leq e^x$ as a starting point. Pólya claims that this proof came to him in a dream, and that it was ‘the best mathematics he had ever dreamt.’ [1, pp. 22–26]

Contents

1	The Weighted Arithmetic–Geometric Mean Inequality	2
1.1	Auxiliary Facts	2
1.2	The Inequality	3
1.3	The Equality Case	3
1.4	The Binary Version	4

1 The Weighted Arithmetic–Geometric Mean Inequality

theory *Weighted-Arithmetic-Geometric-Mean*
imports *Complex-Main*
begin

1.1 Auxiliary Facts

lemma *root-powr-inverse'*: $0 < n \implies 0 \leq x \implies \text{root } n \ x = x \ \text{powr } (1/n)$
<proof>

lemma *powr-sum-distrib-real-right*:
assumes $a \neq 0$
shows $(\prod x \in X. a \ \text{powr } e \ x :: \text{real}) = a \ \text{powr } (\sum x \in X. e \ x)$
<proof>

lemma *powr-sum-distrib-real-left*:
assumes $\bigwedge x. x \in X \implies a \ x \geq 0$
shows $(\prod x \in X. a \ x \ \text{powr } e :: \text{real}) = (\prod x \in X. a \ x) \ \text{powr } e$
<proof>

lemma *prod-ge-pointwise-le-imp-pointwise-eq*:
fixes $f :: 'a \Rightarrow \text{real}$
assumes *finite* X
assumes $ge: \text{prod } f \ X \geq \text{prod } g \ X$
assumes *nonneg*: $\bigwedge x. x \in X \implies f \ x \geq 0$
assumes *pos*: $\bigwedge x. x \in X \implies g \ x > 0$
assumes *le*: $\bigwedge x. x \in X \implies f \ x \leq g \ x$ **and** $x: x \in X$
shows $f \ x = g \ x$
<proof>

lemma *powr-right-real-eq-iff*:
assumes $a \geq (0 :: \text{real})$
shows $a \ \text{powr } x = a \ \text{powr } y \longleftrightarrow a = 0 \vee a = 1 \vee x = y$
<proof>

lemma *powr-left-real-eq-iff*:
assumes $a \geq (0 :: \text{real}) \ b \geq 0 \ x \neq 0$
shows $a \ \text{powr } x = b \ \text{powr } x \longleftrightarrow a = b$
<proof>

lemma *exp-real-eq-one-plus-iff*:
fixes $x :: \text{real}$
shows $\exp \ x = 1 + x \longleftrightarrow x = 0$
<proof>

1.2 The Inequality

We first prove the equality under the assumption that all the a_i and w_i are positive.

lemma *weighted-arithmetic-geometric-mean-pos:*

fixes $a w :: 'a \Rightarrow \text{real}$

assumes *finite X*

assumes *pos1*: $\bigwedge x. x \in X \implies a x > 0$

assumes *pos2*: $\bigwedge x. x \in X \implies w x > 0$

assumes *sum-weights*: $(\sum x \in X. w x) = 1$

shows $(\prod x \in X. a x \text{ powr } w x) \leq (\sum x \in X. w x * a x)$

<proof>

We can now relax the positivity assumptions to non-negativity: if one of the a_i is zero, the theorem becomes trivial (note that $0^0 = 0$ by convention for the real-valued power operator (*powr*)).

Otherwise, we can simply remove all the indices that have weight 0 and apply the above auxiliary version of the theorem.

theorem *weighted-arithmetic-geometric-mean:*

fixes $a w :: 'a \Rightarrow \text{real}$

assumes *finite X*

assumes *nonneg1*: $\bigwedge x. x \in X \implies a x \geq 0$

assumes *nonneg2*: $\bigwedge x. x \in X \implies w x \geq 0$

assumes *sum-weights*: $(\sum x \in X. w x) = 1$

shows $(\prod x \in X. a x \text{ powr } w x) \leq (\sum x \in X. w x * a x)$

<proof>

We can derive the regular arithmetic/geometric mean inequality from this by simply setting all the weights to $\frac{1}{n}$:

corollary *arithmetic-geometric-mean:*

fixes $a :: 'a \Rightarrow \text{real}$

assumes *finite X*

defines $n \equiv \text{card } X$

assumes *nonneg*: $\bigwedge x. x \in X \implies a x \geq 0$

shows $\text{root } n (\prod x \in X. a x) \leq (\sum x \in X. a x) / n$

<proof>

1.3 The Equality Case

Next, we show that weighted arithmetic and geometric mean are equal if and only if all the a_i are equal.

We first prove the more difficult direction as a lemmas and again first assume positivity of all a_i and w_i and will relax this somewhat later.

lemma *weighted-arithmetic-geometric-mean-eq-iff-pos:*

fixes $a w :: 'a \Rightarrow \text{real}$

assumes *finite X*

assumes *pos1*: $\bigwedge x. x \in X \implies a x > 0$
assumes *pos2*: $\bigwedge x. x \in X \implies w x > 0$
assumes *sum-weights*: $(\sum x \in X. w x) = 1$
assumes *eq*: $(\prod x \in X. a x \text{ powr } w x) = (\sum x \in X. w x * a x)$
shows $\forall x \in X. \forall y \in X. a x = a y$
 <proof>

We can now show the full theorem and relax the positivity condition on the a_i to non-negativity. This is possible because if some a_i is zero and the two means coincide, then the product is obviously 0, but the sum can only be 0 if *all* the a_i are 0.

theorem *weighted-arithmetic-geometric-mean-eq-iff*:
fixes $a w :: 'a \Rightarrow \text{real}$
assumes *finite* X
assumes *nonneg1*: $\bigwedge x. x \in X \implies a x \geq 0$
assumes *pos2*: $\bigwedge x. x \in X \implies w x > 0$
assumes *sum-weights*: $(\sum x \in X. w x) = 1$
shows $(\prod x \in X. a x \text{ powr } w x) = (\sum x \in X. w x * a x) \longleftrightarrow X \neq \{\} \wedge (\forall x \in X. \forall y \in X. a x = a y)$
 <proof>

Again, we derive a version for the unweighted arithmetic/geometric mean.

corollary *arithmetic-geometric-mean-eq-iff*:
fixes $a :: 'a \Rightarrow \text{real}$
assumes *finite* X
defines $n \equiv \text{card } X$
assumes *nonneg*: $\bigwedge x. x \in X \implies a x \geq 0$
shows $\text{root } n (\prod x \in X. a x) = (\sum x \in X. a x) / n \longleftrightarrow (\forall x \in X. \forall y \in X. a x = a y)$
 <proof>

1.4 The Binary Version

For convenience, we also derive versions for only two numbers:

corollary *weighted-arithmetic-geometric-mean-binary*:
fixes $w1 w2 x1 x2 :: \text{real}$
assumes $x1 \geq 0 x2 \geq 0 w1 \geq 0 w2 \geq 0 w1 + w2 = 1$
shows $x1 \text{ powr } w1 * x2 \text{ powr } w2 \leq w1 * x1 + w2 * x2$
 <proof>

corollary *weighted-arithmetic-geometric-mean-eq-iff-binary*:
fixes $w1 w2 x1 x2 :: \text{real}$
assumes $x1 \geq 0 x2 \geq 0 w1 > 0 w2 > 0 w1 + w2 = 1$
shows $x1 \text{ powr } w1 * x2 \text{ powr } w2 = w1 * x1 + w2 * x2 \longleftrightarrow x1 = x2$
 <proof>

corollary *arithmetic-geometric-mean-binary*:
fixes $x1 x2 :: \text{real}$

assumes $x1 \geq 0 \ x2 \geq 0$
shows $\text{sqrt}(x1 * x2) \leq (x1 + x2) / 2$
<proof>

corollary *arithmetic-geometric-mean-eq-iff-binary:*

fixes $x1 \ x2 :: \text{real}$

assumes $x1 \geq 0 \ x2 \geq 0$

shows $\text{sqrt}(x1 * x2) = (x1 + x2) / 2 \longleftrightarrow x1 = x2$
<proof>

end

References

- [1] J. M. Steele. *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge University Press, 2004.