

Pólya's Proof of the Weighted Arithmetic–Geometric Mean Inequality

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Abstract

This article provides a formalisation of the Weighted Arithmetic–Geometric Mean Inequality: given non-negative reals a_1, \dots, a_n and non-negative weights w_1, \dots, w_n such that $w_1 + \dots + w_n = 1$, we have

$$\prod_{i=1}^n a_i^{w_i} \leq \sum_{i=1}^n w_i a_i .$$

If the weights are additionally all non-zero, equality holds if and only if $a_1 = \dots = a_n$.

As a corollary with $w_1 = \dots = w_n = \frac{1}{n}$, the regular arithmetic–geometric mean inequality follows, namely that

$$\sqrt[n]{a_1 \dots a_n} \leq \frac{1}{n}(a_1 + \dots + a_n) .$$

I follow Pólya's elegant proof, which uses the inequality $1 + x \leq e^x$ as a starting point. Pólya claims that this proof came to him in a dream, and that it was ‘the best mathematics he had ever dreamt.’ [1, pp. 22–26]

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1 The Weighted Arithmetic–Geometric Mean Inequality

theory *Weighted-Arithmetic-Geometric-Mean*
imports *Complex-Main*
begin

1.1 Auxiliary Facts

lemma *root-powr-inverse'*: $0 < n \implies 0 \leq x \implies \text{root } n \ x = x \ \text{powr } (1/n)$
by (*cases* $x = 0$) (*auto simp: root-powr-inverse*)

lemma *powr-sum-distrib-real-right*:

assumes $a \neq 0$

shows $(\prod x \in X. a \ \text{powr } e \ x :: \text{real}) = a \ \text{powr } (\sum x \in X. e \ x)$

using *assms*

by (*induction* X *rule: infinite-finite-induct*) (*auto simp: powr-add*)

lemma *powr-sum-distrib-real-left*:

assumes $\bigwedge x. x \in X \implies a \ x \geq 0$

shows $(\prod x \in X. a \ x \ \text{powr } e :: \text{real}) = (\prod x \in X. a \ x) \ \text{powr } e$

using *assms*

by (*induction* X *rule: infinite-finite-induct*)

(*auto simp: powr-mult prod-nonneg*)

lemma *prod-ge-pointwise-le-imp-pointwise-eq*:

fixes $f :: 'a \Rightarrow \text{real}$

assumes *finite* X

assumes *ge*: $\text{prod } f \ X \geq \text{prod } g \ X$

assumes *nonneg*: $\bigwedge x. x \in X \implies f \ x \geq 0$

assumes *pos*: $\bigwedge x. x \in X \implies g \ x > 0$

assumes *le*: $\bigwedge x. x \in X \implies f \ x \leq g \ x$ **and** $x: x \in X$

shows $f \ x = g \ x$

proof (*rule ccontr*)

assume $f \ x \neq g \ x$

with *le*[*of* x] **and** x **have** $f \ x < g \ x$

by *auto*

hence $\text{prod } f \ X < \text{prod } g \ X$

using x **and** *le* **and** *nonneg* **and** *pos* **and** $\langle \text{finite } X \rangle$

by (*intro prod-mono-strict*) *auto*

with *ge* **show** *False*

by *simp*

qed

lemma *powr-right-real-eq-iff*:

assumes $a \geq (0 :: \text{real})$

shows $a \ \text{powr } x = a \ \text{powr } y \iff a = 0 \vee a = 1 \vee x = y$

using *assms* **by** (*auto simp: powr-def*)

lemma *powr-left-real-eq-iff*:
assumes $a \geq (0 :: \text{real})$ $b \geq 0$ $x \neq 0$
shows $a \text{ powr } x = b \text{ powr } x \longleftrightarrow a = b$
using *assms* **by** (*auto simp: powr-def*)

lemma *exp-real-eq-one-plus-iff*:
fixes $x :: \text{real}$
shows $\exp x = 1 + x \longleftrightarrow x = 0$
proof (*cases x = 0*)
case *False*
define $f :: \text{real} \Rightarrow \text{real}$ **where** $f = (\lambda x. \exp x - 1 - x)$
have *deriv*: (*f has-field-derivative (exp x - 1)*) (*at x*) **for** x
by (*auto simp: f-def intro!: derivative-eq-intros*)

have $\exists z. z > \min x \ 0 \wedge z < \max x \ 0 \wedge f (\max x \ 0) - f (\min x \ 0) =$
 $(\max x \ 0 - \min x \ 0) * (\exp z - 1)$
using *MVT2*[*of min x 0 max x 0 f lambda. exp x - 1*] *deriv False*
by (*auto simp: min-def max-def*)
then obtain z **where** $z \in \{\min x \ 0 .. < \max x \ 0\}$
 $f (\max x \ 0) - f (\min x \ 0) = (\max x \ 0 - \min x \ 0) * (\exp z - 1)$
by (*auto simp: f-def*)
thus *?thesis* **using** *False*
by (*cases x 0 :: real rule: linorder-cases*) (*auto simp: f-def*)
qed *auto*

1.2 The Inequality

We first prove the equality under the assumption that all the a_i and w_i are positive.

lemma *weighted-arithmetic-geometric-mean-pos*:
fixes $a \ w :: 'a \Rightarrow \text{real}$
assumes *finite X*
assumes *pos1*: $\bigwedge x. x \in X \Longrightarrow a \ x > 0$
assumes *pos2*: $\bigwedge x. x \in X \Longrightarrow w \ x > 0$
assumes *sum-weights*: $(\sum x \in X. w \ x) = 1$
shows $(\prod x \in X. a \ x \text{ powr } w \ x) \leq (\sum x \in X. w \ x * a \ x)$
proof –
note *nonneg1* = *less-imp-le*[*OF pos1*]
note *nonneg2* = *less-imp-le*[*OF pos2*]
define A **where** $A = (\sum x \in X. w \ x * a \ x)$
define r **where** $r = (\lambda x. a \ x / A - 1)$
from *sum-weights* **have** $X \neq \{\}$ **by** *auto*
hence $A \neq 0$
unfolding *A-def* **using** *nonneg1 nonneg2 pos1 pos2* $\langle \text{finite } X \rangle$
by (*subst sum-nonneg-eq-0-iff*) *force+*
moreover from *nonneg1 nonneg2* **have** $A \geq 0$
by (*auto simp: A-def intro!: sum-nonneg*)
ultimately have $A > 0$ **by** *simp*

have $(\prod x \in X. (1 + r x) \text{ powr } w x) = (\prod x \in X. (a x / A) \text{ powr } w x)$
by (*simp add: r-def*)
also have $\dots = (\prod x \in X. a x \text{ powr } w x) / (\prod x \in X. A \text{ powr } w x)$
unfolding *prod-dividef [symmetric]*
using *assms pos2* $\langle A > 0 \rangle$ **by** (*intro prod.cong powr-divide*) (*auto intro: less-imp-le*)
also have $(\prod x \in X. A \text{ powr } w x) = \exp ((\sum x \in X. w x) * \ln A)$
using $\langle A > 0 \rangle$ **and** $\langle \text{finite } X \rangle$ **by** (*simp add: powr-def exp-sum sum-distrib-right*)
also have $(\sum x \in X. w x) = 1$ **by** *fact*
also have $\exp (1 * \ln A) = A$
using $\langle A > 0 \rangle$ **by** *simp*
finally have *lhs*: $(\prod x \in X. (1 + r x) \text{ powr } w x) = (\prod x \in X. a x \text{ powr } w x) / A$.

have $(\prod x \in X. \exp (w x * r x)) = \exp (\sum x \in X. w x * r x)$
using $\langle \text{finite } X \rangle$ **by** (*simp add: exp-sum*)
also have $(\sum x \in X. w x * r x) = (\sum x \in X. a x * w x) / A - 1$
using $\langle A > 0 \rangle$ **by** (*simp add: r-def algebra-simps sum-subtractf sum-divide-distrib sum-weights*)
also have $(\sum x \in X. a x * w x) / A = 1$
using $\langle A > 0 \rangle$ **by** (*simp add: A-def mult.commute*)
finally have *rhs*: $(\prod x \in X. \exp (w x * r x)) = 1$ **by** *simp*

have $(\prod x \in X. a x \text{ powr } w x) / A = (\prod x \in X. (1 + r x) \text{ powr } w x)$
by (*fact lhs [symmetric]*)
also have $(\prod x \in X. (1 + r x) \text{ powr } w x) \leq (\prod x \in X. \exp (w x * r x))$
proof (*intro prod-mono conjI*)
fix x **assume** $x: x \in X$
have $1 + r x \leq \exp (r x)$
by (*rule exp-ge-add-one-self*)
hence $(1 + r x) \text{ powr } w x \leq \exp (r x) \text{ powr } w x$
using *nonneg1* [of x] *nonneg2* [of x] $x \langle A > 0 \rangle$
by (*intro powr-mono2*) (*auto simp: r-def field-simps*)
also have $\dots = \exp (w x * r x)$
by (*simp add: powr-def*)
finally show $(1 + r x) \text{ powr } w x \leq \exp (w x * r x)$.
qed *auto*
also have $(\prod x \in X. \exp (w x * r x)) = 1$ **by** (*fact rhs*)
finally show $(\prod x \in X. a x \text{ powr } w x) \leq A$
using $\langle A > 0 \rangle$ **by** (*simp add: field-simps*)
qed

We can now relax the positivity assumptions to non-negativity: if one of the a_i is zero, the theorem becomes trivial (note that $0^0 = 0$ by convention for the real-valued power operator (*powr*)).

Otherwise, we can simply remove all the indices that have weight 0 and apply the above auxiliary version of the theorem.

theorem *weighted-arithmetic-geometric-mean*:
fixes $a w :: 'a \Rightarrow \text{real}$
assumes *finite X*

assumes *nonneg1*: $\bigwedge x. x \in X \implies a \ x \geq 0$
assumes *nonneg2*: $\bigwedge x. x \in X \implies w \ x \geq 0$
assumes *sum-weights*: $(\sum x \in X. w \ x) = 1$
shows $(\prod x \in X. a \ x \text{ powr } w \ x) \leq (\sum x \in X. w \ x * a \ x)$
proof (*cases* $\exists x \in X. a \ x = 0$)
 case *True*
 hence $(\prod x \in X. a \ x \text{ powr } w \ x) = 0$
 using $\langle \text{finite } X \rangle$ **by** *simp*
 also have $\dots \leq (\sum x \in X. w \ x * a \ x)$
 by (*intro sum-nonneg mult-nonneg-nonneg assms*)
 finally show *?thesis* .
 next
 case *False*
 have $(\sum x \in X - \{x. w \ x = 0\}. w \ x) = (\sum x \in X. w \ x)$
 by (*intro sum.mono-neutral-left assms*) *auto*
 also have $\dots = 1$ **by** *fact*
 finally have *sum-weights'*: $(\sum x \in X - \{x. w \ x = 0\}. w \ x) = 1$.

 have $(\prod x \in X. a \ x \text{ powr } w \ x) = (\prod x \in X - \{x. w \ x = 0\}. a \ x \text{ powr } w \ x)$
 using $\langle \text{finite } X \rangle$ *False* **by** (*intro prod.mono-neutral-right*) *auto*
 also have $\dots \leq (\sum x \in X - \{x. w \ x = 0\}. w \ x * a \ x)$ **using** *assms False*
 by (*intro weighted-arithmetic-geometric-mean-pos sum-weights'*)
 (*auto simp: order.strict-iff-order*)
 also have $\dots = (\sum x \in X. w \ x * a \ x)$
 using $\langle \text{finite } X \rangle$ **by** (*intro sum.mono-neutral-left*) *auto*
 finally show *?thesis* .
 qed

We can derive the regular arithmetic/geometric mean inequality from this by simply setting all the weights to $\frac{1}{n}$:

corollary *arithmetic-geometric-mean*:

fixes *a* :: 'a \Rightarrow real
assumes *finite X*
defines *n* \equiv *card X*
assumes *nonneg*: $\bigwedge x. x \in X \implies a \ x \geq 0$
shows $\text{root } n \ (\prod x \in X. a \ x) \leq (\sum x \in X. a \ x) / n$
proof (*cases* $X = \{\}$)
 case *False*
 with *assms* **have** *n*: $n > 0$
 by *auto*
 have $(\prod x \in X. a \ x \text{ powr } (1 / n)) \leq (\sum x \in X. (1 / n) * a \ x)$
 using *n assms* **by** (*intro weighted-arithmetic-geometric-mean*) *auto*
 also have $(\prod x \in X. a \ x \text{ powr } (1 / n)) = (\prod x \in X. a \ x) \text{ powr } (1 / n)$
 using *nonneg* **by** (*subst powr-sum-distrib-real-left*) *auto*
 also have $\dots = \text{root } n \ (\prod x \in X. a \ x)$
 using $\langle n > 0 \rangle$ *nonneg* **by** (*subst root-powr-inverse'*) (*auto simp: prod-nonneg*)
 also have $(\sum x \in X. (1 / n) * a \ x) = (\sum x \in X. a \ x) / n$
 by (*subst sum-distrib-left [symmetric]*) *auto*
 finally show *?thesis* .

qed (auto simp: n-def)

1.3 The Equality Case

Next, we show that weighted arithmetic and geometric mean are equal if and only if all the a_i are equal.

We first prove the more difficult direction as a lemmas and again first assume positivity of all a_i and w_i and will relax this somewhat later.

lemma *weighted-arithmetic-geometric-mean-eq-iff-pos:*

fixes $a\ w :: 'a \Rightarrow \text{real}$

assumes *finite X*

assumes *pos1*: $\bigwedge x. x \in X \implies a\ x > 0$

assumes *pos2*: $\bigwedge x. x \in X \implies w\ x > 0$

assumes *sum-weights*: $(\sum x \in X. w\ x) = 1$

assumes *eq*: $(\prod x \in X. a\ x\ \text{powr}\ w\ x) = (\sum x \in X. w\ x * a\ x)$

shows $\forall x \in X. \forall y \in X. a\ x = a\ y$

proof –

note *nonneg1* = *less-imp-le*[OF *pos1*]

note *nonneg2* = *less-imp-le*[OF *pos2*]

define *A* **where** $A = (\sum x \in X. w\ x * a\ x)$

define *r* **where** $r = (\lambda x. a\ x / A - 1)$

from *sum-weights* **have** $X \neq \{\}$ **by** *auto*

hence $A \neq 0$

unfolding *A-def* **using** *nonneg1 nonneg2 pos1 pos2* $\langle \text{finite } X \rangle$

by (*subst sum-nonneg-eq-0-iff*) *force+*

moreover from *nonneg1 nonneg2* **have** $A \geq 0$

by (*auto simp: A-def intro!: sum-nonneg*)

ultimately have $A > 0$ **by** *simp*

have *r-ge*: $r\ x \geq -1$ **if** $x: x \in X$ **for** x

using $\langle A > 0 \rangle$ *pos1*[OF x] **by** (*auto simp: r-def field-simps*)

have $(\prod x \in X. (1 + r\ x)\ \text{powr}\ w\ x) = (\prod x \in X. (a\ x / A)\ \text{powr}\ w\ x)$

by (*simp add: r-def*)

also have $\dots = (\prod x \in X. a\ x\ \text{powr}\ w\ x) / (\prod x \in X. A\ \text{powr}\ w\ x)$

unfolding *prod-dividef* [*symmetric*]

using *assms pos2* $\langle A > 0 \rangle$ **by** (*intro prod.cong powr-divide*) (*auto intro: less-imp-le*)

also have $(\prod x \in X. A\ \text{powr}\ w\ x) = \text{exp}\ ((\sum x \in X. w\ x) * \ln A)$

using $\langle A > 0 \rangle$ **and** $\langle \text{finite } X \rangle$ **by** (*simp add: powr-def exp-sum sum-distrib-right*)

also have $(\sum x \in X. w\ x) = 1$ **by** *fact*

also have $\text{exp}\ (1 * \ln A) = A$

using $\langle A > 0 \rangle$ **by** *simp*

finally have *lhs*: $(\prod x \in X. (1 + r\ x)\ \text{powr}\ w\ x) = (\prod x \in X. a\ x\ \text{powr}\ w\ x) / A .$

have $(\prod x \in X. \text{exp}\ (w\ x * r\ x)) = \text{exp}\ (\sum x \in X. w\ x * r\ x)$

using $\langle \text{finite } X \rangle$ **by** (*simp add: exp-sum*)

also have $(\sum x \in X. w\ x * r\ x) = (\sum x \in X. a\ x * w\ x) / A - 1$

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using ⟨A > 0⟩ by (simp add: r-def algebra-simps sum-subtractf sum-divide-distrib
sum-weights)
also have (∑ x∈X. a x * w x) / A = 1
  using ⟨A > 0⟩ by (simp add: A-def mult.commute)
finally have rhs: (∏ x∈X. exp (w x * r x)) = 1 by simp

have a x = A if x: x ∈ X for x
proof -
  have (1 + r x) powr w x = exp (w x * r x)
  proof (rule prod-ge-pointwise-le-imp-pointwise-eq
    [where f = λx. (1 + r x) powr w x and g = λx. exp (w x * r x)])
  show (1 + r x) powr w x ≤ exp (w x * r x) if x: x ∈ X for x
  proof -
    have 1 + r x ≤ exp (r x)
      by (rule exp-ge-add-one-self)
    hence (1 + r x) powr w x ≤ exp (r x) powr w x
      using nonneg1[of x] nonneg2[of x] x ⟨A > 0⟩
      by (intro powr-mono2) (auto simp: r-def field-simps)
    also have ... = exp (w x * r x)
      by (simp add: powr-def)
    finally show (1 + r x) powr w x ≤ exp (w x * r x) .
  qed
next
show (∏ x∈X. (1 + r x) powr w x) ≥ (∏ x∈X. exp (w x * r x))
proof -
  have (∏ x∈X. (1 + r x) powr w x) = (∏ x∈X. a x powr w x) / A
    by (fact lhs)
  also have ... = 1
    using ⟨A ≠ 0⟩ by (simp add: eq A-def)
  also have ... = (∏ x∈X. exp (w x * r x))
    by (simp add: rhs)
  finally show ?thesis by simp
qed
qed (use x ⟨finite X⟩ in auto)

also have exp (w x * r x) = exp (r x) powr w x
  by (simp add: powr-def)
finally have 1 + r x = exp (r x)
  using x pos2[of x] r-ge[of x] by (subst (asm) powr-left-real-eq-iff) auto
hence r x = 0
  using exp-real-eq-one-plus-iff[of r x] by auto
hence a x = A
  using ⟨A > 0⟩ by (simp add: r-def field-simps)
thus ?thesis
  by (simp add: )
qed
thus ∀ x∈X. ∀ y∈X. a x = a y
  by auto
qed

```

We can now show the full theorem and relax the positivity condition on the a_i to non-negativity. This is possible because if some a_i is zero and the two means coincide, then the product is obviously 0, but the sum can only be 0 if *all* the a_i are 0.

theorem *weighted-arithmetic-geometric-mean-eq-iff*:

fixes $a\ w :: 'a \Rightarrow \text{real}$
assumes *finite* X
assumes *nonneg1*: $\bigwedge x. x \in X \implies a\ x \geq 0$
assumes *pos2*: $\bigwedge x. x \in X \implies w\ x > 0$
assumes *sum-weights*: $(\sum x \in X. w\ x) = 1$
shows $(\prod x \in X. a\ x\ \text{powr}\ w\ x) = (\sum x \in X. w\ x * a\ x) \longleftrightarrow X \neq \{\} \wedge (\forall x \in X. \forall y \in X. a\ x = a\ y)$

proof

assume $*$: $X \neq \{\} \wedge (\forall x \in X. \forall y \in X. a\ x = a\ y)$
from $*$ **have** $X \neq \{\}$
by *blast*

from $*$ **obtain** c **where** $c: \bigwedge x. x \in X \implies a\ x = c\ c \geq 0$

proof (*cases* $X = \{\}$)

case *False*

then obtain x **where** $x \in X$ **by** *blast*

thus *?thesis* **using** $*$ *that*[*of* $a\ x$] *nonneg1*[*of* x] **by** *metis*

next

case *True*

thus *?thesis*

using *that*[*of* 1] **by** *auto*

qed

have $(\prod x \in X. a\ x\ \text{powr}\ w\ x) = (\prod x \in X. c\ \text{powr}\ w\ x)$

by (*simp add*: c)

also have $\dots = c$

using *assms* $c \langle X \neq \{\} \rangle$ **by** (*cases* $c = 0$) (*auto simp*: *powr-sum-distrib-real-right*)

also have $\dots = (\sum x \in X. w\ x * a\ x)$

using *sum-weights* **by** (*simp add*: $c(1)$ *flip*: *sum-distrib-left sum-distrib-right*)

finally show $(\prod x \in X. a\ x\ \text{powr}\ w\ x) = (\sum x \in X. w\ x * a\ x)$.

next

assume $*$: $(\prod x \in X. a\ x\ \text{powr}\ w\ x) = (\sum x \in X. w\ x * a\ x)$

have $X \neq \{\}$

using $*$ **by** *auto*

moreover have $(\forall x \in X. \forall y \in X. a\ x = a\ y)$

proof (*cases* $\exists x \in X. a\ x = 0$)

case *False*

with *nonneg1* **have** *pos1*: $\forall x \in X. a\ x > 0$

by *force*

thus *?thesis*

using *weighted-arithmetic-geometric-mean-eq-iff-pos*[*of* $X\ a\ w$] *assms* $*$

by *blast*

next

case *True*

hence $(\prod x \in X. a \ x \ \text{powr} \ w \ x) = 0$
using *assms* **by** *auto*
with $*$ **have** $(\sum x \in X. w \ x \ * \ a \ x) = 0$
by *auto*
also have $?this \longleftrightarrow (\forall x \in X. w \ x \ * \ a \ x = 0)$
using *assms* **by** (*intro sum-nonneg-eq-0-iff mult-nonneg-nonneg*) (*auto intro: less-imp-le*)
finally have $(\forall x \in X. a \ x = 0)$
using *pos2* **by** *force*
thus $?thesis$
by *auto*
qed
ultimately show $X \neq \{\} \wedge (\forall x \in X. \forall y \in X. a \ x = a \ y)$
by *blast*
qed

Again, we derive a version for the unweighted arithmetic/geometric mean.

corollary *arithmetic-geometric-mean-eq-iff*:

fixes $a :: 'a \Rightarrow \text{real}$
assumes *finite X*
defines $n \equiv \text{card } X$
assumes *nonneg*: $\bigwedge x. x \in X \implies a \ x \geq 0$
shows $\text{root } n \ (\prod x \in X. a \ x) = (\sum x \in X. a \ x) / n \longleftrightarrow (\forall x \in X. \forall y \in X. a \ x = a \ y)$
proof (*cases X = \{\}*)
case *False*
with *assms* **have** $n > 0$
by *auto*
have $(\prod x \in X. a \ x \ \text{powr} \ (1 / n)) = (\sum x \in X. (1 / n) * a \ x) \longleftrightarrow$
 $X \neq \{\} \wedge (\forall x \in X. \forall y \in X. a \ x = a \ y)$
using *assms False* **by** (*intro weighted-arithmetic-geometric-mean-eq-iff*) *auto*
also have $(\prod x \in X. a \ x \ \text{powr} \ (1 / n)) = (\prod x \in X. a \ x) \ \text{powr} \ (1 / n)$
using *nonneg* **by** (*subst powr-sum-distrib-real-left*) *auto*
also have $\dots = \text{root } n \ (\prod x \in X. a \ x)$
using $\langle n > 0 \rangle$ *nonneg* **by** (*subst root-powr-inverse'*) (*auto simp: prod-nonneg*)
also have $(\sum x \in X. (1 / n) * a \ x) = (\sum x \in X. a \ x) / n$
by (*subst sum-distrib-left [symmetric]*) *auto*
finally show $?thesis$ **using** *False* **by** *auto*
qed (*auto simp: n-def*)

1.4 The Binary Version

For convenience, we also derive versions for only two numbers:

corollary *weighted-arithmetic-geometric-mean-binary*:

fixes $w1 \ w2 \ x1 \ x2 :: \text{real}$
assumes $x1 \geq 0 \ x2 \geq 0 \ w1 \geq 0 \ w2 \geq 0 \ w1 + w2 = 1$
shows $x1 \ \text{powr} \ w1 \ * \ x2 \ \text{powr} \ w2 \leq w1 \ * \ x1 + w2 \ * \ x2$
proof –
let $?a = \lambda b. \text{if } b \ \text{then } x1 \ \text{else } x2$

let $?w = \lambda b. \text{if } b \text{ then } w1 \text{ else } w2$
from *assms* **have** $(\prod x \in UNIV. ?a \ x \ \text{powr } ?w \ x) \leq (\sum x \in UNIV. ?w \ x * ?a \ x)$
by (*intro weighted-arithmetic-geometric-mean*) (*auto simp add: UNIV-bool*)
thus $?thesis$ **by** (*simp add: UNIV-bool add-ac mult-ac*)
qed

corollary *weighted-arithmetic-geometric-mean-eq-iff-binary*:

fixes $w1 \ w2 \ x1 \ x2 :: \text{real}$
assumes $x1 \geq 0 \ x2 \geq 0 \ w1 > 0 \ w2 > 0 \ w1 + w2 = 1$
shows $x1 \ \text{powr } w1 * x2 \ \text{powr } w2 = w1 * x1 + w2 * x2 \longleftrightarrow x1 = x2$
proof –
let $?a = \lambda b. \text{if } b \text{ then } x1 \text{ else } x2$
let $?w = \lambda b. \text{if } b \text{ then } w1 \text{ else } w2$
from *assms* **have** $(\prod x \in UNIV. ?a \ x \ \text{powr } ?w \ x) = (\sum x \in UNIV. ?w \ x * ?a \ x)$
 $\longleftrightarrow (UNIV :: \text{bool set}) \neq \{\}$ $\wedge (\forall x \in UNIV. \forall y \in UNIV. ?a \ x =$
 $?a \ y)$
by (*intro weighted-arithmetic-geometric-mean-eq-iff*) (*auto simp add: UNIV-bool*)
thus $?thesis$ **by** (*auto simp: UNIV-bool add-ac mult-ac*)
qed

corollary *arithmetic-geometric-mean-binary*:

fixes $x1 \ x2 :: \text{real}$
assumes $x1 \geq 0 \ x2 \geq 0$
shows $\text{sqrt } (x1 * x2) \leq (x1 + x2) / 2$
using *weighted-arithmetic-geometric-mean-binary*[of $x1 \ x2 \ 1/2 \ 1/2$] *assms*
by (*simp add: powr-half-sqrt field-simps real-sqrt-mult*)

corollary *arithmetic-geometric-mean-eq-iff-binary*:

fixes $x1 \ x2 :: \text{real}$
assumes $x1 \geq 0 \ x2 \geq 0$
shows $\text{sqrt } (x1 * x2) = (x1 + x2) / 2 \longleftrightarrow x1 = x2$
using *weighted-arithmetic-geometric-mean-eq-iff-binary*[of $x1 \ x2 \ 1/2 \ 1/2$] *assms*
by (*simp add: powr-half-sqrt field-simps real-sqrt-mult*)

end

References

- [1] J. M. Steele. *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge University Press, 2004.