

Weight-Balanced Trees

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June 24, 2019

Abstract

This theory provides a verified implementation of weight-balanced trees following the work of Hirai and Yamamoto [4] who proved that all parameters in a certain range are valid, i.e. guarantee that insertion and deletion preserve weight-balance. Instead of a general theorem we provide parameterized proofs of preservation of the invariant that work for many (all?) valid parameters.

1 Introduction

Weight-balanced trees (*WB trees*) are a class of binary search trees of logarithmic height. They were invented by Nievergelt and Reingold [5, 6] who called them *trees of bounded balance*. They are parametrized by a constant. Parameters are called *valid* if they guarantee that insertion and deletion preserve the WB invariant. Blum and Mehlhorn [3] later discovered that there is a flaw in Nievergelt and Reingold’s analysis of valid parameters and gave a detailed correctness proof for a modified range of parameters. Adams [1, 2] considered a slightly modified version of WB trees and analyzed which parameters are valid. The Haskell libraries `Data.Set` and `Data.Map` are based on Adams’ papers but it was found that the implementation did not preserve the invariant. This motivated Hirai and Yamamoto [4] to verify the valid parameter range for the original definition of WB tree formally in Coq. They also showed that Adams’ analysis is flawed by giving a counterexample to Adams’ claimed range of valid parameters. Straka [8] analyzes valid parameters for Adam’s variant. Yet another variant of WB trees was considered by Roura [7].

2 Weight-Balanced Trees Have Logarithmic Height

This theory is based on the original definition of weight-balanced trees [5, 6] where the size of the child of a node must be a minimum and a maximum fraction of the size of the node.

theory *Weight_Balanced_Trees_log*

```

imports
  Complex_Main
  HOL-Library.Tree
begin

lemmas neq0_if = less_imp_neq dual_order.strict_implies_not_eq

locale WBT0 =
fixes  $\alpha :: \text{real}$ 
assumes alpha_pos: 0 < \alpha and alpha_ub: \alpha \le 1/2
begin

fun wbt :: 'a tree \Rightarrow bool where
  wbt Leaf = True |
  wbt (Node l _ r) = (wbt l \wedge wbt r \wedge (let ratio = size1 l / (size1 l + size1 r)
    in \alpha \le ratio \wedge ratio \le 1 - \alpha))

lemma height_size1_exp:
  wbt t \Longrightarrow t \neq Leaf \Longrightarrow 2 \le (1-\alpha) ^ (height t - 1) * size1 t
  \langle proof \rangle

lemma height_size1_log: assumes wbt t t \neq Leaf
shows height t \le (log 2 (size1 t) - 1) / log 2 (1/(1-\alpha)) + 1
  \langle proof \rangle

end

end

```

3 Weight Balanced Tree Implementation of Sets

This theory follows Hirai and Yamamoto but we do not prove their general theorem. Instead we provide a short parameterized theory that, when interpreted with valid parameters, will prove perservation of the invariant for these parameters.

```

theory Weight_Balanced_Trees
imports
  HOL-Data_Structures.Isin2
begin

lemma neq_Leaf2_iff: t \neq Leaf \longleftrightarrow (\exists n l a r. t = Node n l a r)
  \langle proof \rangle

type-synonym 'a wbt = ('a, nat) tree

fun size_wbt :: 'a wbt \Rightarrow nat where
  size_wbt Leaf = 0 |

```

$size_wbt (Node _ _ n _) = n$

Smart constructor:

fun $N :: 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a wbt$ **where**
 $N l a r = Node l a (size_wbt l + size_wbt r + 1) r$

Basic Rotations:

fun $rot1L :: 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a wbt$ **where**
 $rot1L A a B b C = N (N A a B) b C$

fun $rot1R :: 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a wbt$ **where**
 $rot1R A a B b C = N A a (N B b C)$

fun $rot2 :: 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a \Rightarrow 'a wbt \Rightarrow 'a wbt$ **where**
 $rot2 A a (Node B1 b _ B2) c C = N (N A a B1) b (N B2 c C)$

3.1 WB trees

Parameters:

Δ determines when a tree needs to be rebalanced

Γ determines whether it needs to be single or double rotation.

We represent rational numbers as pairs: $\Delta = \Delta1/\Delta2$ and $\Gamma = \Gamma1/\Gamma2$.

Hirai and Yamamoto [4] proved that under the following constraints insertion and deletion preserve the WB invariant, i.e. Δ and Γ are *valid*:

definition $valid_params :: nat \Rightarrow nat \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
 $valid_params \Delta1 \Delta2 \Gamma1 \Gamma2 = ($
 $\Delta1 * 2 < \Delta2 * 9 \quad \text{--- right: } \Delta < 4.5 \wedge$
 $\Gamma1 * \Delta2 + \Gamma2 * \Delta2 \leq \Gamma2 * \Delta1 \quad \text{--- left: } \Gamma + 1 \leq \Delta \wedge$
 $\Gamma1 * \Delta1 \geq \Gamma2 * (\Delta1 + \Delta2) \quad \text{--- lower: } \Gamma \geq (\Delta + 1) / \Delta \wedge$
 --- upper:
 $(5*\Delta2 \leq 2*\Delta1 \wedge 1*\Delta1 < 3*\Delta2 \longrightarrow \Gamma1*2 \leq \Gamma2*3)$
 $\quad \text{--- } \Gamma \leq 3/2 \text{ if } 2.5 \leq \Delta < 3 \wedge$
 $(3*\Delta2 \leq 1*\Delta1 \wedge 2*\Delta1 < 7*\Delta2 \longrightarrow \Gamma1*2 \leq \Gamma2*4)$
 $\quad \text{--- } \Gamma \leq 4/2 \text{ if } 3 \leq \Delta < 3.5 \wedge$
 $(7*\Delta2 \leq 2*\Delta1 \wedge 1*\Delta1 < 4*\Delta2 \longrightarrow \Gamma1*3 \leq \Gamma2*4)$
 $\quad \text{--- } \Gamma \leq 4/3 \text{ when } 3.5 \leq \Delta < 4 \wedge$
 $(4*\Delta2 \leq 1*\Delta1 \wedge 2*\Delta1 < 9*\Delta2 \longrightarrow \Gamma1*3 \leq \Gamma2*5)$
 $\quad \text{--- } \Gamma \leq 5/3 \text{ when } 4 \leq \Delta < 4.5$
 $)$

We do not make use of these constraints and do not prove that they guarantee preservation of the invariant. Instead, we provide generic proofs of invariant preservation that work for many (all?) interpretations of locale *WBT* (below) with valid parameters. Further down we demonstrate this by

interpreting *WBT* with a selection of valid parameters. [For some parameters, some *smt* proofs fail because *smt* on *nats* fails although on non-negative *ints* it succeeds, i.e. the goal should be provable. This is a shortcoming of *smt* that is under investigation.]

Locale *WBT* comes with some minimal assumptions ($\Gamma1 > \Gamma2$ and $\Delta1 > \Delta2$) which follow from *valid_params* and from which we conclude some simple lemmas.

```

locale WBT =
fixes  $\Delta1 \Delta2 :: nat$  and  $\Gamma1 \Gamma2 :: nat$ 
assumes Delta_gr1:  $\Delta1 > \Delta2$  and Gamma_gr1:  $\Gamma1 > \Gamma2$ 
begin

```

3.1.1 Balance Indicators

```

fun balanced1 :: 'a wbt  $\Rightarrow$  'a wbt  $\Rightarrow$  bool where
balanced1 t1 t2 = ( $\Delta1 * (size\_wbt\ t1 + 1) \geq \Delta2 * (size\_wbt\ t2 + 1)$ )

```

The global weight-balanced tree invariant:

```

fun wbt :: 'a wbt  $\Rightarrow$  bool where
wbt Leaf = True |
wbt (Node l _ n r) =
  ( $n = size\ l + size\ r + 1 \wedge balanced1\ l\ r \wedge balanced1\ r\ l \wedge wbt\ l \wedge wbt\ r$ )

```

```

lemma size_wbt_eq_size[simp]: wbt t  $\Longrightarrow$  size_wbt t = size t
<proof>

```

```

fun single :: 'a wbt  $\Rightarrow$  'a wbt  $\Rightarrow$  bool where
single t1 t2 = ( $\Gamma1 * (size\_wbt\ t2 + 1) > \Gamma2 * (size\_wbt\ t1 + 1)$ )

```

3.1.2 Code

```

fun rotateL :: 'a wbt  $\Rightarrow$  'a  $\Rightarrow$  'a wbt  $\Rightarrow$  'a wbt where
rotateL A a (Node B b _ C) =
  (if single B C then rot1L A a B b C else rot2 A a B b C)

```

```

fun balanceL :: 'a wbt  $\Rightarrow$  'a  $\Rightarrow$  'a wbt  $\Rightarrow$  'a wbt where
balanceL l a r = (if balanced1 l r then N l a r else rotateL l a r)

```

```

fun rotateR :: 'a wbt  $\Rightarrow$  'a  $\Rightarrow$  'a wbt  $\Rightarrow$  'a wbt where
rotateR (Node A a _ B) b C =
  (if single B A then rot1R A a B b C else rot2 A a B b C)

```

```

fun balanceR :: 'a wbt  $\Rightarrow$  'a  $\Rightarrow$  'a wbt  $\Rightarrow$  'a wbt where
balanceR l a r = (if balanced1 r l then N l a r else rotateR l a r)

```

```

fun insert :: 'a::linorder  $\Rightarrow$  'a wbt  $\Rightarrow$  'a wbt where
insert x Leaf = Node Leaf x 1 Leaf |
insert x (Node l a n r) =

```

```

    (case cmp x a of
      LT ⇒ balanceR (insert x l) a r |
      GT ⇒ balanceL l a (insert x r) |
      EQ ⇒ Node l a n r )

fun split_min :: 'a wbt ⇒ 'a * 'a wbt where
split_min (Node l a _ r) =
  (if l = Leaf then (a,r) else let (x,l') = split_min l in (x, balanceL l' a r))

fun del_max :: 'a wbt ⇒ 'a * 'a wbt where
del_max (Node l a _ r) =
  (if r = Leaf then (a,l) else let (x,r') = del_max r in (x, balanceR l a r'))

fun combine :: 'a wbt ⇒ 'a wbt ⇒ 'a wbt where
combine Leaf Leaf = Leaf |
combine Leaf r = r |
combine l Leaf = l |
combine l r =
  (if size l > size r then
    let (lMax, l') = del_max l in balanceL l' lMax r
  else
    let (rMin, r') = split_min r in balanceR l rMin r')

fun delete :: 'a::linorder ⇒ 'a wbt ⇒ 'a wbt where
delete _ Leaf = Leaf |
delete x (Node l a _ r) =
  (case cmp x a of
    LT ⇒ balanceL (delete x l) a r |
    GT ⇒ balanceR l a (delete x r) |
    EQ ⇒ combine l r)

```

3.2 Functional Correctness Proofs

A WB tree must be of a certain structure if `balanced1` and `single` are `False`.

lemma *not_Leaf_if_not_balanced1*:

assumes \neg `balanced1 l r`

shows $r \neq$ `Leaf`

<proof>

lemma *not_Leaf_if_not_single*:

assumes \neg `single l r`

shows $l \neq$ `Leaf`

<proof>

3.2.1 Inorder Properties

lemma *inorder_rot2*:

$B \neq$ `Leaf` \implies `inorder(rot2 A a B b C) = inorder A @ a # inorder B @ b # inorder C`

<proof>

lemma *inorder_rotateL*:

$r \neq \text{Leaf} \implies \text{inorder}(\text{rotateL } l \ a \ r) = \text{inorder } l \ @ \ a \ \# \ \text{inorder } r$
<proof>

lemma *inorder_rotateR*:

$l \neq \text{Leaf} \implies \text{inorder}(\text{rotateR } l \ a \ r) = \text{inorder } l \ @ \ a \ \# \ \text{inorder } r$
<proof>

lemma *inorder_insert*:

$\text{sorted}(\text{inorder } t) \implies \text{inorder}(\text{insert } x \ t) = \text{ins_list } x \ (\text{inorder } t)$
<proof>

lemma *split_minD*:

$\text{split_min } t = (x, t') \implies t \neq \text{Leaf} \implies x \ \# \ \text{inorder } t' = \text{inorder } t$
<proof>

lemma *del_maxD*:

$\text{del_max } t = (x, t') \implies t \neq \text{Leaf} \implies \text{inorder } t' \ @ \ [x] = \text{inorder } t$
<proof>

lemma *inorder_combine*:

$\text{inorder}(\text{combine } l \ r) = \text{inorder } l \ @ \ \text{inorder } r$
<proof>

lemma *inorder_delete*:

$\text{sorted}(\text{inorder } t) \implies \text{inorder}(\text{delete } x \ t) = \text{del_list } x \ (\text{inorder } t)$
<proof>

3.3 Size Lemmas

3.3.1 Insertion

lemma *size_rot2L[simp]*:

$B \neq \text{Leaf} \implies \text{size}(\text{rot2 } A \ a \ B \ b \ C) = \text{size } A + \text{size } B + \text{size } C + 2$
<proof>

lemma *size_rotateR[simp]*:

$l \neq \text{Leaf} \implies \text{size}(\text{rotateR } l \ a \ r) = \text{size } l + \text{size } r + 1$
<proof>

lemma *size_rotateL[simp]*:

$r \neq \text{Leaf} \implies \text{size}(\text{rotateL } l \ a \ r) = \text{size } l + \text{size } r + 1$
<proof>

lemma *size_length*: $\text{size } t = \text{length } (\text{inorder } t)$

<proof>

lemma *size_insert*: $\text{size } (\text{insert } x \ t) = (\text{if } \text{isin } t \ x \ \text{then } \text{size } t \ \text{else } \text{Suc } (\text{size } t))$

<proof>

3.3.2 Deletion

lemma *size_delete_if_isin*: $isin\ t\ x \implies size\ t = Suc\ (size(delete\ x\ t))$
<proof>

lemma *delete_id_if_wbt_notin*: $wbt\ t \implies \neg\ isin\ t\ x \implies delete\ x\ t = t$
<proof>

lemma *size_split_min*: $t \neq Leaf \implies size\ t = Suc\ (size\ (snd\ (split_min\ t)))$
<proof>

lemma *size_del_max*: $t \neq Leaf \implies size\ t = Suc(size(snd(del_max\ t)))$
<proof>

3.4 Auxiliary Definitions

fun *balanced1_arith* :: $nat \Rightarrow nat \Rightarrow bool$ **where**
balanced1_arith $a\ b = (\Delta1 * (a + 1) \geq \Delta2 * (b + 1))$

fun *balanced2_arith* :: $nat \Rightarrow nat \Rightarrow bool$ **where**
balanced2_arith $a\ b = (balanced1_arith\ a\ b \wedge balanced1_arith\ b\ a)$

fun *singly_balanced_arith* :: $nat \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
singly_balanced_arith $x\ y\ w = (balanced2_arith\ x\ y \wedge balanced2_arith\ (x+y+1)\ w)$

fun *doubly_balanced_arith* :: $nat \Rightarrow nat \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
doubly_balanced_arith $x\ y\ z\ w =$
 $(balanced2_arith\ x\ y \wedge balanced2_arith\ z\ w \wedge balanced2_arith\ (x+y+1)\ (z+w+1))$

end

3.5 Preservation of WB tree Invariant for Concrete Parameters

A number of sample interpretations with valid parameters:

interpretation *WBT* **where**
 $\Delta1 = 25$ **and** $\Delta2 = 10$ **and** $\Gamma1 = 14$ **and** $\Gamma2 = 10$

<proof>

lemma *wbt_insert*:

wbt t \implies *wbt (insert x t)*

<proof>

declare *[[smt_nat_as_int]]*

Show that invariant is preserved by deletion in the left/right subtree:

lemma *wbt_balanceL*:

assumes *wbt (Node l a n r)* *wbt l'* *size l = size l' + 1*

shows *wbt (balanceL l' a' r)*

<proof>

lemma *wbt_balanceR*:

assumes *wbt (Node l a n r)* *wbt r'* *size r = size r' + 1*

shows *wbt (balanceR l a' r')*

<proof>

lemma *wbt_split_min*: *t* \neq *Leaf* \implies *wbt t* \implies *wbt (snd (split_min t))*

<proof>

lemma *wbt_del_max*: *t* \neq *Leaf* \implies *wbt t* \implies *wbt (snd (del_max t))*

<proof>

lemma *wbt_delete*: *wbt t* \implies *wbt (delete x t)*

<proof>

3.6 The final correctness proof

interpretation *S*: *Set_by_Ordered*

where *empty = Leaf* **and** *isin = isin* **and** *insert = insert* **and** *delete = delete*

and *inorder = inorder* **and** *inv = wbt*

<proof>

end

References

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