

# VectorSpace

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## Abstract

I present a formalisation of basic linear algebra based completely on locales, building off HOL-Algebra. It includes the following:

1. basic definitions: linear combinations, span, linear independence
2. linear transformations
3. interpretation of function spaces as vector spaces
4. direct sum of vector spaces, sum of subspaces
5. the replacement theorem
6. existence of bases in finite-dimensional vector spaces, definition of dimension
7. rank-nullity theorem.

Note that some concepts are actually defined and proved for modules as they also apply there.

In the process, I also prove some basic facts about rings, modules, and fields, as well as finite sums in monoids/modules.

Note that infinite-dimensional vector spaces are supported, but dimension is only supported for finite-dimensional vector spaces.

The proofs are standard; the proofs of the replacement theorem and rank-nullity theorem roughly follow the presentation in [?]. The rank-nullity theorem generalises the existing development in [?] (originally using type classes, now using a mix of type classes and locales).

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# 1 Basic facts about rings and modules

```
theory RingModuleFacts
imports Main
  HOL-Algebra.Module
  HOL-Algebra.Coset
```

begin

## 1.1 Basic facts

In a field, every nonzero element has an inverse.

```
lemma (in field) inverse-exists [simp, intro]:
  assumes h1: a ∈ carrier R and h2: a ≠ 0_R
  shows inv_R a ∈ carrier R
proof -
  have 1: Units R = carrier R - {0_R} by (rule field-Units)
  from h1 h2 1 show ?thesis by auto
qed
```

Multiplication by 0 in  $R$  gives 0. (Note that this fact encompasses smult-l-null as this is for module while that is for algebra, so smult-l-null is redundant.)

```
lemma (in module) lmult-0 [simp]:
  assumes 1: m ∈ carrier M
  shows 0_R ⊙_M m = 0_M
proof -
  from 1 have 0: 0_R ⊙_M m ∈ carrier M by simp
  from 1 have 2: 0_R ⊙_M m = (0_R ⊕_R 0_R) ⊙_M m by simp
  from 1 have 3: (0_R ⊕_R 0_R) ⊙_M m = (0_R ⊙_M m) ⊕_M (0_R ⊙_M m)
  using [[simp-trace, simp-trace-depth-limit=3]]
  by (simp add: smult-l-distr del: R.add.r-one R.add.l-one)
```

**from** 2 3 **have** 4:  $\mathbf{0}_R \odot_M m = (\mathbf{0}_R \odot_M m) \oplus_M (\mathbf{0}_R \odot_M m)$  **by** *auto*  
**from** 0 4 **show** ?thesis  
**using** *M.l-neg M.r-neg1* **by** *fastforce*  
**qed**

Multiplication by 0 in  $M$  gives 0.

**lemma** (**in** *module*) *rmult-0* [*simp*]:  
**assumes** 0:  $r \in \text{carrier } R$   
**shows**  $r \odot_M \mathbf{0}_M = \mathbf{0}_M$   
**by** (*metis M.zero-closed R.zero-closed assms lmult-0 r-null smult-assoc1*)

Multiplication by  $-1$  is the same as negation. May be useful as a simp rule.

**lemma** (**in** *module*) *smult-minus-1*:  
**fixes**  $v$   
**assumes** 0:  $v \in \text{carrier } M$   
**shows**  $(\ominus_R \mathbf{1}_R) \odot_M v = (\ominus_M v)$

**proof** –

**from** 0 **have** a0:  $\mathbf{1}_R \odot_M v = v$  **by** *simp*  
**from** 0 **have** 1:  $((\ominus_R \mathbf{1}_R) \oplus_R \mathbf{1}_R) \odot_M v = \mathbf{0}_M$   
**by** (*simp add: R.l-neg*)  
**from** 0 **have** 2:  $((\ominus_R \mathbf{1}_R) \oplus_R \mathbf{1}_R) \odot_M v = (\ominus_R \mathbf{1}_R) \odot_M v \oplus_M \mathbf{1}_R \odot_M v$   
**by** (*simp add: smult-l-distr*)  
**from** 1 2 **show** ?thesis **by** (*metis M.minus-equality R.add.inv-closed*)

*a0* *assms one-closed smult-closed*)

**qed**

The version with equality reversed.

**lemmas** (**in** *module*) *smult-minus-1-back = smult-minus-1* [*THEN sym*]

$-1$  is not 0

**lemma** (**in** *field*) *neg-1-not-0* [*simp*]:  $\ominus_R \mathbf{1}_R \neq \mathbf{0}_R$   
**by** (*metis minus-minus minus-zero one-closed zero-not-one*)

Note *smult-assoc1* is the wrong way around for simplification. This is the reverse of *smult-assoc1*.

**lemma** (**in** *module*) *smult-assoc-simp*:  
 $[[ a \in \text{carrier } R; b \in \text{carrier } R; x \in \text{carrier } M ]] ==>$   
 $a \odot_M (b \odot_M x) = (a \otimes b) \odot_M x$   
**by** (*auto simp add: smult-assoc1*)

**lemmas** (**in** *abelian-group*) *show-r-zero = add.l-cancel-one*

**lemmas** (**in** *abelian-group*) *show-l-zero = add.r-cancel-one*

A nontrivial ring has  $0 \neq 1$ .

```

lemma (in ring) nontrivial-ring [simp]:
  assumes carrier  $R \neq \mathbf{0}_R$ 
  shows  $\mathbf{0}_R \neq \mathbf{1}_R$ 
proof (rule ccontr)
  assume  $1: \neg(\mathbf{0}_R \neq \mathbf{1}_R)$ 
  {
    fix  $r$ 
    assume  $2: r \in \text{carrier } R$ 
    from  $1\ 2$  have  $3: \mathbf{1}_R \otimes_R r = \mathbf{0}_R \otimes_R r$  by auto
    from  $2\ 3$  have  $r = \mathbf{0}_R$  by auto
  }
  from this assms show False by auto
qed

```

Use as simp rule. To show  $a - b = 0$ , it suffices to show  $a = b$ .

```

lemma (in abelian-group) minus-other-side [simp]:
   $\llbracket a \in \text{carrier } G; b \in \text{carrier } G \rrbracket \implies (a \ominus_G b = \mathbf{0}_G) = (a = b)$ 
  by (metis a-minus-def add.inv-closed add.m-comm r-neg r-neg2)

```

## 1.2 Units group

Define the units group  $R^\times$  and show it is actually a group.

```

definition units-group::('a,'b) ring-scheme  $\Rightarrow$  'a monoid
  where units-group  $R = (\text{carrier} = \text{Units } R, \text{mult} = (\lambda x y. x \otimes_R y),$ 
   $\text{one} = \mathbf{1}_R)$ 

```

The units form a group.

```

lemma (in ring) units-form-group: group (units-group  $R$ )
  apply (intro groupI)
  apply (unfold units-group-def, auto)
  apply (intro m-assoc)
  apply auto
  apply (unfold Units-def)
  apply auto
  done

```

The units of a *cring* form a commutative group.

```

lemma (in cring) units-form-cgroup: comm-group (units-group  $R$ )
  apply (intro comm-groupI)
  apply (unfold units-group-def) apply auto
  apply (intro m-assoc) apply auto
  apply (unfold Units-def) apply auto
  apply (rule m-comm) apply auto
  done

```

**end**

## 2 Basic lemmas about functions

**theory** *FunctionLemmas*

**imports** *Main*  
*HOL-Library.FuncSet*  
**begin**

These are used in simplification. Note that the difference from Pi-mem is that the statement about the function comes first, so Isabelle can more easily figure out what  $S$  is.

**lemma** *PiE-mem2*:  $f \in S \rightarrow_E T \implies x \in S \implies f x \in T$   
**unfolding** *PiE-def* **by** *auto*

**lemma** *Pi-mem2*:  $f \in S \rightarrow T \implies x \in S \implies f x \in T$   
**unfolding** *Pi-def* **by** *auto*

**end**

## 3 Sums in monoids

**theory** *MonoidSums*

**imports** *Main*  
*HOL-Algebra.Module*  
*RingModuleFacts*  
*FunctionLemmas*  
**begin**

We build on the finite product simplifications in *FiniteProduct.thy* and the analogous ones for finite sums (see "lemmas" in *Ring.thy*).

Use as an intro rule

**lemma** (**in** *comm-monoid*) *factors-equal*:  
 $\llbracket a=b; c=d \rrbracket \implies a \otimes_G c = b \otimes_G d$   
**by** *simp*

**lemma** (**in** *comm-monoid*) *extend-prod*:  
**fixes**  $a A S$   
**assumes** *fin*: *finite S* **and** *subset*:  $A \subseteq S$  **and**  $a: a \in A \rightarrow \text{carrier } G$   
**shows**  $(\bigotimes_G x \in S. (\text{if } x \in A \text{ then } a x \text{ else } \mathbf{1}_G)) = (\bigotimes_G x \in A. a x)$   
**(is**  $(\bigotimes_G x \in S. ?b x) = (\bigotimes_G x \in A. a x)$   
**proof** –  
**from** *subset* **have**  $\text{uni}: S = A \cup (S - A)$  **by** *auto*  
**from** *assms subset* **show** *?thesis*  
**apply** (*subst uni*)  
**apply** (*subst finprod-Un-disjoint, auto*)

**by** (*auto cong: finprod-cong if-cong elim: finite-subset simp add:Pi-def finite-subset*)

**qed**

Scalar multiplication distributes over scalar multiplication (on left).

**lemma** (*in module*) *finsum-smult*:

$\llbracket c \in \text{carrier } R; g \in A \rightarrow \text{carrier } M \rrbracket \implies$

$(c \odot_M \text{finsum } M g A) = \text{finsum } M (\%x. c \odot_M g x) A$

**proof** (*induct A rule: infinite-finite-induct*)

**case** (*insert a A*)

**from** *insert.hyps insert.prem*s **have** 1:  $\text{finsum } M g (\text{insert } a A) = g a \oplus_M \text{finsum } M g A$

**by** (*intro finsum-insert, auto*)

**from** *insert.hyps insert.prem*s **have** 2:  $(\bigoplus_{Mx \in \text{insert } a A} c \odot_M g x) = c \odot_M g a \oplus_M (\bigoplus_{Mx \in A} c \odot_M g x)$

**by** (*intro finsum-insert, auto*)

**from** *insert.hyps insert.prem*s **show** ?*case*

**by** (*auto simp add:1 2 smult-r-distr*)

**qed** *auto*

Scalar multiplication distributes over scalar multiplication (on right).

**lemma** (*in module*) *finsum-smult-r*:

$\llbracket v \in \text{carrier } M; f \in A \rightarrow \text{carrier } R \rrbracket \implies$

$(\text{finsum } R f A \odot_M v) = \text{finsum } M (\%x. f x \odot_M v) A$

**proof** (*induct A rule: infinite-finite-induct*)

**case** (*insert a A*)

**from** *insert.hyps insert.prem*s **have** 1:  $\text{finsum } R f (\text{insert } a A) = f a \oplus_R \text{finsum } R f A$

**by** (*intro R.finsum-insert, auto*)

**from** *insert.hyps insert.prem*s **have** 2:  $(\bigoplus_{Mx \in \text{insert } a A} f x \odot_M v) = f a \odot_M v \oplus_M (\bigoplus_{Mx \in A} f x \odot_M v)$

**by** (*intro finsum-insert, auto*)

**from** *insert.hyps insert.prem*s **show** ?*case*

**by** (*auto simp add:1 2 smult-l-distr*)

**qed** *auto*

A sequence of lemmas that shows that the product does not depend on the ambient group. Note I had to dig back into the definitions of *foldSet* to show this.

**lemma** *foldSet-not-depend*:

**fixes** *A E*

**assumes** *h1: D ⊆ E*

**shows** *foldSetD D f e ⊆ foldSetD E f e*

**proof** –

**from** *h1* **have** 1:  $\bigwedge x1 x2. (x1, x2) \in \text{foldSetD } D f e \implies (x1, x2) \in \text{foldSetD } E f e$

```

proof –
  fix  $x1\ x2$ 
  assume  $2: (x1,x2) \in \text{foldSetD } D\ f\ e$ 
  from  $h1\ 2$  show  $?thesis\ x1\ x2$ 
  apply (intro foldSetD.induct[where  $?D=D$  and  $?f=f$  and  $?e=e$ 
and  $?x1.0=x1$  and  $?x2.0=x2$ 
  and  $?P = \lambda x1\ x2. ((x1, x2) \in \text{foldSetD } E\ f\ e)$ ])
  apply auto
  apply (intro emptyI, auto)
  by (intro insertI, auto)
qed
from 1 show  $?thesis$  by auto
qed

```

```

lemma foldD-not-depend:
  fixes  $D\ E\ B\ f\ e\ A$ 
  assumes  $h1: LCD\ B\ D\ f$  and  $h2: LCD\ B\ E\ f$  and  $h3: D \subseteq E$  and
 $h4: e \in D$  and  $h5: A \subseteq B$  and  $h6: \text{finite } B$ 
  shows  $\text{foldD } D\ f\ e\ A = \text{foldD } E\ f\ e\ A$ 
proof –
  from assms have  $1: \exists y. (A,y) \in \text{foldSetD } D\ f\ e$ 
  apply (intro finite-imp-foldSetD, auto)
  apply (metis finite-subset)
  by (unfold LCD-def, auto)
  from 1 obtain  $y$  where  $2: (A,y) \in \text{foldSetD } D\ f\ e$  by auto
  from assms 2 have  $3: \text{foldD } D\ f\ e\ A = y$  by (intro LCD.foldD-equality[of
 $B$ ], auto)
  from  $h3$  have  $4: \text{foldSetD } D\ f\ e \subseteq \text{foldSetD } E\ f\ e$  by (rule fold-
Set-not-depend)
  from 2 4 have  $5: (A,y) \in \text{foldSetD } E\ f\ e$  by auto
  from assms 5 have  $6: \text{foldD } E\ f\ e\ A = y$  by (intro LCD.foldD-equality[of
 $B$ ], auto)

  from 3 6 show  $?thesis$  by auto
qed

```

```

lemma (in comm-monoid) finprod-all1[simp]:
  assumes  $all1: \bigwedge a. a \in A \implies f\ a = \mathbf{1}_G$ 
  shows  $(\bigotimes_G a \in A. f\ a) = \mathbf{1}_G$ 

```

```

proof –
  from assms show  $?thesis$ 
  by (simp cong: finprod-cong)
qed

```

```

context abelian-monoid
begin
lemmas summands-equal = add.factors-equal
lemmas extend-sum = add.extend-prod

```

```

lemmas finsum-all0 = add.finprod-all1
end

```

```

end

```

## 4 Linear Combinations

```

theory LinearCombinations

```

```

imports Main

```

```

  HOL-Algebra.Module

```

```

  HOL-Algebra.Coset

```

```

  RingModuleFacts

```

```

  MonoidSums

```

```

  FunctionLemmas

```

```

begin

```

### 4.1 Lemmas for simplification

The following are helpful in certain simplifications (esp. congruence rules). Warning: arbitrary use leads to looping.

```

lemma (in ring) coeff-in-ring:

```

```

   $\llbracket a \in A \rightarrow \text{carrier } R; x \in A \rrbracket \Longrightarrow a \ x \in \text{carrier } R$ 

```

```

by (rule Pi-mem)

```

```

lemma (in ring) coeff-in-ring2:

```

```

   $\llbracket x \in A; a \in A \rightarrow \text{carrier } R \rrbracket \Longrightarrow a \ x \in \text{carrier } R$ 

```

```

by (metis Pi-mem)

```

```

lemma ring-subset-carrier:

```

```

   $\llbracket x \in A; A \subseteq \text{carrier } R \rrbracket \Longrightarrow x \in \text{carrier } R$ 

```

```

by auto

```

```

lemma disj-if:

```

```

   $\llbracket A \cap B = \{\}; x \in B \rrbracket \Longrightarrow (\text{if } x \in A \text{ then } f \ x \text{ else } g \ x) = g \ x$ 

```

```

by auto

```

```

lemmas (in module) sum-simp = ring-subset-carrier

```

### 4.2 Linear combinations

A linear combination is  $\sum_{v \in A} a_v v$ .  $(a_v)_{v \in S}$  is a function  $A \rightarrow K$ , where  $A \subseteq K$ .

```

definition (in module) lincomb::['c  $\Rightarrow$  'a, 'c set]  $\Rightarrow$  'c

```

```

where lincomb a A =  $(\bigoplus_M v \in A. (a \ v \odot_M \ v))$ 

```

```

lemma (in module) summands-valid:

```

**fixes**  $A a$   
**assumes**  $h2: A \subseteq \text{carrier } M$  **and**  $h3: a \in (A \rightarrow \text{carrier } R)$   
**shows**  $\forall v \in A. ((a v) \odot_M v) \in \text{carrier } M$   
**proof** –  
**from** *assms* **show** ?thesis **by** *auto*  
**qed**

**lemma** (**in** *module*) *lincomb-closed* [*simp*, *intro*]:  
**fixes**  $S a$   
**assumes**  $h2: S \subseteq \text{carrier } M$  **and**  $h3: a \in (S \rightarrow \text{carrier } R)$   
**shows** *lincomb*  $a S \in \text{carrier } M$   
**proof** –  
**from**  $h2 h3$  **show** ?thesis **by** (*unfold lincomb-def*, *auto intro: finsum-closed*)  
**qed**

**lemma** (**in** *comm-monoid*) *finprod-cong2*:  
 $[[ A = B;$   
 $!!i. i \in B ==> f i = g i; f \in B \rightarrow \text{carrier } G]] ==>$   
 $\text{finprod } G f A = \text{finprod } G g B$   
**by** (*intro finprod-cong*, *auto*)

**lemmas** (**in** *abelian-monoid*) *finsum-cong2* = *add.finprod-cong2*

**lemma** (**in** *module*) *lincomb-cong*:  
**assumes**  $h2: A=B$  **and**  $h3: A \subseteq \text{carrier } M$   
**and**  $h4: \bigwedge v. v \in A \implies a v = b v$  **and**  $h5: b \in B \rightarrow \text{carrier } R$   
**shows** *lincomb*  $a A = \text{lincomb } b B$   
**using** *assms*  
**by** (*simp cong: finsum-cong2 add: lincomb-def summands-valid ring-subset-carrier*)

**lemma** (**in** *module*) *lincomb-union*:  
**fixes**  $a A B$   
**assumes**  $h1: \text{finite } (A \cup B)$  **and**  $h3: A \cup B \subseteq \text{carrier } M$   
**and**  $h4: A \cap B = \{\}$  **and**  $h5: a \in (A \cup B \rightarrow \text{carrier } R)$   
**shows** *lincomb*  $a (A \cup B) = \text{lincomb } a A \oplus_M \text{lincomb } a B$   
**using** *assms*  
**by** (*auto cong: finsum-cong2 simp add: lincomb-def finsum-Un-disjoint summands-valid ring-subset-carrier*)

This is useful as a simp rule sometimes, for combining linear combinations.

**lemma** (**in** *module*) *lincomb-union2*:  
**fixes**  $a b A B$   
**assumes**  $h1: \text{finite } (A \cup B)$  **and**  $h3: A \cup B \subseteq \text{carrier } M$   
**and**  $h4: A \cap B = \{\}$  **and**  $h5: a \in A \rightarrow \text{carrier } R$  **and**  $h6: b \in B \rightarrow \text{carrier } R$   
**shows** *lincomb*  $a A \oplus_M \text{lincomb } b B = \text{lincomb } (\lambda v. \text{if } (v \in A) \text{ then}$

$a \ v \ \text{else} \ b \ v) \ (A \cup B)$   
 (is lincomb  $a \ A \oplus_M \ \text{lincomb} \ b \ B = \text{lincomb} \ ?c \ (A \cup B)$ )  
**using** *assms*  
**by** (*auto cong: finsum-cong2*  
*simp add: lincomb-def finsum-Un-disjoint summands-valid*  
*ring-subset-carrier disj-if*)

**lemma** (in module) *lincomb-del2*:  
**fixes**  $S \ a \ v$   
**assumes**  $h1: \text{finite } S$  **and**  $h2: S \subseteq \text{carrier } M$  **and**  $h3: a \in (S \rightarrow \text{carrier } R)$  **and**  $h4: v \in S$   
**shows**  $\text{lincomb } a \ S = ((a \ v) \odot_M v) \oplus_M \ \text{lincomb } a \ (S - \{v\})$   
**proof** –  
**from**  $h4$  **have**  $1: S = \{v\} \cup (S - \{v\})$  **by** (*metis insert-Diff insert-is-Un*)  
  
**from** *assms* **show** *?thesis*  
**apply** (*subst 1*)  
**apply** (*subst lincomb-union, auto*)  
**by** (*unfold lincomb-def, auto simp add: coeff-in-ring*)  
**qed**

**lemma** (in module) *lincomb-insert*:  
**fixes**  $S \ a \ v$   
**assumes**  $h1: \text{finite } S$  **and**  $h2: S \subseteq \text{carrier } M$  **and**  $h3: a \in (S \cup \{v\} \rightarrow \text{carrier } R)$  **and**  $h4: v \notin S$  **and**  
 $h5: v \in \text{carrier } M$   
**shows**  $\text{lincomb } a \ (S \cup \{v\}) = ((a \ v) \odot_M v) \oplus_M \ \text{lincomb } a \ S$   
**using** *assms*  
**by** (*auto cong: finsum-cong2*  
*simp add: lincomb-def finsum-Un-disjoint summands-valid*  
*ring-subset-carrier disj-if*)

**lemma** (in module) *lincomb-elim-if* [*simp*]:  
**fixes**  $b \ c \ S$   
**assumes**  $h1: S \subseteq \text{carrier } M$  **and**  $h2: \bigwedge v. v \in S \implies \neg P \ v$  **and**  $h3: c \in S \rightarrow \text{carrier } R$   
**shows**  $\text{lincomb } (\lambda w. \text{if } P \ w \ \text{then } b \ w \ \text{else } c \ w) \ S = \text{lincomb } c \ S$   
**using** *assms*  
**by** (*auto cong: finsum-cong2*  
*simp add: lincomb-def finsum-Un-disjoint summands-valid*  
*ring-subset-carrier disj-if*)

**lemma** (in module) *lincomb-smult*:  
**fixes**  $A \ c$   
**assumes**  $h2: A \subseteq \text{carrier } M$  **and**  $h3: a \in A \rightarrow \text{carrier } R$  **and**  $h4: c \in \text{carrier } R$   
**shows**  $\text{lincomb } (\lambda w. c \otimes_R a \ w) \ A = c \odot_M (\text{lincomb } a \ A)$   
**using** *assms*

by (auto cong: finsum-cong2  
simp add: lincomb-def finsum-Un-disjoint finsum-smult ring-subset-carrier  
disj-if smult-assoc1 coeff-in-ring)

### 4.3 Linear dependence and independence.

A set  $S$  in a module/vectorspace is linearly dependent if there is a finite set  $A \subseteq S$  and coefficients  $(a_v)_{v \in A}$  such that  $\text{sum}_{v \in A} a_v v = 0$  and for some  $v$ ,  $a_v \neq 0$ .

**definition (in module) lin-dep where**

$\text{lin-dep } S = (\exists A \ a \ v. (\text{finite } A \wedge A \subseteq S \wedge (a \in (A \rightarrow \text{carrier } R)) \wedge (\text{lincomb } a \ A = \mathbf{0}_M) \wedge (v \in A) \wedge (a \ v \neq \mathbf{0}_R)))$

**abbreviation (in module) lin-indpt::'c set  $\Rightarrow$  bool**  
**where**  $\text{lin-indpt } S \equiv \neg \text{lin-dep } S$

In the finite case, we can take  $A = S$ . This may be more convenient (e.g., when adding two linear combinations).

**lemma (in module) finite-lin-dep:**

**fixes**  $S$

**assumes**  $\text{finS}: \text{finite } S$  **and**  $\text{ld}: \text{lin-dep } S$  **and**  $\text{inC}: S \subseteq \text{carrier } M$

**shows**  $\exists a \ v. (a \in (S \rightarrow \text{carrier } R)) \wedge (\text{lincomb } a \ S = \mathbf{0}_M) \wedge (v \in S) \wedge (a \ v \neq \mathbf{0}_R)$

**proof** –

**from**  $\text{ld}$  **obtain**  $A \ a \ v$  **where**  $A: (A \subseteq S \wedge (a \in (A \rightarrow \text{carrier } R)) \wedge (\text{lincomb } a \ A = \mathbf{0}_M) \wedge (v \in A) \wedge (a \ v \neq \mathbf{0}_R))$

**by** (unfold lin-dep-def, auto)

**let**  $?b = \lambda w. \text{if } w \in A \text{ then } a \ w \text{ else } \mathbf{0}_R$

**from**  $\text{finS}$   $\text{inC}$   $A$  **have**  $\text{if-in}: (\bigoplus_{M} v \in S. (\text{if } v \in A \text{ then } a \ v \text{ else } \mathbf{0})) \odot_M v = (\bigoplus_{M} v \in S. (\text{if } v \in A \text{ then } a \ v \odot_M v \text{ else } \mathbf{0}_M))$

**apply** auto

**apply** (intro finsum-cong')

**by** (auto simp add: coeff-in-ring)

**from**  $\text{finS}$   $\text{inC}$   $A$  **have**  $b: \text{lincomb } ?b \ S = \mathbf{0}_M$

**apply** (unfold lincomb-def)

**apply** (subst if-in)

**by** (subst extend-sum, auto)

**from**  $A \ b$  **show**  $?thesis$

**apply** (rule-tac  $x = ?b$  in  $exI$ )

**apply** (rule-tac  $x = v$  in  $exI$ )

**by** auto

**qed**

Criteria of linear dependency in a easy format to apply: apply (rule lin-dep-crit)

**lemma (in module) lin-dep-crit:**

**fixes**  $A \ S \ a \ v$

**assumes** *fin*: finite  $A$  **and** *subset*:  $A \subseteq S$  **and** *h1*:  $(a \in (A \rightarrow \text{carrier } R))$  **and** *h2*:  $v \in A$   
**and** *h3*:  $a \neq \mathbf{0}_R$  **and** *h4*:  $(\text{lincomb } a \ A = \mathbf{0}_M)$   
**shows** *lin-dep*  $S$   
**proof** –  
**from** *assms* **show** *?thesis*  
**by** (*unfold lin-dep-def*, *auto*)  
**qed**

If  $\sum_{v \in A} a_v v = 0$  implies  $a_v = 0$  for all  $v \in S$ , then  $A$  is linearly independent.

**lemma** (*in module*) *finite-lin-indpt2*:  
**fixes**  $A$   
**assumes** *A-fin*: finite  $A$  **and** *AinC*:  $A \subseteq \text{carrier } M$  **and**  
*lc0*:  $\bigwedge a. a \in (A \rightarrow \text{carrier } R) \implies (\text{lincomb } a \ A = \mathbf{0}_M) \implies (\forall v \in A. a \ v = \mathbf{0}_R)$   
**shows** *lin-indpt*  $A$   
**proof** (*rule ccontr*)  
**assume**  $\neg \text{lin-indpt } A$   
**from** *A-fin* *AinC* **this obtain**  $a \ v$  **where** *av*:  
 $(a \in (A \rightarrow \text{carrier } R)) \wedge (\text{lincomb } a \ A = \mathbf{0}_M) \wedge (v \in A) \wedge (a \ v \neq \mathbf{0}_R)$   
**by** (*metis finite-lin-dep*)  
**from** *av lc0* **show** *False* **by** *auto*  
**qed**

Any set containing 0 is linearly dependent.

**lemma** (*in module*) *zero-lin-dep*:  
**assumes** *0*:  $\mathbf{0}_M \in S$  **and** *nonzero*:  $\text{carrier } R \neq \{\mathbf{0}_R\}$   
**shows** *lin-dep*  $S$   
**proof** –  
**from** *nonzero* **have** *zero-not-one*:  $\mathbf{0}_R \neq \mathbf{1}_R$  **by** (*rule nontrivial-ring*)  
**from** *0* *zero-not-one* **show** *?thesis*  
**apply** (*unfold lin-dep-def*)  
**apply** (*rule-tac x={0\_M} in exI*)  
**apply** (*rule-tac x=(λv. 1\_R) in exI*)  
**apply** (*rule-tac x=0\_M in exI*)  
**by** (*unfold lincomb-def*, *auto*)  
**qed**

**lemma** (*in module*) *zero-nin-lin-indpt*:  
**assumes** *h2*:  $S \subseteq \text{carrier } M$  **and** *li*:  $\neg(\text{lin-dep } S)$  **and** *nonzero*:  $\text{carrier } R \neq \{\mathbf{0}_R\}$   
**shows**  $\mathbf{0}_M \notin S$   
**proof** (*rule ccontr*)  
**assume** *a1*:  $\neg(\mathbf{0}_M \notin S)$   
**from** *a1* **have** *a2*:  $\mathbf{0}_M \in S$  **by** *auto*  
**from** *a2 nonzero* **have** *ld*: *lin-dep*  $S$  **by** (*rule zero-lin-dep*)  
**from** *li ld* **show** *False* **by** *auto*  
**qed**

The *span* of  $S$  is the set of linear combinations with  $A \subseteq S$ .

**definition** (in module) *span::'c set  $\Rightarrow$  'c set*  
**where**  $\text{span } S = \{\text{lincomb } a \ A \mid a \ A. \text{ finite } A \wedge A \subseteq S \wedge a \in (A \rightarrow \text{carrier } R)\}$

The *span* interpreted as a module or vectorspace.

**abbreviation** (in module) *span-vs::'c set  $\Rightarrow$  ('a,'c,'d) module-scheme*

**where**  $\text{span-vs } S \equiv M \ (\text{carrier} := \text{span } S)$

In the finite case, we can take  $A = S$  without loss of generality.

**lemma** (in module) *finite-span:*

**assumes** *fin: finite S and inC:  $S \subseteq \text{carrier } M$*

**shows**  $\text{span } S = \{\text{lincomb } a \ S \mid a. a \in (S \rightarrow \text{carrier } R)\}$

**proof** (rule equalityI)

```

{
  fix A a
  assume subset:  $A \subseteq S$  and a:  $a \in A \rightarrow \text{carrier } R$ 
  let ?b = ( $\lambda v. \text{if } v \in A \text{ then } a \ v \text{ else } \mathbf{0}$ )
  from fin inC subset a have if-in:  $(\bigoplus_{M v \in S}. ?b \ v \ \odot_M \ v) =$ 
 $(\bigoplus_{M v \in S}. (\text{if } v \in A \text{ then } a \ v \ \odot_M \ v \text{ else } \mathbf{0}_M))$ 
  apply (intro finsum-cong')
  by (auto simp add: coeff-in-ring)
  from fin inC subset a have  $\exists b. \text{lincomb } a \ A = \text{lincomb } b \ S \wedge b \in$ 
 $S \rightarrow \text{carrier } R$ 
  apply (rule-tac  $x = ?b$  in exI)
  apply (unfold lincomb-def, auto)
  apply (subst if-in)
  by (subst extend-sum, auto)
}
from this show  $\text{span } S \subseteq \{\text{lincomb } a \ S \mid a. a \in S \rightarrow \text{carrier } R\}$ 
  by (unfold span-def, auto)
next
from fin show  $\{\text{lincomb } a \ S \mid a. a \in S \rightarrow \text{carrier } R\} \subseteq \text{span } S$ 
  by (unfold span-def, auto)
qed

```

If  $v \in \text{span } S$ , then we can find a linear combination. This is in an easy to apply format (e.g. obtain a A where...)

**lemma** (in module) *in-span:*

**fixes**  $S \ v$

**assumes**  $h2: S \subseteq \text{carrier } V$  and  $h3: v \in \text{span } S$

**shows**  $\exists a \ A. (A \subseteq S \wedge (a \in A \rightarrow \text{carrier } R) \wedge (\text{lincomb } a \ A = v))$

**proof** –

**from**  $h2 \ h3$  **show** *?thesis*

**apply** (unfold span-def)

**by** *auto*

**qed**

In the finite case, we can take  $A = S$ .

**lemma** (in module) *finite-in-span*:

fixes  $S v$

assumes  $fin$ : finite  $S$  and  $h2$ :  $S \subseteq \text{carrier } M$  and  $h3$ :  $v \in \text{span } S$

shows  $\exists a. (a \in S \rightarrow \text{carrier } R) \wedge (\text{lincomb } a S = v)$

**proof** –

from  $fin$   $h2$  have  $fin\text{-span}$ :  $\text{span } S = \{\text{lincomb } a S \mid a. a \in S \rightarrow \text{carrier } R\}$  by (rule *finite-span*)

from  $h3$   $fin\text{-span}$  show ?thesis by auto

**qed**

If a subset is linearly independent, then any linear combination that is 0 must have a nonzero coefficient outside that set.

**lemma** (in module) *lincomb-must-include*:

fixes  $A S T b v$

assumes  $inC$ :  $T \subseteq \text{carrier } M$  and  $li$ : *lin-indpt*  $S$  and  $Ssub$ :  $S \subseteq T$   
and  $Ssub$ :  $A \subseteq T$

and  $fin$ : finite  $A$

and  $b$ :  $b \in A \rightarrow \text{carrier } R$  and  $lc$ :  $\text{lincomb } b A = \mathbf{0}_M$  and  $v\text{-in}$ :  $v \in A$

and  $nz\text{-coeff}$ :  $b v \neq \mathbf{0}_R$

shows  $\exists w \in A - S. b w \neq \mathbf{0}_R$

**proof** (rule *ccontr*)

assume 0:  $\neg(\exists w \in A - S. b w \neq \mathbf{0}_R)$

from 0 have 1:  $\bigwedge w. w \in A - S \implies b w = \mathbf{0}_R$  by auto

have  $Auni$ :  $A = (S \cap A) \cup (A - S)$  by auto

from  $fin$   $b$   $Ssub$   $inC$  1 have 2:  $\text{lincomb } b A = \text{lincomb } b (S \cap A)$

apply (subst  $Auni$ )

apply (subst *lincomb-union*, auto)

apply (unfold *lincomb-def*)

apply (subst (2) *finsum-all0*, auto)

by (subst *show-r-zero*, auto *intro!*: *finsum-closed*)

from 1 2 *assms* have  $ld$ : *lin-dep*  $S$

apply (*intro lin-dep-crit*[where ? $A=S \cap A$  and ? $a=b$  and ? $v=v$ ])

by auto

from  $ld$   $li$  show *False* by auto

**qed**

A generating set is a set such that the span of  $S$  is all of  $M$ .

**abbreviation** (in module) *gen-set::'c set  $\Rightarrow$  bool*

where *gen-set*  $S \equiv (\text{span } S = \text{carrier } M)$

## 4.4 Submodules

**lemma** *module-criteria*:

fixes  $R$  and  $M$

assumes *cring*: *cring*  $R$

and *zero*:  $\mathbf{0}_M \in \text{carrier } M$

**and add:**  $\forall v w. v \in \text{carrier } M \wedge w \in \text{carrier } M \longrightarrow v \oplus_M w \in \text{carrier } M$   
**and neg:**  $\forall v \in \text{carrier } M. (\exists \text{neg-}v \in \text{carrier } M. v \oplus_M \text{neg-}v = \mathbf{0}_M)$   
**and smult:**  $\forall c v. c \in \text{carrier } R \wedge v \in \text{carrier } M \longrightarrow c \odot_M v \in \text{carrier } M$   
**and comm:**  $\forall v w. v \in \text{carrier } M \wedge w \in \text{carrier } M \longrightarrow v \oplus_M w = w \oplus_M v$   
**and assoc:**  $\forall v w x. v \in \text{carrier } M \wedge w \in \text{carrier } M \wedge x \in \text{carrier } M \longrightarrow (v \oplus_M w) \oplus_M x = v \oplus_M (w \oplus_M x)$   
**and add-id:**  $\forall v \in \text{carrier } M. (v \oplus_M \mathbf{0}_M = v)$   
**and compat:**  $\forall a b v. a \in \text{carrier } R \wedge b \in \text{carrier } R \wedge v \in \text{carrier } M \longrightarrow (a \otimes_R b) \odot_M v = a \odot_M (b \odot_M v)$   
**and smult-id:**  $\forall v \in \text{carrier } M. (\mathbf{1}_R \odot_M v = v)$   
**and dist-f:**  $\forall a b v. a \in \text{carrier } R \wedge b \in \text{carrier } R \wedge v \in \text{carrier } M \longrightarrow (a \oplus_R b) \odot_M v = (a \odot_M v) \oplus_M (b \odot_M v)$   
**and dist-add:**  $\forall a v w. a \in \text{carrier } R \wedge v \in \text{carrier } M \wedge w \in \text{carrier } M \longrightarrow a \odot_M (v \oplus_M w) = (a \odot_M v) \oplus_M (a \odot_M w)$   
**shows module R M**  
**proof –**  
**from assms have 2: abelian-group M**  
**by (intro abelian-groupI, auto)**  
**from assms have 3: module-axioms R M**  
**by (unfold module-axioms-def, auto)**  
**from 2 3 cring show ?thesis**  
**by (unfold module-def module-def, auto)**  
**qed**

A submodule is  $N \subseteq M$  that is closed under addition and scalar multiplication, and contains 0 (so is not empty).

**locale submodule =**  
**fixes R and N and M (structure)**  
**assumes module: module R M**  
**and subset:  $N \subseteq \text{carrier } M$**   
**and m-closed [intro, simp]:  $\llbracket v \in N; w \in N \rrbracket \Longrightarrow v \oplus w \in N$**   
**and zero-closed [simp]:  $\mathbf{0} \in N$**   
**and smult-closed [intro, simp]:  $\llbracket c \in \text{carrier } R; v \in N \rrbracket \Longrightarrow c \odot v \in N$**

**abbreviation (in module) md::'c set  $\Rightarrow$  ('a, 'c, 'd) module-scheme**  
**where md N  $\equiv$  M( $\downarrow$ carrier :=N)**

**lemma (in module) carrier-vs-is-self [simp]:**  
**carrier (md N) = N**  
**by auto**

**lemma (in module) submodule-is-module:**  
**fixes N::'c set**  
**assumes 0: submodule R N M**  
**shows module R (md N)**

```

proof (unfold module-def, auto)
  show 1: cring R..
next
  from assms show 2: abelian-group (md N)
    apply (unfold submodule-def)
    apply (intro abelian-groupI, auto)
      apply (metis (no-types, opaque-lifting) M.add.m-assoc contra-subsetD)
    apply (metis (no-types, opaque-lifting) M.add.m-comm contra-subsetD)
    apply (rename-tac v)

```

The inverse of  $v$  under addition is  $-v$

```

  apply (rule-tac  $x = \ominus_M v$  in bxI)
  apply (metis M.l-neg contra-subsetD)
  by (metis R.add.inv-closed one-closed smult-minus-1 subset-iff)
next
  from assms show 3: module-axioms R (md N)
    apply (unfold module-axioms-def submodule-def, auto)
    apply (metis (no-types, opaque-lifting) smult-l-distr contra-subsetD)
    apply (metis (no-types, opaque-lifting) smult-r-distr contra-subsetD)
    by (metis (no-types, opaque-lifting) smult-assoc1 contra-subsetD)
qed

```

$$N_1 + N_2 = \{x + y \mid x \in N_1, y \in N_2\}$$

**definition** (in *module*) *submodule-sum*:: [ $'c$  set,  $'c$  set]  $\Rightarrow$   $'c$  set  
**where** *submodule-sum*  $N1$   $N2$  =  $(\lambda (x,y). x \oplus_M y) \{ (x,y). x \in N1 \wedge y \in N2 \}$

A module homomorphism  $M \rightarrow N$  preserves addition and scalar multiplication.

**definition** *module-hom*:: [ $'a$ ,  $'c0$ ] *ring-scheme*,  
 $( 'a, 'b1, 'c1)$  *module-scheme*,  $( 'a, 'b2, 'c2)$  *module-scheme*]  $\Rightarrow$   $( 'b1 \Rightarrow 'b2)$   
*set*  
**where** *module-hom*  $R$   $M$   $N$  =  $\{f.$   
 $((f \in \text{carrier } M \rightarrow \text{carrier } N)$   
 $\wedge (\forall m1\ m2. m1 \in \text{carrier } M \wedge m2 \in \text{carrier } M \longrightarrow f (m1 \oplus_M m2)$   
 $= (f\ m1) \oplus_N (f\ m2))$   
 $\wedge (\forall r\ m. r \in \text{carrier } R \wedge m \in \text{carrier } M \longrightarrow f (r \odot_M m) = r \odot_N (f\ m))\}$

**lemma** *module-hom-closed*:  $f \in \text{module-hom } R\ M\ N \implies f \in \text{carrier } M \rightarrow \text{carrier } N$

**by** (unfold module-hom-def, auto)

**lemma** *module-hom-add*:  $\llbracket f \in \text{module-hom } R\ M\ N; m1 \in \text{carrier } M; m2 \in \text{carrier } M \rrbracket \implies f (m1 \oplus_M m2) = (f\ m1) \oplus_N (f\ m2)$   
**by** (unfold module-hom-def, auto)

**lemma** *module-hom-smult*:  $\llbracket f \in \text{module-hom } R \ M \ N; r \in \text{carrier } R; m \in \text{carrier } M \rrbracket \implies f (r \odot_M m) = r \odot_N (f m)$   
**by** (*unfold module-hom-def, auto*)

**locale** *mod-hom* =  
 $M^?$ : *module*  $R \ M + N^?$ : *module*  $R \ N$   
**for**  $R$  **and**  $M$  **and**  $N +$   
**fixes**  $f$   
**assumes**  $f\text{-hom}$ :  $f \in \text{module-hom } R \ M \ N$   
**notes**  $f\text{-add}$  [*simp*] = *module-hom-add* [*OF f-hom*]  
**and**  $f\text{-smult}$  [*simp*] = *module-hom-smult* [*OF f-hom*]

Some basic simplification rules for module homomorphisms.

**context** *mod-hom*  
**begin**

**lemma**  $f\text{-im}$  [*simp, intro*]:  
**assumes**  $v \in \text{carrier } M$   
**shows**  $f v \in \text{carrier } N$   
**proof** –  
**have**  $0$ : *mod-hom*  $R \ M \ N \ f..$   
**from**  $0$  *assms* **show** *?thesis*  
**apply** (*unfold mod-hom-def module-hom-def mod-hom-axioms-def Pi-def*)  
**by** *auto*  
**qed**

**definition**  $im$ :: 'e *set*  
**where**  $im = f'(\text{carrier } M)$

**definition**  $ker$ :: 'c *set*  
**where**  $ker = \{v. v \in \text{carrier } M \ \& \ f v = \mathbf{0}_N\}$

**lemma**  $f0\text{-is-0}$  [*simp*]:  $f \ \mathbf{0}_M = \mathbf{0}_N$   
**proof** –  
**have**  $1$ :  $f \ \mathbf{0}_M = f (\mathbf{0}_R \odot_M \mathbf{0}_M)$  **by** *simp*  
**have**  $2$ :  $f (\mathbf{0}_R \odot_M \mathbf{0}_M) = \mathbf{0}_N$   
**using**  $M.M.zero\text{-closed}$   $N.lmult\text{-0}$   $R.zero\text{-closed}$   $f\text{-im}$   $f\text{-smult}$  **by**  
*presburger*  
**from**  $1 \ 2$  **show** *?thesis* **by** *auto*  
**qed**

**lemma**  $f\text{-neg}$  [*simp*]:  $v \in \text{carrier } M \implies f (\ominus_M v) = \ominus_N f v$   
**by** (*simp flip: M.smult-minus-1 N.smult-minus-1*)

**lemma**  $f\text{-minus}$  [*simp*]:  $\llbracket v \in \text{carrier } M; w \in \text{carrier } M \rrbracket \implies f (v \ominus_M w) = f v \ominus_N f w$   
**by** (*simp add: a-minus-def*)

**lemma** *ker-is-submodule: submodule R ker M*  
**proof** –  
**have** 0: *mod-hom R M N f..*  
**from** 0 **have** 1: *module R M* **by** (*unfold mod-hom-def, auto*)  
**show** ?thesis  
**by** (*rule submodule.intro, auto simp add: ker-def, rule 1*)  
**qed**

**lemma** *im-is-submodule: submodule R im N*  
**proof** –  
**have** 1:  $im \subseteq carrier\ N$  **by** (*auto simp add: im-def image-def mod-hom-def module-hom-def f-im*)  
**have** 2:  $\bigwedge w1\ w2. \llbracket w1 \in im; w2 \in im \rrbracket \implies w1 \oplus_N w2 \in im$   
**proof** –  
**fix**  $w1\ w2$   
**assume**  $w1: w1 \in im$  **and**  $w2: w2 \in im$   
**from**  $w1$  **obtain**  $v1$  **where** 3:  $v1 \in carrier\ M \wedge f\ v1 = w1$  **by**  
(*unfold im-def, auto*)  
**from**  $w2$  **obtain**  $v2$  **where** 4:  $v2 \in carrier\ M \wedge f\ v2 = w2$  **by**  
(*unfold im-def, auto*)  
**from** 3 4 **have** 5:  $f\ (v1 \oplus_M v2) = w1 \oplus_N w2$  **by** *simp*  
**from** 3 4 **have** 6:  $v1 \oplus_M v2 \in carrier\ M$  **by** *simp*  
**from** 5 6 **have** 7:  $\exists x \in carrier\ M. w1 \oplus_N w2 = f\ x$  **by** *metis*  
**from** 7 **show** ?thesis  $w1\ w2$  **by** (*unfold im-def image-def, auto*)  
**qed**  
**have** 3:  $0_N \in im$   
**proof** –  
**have** 8:  $f\ 0_M = 0_N \wedge 0_M \in carrier\ M$  **by** *auto*  
**from** 8 **have** 9:  $\exists x \in carrier\ M. 0_N = f\ x$  **by** *metis*  
**from** 9 **show** ?thesis **by** (*unfold im-def image-def, auto*)  
**qed**  
**have** 4:  $\bigwedge c\ w. \llbracket c \in carrier\ R; w \in im \rrbracket \implies c \odot_N w \in im$   
**proof** –  
**fix**  $c\ w$   
**assume**  $c: c \in carrier\ R$  **and**  $w: w \in im$   
**from**  $w$  **obtain**  $v$  **where** 10:  $v \in carrier\ M \wedge f\ v = w$  **by** (*unfold im-def, auto*)  
**from**  $c$  10 **have** 11:  $f\ (c \odot_M v) = c \odot_N w \wedge (c \odot_M v \in carrier\ M)$   
**by** *auto*  
**from** 11 **have** 12:  $\exists v1 \in carrier\ M. c \odot_N w = f\ v1$  **by** *metis*  
**from** 12 **show** ?thesis  $c\ w$  **by** (*unfold im-def image-def, auto*)  
**qed**  
**from** 1 2 3 4 **show** ?thesis **by** (*unfold-locals, auto*)  
**qed**

**lemma** (*in mod-hom*) *f-ker:*  
 $v \in ker \implies f\ v = 0_N$   
**by** (*unfold ker-def, auto*)  
**end**

We will show that for any set  $S$ , the space of functions  $S \rightarrow K$  forms a vector space.

**definition** (in *ring*) *func-space*:: 'z set⇒('a,('z ⇒ 'a)) module  
**where** *func-space*  $S =$  ( $\text{carrier} = S \rightarrow_E \text{carrier } R,$   
 $\text{mult} = (\lambda f g. \text{restrict } (\lambda v. \mathbf{0}_R) S),$   
 $\text{one} = \text{restrict } (\lambda v. \mathbf{0}_R) S,$   
 $\text{zero} = \text{restrict } (\lambda v. \mathbf{0}_R) S,$   
 $\text{add} = (\lambda f g. \text{restrict } (\lambda v. f v \oplus_R g v) S),$   
 $\text{smult} = (\lambda c f. \text{restrict } (\lambda v. c \otimes_R f v) S)$ )

**lemma** (in *cring*) *func-space-is-module*:

**fixes**  $S$

**shows** module  $R$  (*func-space*  $S$ )

**proof** –

**have**  $0$ : *cring*  $R..$

**from**  $0$  **show** ?thesis

**apply** (*auto intro!*: *module-criteria simp add: func-space-def*)

**apply** (*auto simp add: module-def*)

**apply** (*rename-tac f*)

**apply** (*rule-tac x=restrict* ( $\lambda v'. \ominus_R (f v')$ )  $S$  **in** *bestI*)

**apply** (*auto simp add: restrict-def cong: if-cong split: if-split-asm,*  
*auto*)

**apply** (*auto simp add: a-ac PiE-mem2 r-neg*)

**apply** (*unfold PiE-def extensional-def Pi-def*)

**by** (*auto simp add: m-assoc l-distr r-distr*)

**qed**

Note: one can define  $M^n$  from this.

A linear combination is a module homomorphism from the space of coefficients to the module,  $(a_v) \mapsto \sum_{v \in S} a_v v$ .

**lemma** (in *module*) *lincomb-is-mod-hom*:

**fixes**  $S$

**assumes**  $h$ : *finite*  $S$  **and**  $h2$ :  $S \subseteq \text{carrier } M$

**shows** *mod-hom*  $R$  (*func-space*  $S$ )  $M$  ( $\lambda a. \text{lincomb } a S$ )

**proof** –

**have**  $0$ : *module*  $R M..$

{

**fix**  $m1 m2$

**assume**  $m1$ :  $m1 \in S \rightarrow_E \text{carrier } R$  **and**  $m2$ :  $m2 \in S \rightarrow_E \text{carrier}$

$R$

**from**  $h h2 m1 m2$  **have**  $a1$ :  $(\bigoplus_{M^v \in S. (\lambda v \in S. m1 v \oplus_R m2 v)} v \odot_M v) =$

$(\bigoplus_{M^v \in S. m1 v \odot_M v \oplus_M m2 v \odot_M v)$

**by** (*intro finsum-cong'*, *auto simp add: smult-l-distr PiE-mem2*)

**from**  $h h2 m1 m2$  **have**  $a2$ :  $(\bigoplus_{M^v \in S. m1 v \odot_M v \oplus_M m2 v \odot_M v) =$

$(\bigoplus_{M^v \in S. m1 v \odot_M v) \oplus_M (\bigoplus_{M^v \in S. m2 v \odot_M v)$

**by** (*intro finsum-addf, auto*)

```

from a1 a2 have ( $\bigoplus_{Mv \in S}. (\lambda v \in S. m1\ v \oplus m2\ v)\ v \odot_M v$ ) =
  ( $\bigoplus_{Mv \in S}. m1\ v \odot_M v$ )  $\oplus_M$  ( $\bigoplus_{Mv \in S}. m2\ v \odot_M v$ ) by auto
}
hence 1:  $\bigwedge m1\ m2.$ 
   $m1 \in S \rightarrow_E \text{carrier } R \implies$ 
   $m2 \in S \rightarrow_E \text{carrier } R \implies$  ( $\bigoplus_{Mv \in S}. (\lambda v \in S. m1\ v \oplus m2\ v)\ v$ 
 $\odot_M v$ ) =
  ( $\bigoplus_{Mv \in S}. m1\ v \odot_M v$ )  $\oplus_M$  ( $\bigoplus_{Mv \in S}. m2\ v \odot_M v$ ) by auto
{
  fix r m
  assume r:  $r \in \text{carrier } R$  and m:  $m \in S \rightarrow_E \text{carrier } R$ 
  from h h2 r m have b1:  $r \odot_M (\bigoplus_{Mv \in S}. m\ v \odot_M v) = (\bigoplus_{Mv \in S}. r \odot_M (m\ v \odot_M v))$ 
  by (intro finsum-smult, auto)
  from h h2 r m have b2: ( $\bigoplus_{Mv \in S}. (\lambda v \in S. r \otimes m\ v)\ v \odot_M v$ ) =
   $r \odot_M (\bigoplus_{Mv \in S}. m\ v \odot_M v)$ 
  apply (subst b1)
  apply (intro finsum-cong', auto)
  by (subst smult-assoc1, auto)
}
hence 2:  $\bigwedge r\ m. r \in \text{carrier } R \implies$ 
   $m \in S \rightarrow_E \text{carrier } R \implies$  ( $\bigoplus_{Mv \in S}. (\lambda v \in S. r \otimes m\ v)\ v \odot_M v$ )
=  $r \odot_M (\bigoplus_{Mv \in S}. m\ v \odot_M v)$ 
by auto
from h h2 0 1 2 show ?thesis
apply (unfold mod-hom-def, auto)
apply (rule func-space-is-module)
apply (unfold mod-hom-axioms-def module-hom-def, auto)
apply (rule lincomb-closed, unfold func-space-def, auto)
apply (unfold lincomb-def)
by auto
qed

```

**lemma** (*in module*) *lincomb-sum*:

**assumes** *A-fin*: *finite A* **and** *AinC*:  $A \subseteq \text{carrier } M$  **and** *a-fun*:  $a \in A \rightarrow \text{carrier } R$  **and**

*b-fun*:  $b \in A \rightarrow \text{carrier } R$

**shows** *lincomb* ( $\lambda v. a\ v \oplus_R b\ v$ )  $A = \text{lincomb } a\ A \oplus_M \text{lincomb } b\ A$

**proof** –

**from** *A-fin* *AinC* **interpret** *mh*: *mod-hom R func-space A M* ( $\lambda a.$  *lincomb a A*) **by** (*rule*

*lincomb-is-mod-hom*)

**let** *?a=restrict a A*

**let** *?b=restrict b A*

**from** *a-fun b-fun A-fin AinC*

**have** 1: *LinearCombinations.module.lincomb M* (*?a* $\oplus$ (*LinearCombinations.ring.func-space R A*) *?b*)  $A$

= *LinearCombinations.module.lincomb M* ( $\lambda x. a\ x \oplus_R b\ x$ )  $A$

```

    by (auto simp add: func-space-def Pi-iff restrict-apply' cong: lincomb-cong)
  from a-fun b-fun A-fin AinC
  have 2: LinearCombinations.module.lincomb M ?a A  $\oplus_M$ 
    LinearCombinations.module.lincomb M ?b A = LinearCombinations.module.lincomb M a A  $\oplus_M$ 
    LinearCombinations.module.lincomb M b A
  by (simp-all add: sum-simp cong: lincomb-cong)
  from a-fun b-fun have ainC: ?a $\in$ carrier (LinearCombinations.ring.func-space R A)
  and binC: ?b $\in$ carrier (LinearCombinations.ring.func-space R A)
  by (unfold func-space-def, auto)
  from ainC binC have LinearCombinations.module.lincomb M (?a $\oplus$ (LinearCombinations.ring.func-space ?b) A)
    = LinearCombinations.module.lincomb M ?a A  $\oplus_M$ 
    LinearCombinations.module.lincomb M ?b A
  by (simp cong: lincomb-cong)
  with 1 2 show ?thesis by auto
qed

```

The negative of a function is just pointwise negation.

```

lemma (in cring) func-space-neg:
  fixes f
  assumes f $\in$  carrier (func-space S)
  shows  $\ominus_{func-space S} f = (\lambda v. \text{if } (v\in S) \text{ then } \ominus_R f v \text{ else undefined})$ 
proof -
  interpret fs: module R func-space S by (rule func-space-is-module)
  from assms show ?thesis
  apply (intro fs.minus-equality)
  apply (unfold func-space-def PiE-def extensional-def)
  apply auto
  apply (intro restrict-ext, auto)
  by (simp add: l-neg coeff-in-ring)
qed

```

Ditto for subtraction. Note the above is really a special case, when a is the 0 function.

```

lemma (in module) lincomb-diff:
  assumes A-fin: finite A and AinC: A $\subseteq$ carrier M and a-fun: a $\in$ A $\rightarrow$ carrier R and
    b-fun: b $\in$ A $\rightarrow$ carrier R
  shows lincomb ( $\lambda v. a v \ominus_R b v$ ) A = lincomb a A  $\ominus_M$  lincomb b A
proof -
  from A-fin AinC interpret mh: mod-hom R func-space A M ( $\lambda a. \text{lincomb } a A$ ) by (rule lincomb-is-mod-hom)
  let ?a=restrict a A
  let ?b=restrict b A

```

```

from a-fun b-fun have ainC: ?a∈carrier (LinearCombinations.ring.func-space
R A)
and binC: ?b∈carrier (LinearCombinations.ring.func-space R A)
by (unfold func-space-def, auto)
from a-fun b-fun ainC binC A-fin AinC
have 1: LinearCombinations.module.lincomb M (?a⊖(func-space A)
?b) A
= LinearCombinations.module.lincomb M (λx. a x ⊖R b x) A
apply (subst mh.M.M.minus-eq)
apply (intro lincomb-cong, auto)
apply (subst func-space-neg, auto)
apply (simp add: restrict-def func-space-def)
by (subst R.minus-eq, auto)
from a-fun b-fun A-fin AinC
have 2: LinearCombinations.module.lincomb M ?a A ⊖M
LinearCombinations.module.lincomb M ?b A = LinearCombina-
tions.module.lincomb M a A ⊖M
LinearCombinations.module.lincomb M b A
by (simp cong: lincomb-cong)
from ainC binC have LinearCombinations.module.lincomb M (?a⊖(LinearCombinations.ring.func-space
?b) A)
= LinearCombinations.module.lincomb M ?a A ⊖M
LinearCombinations.module.lincomb M ?b A
by (simp cong: lincomb-cong)
with 1 2 show ?thesis by auto
qed

```

The union of nested submodules is a submodule. We will use this to show that span of any set is a submodule.

**lemma** (in module) nested-union-vs:

```

fixes I N N'
assumes subm:  $\bigwedge i. i \in I \implies \text{submodule } R (N i) M$ 
and max-exists:  $\bigwedge i j. i \in I \implies j \in I \implies (\exists k. k \in I \wedge N i \subseteq N k \wedge N j \subseteq N k)$ 

```

```

and uni:  $N' = (\bigcup i \in I. N i)$ 

```

```

and ne:  $I \neq \{\}$ 

```

```

shows submodule R N' M

```

**proof** –

```

have 1: module R M..

```

```

from subm have all-in:  $\bigwedge i. i \in I \implies N i \subseteq \text{carrier } M$ 

```

```

by (unfold submodule-def, auto)

```

```

from uni all-in have 2:  $\bigwedge x. x \in N' \implies x \in \text{carrier } M$ 

```

```

by auto

```

```

from uni have 3:  $\bigwedge v w. v \in N' \implies w \in N' \implies v \oplus_M w \in N'$ 

```

**proof** –

```

fix v w

```

```

assume v:  $v \in N'$  and w:  $w \in N'$ 

```

```

from uni v w obtain i j where i:  $i \in I \wedge v \in N i$  and j:  $j \in I \wedge w \in N j$  by auto

```

```

    from max-exists i j obtain k where k: k ∈ I ∧ N i ⊆ N k ∧ N j
    ⊆ N k by presburger
    from v w i j k have v2: v ∈ N k and w2: w ∈ N k by auto
    from v2 w2 k subm[of k] have vw: v ⊕M w ∈ N k apply (unfold
    submodule-def) by auto
    from k vw uni show ?thesis v w by auto
  qed
  have 4: 0M ∈ N'
  proof -
    from ne obtain i where i: i ∈ I by auto
    from i subm have zi: 0M ∈ N i by (unfold submodule-def, auto)
    from i zi uni show ?thesis by auto
  qed
  from uni subm have 5: ∧ c v. c ∈ carrier R ⇒ v ∈ N' ⇒ c ⊙M
  v ∈ N'
    by (unfold submodule-def, auto)
  from 1 2 3 4 5 show ?thesis by (unfold submodule-def, auto)
  qed

```

lemma (in module) span-is-monotone:

```

  fixes S T
  assumes subs: S ⊆ T
  shows span S ⊆ span T
  proof -
    from subs show ?thesis
      by (unfold span-def, auto)
  qed

```

lemma (in module) span-is-submodule:

```

  fixes S
  assumes h2: S ⊆ carrier M
  shows submodule R (span S) M
  proof (cases S={})
    case True
    moreover have module R M..
    ultimately show ?thesis apply (unfold submodule-def span-def lin-
    comb-def, auto) done
  next
    case False
    show ?thesis
    proof (rule nested-union-vs[where ?I={F. F ⊆ S ∧ finite F} and
    ?N=λF. span F and ?N'=span S])
      show ∧ F. F ∈ {F. F ⊆ S ∧ finite F} ⇒ submodule R (span F)
      M
    proof -
      fix F
      assume F: F ∈ {F. F ⊆ S ∧ finite F}
      from F have h1: finite F by auto
    qed
  qed

```

```

from  $F$   $h2$  have  $inC: F \subseteq carrier\ M$  by auto
from  $h1$   $inC$  interpret  $mh: mod-hom\ R\ (func-space\ F)\ M$  ( $\lambda a.$ 
lincomb a F)
by (rule lincomb-is-mod-hom)
from  $h1$   $inC$  have  $1: mh.im = span\ F$ 
apply (unfold mh.im-def)
apply (unfold func-space-def, simp)
apply (subst finite-span, auto)
apply (unfold image-def, auto)
apply (rule-tac x=restrict a F in bexI)
by (auto intro!: lincomb-cong)
from  $1$  show submodule R (span F) M by (metis mh.im-is-submodule)
qed
next
show  $\bigwedge i\ j. i \in \{F. F \subseteq S \wedge finite\ F\} \implies$ 
 $j \in \{F. F \subseteq S \wedge finite\ F\} \implies$ 
 $\exists k. k \in \{F. F \subseteq S \wedge finite\ F\} \wedge span\ i \subseteq span\ k \wedge span\ j$ 
 $\subseteq span\ k$ 
proof  $-$ 
fix  $i\ j$ 
assume  $i: i \in \{F. F \subseteq S \wedge finite\ F\}$  and  $j: j \in \{F. F \subseteq S \wedge$ 
finite F
from  $i\ j$  show ?thesis i j
apply (rule-tac x=i∪j in exI)
apply (auto del: subsetI)
by (intro span-is-monotone, auto del: subsetI)+
qed
next
show  $span\ S = (\bigcup i \in \{F. F \subseteq S \wedge finite\ F\}. span\ i)$ 
by (unfold span-def, auto)
next
have  $ne: S \neq \{\}$  by fact
from  $ne$  show  $\{F. F \subseteq S \wedge finite\ F\} \neq \{\}$  by auto
qed
qed

```

A finite sum does not depend on the ambient module. This can be done for monoid, but "submonoid" isn't currently defined. (It can be copied, however, for groups...) This lemma requires a somewhat annoying lemma `foldD-not-depend`. Then we show that linear combinations, linear independence, span do not depend on the ambient module.

**lemma** (*in module*) *finsum-not-depend*:

```

fixes  $a\ A\ N$ 
assumes  $h1: finite\ A$  and  $h2: A \subseteq N$  and  $h3: submodule\ R\ N\ M$ 
and  $h4: f: A \rightarrow N$ 
shows  $(\bigoplus_{(md\ N)} v \in A. f\ v) = (\bigoplus_M v \in A. f\ v)$ 
proof  $-$ 

```

```

from h1 h2 h3 h4 show ?thesis
  apply (unfold finsum-def finprod-def)
  apply simp
  apply (intro foldD-not-depend[where ?B=A])
    apply (unfold submodule-def LCD-def, auto)
  apply (meson M.add.m-lcomm PiE subsetCE)+
  done
qed

```

```

lemma (in module) lincomb-not-depend:
  fixes a A N
  assumes h1: finite A and h2:  $A \subseteq N$  and h3: submodule R N M
  and h4:  $a: A \rightarrow \text{carrier } R$ 
  shows lincomb a A = module.lincomb (md N) a A
proof –
  from h3 interpret N: module R (md N) by (rule submodule-is-module)
  have 3:  $N = \text{carrier } (md N)$  by auto
  have 4: (smult M) = (smult (md N)) by auto
  from h1 h2 h3 h4 have  $(\bigoplus_{(md N)} v \in A. a v \odot_M v) = (\bigoplus_{M} v \in A. a v \odot_M v)$ 
  apply (intro finsum-not-depend)
  using N.summands-valid by auto
  from this show ?thesis by (unfold lincomb-def N.lincomb-def, simp)
qed

```

```

lemma (in module) span-li-not-depend:
  fixes S N
  assumes h2:  $S \subseteq N$  and h3: submodule R N M
  shows module.span R (md N) S = module.span R M S
  and module.lin-dep R (md N) S = module.lin-dep R M S
proof –
  from h3 interpret w: module R (md N) by (rule submodule-is-module)
  from h2 have 1: submodule R (module.span R (md N) S) (md N)
  by (intro w.span-is-submodule, simp)
  have 3:  $\bigwedge a A. (\text{finite } A \wedge A \subseteq S \wedge a \in A \rightarrow \text{carrier } R \implies$ 
     $\text{module.lincomb } M a A = \text{module.lincomb } (md N) a A)$ 
  proof –
  fix a A
  assume 31:  $\text{finite } A \wedge A \subseteq S \wedge a \in A \rightarrow \text{carrier } R$ 
  from assms 31 show ?thesis a A
  by (intro lincomb-not-depend, auto)
qed
  from 3 show 4: module.span R (md N) S = module.span R M S
  apply (unfold span-def w.span-def)
  apply auto
  by (metis)
  have zeros:  $\mathbf{0}_{md N} = \mathbf{0}_M$  by auto
  from assms 3 show 5: module.lin-dep R (md N) S = module.lin-dep
  R M S

```

```

    apply (unfold lin-dep-def w.lin-dep-def)
    apply (subst zeros)
    by metis
qed

lemma (in module) span-is-subset:
  fixes S N
  assumes h2:  $S \subseteq N$  and h3: submodule R N M
  shows span S  $\subseteq$  N
proof -
  from h3 interpret w: module R (md N) by (rule submodule-is-module)
  from h2 have 1: submodule R (module.span R (md N) S) (md N)
    by (intro w.span-is-submodule, simp)
  from assms have 4: module.span R (md N) S = module.span R M
  S
    by (rule span-li-not-depend)
  from 1 4 have 5: submodule R (module.span R M S) (md N) by
  auto
  from 5 show ?thesis by (unfold submodule-def, simp)
qed

```

```

lemma (in module) span-is-subset2:
  fixes S
  assumes h2:  $S \subseteq \text{carrier } M$ 
  shows span S  $\subseteq$  carrier M
proof -
  have 0: module R M..
  from 0 have h3: submodule R (carrier M) M by (unfold submodule-def, auto)
  from h2 h3 show ?thesis by (rule span-is-subset)
qed

```

```

lemma (in module) in-own-span:
  fixes S
  assumes inC:  $S \subseteq \text{carrier } M$ 
  shows S  $\subseteq$  span S
proof -
  from inC show ?thesis
    apply (unfold span-def, auto)
    apply (rename-tac v)
    apply (rule-tac x=( $\lambda w.$  if (w=v) then  $\mathbf{1}_R$  else  $\mathbf{0}_R$ ) in exI)
    apply (rule-tac x={v} in exI)
    apply (unfold lincomb-def)
    by auto
qed

```

```

lemma (in module) supset-ld-is-ld:
  fixes A B

```

```

assumes ld: lin-dep A and sub:  $A \subseteq B$ 
shows lin-dep B
proof –
from ld obtain A' a v where 1: (finite A'  $\wedge$   $A' \subseteq A$   $\wedge$  ( $a \in (A' \rightarrow \text{carrier } R)$ )  $\wedge$  (lincomb a A' =  $\mathbf{0}_M$ )  $\wedge$  ( $v \in A'$ )  $\wedge$  ( $a \neq \mathbf{0}_R$ ))
  by (unfold lin-dep-def, auto)
from 1 sub show ?thesis
  apply (unfold lin-dep-def)
  apply (rule-tac  $x=A'$  in exI)
  apply (rule-tac  $x=a$  in exI)
  apply (rule-tac  $x=v$  in exI)
  by auto
qed

```

```

lemma (in module) subset-li-is-li:
  fixes A B
  assumes li: lin-indpt A and sub:  $B \subseteq A$ 
  shows lin-indpt B
proof (rule ccontr)
  assume ld:  $\neg$ lin-indpt B
  from ld sub have ldA: lin-dep A by (metis supset-ld-is-ld)
  from li ldA show False by auto
qed

```

```

lemma (in mod-hom) hom-sum:
  fixes A B g
  assumes h2:  $A \subseteq \text{carrier } M$  and h3:  $g:A \rightarrow \text{carrier } M$ 
  shows  $f (\bigoplus_M a \in A. g a) = (\bigoplus_N a \in A. f (g a))$ 
proof –
  from h2 h3 show ?thesis
  proof (induct A rule: infinite-finite-induct)
    case (insert a A)
    then have  $(\bigoplus_N a \in \text{insert } a A. f (g a)) = f (g a) \oplus_N (\bigoplus_N a \in A. f (g a))$ 
    by (intro finsum-insert, auto)
    with insert.premis insert.hyps show ?case
    by simp
  qed auto
qed

```

end

## 5 The direct sum of modules.

```

theory SumSpaces
imports Main
  HOL-Algebra.Module
  HOL-Algebra.Coset

```

*RingModuleFacts*  
*MonoidSums*  
*FunctionLemmas*  
*LinearCombinations*

**begin**

We define the direct sum  $M_1 \oplus M_2$  of 2 vector spaces as the set  $M_1 \times M_2$  under componentwise addition and scalar multiplication.

**definition** *direct-sum*:: ('a,'b, 'd) module-scheme  $\Rightarrow$  ('a, 'c, 'e) module-scheme  $\Rightarrow$  ('a, ('b $\times$ 'c)) module

**where** *direct-sum* M1 M2 = ( $\downarrow$ carrier = carrier M1  $\times$  carrier M2,  
mult = ( $\lambda$  v w. ( $\mathbf{0}_{M_1}$ ,  $\mathbf{0}_{M_2}$ )),  
one = ( $\mathbf{0}_{M_1}$ ,  $\mathbf{0}_{M_2}$ ),  
zero = ( $\mathbf{0}_{M_1}$ ,  $\mathbf{0}_{M_2}$ ),  
add = ( $\lambda$  v w. (fst v  $\oplus_{M_1}$  fst w, snd v  $\oplus_{M_2}$  snd w)),  
smult = ( $\lambda$  c v. (c  $\odot_{M_1}$  fst v, c  $\odot_{M_2}$  snd v)))

**lemma** *direct-sum-is-module*:

**fixes** R M1 M2

**assumes** h1: module R M1 **and** h2: module R M2

**shows** module R (direct-sum M1 M2)

**proof** –

**from** h1 **have** 1: cring R **by** (unfold module-def, auto)

**from** h1 **interpret** v1: module R M1 **by** auto

**from** h2 **interpret** v2: module R M2 **by** auto

**from** h1 h2 **have** 2: abelian-group (direct-sum M1 M2)

**apply** (intro abelian-groupI, auto)

**apply** (unfold direct-sum-def, auto)

**by** (auto simp add: v1.a-ac v2.a-ac)

**from** h1 h2 **assms** **have** 3: module-axioms R (direct-sum M1 M2)

**apply** (unfold module-axioms-def, auto)

**apply** (unfold direct-sum-def, auto)

**by** (auto simp add: v1.smult-l-distr v2.smult-l-distr v1.smult-r-distr  
v2.smult-r-distr

v1.smult-assoc1 v2.smult-assoc1)

**from** 1 2 3 **show** ?thesis **by** (unfold module-def, auto)

**qed**

**definition** *inj1*:: ('a,'b) module  $\Rightarrow$  ('a, 'c) module  $\Rightarrow$  ('b $\Rightarrow$ ('b $\times$ 'c))

**where** *inj1* M1 M2 = ( $\lambda$ v. (v,  $\mathbf{0}_{M_2}$ ))

**definition** *inj2*:: ('a,'b) module  $\Rightarrow$  ('a, 'c) module  $\Rightarrow$  ('c $\Rightarrow$ ('b $\times$ 'c))

**where** *inj2* M1 M2 = ( $\lambda$ v. ( $\mathbf{0}_{M_1}$ , v))

**lemma** *inj1-hom*:

**fixes** R M1 M2

**assumes** h1: module R M1 **and** h2: module R M2

**shows** mod-hom R M1 (direct-sum M1 M2) (*inj1* M1 M2)

```

proof –
  from h1 interpret v1:module R M1 by auto
  from h2 interpret v2:module R M2 by auto
  from h1 h2 show ?thesis
    apply (unfold mod-hom-def module-hom-def mod-hom-axioms-def
inj1-def, auto)
      apply (rule direct-sum-is-module, auto)
      by (unfold direct-sum-def, auto)
qed

```

```

lemma inj2-hom:
  fixes R M1 M2
  assumes h1: module R M1 and h2: module R M2
  shows mod-hom R M2 (direct-sum M1 M2) (inj2 M1 M2)

```

```

proof –
  from h1 interpret v1:module R M1 by auto
  from h2 interpret v2:module R M2 by auto
  from h1 h2 show ?thesis
    apply (unfold mod-hom-def module-hom-def mod-hom-axioms-def
inj2-def, auto)
      apply (rule direct-sum-is-module, auto)
      by (unfold direct-sum-def, auto)
qed

```

For submodules  $M_1, M_2 \subseteq M$ , the map  $M_1 \oplus M_2 \rightarrow M$  given by  $(m_1, m_2) \mapsto m_1 + m_2$  is linear.

```

lemma (in module) sum-map-hom:
  fixes M1 M2
  assumes h1: submodule R M1 M and h2: submodule R M2 M
  shows mod-hom R (direct-sum (md M1) (md M2)) M ( $\lambda v. (fst v)$ 
 $\oplus_M (snd v)$ )

```

```

proof –
  have 0: module R M..
  from h1 have 1: module R (md M1) by (rule submodule-is-module)
  from h2 have 2: module R (md M2) by (rule submodule-is-module)
  from h1 interpret w1: module R (md M1) by (rule submodule-is-module)
  from h2 interpret w2: module R (md M2) by (rule submodule-is-module)
  from 0 h1 h2 1 2 show ?thesis
    apply (unfold mod-hom-def mod-hom-axioms-def module-hom-def,
auto)
      apply (rule direct-sum-is-module, auto)
      apply (unfold direct-sum-def, auto)
      apply (unfold submodule-def, auto)
      by (auto simp add: a-ac smult-r-distr ring-subset-carrier)

```

**qed**

```

lemma (in module) sum-is-submodule:
  fixes N1 N2

```

**assumes**  $h1$ : submodule  $R$   $N1$   $M$  **and**  $h2$ : submodule  $R$   $N2$   $M$   
**shows** submodule  $R$  (submodule-sum  $N1$   $N2$ )  $M$   
**proof** –  
**from**  $h1$   $h2$  **interpret**  $l$ : mod-hom  $R$  (direct-sum (md  $N1$ ) (md  $N2$ ))  
 $M$  ( $\lambda$   $v$ . (fst  $v$ )  $\oplus_M$  (snd  $v$ ))  
**by** (rule sum-map-hom)  
**have**  $1$ :  $l.im = \text{submodule-sum } N1 \ N2$   
**apply** (unfold  $l.im\text{-def}$  submodule-sum-def)  
**apply** (unfold direct-sum-def, auto)  
**by** (unfold image-def, auto)  
**have**  $2$ : submodule  $R$  ( $l.im$ )  $M$  **by** (rule  $l.im\text{-is-submodule}$ )  
**from**  $1$   $2$  **show** ?thesis **by** auto  
**qed**

**lemma** (in module) in-sum:  
**fixes**  $N1$   $N2$   
**assumes**  $h1$ : submodule  $R$   $N1$   $M$  **and**  $h2$ : submodule  $R$   $N2$   $M$   
**shows**  $N1 \subseteq \text{submodule-sum } N1 \ N2$   
**proof** –  
**from**  $h1$   $h2$  **show** ?thesis  
**apply** auto  
**apply** (unfold submodule-sum-def image-def, auto)  
**apply** (rename-tac  $v$ )  
**apply** (rule-tac  $x=v$  in  $bxI$ )  
**apply** (rule-tac  $x=0_M$  in  $bxI$ )  
**by** (unfold submodule-def, auto)  
**qed**

**lemma** (in module) msum-comm:  
**fixes**  $N1$   $N2$   
**assumes**  $h1$ : submodule  $R$   $N1$   $M$  **and**  $h2$ : submodule  $R$   $N2$   $M$   
**shows** (submodule-sum  $N1$   $N2$ ) = (submodule-sum  $N2$   $N1$ )  
**proof** –  
**from**  $h1$   $h2$  **show** ?thesis  
**apply** (unfold submodule-sum-def image-def, auto)  
**apply** (unfold submodule-def)  
**apply** (rename-tac  $v$   $w$ )  
**by** (metis (full-types)  $M.add.m\text{-comm}$  subsetD)+  
**qed**

If  $M_1, M_2 \subseteq M$  are submodules, then  $M_1 + M_2$  is the minimal subspace such that both  $M_1 \subseteq M$  and  $M_2 \subseteq M$ .

**lemma** (in module) sum-is-minimal:  
**fixes**  $N$   $N1$   $N2$   
**assumes**  $h1$ : submodule  $R$   $N1$   $M$  **and**  $h2$ : submodule  $R$   $N2$   $M$  **and**  
 $h3$ : submodule  $R$   $N$   $M$   
**shows** (submodule-sum  $N1$   $N2$ )  $\subseteq N \iff N1 \subseteq N \wedge N2 \subseteq N$

```

proof –
  have 1: (submodule-sum N1 N2) ⊆ N ⇒ N1 ⊆ N ∧ N2 ⊆ N
  proof –
    assume 10: (submodule-sum N1 N2) ⊆ N
    from h1 h2 have 11: N1 ⊆ submodule-sum N1 N2 by (rule in-sum)
    from h2 h1 have 12: N2 ⊆ submodule-sum N2 N1 by (rule in-sum)
    from 12 h1 h2 have 13: N2 ⊆ submodule-sum N1 N2 by (metis
msum-comm)
    from 10 11 13 show ?thesis by auto
  qed
  have 2: N1 ⊆ N ∧ N2 ⊆ N ⇒ (submodule-sum N1 N2) ⊆ N
  proof –
    assume 19: N1 ⊆ N ∧ N2 ⊆ N
    {
    fix v
    assume 20: v ∈ submodule-sum N1 N2
    from 20 obtain w1 w2 where 21: w1 ∈ N1 and 22: w2 ∈ N2 and
23: v = w1 ⊕M w2
      by (unfold submodule-sum-def image-def, auto)
    from 19 21 22 23 h3 have v ∈ N
      apply (unfold submodule-def, auto)
      by (metis (poly-guards-query) contra-subsetD)
    }
    thus ?thesis
      by (metis subset-iff)
  qed
from 1 2 show ?thesis by metis
qed

```

$\text{span}A \cup B = \text{span}A + \text{span}B$

**lemma** (in module) span-union-is-sum:

**fixes** A B

**assumes** h2:  $A \subseteq \text{carrier } M$  **and** h3:  $B \subseteq \text{carrier } M$

**shows**  $\text{span } (A \cup B) = \text{submodule-sum } (\text{span } A) (\text{span } B)$

**proof**–

**let** ?AplusB = submodule-sum (span A) (span B)

**from** h2 **have** s0: submodule R (span A) M **by** (rule span-is-submodule)

**from** h3 **have** s1: submodule R (span B) M **by** (rule span-is-submodule)

**from** s0 **have** s0-1:  $(\text{span } A) \subseteq \text{carrier } M$  **by** (unfold submodule-def, auto)

**from** s1 **have** s1-1:  $(\text{span } B) \subseteq \text{carrier } M$  **by** (unfold submodule-def, auto)

**from** h2 h3 **have** 1:  $A \cup B \subseteq \text{carrier } M$  **by** auto

**from** 1 **have** 2: submodule R (span (A ∪ B)) M **by** (rule span-is-submodule)

**from** s0 s1 **have** 3: submodule R ?AplusB M **by** (rule sum-is-submodule)

**have** c1:  $\text{span } (A \cup B) \subseteq ?AplusB$

**proof** –

```

from h2 have a1:  $A \subseteq \text{span } A$  by (rule in-own-span)
from s0 s1 have a2:  $\text{span } A \subseteq ?\text{Aplus}B$  by (rule in-sum)
from a1 a2 have a3:  $A \subseteq ?\text{Aplus}B$  by auto

from h3 have b1:  $B \subseteq \text{span } B$  by (rule in-own-span)
from s1 s0 have b2:  $\text{span } B \subseteq ?\text{Aplus}B$  by (metis in-sum msum-comm)

from b1 b2 have b3:  $B \subseteq ?\text{Aplus}B$  by auto
from a3 b3 have 5:  $A \cup B \subseteq ?\text{Aplus}B$  by auto

from 5 3 show ?thesis by (rule span-is-subset)
qed
have c2:  $?AplusB \subseteq \text{span } (A \cup B)$ 
proof -
  have 11:  $A \subseteq A \cup B$  by auto
  have 12:  $B \subseteq A \cup B$  by auto
  from 11 have 21:  $\text{span } A \subseteq \text{span } (A \cup B)$  by (rule span-is-monotone)
  from 12 have 22:  $\text{span } B \subseteq \text{span } (A \cup B)$  by (rule span-is-monotone)
  from s0 s1 21 22 show ?thesis by (auto simp add: sum-is-minimal)
qed
from c1 c2 show ?thesis by auto
qed

end

```

## 6 Basic theory of vector spaces, using locales

```

theory VectorSpace
imports Main
  HOL-Algebra.Module
  HOL-Algebra.Coset
  RingModuleFacts
  MonoidSums
  LinearCombinations
  SumSpaces
begin

```

### 6.1 Basic definitions and facts carried over from modules

A *vectorspace* is a module where the ring is a field. Note that we switch notation from  $(R, M)$  to  $(K, V)$ .

```

locale vectorspace =
  module?: module  $K$   $V$  + field?: field  $K$ 
for  $K$  and  $V$ 

```

A *subspace* of a vectorspace is a nonempty subset that is closed

under addition and scalar multiplication. These properties have already been defined in submodule. Caution:  $W$  is a set, while  $V$  is a module record. To get  $W$  as a vectorspace, write  $vs\ W$ .

```

locale subspace =
  fixes  $K$  and  $W$  and  $V$  (structure)
  assumes  $vs$ : vectorspace  $K$   $V$ 
  and  $submod$ : submodule  $K$   $W$   $V$ 

```

```

lemma (in vectorspace) is-module[simp]:
   $subspace\ K\ W\ V \implies submodule\ K\ W\ V$ 
by (unfold subspace-def, auto)

```

We introduce some basic facts and definitions copied from module. We introduce some abbreviations, to match convention.

```

abbreviation (in vectorspace)  $vs::'c\ set \Rightarrow ('a, 'c, 'd)\ module-scheme$ 
  where  $vs\ W \equiv V(\text{carrier} := W)$ 

```

```

lemma (in vectorspace) carrier-vs-is-self [simp]:
   $carrier\ (vs\ W) = W$ 
by auto

```

```

lemma (in vectorspace) subspace-is-vs:
  fixes  $W::'c\ set$ 
  assumes  $0$ : subspace  $K$   $W$   $V$ 
  shows vectorspace  $K$  ( $vs\ W$ )
proof –
  from  $0$  show ?thesis
  apply (unfold vectorspace-def subspace-def, auto)
  by (intro submodule-is-module, auto)
qed

```

```

abbreviation (in module)  $subspace-sum::['c\ set, 'c\ set] \Rightarrow 'c\ set$ 
  where  $subspace-sum\ W1\ W2 \equiv submodule-sum\ W1\ W2$ 

```

```

lemma (in vectorspace) vs-zero-lin-dep:
  assumes  $h2$ :  $S \subseteq carrier\ V$  and  $h3$ : lin-indpt  $S$ 
  shows  $\mathbf{0}_V \notin S$ 
proof –
  have  $vs$ : vectorspace  $K$   $V..$ 
  from  $vs$  have  $nonzero$ :  $carrier\ K \neq \{\mathbf{0}_K\}$ 
  by (metis one-zeroI zero-not-one)
  from  $h2\ h3\ nonzero$  show ?thesis by (rule zero-nin-lin-indpt)
qed

```

A *linear-map* is a module homomorphism between 2 vectorspaces over the same field.

```

locale linear-map =

```

$V?$ : *vectorspace*  $K$   $V$  +  $W?$ : *vectorspace*  $K$   $W$   
+ *mod-hom?*: *mod-hom*  $K$   $V$   $W$   $T$   
**for**  $K$  **and**  $V$  **and**  $W$  **and**  $T$

**context** *linear-map*

**begin**

**lemmas**  $T\text{-hom} = f\text{-hom}$

**lemmas**  $T\text{-add} = f\text{-add}$

**lemmas**  $T\text{-smult} = f\text{-smult}$

**lemmas**  $T\text{-im} = f\text{-im}$

**lemmas**  $T\text{-neg} = f\text{-neg}$

**lemmas**  $T\text{-minus} = f\text{-minus}$

**lemmas**  $T\text{-ker} = f\text{-ker}$

**abbreviation**  $\text{im}T:: 'e$  *set*

**where**  $\text{im}T \equiv \text{mod-hom.im}$

**abbreviation**  $\text{ker}T:: 'c$  *set*

**where**  $\text{ker}T \equiv \text{mod-hom.ker}$

**lemmas**  $T0\text{-is-}0[\text{simp}] = f0\text{-is-}0$

**lemma**  $\text{ker}T\text{-is-subspace}$ : *subspace*  $K$   $\text{ker} V$

**proof** –

**have**  $vs$ : *vectorspace*  $K$   $V$ ..

**from**  $vs$  **show** *?thesis*

**apply** (*unfold subspace-def*, *auto*)

**by** (*rule ker-is-submodule*)

**qed**

**lemma**  $\text{im}T\text{-is-subspace}$ : *subspace*  $K$   $\text{im}T W$

**proof** –

**have**  $vs$ : *vectorspace*  $K$   $W$ ..

**from**  $vs$  **show** *?thesis*

**apply** (*unfold subspace-def*, *auto*)

**by** (*rule im-is-submodule*)

**qed**

**end**

**lemma** *vs-criteria*:

**fixes**  $K$  **and**  $V$

**assumes** *field*: *field*  $K$

**and** *zero*:  $\mathbf{0}_V \in \text{carrier } V$

**and** *add*:  $\forall v w. v \in \text{carrier } V \wedge w \in \text{carrier } V \longrightarrow v \oplus_V w \in \text{carrier } V$

$V$

**and** *neg*:  $\forall v \in \text{carrier } V. (\exists \text{neg-}v \in \text{carrier } V. v \oplus_V \text{neg-}v = \mathbf{0}_V)$

**and** *smult*:  $\forall c v. c \in \text{carrier } K \wedge v \in \text{carrier } V \longrightarrow c \odot_V v \in \text{carrier } V$

$V$

**and** *comm*:  $\forall v w. v \in \text{carrier } V \wedge w \in \text{carrier } V \longrightarrow v \oplus_V w = w \oplus_V v$

$v$   
**and** *assoc*:  $\forall v w x. v \in \text{carrier } V \wedge w \in \text{carrier } V \wedge x \in \text{carrier } V$   
 $V \longrightarrow (v \oplus_V w) \oplus_V x = v \oplus_V (w \oplus_V x)$   
**and** *add-id*:  $\forall v \in \text{carrier } V. (v \oplus_V \mathbf{0}_V = v)$   
**and** *compat*:  $\forall a b v. a \in \text{carrier } K \wedge b \in \text{carrier } K \wedge v \in \text{carrier } V$   
 $V \longrightarrow (a \otimes_K b) \odot_V v = a \odot_V (b \odot_V v)$   
**and** *smult-id*:  $\forall v \in \text{carrier } V. (\mathbf{1}_K \odot_V v = v)$   
**and** *dist-f*:  $\forall a b v. a \in \text{carrier } K \wedge b \in \text{carrier } K \wedge v \in \text{carrier } V$   
 $V \longrightarrow (a \oplus_K b) \odot_V v = (a \odot_V v) \oplus_V (b \odot_V v)$   
**and** *dist-add*:  $\forall a v w. a \in \text{carrier } K \wedge v \in \text{carrier } V \wedge w \in \text{carrier } V$   
 $V \longrightarrow a \odot_V (v \oplus_V w) = (a \odot_V v) \oplus_V (a \odot_V w)$   
**shows** *vectorspace*  $K V$   
**proof** –  
**from** *field* **have** 1: *cring*  $K$  **by** (*unfold field-def domain-def, auto*)  
**from** *assms* 1 **have** 2: *module*  $K V$  **by** (*intro module-criteria, auto*)  
**from** *field* 2 **show** ?thesis **by** (*unfold vectorspace-def module-def, auto*)  
**qed**

For any set  $S$ , the space of functions  $S \rightarrow K$  forms a vector space.

**lemma** (*in vectorspace*) *func-space-is-vs*:

**fixes**  $S$

**shows** *vectorspace*  $K$  (*func-space*  $S$ )

**proof** –

**have** 0: *field*  $K$ .

**have** 1: *module*  $K$  (*func-space*  $S$ ) **by** (*rule func-space-is-module*)

**from** 0 1 **show** ?thesis **by** (*unfold vectorspace-def module-def, auto*)

**qed**

**lemma** *direct-sum-is-vs*:

**fixes**  $K V1 V2$

**assumes** *h1*: *vectorspace*  $K V1$  **and** *h2*: *vectorspace*  $K V2$

**shows** *vectorspace*  $K$  (*direct-sum*  $V1 V2$ )

**proof** –

**from** *h1 h2* **have** *mod*: *module*  $K$  (*direct-sum*  $V1 V2$ ) **by** (*unfold vectorspace-def, intro direct-sum-is-module, auto*)

**from** *mod h1* **show** ?thesis **by** (*unfold vectorspace-def, auto*)

**qed**

**lemma** *inj1-linear*:

**fixes**  $K V1 V2$

**assumes** *h1*: *vectorspace*  $K V1$  **and** *h2*: *vectorspace*  $K V2$

**shows** *linear-map*  $K V1$  (*direct-sum*  $V1 V2$ ) (*inj1*  $V1 V2$ )

**proof** –

**from** *h1 h2* **have** *mod*: *mod-hom*  $K V1$  (*direct-sum*  $V1 V2$ ) (*inj1*  $V1 V2$ ) **by** (*unfold vectorspace-def, intro inj1-hom, auto*)

**from** *mod h1 h2* **show** ?thesis

by (unfold linear-map-def vectorspace-def , auto, intro direct-sum-is-module, auto)  
qed

lemma inj2-linear:

fixes K V1 V2  
 assumes h1: vectorspace K V1 and h2: vectorspace K V2  
 shows linear-map K V2 (direct-sum V1 V2) (inj2 V1 V2)  
 proof –  
 from h1 h2 have mod: mod-hom K V2 (direct-sum V1 V2) (inj2 V1 V2) by (unfold vectorspace-def, intro inj2-hom, auto)  
 from mod h1 h2 show ?thesis  
 by (unfold linear-map-def vectorspace-def , auto, intro direct-sum-is-module, auto)  
 qed

For subspaces  $V_1, V_2 \subseteq V$ , the map  $V_1 \oplus V_2 \rightarrow V$  given by  $(v_1, v_2) \mapsto v_1 + v_2$  is linear.

lemma (in vectorspace) sum-map-linear:

fixes V1 V2  
 assumes h1: subspace K V1 V and h2: subspace K V2 V  
 shows linear-map K (direct-sum (vs V1) (vs V2)) V ( $\lambda v. (fst v) \oplus_V (snd v)$ )  
 proof –  
 from h1 h2 have mod: mod-hom K (direct-sum (vs V1) (vs V2)) V ( $\lambda v. (fst v) \oplus_V (snd v)$ )  
 by (intro sum-map-hom, unfold subspace-def, auto)  
 from mod h1 h2 show ?thesis  
 apply (unfold linear-map-def, auto) apply (intro direct-sum-is-vs subspace-is-vs, auto)..  
 qed

lemma (in vectorspace) sum-is-subspace:

fixes W1 W2  
 assumes h1: subspace K W1 V and h2: subspace K W2 V  
 shows subspace K (subspace-sum W1 W2) V  
 proof –  
 from h1 h2 have mod: submodule K (submodule-sum W1 W2) V  
 by (intro sum-is-submodule, unfold subspace-def, auto)  
 from mod h1 h2 show ?thesis  
 by (unfold subspace-def, auto)  
 qed

If  $W_1, W_2 \subseteq V$  are subspaces,  $W_1 \subseteq W_1 + W_2$

lemma (in vectorspace) in-sum-vs:

fixes W1 W2  
 assumes h1: subspace K W1 V and h2: subspace K W2 V  
 shows  $W_1 \subseteq \text{subspace-sum } W_1 W_2$   
 proof –

**from**  $h1$   $h2$  **show** *?thesis* **by** (*intro in-sum, unfold subspace-def, auto*)  
**qed**

**lemma** (*in vectorspace*) *vsum-comm*:  
**fixes**  $W1$   $W2$   
**assumes**  $h1$ : *subspace*  $K$   $W1$   $V$  **and**  $h2$ : *subspace*  $K$   $W2$   $V$   
**shows** (*subspace-sum*  $W1$   $W2$ ) = (*subspace-sum*  $W2$   $W1$ )  
**proof** –  
**from**  $h1$   $h2$  **show** *?thesis* **by** (*intro msum-comm, unfold subspace-def, auto*)  
**qed**

If  $W_1, W_2 \subseteq V$  are subspaces, then  $W_1 + W_2$  is the minimal subspace such that both  $W_1 \subseteq W$  and  $W_2 \subseteq W$ .

**lemma** (*in vectorspace*) *vsum-is-minimal*:  
**fixes**  $W$   $W1$   $W2$   
**assumes**  $h1$ : *subspace*  $K$   $W1$   $V$  **and**  $h2$ : *subspace*  $K$   $W2$   $V$  **and**  $h3$ : *subspace*  $K$   $W$   $V$   
**shows** (*subspace-sum*  $W1$   $W2$ )  $\subseteq W \iff W1 \subseteq W \wedge W2 \subseteq W$   
**proof** –  
**from**  $h1$   $h2$   $h3$  **show** *?thesis* **by** (*intro sum-is-minimal, unfold subspace-def, auto*)  
**qed**

**lemma** (*in vectorspace*) *span-is-subspace*:  
**fixes**  $S$   
**assumes**  $h2$ :  $S \subseteq \text{carrier } V$   
**shows** *subspace*  $K$  (*span*  $S$ )  $V$   
**proof** –  
**have**  $0$ : *vectorspace*  $K$   $V$ ..  
**from**  $h2$  **have**  $1$ : *submodule*  $K$  (*span*  $S$ )  $V$  **by** (*rule span-is-submodule*)  
**from**  $0$   $1$  **show** *?thesis* **by** (*unfold subspace-def mod-hom-def linear-map-def, auto*)  
**qed**

### 6.1.1 Facts specific to vector spaces

If  $av = w$  and  $a \neq 0$ ,  $v = a^{-1}w$ .

**lemma** (*in vectorspace*) *mult-inverse*:  
**assumes**  $h1$ :  $a \in \text{carrier } K$  **and**  $h2$ :  $v \in \text{carrier } V$  **and**  $h3$ :  $a \odot_V v = w$  **and**  $h4$ :  $a \neq 0_K$   
**shows**  $v = (\text{inv}_K a) \odot_V w$   
**proof** –  
**from**  $h1$   $h2$   $h3$  **have**  $1$ :  $w \in \text{carrier } V$  **by** *auto*  
**from**  $h3$   $1$  **have**  $2$ :  $(\text{inv}_K a) \odot_V (a \odot_V v) = (\text{inv}_K a) \odot_V w$  **by** *auto*  
**from**  $h1$   $h4$  **have**  $3$ :  $\text{inv}_K a \in \text{carrier } K$  **by** *auto*  
**interpret**  $g$ : *group* (*units-group*  $K$ ) **by** (*rule units-form-group*)

**have**  $f$ : *field*  $K$ ..  
**from**  $f$   $h1$   $h4$  **have**  $4$ :  $a \in \text{Units } K$   
**by** (*unfold field-def field-axioms-def, simp*)  
**from**  $4$   $h1$   $h4$  **have**  $5$ :  $((\text{inv}_K a) \otimes_K a) = \mathbf{1}_K$   
**by** (*intro Units-l-inv, auto*)  
**from**  $5$  **have**  $6$ :  $(\text{inv}_K a) \odot_V (a \odot_V v) = v$   
**proof** –  
**from**  $h1$   $h2$   $h4$  **have**  $7$ :  $(\text{inv}_K a) \odot_V (a \odot_V v) = (\text{inv}_K a \otimes_K a)$   
 $\odot_V v$  **by** (*auto simp add: smult-assoc1*)  
**from**  $5$   $h2$  **have**  $8$ :  $(\text{inv}_K a \otimes_K a) \odot_V v = v$  **by** *auto*  
**from**  $7$   $8$  **show** *?thesis* **by** *auto*  
**qed**  
**from**  $2$   $6$  **show** *?thesis* **by** *auto*  
**qed**

If  $w \in S$  and  $\sum_{w \in S} a_w w = 0$ , we have  $v = \sum_{w \notin S} a_w^{-1} a_w w$

**lemma** (*in vectorspace*) *lincomb-isolate*:

**fixes**  $A$   $v$   
**assumes**  $h1$ : *finite*  $A$  **and**  $h2$ :  $A \subseteq \text{carrier } V$  **and**  $h3$ :  $a \in A \rightarrow \text{carrier } K$  **and**  $h4$ :  $v \in A$   
**and**  $h5$ :  $a \ v \neq \mathbf{0}_K$  **and**  $h6$ : *lincomb*  $a$   $A = \mathbf{0}_V$   
**shows**  $v = \text{lincomb } (\lambda w. \ominus_K (\text{inv}_K (a \ v)) \otimes_K a \ w) (A - \{v\})$  **and**  $v \in \text{span } (A - \{v\})$   
**proof** –  
**from**  $h1$   $h2$   $h3$   $h4$  **have**  $1$ :  $\text{lincomb } a \ A = ((a \ v) \odot_V v) \oplus_V \text{lincomb } a \ (A - \{v\})$   
**by** (*rule lincomb-del2*)  
**from**  $1$  **have**  $2$ :  $\mathbf{0}_V = ((a \ v) \odot_V v) \oplus_V \text{lincomb } a \ (A - \{v\})$  **by** (*simp add: h6*)  
**from**  $h1$   $h2$   $h3$  **have**  $5$ :  $\text{lincomb } a \ (A - \{v\}) \in \text{carrier } V$  **by** *auto*  
**from**  $2$   $h1$   $h2$   $h3$   $h4$  **have**  $3$ :  $\ominus_V \text{lincomb } a \ (A - \{v\}) = ((a \ v) \odot_V v)$   
**by** (*auto intro!: M.minus-equality*)  
**have**  $6$ :  $v = (\ominus_K (\text{inv}_K (a \ v))) \odot_V \text{lincomb } a \ (A - \{v\})$   
**proof** –  
**from**  $h2$   $h3$   $h4$   $h5$   $3$  **have**  $7$ :  $v = \text{inv}_K (a \ v) \odot_V (\ominus_V \text{lincomb } a \ (A - \{v\}))$   
**by** (*intro mult-inverse, auto*)  
**from** *assms* **have**  $8$ :  $\text{inv}_K (a \ v) \in \text{carrier } K$  **by** *auto*  
**from** *assms*  $5$   $8$  **have**  $9$ :  $\text{inv}_K (a \ v) \odot_V (\ominus_V \text{lincomb } a \ (A - \{v\})) = (\ominus_K (\text{inv}_K (a \ v))) \odot_V \text{lincomb } a \ (A - \{v\})$   
**by** (*simp add: smult-assoc-simp smult-minus-1-back r-minus*)  
**from**  $7$   $9$  **show** *?thesis* **by** *auto*  
**qed**  
**from**  $h1$  **have**  $10$ : *finite*  $(A - \{v\})$  **by** *auto*  
**from** *assms* **have**  $11$ :  $(\ominus_K (\text{inv}_K (a \ v))) \in \text{carrier } K$  **by** *auto*  
**from** *assms* **have**  $12$ :  $\text{lincomb } (\lambda w. \ominus_K (\text{inv}_K (a \ v)) \otimes_K a \ w) (A - \{v\}) = (\ominus_K (\text{inv}_K (a \ v))) \odot_V \text{lincomb } a \ (A - \{v\})$

by (intro lincomb-smult, auto)  
 from 6 12 show  $v = \text{lincomb } (\lambda w. \ominus_K(\text{inv}_K(a v)) \otimes_K a w) (A - \{v\})$   
 by auto  
 with *assms* show  $v \in \text{span } (A - \{v\})$   
 unfolding *span-def*  
 by (force simp add: 11 ring-subset-carrier)  
 qed

The map  $(S \rightarrow K) \mapsto V$  given by  $(a_v)_{v \in S} \mapsto \sum_{v \in S} a_v v$  is linear.

**lemma** (in *vectorspace*) *lincomb-is-linear*:  
 fixes  $S$   
 assumes  $h$ : *finite*  $S$  and  $h2$ :  $S \subseteq \text{carrier } V$   
 shows *linear-map*  $K$  (*func-space*  $S$ )  $V$  ( $\lambda a. \text{lincomb } a S$ )  
**proof** –  
 have 0: *vectorspace*  $K V$ .  
 from  $h$   $h2$  have 1: *mod-hom*  $K$  (*func-space*  $S$ )  $V$  ( $\lambda a. \text{lincomb } a S$ )  
 by (*rule lincomb-is-mod-hom*)  
 from 0 1 show *?thesis* by (*unfold vectorspace-def mod-hom-def linear-map-def, auto*)  
 qed

## 6.2 Basic facts about span and linear independence

If  $S$  is linearly independent, then  $v \in \text{span } S$  iff  $S \cup \{v\}$  is linearly dependent.

**theorem** (in *vectorspace*) *lin-dep-iff-in-span*:  
 fixes  $A v S$   
 assumes  $h1$ :  $S \subseteq \text{carrier } V$  and  $h2$ : *lin-indpt*  $S$  and  $h3$ :  $v \in \text{carrier } V$  and  $h4$ :  $v \notin S$   
 shows  $v \in \text{span } S \iff \text{lin-dep } (S \cup \{v\})$   
**proof** –  
 let  $?T = S \cup \{v\}$   
 have 0:  $v \in ?T$  by *auto*  
 from  $h1$   $h3$  have  $h1-1$ :  $?T \subseteq \text{carrier } V$  by *auto*  
 have  $a1$ : *lin-dep*  $?T \implies v \in \text{span } S$   
**proof** –  
 assume  $a11$ : *lin-dep*  $?T$   
 from  $a11$  obtain  $a w A$  where  $a$ : (*finite*  $A \wedge A \subseteq ?T \wedge (a \in (A \rightarrow \text{carrier } K)) \wedge (\text{lincomb } a A = \mathbf{0}_V) \wedge (w \in A) \wedge (a w \neq \mathbf{0}_K)$ )  
 by (*metis lin-dep-def*)  
 from *assms*  $a$  have  $nz2$ :  $\exists v \in A - S. a v \neq \mathbf{0}_K$   
 by (*intro lincomb-must-include*[**where**  $?v = w$  and  $?T = S \cup \{v\}$ ], *auto*)  
 from  $a$   $nz2$  have *singleton*:  $\{v\} = A - S$  by *auto*  
 from *singleton*  $nz2$  have  $nz3$ :  $a v \neq \mathbf{0}_K$  by *auto*  
  
 let  $?b = (\lambda w. \ominus_K(\text{inv}_K(a v)) \otimes_K (a w))$   
 from *singleton* have  $A \text{ins}$ :  $(A \cap S) = A - \{v\}$  by *auto*

```

from assms a singleton nz3 have a31: v = lincomb ?b (A ∩ S)
  apply (subst Ains)
  by (intro lincomb-isolate(1), auto)
from a a31 nz3 singleton show ?thesis
  apply (unfold span-def, auto)
  apply (rule-tac x=?b in exI)
  apply (rule-tac x=A ∩ S in exI)
  by (auto intro!: m-closed)
qed
have a2: v ∈ (span S) ⇒ lin-dep ?T
proof –
  assume inspan: v ∈ (span S)
  from inspan obtain a A where a: A ⊆ S ∧ finite A ∧ (v = lincomb
a A) ∧ a ∈ A → carrier K by (simp add: span-def, auto)
  let ?b = λ w. if (w=v) then (⊖K 1K) else a w
  have lc0: lincomb ?b (A ∪ {v}) = 0V
  proof –
    from assms a have lc-ins: lincomb ?b (A ∪ {v}) = ((?b v) ⊙V v)
    ⊕V lincomb ?b A
    by (intro lincomb-insert, auto)
    from assms a have lc-elim: lincomb ?b A = lincomb a A by (intro
lincomb-elim-if, auto)
    from assms lc-ins lc-elim a show ?thesis by (simp add: M.l-neg
smult-minus-1)
  qed
  from a lc0 show ?thesis
  apply (unfold lin-dep-def)
  apply (rule-tac x=A ∪ {v} in exI)
  apply (rule-tac x=?b in exI)
  apply (rule-tac x=v in exI)
  by auto
qed
from a1 a2 show ?thesis by auto
qed

```

If  $v \in \text{span} A$  then  $\text{span} A = \text{span}(A \cup \{v\})$

**lemma** (*in vectorspace*) *already-in-span:*

```

fixes v A
assumes inC: A ⊆ carrier V and inspan: v ∈ span A
shows span A = span (A ∪ {v})
proof –
  from inC inspan have dir1: span A ⊆ span (A ∪ {v}) by (intro
span-is-monotone, auto)

```

```

  from inC have inown: A ⊆ span A by (rule in-own-span)
  from inC have subm: submodule K (span A) V by (rule span-is-submodule)
  from inown inspan subm have dir2: span (A ∪ {v}) ⊆ span A by
(intro span-is-subset, auto)

```

from *dir1 dir2* show *?thesis* by *auto*  
qed

### 6.3 The Replacement Theorem

If  $A, B \subseteq V$  are finite,  $A$  is linearly independent,  $B$  generates  $W$ , and  $A \subseteq W$ , then there exists  $C \subseteq V$  disjoint from  $A$  such that  $\text{span}(A \cup C) = W$  and  $|C| \leq |B| - |A|$ . In other words, we can complete any linearly independent set to a generating set of  $W$  by adding at most  $|B| - |A|$  more elements.

**theorem** (in *vectorspace*) *replacement*:

fixes  $A B$   
 assumes  $h1$ : *finite*  $A$   
   and  $h2$ : *finite*  $B$   
   and  $h3$ :  $B \subseteq \text{carrier } V$   
   and  $h4$ : *lin-indpt*  $A$   
   and  $h5$ :  $A \subseteq \text{span } B$   
 shows  $\exists C. \text{finite } C \wedge C \subseteq \text{carrier } V \wedge C \subseteq \text{span } B \wedge C \cap A = \{\} \wedge \text{int}(\text{card } C) \leq (\text{int}(\text{card } B)) - (\text{int}(\text{card } A)) \wedge (\text{span}(A \cup C) = \text{span } B)$   
 (is  $\exists C. ?P A B C$ )

using  $h1 h2 h3 h4 h5$

**proof** (*induct card A arbitrary: A B*)

case 0  
 from 0.prem(1) 0.hyps have  $a0: A = \{\}$  by *auto*  
 from 0.prem(3) have  $a3: B \subseteq \text{span } B$  by (*rule in-own-span*)  
 from  $a0 a3$  0.prem show *?case* by (*rule-tac x=B in exI, auto*)  
 next  
 case (*Suc m*)  
 let  $?W = \text{span } B$   
 from *Suc.prem(3)* have  $\text{Bin } C: \text{span } B \subseteq \text{carrier } V$  by (*rule span-is-subset2*)  
  
 from *Suc.prem Suc.hyps Bin C* have  $A: \text{finite } A \text{ lin-indpt } A \ A \subseteq \text{span } B$   
 $\text{Suc } m = \text{card } A \ A \subseteq \text{carrier } V$   
 by *auto*  
  
 from *Suc.prem* have  $B: \text{finite } B \ B \subseteq \text{carrier } V$  by *auto*  
  
 from *Suc.hyps(2)* obtain  $v$  where  $v: v \in A$  by *fastforce*  
 let  $?A' = A - \{v\}$   
  
 from  $A(2)$  have  $\text{li } A': \text{lin-indpt } ?A'$   
 apply (*intro subset-li-is-li[of A ?A']*)  
 by *auto*  
 from  $v \text{ li } A' \text{ Suc.prem Suc.hyps(2)}$  have  $\exists C'. ?P ?A' B C'$   
 apply (*intro Suc.hyps(1)*)  
 by *auto*

**from this obtain  $C'$  where  $C': ?P ?A' B C'$  by auto**

**show  $?case$**

**proof (cases  $v \in C'$ )**

**case True**

**have  $vinC'$ :  $v \in C'$  by fact**

**from  $vinC' v$  have seteq:  $A - \{v\} \cup C' = A \cup (C' - \{v\})$  by auto**

**from  $C' seteq$  have spaneq:  $span (A \cup (C' - \{v\})) = span (B)$**

**by algebra**

**from  $Suc.premS Suc.hyps C' vinC' v spaneq$  show  $?thesis$**

**apply (rule-tac  $x = C' - \{v\}$  in  $exI$ )**

**apply (subgoal-tac card  $C' > 0$ )**

**by auto**

**next**

**case False**

**have  $f$ :  $v \notin C'$  by fact**

**from  $A v C'$  have  $\exists a. a \in (?A' \cup C') \rightarrow carrier K \wedge lincomb a (?A' \cup C') = v$**

**by (intro finite-in-span, auto)**

**from this obtain  $a$  where  $a: a \in (?A' \cup C') \rightarrow carrier K \wedge v = lincomb a (?A' \cup C')$  by metis**

**let  $?b = (\lambda w. if (w = v) then \ominus_K \mathbf{1}_K else a w)$**

**from  $a$  have  $b: ?b \in A \cup C' \rightarrow carrier K$  by auto**

**from  $v$  have rewrite-ins:  $A \cup C' = (?A' \cup C') \cup \{v\}$  by auto**

**from  $f$  have  $v \notin ?A' \cup C'$  by auto**

**from this  $A C' v a f$  have lcb:  $lincomb ?b (A \cup C') = \mathbf{0}_V$**

**apply (subst rewrite-ins)**

**apply (subst lincomb-insert)**

**apply (simp-all add: ring-subset-carrier coeff-in-ring)**

**apply (auto split: if-split-asm)**

**apply (subst lincomb-elim-if)**

**by (auto simp add: smult-minus-1 l-neg ring-subset-carrier)**

**from  $C' f$  have rewrite-minus:  $C' = (A \cup C') - A$  by auto**

**from  $A C' b lcb v$  have exw:  $\exists w \in C'. ?b w \neq \mathbf{0}_K$**

**apply (subst rewrite-minus)**

**apply (intro lincomb-must-include[where  $?T = A \cup C'$  and  $?v = v$ ])**

**by auto**

**from exw obtain  $w$  where  $w: w \in C' ?b w \neq \mathbf{0}_K$  by auto**

**from  $A C' w f b lcb$  have w-in:  $w \in span ((A \cup C') - \{w\})$**

**apply (intro lincomb-isolate[where  $a = ?b$ ])**

**by auto**

**have spaneq2:  $span (A \cup (C' - \{w\})) = span B$**

**proof -**

**have 1:  $span (?A' \cup C') = span (A \cup C')$**

**proof -**

**from  $A C' v$  have m1:  $span (?A' \cup C') = span ((?A' \cup C') \cup \{v\})$**

**apply (intro already-in-span)**

```

      by auto
      from f m1 show ?thesis by (metis rewrite-ins)
    qed
  have 2: span (A ∪ (C' - {w})) = span (A ∪ C')
  proof -
    from C' w(1) f have b60: A ∪ (C' - {w}) = (A ∪ C') - {w} by
auto
    from w(1) have b61: A ∪ C' = (A ∪ C' - {w}) ∪ {w} by auto
    from A C' w-in show ?thesis
      apply (subst b61)
      apply (subst b60)
      apply (intro already-in-span)
      by auto
    qed
  from C' 1 2 show ?thesis by auto
  qed
  from A C' w f v spaneq2 show ?thesis
    apply (rule-tac x=C' - {w} in exI)
    apply (subgoal-tac card C' > 0)
    by auto
  qed
  qed

```

## 6.4 Defining dimension and bases.

Finite dimensional is defined as having a finite generating set.

**definition** (in *vectorspace*) *fin-dim*:: *bool*  
 where *fin-dim* = ( $\exists A. ((\text{finite } A) \wedge (A \subseteq \text{carrier } V) \wedge (\text{gen-set } A))$ )

The dimension is the size of the smallest generating set. For equivalent characterizations see below.

**definition** (in *vectorspace*) *dim*:: *nat*  
 where *dim* = (*LEAST* *n*. ( $\exists A. ((\text{finite } A) \wedge (\text{card } A = n) \wedge (A \subseteq \text{carrier } V) \wedge (\text{gen-set } A))$ ))

A *basis* is a linearly independent generating set.

**definition** (in *vectorspace*) *basis*:: '*c set*  $\Rightarrow$  *bool*  
 where *basis* *A* = ( $(\text{lin-indpt } A) \wedge (\text{gen-set } A) \wedge (A \subseteq \text{carrier } V)$ )

From the replacement theorem, any linearly independent set is smaller than any generating set.

**lemma** (in *vectorspace*) *li-smaller-than-gen*:  
 fixes *A B*  
 assumes *h1*: *finite A* and *h2*: *finite B* and *h3*: *A*  $\subseteq$  *carrier V* and  
*h4*: *B*  $\subseteq$  *carrier V*  
 and *h5*: *lin-indpt A* and *h6*: *gen-set B*  
 shows *card A*  $\leq$  *card B*

**proof** –  
**from**  $h3\ h6$  **have**  $1: A \subseteq \text{span } B$  **by** *auto*  
**from**  $h1\ h2\ h4\ h5\ 1$  **obtain**  $C$  **where**  
 $2: \text{finite } C \wedge C \subseteq \text{carrier } V \wedge C \subseteq \text{span } B \wedge C \cap A = \{\} \wedge \text{int } (\text{card } C) \leq \text{int } (\text{card } B) - \text{int } (\text{card } A) \wedge (\text{span } (A \cup C) = \text{span } B)$   
**by** (*metis replacement*)  
**from**  $2$  **show** *?thesis* **by** *arith*  
**qed**

The dimension is the cardinality of any basis. (In particular, all bases are the same size.)

**lemma** (**in** *vectorspace*) *dim-basis*:

**fixes**  $A$   
**assumes**  $\text{fin}: \text{finite } A$  **and**  $h2: \text{basis } A$   
**shows**  $\text{dim} = \text{card } A$

**proof** –

**have**  $0: \bigwedge B\ m. ((\text{finite } B) \wedge (\text{card } B = m) \wedge (B \subseteq \text{carrier } V) \wedge (\text{gen-set } B)) \implies \text{card } A \leq m$

**proof** –

**fix**  $B\ m$

**assume**  $1: ((\text{finite } B) \wedge (\text{card } B = m) \wedge (B \subseteq \text{carrier } V) \wedge (\text{gen-set } B))$

**from**  $1\ \text{fin}\ h2$  **have**  $2: \text{card } A \leq \text{card } B$

**apply** (*unfold basis-def*)

**apply** (*intro li-smaller-than-gen*)

**by** *auto*

**from**  $1\ 2$  **show** *?thesis*  $B\ m$  **by** *auto*

**qed**

**from**  $\text{fin}\ h2\ 0$  **show** *?thesis*

**apply** (*unfold dim-def basis-def*)

**apply** (*intro Least-equality*)

**apply** (*rule-tac x=A in exI*)

**by** *auto*

**qed**

A *maximal* set with respect to  $P$  is such that if  $B \supseteq A$  and  $P$  is also satisfied for  $B$ , then  $B = A$ .

**definition** *maximal*:: $'a\ \text{set} \Rightarrow ('a\ \text{set} \Rightarrow \text{bool}) \Rightarrow \text{bool}$

**where**  $\text{maximal } A\ P = ((P\ A) \wedge (\forall B. B \supseteq A \wedge P\ B \longrightarrow B = A))$

A *minimal* set with respect to  $P$  is such that if  $B \subseteq A$  and  $P$  is also satisfied for  $B$ , then  $B = A$ .

**definition** *minimal*:: $'a\ \text{set} \Rightarrow ('a\ \text{set} \Rightarrow \text{bool}) \Rightarrow \text{bool}$

**where**  $\text{minimal } A\ P = ((P\ A) \wedge (\forall B. B \subseteq A \wedge P\ B \longrightarrow B = A))$

A maximal linearly independent set is a generating set.

**lemma** (**in** *vectorspace*) *max-li-is-gen*:

**fixes**  $A$

```

assumes h1: maximal A ( $\lambda S. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S$ )
shows gen-set A
proof (rule ccontr)
  assume 0:  $\neg(\text{gen-set } A)$ 
  from h1 have 1:  $A \subseteq \text{carrier } V \wedge \text{lin-indpt } A$  by (unfold maximal-def, auto)
  from 1 have 2:  $\text{span } A \subseteq \text{carrier } V$  by (intro span-is-subset2, auto)
  from 0 1 2 have 3:  $\exists v. v \in \text{carrier } V \wedge v \notin (\text{span } A)$ 
    by auto
  from 3 obtain v where 4:  $v \in \text{carrier } V \wedge v \notin (\text{span } A)$  by auto
  have 5:  $v \notin A$ 
  proof –
    from h1 1 have 51:  $A \subseteq \text{span } A$  apply (intro in-own-span) by auto
    from 4 51 show ?thesis by auto
  qed
  from lin-dep-iff-in-span have 6:  $\bigwedge S v. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S$ 
 $\wedge v \in \text{carrier } V \wedge v \notin S$ 
     $\wedge v \notin \text{span } S \implies (\text{lin-indpt } (S \cup \{v\}))$  by auto
  from 1 4 5 have 7:  $\text{lin-indpt } (A \cup \{v\})$  apply (intro 6) by auto

  have 9:  $\neg(\text{maximal } A (\lambda S. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S))$ 
  proof –
    from 1 4 5 7 have 8:  $(\exists B. A \subseteq B \wedge B \subseteq \text{carrier } V \wedge \text{lin-indpt } B \wedge B \neq A)$ 
    apply (rule-tac x=A $\cup\{v\}$  in exI)
    by auto
    from 8 show ?thesis
    apply (unfold maximal-def)
    by simp
  qed
from h1 9 show False by auto
qed

```

A minimal generating set is linearly independent.

**lemma** (in *vectorspace*) *min-gen-is-li*:

```

fixes A
assumes h1: minimal A ( $\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S$ )
shows lin-indpt A
proof (rule ccontr)
  assume 0:  $\neg \text{lin-indpt } A$ 
  from h1 have 1:  $A \subseteq \text{carrier } V \wedge \text{gen-set } A$  by (unfold minimal-def, auto)
  from 1 have 2:  $\text{span } A = \text{carrier } V$  by auto
  from 0 1 obtain a v A' where
    3:  $\text{finite } A' \wedge A' \subseteq A \wedge a \in A' \rightarrow \text{carrier } K \wedge \text{LinearCombinations.module.lincomb } V \ a \ A' = \mathbf{0}_V \wedge v \in A' \wedge a \ v \neq \mathbf{0}_K$ 
    by (unfold lin-dep-def, auto)
  have 4:  $\text{gen-set } (A - \{v\})$ 
  proof –

```

```

from 1 3 have 5:  $v \in \text{span } (A' - \{v\})$ 
  apply (intro lincomb-isolate[where  $a=a$  and  $v=v$ ])
    by auto
from 3 5 have 51:  $v \in \text{span } (A - \{v\})$ 
  apply (intro subsetD[where  $?A = \text{span } (A' - \{v\})$  and  $?B = \text{span } (A - \{v\})$  and  $?c=v$ ])
    by (intro span-is-monotone, auto)
from 1 have 6:  $A \subseteq \text{span } A$  apply (intro in-own-span) by auto
from 1 51 have 7:  $\text{span } (A - \{v\}) = \text{span } ((A - \{v\}) \cup \{v\})$  apply
(intro already-in-span) by auto
from 3 have 8:  $A = ((A - \{v\}) \cup \{v\})$  by auto
from 2 7 8 have 9:  $\text{span } (A - \{v\}) = \text{carrier } V$  by auto
from 9 show ?thesis by auto
qed
have 10:  $\neg(\text{minimal } A (\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S))$ 
proof -
  from 1 3 4 have 11:  $(\exists B. A \supseteq B \wedge B \subseteq \text{carrier } V \wedge \text{gen-set } B \wedge B \neq A)$ 
    apply (rule-tac  $x=A - \{v\}$  in  $exI$ )
      by auto
    from 11 show ?thesis
      apply (unfold minimal-def)
        by auto
    qed
from h1 10 show False by auto
qed

```

Given that some finite set satisfies  $P$ , there is a minimal set that satisfies  $P$ .

```

lemma minimal-exists:
  fixes A P
  assumes h1: finite A and h2: P A
  shows  $\exists B. B \subseteq A \wedge \text{minimal } B P$ 
using h1 h2
proof (induct card A arbitrary: A rule: less-induct)
case (less A)
  show ?case
  proof (cases card A = 0)
  case True
    from True less.hyps less.prem1 show ?thesis
      apply (rule-tac  $x=\{\}$  in  $exI$ )
        apply (unfold minimal-def)
          by auto
  next
case False
  show ?thesis
  proof (cases minimal A P)
  case True
    then show ?thesis

```

```

      apply (rule-tac x=A in exI)
      by auto
    next
  case False
  have 2:  $\neg \text{minimal } A \ P$  by fact
  from less.prem1 2 have 3:  $\exists B. P \ B \wedge B \subseteq A \wedge B \neq A$ 
  apply (unfold minimal-def)
  by auto
  from 3 obtain B where 4:  $P \ B \wedge B \subseteq A \wedge B \neq A$  by auto
  from 4 have 5:  $\text{card } B < \text{card } A$  by (metis less.prem1)
  psubset-card-mono
  from less.hyps less.prem1 3 4 5 have 6:  $\exists C \subseteq B. \text{minimal } C \ P$ 
  apply (intro less.hyps)
  apply auto
  by (metis rev-finite-subset)
  from 6 obtain C where 7:  $C \subseteq B \wedge \text{minimal } C \ P$  by auto
  from 4 7 show ?thesis
  apply (rule-tac x=C in exI)
  apply (unfold minimal-def)
  by auto
qed
qed
qed

```

If  $V$  is finite-dimensional, then any linearly independent set is finite.

```

lemma (in vectorspace) fin-dim-li-fin:
  assumes fd: fin-dim and li: lin-indpt A and inC:  $A \subseteq \text{carrier } V$ 
  shows fin: finite A
proof (rule ccontr)
  assume A:  $\neg \text{finite } A$ 
  from fd obtain C where C:  $\text{finite } C \wedge C \subseteq \text{carrier } V \wedge \text{gen-set } C$ 
  by (unfold fin-dim-def, auto)
  from A obtain B where B:  $B \subseteq A \wedge \text{finite } B \wedge \text{card } B = \text{card } C + 1$ 
  by (metis infinite-arbitrarily-large)
  from B li have liB: lin-indpt B
  by (intro subset-li-is-li[where ?A=A and ?B=B], auto)
  from B C liB inC have card B  $\leq \text{card } C$  by (intro li-smaller-than-gen, auto)
  from this B show False by auto
qed

```

If  $V$  is finite-dimensional (has a finite generating set), then a finite basis exists.

```

lemma (in vectorspace) finite-basis-exists:
  assumes h1: fin-dim
  shows  $\exists \beta. \text{finite } \beta \wedge \text{basis } \beta$ 
proof -

```

**from** *h1* **obtain**  $A$  **where**  $1: \text{finite } A \wedge A \subseteq \text{carrier } V \wedge \text{gen-set } A$   
**by** (*metis fin-dim-def*)  
**hence**  $2: \exists \beta. \beta \subseteq A \wedge \text{minimal } \beta (\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S)$   
**apply** (*intro minimal-exists*)  
**by** *auto*  
**then obtain**  $\beta$  **where**  $3: \beta \subseteq A \wedge \text{minimal } \beta (\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S)$  **by** *auto*  
**hence**  $4: \text{lin-indpt } \beta$  **apply** (*intro min-gen-is-li*) **by** *auto*  
**moreover from**  $3$  **have**  $5: \text{gen-set } \beta \wedge \beta \subseteq \text{carrier } V$  **apply** (*unfold minimal-def*) **by** *auto*  
**moreover from**  $1$   $3$  **have**  $6: \text{finite } \beta$  **by** (*auto simp add: finite-subset*)  
**ultimately show** *?thesis* **apply** (*unfold basis-def*) **by** *auto*  
**qed**

The proof is as follows.

1. Because  $V$  is finite-dimensional, there is a finite generating set (we took this as our definition of finite-dimensional).
2. Hence, there is a minimal  $\beta \subseteq A$  such that  $\beta$  generates  $V$ .
3.  $\beta$  is linearly independent because a minimal generating set is linearly independent.

Finally,  $\beta$  is a basis because it is both generating and linearly independent.

Any linearly independent set has cardinality at most equal to the dimension.

**lemma** (*in vectorspace*) *li-le-dim*:

**fixes**  $A$

**assumes**  $fd: \text{fin-dim}$  **and**  $c: A \subseteq \text{carrier } V$  **and**  $l: \text{lin-indpt } A$

**shows**  $\text{finite } A \text{ card } A \leq \text{dim}$

**proof** –

**from**  $fd$   $c$   $l$  **show**  $fa: \text{finite } A$  **by** (*intro fin-dim-li-fin, auto*)

**from**  $fd$  **obtain**  $\beta$  **where**  $1: \text{finite } \beta \wedge \text{basis } \beta$

**by** (*metis finite-basis-exists*)

**from** *assms*  $fa$   $1$  **have**  $2: \text{card } A \leq \text{card } \beta$

**apply** (*intro li-smaller-than-gen, auto*)

**by** (*unfold basis-def, auto*)

**from** *assms*  $1$  **have**  $3: \text{dim} = \text{card } \beta$  **by** (*intro dim-basis, auto*)

**from**  $2$   $3$  **show**  $\text{card } A \leq \text{dim}$  **by** *auto*

**qed**

Any generating set has cardinality at least equal to the dimension.

**lemma** (*in vectorspace*) *gen-ge-dim*:

**fixes**  $A$

**assumes**  $fa: \text{finite } A$  **and**  $c: A \subseteq \text{carrier } V$  **and**  $l: \text{gen-set } A$

**shows**  $\text{card } A \geq \text{dim}$

**proof** –  
**from** *assms* **have** *fd: fin-dim* **by** (*unfold fin-dim-def, auto*)  
**from** *fd* **obtain**  $\beta$  **where**  $1: \text{finite } \beta \wedge \text{basis } \beta$  **by** (*metis finite-basis-exists*)  
**from** *assms 1* **have**  $2: \text{card } A \geq \text{card } \beta$   
**apply** (*intro li-smaller-than-gen, auto*)  
**by** (*unfold basis-def, auto*)  
**from** *assms 1* **have**  $3: \text{dim} = \text{card } \beta$  **by** (*intro dim-basis, auto*)  
**from**  $2\ 3$  **show** *?thesis* **by** *auto*  
**qed**

If there is an upper bound on the cardinality of sets satisfying  $P$ , then there is a maximal set satisfying  $P$ .

**lemma** *maximal-exists*:  
**fixes**  $P\ B\ N$   
**assumes** *maxc*:  $\bigwedge A. P\ A \implies \text{finite } A \wedge (\text{card } A \leq N)$  **and**  $b: P\ B$   
**shows**  $\exists A. \text{finite } A \wedge \text{maximal } A\ P$

**proof** –  
**let**  $?S = \{\text{card } A \mid A. P\ A\}$   
**let**  $?n = \text{Max } ?S$   
**from** *maxc* **have**  $1: \text{finite } ?S$   
**apply** (*simp add: finite-nat-set-iff-bounded-le*) **by** *auto*  
**from**  $1$  **have**  $2: ?n \in ?S$   
**by** (*metis (mono-tags, lifting) Collect-empty-eq Max-in b*)  
**from** *assms 2* **have**  $3: \exists A. P\ A \wedge \text{finite } A \wedge \text{card } A = ?n$   
**by** *auto*  
**from**  $3$  **obtain**  $A$  **where**  $4: P\ A \wedge \text{finite } A \wedge \text{card } A = ?n$  **by** *auto*  
**from**  $1$  *maxc* **have**  $5: \bigwedge A. P\ A \implies \text{finite } A \wedge (\text{card } A \leq ?n)$   
**by** (*metis (mono-tags, lifting) Max.coboundedI mem-Collect-eq*)  
**from**  $4\ 5$  **have**  $6: \text{maximal } A\ P$   
**apply** (*unfold maximal-def*)  
**by** (*metis card-seteq*)  
**from**  $4\ 6$  **show** *?thesis* **by** *auto*  
**qed**

Any maximal linearly independent set is a basis.

**lemma** (*in vectorspace*) *max-li-is-basis*:  
**fixes**  $A$   
**assumes**  $h1: \text{maximal } A (\lambda S. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S)$   
**shows** *basis*  $A$

**proof** –  
**from**  $h1$  **have**  $1: \text{gen-set } A$  **by** (*rule max-li-is-gen*)  
**from** *assms 1* **show** *?thesis* **by** (*unfold basis-def maximal-def, auto*)  
**qed**

Any minimal linearly independent set is a generating set.

**lemma** (*in vectorspace*) *min-gen-is-basis*:  
**fixes**  $A$   
**assumes**  $h1: \text{minimal } A (\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S)$

**shows** *basis A*  
**proof** –  
**from** *h1* **have** *1: lin-indpt A* **by** (*rule min-gen-is-li*)  
**from** *assms 1* **show** *?thesis* **by** (*unfold basis-def minimal-def, auto*)  
**qed**

Any linearly independent set with cardinality at least the dimension is a basis.

**lemma** (*in vectorspace*) *dim-li-is-basis*:  
**fixes** *A*  
**assumes** *fd: fin-dim* **and** *fa: finite A* **and** *ca:  $A \subseteq \text{carrier } V$*  **and** *li: lin-indpt A*  
**and** *d: card A  $\geq$  dim*  
**shows** *basis A*  
**proof** –  
**from** *fd* **have** *0:  $\bigwedge S. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S \implies \text{finite } S \wedge \text{card } S \leq \text{dim}$*   
**by** (*auto intro: li-le-dim*)

**from** *0 assms* **have** *h1: finite A  $\wedge$  maximal A ( $\lambda S. S \subseteq \text{carrier } V \wedge \text{lin-indpt } S$ )*  
**apply** (*unfold maximal-def*)  
**apply** *auto*  
**by** (*metis card-seteq eq-iff*)  
**from** *h1* **show** *?thesis* **by** (*auto intro: max-li-is-basis*)  
**qed**

Any generating set with cardinality at most the dimension is a basis.

**lemma** (*in vectorspace*) *dim-gen-is-basis*:  
**fixes** *A*  
**assumes** *fa: finite A* **and** *ca:  $A \subseteq \text{carrier } V$*  **and** *li: gen-set A*  
**and** *d: card A  $\leq$  dim*  
**shows** *basis A*  
**proof** –  
**have** *0:  $\bigwedge S. \text{finite } S \wedge S \subseteq \text{carrier } V \wedge \text{gen-set } S \implies \text{card } S \geq \text{dim}$*   
**by** (*intro gen-ge-dim, auto*)

**from** *0 assms* **have** *h1: minimal A ( $\lambda S. \text{finite } S \wedge S \subseteq \text{carrier } V \wedge \text{gen-set } S$ )*  
**apply** (*unfold minimal-def*)  
**apply** *auto*  
**by** (*metis card-seteq eq-iff*)

**from** *h1* **have** *h:  $\bigwedge B. B \subseteq A \wedge B \subseteq \text{carrier } V \wedge \text{LinearCombinations.module.gen-set } K \ V \ B \implies B = A$*   
**proof** –  
**fix** *B*

```

assume asm:  $B \subseteq A \wedge B \subseteq \text{carrier } V \wedge \text{LinearCombinations.module.gen-set}$ 
K V B
from asm h1 have finite B
apply (unfold minimal-def)
apply (intro finite-subset[where ?A=B and ?B=A])
by auto
from h1 asm this show ?thesis B apply (unfold minimal-def) by
simp
qed
from h1 h have h2: minimal A ( $\lambda S. S \subseteq \text{carrier } V \wedge \text{gen-set } S$ )
apply (unfold minimal-def)
by presburger
from h2 show ?thesis by (rule min-gen-is-basis)
qed

```

$\beta$  is a basis iff for all  $v \in V$ , there exists a unique  $(a_v)_{v \in S}$  such that  $\sum_{v \in S} a_v v = v$ .

**lemma** (in *vectorspace*) *basis-criterion*:

```

assumes A-fin: finite A and AinC:  $A \subseteq \text{carrier } V$ 
shows basis A  $\longleftrightarrow$  ( $\forall v. v \in \text{carrier } V \longrightarrow (\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v)$ )

```

**proof** –

```

have 1:  $\neg(\forall v. v \in \text{carrier } V \longrightarrow (\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v)) \implies \neg \text{basis } A$ 

```

**proof** –

```

assume  $\neg(\forall v. v \in \text{carrier } V \longrightarrow (\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v))$ 

```

```

then obtain v where v:  $v \in \text{carrier } V \wedge \neg(\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v)$  by metis

```

```

from v have vinC:  $v \in \text{carrier } V$  by auto

```

```

from v have  $\neg(\exists a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v) \vee (\exists a b.$ 

```

```

 $a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v \wedge b \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } b A = v$ 

```

```

 $\wedge a \neq b)$  by metis

```

```

then show ?thesis

```

**proof**

```

assume a:  $\neg(\exists a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v)$ 

```

```

from A-fin AinC have  $\bigwedge a. a \in A \rightarrow \text{carrier } K \implies \text{lincomb } a A = \text{lincomb } (\text{restrict } a A) A$ 

```

```

unfolding lincomb-def restrict-def

```

```

by (simp cong: finsum-cong add: ring-subset-carrier coeff-in-ring)

```

```

with a have  $\neg(\exists a. a \in A \rightarrow \text{carrier } K \wedge \text{lincomb } a A = v)$  by

```

*auto*

```

with A-fin AinC have  $v \notin \text{span } A$ 

```

```

using finite-in-span by blast

```

```

with AinC v show  $\neg(\text{basis } A)$  by (unfold basis-def, auto)

```

**next**

```

assume a2: ( $\exists a b$ .
   $a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A = v \wedge b \in A \rightarrow_E \text{carrier } K \wedge$ 
   $\text{lincomb } b A = v$ 
   $\wedge a \neq b$ )
then obtain a b where ab:  $a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } a A$ 
 $= v \wedge b \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb } b A = v$ 
   $\wedge a \neq b$  by metis
from ab obtain w where w:  $w \in A \wedge a w \neq b w$  apply (unfold
  PiE-def, auto)
by (metis extensionalityI)
let ?c= $\lambda x$ . (if  $x \in A$  then  $(a x) \ominus_K (b x)$  else undefined)
from ab have a-fun:  $a \in A \rightarrow \text{carrier } K$ 
  and b-fun:  $b \in A \rightarrow \text{carrier } K$ 
by (unfold PiE-def, auto)
from w a-fun b-fun have abinC:  $a w \in \text{carrier } K \wedge b w \in \text{carrier } K$ 
by auto

from abinC w have nz:  $a w \ominus_K b w \neq \mathbf{0}_K$ 
by auto
from A-fin AinC a-fun b-fun ab vinC have a-b:
  LinearCombinations.module.lincomb V ( $\lambda x$ . if  $x \in A$  then  $a x \ominus_K$ 
   $b x$  else undefined)  $A = \mathbf{0}_V$ 
by (simp cong: lincomb-cong add: coeff-in-ring lincomb-diff)
from A-fin AinC ab w v nz a-b have lin-dep A
apply (intro lin-dep-crit[where ?A=A and ?a=?c and ?v=w])
apply (auto simp add: PiE-def)
by auto
thus  $\neg \text{basis } A$  by (unfold basis-def, auto)
qed
qed
have 2: ( $\forall v. v \in \text{carrier } V \rightarrow (\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge \text{lincomb}$ 
 $a A = v) \implies \text{basis } A$ )
proof -
assume b1: ( $\forall v. v \in \text{carrier } V \rightarrow (\exists! a. a \in A \rightarrow_E \text{carrier } K \wedge$ 
 $\text{lincomb } a A = v)$ )
  (is ( $\forall v. v \in \text{carrier } V \rightarrow (\exists! a. ?Q a v)$ ))
from b1 have b2: ( $\forall v. v \in \text{carrier } V \rightarrow (\exists a. a \in A \rightarrow \text{carrier}$ 
 $K \wedge \text{lincomb } a A = v)$ )
apply (unfold PiE-def)
by blast
from A-fin AinC b2 have gen-set A
apply (unfold span-def)
by blast
from b1 have A-li: lin-indpt A
proof -
let ?z= $\lambda x$ . (if  $x \in A$  then  $\mathbf{0}_K$  else undefined)
from A-fin AinC have zero: ?Q ?z  $\mathbf{0}_V$ 
by (unfold PiE-def extensional-def lincomb-def, auto simp add:
  ring-subset-carrier)

```

```

from A-fin AinC show ?thesis
proof (rule finite-lin-indpt2)
  fix a
  assume a-fun: a ∈ A → carrier K and
    lc-a: LinearCombinations.module.lincomb V a A = 0_V
  from a-fun have a-res: restrict a A ∈ A →E carrier K by auto
  from a-fun A-fin AinC lc-a have
    lc-a-res: LinearCombinations.module.lincomb V (restrict a A)
A = 0_V
  apply (unfold lincomb-def restrict-def)
  by (simp cong: finsum-cong2 add: coeff-in-ring ring-subset-carrier)
  from a-fun a-res lc-a-res zero b1 have restrict a A = ?z by
auto
  from this show  $\forall v \in A. a v = \mathbf{0}_K$ 
  apply (unfold restrict-def)
  by meson
  qed
qed
have A-gen: gen-set A
proof –
  from AinC have dir1: span A ⊆ carrier V by (rule span-is-subset2)
  have dir2: carrier V ⊆ span A
  proof (auto)
  fix v
  assume v: v ∈ carrier V
  from v b2 obtain a where a ∈ A → carrier K ∧ lincomb a A
  = v by auto
  from this A-fin AinC show v ∈ span A by (subst finite-span,
auto)
  qed
  from dir1 dir2 show ?thesis by auto
  qed
  from A-li A-gen AinC show basis A by (unfold basis-def, auto)
  qed
from 1 2 show ?thesis by satx
qed

```

```

lemma (in linear-map) surj-imp-imT-carrier:
  assumes surj: Tc (carrier V) = carrier W
  shows (imT) = carrier W
  by (simp add: surj im-def)

```

## 6.5 The rank-nullity (dimension) theorem

If  $V$  is finite-dimensional and  $T : V \rightarrow W$  is a linear map, then  $\dim(\text{im}(T)) + \dim(\text{ker}(T)) = \dim V$ . Moreover, we prove that if  $T$  is surjective linear-map between  $V$  and  $W$ , where  $V$  is finite-dimensional, then also  $W$  is finite-dimensional.

**theorem** (in *linear-map*) *rank-nullity-main*:  
**assumes** *fd*:  $V.\text{fin-dim}$   
**shows**  $(\text{vectorspace.dim } K (W.\text{vs } \text{im}T)) + (\text{vectorspace.dim } K (V.\text{vs } \text{ker}T)) = V.\text{dim}$   
 $T \text{ ' } (\text{carrier } V) = \text{carrier } W \implies W.\text{fin-dim}$   
**proof** –  
— First interpret  $\text{ker}T$ ,  $\text{im}T$  as vectorspaces  
**have** *subs-ker*: *subspace*  $K \text{ ker}T V$  **by** (*intro kerT-is-subspace*)  
**from** *subs-ker* **have** *vs-ker*: *vectorspace*  $K (V.\text{vs } \text{ker}T)$  **by** (*rule V.subspace-is-vs*)  
**from** *vs-ker* **interpret** *ker*: *vectorspace*  $K (V.\text{vs } \text{ker}T)$  **by** *auto*  
**have** *kerInC*:  $\text{ker}T \subseteq \text{carrier } V$  **by** (*unfold ker-def, auto*)  
  
**have** *subs-im*: *subspace*  $K \text{ im}T W$  **by** (*intro imT-is-subspace*)  
**from** *subs-im* **have** *vs-im*: *vectorspace*  $K (W.\text{vs } \text{im}T)$  **by** (*rule W.subspace-is-vs*)  
**from** *vs-im* **interpret** *im*: *vectorspace*  $K (W.\text{vs } \text{im}T)$  **by** *auto*  
**have** *imInC*:  $\text{im}T \subseteq \text{carrier } W$  **by** (*unfold im-def, auto*)  
  
**have** *zero-same[simp]*:  $\mathbf{0}_{V.\text{vs } \text{ker}T} = \mathbf{0}_V$  **apply** (*unfold ker-def*) **by** *auto*  
— Show  $\text{ker } T$  has a finite basis. This is not obvious. Show that any linearly independent set has size at most that of  $V$ . There exists a maximal linearly independent set, which is the basis.  
**have** *every-li-small*:  $\bigwedge A. (A \subseteq \text{ker}T) \wedge \text{ker.lin-indpt } A \implies \text{finite } A \wedge \text{card } A \leq V.\text{dim}$   
**proof** –  
**fix**  $A$   
**assume** *eli-asm*:  $(A \subseteq \text{ker}T) \wedge \text{ker.lin-indpt } A$   
  
**note**  $V.\text{module.span-li-not-depend}(2)[\text{where } ?N=\text{ker}T \text{ and } ?S=A]$   
  
**from** *this subs-ker fd eli-asm kerInC* **show** *?thesis A*  
**apply** (*intro conjI*)  
**by** (*auto intro!*:  $V.\text{li-le-dim}$ )  
**qed**  
**from** *every-li-small* **have** *exA*:  
 $\exists A. \text{finite } A \wedge \text{maximal } A (\lambda S. S \subseteq \text{carrier } (V.\text{vs } \text{ker}T) \wedge \text{ker.lin-indpt } S)$   
**apply** (*intro maximal-exists[where ?N=V.dim and ?B={}]*)  
**apply** *auto*  
**by** (*unfold ker.lin-dep-def, auto*)  
**from** *exA* **obtain**  $A$  **where**  $A$ : *finite A*  $\wedge$  *maximal A*  $(\lambda S. S \subseteq \text{carrier } (V.\text{vs } \text{ker}T) \wedge \text{ker.lin-indpt } S)$   
**by** *blast*  
**hence** *finA*: *finite A* **and** *Ainker*:  $A \subseteq \text{carrier } (V.\text{vs } \text{ker}T)$  **and** *AinC*:  $A \subseteq \text{carrier } V$   
**by** (*unfold maximal-def ker-def, auto*)  
— We obtain the basis  $A$  of  $\text{ker}T$ . It is also linearly independent when

considered in  $V$  rather than  $\ker T$

```

from  $A$  have  $Abasis: \ker.basis\ A$ 
  by ( $intro\ \ker.max-li-is-basis, auto$ )
from  $subs-ker\ Abasis$  have  $spanA: V.module.span\ A = \ker T$ 
  apply ( $unfold\ \ker.basis-def$ )
  by ( $subst\ sym[OF\ V.module.span-li-not-depend(1)]$  where  $?N = \ker T$ ),
 $auto$ )
from  $Abasis$  have  $Akerli: \ker.lin-indpt\ A$ 
  apply ( $unfold\ \ker.basis-def$ )
  by  $auto$ 
from  $subs-ker\ Ainker\ Akerli$  have  $Ali: V.module.lin-indpt\ A$ 
  by ( $auto\ simp\ add: V.module.span-li-not-depend(2)$ )

```

Use the replacement theorem to find  $C$  such that  $A \cup C$  is a basis of  $V$ .

```

from  $fd$  obtain  $B$  where  $B: finite\ B \wedge V.basis\ B$  by ( $metis\ V.finite-basis-exists$ )
from  $B$  have  $Bfin: finite\ B$  and  $Bbasis: V.basis\ B$  by  $auto$ 
from  $B$  have  $Bcard: V.dim = card\ B$  by ( $intro\ V.dim-basis, auto$ )
from  $Bbasis$  have  $62: V.module.span\ B = carrier\ V$ 
  by ( $unfold\ V.basis-def, auto$ )
from  $A\ Abasis\ Ali\ B\ vs-ker$  have  $\exists C. finite\ C \wedge C \subseteq carrier\ V \wedge$ 
 $C \subseteq V.module.span\ B \wedge C \cap A = \{\}$ 
   $\wedge int\ (card\ C) \leq (int\ (card\ B)) - (int\ (card\ A)) \wedge (V.module.span$ 
 $(A \cup C) = V.module.span\ B)$ 
  apply ( $intro\ V.replacement$ )
  apply ( $unfold\ vectorspace.basis-def\ V.basis-def$ )
  by ( $unfold\ ker-def, auto$ )

```

From replacement we got  $|C| \leq |B| - |A|$ . Equality must actually hold, because no generating set can be smaller than  $B$ . Now  $A \cup C$  is a maximal generating set, hence a basis; its cardinality equals the dimension.

We claim that  $T(C)$  is basis for  $\text{im}(T)$ .

```

then obtain  $C$  where  $C: finite\ C \wedge C \subseteq carrier\ V \wedge C \subseteq V.module.span$ 
 $B \wedge C \cap A = \{\}$ 
   $\wedge int\ (card\ C) \leq (int\ (card\ B)) - (int\ (card\ A)) \wedge (V.module.span$ 
 $(A \cup C) = V.module.span\ B)$  by  $auto$ 
hence  $Cfin: finite\ C$  and  $CinC: C \subseteq carrier\ V$  and  $CinSpanB:$ 
 $C \subseteq V.module.span\ B$  and  $CAdis: C \cap A = \{\}$ 
  and  $Ccard: int\ (card\ C) \leq (int\ (card\ B)) - (int\ (card\ A))$ 
  and  $ACspanB: (V.module.span\ (A \cup C) = V.module.span\ B)$  by
 $auto$ 
from  $C$  have  $cardLe: card\ A + card\ C \leq card\ B$  by  $auto$ 
from  $B\ C$  have  $ACgen: V.module.gen-set\ (A \cup C)$  apply ( $unfold$ 
 $V.basis-def$ ) by  $auto$ 
from  $finA\ C\ ACgen\ AinC\ B$  have  $cardGe: card\ (A \cup C) \geq card\ B$ 
by ( $intro\ V.li-smaller-than-gen, unfold\ V.basis-def, auto$ )
from  $finA\ C$  have  $cardUn: card\ (A \cup C) \leq card\ A + card\ C$ 
  by ( $metis\ Int-commute\ card-Un-disjoint\ le-refl$ )
from  $cardLe\ cardUn\ cardGe\ Bcard$  have  $cardEq:$ 

```

```

    card (A∪C) = card A + card C
    card (A∪C) = card B
    card (A∪C) = V.dim
  by auto
  from Abasis C cardEq have disj: A∩C={ } by auto
  from finA AinC C cardEq 62 have ACfin: finite (A∪C) and ACba-
sis: V.basis (A∪C)
  by (auto intro!: V.dim-gen-is-basis)
  have lm: linear-map K V W T..

```

Let  $C'$  be the image of  $C$  under  $T$ . We will show  $C'$  is a basis for  $\text{im}(T)$ .

```

  let ?C' = T`C
  from Cfin have C'fin: finite ?C' by auto
  from AinC C have cim: ?C'⊆imT by (unfold im-def, auto)

```

"There is a subtle detail: we first have to show  $T$  is injective on  $C$ .

We establish that no nontrivial linear combination of  $C$  can have image 0 under  $T$ , because that would mean it is a linear combination of  $A$ , giving that  $A \cup C$  is linearly dependent, contradiction. We use this result in 2 ways: (1) if  $T$  is not injective on  $C$ , then we obtain  $v, w \in C$  such that  $v - w$  is in the kernel, contradiction, (2) if  $T(C)$  is linearly dependent, taking the inverse image of that linear combination gives a linear combination of  $C$  in the kernel, contradiction. Hence  $T$  is injective on  $C$  and  $T(C)$  is linearly independent.

```

  have lc-in-ker: ⋀ d D v. [D⊆C; d∈D→carrier K; T (V.module.lincomb
d D) = 0_W;
  v∈D; d v ≠0_K]⇒False
  proof -
    fix d D v
    assume D: D⊆C and d: d∈D→carrier K and T0: T (V.module.lincomb
d D) = 0_W
    and v: v∈D and dvnz: d v ≠0_K
    from D Cfin have Dfin: finite D by (auto intro: finite-subset)
    from D C in C have DinC: D⊆carrier V by auto
    from T0 d Dfin DinC have lc-d: V.module.lincomb d D∈kerT
    by (unfold ker-def, auto)
    from lc-d spanA AinC have ∃ a' A'. A'⊆A ∧ a'∈A'→carrier K
  ∧
    V.module.lincomb a' A' = V.module.lincomb d D
    by (intro V.module.in-span, auto)
    then obtain a' A' where a': A'⊆A ∧ a'∈A'→carrier K ∧
    V.module.lincomb d D = V.module.lincomb a' A'
    by metis
    hence A'sub: A'⊆A and a'fun: a'∈A'→carrier K
    and a'-lc: V.module.lincomb d D = V.module.lincomb a' A' by
auto
    from a' finA Dfin have A'fin: finite (A') by (auto intro: fi-
nite-subset)

```

```

from AinC A'sub have A'inC: A'⊆carrier V by auto
let ?e = (λv. if v ∈ A' then a' v else ⊖K1K⊗K d v)
from a'fun d have e-fun: ?e ∈ A' ∪ D → carrier K
  apply (unfold Pi-def)
  by auto
from
  A'fin Dfin
  A'inC DinC
  a'fun d e-fun
  disj D A'sub
have lccomp1:
  V.module.lincomb a' A' ⊕V ⊖K1K⊗V V.module.lincomb d D =
  V.module.lincomb (λv. if v ∈ A' then a' v else ⊖K1K⊗K d v)
(A' ∪ D)
  apply (subst sym[OF V.module.lincomb-smult])
  apply (simp-all)
  apply (subst V.module.lincomb-union2)
  by (auto)
from
  A'fin
  A'inC
  a'fun
have lccomp2:
  V.module.lincomb a' A' ⊕V ⊖K1K⊗V V.module.lincomb d D =
  0V
  by (simp add: a'-lc
  V.module.smult-minus-1 V.module.M.r-neg)
  from lccomp1 lccomp2 have lc0: V.module.lincomb (λv. if v ∈ A'
then a' v else ⊖K1K⊗K d v) (A' ∪ D)
  = 0V by auto
from disj a' v D have v-nin: v ∉ A' by auto
from A'fin Dfin
  A'inC DinC
  e-fun d
  A'sub D disj
  v dvmz
  lc0
have AC-ld: V.module.lin-dep (A ∪ C)
  apply (intro V.module.lin-dep-crit[where ?A=A' ∪ D and
  ?S=A ∪ C and ?a=λv. if v ∈ A' then a' v else ⊖K1K⊗K d v and
  ?v=v])
  by (auto dest: integral)
from AC-ld ACbasis show False by (unfold V.basis-def, auto)
qed
have C'-card: inj-on T C card C = card ?C'
proof –
  show inj-on T C
  proof (rule ccontr)
    assume ¬inj-on T C

```

**then obtain**  $v w$  **where**  $v \in C \ w \in C \ v \neq w \ T \ v = T \ w$  **by** (*unfold inj-on-def, auto*)  
**from** *this CinC* **show** *False*  
**apply** (*intro lc-in-ker*[**where**  $?D1 = \{v, w\}$  **and**  $?d1 = \lambda x. \text{if } x = v$   
*then*  $\mathbf{1}_K$  *else*  $\ominus_K \mathbf{1}_K$   
**and**  $?v1 = v$ ])  
**by** (*auto simp add: V.module.lincomb-def hom-sum ring-subset-carrier*  
 $W.module.smult-minus-1 \ r\text{-neg} \ T\text{-im}$ )  
**qed**  
**from** *this Cfin* **show**  $\text{card } C = \text{card } ?C'$   
**by** (*metis card-image*)  
**qed**  
**let**  $?f = \text{the-inv-into } C \ T$   
**have**  $f: \bigwedge x. x \in C \implies ?f (T \ x) = x \ \bigwedge y. y \in ?C' \implies T (?f \ y) = y$   
**apply** (*insert C'-card(1)*)  
**apply** (*metis the-inv-into-f-f*)  
**by** (*metis f-the-inv-into-f*)  
  
**have**  $C'\text{-li}: \text{im.lin-indpt } ?C'$   
**proof** (*rule ccontr*)  
**assume**  $Cld: \neg \text{im.lin-indpt } ?C'$   
**from**  $Cld$  *cim subs-im* **have**  $CldW: W.module.lin-dep ?C'$   
**apply** (*subst sym*[*OF*  $W.module.span-li-not-depend(2)$ ][**where**  
 $?S = T' C$  **and**  $?N = \text{im } T$ ])  
**by** *auto*  
**from**  $C \ CldW$  **have**  $\exists c' \ v'. (c' \in (?C' \rightarrow \text{carrier } K)) \wedge (W.module.lincomb$   
 $c' \ ?C' = \mathbf{0}_W)$   
 $\wedge (v' \in ?C') \wedge (c' \ v' \neq \mathbf{0}_K)$  **by** (*intro W.module.finite-lin-dep,*  
*auto*)  
**then obtain**  $c' \ v'$  **where**  $c': (c' \in (?C' \rightarrow \text{carrier } K)) \wedge (W.module.lincomb$   
 $c' \ ?C' = \mathbf{0}_W)$   
 $\wedge (v' \in ?C') \wedge (c' \ v' \neq \mathbf{0}_K)$  **by** *auto*  
**hence**  $c'\text{fun}: (c' \in (?C' \rightarrow \text{carrier } K))$  **and**  $c'\text{lc}: (W.module.lincomb$   
 $c' \ ?C' = \mathbf{0}_W)$  **and**  
 $v': (v' \in ?C')$  **and**  $cvnz: (c' \ v' \neq \mathbf{0}_K)$  **by** *auto*

We take the inverse image of  $C'$  under  $T$  to get a linear combination of  $C$  that is in the kernel and hence a linear combination of  $A$ . This contradicts  $A \cup C$  being linearly independent.

**let**  $?c = \lambda v. c' (T \ v)$   
**from**  $c'\text{fun}$  **have**  $c\text{-fun}: ?c \in C \rightarrow \text{carrier } K$  **by** *auto*  
**from** *Cfin*  
 $c\text{-fun } c'\text{fun}$   
 $C'\text{-card}$   
 $CinC$   
 $f$   
 $c'\text{lc}$   
**have**  $T (V.module.lincomb ?c \ C) = \mathbf{0}_W$

```

apply (unfold V.module.lincomb-def W.module.lincomb-def)
apply (subst hom-sum, auto)
apply (simp cong: finsum-cong add: ring-subset-carrier coeff-in-ring)
  apply (subst finsum-reindex[where ?f= $\lambda w. c' w \odot_W w$  and
?h=T and ?A=C, THEN sym])
    by auto
  with f c'fun cvnz v' show False
    by (intro lc-in-ker[where ?D1=C and ?d1=?c and ?v1=?f v'],
auto)
qed
have C'-gen: im.gen-set ?C'
proof -
  have C'-span: span ?C' = imT
  proof (rule equalityI)
    from cim subs-im show W.module.span ?C'  $\subseteq$  imT
    by (intro span-is-subset, unfold subspace-def, auto)
  next
  show imT  $\subseteq$  W.module.span ?C'
  proof (auto)
    fix w
    assume w: w  $\in$  imT
    from this finA Cfin AinC CinC obtain v where v-inC:
v  $\in$  carrier V and w-eq-T-v: w = T v
      by (unfold im-def image-def, auto)
    from finA Cfin AinC CinC v-inC ACgen have  $\exists a. a \in AUC$ 
 $\rightarrow$  carrier K  $\wedge$  V.module.lincomb a (AUC) = v
      by (intro V.module.finite-in-span, auto)
    then obtain a where
      a-fun: a  $\in$  AUC  $\rightarrow$  carrier K and
      lc-a-v: v = V.module.lincomb a (AUC)
    by auto
    let ?a'= $\lambda v. a$  (?f v)
    from finA Cfin AinC CinC a-fun disj Ainker f C'-card have
Tv: T v = W.module.lincomb ?a' ?C'
      apply (subst lc-a-v)
      apply (subst V.module.lincomb-union, simp-all)
      apply (unfold lincomb-def V.module.lincomb-def)
      apply (subst hom-sum, auto)
      apply (simp add: subsetD coeff-in-ring
hom-sum
T-ker
)
      apply (subst finsum-reindex[where ?h=T and ?f= $\lambda v. ?a'$ 
v  $\odot_W v$ ], auto)
      by (auto cong: finsum-cong simp add: coeff-in-ring ring-subset-carrier)
    from a-fun f have a'-fun: ?a'  $\in$  ?C'  $\rightarrow$  carrier K by auto
    from C'fin CinC this w-eq-T-v a'-fun Tv show w  $\in$  LinearCom-
binations.module.span K W (T ' C)
      by (subst finite-span, auto)

```

```

    qed
  qed
  from this subs-im CinC show ?thesis
    apply (subst span-li-not-depend(1))
    by (unfold im-def subspace-def, auto)
  qed
  from C'-li C'-gen C cim have C'-basis: im.basis (T'C)
    by (unfold im.basis-def, auto)
  have C-card-im: card C = (vectorspace.dim K (W.vs imT))
    using C'-basis C'-card(2) C'fin im.dim-basis by auto
  from finA Abasis have ker.dim = card A by (rule ker.dim-basis)
  note * = this C-card-im cardEq
  show (vectorspace.dim K (W.vs imT)) + (vectorspace.dim K (V.vs
kerT)) = V.dim using * by auto
  assume T' (carrier V) = carrier W
  from * surj-imp-imT-carrier[OF this]
  show W.fin-dim using C'-basis C'fin unfolding W.fin-dim-def
im.basis-def by auto
  qed

theorem (in linear-map) rank-nullity:
  assumes fd: V.fin-dim
  shows (vectorspace.dim K (W.vs imT)) + (vectorspace.dim K (V.vs
kerT)) = V.dim
  by (rule rank-nullity-main[OF fd])

end

```