

# Multi-Head Monitoring of Metric Dynamic Logic

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## Abstract

Runtime monitoring (or runtime verification) is an approach to checking compliance of a system’s execution with a specification (e.g., a temporal formula). The system’s execution is logged into a *trace*—a sequence of time-points, each consisting of a time-stamp and observed events. A *monitor* is an algorithm that produces *verdicts* on the satisfaction of a temporal formula on a trace.

We formalize the time-stamps as an abstract algebraic structure satisfying certain assumptions. Instances of this structure include natural numbers, real numbers, and lexicographic combinations of them. We also include the formalization of a conversion from the abstract time domain introduced by Koymans [1] to our time-stamps.

We formalize a monitoring algorithm for metric dynamic logic, an extension of metric temporal logic with regular expressions. The monitor computes whether a given formula is satisfied at every position in an input trace of time-stamped events. Our monitor follows the multi-head paradigm: it reads the input simultaneously at multiple positions and moves its reading heads asynchronously. This mode of operation results in unprecedented time and space complexity guarantees for metric dynamic logic: The monitor’s amortized time complexity to process a time-point and the monitor’s space complexity neither depends on the event-rate, i.e., the number of events within a fixed time-unit, nor on the numeric constants occurring in the quantitative temporal constraints in the given formula.

The multi-head monitoring algorithm for metric dynamic logic is reported in our paper “Multi-Head Monitoring of Metric Dynamic Logic” [2] published at ATVA 2020. We have also formalized unpublished specialized algorithms for the temporal operators of metric temporal logic.

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theory *Timestamp*

```
imports HOL-Library.Extended_Nat HOL-Library.Extended_Real
begin
```

```
class embed_nat =
  fixes  $\iota :: \text{nat} \Rightarrow 'a$ 
```

```
class tfin =
  fixes  $tfin :: 'a \text{ set}$ 
```

```
class timestamp = comm_monoid_add + semilattice_sup + embed_nat + tfin +
  assumes  $\iota\_mono: \bigwedge i j. i \leq j \implies \iota i \leq \iota j$ 
```

```

and  $\iota\_tfin$ :  $\bigwedge i. \iota i \in tfin$ 
and  $\iota\_progressing$ :  $x \in tfin \implies \exists j. \neg \iota j \leq \iota i + x$ 
and  $zero\_tfin$ :  $0 \in tfin$ 
and  $tfin\_closed$ :  $c \in tfin \implies d \in tfin \implies c + d \in tfin$ 
and  $add\_mono$ :  $c \leq d \implies a + c \leq a + d$ 
and  $add\_pos$ :  $a \in tfin \implies 0 < c \implies a < a + c$ 
begin

lemma  $add\_mono\_comm$ :
  fixes  $a :: 'a$ 
  shows  $c \leq d \implies c + a \leq d + a$ 
  by (auto simp: add.commute add_mono)

end

instantiation  $option$  :: (timestamp) timestamp
begin

definition  $tfin\_option$  :: ' $a$  option set where
   $tfin\_option = Some \ 'tfin$ 

definition  $\iota\_option$  ::  $nat \Rightarrow 'a$  option where
   $\iota\_option = Some \circ \iota$ 

definition  $zero\_option$  :: ' $a$  option where
   $zero\_option = Some \ 0$ 

definition  $plus\_option$  :: ' $a$  option  $\Rightarrow$  ' $a$  option  $\Rightarrow$  ' $a$  option where
   $plus\_option \ x \ y = (case \ x \ of \ None \Rightarrow \ None \ | \ Some \ x' \Rightarrow (case \ y \ of \ None \Rightarrow \ None \ | \ Some \ y' \Rightarrow \ Some \ (x' + y')))$ 

definition  $sup\_option$  :: ' $a$  option  $\Rightarrow$  ' $a$  option  $\Rightarrow$  ' $a$  option where
   $sup\_option \ x \ y = (case \ x \ of \ None \Rightarrow \ None \ | \ Some \ x' \Rightarrow (case \ y \ of \ None \Rightarrow \ None \ | \ Some \ y' \Rightarrow \ Some \ (sup \ x' \ y')))$ 

definition  $less\_option$  :: ' $a$  option  $\Rightarrow$  ' $a$  option  $\Rightarrow$  bool where
   $less\_option \ x \ y = (case \ x \ of \ None \Rightarrow \ False \ | \ Some \ x' \Rightarrow (case \ y \ of \ None \Rightarrow \ True \ | \ Some \ y' \Rightarrow \ x' < y'))$ 

definition  $less\_eq\_option$  :: ' $a$  option  $\Rightarrow$  ' $a$  option  $\Rightarrow$  bool where
   $less\_eq\_option \ x \ y = (case \ x \ of \ None \Rightarrow \ x = y \ | \ Some \ x' \Rightarrow (case \ y \ of \ None \Rightarrow \ True \ | \ Some \ y' \Rightarrow \ x' \leq y'))$ 

instance
  apply standard
    apply (auto simp: plus_option_def add.assoc split: option.splits)[1]
    apply (auto simp: plus_option_def add.commute split: option.splits)[1]
    apply (auto simp: zero_option_def plus_option_def split: option.splits)[1]
    apply (auto simp: less_option_def less_eq_option_def split: option.splits)[1]
    apply (auto simp: less_eq_option_def split: option.splits)[3]
    apply (auto simp: sup_option_def less_eq_option_def split: option.splits)[3]
    apply (auto simp:  $\iota\_option\_def$  less_eq_option_def intro:  $\iota\_mono$ )[1]
    apply (auto simp:  $tfin\_option\_def$   $\iota\_option\_def$  intro:  $\iota\_tfin$ )[1]
    apply (auto simp:  $tfin\_option\_def$   $\iota\_option\_def$  plus_option_def less_eq_option_def intro:  $\iota\_progressing$ )[1]
    apply (auto simp:  $tfin\_option\_def$  zero_option_def intro: zero_tfin)[1]
    apply (auto simp:  $tfin\_option\_def$  plus_option_def intro: tfin_closed)[1]
    apply (auto simp: plus_option_def less_eq_option_def intro: add_mono split: option.splits)[1]

```

```

    apply (auto simp: tfin_option_def zero_option_def plus_option_def less_option_def intro: add_pos
split: option.splits)
  done

end

instantiation enat :: timestamp
begin

definition tfin_enat :: enat set where
  tfin_enat = UNIV - {∞}

definition υ_enat :: nat ⇒ enat where
  υ_enat n = n

instance
  by standard (auto simp add: υ_enat_def tfin_enat_def dest!: leD)

end

instantiation ereal :: timestamp
begin

definition υ_ereal :: nat ⇒ ereal where
  υ_ereal n = ereal n

definition tfin_ereal :: ereal set where
  tfin_ereal = UNIV - {-∞, ∞}

lemma ereal_add_pos:
  fixes a :: ereal
  shows a ∈ tfin ⇒ 0 < c ⇒ a < a + c
  by (auto simp: tfin_ereal_def) (metis add.right_neutral ereal_add_cancel_left ereal_le_add_self order_less_le)

instance
  by standard (auto simp add: υ_ereal_def tfin_ereal_def add.commute ereal_add_le_add_iff2 not_le less_PInf_Ex_of_nat ereal_less_ereal_Ex reals_Archimedean2 intro: ereal_add_pos)

end

class timestamp_total = timestamp +
  assumes timestamp_total: a ≤ b ∨ b ≤ a
  assumes timestamp_tfin_le_not_tfin: 0 ≤ a ⇒ a ∈ tfin ⇒ 0 ≤ b ⇒ b ∉ tfin ⇒ a ≤ b
begin

lemma add_not_tfin: 0 ≤ a ⇒ a ∈ tfin ⇒ a ≤ c ⇒ c ∈ tfin ⇒ 0 ≤ b ⇒ b ∉ tfin ⇒ c < a + b
  by (metis add_0_left local.add_mono_comm timestamp_tfin_le_not_tfin dual_order.order_iff_strict dual_order.strict_trans1)

end

instantiation enat :: timestamp_total
begin

instance
  by standard (auto simp: tfin_enat_def)

```

```

end

instantiation ereal :: timestamp_total
begin

instance
  by standard (auto simp: tfin_ereal_def)

end

class timestamp_strict = timestamp +
  assumes add_mono_strict:  $c < d \implies a + c < a + d$ 

class timestamp_total_strict = timestamp_total + timestamp_strict

instantiation nat :: timestamp_total_strict
begin

definition tfin_nat :: nat set where
  tfin_nat = UNIV

definition  $\iota\_nat$  ::  $nat \Rightarrow nat$  where
   $\iota\_nat\ n = n$ 

instance
  by standard (auto simp: tfin_nat_def \iota_nat_def dest!: leD)

end

instantiation real :: timestamp_total_strict
begin

definition tfin_real :: real set where tfin_real = UNIV

definition  $\iota\_real$  ::  $nat \Rightarrow real$  where  $\iota\_real\ n = real\ n$ 

instance
  by standard (auto simp: tfin_real_def \iota_real_def not_le reals_Archimedean2)

end

instantiation prod :: (comm_monoid_add, comm_monoid_add) comm_monoid_add
begin

definition zero_prod :: 'a × 'b where
  zero_prod = (0, 0)

fun plus_prod :: 'a × 'b ⇒ 'a × 'b ⇒ 'a × 'b where
  (a, b) + (c, d) = (a + c, b + d)

instance
  by standard (auto simp: zero_prod_def ac_simps)

end

end

```

# 1 Intervals

```
typedef (overloaded) ('a :: timestamp)  $\mathcal{I} = \{(i :: 'a, j :: 'a, lei :: bool, lej :: bool). 0 \leq i \wedge i \leq j \wedge i \in tfin \wedge \neg(j = 0 \wedge \neg lej)\}$ 
  by (intro exI[of_ (0, 0, True, True)]) (auto intro: zero_tfin)
```

```
setup_lifting type_definition  $\mathcal{I}$ 
```

```
instantiation  $\mathcal{I} :: (timestamp)$  equal begin
```

```
lift_definition equal  $\mathcal{I} :: 'a \mathcal{I} \Rightarrow 'a \mathcal{I} \Rightarrow bool$  is (=) .
```

```
instance by standard (transfer, auto)
```

```
end
```

```
lift_definition right :: 'a :: timestamp  $\mathcal{I} \Rightarrow 'a$  is fst  $\circ$  snd .
```

```
lift_definition memL :: 'a :: timestamp  $\Rightarrow 'a \Rightarrow 'a \mathcal{I} \Rightarrow bool$  is
   $\lambda t t' (a, b, lei, lej). \text{if } lei \text{ then } t + a \leq t' \text{ else } t + a < t' .$ 
```

```
lift_definition memR :: 'a :: timestamp  $\Rightarrow 'a \Rightarrow 'a \mathcal{I} \Rightarrow bool$  is
   $\lambda t t' (a, b, lei, lej). \text{if } lej \text{ then } t' \leq t + b \text{ else } t' < t + b .$ 
```

```
definition mem :: 'a :: timestamp  $\Rightarrow 'a \Rightarrow 'a \mathcal{I} \Rightarrow bool$  where
  mem t t' I  $\longleftrightarrow$  memL t t' I  $\wedge$  memR t t' I
```

```
lemma memL_mono: memL t t' I  $\Longrightarrow t'' \leq t \Longrightarrow$  memL t'' t' I
  by transfer (auto simp: add.commute order_le_less_subst2 order_subst2 add_mono split: if_splits)
```

```
lemma memL_mono': memL t t' I  $\Longrightarrow t' \leq t'' \Longrightarrow$  memL t t'' I
  by transfer (auto split: if_splits)
```

```
lemma memR_mono: memR t t' I  $\Longrightarrow t \leq t'' \Longrightarrow$  memR t'' t' I
  apply transfer
  apply (simp split: prod.splits)
  apply (meson add_mono_comm dual_order.trans order_less_le_trans)
  done
```

```
lemma memR_mono': memR t t' I  $\Longrightarrow t'' \leq t' \Longrightarrow$  memR t t'' I
  by transfer (auto split: if_splits)
```

```
lemma memR_dest: memR t t' I  $\Longrightarrow t' \leq t + right$  I
  by transfer (auto split: if_splits)
```

```
lemma memR_tfin_refl:
  assumes fin: t  $\in$  tfin
  shows memR t t I
  by (transfer fixing: t) (force split: if_splits intro: order_trans[OF add_mono, where ?x=t and ?a1=t
and ?c1=0] add_pos[OF fin])
```

```
lemma right_I_add_mono:
  fixes x :: 'a :: timestamp
  shows x  $\leq$  x + right I
  by transfer (auto split: if_splits intro: order_trans[OF add_mono, of __ 0])
```

```
lift_definition interval :: 'a :: timestamp  $\Rightarrow 'a \Rightarrow bool \Rightarrow bool \Rightarrow 'a \mathcal{I}$  is
   $\lambda i j lei lej. (\text{if } 0 \leq i \wedge i \leq j \wedge i \in tfin \wedge \neg(j = 0 \wedge \neg lej) \text{ then } (i, j, lei, lej) \text{ else Code.abort (STR$ 
  "malformed interval")  $(\lambda \_ . (0, 0, True, True))$ )
```

by (auto intro: zero\_tfin)

**lemma**  $Rep\_I I = (l, r, b1, b2) \implies memL\ 0\ 0\ I \longleftrightarrow l = 0 \wedge b1$   
by transfer auto

**lift\_definition** dropL :: 'a :: timestamp  $\mathcal{I} \Rightarrow 'a\ \mathcal{I}$  is  
 $\lambda(l, r, b1, b2). (0, r, True, b2)$   
by (auto intro: zero\_tfin)

**lemma** memL\_dropL:  $t \leq t' \implies memL\ t\ t' (dropL\ I)$   
by transfer auto

**lemma** memR\_dropL:  $memR\ t\ t' (dropL\ I) = memR\ t\ t'\ I$   
by transfer auto

**lift\_definition** flipL :: 'a :: timestamp  $\mathcal{I} \Rightarrow 'a\ \mathcal{I}$  is  
 $\lambda(l, r, b1, b2). \text{if } \neg(l = 0 \wedge b1) \text{ then } (0, l, True, \neg b1) \text{ else } Code.abort\ (STR\ \text{"invalid flipL"})\ (\lambda_. (0,$   
 $0, True, True))$   
by (auto intro: zero\_tfin split: if\_splits)

**lemma** memL\_flipL:  $t \leq t' \implies memL\ t\ t' (flipL\ I)$   
by transfer (auto split: if\_splits)

**lemma** memR\_flipLD:  $\neg memL\ 0\ 0\ I \implies memR\ t\ t' (flipL\ I) \implies \neg memL\ t\ t'\ I$   
by transfer (auto split: if\_splits)

**lemma** memR\_flipLI:  
fixes  $t :: 'a :: timestamp$   
shows  $(\bigwedge u\ v. (u :: 'a :: timestamp) \leq v \vee v \leq u) \implies \neg memL\ t\ t'\ I \implies memR\ t\ t' (flipL\ I)$   
by transfer (force split: if\_splits)

**lemma**  $t \in tfin \implies memL\ 0\ 0\ I \longleftrightarrow memL\ t\ t\ I$   
apply transfer  
apply (simp split: prod\_splits)  
apply (metis add.right\_neutral add\_pos antisym\_conv2 dual\_order.eq\_iff order\_less\_imp\_not\_less)  
done

**definition** full ( $I :: ('a :: timestamp)\ \mathcal{I}$ )  $\longleftrightarrow (\forall t\ t'. 0 \leq t \wedge t \leq t' \wedge t \in tfin \wedge t' \in tfin \longrightarrow mem\ t\ t'\ I)$

**lemma** memL 0 0 ( $I :: ('a :: timestamp\_total)\ \mathcal{I}$ )  $\implies right\ I \notin tfin \implies full\ I$   
unfolding full\_def mem\_def  
by transfer (fastforce split: if\_splits dest: add\_not\_tfin)

## 2 Infinite Traces

**inductive** sorted\_list :: 'a :: order list  $\Rightarrow bool$  where  
[intro]: sorted\_list []  
| [intro]: sorted\_list [x]  
| [intro]:  $x \leq y \implies sorted\_list\ (y\ \# ys) \implies sorted\_list\ (x\ \# y\ \# ys)$

**lemma** sorted\_list\_app: sorted\_list xs  $\implies (\bigwedge x. x \in set\ xs \implies x \leq y) \implies sorted\_list\ (xs\ @ [y])$   
by (induction xs rule: sorted\_list.induct) auto

**lemma** sorted\_list\_drop: sorted\_list xs  $\implies sorted\_list\ (drop\ n\ xs)$   
**proof** (induction xs arbitrary: n rule: sorted\_list.induct)  
case (2 x n)

```

then show ?case
  by (cases n) auto
next
  case (∃ x y ys n)
  then show ?case
    by (cases n) auto
qed auto

lemma sorted_list_ConsD: sorted_list (x # xs) ⇒ sorted_list xs
  by (auto elim: sorted_list.cases)

lemma sorted_list_Cons_nth: sorted_list (x # xs) ⇒ j < length xs ⇒ x ≤ xs ! j
  by (induction x # xs arbitrary: x xs j rule: sorted_list.induct)
    (fastforce simp: nth_Cons split: nat.splits)+

lemma sorted_list_atD: sorted_list xs ⇒ i ≤ j ⇒ j < length xs ⇒ xs ! i ≤ xs ! j
proof (induction xs arbitrary: i j rule: sorted_list.induct)
  case (2 x i j)
  then show ?case
    by (cases i) auto
next
  case (∃ x y ys i j)
  have x ≤ (x # y # ys) ! j
    using 3(5) sorted_list_Cons_nth[OF sorted_list.intros(3)[OF 3(1,2)]]
    by (auto simp: nth_Cons split: nat.splits)
  then show ?case
    using 3
    by (cases i) auto
qed auto

coinductive ssorted :: 'a :: order stream ⇒ bool where
  shd s ≤ shd (stl s) ⇒ ssorted (stl s) ⇒ ssorted s

lemma ssorted_siterate[simp]: (∧n. n ≤ f n) ⇒ ssorted (siterate f n)
  by (coinduction arbitrary: n) auto

lemma ssortedD: ssorted s ⇒ s !! i ≤ stl s !! i
  by (induct i arbitrary: s) (auto elim: ssorted.cases)

lemma ssorted_sdrop: ssorted s ⇒ ssorted (sdrop i s)
  by (coinduction arbitrary: i s) (auto elim: ssorted.cases ssortedD)

lemma ssorted_monoD: ssorted s ⇒ i ≤ j ⇒ s !! i ≤ s !! j
proof (induct j - i arbitrary: j)
  case (Suc x)
  from Suc(1)[of j - 1] Suc(2-4) ssortedD[of s j - 1]
  show ?case by (cases j) (auto simp: le_Suc_eq Suc_diff_le)
qed simp

lemma sorted_stake: ssorted s ⇒ sorted_list (stake i s)
proof (induct i arbitrary: s)
  case (Suc i)
  then show ?case
    by (cases i) (auto elim: ssorted.cases)
qed auto

lemma ssorted_monoI: ∀ i j. i ≤ j → s !! i ≤ s !! j ⇒ ssorted s
  by (coinduction arbitrary: s)

```

```

(auto dest: spec2[of _ Suc _ Suc _] spec2[of _ 0 Suc 0])

lemma sorted_iff_mono: sorted s  $\longleftrightarrow$  ( $\forall i j. i \leq j \longrightarrow s !! i \leq s !! j$ )
  using sorted_monoI sorted_monoD by metis

typedef (overloaded) ('a, 'b :: timestamp) trace = {s :: ('a set  $\times$  'b) stream.
  sorted (smap snd s)  $\wedge$  ( $\forall x. x \in \text{snd } s \longrightarrow x \in \text{tfin}$ )  $\wedge$  ( $\forall i x. x \in \text{tfin} \longrightarrow (\exists j. \neg \text{snd } (s !! j) \leq$ 
   $\text{snd } (s !! i) + x)$ )}
  by (auto simp:  $\iota$ _mono  $\iota$ _tfin  $\iota$ _progressing stream.set_map
    intro!: exI[of _ smap ( $\lambda n. \{\}, \iota n$ ) nats] sorted_monoI)

setup_lifting type_definition_trace

lift_definition  $\Gamma$  :: ('a, 'b :: timestamp) trace  $\Rightarrow$  nat  $\Rightarrow$  'a set is
   $\lambda s i. \text{fst } (s !! i)$  .
lift_definition  $\tau$  :: ('a, 'b :: timestamp) trace  $\Rightarrow$  nat  $\Rightarrow$  'b is
   $\lambda s i. \text{snd } (s !! i)$  .

lemma  $\tau$ _mono[simp]:  $i \leq j \Longrightarrow \tau s i \leq \tau s j$ 
  by transfer (auto simp: sorted_iff_mono)

lemma  $\tau$ _fin:  $\tau \sigma i \in \text{tfin}$ 
  by transfer auto

lemma ex_lt_ $\tau$ :  $x \in \text{tfin} \Longrightarrow \exists j. \neg \tau s j \leq \tau s i + x$ 
  by transfer auto

lemma le_ $\tau$ _less:  $\tau \sigma i \leq \tau \sigma j \Longrightarrow j < i \Longrightarrow \tau \sigma i = \tau \sigma j$ 
  by (simp add: antisym)

lemma less_ $\tau$ D:  $\tau \sigma i < \tau \sigma j \Longrightarrow i < j$ 
  by (meson  $\tau$ _mono less_le_not_le not_le_imp_less)

theory MDL
  imports Interval Trace
begin



### 3 Formulas and Satisfiability



declare [[typedef_overloaded]]

datatype ('a, 't :: timestamp) formula = Bool bool | Atom 'a | Neg ('a, 't) formula |
  Bin bool  $\Rightarrow$  bool  $\Rightarrow$  bool ('a, 't) formula ('a, 't) formula |
  Prev 't  $\mathcal{I}$  ('a, 't) formula | Next 't  $\mathcal{I}$  ('a, 't) formula |
  Since ('a, 't) formula 't  $\mathcal{I}$  ('a, 't) formula |
  Until ('a, 't) formula 't  $\mathcal{I}$  ('a, 't) formula |
  MatchP 't  $\mathcal{I}$  ('a, 't) regex | MatchF 't  $\mathcal{I}$  ('a, 't) regex
  and ('a, 't) regex = Lookahead ('a, 't) formula | Symbol ('a, 't) formula |
  Plus ('a, 't) regex ('a, 't) regex | Times ('a, 't) regex ('a, 't) regex |
  Star ('a, 't) regex

fun eps :: ('a, 't :: timestamp) regex  $\Rightarrow$  bool where
  eps (Lookahead phi) = True
| eps (Symbol phi) = False
| eps (Plus r s) = (eps r  $\vee$  eps s)
| eps (Times r s) = (eps r  $\wedge$  eps s)
| eps (Star r) = True

```

```

fun atms :: ('a, 't :: timestamp) regex  $\Rightarrow$  ('a, 't) formula set where
  atms (Lookahead phi) = {phi}
| atms (Symbol phi) = {phi}
| atms (Plus r s) = atms r  $\cup$  atms s
| atms (Times r s) = atms r  $\cup$  atms s
| atms (Star r) = atms r

```

```

lemma size_atms[termination_simp]: phi  $\in$  atms r  $\Longrightarrow$  size phi < size r
by (induction r) auto

```

```

fun wf_fm1a :: ('a, 't :: timestamp) formula  $\Rightarrow$  bool
and wf_regex :: ('a, 't) regex  $\Rightarrow$  bool where
  wf_fm1a (Bool b) = True
| wf_fm1a (Atom a) = True
| wf_fm1a (Neg phi) = wf_fm1a phi
| wf_fm1a (Bin f phi psi) = (wf_fm1a phi  $\wedge$  wf_fm1a psi)
| wf_fm1a (Prev I phi) = wf_fm1a phi
| wf_fm1a (Next I phi) = wf_fm1a phi
| wf_fm1a (Since phi I psi) = (wf_fm1a phi  $\wedge$  wf_fm1a psi)
| wf_fm1a (Until phi I psi) = (wf_fm1a phi  $\wedge$  wf_fm1a psi)
| wf_fm1a (MatchP I r) = (wf_regex r  $\wedge$  ( $\forall$  phi  $\in$  atms r. wf_fm1a phi))
| wf_fm1a (MatchF I r) = (wf_regex r  $\wedge$  ( $\forall$  phi  $\in$  atms r. wf_fm1a phi))
| wf_regex (Lookahead phi) = False
| wf_regex (Symbol phi) = wf_fm1a phi
| wf_regex (Plus r s) = (wf_regex r  $\wedge$  wf_regex s)
| wf_regex (Times r s) = (wf_regex s  $\wedge$  ( $\neg$ eps s  $\vee$  wf_regex r))
| wf_regex (Star r) = wf_regex r

```

```

fun progress :: ('a, 't :: timestamp) formula  $\Rightarrow$  't list  $\Rightarrow$  nat where
  progress (Bool b) ts = length ts
| progress (Atom a) ts = length ts
| progress (Neg phi) ts = progress phi ts
| progress (Bin f phi psi) ts = min (progress phi ts) (progress psi ts)
| progress (Prev I phi) ts = min (length ts) (Suc (progress phi ts))
| progress (Next I phi) ts = (case progress phi ts of 0  $\Rightarrow$  0 | Suc k  $\Rightarrow$  k)
| progress (Since phi I psi) ts = min (progress phi ts) (progress psi ts)
| progress (Until phi I psi) ts = (if length ts = 0 then 0 else
  (let k = min (length ts - 1) (min (progress phi ts) (progress psi ts)) in
  Min {j. 0  $\leq$  j  $\wedge$  j  $\leq$  k  $\wedge$  memR (ts ! j) (ts ! k) I}))
| progress (MatchP I r) ts = Min (( $\lambda$ f. progress f ts) 'atms r)
| progress (MatchF I r) ts = (if length ts = 0 then 0 else
  (let k = min (length ts - 1) (Min (( $\lambda$ f. progress f ts) 'atms r)) in
  Min {j. 0  $\leq$  j  $\wedge$  j  $\leq$  k  $\wedge$  memR (ts ! j) (ts ! k) I}))

```

```

fun bounded_future_fm1a :: ('a, 't :: timestamp) formula  $\Rightarrow$  bool
and bounded_future_regex :: ('a, 't) regex  $\Rightarrow$  bool where
  bounded_future_fm1a (Bool b)  $\longleftrightarrow$  True
| bounded_future_fm1a (Atom a)  $\longleftrightarrow$  True
| bounded_future_fm1a (Neg phi)  $\longleftrightarrow$  bounded_future_fm1a phi
| bounded_future_fm1a (Bin f phi psi)  $\longleftrightarrow$  bounded_future_fm1a phi  $\wedge$  bounded_future_fm1a psi
| bounded_future_fm1a (Prev I phi)  $\longleftrightarrow$  bounded_future_fm1a phi
| bounded_future_fm1a (Next I phi)  $\longleftrightarrow$  bounded_future_fm1a phi
| bounded_future_fm1a (Since phi I psi)  $\longleftrightarrow$  bounded_future_fm1a phi  $\wedge$  bounded_future_fm1a psi
| bounded_future_fm1a (Until phi I psi)  $\longleftrightarrow$  bounded_future_fm1a phi  $\wedge$  bounded_future_fm1a psi  $\wedge$ 
right I  $\in$  tfin
| bounded_future_fm1a (MatchP I r)  $\longleftrightarrow$  bounded_future_regex r
| bounded_future_fm1a (MatchF I r)  $\longleftrightarrow$  bounded_future_regex r  $\wedge$  right I  $\in$  tfin
| bounded_future_regex (Lookahead phi)  $\longleftrightarrow$  bounded_future_fm1a phi

```

| *bounded\_future\_regex* (*Symbol phi*)  $\longleftrightarrow$  *bounded\_future\_fm1a* *phi*  
| *bounded\_future\_regex* (*Plus r s*)  $\longleftrightarrow$  *bounded\_future\_regex* *r*  $\wedge$  *bounded\_future\_regex* *s*  
| *bounded\_future\_regex* (*Times r s*)  $\longleftrightarrow$  *bounded\_future\_regex* *r*  $\wedge$  *bounded\_future\_regex* *s*  
| *bounded\_future\_regex* (*Star r*)  $\longleftrightarrow$  *bounded\_future\_regex* *r*

**lemmas** *regex\_induct*[*case\_names Lookahead Symbol Plus Times Star, induct type: regex*] =  
*regex.induct*[*of*  $\lambda$ \_. *True, simplified*]

**definition** *Once I phi*  $\equiv$  *Since* (*Bool True*) *I phi*

**definition** *Historically I phi*  $\equiv$  *Neg* (*Once I* (*Neg phi*))

**definition** *Eventually I phi*  $\equiv$  *Until* (*Bool True*) *I phi*

**definition** *Always I phi*  $\equiv$  *Neg* (*Eventually I* (*Neg phi*))

**fun** *rderive* :: ('a, 't :: *timestamp*) *regex*  $\Rightarrow$  ('a, 't) *regex* **where**  
*rderive* (*Lookahead phi*) = *Lookahead* (*Bool False*)  
| *rderive* (*Symbol phi*) = *Lookahead phi*  
| *rderive* (*Plus r s*) = *Plus* (*rderive r*) (*rderive s*)  
| *rderive* (*Times r s*) = (*if eps s then Plus* (*rderive r*) (*Times r* (*rderive s*)) *else Times r* (*rderive s*))  
| *rderive* (*Star r*) = *Times* (*Star r*) (*rderive r*)

**lemma** *atms\_rderive*: *phi*  $\in$  *atms* (*rderive r*)  $\Longrightarrow$  *phi*  $\in$  *atms r*  $\vee$  *phi* = *Bool False*  
**by** (*induction r*) (*auto split: if\_splits*)

**lemma** *size\_formula\_positive*: *size* (*phi* :: ('a, 't :: *timestamp*) *formula*)  $>$  0  
**by** (*induction phi*) *auto*

**lemma** *size\_regex\_positive*: *size* (*r* :: ('a, 't :: *timestamp*) *regex*)  $>$  *Suc 0*  
**by** (*induction r*) (*auto intro: size\_formula\_positive*)

**lemma** *size\_rderive*[*termination\_simp*]: *phi*  $\in$  *atms* (*rderive r*)  $\Longrightarrow$  *size phi*  $<$  *size r*  
**by** (*drule atms\_rderive*) (*auto intro: size\_atms size\_regex\_positive*)

**locale** *MDL* =  
**fixes**  $\sigma$  :: ('a, 't :: *timestamp*) *trace*  
**begin**

**fun** *sat* :: ('a, 't) *formula*  $\Rightarrow$  *nat*  $\Rightarrow$  *bool*  
**and** *match* :: ('a, 't) *regex*  $\Rightarrow$  (*nat*  $\times$  *nat*) *set* **where**  
*sat* (*Bool b*) *i* = *b*  
| *sat* (*Atom a*) *i* = ( $a \in \Gamma \sigma i$ )  
| *sat* (*Neg phi*) *i* = ( $\neg$  *sat phi i*)  
| *sat* (*Bin f phi psi*) *i* = (*f* (*sat phi i*) (*sat psi i*))  
| *sat* (*Prev I phi*) *i* = (*case i of 0*  $\Rightarrow$  *False* | *Suc j*  $\Rightarrow$  *mem* ( $\tau \sigma j$ ) ( $\tau \sigma i$ ) *I*  $\wedge$  *sat phi j*)  
| *sat* (*Next I phi*) *i* = (*mem* ( $\tau \sigma i$ ) ( $\tau \sigma$  (*Suc i*))) *I*  $\wedge$  *sat phi* (*Suc i*)  
| *sat* (*Since phi I psi*) *i* = ( $\exists j \leq i. \text{mem } (\tau \sigma j) (\tau \sigma i) I \wedge \text{sat } \psi j \wedge (\forall k \in \{j..i\}. \text{sat } \phi k)$ )  
| *sat* (*Until phi I psi*) *i* = ( $\exists j \geq i. \text{mem } (\tau \sigma i) (\tau \sigma j) I \wedge \text{sat } \psi j \wedge (\forall k \in \{i..j\}. \text{sat } \phi k)$ )  
| *sat* (*MatchP I r*) *i* = ( $\exists j \leq i. \text{mem } (\tau \sigma j) (\tau \sigma i) I \wedge (j, \text{Suc } i) \in \text{match } r$ )  
| *sat* (*MatchF I r*) *i* = ( $\exists j \geq i. \text{mem } (\tau \sigma i) (\tau \sigma j) I \wedge (i, \text{Suc } j) \in \text{match } r$ )  
| *match* (*Lookahead phi*) =  $\{(i, i) \mid i. \text{sat } \phi i\}$   
| *match* (*Symbol phi*) =  $\{(i, \text{Suc } i) \mid i. \text{sat } \phi i\}$   
| *match* (*Plus r s*) = *match r*  $\cup$  *match s*  
| *match* (*Times r s*) = *match r* *O* *match s*  
| *match* (*Star r*) = *rtrancl* (*match r*)

**lemma** *sat* (*Prev I* (*Bool False*)) *i*  $\longleftrightarrow$  *sat* (*Bool False*) *i*  
*sat* (*Next I* (*Bool False*)) *i*  $\longleftrightarrow$  *sat* (*Bool False*) *i*  
*sat* (*Since phi I* (*Bool False*)) *i*  $\longleftrightarrow$  *sat* (*Bool False*) *i*  
*sat* (*Until phi I* (*Bool False*)) *i*  $\longleftrightarrow$  *sat* (*Bool False*) *i*

```

apply (auto split: nat.splits)
done

lemma prev_rewrite: sat (Prev I  $\varphi$ ) i  $\longleftrightarrow$  sat (MatchP I (Times (Symbol  $\varphi$ ) (Symbol (Bool True)))) i
apply (auto split: nat.splits)
subgoal for j
  by (fastforce intro: exI[of _ j])
done

lemma next_rewrite: sat (Next I  $\varphi$ ) i  $\longleftrightarrow$  sat (MatchF I (Times (Symbol (Bool True)) (Symbol  $\varphi$ ))) i
by (fastforce intro: exI[of _ Suc i])

lemma trancl_Base:  $\{(i, \text{Suc } i) \mid i. P\ i\}^* = \{(i, j). i \leq j \wedge (\forall k \in \{i..<j\}. P\ k)\}$ 
proof -
  have  $(x, y) \in \{(i, j). i \leq j \wedge (\forall k \in \{i..<j\}. P\ k)\}$ 
    if  $(x, y) \in \{(i, \text{Suc } i) \mid i. P\ i\}^*$  for x y
    using that by (induct rule: rtrancl_induct) (auto simp: less_Suc_eq)
  moreover have  $(x, y) \in \{(i, \text{Suc } i) \mid i. P\ i\}^*$ 
    if  $(x, y) \in \{(i, j). i \leq j \wedge (\forall k \in \{i..<j\}. P\ k)\}$  for x y
    using that unfolding mem_Collect_eq prod.case Ball_def
    by (induct y arbitrary: x)
      (auto 0 3 simp: le_Suc_eq intro: rtrancl_into_rtrancl[rotated])
  ultimately show ?thesis by blast
qed

lemma Ball_atLeastLessThan_reindex:
   $(\forall k \in \{j..<i\}. P\ (\text{Suc } k)) = (\forall k \in \{j<..i\}. P\ k)$ 
by (auto simp: less_eq_Suc_le less_eq_nat.simps split: nat.splits)

lemma since_rewrite: sat (Since  $\varphi$  I  $\psi$ ) i  $\longleftrightarrow$  sat (MatchP I (Times (Symbol  $\psi$ ) (Star (Symbol  $\varphi$ )))) i
proof (rule iffI)
  assume sat (Since  $\varphi$  I  $\psi$ ) i
  then obtain j where j_def:  $j \leq i$  mem  $(\tau\ \sigma\ j)$   $(\tau\ \sigma\ i)$  I sat  $\psi\ j$ 
     $\forall k \in \{j..<i\}. \text{sat } \varphi\ (\text{Suc } k)$ 
    by auto
  have  $k \in \{\text{Suc } j..<\text{Suc } i\} \implies (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$  for k
    using j_def(4)
    by (cases k) auto
  then have  $(\text{Suc } j, \text{Suc } i) \in (\text{match } (\text{Symbol } \varphi))^*$ 
    using j_def(1) trancl_Base
    by auto
  then show sat (MatchP I (Times (Symbol  $\psi$ ) (Star (Symbol  $\varphi$ )))) i
    using j_def(1,2,3)
    by auto
next
  assume sat (MatchP I (Times (Symbol  $\psi$ ) (Star (Symbol  $\varphi$ )))) i
  then obtain j where j_def:  $j \leq i$  mem  $(\tau\ \sigma\ j)$   $(\tau\ \sigma\ i)$  I  $(\text{Suc } j, \text{Suc } i) \in (\text{match } (\text{Symbol } \varphi))^* \text{ sat } \psi\ j$ 
    by auto
  have  $\bigwedge k. k \in \{\text{Suc } j..<\text{Suc } i\} \implies (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$ 
    using j_def(3) trancl_Base[of  $\lambda k. (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$ ]
    by simp
  then have  $\forall k \in \{j..<i\}. \text{sat } \varphi\ (\text{Suc } k)$ 
    by auto
  then show sat (Since  $\varphi$  I  $\psi$ ) i
    using j_def(1,2,4) Ball_atLeastLessThan_reindex[of j i sat  $\varphi$ ]
    by auto
qed

```

**lemma** *until\_rewrite*:  $\text{sat } (\text{Until } \varphi \ I \ \psi) \ i \longleftrightarrow \text{sat } (\text{MatchF } I \ (\text{Times } (\text{Star } (\text{Symbol } \varphi)) \ (\text{Symbol } \psi))) \ i$   
**proof** (*rule iffI*)

**assume**  $\text{sat } (\text{Until } \varphi \ I \ \psi) \ i$   
**then obtain**  $j$  **where**  $j\_def: j \geq i \text{ mem } (\tau \ \sigma \ i) \ (\tau \ \sigma \ j) \ I \ \text{sat } \psi \ j$   
 $\forall k \in \{i..<j\}. \text{sat } \varphi \ k$   
**by** *auto*  
**have**  $k \in \{i..<j\} \implies (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$  **for**  $k$   
**using**  $j\_def(4)$   
**by** *auto*  
**then have**  $(i, j) \in (\text{match } (\text{Symbol } \varphi))^*$   
**using**  $j\_def(1)$  *trancl\_Base*  
**by** *simp*  
**then show**  $\text{sat } (\text{MatchF } I \ (\text{Times } (\text{Star } (\text{Symbol } \varphi)) \ (\text{Symbol } \psi))) \ i$   
**using**  $j\_def(1,2,3)$   
**by** *auto*

**next**

**assume**  $\text{sat } (\text{MatchF } I \ (\text{Times } (\text{Star } (\text{Symbol } \varphi)) \ (\text{Symbol } \psi))) \ i$   
**then obtain**  $j$  **where**  $j\_def: j \geq i \text{ mem } (\tau \ \sigma \ i) \ (\tau \ \sigma \ j) \ I \ (i, j) \in (\text{match } (\text{Symbol } \varphi))^* \ \text{sat } \psi \ j$   
**by** *auto*  
**have**  $\bigwedge k. k \in \{i..<j\} \implies (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$   
**using**  $j\_def(3)$  *trancl\_Base*[of  $\lambda k. (k, \text{Suc } k) \in \text{match } (\text{Symbol } \varphi)$ ]  
**by** *auto*  
**then have**  $\forall k \in \{i..<j\}. \text{sat } \varphi \ k$   
**by** *simp*  
**then show**  $\text{sat } (\text{Until } \varphi \ I \ \psi) \ i$   
**using**  $j\_def(1,2,4)$   
**by** *auto*

**qed**

**lemma** *match\_le*:  $(i, j) \in \text{match } r \implies i \leq j$

**proof** (*induction r arbitrary: i j*)

**case**  $(\text{Times } r \ s)$

**then show** *?case* **using** *order.trans* **by** *fastforce*

**next**

**case**  $(\text{Star } r)$

**from** *Star.prem*s **show** *?case*

**unfolding** *match.simps*

**by** (*induct i j rule: rtrancl.induct*) (*force dest: Star.IH*)+

**qed** *auto*

**lemma** *match\_Times*:  $(i, i + n) \in \text{match } (\text{Times } r \ s) \longleftrightarrow$

$(\exists k \leq n. (i, i + k) \in \text{match } r \wedge (i + k, i + n) \in \text{match } s)$

**using** *match\_le* **by** *auto* (*metis le\_iff\_add nat\_add\_left\_cancel\_le*)

**lemma** *rtrancl\_unfold*:  $(x, z) \in \text{rtrancl } R \implies$

$x = z \vee (\exists y. (x, y) \in R \wedge x \neq y \wedge (y, z) \in \text{rtrancl } R)$

**by** (*induction x z rule: rtrancl.induct*) *auto*

**lemma** *rtrancl\_unfold'*:  $(x, z) \in \text{rtrancl } R \implies$

$x = z \vee (\exists y. (x, y) \in \text{rtrancl } R \wedge y \neq z \wedge (y, z) \in R)$

**by** (*induction x z rule: rtrancl.induct*) *auto*

**lemma** *match\_Star*:  $(i, i + \text{Suc } n) \in \text{match } (\text{Star } r) \longleftrightarrow$

$(\exists k \leq n. (i, i + 1 + k) \in \text{match } r \wedge (i + 1 + k, i + \text{Suc } n) \in \text{match } (\text{Star } r))$

**proof** (*rule iffI*)

**assume** *assms*:  $(i, i + \text{Suc } n) \in \text{match } (\text{Star } r)$

**obtain**  $k$  **where**  $k\_def: (i, k) \in \text{local.match } r \ i \leq k \ i \neq k$

$(k, i + \text{Suc } n) \in (\text{local.match } r)^*$

```

    using rtrancl_unfold[OF assms[unfolded match.simps]] match_le by auto
  from k_def(4) have (k, i + Suc n) ∈ match (Star r)
    unfolding match.simps by simp
  then have k_le: k ≤ i + Suc n
    using match_le by blast
  from k_def(2,3) obtain k' where k'_def: k = i + Suc k'
    by (metis Suc_diff_Suc le_add_diff_inverse le_neq_implies_less)
  show ∃ k ≤ n. (i, i + 1 + k) ∈ match r ∧ (i + 1 + k, i + Suc n) ∈ match (Star r)
    using k_def k_le unfolding k'_def by auto
next
  assume assms: ∃ k ≤ n. (i, i + 1 + k) ∈ match r ∧
    (i + 1 + k, i + Suc n) ∈ match (Star r)
  then show (i, i + Suc n) ∈ match (Star r)
    by (induction n) auto
qed

lemma match_refl_eps: (i, i) ∈ match r ⟹ eps r
proof (induction r)
  case (Times r s)
  then show ?case
    using match_Times[where ?i=i and ?n=0]
    by auto
qed auto

lemma wf_regex_eps_match: wf_regex r ⟹ eps r ⟹ (i, i) ∈ match r
  by (induction r arbitrary: i) auto

lemma match_Star_unfold: i < j ⟹ (i, j) ∈ match (Star r) ⟹ ∃ k ∈ {i..<j}. (i, k) ∈ match (Star r) ∧ (k, j) ∈ match r
  using rtrancl_unfold'[of i j match r] match_le[of _ j r] match_le[of i _ Star r]
  by auto (meson atLeastLessThan_iff order_le_less)

lemma match_rderive: wf_regex r ⟹ i ≤ j ⟹ (i, Suc j) ∈ match r ⟷ (i, j) ∈ match (rderive r)
proof (induction r arbitrary: i j)
  case (Times r1 r2)
  then show ?case
    using match_refl_eps[of Suc j r2] match_le[of _ Suc j r2]
    apply (auto)
      apply (metis le_Suc_eq relcomp.simps)
      apply (meson match_le relcomp.simps)
      apply (metis le_SucE relcomp.simps)
      apply (meson relcomp.relcompI wf_regex_eps_match)
      apply (meson match_le relcomp.simps)
      apply (metis le_SucE relcomp.simps)
      apply (meson match_le relcomp.simps)
    done
  next
  case (Star r)
  then show ?case
    using match_Star_unfold[of i Suc j r]
    by auto (meson match_le rtrancl.simps)
qed auto

end

lemma atms_nonempty: atms r ≠ {}
  by (induction r) auto

```

```

lemma atms_finite: finite (atms r)
  by (induction r) auto

lemma progress_le_ts:
  assumes  $\bigwedge t. t \in \text{set } ts \implies t \in \text{tfin}$ 
  shows progress phi ts  $\leq$  length ts
  using assms
proof (induction phi ts rule: progress.induct)
case (8 phi I psi ts)
  have  $ts \neq [] \implies \text{Min } \{j. j \leq \min(\text{length } ts - \text{Suc } 0) (\min(\text{progress } \text{phi } ts) (\text{progress } \text{psi } ts)) \wedge$ 
     $\text{memR } (ts ! j) (ts ! \min(\text{length } ts - \text{Suc } 0) (\min(\text{progress } \text{phi } ts) (\text{progress } \text{psi } ts))) I\}$ 
     $\leq \text{length } ts$ 
  apply (rule le_trans[OF Min_le[where ?x=min (length ts - Suc 0) (min (progress phi ts) (progress
psi ts))]])
    apply (auto simp: in_set_conv_nth intro!: memR_tfin_refl 8(3))
    apply (metis One_nat_def diff_less length_greater_0_conv less_numerical_extra(1) min.commute
min.strict_coboundedI2)
  done
  then show ?case
    by auto
next
case (9 I r ts)
  then show ?case
    using atms_nonempty[of r] atms_finite[of r]
    by auto (meson Min_le dual_order.trans finite_imageI image_iff)
next
case (10 I r ts)
  have  $ts \neq [] \implies \text{Min } \{j. j \leq \min(\text{length } ts - \text{Suc } 0) (\text{MIN } f \in \text{atms } r. \text{progress } f \text{ } ts) \wedge$ 
     $\text{memR } (ts ! j) (ts ! \min(\text{length } ts - \text{Suc } 0) (\text{MIN } f \in \text{atms } r. \text{progress } f \text{ } ts)) I\}$ 
     $\leq \text{length } ts$ 
  apply (rule le_trans[OF Min_le[where ?x=min (length ts - Suc 0) (Min (( $\lambda f. \text{progress } f \text{ } ts$ ) ' atms
r))]])
    apply (auto simp: in_set_conv_nth intro!: memR_tfin_refl 10(2))
    apply (metis One_nat_def diff_less length_greater_0_conv less_numerical_extra(1) min.commute
min.strict_coboundedI2)
  done
  then show ?case
    by auto
qed (auto split: nat.splits)

end
theory Metric_Point_Structure
  imports Interval
begin

```

```

class metric_domain = plus + zero + ord +
  assumes  $\Delta 1: x + x' = x' + x$ 
  and  $\Delta 2: (x + x') + x'' = x + (x' + x'')$ 
  and  $\Delta 3: x + 0 = x$ 
  and  $\Delta 3': x = 0 + x$ 
  and  $\Delta 4: x + x' = x + x'' \implies x' = x''$ 
  and  $\Delta 4': x + x'' = x' + x'' \implies x = x'$ 
  and  $\Delta 5: x + x' = 0 \implies x = 0$ 
  and  $\Delta 5': x + x' = 0 \implies x' = 0$ 

```

```

    and  $\Delta 6$ :  $\exists x''. x = x' + x'' \vee x' = x + x''$ 
    and metric_domain_le_def:  $x \leq x' \iff (\exists x''. x' = x + x'')$ 
    and metric_domain_lt_def:  $x < x' \iff (\exists x''. x'' \neq 0 \wedge x' = x + x'')$ 
begin

lemma metric_domain_pos:  $x \geq 0$ 
  using  $\Delta 3'$  local.metric_domain_le_def by auto

lemma less_eq_le_neg:  $x < x' \iff (x \leq x' \wedge x \neq x')$ 
  apply (auto simp: metric_domain_le_def metric_domain_lt_def)
  apply (metis  $\Delta 3$   $\Delta 4$ )
  apply (metis  $\Delta 3$ )
done

end

class metric_domain_timestamp = metric_domain + sup + embed_nat + tfin +
  assumes metric_domain_sup_def:  $\text{sup } x x' = (\text{if } x \leq x' \text{ then } x' \text{ else } x)$ 
    and metric_domain_ι_mono:  $\bigwedge i j. i \leq j \implies \iota i \leq \iota j$ 
    and metric_domain_ι_progressing:  $\exists j. \neg \iota j \leq \iota i + x$ 
    and metric_domain_tfin_def: tfin = UNIV

subclass (in metric_domain_timestamp) timestamp
  apply unfold_locales
    apply (auto simp:  $\Delta 2$ )[1]
    apply (auto simp:  $\Delta 1$ )[1]
    apply (auto simp:  $\Delta 3'$ [symmetric])[1]
  subgoal for  $x y$ 
    apply (auto simp: metric_domain_le_def metric_domain_lt_def)
    apply (metis  $\Delta 2$   $\Delta 3$   $\Delta 4$   $\Delta 5$ )
    apply (metis  $\Delta 2$   $\Delta 3$ )
  done
  using  $\Delta 6$  apply (auto simp: metric_domain_le_def)[1]
  using  $\Delta 2$  apply (auto simp: metric_domain_le_def)[1]
  subgoal for  $x y$ 
    apply (auto simp: metric_domain_le_def metric_domain_lt_def)
    apply (metis  $\Delta 2$   $\Delta 3$   $\Delta 4$   $\Delta 5$ )
  done
  using  $\Delta 6$  apply (fastforce simp: metric_domain_le_def metric_domain_sup_def)
  using  $\Delta 6$  apply (fastforce simp: metric_domain_le_def metric_domain_sup_def)
    apply (auto simp: metric_domain_le_def metric_domain_sup_def)[1]
  using metric_domain_ι_mono apply (auto simp: metric_domain_le_def)[1]
    apply (auto simp: metric_domain_tfin_def)[1]
  using metric_domain_ι_progressing apply (auto simp: metric_domain_le_def)[1]
    apply (auto simp: metric_domain_tfin_def)[2]
  using  $\Delta 2$  apply (auto simp: metric_domain_le_def)[1]
  using  $\Delta 1$   $\Delta 3$  apply (auto simp: metric_domain_lt_def)
done

locale metric_point_structure =
  fixes  $d :: 't :: \{\text{order}\} \Rightarrow 't \Rightarrow 'd :: \text{metric\_domain\_timestamp}$ 
  assumes  $d1: d t t' = 0 \iff t = t'$ 
    and  $d2: d t t' = d t' t$ 
    and  $d3: t < t' \implies t' < t'' \implies d t t'' = d t t' + d t' t''$ 

```

**and**  $d3': t < t' \implies t' < t'' \implies d t'' t = d t'' t' + d t' t$   
**begin**

**lemma** *metric\_point\_structure\_memL\_aux*:  $t0 \leq t \implies t \leq t' \implies x \leq d t t' \iff (d t0 t + x \leq d t0 t')$

**apply** (*rule iffI*)  
**apply** (*metis*  $\Delta 1$  *add\_0 add\_mono\_comm d1 d3 order\_le\_less*)  
**apply** (*cases*  $t0 < t$ ; *cases*  $t < t'$ )  
**apply** (*auto simp: metric\_domain\_le\_def*)  
**apply** (*metis*  $\Delta 4$  *ab\_semigroup\_add\_class.add\_ac(1) d3*)  
**apply** (*metis comm\_monoid\_add\_class.add\_0 group\_cancel.add1 metric\_domain\_lt\_def nless\_le*)  
**apply** (*metis*  $\Delta 3'$  *d1*)  
**apply** (*metis add\_0 d1*)  
**done**

**lemma** *metric\_point\_structure\_memL\_strict\_aux*:  $t0 \leq t \implies t \leq t' \implies x < d t t' \iff (d t0 t + x < d t0 t')$

**using** *metric\_point\_structure\_memL\_aux*[*of*  $t0 t t' x$ ]  
**apply** *auto*  
**apply** (*metis* (*no\_types, lifting*)  $\Delta 1 \Delta 3 \Delta 4$  *antisym\_conv2 d1 d3*)  
**apply** (*metis*  $\Delta 1$  *add\_0 d1 d3 order.order\_iff\_strict order\_less\_irrefl*)  
**done**

**lemma** *metric\_point\_structure\_memR\_aux*:  $t0 \leq t \implies t \leq t' \implies d t t' \leq x \iff (d t0 t' \leq d t0 t + x)$

**apply** *auto*  
**apply** (*metis*  $\Delta 1 \Delta 3$  *d1 d3 order\_le\_less add\_mono*)  
**apply** (*smt* (*verit, ccfv\_threshold*)  $\Delta 1 \Delta 2 \Delta 3 \Delta 4$  *d1 d3 metric\_domain\_le\_def order\_le\_less*)  
**done**

**lemma** *metric\_point\_structure\_memR\_strict\_aux*:  $t0 \leq t \implies t \leq t' \implies d t t' < x \iff (d t0 t' < d t0 t + x)$

**by** (*auto simp add: metric\_point\_structure\_memL\_aux metric\_point\_structure\_memR\_aux less\_le\_not\_le*)

**lemma** *metric\_point\_structure\_le\_mem*:  $t0 \leq t \implies t \leq t' \implies d t t' \leq x \iff \text{mem } (d t0 t) (d t0 t')$   
*(interval 0 x True True)*

**unfolding** *mem\_def*  
**apply** (*transfer fixing: d*)  
**using** *metric\_point\_structure\_memR\_aux*  
**apply** (*auto simp: metric\_domain\_le\_def metric\_domain\_tfin\_def*)  
**apply** (*metis add.right\_neutral d3 order.order\_iff\_strict*)  
**done**

**lemma** *metric\_point\_structure\_lt\_mem*:  $t0 \leq t \implies t \leq t' \implies 0 < x \implies d t t' < x \iff \text{mem } (d t0 t) (d t0 t')$   
*(interval 0 x True False)*

**unfolding** *mem\_def*  
**apply** (*transfer fixing: d*)  
**using** *metric\_point\_structure\_memR\_strict\_aux*  
**apply** (*auto simp: metric\_domain\_tfin\_def*)  
**apply** (*metis*  $\Delta 3$  *metric\_domain\_pos metric\_point\_structure\_memL\_aux*)  
**done**

**lemma** *metric\_point\_structure\_eq\_mem*:  $t0 \leq t \implies t \leq t' \implies d t t' = x \iff \text{mem } (d t0 t) (d t0 t')$   
*(interval x x True True)*

**unfolding** *mem\_def*  
**apply** (*transfer fixing: d*)  
**subgoal for**  $t0 t t' x$   
**using** *metric\_point\_structure\_memL\_aux*[*of*  $t0 t t' x$ ] *metric\_point\_structure\_memR\_aux*[*of*  $t0 t t'$ ]

```

x] metric_domain_pos
  by (auto simp: metric_domain_tfin_def)
done

lemma metric_point_structure_ge_mem:  $t_0 \leq t \implies t \leq t' \implies x \leq d t t' \iff \text{mem } (\text{Some } (d t_0 t))$ 
(Some (d t_0 t')) (interval (Some x) None True True)
  unfolding mem_def
  apply (transfer fixing: d)
  using metric_point_structure_memL_aux by (auto simp: tfin_option_def zero_option_def plus_option_def
less_eq_option_def metric_domain_le_def metric_domain_tfin_def split: option.splits)

lemma metric_point_structure_gt_mem:  $t_0 \leq t \implies t \leq t' \implies x < d t t' \iff \text{mem } (\text{Some } (d t_0 t))$ 
(Some (d t_0 t')) (interval (Some x) None False True)
  unfolding mem_def
  apply (transfer fixing: d)
  using metric_point_structure_memL_strict_aux by (auto simp: tfin_option_def zero_option_def
plus_option_def less_option_def less_eq_option_def metric_domain_le_def metric_domain_tfin_def
split: option.splits)

end

instantiation nat :: metric_domain_timestamp
begin

instance
  apply standard
    apply auto[8]
    apply (meson less_eqE timestamp_total)
  using nat_le_iff_add apply blast
  using less_imp_add_positive apply auto[1]
    apply (auto simp: sup_max)[1]
    apply (auto simp:  $\iota$ _nat_def)[1]
  subgoal for i x
    apply (auto simp:  $\iota$ _nat_def)
    using add_le_same_cancel1 by blast
  apply (auto simp: tfin_nat_def)
done

end

interpretation nat_metric_point_structure: metric_point_structure  $\lambda t :: \text{nat}. \lambda t'. \text{if } t \leq t' \text{ then } t' - t$ 
else  $t - t'$ 
  by unfold_locales auto

end
theory NFA
  imports HOL-Library.IArray
begin

type_synonym state = nat

datatype transition = eps_trans state nat | symb_trans state | split_trans state state

fun state_set :: transition  $\Rightarrow$  state set where
  state_set (eps_trans s _) = {s}
| state_set (symb_trans s) = {s}
| state_set (split_trans s s') = {s, s'}

```

**fun** *fmla\_set* :: *transition*  $\Rightarrow$  *nat set* **where**  
*fmla\_set* (*eps\_trans* \_ *n*) = {*n*}  
| *fmla\_set* \_ = {}

**lemma** *rtranclp\_closed*:  $rtranclp\ R\ q\ q' \Longrightarrow X = X \cup \{q'. \exists q \in X. R\ q\ q'\} \Longrightarrow$   
 $q \in X \Longrightarrow q' \in X$   
**by** (*induction* *q* *q'* *rule*: *rtranclp.induct*) *auto*

**lemma** *rtranclp\_closed\_sub*:  $rtranclp\ R\ q\ q' \Longrightarrow \{q'. \exists q \in X. R\ q\ q'\} \subseteq X \Longrightarrow$   
 $q \in X \Longrightarrow q' \in X$   
**by** (*induction* *q* *q'* *rule*: *rtranclp.induct*) *auto*

**lemma** *rtranclp\_closed\_sub'*:  $rtranclp\ R\ q\ q' \Longrightarrow q' = q \vee (\exists q''. R\ q\ q'' \wedge rtranclp\ R\ q''\ q')$   
**using** *converse\_rtranclpE* **by** *force*

**lemma** *rtranclp\_step*:  $rtranclp\ R\ q\ q'' \Longrightarrow (\bigwedge q'. R\ q\ q' \longleftrightarrow q' \in X) \Longrightarrow$   
 $q = q'' \vee (\exists q' \in X. R\ q\ q' \wedge rtranclp\ R\ q'\ q'')$   
**by** (*induction* *q* *q''* *rule*: *rtranclp.induct*)  
(*auto intro*: *rtranclp.rtrancl\_into\_rtrancl*)

**lemma** *rtranclp\_unfold*:  $rtranclp\ R\ x\ z \Longrightarrow x = z \vee (\exists y. R\ x\ y \wedge rtranclp\ R\ y\ z)$   
**by** (*induction* *x* *z* *rule*: *rtranclp.induct*) *auto*

**context** *fixes*

*q0* :: *state* **and**

*qf* :: *state* **and**

*transs* :: *transition list*

**begin**

**qualified definition** *SQ* :: *state set* **where**

*SQ* = {*q0*..*q0* + *length transs*}

**lemma** *q\_in\_SQ*[*code\_unfold*]:  $q \in SQ \longleftrightarrow q0 \leq q \wedge q < q0 + \text{length transs}$   
**by** (*auto simp*: *SQ\_def*)

**lemma** *finite\_SQ*: *finite SQ*  
**by** (*auto simp add*: *SQ\_def*)

**lemma** *transs\_q\_in\_set*:  $q \in SQ \Longrightarrow \text{transs} ! (q - q0) \in \text{set transs}$   
**by** (*auto simp add*: *SQ\_def*)

**qualified definition** *Q* :: *state set* **where**

*Q* = *SQ*  $\cup$  {*qf*}

**lemma** *finite\_Q*: *finite Q*  
**by** (*auto simp add*: *Q\_def SQ\_def*)

**lemma** *SQ\_sub\_Q*:  $SQ \subseteq Q$   
**by** (*auto simp add*: *SQ\_def Q\_def*)

**qualified definition** *nfa\_fmla\_set* :: *nat set* **where**

*nfa\_fmla\_set* =  $\bigcup$  (*fmla\_set* ' *set transs*)

**qualified definition**  $step\_eps :: bool\ list \Rightarrow state \Rightarrow state \Rightarrow bool$  **where**  
 $step\_eps\ bs\ q\ q' \longleftrightarrow q \in SQ \wedge$   
 $(case\ transs\ !\ (q - q0)\ of\ eps\_trans\ p\ n \Rightarrow n < length\ bs \wedge bs\ !\ n \wedge p = q'$   
 $\mid split\_trans\ p\ p' \Rightarrow p = q' \vee p' = q' \mid \_ \Rightarrow False)$

**lemma**  $step\_eps\_dest: step\_eps\ bs\ q\ q' \Longrightarrow q \in SQ$   
**by**  $(auto\ simp\ add: step\_eps\_def)$

**lemma**  $step\_eps\_mono: step\_eps\ []\ q\ q' \Longrightarrow step\_eps\ bs\ q\ q'$   
**by**  $(auto\ simp: step\_eps\_def\ split: transition.splits)$

**qualified definition**  $step\_eps\_sucs :: bool\ list \Rightarrow state \Rightarrow state\ set$  **where**  
 $step\_eps\_sucs\ bs\ q = (if\ q \in SQ\ then$   
 $(case\ transs\ !\ (q - q0)\ of\ eps\_trans\ p\ n \Rightarrow if\ n < length\ bs \wedge bs\ !\ n\ then\ \{p\}\ else\ \{\})$   
 $\mid split\_trans\ p\ p' \Rightarrow \{p, p'\} \mid \_ \Rightarrow \{\})\ else\ \{\})$

**lemma**  $step\_eps\_sucs\_sound: q' \in step\_eps\_sucs\ bs\ q \longleftrightarrow step\_eps\ bs\ q\ q'$   
**by**  $(auto\ simp\ add: step\_eps\_sucs\_def\ step\_eps\_def\ split: transition.splits)$

**qualified definition**  $step\_eps\_set :: bool\ list \Rightarrow state\ set \Rightarrow state\ set$  **where**  
 $step\_eps\_set\ bs\ R = \bigcup (step\_eps\_sucs\ bs\ ` R)$

**lemma**  $step\_eps\_set\_sound: step\_eps\_set\ bs\ R = \{q'. \exists q \in R. step\_eps\ bs\ q\ q'\}$   
**using**  $step\_eps\_sucs\_sound$  **by**  $(auto\ simp\ add: step\_eps\_set\_def)$

**lemma**  $step\_eps\_set\_mono: R \subseteq S \Longrightarrow step\_eps\_set\ bs\ R \subseteq step\_eps\_set\ bs\ S$   
**by**  $(auto\ simp\ add: step\_eps\_set\_def)$

**qualified definition**  $step\_eps\_closure :: bool\ list \Rightarrow state \Rightarrow state \Rightarrow bool$  **where**  
 $step\_eps\_closure\ bs = (step\_eps\ bs)^{**}$

**lemma**  $step\_eps\_closure\_dest: step\_eps\_closure\ bs\ q\ q' \Longrightarrow q \neq q' \Longrightarrow q \in SQ$   
**unfolding**  $step\_eps\_closure\_def$   
**apply**  $(induction\ q\ q'\ rule: rtranclp.induct)$  **using**  $step\_eps\_dest$  **by**  $auto$

**qualified definition**  $step\_eps\_closure\_set :: state\ set \Rightarrow bool\ list \Rightarrow state\ set$  **where**  
 $step\_eps\_closure\_set\ R\ bs = \bigcup ((\lambda q. \{q'. step\_eps\_closure\ bs\ q\ q'\}) ` R)$

**lemma**  $step\_eps\_closure\_set\_refl: R \subseteq step\_eps\_closure\_set\ R\ bs$   
**by**  $(auto\ simp\ add: step\_eps\_closure\_set\_def\ step\_eps\_closure\_def)$

**lemma**  $step\_eps\_closure\_set\_mono: R \subseteq S \Longrightarrow step\_eps\_closure\_set\ R\ bs \subseteq step\_eps\_closure\_set\ S\ bs$   
**by**  $(auto\ simp\ add: step\_eps\_closure\_set\_def)$

**lemma**  $step\_eps\_closure\_set\_empty: step\_eps\_closure\_set\ \{\}\ bs = \{\}$   
**by**  $(auto\ simp\ add: step\_eps\_closure\_set\_def)$

**lemma**  $step\_eps\_closure\_set\_mono': step\_eps\_closure\_set\ R\ [] \subseteq step\_eps\_closure\_set\ R\ bs$   
**by**  $(auto\ simp: step\_eps\_closure\_set\_def\ step\_eps\_closure\_def)\ (metis\ mono\_rtranclp\ step\_eps\_mono)$

**lemma**  $step\_eps\_closure\_set\_split: step\_eps\_closure\_set\ (R \cup S)\ bs =$   
 $step\_eps\_closure\_set\ R\ bs \cup step\_eps\_closure\_set\ S\ bs$

by (auto simp add: step\_eps\_closure\_set\_def)

**lemma** step\_eps\_closure\_set\_Un: step\_eps\_closure\_set ( $\bigcup x \in X. R x$ ) bs =  
( $\bigcup x \in X. step\_eps\_closure\_set (R x) bs$ )

by (auto simp add: step\_eps\_closure\_set\_def)

**lemma** step\_eps\_closure\_set\_idem: step\_eps\_closure\_set (step\_eps\_closure\_set R bs) bs =  
step\_eps\_closure\_set R bs

unfolding step\_eps\_closure\_set\_def step\_eps\_closure\_def by auto

**lemma** step\_eps\_closure\_set\_flip:

assumes step\_eps\_closure\_set R bs =  $R \cup S$

shows step\_eps\_closure\_set S bs  $\subseteq R \cup S$

using step\_eps\_closure\_set\_idem[of R bs, unfolded assms, unfolded step\_eps\_closure\_set\_split]

by auto

**lemma** step\_eps\_closure\_set\_unfold: ( $\bigwedge q'. step\_eps\ bs\ q\ q' \longleftrightarrow q' \in X$ )  $\implies$

step\_eps\_closure\_set {q} bs = {q}  $\cup$  step\_eps\_closure\_set X bs

unfolding step\_eps\_closure\_set\_def step\_eps\_closure\_def

using rtranclp\_step[of step\_eps bs q]

by (auto simp add: converse\_rtranclp\_into\_rtranclp)

**lemma** step\_step\_eps\_closure: step\_eps bs q q'  $\implies q \in R \implies q' \in step\_eps\_closure\_set R bs$

unfolding step\_eps\_closure\_set\_def step\_eps\_closure\_def by auto

**lemma** step\_eps\_closure\_set\_code[code]:

step\_eps\_closure\_set R bs =

(let R' =  $R \cup step\_eps\_set\ bs\ R$  in if  $R = R'$  then R else step\_eps\_closure\_set R' bs)

using rtranclp\_closed

by (auto simp add: step\_eps\_closure\_set\_def step\_eps\_closure\_def step\_eps\_set\_sound Let\_def)

**lemma** step\_eps\_closure\_empty: step\_eps\_closure bs q q'  $\implies (\bigwedge q'. \neg step\_eps\ bs\ q\ q') \implies q = q'$

unfolding step\_eps\_closure\_def by (induction q q' rule: rtranclp.induct) auto

**lemma** step\_eps\_closure\_set\_step\_id: ( $\bigwedge q\ q'. q \in R \implies \neg step\_eps\ bs\ q\ q'$ )  $\implies$

step\_eps\_closure\_set R bs = R

using step\_eps\_closure\_empty step\_eps\_closure\_set\_refl unfolding step\_eps\_closure\_set\_def by

blast

**qualified definition** step\_symb :: state  $\Rightarrow$  state  $\Rightarrow$  bool **where**

step\_symb q q'  $\longleftrightarrow q \in SQ \wedge$

(case transs ! (q - q0) of symb\_trans p  $\Rightarrow p = q' \mid \_ \Rightarrow False$ )

**lemma** step\_symb\_dest: step\_symb q q'  $\implies q \in SQ$

by (auto simp add: step\_symb\_def)

**qualified definition** step\_symb\_sucs :: state  $\Rightarrow$  state set **where**

step\_symb\_sucs q = (if q  $\in SQ$  then

(case transs ! (q - q0) of symb\_trans p  $\Rightarrow \{p\} \mid \_ \Rightarrow \{\}$ ) else  $\{\}$ )

**lemma** step\_symb\_sucs\_sound: q'  $\in step\_symb\_sucs\ q \longleftrightarrow step\_symb\ q\ q'$

by (auto simp add: step\_symb\_sucs\_def step\_symb\_def split: transition.splits)

**qualified definition**  $step\_symb\_set :: state\ set \Rightarrow state\ set$  **where**  
 $step\_symb\_set\ R = \{q'. \exists q \in R. step\_symb\ q\ q'\}$

**lemma**  $step\_symb\_set\_mono: R \subseteq S \Longrightarrow step\_symb\_set\ R \subseteq step\_symb\_set\ S$   
**by** (auto simp add: step\_symb\_set\_def)

**lemma**  $step\_symb\_set\_empty: step\_symb\_set\ \{\} = \{\}$   
**by** (auto simp add: step\_symb\_set\_def)

**lemma**  $step\_symb\_set\_proj: step\_symb\_set\ R = step\_symb\_set\ (R \cap SQ)$   
**using** step\_symb\_dest **by** (auto simp add: step\_symb\_set\_def)

**lemma**  $step\_symb\_set\_split: step\_symb\_set\ (R \cup S) = step\_symb\_set\ R \cup step\_symb\_set\ S$   
**by** (auto simp add: step\_symb\_set\_def)

**lemma**  $step\_symb\_set\_Un: step\_symb\_set\ (\bigcup x \in X. R\ x) = (\bigcup x \in X. step\_symb\_set\ (R\ x))$   
**by** (auto simp add: step\_symb\_set\_def)

**lemma**  $step\_symb\_set\_code[code]: step\_symb\_set\ R = \bigcup (step\_symb\_sucs\ 'R)$   
**using** step\_symb\_sucs\_sound **by** (auto simp add: step\_symb\_set\_def)

**qualified definition**  $delta :: state\ set \Rightarrow bool\ list \Rightarrow state\ set$  **where**  
 $delta\ R\ bs = step\_symb\_set\ (step\_eps\_closure\_set\ R\ bs)$

**lemma**  $delta\_eps: delta\ (step\_eps\_closure\_set\ R\ bs)\ bs = delta\ R\ bs$   
**unfolding** delta\_def step\_eps\_closure\_set\_idem **by** (rule refl)

**lemma**  $delta\_eps\_split:$   
**assumes**  $step\_eps\_closure\_set\ R\ bs = R \cup S$   
**shows**  $delta\ R\ bs = step\_symb\_set\ R \cup delta\ S\ bs$   
**unfolding** delta\_def **assms** step\_symb\_set\_split  
**using** step\_symb\_set\_mono[OF step\_eps\_closure\_set\_flip[OF assms], unfolded step\_symb\_set\_split]  
 $step\_symb\_set\_mono$ [OF step\_eps\_closure\_set\_refl] **by** auto

**lemma**  $delta\_split: delta\ (R \cup S)\ bs = delta\ R\ bs \cup delta\ S\ bs$   
**by** (auto simp add: delta\_def step\_symb\_set\_split step\_eps\_closure\_set\_split)

**lemma**  $delta\_Un: delta\ (\bigcup x \in X. R\ x)\ bs = (\bigcup x \in X. delta\ (R\ x)\ bs)$   
**unfolding** delta\_def step\_eps\_closure\_set\_Un step\_symb\_set\_Un **by** simp

**lemma**  $delta\_step\_symb\_set\_absorb: delta\ R\ bs = delta\ R\ bs \cup step\_symb\_set\ R$   
**using** step\_eps\_closure\_set\_refl **by** (auto simp add: delta\_def step\_symb\_set\_def)

**lemma**  $delta\_sub\_eps\_mono:$   
**assumes**  $S \subseteq step\_eps\_closure\_set\ R\ bs$   
**shows**  $delta\ S\ bs \subseteq delta\ R\ bs$   
**unfolding** delta\_def  
**using** step\_symb\_set\_mono[OF step\_eps\_closure\_set\_mono[OF assms, of bs,  
 $unfolded\ step\_eps\_closure\_set\_idem$ ]] **by** simp

**qualified definition**  $run :: state\ set \Rightarrow bool\ list\ list \Rightarrow state\ set$  **where**  
 $run\ R\ bss = foldl\ delta\ R\ bss$

**lemma** *run\_eps\_split*:  
**assumes** *step\_eps\_closure\_set*  $R\ bs = R \cup S$  *step\_symb\_set*  $R = \{\}$   
**shows**  $\text{run } R\ (bs \# bss) = \text{run } S\ (bs \# bss)$   
**unfolding** *run\_def* *foldl\_simps* *delta\_eps\_split* [*OF* *assms*(1), *unfolded* *assms*(2)]  
**by** *auto*

**lemma** *run\_empty*:  $\text{run } \{\} bss = \{\}$   
**unfolding** *run\_def*  
**by** (*induction* *bss*)  
*(auto simp add: delta\_def step\_symb\_set\_empty step\_eps\_closure\_set\_empty)*

**lemma** *run\_Nil*:  $\text{run } R\ [] = R$   
**by** (*auto simp add: run\_def*)

**lemma** *run\_Cons*:  $\text{run } R\ (bs \# bss) = \text{run } (\text{delta } R\ bs)\ bss$   
**unfolding** *run\_def* **by** *simp*

**lemma** *run\_split*:  $\text{run } (R \cup S)\ bss = \text{run } R\ bss \cup \text{run } S\ bss$   
**unfolding** *run\_def*  
**by** (*induction* *bss* *arbitrary*:  $R\ S$ ) (*auto simp add: delta\_split*)

**lemma** *run\_Un*:  $\text{run } (\bigcup x \in X. R\ x)\ bss = (\bigcup x \in X. \text{run } (R\ x)\ bss)$   
**unfolding** *run\_def*  
**by** (*induction* *bss* *arbitrary*:  $R$ ) (*auto simp add: delta\_Un*)

**lemma** *run\_comp*:  $\text{run } R\ (bss @ css) = \text{run } (\text{run } R\ bss)\ css$   
**unfolding** *run\_def* **by** *simp*

**qualified definition** *accept\_eps* :: *state set*  $\Rightarrow$  *bool list*  $\Rightarrow$  *bool* **where**  
*accept\_eps*  $R\ bs \longleftrightarrow (qf \in \text{step\_eps\_closure\_set } R\ bs)$

**lemma** *step\_eps\_accept\_eps*:  $\text{step\_eps } bs\ q\ qf \Longrightarrow q \in R \Longrightarrow \text{accept\_eps } R\ bs$   
**unfolding** *accept\_eps\_def* **using** *step\_step\_eps\_closure* **by** *simp*

**lemma** *accept\_eps\_empty*:  $\text{accept\_eps } \{\} bs \longleftrightarrow \text{False}$   
**by** (*auto simp add: accept\_eps\_def step\_eps\_closure\_set\_def*)

**lemma** *accept\_eps\_split*:  $\text{accept\_eps } (R \cup S)\ bs \longleftrightarrow \text{accept\_eps } R\ bs \vee \text{accept\_eps } S\ bs$   
**by** (*auto simp add: accept\_eps\_def step\_eps\_closure\_set\_split*)

**lemma** *accept\_eps\_Un*:  $\text{accept\_eps } (\bigcup x \in X. R\ x)\ bs \longleftrightarrow (\exists x \in X. \text{accept\_eps } (R\ x)\ bs)$   
**by** (*auto simp add: accept\_eps\_def step\_eps\_closure\_set\_def*)

**qualified definition** *accept* :: *state set*  $\Rightarrow$  *bool* **where**  
*accept*  $R \longleftrightarrow \text{accept\_eps } R\ []$

**qualified definition** *run\_accept\_eps* :: *state set*  $\Rightarrow$  *bool list list*  $\Rightarrow$  *bool list*  $\Rightarrow$  *bool* **where**  
*run\_accept\_eps*  $R\ bss\ bs = \text{accept\_eps } (\text{run } R\ bss)\ bs$

**lemma** *run\_accept\_eps\_empty*:  $\neg \text{run\_accept\_eps } \{\} bss\ bs$   
**unfolding** *run\_accept\_eps\_def* *run\_empty* *accept\_eps\_empty* **by** *simp*

**lemma** *run\_accept\_eps\_Nil*:  $\text{run\_accept\_eps } R\ []\ cs \longleftrightarrow \text{accept\_eps } R\ cs$   
**by** (*auto simp add: run\_accept\_eps\_def run\_Nil*)

**lemma** *run\_accept\_eps\_Cons*:  $\text{run\_accept\_eps } R \text{ (bs \# bss) cs} \longleftrightarrow \text{run\_accept\_eps (delta R bs) bss cs}$

**by** (*auto simp add: run\_accept\_eps\_def run\_Cons*)

**lemma** *run\_accept\_eps\_Cons\_delta\_cong*:  $\text{delta } R \text{ bs} = \text{delta } S \text{ bs} \implies \text{run\_accept\_eps } R \text{ (bs \# bss) cs} \longleftrightarrow \text{run\_accept\_eps } S \text{ (bs \# bss) cs}$

**unfolding** *run\_accept\_eps\_Cons* **by** *auto*

**lemma** *run\_accept\_eps\_Nil\_eps*:  $\text{run\_accept\_eps (step\_eps\_closure\_set } R \text{ bs) [] bs} \longleftrightarrow \text{run\_accept\_eps } R \text{ [] bs}$

**unfolding** *run\_accept\_eps\_Nil accept\_eps\_def step\_eps\_closure\_set\_idem* **by** (*rule refl*)

**lemma** *run\_accept\_eps\_Cons\_eps*:  $\text{run\_accept\_eps (step\_eps\_closure\_set } R \text{ cs) (cs \# css) bs} \longleftrightarrow \text{run\_accept\_eps } R \text{ (cs \# css) bs}$

**unfolding** *run\_accept\_eps\_Cons delta\_eps* **by** (*rule refl*)

**lemma** *run\_accept\_eps\_Nil\_eps\_split*:

**assumes** *step\_eps\_closure\_set*  $R \text{ bs} = R \cup S$  *step\_symb\_set*  $R = \{\}$   $qf \notin R$

**shows**  $\text{run\_accept\_eps } R \text{ [] bs} = \text{run\_accept\_eps } S \text{ [] bs}$

**unfolding** *Nil run\_accept\_eps\_Nil accept\_eps\_def assms(1)*

**using** *assms(3) step\_eps\_closure\_set\_refl step\_eps\_closure\_set\_flip[OF assms(1)]* **by** *auto*

**lemma** *run\_accept\_eps\_Cons\_eps\_split*:

**assumes** *step\_eps\_closure\_set*  $R \text{ cs} = R \cup S$  *step\_symb\_set*  $R = \{\}$   $qf \notin R$

**shows**  $\text{run\_accept\_eps } R \text{ (cs \# css) bs} = \text{run\_accept\_eps } S \text{ (cs \# css) bs}$

**unfolding** *run\_accept\_eps\_def Cons run\_eps\_split[OF assms(1,2)]* **by** (*rule refl*)

**lemma** *run\_accept\_eps\_split*:  $\text{run\_accept\_eps (} R \cup S \text{) bss bs} \longleftrightarrow$

$\text{run\_accept\_eps } R \text{ bss bs} \vee \text{run\_accept\_eps } S \text{ bss bs}$

**unfolding** *run\_accept\_eps\_def run\_split accept\_eps\_split* **by** *auto*

**lemma** *run\_accept\_eps\_Un*:  $\text{run\_accept\_eps } (\bigcup x \in X. R \ x) \text{ bss bs} \longleftrightarrow$

$(\exists x \in X. \text{run\_accept\_eps } (R \ x) \text{ bss bs})$

**unfolding** *run\_accept\_eps\_def run\_Un accept\_eps\_Un* **by** *simp*

**qualified definition** *run\_accept* :: *state set*  $\Rightarrow$  *bool list list*  $\Rightarrow$  *bool* **where**

*run\_accept*  $R \text{ bss} = \text{accept (run } R \text{ bss)}$

**end**

**definition** *iarray\_of\_list*  $xs = \text{IArray } xs$

**context** *fixes*

*transs* :: *transition iarray*

**and** *len* :: *nat*

**begin**

**qualified definition** *step\_eps'* :: *bool iarray*  $\Rightarrow$  *state*  $\Rightarrow$  *state*  $\Rightarrow$  *bool* **where**

*step\_eps'*  $bs \ q \ q' \longleftrightarrow q < \text{len} \wedge$

$(\text{case } \text{transs} \ !\! \ q \ \text{of } \text{eps\_trans } p \ n \Rightarrow n < \text{IArray.length } bs \wedge bs \ !\! \ n \wedge p = q'$

$\mid \text{split\_trans } p \ p' \Rightarrow p = q' \vee p' = q' \mid \_ \Rightarrow \text{False})$

**qualified definition** *step\_eps\_closure'* :: *bool iarray*  $\Rightarrow$  *state*  $\Rightarrow$  *state*  $\Rightarrow$  *bool* **where**

*step\_eps\_closure'*  $bs = (\text{step\_eps}' \ bs)^*$

**qualified definition** *step\_eps\_sucs'* :: *bool iarray*  $\Rightarrow$  *state*  $\Rightarrow$  *state set* **where**

*step\_eps\_sucs'*  $bs \ q = (\text{if } q < \text{len} \ \text{then}$

(*case transs !! q of eps\_trans p n ⇒ if n < IArray.length bs ∧ bs !! n then {p} else {}*  
| *split\_trans p p' ⇒ {p, p'} | \_ ⇒ {}*) else {})

**lemma** *step\_eps\_sucs'\_sound*:  $q' \in \text{step\_eps\_sucs}' \text{ bs } q \longleftrightarrow \text{step\_eps}' \text{ bs } q \text{ } q'$   
**by** (*auto simp add: step\_eps\_sucs'\_def step\_eps'\_def split: transition.splits*)

**qualified definition** *step\_eps\_set'* :: *bool iarray ⇒ state set ⇒ state set where*  
*step\_eps\_set' bs R =  $\bigcup$ (step\_eps\_sucs' bs ' R)*

**lemma** *step\_eps\_set'\_sound*:  $\text{step\_eps\_set}' \text{ bs } R = \{q'. \exists q \in R. \text{step\_eps}' \text{ bs } q \text{ } q'\}$   
**using** *step\_eps\_sucs'\_sound* **by** (*auto simp add: step\_eps\_set'\_def*)

**qualified definition** *step\_eps\_closure\_set'* :: *state set ⇒ bool iarray ⇒ state set where*  
*step\_eps\_closure\_set' R bs =  $\bigcup$ (( $\lambda q. \{q'. \text{step\_eps\_closure}' \text{ bs } q \text{ } q'\}$ ) ' R)*

**lemma** *step\_eps\_closure\_set'\_code*[*code*]:  
*step\_eps\_closure\_set' R bs =*  
(*let R' = R  $\cup$  step\_eps\_set' bs R in if R = R' then R else step\_eps\_closure\_set' R' bs*)  
**using** *rtranclp\_closed*  
**by** (*auto simp add: step\_eps\_closure\_set'\_def step\_eps\_closure'\_def step\_eps\_set'\_sound Let\_def*)

**qualified definition** *step\_symb\_sucs'* :: *state ⇒ state set where*  
*step\_symb\_sucs' q = (if q < len then*  
(*case transs !! q of symb\_trans p ⇒ {p} | \_ ⇒ {}*) else {})

**qualified definition** *step\_symb\_set'* :: *state set ⇒ state set where*  
*step\_symb\_set' R =  $\bigcup$ (step\_symb\_sucs' ' R)*

**qualified definition** *delta'* :: *state set ⇒ bool iarray ⇒ state set where*  
*delta' R bs = step\_symb\_set' (step\_eps\_closure\_set' R bs)*

**qualified definition** *accept\_eps'* :: *state set ⇒ bool iarray ⇒ bool where*  
*accept\_eps' R bs  $\longleftrightarrow$  (len  $\in$  step\_eps\_closure\_set' R bs)*

**qualified definition** *accept'* :: *state set ⇒ bool where*  
*accept' R  $\longleftrightarrow$  accept\_eps' R (iarray\_of\_list [])*

**qualified definition** *run'* :: *state set ⇒ bool iarray list ⇒ state set where*  
*run' R bss = foldl delta' R bss*

**qualified definition** *run\_accept\_eps'* :: *state set ⇒ bool iarray list ⇒ bool iarray ⇒ bool where*  
*run\_accept\_eps' R bss bs = accept\_eps' (run' R bss) bs*

**qualified definition** *run\_accept'* :: *state set ⇒ bool iarray list ⇒ bool where*  
*run\_accept' R bss = accept' (run' R bss)*

**end**

**locale** *nfa\_array* =  
**fixes** *transs* :: *transition list*  
**and** *transs'* :: *transition iarray*  
**and** *len* :: *nat*  
**assumes** *transs\_eq*: *transs' = IArray transs*  
**and** *len\_def*: *len = length transs*  
**begin**

**abbreviation** *step\_eps*  $\equiv$  *NFA.step\_eps 0 transs*

**abbreviation** *step\_eps'*  $\equiv$  *NFA.step\_eps' transs' len*

**abbreviation**  $step\_eps\_closure \equiv NFA.step\_eps\_closure\ 0\ transs$   
**abbreviation**  $step\_eps\_closure' \equiv NFA.step\_eps\_closure'\ transs'\ len$   
**abbreviation**  $step\_eps\_sucs \equiv NFA.step\_eps\_sucs\ 0\ transs$   
**abbreviation**  $step\_eps\_sucs' \equiv NFA.step\_eps\_sucs'\ transs'\ len$   
**abbreviation**  $step\_eps\_set \equiv NFA.step\_eps\_set\ 0\ transs$   
**abbreviation**  $step\_eps\_set' \equiv NFA.step\_eps\_set'\ transs'\ len$   
**abbreviation**  $step\_eps\_closure\_set \equiv NFA.step\_eps\_closure\_set\ 0\ transs$   
**abbreviation**  $step\_eps\_closure\_set' \equiv NFA.step\_eps\_closure\_set'\ transs'\ len$   
**abbreviation**  $step\_symb\_sucs \equiv NFA.step\_symb\_sucs\ 0\ transs$   
**abbreviation**  $step\_symb\_sucs' \equiv NFA.step\_symb\_sucs'\ transs'\ len$   
**abbreviation**  $step\_symb\_set \equiv NFA.step\_symb\_set\ 0\ transs$   
**abbreviation**  $step\_symb\_set' \equiv NFA.step\_symb\_set'\ transs'\ len$   
**abbreviation**  $delta \equiv NFA.delta\ 0\ transs$   
**abbreviation**  $delta' \equiv NFA.delta'\ transs'\ len$   
**abbreviation**  $accept\_eps \equiv NFA.accept\_eps\ 0\ len\ transs$   
**abbreviation**  $accept\_eps' \equiv NFA.accept\_eps'\ transs'\ len$   
**abbreviation**  $accept \equiv NFA.accept\ 0\ len\ transs$   
**abbreviation**  $accept' \equiv NFA.accept'\ transs'\ len$   
**abbreviation**  $run \equiv NFA.run\ 0\ transs$   
**abbreviation**  $run' \equiv NFA.run'\ transs'\ len$   
**abbreviation**  $run\_accept\_eps \equiv NFA.run\_accept\_eps\ 0\ len\ transs$   
**abbreviation**  $run\_accept\_eps' \equiv NFA.run\_accept\_eps'\ transs'\ len$   
**abbreviation**  $run\_accept \equiv NFA.run\_accept\ 0\ len\ transs$   
**abbreviation**  $run\_accept' \equiv NFA.run\_accept'\ transs'\ len$

**lemma**  $q\_in\_SQ: q \in NFA.SQ\ 0\ transs \longleftrightarrow q < len$   
**using**  $len\_def$   
**by**  $(auto\ simp: NFA.SQ\_def)$

**lemma**  $step\_eps'\_eq: bs' = IArray\ bs \implies step\_eps\ bs\ q\ q' \longleftrightarrow step\_eps'\ bs'\ q\ q'$   
**by**  $(auto\ simp: NFA.step\_eps\_def\ NFA.step\_eps'\_def\ q\_in\_SQ\ transs\_eq\ split: transition.splits)$

**lemma**  $step\_eps\_closure'\_eq: bs' = IArray\ bs \implies step\_eps\_closure\ bs\ q\ q' \longleftrightarrow step\_eps\_closure'\ bs'\ q\ q'$   
**proof** –  
**assume**  $lassm: bs' = IArray\ bs$   
**have**  $step\_eps\_eq\_folded: step\_eps\ bs = step\_eps'\ bs'$   
**using**  $step\_eps'\_eq[OF\ lassm]$   
**by**  $auto$   
**show**  $?thesis$   
**by**  $(auto\ simp: NFA.step\_eps\_closure\_def\ NFA.step\_eps\_closure'\_def\ step\_eps\_eq\_folded)$   
**qed**

**lemma**  $step\_eps\_sucs'\_eq: bs' = IArray\ bs \implies step\_eps\_sucs\ bs\ q = step\_eps\_sucs'\ bs'\ q$   
**by**  $(auto\ simp: NFA.step\_eps\_sucs\_def\ NFA.step\_eps\_sucs'\_def\ q\_in\_SQ\ transs\_eq\ split: transition.splits)$

**lemma**  $step\_eps\_set'\_eq: bs' = IArray\ bs \implies step\_eps\_set\ bs\ R = step\_eps\_set'\ bs'\ R$   
**by**  $(auto\ simp: NFA.step\_eps\_set\_def\ NFA.step\_eps\_set'\_def\ step\_eps\_sucs'\_eq)$

**lemma**  $step\_eps\_closure\_set'\_eq: bs' = IArray\ bs \implies step\_eps\_closure\_set\ R\ bs = step\_eps\_closure\_set'\ R\ bs'$   
**by**  $(auto\ simp: NFA.step\_eps\_closure\_set\_def\ NFA.step\_eps\_closure\_set'\_def\ step\_eps\_closure'\_eq)$

**lemma**  $step\_symb\_sucs'\_eq: bs' = IArray\ bs \implies step\_symb\_sucs\ R = step\_symb\_sucs'\ R$   
**by**  $(auto\ simp: NFA.step\_symb\_sucs\_def\ NFA.step\_symb\_sucs'\_def\ q\_in\_SQ\ transs\_eq\ split: transition.splits)$

**lemma** *step\_symb\_set'\_eq*:  $bs' = IArray\ bs \implies step\_symb\_set\ R = step\_symb\_set'\ R$   
**by** (*auto simp*: *step\_symb\_set\_code NFA.step\_symb\_set'\_def step\_symb\_sucs'\_eq*)

**lemma** *delta'\_eq*:  $bs' = IArray\ bs \implies delta\ R\ bs = delta'\ R\ bs'$   
**by** (*auto simp*: *NFA.delta\_def NFA.delta'\_def step\_eps\_closure\_set'\_eq step\_symb\_set'\_eq*)

**lemma** *accept\_eps'\_eq*:  $bs' = IArray\ bs \implies accept\_eps\ R\ bs = accept\_eps'\ R\ bs'$   
**by** (*auto simp*: *NFA.accept\_eps\_def NFA.accept\_eps'\_def step\_eps\_closure\_set'\_eq*)

**lemma** *accept'\_eq*:  $accept\ R = accept'\ R$   
**by** (*auto simp*: *NFA.accept\_def NFA.accept'\_def accept\_eps'\_eq iarray\_of\_list\_def*)

**lemma** *run'\_eq*:  $bss' = map\ IArray\ bss \implies run\ R\ bss = run'\ R\ bss'$   
**by** (*induction bss arbitrary: R bss'*) (*auto simp*: *NFA.run\_def NFA.run'\_def delta'\_eq*)

**lemma** *run\_accept\_eps'\_eq*:  $bss' = map\ IArray\ bss \implies bs' = IArray\ bs \implies$   
 $run\_accept\_eps\ R\ bss\ bs \longleftrightarrow run\_accept\_eps'\ R\ bss'\ bs'$   
**by** (*auto simp*: *NFA.run\_accept\_eps\_def NFA.run\_accept\_eps'\_def accept\_eps'\_eq run'\_eq*)

**lemma** *run\_accept'\_eq*:  $bss' = map\ IArray\ bss \implies$   
 $run\_accept\ R\ bss \longleftrightarrow run\_accept'\ R\ bss'$   
**by** (*auto simp*: *NFA.run\_accept\_def NFA.run\_accept'\_def run'\_eq accept'\_eq*)

**end**

**locale** *nfa* =  
**fixes**  $q0 :: nat$   
**and**  $qf :: nat$   
**and**  $transs :: transition\ list$   
**assumes**  $state\_closed: \bigwedge t. t \in set\ transs \implies state\_set\ t \subseteq NFA.Q\ q0\ qf\ transs$   
**and**  $transs\_not\_Nil: transs \neq []$   
**and**  $qf\_not\_in\_SQ: qf \notin NFA.SQ\ q0\ transs$   
**begin**

**abbreviation**  $SQ \equiv NFA.SQ\ q0\ transs$   
**abbreviation**  $Q \equiv NFA.Q\ q0\ qf\ transs$   
**abbreviation**  $nfa\_fmla\_set \equiv NFA.nfa\_fmla\_set\ transs$   
**abbreviation**  $step\_eps \equiv NFA.step\_eps\ q0\ transs$   
**abbreviation**  $step\_eps\_sucs \equiv NFA.step\_eps\_sucs\ q0\ transs$   
**abbreviation**  $step\_eps\_set \equiv NFA.step\_eps\_set\ q0\ transs$   
**abbreviation**  $step\_eps\_closure \equiv NFA.step\_eps\_closure\ q0\ transs$   
**abbreviation**  $step\_eps\_closure\_set \equiv NFA.step\_eps\_closure\_set\ q0\ transs$   
**abbreviation**  $step\_symb \equiv NFA.step\_symb\ q0\ transs$   
**abbreviation**  $step\_symb\_sucs \equiv NFA.step\_symb\_sucs\ q0\ transs$   
**abbreviation**  $step\_symb\_set \equiv NFA.step\_symb\_set\ q0\ transs$   
**abbreviation**  $delta \equiv NFA.delta\ q0\ transs$   
**abbreviation**  $run \equiv NFA.run\ q0\ transs$   
**abbreviation**  $accept\_eps \equiv NFA.accept\_eps\ q0\ qf\ transs$   
**abbreviation**  $run\_accept\_eps \equiv NFA.run\_accept\_eps\ q0\ qf\ transs$

**lemma** *Q\_diff\_qf\_SQ*:  $Q - \{qf\} = SQ$   
**using** *qf\_not\_in\_SQ* **by** (*auto simp add: NFA.Q\_def*)

**lemma** *q0\_sub\_SQ*:  $\{q0\} \subseteq SQ$   
**using** *transs\_not\_Nil* **by** (*auto simp add: NFA.SQ\_def*)

**lemma** *q0\_sub\_Q*:  $\{q0\} \subseteq Q$   
**using** *q0\_sub\_SQ SQ\_sub\_Q* **by** *auto*

```

lemma step_eps_closed: step_eps bs q q'  $\implies$  q' ∈ Q
  using transs_q_in_set_state_closed
  by (fastforce simp add: NFA.step_eps_def split: transition.splits)

lemma step_eps_set_closed: step_eps_set bs R  $\subseteq$  Q
  using step_eps_closed by (auto simp add: step_eps_set_sound)

lemma step_eps_closure_closed: step_eps_closure bs q q'  $\implies$  q ≠ q' ⟹ q' ∈ Q
  unfolding NFA.step_eps_closure_def
  apply (induction q q' rule: rtranclp.induct) using step_eps_closed by auto

lemma step_eps_closure_set_closed_union: step_eps_closure_set R bs  $\subseteq$  R ∪ Q
  using step_eps_closure_closed by (auto simp add: NFA.step_eps_closure_set_def NFA.step_eps_closure_def)

lemma step_eps_closure_set_closed: R  $\subseteq$  Q  $\implies$  step_eps_closure_set R bs  $\subseteq$  Q
  using step_eps_closure_set_closed_union by auto

lemma step_symb_closed: step_symb q q'  $\implies$  q' ∈ Q
  using transs_q_in_set_state_closed
  by (fastforce simp add: NFA.step_symb_def split: transition.splits)

lemma step_symb_set_closed: step_symb_set R  $\subseteq$  Q
  using step_symb_closed by (auto simp add: NFA.step_symb_set_def)

lemma step_symb_set_qf: step_symb_set {qf} = {}
  using qf_not_in_SQ step_symb_set_proj[of _ _ {qf}] step_symb_set_empty by auto

lemma delta_closed: delta R bs  $\subseteq$  Q
  using step_symb_set_closed by (auto simp add: NFA.delta_def)

lemma run_closed_Cons: run R (bs # bss)  $\subseteq$  Q
  unfolding NFA.run_def
  using delta_closed by (induction bss arbitrary: R bs) auto

lemma run_closed: R  $\subseteq$  Q  $\implies$  run R bss  $\subseteq$  Q
  using run_Nil run_closed_Cons by (cases bss) auto

lemma step_eps_qf: step_eps bs qf q  $\longleftrightarrow$  False
  using qf_not_in_SQ step_eps_dest by force

lemma step_symb_qf: step_symb qf q  $\longleftrightarrow$  False
  using qf_not_in_SQ step_symb_dest by force

lemma step_eps_closure_qf: step_eps_closure bs q q'  $\implies$  q = qf ⟹ q = q'
  unfolding NFA.step_eps_closure_def
  apply (induction q q' rule: rtranclp.induct) using step_eps_qf by auto

lemma step_eps_closure_set_qf: step_eps_closure_set {qf} bs = {qf}
  using step_eps_closure_qf unfolding NFA.step_eps_closure_set_def NFA.step_eps_closure_def by
auto

lemma delta_qf: delta {qf} bs = {}
  using step_eps_closure_qf step_symb_qf
  by (auto simp add: NFA.delta_def NFA.step_symb_set_def NFA.step_eps_closure_set_def)

```

**lemma** *run\_qf\_many*:  $\text{run } \{qf\} (bs \# bss) = \{\}$   
**unfolding** *run\_Cons delta\_qf run\_empty* **by** (*rule refl*)

**lemma** *run\_accept\_eps\_qf\_many*:  $\text{run\_accept\_eps } \{qf\} (bs \# bss) cs \longleftrightarrow \text{False}$   
**unfolding** *NFA.run\_accept\_eps\_def* **using** *run\_qf\_many accept\_eps\_empty* **by** *simp*

**lemma** *run\_accept\_eps\_qf\_one*:  $\text{run\_accept\_eps } \{qf\} [] bs \longleftrightarrow \text{True}$   
**unfolding** *NFA.run\_accept\_eps\_def NFA.accept\_eps\_def* **using** *run\_Nil step\_eps\_closure\_set\_refl*  
**by** *force*

**end**

**locale** *nfa\_cong* = *nfa* *q0* *qf* *transs* + *nfa'*: *nfa* *q0'* *qf'* *transs'*  
**for** *q0* *q0'* *qf* *qf'* *transs* *transs'* +  
**assumes** *SQ\_sub*:  $nfa'.SQ \subseteq SQ$  **and**  
*qf\_eq*:  $qf = qf'$  **and**  
*transs\_eq*:  $\bigwedge q. q \in nfa'.SQ \implies \text{transs } ! (q - q0) = \text{transs}' ! (q - q0')$   
**begin**

**lemma** *q\_Q\_SQ\_nfa'\_SQ*:  $q \in nfa'.Q \implies q \in SQ \longleftrightarrow q \in nfa'.SQ$   
**using** *SQ\_sub qf\_not\_in\_SQ qf\_eq* **by** (*auto simp add: NFA.Q\_def*)

**lemma** *step\_eps\_cong*:  $q \in nfa'.Q \implies \text{step\_eps } bs \ q \ q' \longleftrightarrow nfa'.\text{step\_eps } bs \ q \ q'$   
**using** *q\_Q\_SQ\_nfa'\_SQ transs\_eq* **by** (*auto simp add: NFA.step\_eps\_def*)

**lemma** *eps\_nfa'\_step\_eps\_closure*:  $\text{step\_eps\_closure } bs \ q \ q' \implies q \in nfa'.Q \implies$   
 $q' \in nfa'.Q \wedge nfa'.\text{step\_eps\_closure } bs \ q \ q'$   
**unfolding** *NFA.step\_eps\_closure\_def*  
**apply** (*induction q q' rule: rtranclp.induct*)  
**using** *nfa'.step\_eps\_closure\_closed step\_eps\_cong* **by** (*auto simp add: NFA.step\_eps\_closure\_def*)

**lemma** *nfa'\_eps\_step\_eps\_closure*:  $nfa'.\text{step\_eps\_closure } bs \ q \ q' \implies q \in nfa'.Q \implies$   
 $q' \in nfa'.Q \wedge \text{step\_eps\_closure } bs \ q \ q'$   
**unfolding** *NFA.step\_eps\_closure\_def*  
**apply** (*induction q q' rule: rtranclp.induct*)  
**using** *nfa'.step\_eps\_closure\_closed step\_eps\_cong*  
**by** (*auto simp add: NFA.step\_eps\_closure\_def intro: rtranclp.intros(2)*)

**lemma** *step\_eps\_closure\_set\_cong*:  $R \subseteq nfa'.Q \implies \text{step\_eps\_closure\_set } R \ bs =$   
 $nfa'.\text{step\_eps\_closure\_set } R \ bs$   
**using** *eps\_nfa'\_step\_eps\_closure nfa'\_eps\_step\_eps\_closure*  
**by** (*fastforce simp add: NFA.step\_eps\_closure\_set\_def*)

**lemma** *step\_symb\_cong*:  $q \in nfa'.Q \implies \text{step\_symb } q \ q' \longleftrightarrow nfa'.\text{step\_symb } q \ q'$   
**using** *q\_Q\_SQ\_nfa'\_SQ transs\_eq* **by** (*auto simp add: NFA.step\_symb\_def*)

**lemma** *step\_symb\_set\_cong*:  $R \subseteq nfa'.Q \implies \text{step\_symb\_set } R = nfa'.\text{step\_symb\_set } R$   
**using** *step\_symb\_cong* **by** (*auto simp add: NFA.step\_symb\_set\_def*)

**lemma** *delta\_cong*:  $R \subseteq nfa'.Q \implies \text{delta } R \ bs = nfa'.\text{delta } R \ bs$   
**using** *step\_symb\_set\_cong nfa'.step\_eps\_closure\_set\_closed*  
**by** (*auto simp add: NFA.delta\_def step\_eps\_closure\_set\_cong*)

**lemma** *run\_cong*:  $R \subseteq nfa'.Q \implies \text{run } R \ bss = nfa'.\text{run } R \ bss$   
**unfolding** *NFA.run\_def*  
**using** *nfa'.delta\_closed delta\_cong* **by** (*induction bss arbitrary: R*) *auto*

**lemma** *accept\_eps\_cong*:  $R \subseteq nfa'.Q \implies \text{accept\_eps } R \ bs \longleftrightarrow nfa'.\text{accept\_eps } R \ bs$

**unfolding**  $NFA.accept\_eps\_def$  **using**  $step\_eps\_closure\_set\_cong$   $qf\_eq$  **by**  $auto$

**lemma**  $run\_accept\_eps\_cong$ :

**assumes**  $R \subseteq nfa'.Q$

**shows**  $run\_accept\_eps R bss bs \longleftrightarrow nfa'.run\_accept\_eps R bss bs$

**unfolding**  $NFA.run\_accept\_eps\_def$   $run\_cong[OF\ assms]$

$accept\_eps\_cong[OF\ nfa'.run\_closed[OF\ assms]]$  **by**  $simp$

**end**

**fun**  $list\_split :: 'a\ list \Rightarrow ('a\ list \times 'a\ list)\ set$  **where**

$list\_split [] = \{\}$

$| list\_split (x \# xs) = \{([], x \# xs)\} \cup (\bigcup (ys, zs) \in list\_split\ xs. \{(x \# ys, zs)\})$

**lemma**  $list\_split\_unfold$ :  $(\bigcup (ys, zs) \in list\_split (x \# xs). f\ ys\ zs) =$

$f [] (x \# xs) \cup (\bigcup (ys, zs) \in list\_split\ xs. f (x \# ys) zs)$

**by**  $(induction\ xs)\ auto$

**lemma**  $list\_split\_def$ :  $list\_split\ xs = (\bigcup n < length\ xs. \{(take\ n\ xs, drop\ n\ xs)\})$

**using**  $less\_Suc\_eq\_0\_disj$  **by**  $(induction\ xs\ rule: list\_split.induct)\ auto+$

**locale**  $nfa\_cong' = nfa\ q0\ qf\ transs + nfa'$ :  $nfa\ q0' qf' transs'$

**for**  $q0\ q0' qf\ qf' transs\ transs' +$

**assumes**  $SQ\_sub$ :  $nfa'.SQ \subseteq SQ$  **and**

$qf'\_in\_SQ$ :  $qf' \in SQ$  **and**

$transs\_eq$ :  $\bigwedge q. q \in nfa'.SQ \implies transs ! (q - q0) = transs' ! (q - q0')$

**begin**

**lemma**  $nfa'\_Q\_sub\_Q$ :  $nfa'.Q \subseteq Q$

**unfolding**  $NFA.Q\_def$  **using**  $SQ\_sub\ qf'\_in\_SQ$  **by**  $auto$

**lemma**  $q\_SQ\_SQ\_nfa'\_SQ$ :  $q \in nfa'.SQ \implies q \in SQ \longleftrightarrow q \in nfa'.SQ$

**using**  $SQ\_sub$  **by**  $auto$

**lemma**  $step\_eps\_cong\_SQ$ :  $q \in nfa'.SQ \implies step\_eps\ bs\ q\ q' \longleftrightarrow nfa'.step\_eps\ bs\ q\ q'$

**using**  $q\_SQ\_SQ\_nfa'\_SQ\ transs\_eq$  **by**  $(auto\ simp\ add: NFA.step\_eps\_def)$

**lemma**  $step\_eps\_cong\_Q$ :  $q \in nfa'.Q \implies nfa'.step\_eps\ bs\ q\ q' \implies step\_eps\ bs\ q\ q'$

**using**  $SQ\_sub\ transs\_eq$  **by**  $(auto\ simp\ add: NFA.step\_eps\_def)$

**lemma**  $nfa'\_step\_eps\_closure\_cong$ :  $nfa'.step\_eps\_closure\ bs\ q\ q' \implies q \in nfa'.Q \implies$

$step\_eps\_closure\ bs\ q\ q'$

**unfolding**  $NFA.step\_eps\_closure\_def$

**apply**  $(induction\ q\ q'\ rule: rtranclp.induct)$

**using**  $NFA.Q\_def\ NFA.step\_eps\_closure\_def$

**by**  $(auto\ simp\ add: rtranclp.rtrancl\_into\_rtrancl\ step\_eps\_cong\_SQ\ step\_eps\_dest)$

**lemma**  $nfa'\_step\_eps\_closure\_set\_sub$ :  $R \subseteq nfa'.Q \implies nfa'.step\_eps\_closure\_set\ R\ bs \subseteq$

$step\_eps\_closure\_set\ R\ bs$

**unfolding**  $NFA.step\_eps\_closure\_set\_def$

**using**  $nfa'\_step\_eps\_closure\_cong$  **by**  $fastforce$

**lemma**  $eps\_nfa'\_step\_eps\_closure\_cong$ :  $step\_eps\_closure\ bs\ q\ q' \implies q \in nfa'.Q \implies$

$(q' \in nfa'.Q \wedge nfa'.step\_eps\_closure\ bs\ q\ q') \vee$

$(nfa'.step\_eps\_closure\ bs\ q\ qf' \wedge step\_eps\_closure\ bs\ qf'\ q')$

**unfolding**  $NFA.step\_eps\_closure\_def$

**apply**  $(induction\ q\ q'\ rule: rtranclp.induct)$

**using**  $nfa'.step\_eps\_closure\_closed\ nfa'.step\_eps\_closed\ step\_eps\_cong\_SQ\ NFA.Q\_def$

by (auto simp add: intro: rtranclp.rtrancl\_into\_rtrancl) fastforce+

**lemma** *nfa'\_eps\_step\_eps\_closure\_cong*:  $nfa'.step\_eps\_closure\ bs\ q\ q' \implies q \in nfa'.Q \implies q' \in nfa'.Q \wedge step\_eps\_closure\ bs\ q\ q'$   
**unfolding** *NFA.step\_eps\_closure\_def*  
**apply** (*induction*  $q\ q'$  *rule*: *rtranclp.induct*)  
**using** *nfa'.step\_eps\_closed step\_eps\_cong\_Q*  
**by** (*auto* *intro*: *rtranclp.intros(2)*)

**lemma** *step\_eps\_closure\_set\_cong\_reach*:  $R \subseteq nfa'.Q \implies qf' \in nfa'.step\_eps\_closure\_set\ R\ bs \implies step\_eps\_closure\_set\ R\ bs = nfa'.step\_eps\_closure\_set\ R\ bs \cup step\_eps\_closure\_set\ \{qf'\}\ bs$   
**using** *eps\_nfa'\_step\_eps\_closure\_cong nfa'\_eps\_step\_eps\_closure\_cong rtranclp\_trans[of step\_eps bs]*  
**unfolding** *NFA.step\_eps\_closure\_set\_def NFA.step\_eps\_closure\_def*  
**by** *auto* *blast+*

**lemma** *step\_eps\_closure\_set\_cong\_unreach*:  $R \subseteq nfa'.Q \implies qf' \notin nfa'.step\_eps\_closure\_set\ R\ bs \implies step\_eps\_closure\_set\ R\ bs = nfa'.step\_eps\_closure\_set\ R\ bs$   
**using** *eps\_nfa'\_step\_eps\_closure\_cong nfa'\_eps\_step\_eps\_closure\_cong*  
**unfolding** *NFA.step\_eps\_closure\_set\_def NFA.step\_eps\_closure\_def*  
**by** *auto* *blast+*

**lemma** *step\_symb\_cong\_SQ*:  $q \in nfa'.SQ \implies step\_symb\ q\ q' \longleftrightarrow nfa'.step\_symb\ q\ q'$   
**using**  $q\_SQ\_SQ\_nfa'\_SQ\ transs\_eq$  **by** (*auto* *simp* *add*: *NFA.step\_symb\_def*)

**lemma** *step\_symb\_cong\_Q*:  $nfa'.step\_symb\ q\ q' \implies step\_symb\ q\ q'$   
**using** *SQ\_sub transs\_eq* **by** (*auto* *simp* *add*: *NFA.step\_symb\_def*)

**lemma** *step\_symb\_set\_cong\_SQ*:  $R \subseteq nfa'.SQ \implies step\_symb\_set\ R = nfa'.step\_symb\_set\ R$   
**using** *step\_symb\_cong\_SQ* **by** (*auto* *simp* *add*: *NFA.step\_symb\_set\_def*)

**lemma** *step\_symb\_set\_cong\_Q*:  $nfa'.step\_symb\_set\ R \subseteq step\_symb\_set\ R$   
**using** *step\_symb\_cong\_Q* **by** (*auto* *simp* *add*: *NFA.step\_symb\_set\_def*)

**lemma** *delta\_cong\_unreach*:  
**assumes**  $R \subseteq nfa'.Q \neg nfa'.accept\_eps\ R\ bs$   
**shows**  $delta\ R\ bs = nfa'.delta\ R\ bs$   
**proof** –  
**have**  $nfa'.step\_eps\_closure\_set\ R\ bs \subseteq nfa'.SQ$   
**using** *nfa'.step\_eps\_closure\_set\_closed[OF assms(1), unfolded NFA.Q\_def]*  
*assms(2)[unfolded NFA.accept\_eps\_def]* **by** *auto*  
**then show** *?thesis*  
**unfolding** *NFA.accept\_eps\_def NFA.delta\_def* **using** *step\_symb\_set\_cong\_SQ*  
*step\_eps\_closure\_set\_cong\_unreach[OF assms(1) assms(2)[unfolded NFA.accept\_eps\_def]]*  
**by** *auto*  
**qed**

**lemma** *nfa'\_delta\_sub\_delta*:  
**assumes**  $R \subseteq nfa'.Q$   
**shows**  $nfa'.delta\ R\ bs \subseteq delta\ R\ bs$   
**unfolding** *NFA.delta\_def*  
**using** *step\_symb\_set\_mono[OF nfa'\_step\_eps\_closure\_set\_sub[OF assms]] step\_symb\_set\_cong\_Q*  
**by** *fastforce*

**lemma** *delta\_cong\_reach*:  
**assumes**  $R \subseteq nfa'.Q\ nfa'.accept\_eps\ R\ bs$   
**shows**  $delta\ R\ bs = nfa'.delta\ R\ bs \cup delta\ \{qf'\}\ bs$   
**proof** (*rule* *set\_eqI*, *rule* *iffI*)

```

fix q
assume assm:  $q \in \text{delta } R \text{ } bs$ 
have nfa'.step_eps_closure_set R bs =
  nfa'.step_eps_closure_set R bs - {qf'}  $\cup$  {qf'}
  using assms(2)[unfolded NFA.accept_eps_def] by auto
from assm have  $q \in \text{step\_symb\_set } (nfa'.\text{step\_eps\_closure\_set } R \text{ } bs - \{qf'\}) \cup$ 
  step\_symb\_set {qf'}  $\cup$  delta {qf'} bs
  unfolding NFA.delta_def step_eps_closure_set_cong_reach[OF assms(1)
    assms(2)[unfolded NFA.accept_eps_def]] step_symb_set_split[symmetric]
    nfa'_eps_diff_Un[symmetric] by simp
then have  $q \in \text{step\_symb\_set } (nfa'.\text{step\_eps\_closure\_set } R \text{ } bs - \{qf'\}) \cup$  delta {qf'} bs
  using step_symb_set_mono[of {qf'} step_eps_closure_set {qf'} bs,
    OF step_eps_closure_set_refl, unfolded NFA.delta_def[symmetric]]
    delta_step_symb_set_absorb by blast
then show  $q \in nfa'.\text{delta } R \text{ } bs \cup \text{delta } \{qf'\} \text{ } bs$ 
  unfolding NFA.delta_def
  using nfa'.step_eps_closure_set_closed[OF assms(1), unfolded NFA.Q_def]
    step_symb_set_cong_SQ[of nfa'.step_eps_closure_set R bs - {qf'}]
    step_symb_set_mono by blast
next
fix q
assume  $q \in nfa'.\text{delta } R \text{ } bs \cup \text{delta } \{qf'\} \text{ } bs$ 
then show  $q \in \text{delta } R \text{ } bs$ 
  using nfa'_delta_sub_delta[OF assms(1)] delta_sub_eps_mono[of {qf'}  $\_ \_$  R bs]
    assms(2)[unfolded NFA.accept_eps_def] nfa'_step_eps_closure_set_sub[OF assms(1)]
  by fastforce
qed

lemma run_cong:
assumes  $R \subseteq nfa'.Q$ 
shows  $\text{run } R \text{ } bss = nfa'.\text{run } R \text{ } bss \cup (\bigcup (css, css') \in \text{list\_split } bss.$ 
  if nfa'.run_accept_eps R css (hd css') then run {qf'} css' else {} $)$ 
using assms
proof (induction bss arbitrary: R rule: list_split.induct)
case 1
then show ?case
  using run_Nil by simp
next
case (2 x xs)
show ?case
  apply (cases nfa'.accept_eps R x)
  unfolding run_Cons delta_cong_reach[OF 2(2)]
    delta_cong_unreach[OF 2(2)] run_split_run_accept_eps_Nil run_accept_eps_Cons
    list_split_unfold[of  $\lambda$  ys zs. if nfa'.run_accept_eps R ys (hd zs)
    then run {qf'} zs else {}] x xs] using 2(1)[of nfa'.delta R x,
    OF nfa'.delta_closed, unfolded run_accept_eps_Nil] by auto
qed

lemma run_cong_Cons_sub:
assumes  $R \subseteq nfa'.Q$   $\text{delta } \{qf'\} \text{ } bs \subseteq nfa'.\text{delta } R \text{ } bs$ 
shows  $\text{run } R \text{ } (bs \# bss) = nfa'.\text{run } R \text{ } (bs \# bss) \cup$ 
   $(\bigcup (css, css') \in \text{list\_split } bss.$ 
  if nfa'.run_accept_eps (nfa'.delta R bs) css (hd css') then run {qf'} css' else {} $)$ 
unfolding run_Cons using run_cong[OF nfa'.delta_closed]
  delta_cong_reach[OF assms(1)] delta_cong_unreach[OF assms(1)]
by (cases nfa'.accept_eps R bs) (auto simp add: Un_absorb2[OF assms(2)])

lemma accept_eps_nfa'_run:

```

**assumes**  $R \subseteq nfa'.Q$   
**shows**  $accept\_eps (nfa'.run R bss) bs \longleftrightarrow$   
 $nfa'.accept\_eps (nfa'.run R bss) bs \wedge accept\_eps (run \{qf'\} []) bs$   
**unfolding**  $NFA.accept\_eps\_def$   $run\_Nil$   
**using**  $step\_eps\_closure\_set\_cong\_reach[OF nfa'.run\_closed[OF assms]]$   
 $step\_eps\_closure\_set\_cong\_unreach[OF nfa'.run\_closed[OF assms]]$   $qf\_not\_in\_SQ$   
 $qf'\_in\_SQ$   $nfa'.step\_eps\_closure\_set\_closed[OF nfa'.run\_closed[OF assms]]$ ,  
 $unfolded$   $NFA.Q\_def$   $SQ\_sub$   
**by**  $(cases\ qf' \in nfa'.step\_eps\_closure\_set (nfa'.run R bss) bs)$   $fastforce+$

**lemma**  $run\_accept\_eps\_cong$ :

**assumes**  $R \subseteq nfa'.Q$   
**shows**  $run\_accept\_eps R bss bs \longleftrightarrow (nfa'.run\_accept\_eps R bss bs \wedge run\_accept\_eps \{qf'\} [] bs) \vee$   
 $(\exists (css, css') \in list\_split\ bss. nfa'.run\_accept\_eps R css (hd\ css') \wedge$   
 $run\_accept\_eps \{qf'\} css' bs)$   
**unfolding**  $NFA.run\_accept\_eps\_def$   $run\_cong[OF assms]$   $accept\_eps\_split$   
 $accept\_eps\_Un$   $accept\_eps\_nfa'\_run[OF assms]$   
**using**  $accept\_eps\_empty$  **by**  $(auto\ split: if\_splits)+$

**lemma**  $run\_accept\_eps\_cong\_Cons\_sub$ :

**assumes**  $R \subseteq nfa'.Q$   $delta \{qf'\} bs \subseteq nfa'.delta R bs$   
**shows**  $run\_accept\_eps R (bs \# bss) cs \longleftrightarrow$   
 $(nfa'.run\_accept\_eps R (bs \# bss) cs \wedge run\_accept\_eps \{qf'\} [] cs) \vee$   
 $(\exists (css, css') \in list\_split\ bss. nfa'.run\_accept\_eps (nfa'.delta R bs) css (hd\ css') \wedge$   
 $run\_accept\_eps \{qf'\} css' cs)$   
**unfolding**  $NFA.run\_accept\_eps\_def$   $run\_cong\_Cons\_sub[OF assms]$   
 $accept\_eps\_split$   $accept\_eps\_Un$   $accept\_eps\_nfa'\_run[OF assms(1)]$   
**using**  $accept\_eps\_empty$  **by**  $(auto\ split: if\_splits)+$

**lemmas**  $run\_accept\_eps\_cong\_Cons\_sub\_simp =$

$run\_accept\_eps\_cong\_Cons\_sub[unfolded\ list\_split\_def, simplified,$   
 $unfolded\ run\_accept\_eps\_Cons[symmetric]]$   $take\_Suc\_Cons[symmetric]$

**end**

**locale**  $nfa\_cong\_Plus = nfa\_cong\ q0\ q0'\ qf\ qf'\ transs\ transs' +$   
 $right: nfa\_cong\ q0\ q0''\ qf\ qf'' transs\ transs''$   
**for**  $q0\ q0'\ q0''\ qf\ qf'\ qf'' transs\ transs'\ transs'' +$   
**assumes**  $step\_eps\_q0: step\_eps\ bs\ q0\ q \longleftrightarrow q \in \{q0', q0''\}$  **and**  
 $step\_symb\_q0: \neg step\_symb\ q0\ q$   
**begin**

**lemma**  $step\_symb\_set\_q0: step\_symb\_set \{q0\} = \{\}$

**unfolding**  $NFA.step\_symb\_set\_def$  **using**  $step\_symb\_q0$  **by**  $simp$

**lemma**  $qf\_not\_q0: qf \notin \{q0\}$

**using**  $qf\_not\_in\_SQ$   $q0\_sub\_SQ$  **by**  $auto$

**lemma**  $step\_eps\_closure\_set\_q0: step\_eps\_closure\_set \{q0\} bs = \{q0\} \cup$

$(nfa'.step\_eps\_closure\_set \{q0'\} bs \cup right.nfa'.step\_eps\_closure\_set \{q0''\} bs)$

**using**  $step\_eps\_closure\_set\_unfold[OF step\_eps\_q0]$

$insert\_is\_Un[of\ q0'\ \{q0''\}]$

$step\_eps\_closure\_set\_split[of\ \_ \{q0'\} \{q0''\}]$

$step\_eps\_closure\_set\_cong[OF\ nfa'.q0\_sub\_Q]$

$right.step\_eps\_closure\_set\_cong[OF\ right.nfa'.q0\_sub\_Q]$

**by**  $auto$

**lemmas**  $run\_accept\_eps\_Nil\_cong =$

```

run_accept_eps_Nil_eps_split[OF step_eps_closure_set_q0 step_symb_set_q0 qf_not_q0,
  unfolded run_accept_eps_split
run_accept_eps_cong[OF nfa'.step_eps_closure_set_closed[OF nfa'.q0_sub_Q]]
right.run_accept_eps_cong[OF right.nfa'.step_eps_closure_set_closed[OF right.nfa'.q0_sub_Q]]
run_accept_eps_Nil_eps]

```

```

lemmas run_accept_eps_Cons_cong =
run_accept_eps_Cons_eps_split[OF step_eps_closure_set_q0 step_symb_set_q0 qf_not_q0,
  unfolded run_accept_eps_split
run_accept_eps_cong[OF nfa'.step_eps_closure_set_closed[OF nfa'.q0_sub_Q]]
right.run_accept_eps_cong[OF right.nfa'.step_eps_closure_set_closed[OF right.nfa'.q0_sub_Q]]
run_accept_eps_Cons_eps]

```

```

lemma run_accept_eps_cong: run_accept_eps {q0} bss bs  $\longleftrightarrow$ 
(nfa'.run_accept_eps {q0'} bss bs  $\vee$  right.nfa'.run_accept_eps {q0''} bss bs)
using run_accept_eps_Nil_cong run_accept_eps_Cons_cong by (cases bss) auto

```

**end**

```

locale nfa_cong_Times = nfa_cong' q0 q0' qf q0' transs transs' +
  right: nfa_cong q0 q0' qf qf transs transs''
  for q0 q0' qf transs transs' transs''
begin

```

```

lemmas run_accept_eps_cong =
run_accept_eps_cong[OF nfa'.q0_sub_Q, unfolded
  right.run_accept_eps_cong[OF right.nfa'.q0_sub_Q], unfolded list_split_def, simplified]

```

**end**

```

locale nfa_cong_Star = nfa_cong' q0 q0' qf q0 transs transs'
  for q0 q0' qf transs transs' +
assumes step_eps_q0: step_eps bs q0 q  $\longleftrightarrow$  q  $\in$  {q0', qf} and
  step_symb_q0:  $\neg$ step_symb q0 q
begin

```

```

lemma step_symb_set_q0: step_symb_set {q0} = {}
unfolding NFA.step_symb_set_def using step_symb_q0 by simp

```

```

lemma run_accept_eps_Nil: run_accept_eps {q0} [] bs
unfolding NFA.run_accept_eps_def NFA.run_def using step_eps_accept_eps step_eps_q0 by fastforce

```

```

lemma rtranclp_step_eps_q0_q0': (step_eps bs)** q q'  $\implies$  q = q0  $\implies$ 
q'  $\in$  {q0, qf}  $\vee$  (q'  $\in$  nfa'.SQ  $\wedge$  (nfa'.step_eps bs)** q0' q')
apply (induction q q' rule: rtranclp.induct)
using step_eps_q0 step_eps_dest qf_not_in_SQ step_eps_cong_SQ nfa'.q0_sub_SQ
  nfa'.step_eps_closed[unfolded NFA.Q_def] by fastforce+

```

```

lemma step_eps_closure_set_q0: step_eps_closure_set {q0} bs  $\subseteq$  {q0, qf}  $\cup$ 
(nfa'.step_eps_closure_set {q0'} bs  $\cap$  nfa'.SQ)
unfolding NFA.step_eps_closure_set_def NFA.step_eps_closure_def
using rtranclp_step_eps_q0_q0' by auto

```

```

lemma delta_sub_nfa'_delta: delta {q0} bs  $\subseteq$  nfa'.delta {q0'} bs
unfolding NFA.delta_def
using step_symb_set_mono[OF step_eps_closure_set_q0, unfolded step_symb_set_q0
  step_symb_set_qf step_symb_set_split insert_is_Un[of q0 {qf}]]

```

```

    step_symb_set_cong_SQ
  by (metis boolean_algebra_cancel.sup0 inf_le2 step_symb_set_proj step_symb_set_q0
      step_symb_set_qf sup_commute)

lemma step_eps_closure_set_q0_split: step_eps_closure_set {q0} bs = {q0, qf} ∪
  step_eps_closure_set {q0'} bs
  unfolding NFA.step_eps_closure_set_def NFA.step_eps_closure_def
  using step_eps_qf step_eps_q0
  apply (auto)
  apply (metis rtranclp_unfold)
  by (metis r_into_rtranclp rtranclp.rtrancl_into_rtrancl rtranclp_idemp)

lemma delta_q0_q0': delta {q0} bs = delta {q0'} bs
  unfolding NFA.delta_def step_eps_closure_set_q0_split step_symb_set_split
  unfolding NFA.delta_def[symmetric]
  unfolding NFA.step_symb_set_def
  using step_symb_q0 step_symb_dest qf_not_in_SQ
  by fastforce

lemmas run_accept_eps_cong_Cons =
  run_accept_eps_cong_Cons_sub_simp[OF nfa'.q0_sub_Q delta_sub_nfa'_delta,
  unfolded run_accept_eps_Cons_delta_cong[OF delta_q0_q0', symmetric]]

end

end

theory Window
  imports HOL-Library.AList HOL-Library.Mapping HOL-Library.While_Combinator Timestamp
begin

type_synonym ('a, 'b) mmap = ('a × 'b) list

inductive chain_le :: 'd :: timestamp list ⇒ bool where
  chain_le_Nil: chain_le []
| chain_le_singleton: chain_le [x]
| chain_le_cons: chain_le (y # xs) ⇒ x ≤ y ⇒ chain_le (x # y # xs)

lemma chain_le_app: chain_le (zs @ [z]) ⇒ z ≤ w ⇒ chain_le ((zs @ [z]) @ [w])
  apply (induction zs @ [z] arbitrary: zs rule: chain_le.induct)
  apply (auto intro: chain_le.intros)[2]
  subgoal for y xs x zs
  apply (cases zs)
  apply (auto)
  apply (metis append.assoc append_Cons append_Nil chain_le_cons)
  done
done

inductive reaches_on :: ('e ⇒ ('e × 'f) option) ⇒ 'e ⇒ 'f list ⇒ 'e ⇒ bool
  for run :: 'e ⇒ ('e × 'f) option where
  reaches_on_run s [] s
| run s = Some (s', v) ⇒ reaches_on run s' vs s'' ⇒ reaches_on run s (v # vs) s''

lemma reaches_on_init_Some: reaches_on r s xs s' ⇒ r s' ≠ None ⇒ r s ≠ None
  by (auto elim: reaches_on.cases)

lemma reaches_on_split: reaches_on run s vs s' ⇒ i < length vs ⇒

```

$\exists s'' s'''. \text{reaches\_on run } s \text{ (take } i \text{ vs) } s'' \wedge \text{run } s'' = \text{Some } (s''', \text{vs } ! \ i) \wedge \text{reaches\_on run } s''' \text{ (drop (Suc } i \text{) vs) } s'$

**proof** (induction s vs s' arbitrary: i rule: reaches\_on.induct)

**case** (2 s s' v vs s')

**show** ?case

**using** 2(1,2)

**proof** (cases i)

**case** (Suc n)

**show** ?thesis

**using** 2

**by** (fastforce simp: Suc intro: reaches\_on.intros)

**qed** (auto intro: reaches\_on.intros)

**qed** auto

**lemma** reaches\_on\_split': reaches\_on run s vs s'  $\implies i \leq \text{length vs} \implies$

$\exists s'' . \text{reaches\_on run } s \text{ (take } i \text{ vs) } s'' \wedge \text{reaches\_on run } s'' \text{ (drop } i \text{ vs) } s'$

**proof** (induction s vs s' arbitrary: i rule: reaches\_on.induct)

**case** (2 s s' v vs s')

**show** ?case

**using** 2(1,2)

**proof** (cases i)

**case** (Suc n)

**show** ?thesis

**using** 2

**by** (fastforce simp: Suc intro: reaches\_on.intros)

**qed** (auto intro: reaches\_on.intros)

**qed** (auto intro: reaches\_on.intros)

**lemma** reaches\_on\_split\_app: reaches\_on run s (vs @ vs') s'  $\implies$

$\exists s'' . \text{reaches\_on run } s \text{ vs } s'' \wedge \text{reaches\_on run } s'' \text{ vs' } s'$

**using** reaches\_on\_split'[where i=length vs, of run s vs @ vs' s']

**by** auto

**lemma** reaches\_on\_inj: reaches\_on run s vs t  $\implies \text{reaches\_on run } s \text{ vs' } t' \implies$

$\text{length vs} = \text{length vs}' \implies \text{vs} = \text{vs}' \wedge t = t'$

**apply** (induction s vs t arbitrary: vs' t' rule: reaches\_on.induct)

**apply** (auto elim: reaches\_on.cases)[1]

**subgoal** for s s' v vs s'' vs' t'

**apply** (rule reaches\_on.cases[of run s' vs s']; rule reaches\_on.cases[of run s vs' t'])

**apply** assumption+

**apply** auto[2]

**apply** fastforce

**apply** (metis length\_0\_conv list.discI)

**apply** (metis Pair\_inject length\_Cons nat.inject option.inject)

**done**

**done**

**lemma** reaches\_on\_split\_last: reaches\_on run s (xs @ [x]) s''  $\implies$

$\exists s' . \text{reaches\_on run } s \text{ xs } s' \wedge \text{run } s' = \text{Some } (s'', x)$

**apply** (induction s xs @ [x] s'' arbitrary: xs x rule: reaches\_on.induct)

**apply** simp

**subgoal** for s s' v vs s'' xs x

**by** (cases vs rule: rev\_cases) (fastforce elim: reaches\_on.cases intro: reaches\_on.intros)+

**done**

**lemma** reaches\_on\_rev\_induct[consumes 1]: reaches\_on run s vs s'  $\implies$

$(\bigwedge s . P s \ \square \ s) \implies$

$(\bigwedge s s' v \text{ vs } s'' . \text{reaches\_on run } s \text{ vs } s' \implies P s \text{ vs } s' \implies \text{run } s' = \text{Some } (s'', v) \implies$

```

    P s (vs @ [v]) s'' =>
    P s vs s'
proof (induction vs arbitrary: s s' rule: rev_induct)
  case (snoc x xs)
  from snoc(2) obtain s'' where s''_def: reaches_on run s xs s'' run s'' = Some (s', x)
  using reaches_on_split_last
  by fast
  show ?case
  using snoc(4)[OF s''_def(1) _ s''_def(2)] snoc(1)[OF s''_def(1) snoc(3,4)]
  by auto
qed (auto elim: reaches_on.cases)

lemma reaches_on_app: reaches_on run s vs s' => run s' = Some (s'', v) =>
  reaches_on run s (vs @ [v]) s''
by (induction s vs s' rule: reaches_on.induct) (auto intro: reaches_on.intros)

lemma reaches_on_trans: reaches_on run s vs s' => reaches_on run s' vs' s'' =>
  reaches_on run s (vs @ vs') s''
by (induction s vs s' rule: reaches_on.induct) (auto intro: reaches_on.intros)

lemma reaches_onD: reaches_on run s ((t, b) # vs) s' =>
  ∃ s''. run s = Some (s'', (t, b)) ∧ reaches_on run s'' vs s'
by (auto elim: reaches_on.cases)

lemma reaches_on_setD: reaches_on run s vs s' => x ∈ set vs =>
  ∃ vs' vs'' s''. reaches_on run s (vs' @ [x]) s'' ∧ reaches_on run s'' vs'' s' ∧ vs = vs' @ x # vs''
proof (induction s vs s' rule: reaches_on_rev_induct)
  case (2 s s' v vs s'')
  show ?case
  proof (cases x ∈ set vs)
  case True
  obtain vs' vs'' s''' where split_def: reaches_on run s (vs' @ [x]) s'''
    reaches_on run s''' vs'' s' vs = vs' @ x # vs''
  using 2(3)[OF True]
  by auto
  show ?thesis
  using split_def(1,3) reaches_on_app[OF split_def(2) 2(2)]
  by auto
  next
  case False
  have x_v: x = v
  using 2(4) False
  by auto
  show ?thesis
  unfolding x_v
  using reaches_on_app[OF 2(1,2)] reaches_on.intros(1)[of run s']
  by auto
  qed
qed auto

lemma reaches_on_len: ∃ vs s'. reaches_on run s vs s' ∧ (length vs = n ∨ run s' = None)
proof (induction n arbitrary: s)
  case (Suc n)
  show ?case
  proof (cases run s)
  case (Some x)
  obtain s' v where x_def: x = (s', v)
  by (cases x) auto

```

```

obtain vs s'' where s''_def: reaches_on run s' vs s'' length vs = n ∨ run s'' = None
  using Suc[of s']
  by auto
show ?thesis
  using reaches_on.intros(2)[OF Some[unfolded x_def] s''_def(1)] s''_def(2)
  by fastforce
qed (auto intro: reaches_on.intros)
qed (auto intro: reaches_on.intros)

lemma reaches_on_NilD: reaches_on run q [] q' ⇒ q = q'
  by (auto elim: reaches_on.cases)

lemma reaches_on_ConsD: reaches_on run q (x # xs) q' ⇒ ∃ q''. run q = Some (q'', x) ∧ reaches_on
run q'' xs q'
  by (auto elim: reaches_on.cases)

inductive reaches :: ('e ⇒ ('e × 'f) option) ⇒ 'e ⇒ nat ⇒ 'e ⇒ bool
  for run :: 'e ⇒ ('e × 'f) option where
    reaches run s 0 s
  | run s = Some (s', v) ⇒ reaches run s' n s'' ⇒ reaches run s (Suc n) s''

lemma reaches_Suc_split_last: reaches run s (Suc n) s' ⇒ ∃ s'' x. reaches run s n s'' ∧ run s'' = Some
(s', x)
proof (induction n arbitrary: s)
  case (Suc n)
  obtain s'' x where s''_def: run s = Some (s'', x) reaches run s'' (Suc n) s'
    using Suc(2)
    by (auto elim: reaches.cases)
  show ?case
    using s''_def(1) Suc(1)[OF s''_def(2)]
    by (auto intro: reaches.intros)
qed (auto elim!: reaches.cases intro: reaches.intros)

lemma reaches_invar: reaches f x n y ⇒ P x ⇒ (∧ z z' v. P z ⇒ f z = Some (z', v) ⇒ P z') ⇒
P y
  by (induction x n y rule: reaches.induct) auto

lemma reaches_cong: reaches f x n y ⇒ P x ⇒ (∧ z z' v. P z ⇒ f z = Some (z', v) ⇒ P z') ⇒
(∧ z. P z ⇒ f' (g z) = map_option (apfst g) (f z)) ⇒ reaches f' (g x) n (g y)
  by (induction x n y rule: reaches.induct) (auto intro: reaches.intros)

lemma reaches_on_n: reaches_on run s vs s' ⇒ reaches run s (length vs) s'
  by (induction s vs s' rule: reaches_on.induct) (auto intro: reaches.intros)

lemma reaches_on: reaches run s n s' ⇒ ∃ vs. reaches_on run s vs s' ∧ length vs = n
  by (induction s n s' rule: reaches.induct) (auto intro: reaches_on.intros)

definition ts_at :: ('d × 'b) list ⇒ nat ⇒ 'd where
  ts_at rho i = fst (rho ! i)

definition bs_at :: ('d × 'b) list ⇒ nat ⇒ 'b where
  bs_at rho i = snd (rho ! i)

fun sub_bs :: ('d × 'b) list ⇒ nat × nat ⇒ 'b list where
  sub_bs rho (i, j) = map (bs_at rho) [i..<j]

definition steps :: ('c ⇒ 'b ⇒ 'c) ⇒ ('d × 'b) list ⇒ 'c ⇒ nat × nat ⇒ 'c where
  steps step rho q ij = foldl step q (sub_bs rho ij)

```

**definition**  $acc :: ('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('c \Rightarrow bool) \Rightarrow ('d \times 'b) list \Rightarrow 'c \Rightarrow nat \times nat \Rightarrow bool$  **where**  
 $acc\ step\ accept\ rho\ q\ ij = accept\ (steps\ step\ rho\ q\ ij)$

**definition**  $sup\_acc :: ('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('c \Rightarrow bool) \Rightarrow ('d \times 'b) list \Rightarrow 'c \Rightarrow nat \Rightarrow nat \Rightarrow ('d \times nat) option$  **where**  
 $sup\_acc\ step\ accept\ rho\ q\ i\ j =$   
 $(let\ L' = \{l \in \{i..j\}. acc\ step\ accept\ rho\ q\ (i, Suc\ l)\};\ m = Max\ L'\ in$   
 $if\ L' = \{\} then\ None\ else\ Some\ (ts\_at\ rho\ m, m))$

**definition**  $sup\_leadsto :: ('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('d \times 'b) list \Rightarrow nat \Rightarrow nat \Rightarrow 'c \Rightarrow 'd option$  **where**  
 $sup\_leadsto\ init\ step\ rho\ i\ j\ q =$   
 $(let\ L' = \{l.\ l < i \wedge steps\ step\ rho\ init\ (l, j) = q\};\ m = Max\ L'\ in$   
 $if\ L' = \{\} then\ None\ else\ Some\ (ts\_at\ rho\ m))$

**definition**  $mmap\_keys :: ('a, 'b) mmap \Rightarrow 'a set$  **where**  
 $mmap\_keys\ kvs = set\ (map\ fst\ kvs)$

**definition**  $mmap\_lookup :: ('a, 'b) mmap \Rightarrow 'a \Rightarrow 'b option$  **where**  
 $mmap\_lookup = map\_of$

**definition**  $valid\_s :: ('c \Rightarrow ('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('c \times 'b, 'c) mapping \Rightarrow ('c \Rightarrow bool) \Rightarrow ('d \times 'b) list \Rightarrow nat \Rightarrow nat \Rightarrow nat \Rightarrow ('c, 'c \times ('d \times nat) option) mmap \Rightarrow bool$  **where**  
 $valid\_s\ init\ step\ st\ accept\ rho\ u\ i\ j\ s \equiv$   
 $(\forall q\ bs.\ case\ Mapping.lookup\ st\ (q, bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v) \wedge$   
 $(mmap\_keys\ s = \{q.\ (\exists l \leq u.\ steps\ step\ rho\ init\ (l, i) = q)\} \wedge distinct\ (map\ fst\ s) \wedge$   
 $(\forall q.\ case\ mmap\_lookup\ s\ q\ of\ None \Rightarrow True$   
 $\mid Some\ (q', tstp) \Rightarrow steps\ step\ rho\ q\ (i, j) = q' \wedge tstp = sup\_acc\ step\ accept\ rho\ q\ i\ j))$

**record**  $( 'b, 'c, 'd, 't, 'e) args =$   
 $w\_init :: 'c$   
 $w\_step :: 'c \Rightarrow 'b \Rightarrow 'c$   
 $w\_accept :: 'c \Rightarrow bool$   
 $w\_run\_t :: 't \Rightarrow ('t \times 'd) option$   
 $w\_read\_t :: 't \Rightarrow 'd option$   
 $w\_run\_sub :: 'e \Rightarrow ('e \times 'b) option$

**record**  $( 'b, 'c, 'd, 't, 'e) window =$   
 $w\_st :: ('c \times 'b, 'c) mapping$   
 $w\_ac :: ('c, bool) mapping$   
 $w\_i :: nat$   
 $w\_ti :: 't$   
 $w\_si :: 'e$   
 $w\_j :: nat$   
 $w\_tj :: 't$   
 $w\_sj :: 'e$   
 $w\_s :: ('c, 'c \times ('d \times nat) option) mmap$   
 $w\_e :: ('c, 'd) mmap$

**copy\_bnf**  $(dead\ 'b, dead\ 'c, dead\ 'd, dead\ 't, 'e, dead\ 'ext) window\_ext$

**fun**  $reach\_window :: ('b, 'c, 'd, 't, 'e) args \Rightarrow 't \Rightarrow 'e \Rightarrow ('d \times 'b) list \Rightarrow nat \times 't \times 'e \times nat \times 't \times 'e \Rightarrow bool$  **where**  
 $reach\_window\ args\ t0\ sub\ rho\ (i, ti, si, j, tj, sj) \longleftrightarrow i \leq j \wedge length\ rho = j \wedge$   
 $reaches\_on\ (w\_run\_t\ args)\ t0\ (take\ i\ (map\ fst\ rho))\ ti \wedge$   
 $reaches\_on\ (w\_run\_t\ args)\ ti\ (drop\ i\ (map\ fst\ rho))\ tj \wedge$

reaches\_on (w\_run\_sub args) sub (take i (map snd rho)) si  $\wedge$   
reaches\_on (w\_run\_sub args) si (drop i (map snd rho)) sj

**lemma** reach\_windowI: reaches\_on (w\_run\_t args) t0 (take i (map fst rho)) ti  $\implies$   
reaches\_on (w\_run\_sub args) sub (take i (map snd rho)) si  $\implies$   
reaches\_on (w\_run\_t args) t0 (map fst rho) tj  $\implies$   
reaches\_on (w\_run\_sub args) sub (map snd rho) sj  $\implies$   
 $i \leq \text{length } \rho \implies \text{length } \rho = j \implies$   
reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  
**by** auto (metis reaches\_on\_split[of \_ \_ \_ \_ i] length\_map reaches\_on\_inj)+

**lemma** reach\_window\_shift:

**assumes** reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  $i < j$   
w\_run\_t args ti = Some (ti', t) w\_run\_sub args si = Some (si', s)  
**shows** reach\_window args t0 sub rho (Suc i, ti', si', j, tj, sj)  
**using** reaches\_on\_app[of w\_run\_t args t0 take i (map fst rho) ti ti' t]  
reaches\_on\_app[of w\_run\_sub args sub take i (map snd rho) si si' s] **assms**  
**apply** (auto)  
**apply** (smt append\_take\_drop\_id id\_take\_nth\_drop length\_map list.discI list.inject  
option.inject reaches\_on.cases same\_append\_eq snd\_conv take\_Suc\_conv\_app\_nth)  
**apply** (smt Cons\_nth\_drop\_Suc fst\_conv length\_map list.discI list.inject option.inject  
reaches\_on.cases)  
**apply** (smt append\_take\_drop\_id id\_take\_nth\_drop length\_map list.discI list.inject  
option.inject reaches\_on.cases same\_append\_eq snd\_conv take\_Suc\_conv\_app\_nth)  
**apply** (smt Cons\_nth\_drop\_Suc fst\_conv length\_map list.discI list.inject option.inject  
reaches\_on.cases)  
**done**

**lemma** reach\_window\_run\_ti: reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  $\implies$   
 $i < j \implies \exists ti'. \text{reaches\_on } (w\_run\_t \text{ args}) t0 \text{ (take } i \text{ (map fst rho)) } ti \wedge$   
w\_run\_t args ti = Some (ti', ts\_at rho i)  $\wedge$   
reaches\_on (w\_run\_t args) ti' (drop (Suc i) (map fst rho)) tj  
**apply** (auto simp: ts\_at\_def elim!: reaches\_on.cases[of w\_run\_t args ti drop i (map fst rho)])  
**using** nth\_via\_drop **apply** fastforce  
**by** (metis Cons\_nth\_drop\_Suc length\_map list.inject)

**lemma** reach\_window\_run\_si: reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  $\implies$   
 $i < j \implies \exists si'. \text{reaches\_on } (w\_run\_sub \text{ args}) \text{ sub (take } i \text{ (map snd rho)) } si \wedge$   
w\_run\_sub args si = Some (si', bs\_at rho i)  $\wedge$   
reaches\_on (w\_run\_sub args) si' (drop (Suc i) (map snd rho)) sj  
**apply** (auto simp: bs\_at\_def elim!: reaches\_on.cases[of w\_run\_sub args si drop i (map snd rho)])  
**using** nth\_via\_drop **apply** fastforce  
**by** (metis Cons\_nth\_drop\_Suc length\_map list.inject)

**lemma** reach\_window\_run\_tj: reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  $\implies$   
reaches\_on (w\_run\_t args) t0 (map fst rho) tj  
**using** reaches\_on\_trans  
**by** fastforce

**lemma** reach\_window\_run\_sj: reach\_window args t0 sub rho (i, ti, si, j, tj, sj)  $\implies$   
reaches\_on (w\_run\_sub args) sub (map snd rho) sj  
**using** reaches\_on\_trans  
**by** fastforce

**lemma** reach\_window\_shift\_all: reach\_window args t0 sub rho (i, si, ti, j, sj, tj)  $\implies$   
reach\_window args t0 sub rho (j, sj, tj, j, sj, tj)  
**using** reach\_window\_run\_tj[of args t0 sub rho] reach\_window\_run\_sj[of args t0 sub rho]  
**by** (auto intro: reaches\_on.intros)

**lemma** *reach\_window\_app*:  $\text{reach\_window } \text{args } t0 \text{ sub } \rho \ (i, si, ti, j, tj, sj) \implies$   
 $\text{w\_run\_t } \text{args } tj = \text{Some } (tj', x) \implies \text{w\_run\_sub } \text{args } sj = \text{Some } (sj', y) \implies$   
 $\text{reach\_window } \text{args } t0 \text{ sub } (\rho @ [(x, y)]) \ (i, si, ti, \text{Suc } j, tj', sj')$   
**by** (*fastforce simp add: reaches\_on\_app*)

**fun** *init\_args* ::  $('c \times ('c \Rightarrow 'b \Rightarrow 'c) \times ('c \Rightarrow \text{bool})) \Rightarrow$   
 $((t \Rightarrow ('t \times 'd) \text{ option}) \times ('t \Rightarrow 'd \text{ option})) \Rightarrow$   
 $('e \Rightarrow ('e \times 'b) \text{ option}) \Rightarrow ('b, 'c, 'd, 't, 'e) \text{ args where}$   
 $\text{init\_args } (\text{init}, \text{step}, \text{accept}) \ (\text{run\_t}, \text{read\_t}) \ \text{run\_sub} =$   
 $(\text{w\_init} = \text{init}, \text{w\_step} = \text{step}, \text{w\_accept} = \text{accept}, \text{w\_run\_t} = \text{run\_t}, \text{w\_read\_t} = \text{read\_t}, \text{w\_run\_sub}$   
 $= \text{run\_sub})$

**fun** *init\_window* ::  $('b, 'c, 'd, 't, 'e) \text{ args} \Rightarrow 't \Rightarrow 'e \Rightarrow ('b, 'c, 'd, 't, 'e) \text{ window where}$   
 $\text{init\_window } \text{args } t0 \text{ sub} = (\text{w\_st} = \text{Mapping.empty}, \text{w\_ac} = \text{Mapping.empty},$   
 $\text{w\_i} = 0, \text{w\_ti} = t0, \text{w\_si} = \text{sub}, \text{w\_j} = 0, \text{w\_tj} = t0, \text{w\_sj} = \text{sub},$   
 $\text{w\_s} = [(\text{w\_init } \text{args}, (\text{w\_init } \text{args}, \text{None}))], \text{w\_e} = [])$

**definition** *valid\_window* ::  $('b, 'c, 'd :: \text{timestamp}, 't, 'e) \text{ args} \Rightarrow 't \Rightarrow 'e \Rightarrow ('d \times 'b) \text{ list} \Rightarrow$   
 $('b, 'c, 'd, 't, 'e) \text{ window} \Rightarrow \text{bool where}$   
 $\text{valid\_window } \text{args } t0 \text{ sub } \rho \ w \iff$   
 $(\text{let } \text{init} = \text{w\_init } \text{args}; \text{step} = \text{w\_step } \text{args}; \text{accept} = \text{w\_accept } \text{args};$   
 $\text{run\_t} = \text{w\_run\_t } \text{args}; \text{run\_sub} = \text{w\_run\_sub } \text{args};$   
 $\text{st} = \text{w\_st } w; \text{ac} = \text{w\_ac } w;$   
 $i = \text{w\_i } w; \text{ti} = \text{w\_ti } w; \text{si} = \text{w\_si } w; j = \text{w\_j } w; \text{tj} = \text{w\_tj } w; \text{sj} = \text{w\_sj } w;$   
 $s = \text{w\_s } w; e = \text{w\_e } w \text{ in}$   
 $(\text{reach\_window } \text{args } t0 \text{ sub } \rho \ (i, ti, si, j, tj, sj) \wedge$   
 $(\forall i \ j. i \leq j \wedge j < \text{length } \rho \longrightarrow \text{ts\_at } \rho \ i \leq \text{ts\_at } \rho \ j) \wedge$   
 $(\forall q. \text{case } \text{Mapping.lookup } \text{ac } q \text{ of } \text{None} \Rightarrow \text{True} \mid \text{Some } v \Rightarrow \text{accept } q = v) \wedge$   
 $(\forall q. \text{mmap\_lookup } e \ q = \text{sup\_leadsto } \text{init } \text{step } \rho \ i \ j \ q) \wedge \text{distinct } (\text{map } \text{fst } e) \wedge$   
 $\text{valid\_s } \text{init } \text{step } \text{st } \text{accept } \rho \ i \ j \ s))$

**lemma** *valid\_init\_window*:  $\text{valid\_window } \text{args } t0 \text{ sub } [] \ (\text{init\_window } \text{args } t0 \text{ sub})$   
**by** (*auto simp: valid\_window\_def mmap\_keys\_def mmap\_lookup\_def sup\_leadsto\_def*  
 $\text{valid\_s\_def steps\_def sup\_acc\_def intro: reaches\_on.intros split: option.splits}$ )

**lemma** *steps\_app\_cong*:  $j \leq \text{length } \rho \implies \text{steps } \text{step } (\rho @ [x]) \ q \ (i, j) =$   
 $\text{steps } \text{step } \rho \ q \ (i, j)$

**proof** –

**assume**  $j \leq \text{length } \rho$

**then have** *map\_cong*:  $\text{map } (\text{bs\_at } (\rho @ [x])) \ [i..<j] = \text{map } (\text{bs\_at } \rho) \ [i..<j]$

**by** (*auto simp: bs\_at\_def nth\_append*)

**show** *?thesis*

**by** (*auto simp: steps\_def map\_cong*)

**qed**

**lemma** *acc\_app\_cong*:  $j < \text{length } \rho \implies \text{acc } \text{step } \text{accept } (\rho @ [x]) \ q \ (i, j) =$   
 $\text{acc } \text{step } \text{accept } \rho \ q \ (i, j)$

**by** (*auto simp: acc\_def bs\_at\_def nth\_append steps\_app\_cong*)

**lemma** *sup\_acc\_app\_cong*:  $j \leq \text{length } \rho \implies \text{sup\_acc } \text{step } \text{accept } (\rho @ [x]) \ q \ i \ j =$   
 $\text{sup\_acc } \text{step } \text{accept } \rho \ q \ i \ j$

**apply** (*auto simp: sup\_acc\_def Let\_def ts\_at\_def nth\_append acc\_def*)

**apply** (*metis (mono\_tags, opaque\_lifting) less\_eq\_Suc\_le order\_less\_le\_trans steps\_app\_cong*) +  
**done**

**lemma** *sup\_acc\_concat\_cong*:  $j \leq \text{length } \rho \implies \text{sup\_acc } \text{step } \text{accept } (\rho @ \rho') \ q \ i \ j =$   
 $\text{sup\_acc } \text{step } \text{accept } \rho \ q \ i \ j$

```

apply (induction rho' rule: rev_induct)
apply auto
apply (smt append.assoc le_add1 le_trans length_append sup_acc_app_cong)
done

lemma sup_leadsto_app_cong:  $i \leq j \implies j \leq \text{length } \rho \implies$ 
  sup_leadsto_init_step ( $\rho @ [x]$ )  $i j q = \text{sup\_leadsto\_init\_step } \rho i j q$ 
proof -
  assume assms:  $i \leq j \wedge j \leq \text{length } \rho$ 
  define L' where L' = {l.  $l < i \wedge \text{steps\_step } \rho \text{ init } (l, j) = q$ }
  define L'' where L'' = {l.  $l < i \wedge \text{steps\_step } (\rho @ [x]) \text{ init } (l, j) = q$ }
  show ?thesis
  using assms
  by (cases L' = {})
  (auto simp: sup_leadsto_def L'_def L''_def ts_at_def nth_append_steps_app_cong)
qed

lemma chain_le:
  fixes xs :: 'd :: timestamp list
  shows chain_le xs  $\implies i \leq j \implies j < \text{length } xs \implies xs ! i \leq xs ! j$ 
proof (induction xs arbitrary: i j rule: chain_le.induct)
  case (chain_le_cons y xs x)
  then show ?case
  proof (cases i)
  case 0
  then show ?thesis
  using chain_le_cons
  apply (cases j)
  apply auto
  apply (metis (no_types, lifting) le_add1 le_add_same_cancel1 le_less less_le_trans nth_Cons_0)
  done
  qed auto
qed auto

lemma steps_refl[simp]: steps_step_rho q (i, i) = q
  unfolding steps_def by auto

lemma steps_split:  $i < j \implies \text{steps\_step\_rho } q (i, j) =$ 
  steps_step_rho (step q (bs_at rho i)) (Suc i, j)
  unfolding steps_def by (simp add: upt_rec)

lemma steps_app:  $i \leq j \implies \text{steps\_step\_rho } q (i, j + 1) =$ 
  step (steps_step_rho q (i, j)) (bs_at rho j)
  unfolding steps_def by auto

lemma steps_appE:  $i \leq j \implies \text{steps\_step\_rho } q (i, \text{Suc } j) = q' \implies$ 
 $\exists q''. \text{steps\_step\_rho } q (i, j) = q'' \wedge q' = \text{step } q'' (\text{bs\_at } \rho j)$ 
  unfolding steps_def sub_bs.simps by auto

lemma steps_comp:  $i \leq l \implies l \leq j \implies \text{steps\_step\_rho } q (i, l) = q' \implies$ 
  steps_step_rho q' (l, j) = q''  $\implies \text{steps\_step\_rho } q (i, j) = q''$ 
proof -
  assume assms:  $i \leq l \wedge l \leq j \implies \text{steps\_step\_rho } q (i, l) = q' \implies \text{steps\_step\_rho } q' (l, j) = q''$ 
  have range_app:  $[i..<l] @ [l..<j] = [i..<j]$ 
  using assms(1,2)
  by (metis le_Suc_ex upt_add_eq_append)
  have q' = foldl step q (map (bs_at rho) [i..<l])
  using assms(3) unfolding steps_def by auto

```

**moreover have**  $q'' = \text{foldl step } q' (\text{map } (bs\_at \text{ rho}) [l..<j])$   
**using**  $assms(4)$  **unfolding**  $steps\_def$  **by**  $auto$   
**ultimately have**  $q'' = \text{foldl step } q (\text{map } (bs\_at \text{ rho}) ([i..<l] @ [l..<j]))$   
**by**  $auto$   
**then show**  $?thesis$   
**unfolding**  $steps\_def$   $range\_app$  **by**  $auto$   
**qed**

**lemma**  $sup\_acc\_SomeI$ :  $acc \text{ step } accept \text{ rho } q (i, Suc \ l) \implies l \in \{i..<j\} \implies$   
 $\exists tp. sup\_acc \text{ step } accept \text{ rho } q \ i \ j = Some (ts\_at \text{ rho } tp, tp) \wedge l \leq tp \wedge tp < j$   
**proof** –  
**assume**  $assms$ :  $acc \text{ step } accept \text{ rho } q (i, Suc \ l) \ l \in \{i..<j\}$   
**define**  $L$  **where**  $L = \{l \in \{i..<j\}. acc \text{ step } accept \text{ rho } q (i, Suc \ l)\}$   
**have**  $L\_props$ :  $finite \ L \ L \neq \{\} \ l \in L$   
**using**  $assms$  **unfolding**  $L\_def$  **by**  $auto$   
**then show**  $\exists tp. sup\_acc \text{ step } accept \text{ rho } q \ i \ j = Some (ts\_at \text{ rho } tp, tp) \wedge l \leq tp \wedge tp < j$   
**using**  $L\_def \ L\_props$   
**by**  $(auto \ simp \ add: sup\_acc\_def)$   
 $(smt \ L\_props(1) \ L\_props(2) \ Max\_ge \ Max\_in \ mem\_Collect\_eq)$   
**qed**

**lemma**  $sup\_acc\_Some\_ts$ :  $sup\_acc \text{ step } accept \text{ rho } q \ i \ j = Some (ts, tp) \implies ts = ts\_at \text{ rho } tp$   
**by**  $(auto \ simp \ add: sup\_acc\_def \ Let\_def \ split: if\_splits)$

**lemma**  $sup\_acc\_SomeE$ :  $sup\_acc \text{ step } accept \text{ rho } q \ i \ j = Some (ts, tp) \implies$   
 $tp \in \{i..<j\} \wedge acc \text{ step } accept \text{ rho } q (i, Suc \ tp)$   
**proof** –  
**assume**  $assms$ :  $sup\_acc \text{ step } accept \text{ rho } q \ i \ j = Some (ts, tp)$   
**define**  $L$  **where**  $L = \{l \in \{i..<j\}. acc \text{ step } accept \text{ rho } q (i, Suc \ l)\}$   
**have**  $L\_props$ :  $finite \ L \ L \neq \{\} \ Max \ L = tp$   
**unfolding**  $L\_def$  **using**  $assms$   
**by**  $(auto \ simp \ add: sup\_acc\_def \ Let\_def \ split: if\_splits)$   
**show**  $?thesis$   
**using**  $Max\_in[OF \ L\_props(1,2)]$  **unfolding**  $L\_props(3)$  **unfolding**  $L\_def$  **by**  $auto$   
**qed**

**lemma**  $sup\_acc\_NoneE$ :  $l \in \{i..<j\} \implies sup\_acc \text{ step } accept \text{ rho } q \ i \ j = None \implies$   
 $\neg acc \text{ step } accept \text{ rho } q (i, Suc \ l)$   
**by**  $(auto \ simp \ add: sup\_acc\_def \ Let\_def \ split: if\_splits)$

**lemma**  $sup\_acc\_same$ :  $sup\_acc \text{ step } accept \text{ rho } q \ i \ i = None$   
**by**  $(auto \ simp \ add: sup\_acc\_def)$

**lemma**  $sup\_acc\_None\_restrict$ :  $i \leq j \implies sup\_acc \text{ step } accept \text{ rho } q \ i \ j = None \implies$   
 $sup\_acc \text{ step } accept \text{ rho } (step \ q \ (bs\_at \ \text{rho} \ i)) (Suc \ i) \ j = None$   
**using**  $steps\_split$   
**apply**  $(auto \ simp \ add: sup\_acc\_def \ Let\_def \ acc\_def \ split: if\_splits)$   
**apply**  $(smt (z3) \ lessI \ less\_imp\_le\_nat \ order\_less\_le\_trans \ steps\_split)$   
**done**

**lemma**  $sup\_acc\_ext\_idle$ :  $i \leq j \implies \neg acc \text{ step } accept \text{ rho } q (i, Suc \ j) \implies$   
 $sup\_acc \text{ step } accept \text{ rho } q \ i (Suc \ j) = sup\_acc \text{ step } accept \text{ rho } q \ i \ j$   
**proof** –  
**assume**  $assms$ :  $i \leq j \ \neg acc \text{ step } accept \text{ rho } q (i, Suc \ j)$   
**define**  $L$  **where**  $L = \{l \in \{i..<j\}. acc \text{ step } accept \text{ rho } q (i, Suc \ l)\}$   
**define**  $L'$  **where**  $L' = \{l \in \{i..<Suc \ j\}. acc \text{ step } accept \text{ rho } q (i, Suc \ l)\}$   
**have**  $L \ L'$ :  $L = L'$   
**unfolding**  $L\_def \ L'\_def$  **using**  $assms(2)$   $less\_antisym$  **by**  $fastforce$

**show**  $\text{sup\_acc step accept rho q } i \text{ (Suc } j) = \text{sup\_acc step accept rho q } i \text{ } j$   
**using**  $L\_def \ L'_def \ L\_L'$  **by** (auto simp add: sup\_acc\_def)

**qed**

**lemma**  $\text{sup\_acc\_comp\_Some\_ge: } i \leq l \implies l \leq j \implies tp \geq l \implies$   
 $\text{sup\_acc step accept rho (steps step rho q (i, l)) } l \text{ } j = \text{Some (ts, tp)} \implies$   
 $\text{sup\_acc step accept rho q } i \text{ } j = \text{sup\_acc step accept rho (steps step rho q (i, l)) } l \text{ } j$

**proof** –

**assume**  $\text{assms: } i \leq l \ l \leq j \ \text{sup\_acc step accept rho (steps step rho q (i, l)) } l \text{ } j =$   
 $\text{Some (ts, tp)} \ tp \geq l$

**define**  $L$  **where**  $L = \{l \in \{i..<j\}. \text{acc step accept rho q (i, Suc } l)\}$

**define**  $L'$  **where**  $L' = \{l' \in \{l..<j\}. \text{acc step accept rho (steps step rho q (i, l)) (l, Suc } l')\}$

**have**  $L'_props: \text{finite } L' \ L' \neq \{\}$   $tp = \text{Max } L' \ ts = \text{ts\_at rho } tp$

**using**  $\text{assms(3) unfolding } L'_def$

**by** (auto simp add: sup\_acc\_def Let\_def split: if\_splits)

**have**  $tp\_in\_L': tp \in L'$

**using**  $\text{Max\_in[OF } L'_props(1,2)] \ \text{unfolding } L'_props(3)$  .

**then have**  $tp\_in\_L: tp \in L$

**unfolding**  $L\_def \ L'_def$  **using**  $\text{assms(1) steps\_comp[OF assms(1,2), of step rho]}$

**apply** (auto simp add: acc\_def)

**using**  $\text{steps\_comp}$

**by** (metis le\_SucI)

**have**  $L\_props: \text{finite } L \ L \neq \{\}$

**using**  $L\_def \ tp\_in\_L$  **by** auto

**have**  $\bigwedge l'. l' \in L \implies l' \leq tp$

**proof** –

**fix**  $l'$

**assume**  $\text{asm: } l' \in L$

**show**  $l' \leq tp$

**proof** (cases  $l' < l$ )

**case**  $\text{True}$

**then show**  $?thesis$

**using**  $\text{assms(4) by auto}$

**next**

**case**  $\text{False}$

**then have**  $l' \in L'$

**using**  $\text{asm}$

**unfolding**  $L\_def \ L'_def$

**by** (auto simp add: acc\_def) (metis  $\text{assms(1) less\_imp\_le\_nat not\_less\_eq steps\_comp}$ )

**then show**  $?thesis$

**using**  $\text{Max\_eq\_iff[OF } L'_props(1,2)] \ L'_props(3)$  **by** auto

**qed**

**qed**

**then have**  $\text{Max } L = tp$

**using**  $\text{Max\_eq\_iff[OF } L\_props] \ tp\_in\_L$  **by** auto

**then have**  $\text{sup\_acc step accept rho q } i \text{ } j = \text{Some (ts, tp)}$

**using**  $L\_def \ L\_props(2)$  **unfolding**  $L'_props(4)$

**by** (auto simp add: sup\_acc\_def)

**then show**  $\text{sup\_acc step accept rho q } i \text{ } j = \text{sup\_acc step accept rho (steps step rho q (i, l)) } l \text{ } j$

**using**  $\text{assms(3) by auto}$

**qed**

**lemma**  $\text{sup\_acc\_comp\_None: } i \leq l \implies l \leq j \implies$   
 $\text{sup\_acc step accept rho (steps step rho q (i, l)) } l \text{ } j = \text{None} \implies$   
 $\text{sup\_acc step accept rho q } i \text{ } j = \text{sup\_acc step accept rho q } i \text{ } l$

**proof** (induction  $j - l$  arbitrary:  $l$ )

**case** (Suc  $n$ )

**have**  $i\_lt\_j: i < j$

```

using Suc by auto
have l_lt_j: l < j
using Suc by auto
have ¬acc step accept rho q (i, Suc l)
using sup_acc_NoneE[of l l j step accept rho steps step rho q (i, l)] Suc(2-)
by (auto simp add: acc_def steps_def)
then have sup_acc step accept rho q i (l + 1) = sup_acc step accept rho q i l
using sup_acc_ext_idle[OF Suc(3)] by auto
moreover have sup_acc step accept rho (steps step rho q (i, l + 1)) (l + 1) j = None
using sup_acc_None_restrict[OF Suc(4,5)] steps_app[OF Suc(3), of step rho]
by auto
ultimately show ?case
using Suc(1)[of l + 1] Suc(2,3,4,5)
by auto
qed (auto simp add: sup_acc_same)

```

```

lemma sup_acc_ext: i ≤ j ⇒ acc step accept rho q (i, Suc j) ⇒
sup_acc step accept rho q i (Suc j) = Some (ts_at rho j, j)

```

```

proof -
assume assms: i ≤ j acc step accept rho q (i, Suc j)
define L' where L' = {l ∈ {i..<j + 1}. acc step accept rho q (i, Suc l)}
have j_in_L': finite L' L' ≠ {} j ∈ L'
using assms unfolding L'_def by auto
have j_is_Max: Max L' = j
using Max_eq_iff[OF j_in_L'(1,2)] j_in_L'(3)
by (auto simp add: L'_def)
show sup_acc step accept rho q i (Suc j) = Some (ts_at rho j, j)
using L'_def j_is_Max j_in_L'(2)
by (auto simp add: sup_acc_def)
qed

```

```

lemma sup_acc_None: i < j ⇒ sup_acc step accept rho q i j = None ⇒
sup_acc step accept rho (step q (bs_at rho i)) (i + 1) j = None
using steps_split[of _ _ step rho]
by (auto simp add: sup_acc_def Let_def acc_def split: if_splits)

```

```

lemma sup_acc_i: i < j ⇒ sup_acc step accept rho q i j = Some (ts, i) ⇒
sup_acc step accept rho (step q (bs_at rho i)) (Suc i) j = None

```

```

proof (rule ccontr)
assume assms: i < j sup_acc step accept rho q i j = Some (ts, i)
sup_acc step accept rho (step q (bs_at rho i)) (Suc i) j ≠ None
from assms(3) obtain l where l_def: l ∈ {Suc i..<j}
acc step accept rho (step q (bs_at rho i)) (Suc i, Suc l)
by (auto simp add: sup_acc_def Let_def split: if_splits)
define L' where L' = {l ∈ {i..<j}. acc step accept rho q (i, Suc l)}
from assms(2) have L'_props: finite L' L' ≠ {} Max L' = i
by (auto simp add: sup_acc_def L'_def Let_def split: if_splits)
have i_lt_l: i < l
using l_def(1) by auto
from l_def have l ∈ L'
unfolding L'_def acc_def using steps_split[OF i_lt_l, of step rho] by (auto simp: steps_def)
then show False
using l_def(1) L'_props Max_ge i_lt_l not_le by auto
qed

```

```

lemma sup_acc_l: i < j ⇒ i ≠ l ⇒ sup_acc step accept rho q i j = Some (ts, l) ⇒
sup_acc step accept rho q i j = sup_acc step accept rho (step q (bs_at rho i)) (Suc i) j
proof -

```

```

assume assms:  $i < j \wedge i \neq l \sup\_acc \ step \ accept \ rho \ q \ i \ j = Some \ (ts, l)$ 
define L where  $L = \{l \in \{i..<j\}. \ acc \ step \ accept \ rho \ q \ (i, \ Suc \ l)\}$ 
define L' where  $L' = \{l \in \{Suc \ i..<j\}. \ acc \ step \ accept \ rho \ (step \ q \ (bs\_at \ rho \ i)) \ (Suc \ i, \ Suc \ l)\}$ 
from assms(3) have L_props:  $finite \ L \ L \neq \{\} \ l = Max \ L$ 
 $\sup\_acc \ step \ accept \ rho \ q \ i \ j = Some \ (ts\_at \ rho \ l, l)$ 
by (auto simp add: sup_acc_def L_def Let_def split: if_splits)
have  $l\_in\_L: l \in L$ 
using Max_in[OF L_props(1,2)] L_props(3) by auto
then have  $i\_lt\_l: i < l$ 
unfolding L_def using assms(2) by auto
have  $l\_in\_L': finite \ L' \ L' \neq \{\} \ l \in L'$ 
using steps_split[OF  $i\_lt\_l$ , of step rho q]  $l\_in\_L$  assms(2)
unfolding L_def L'_def acc_def by (auto simp: steps_def)
have  $\bigwedge l'. l' \in L' \implies l' \leq l$ 
proof -
  fix l'
  assume assms:  $l' \in L'$ 
  have  $i\_lt\_l': i < l'$ 
  using assms unfolding L'_def by auto
  have  $l' \in L$ 
  using steps_split[OF  $i\_lt\_l'$ , of step rho] assms unfolding L_def L'_def acc_def by (auto simp: steps_def)
  then show  $l' \leq l$ 
  using L_props by simp
qed
then have  $l\_sup\_L': Max \ L' = l$ 
using Max_eq_iff[OF  $l\_in\_L'(1,2)$ ]  $l\_in\_L'(3)$  by auto
then show  $\sup\_acc \ step \ accept \ rho \ q \ i \ j =$ 
 $\sup\_acc \ step \ accept \ rho \ (step \ q \ (bs\_at \ rho \ i)) \ (Suc \ i) \ j$ 
unfolding L_props(4)
unfolding sup_acc_def Let_def
using L'_def l_in_L'(2,3) L_props
unfolding Suc_eq_plus1 by auto
qed

```

```

lemma sup_leadsto_idle:  $i < j \implies steps \ step \ rho \ init \ (i, j) \neq q \implies$ 
 $\sup\_leadsto \ init \ step \ rho \ i \ j \ q = \sup\_leadsto \ init \ step \ rho \ (i + 1) \ j \ q$ 
proof -
assume assms:  $i < j \ steps \ step \ rho \ init \ (i, j) \neq q$ 
define L where  $L = \{l. l < i \wedge steps \ step \ rho \ init \ (l, j) = q\}$ 
define L' where  $L' = \{l. l < i + 1 \wedge steps \ step \ rho \ init \ (l, j) = q\}$ 
have  $L\_L': L = L'$ 
unfolding L_def L'_def using assms(2) less_antisym
by fastforce
show  $\sup\_leadsto \ init \ step \ rho \ i \ j \ q = \sup\_leadsto \ init \ step \ rho \ (i + 1) \ j \ q$ 
using L_def L'_def L\_L'
by (auto simp add: sup_leadsto_def)
qed

```

```

lemma sup_leadsto_SomeI:  $l < i \implies steps \ step \ rho \ init \ (l, j) = q \implies$ 
 $\exists l'. \sup\_leadsto \ init \ step \ rho \ i \ j \ q = Some \ (ts\_at \ rho \ l') \wedge l \leq l' \wedge l' < i$ 
proof -
assume assms:  $l < i \ steps \ step \ rho \ init \ (l, j) = q$ 
define L' where  $L' = \{l. l < i \wedge steps \ step \ rho \ init \ (l, j) = q\}$ 
have fin_L':  $finite \ L'$ 
unfolding L'_def by auto
moreover have L'_nonempty:  $L' \neq \{\}$ 
using assms unfolding L'_def

```

by (auto simp add: sup\_leadsto\_def split: if\_splits)  
 ultimately have  $\text{Max } L' \in L'$   
 using Max\_in by auto  
 then show  $\exists l'. \text{sup\_leadsto init step rho } i j q = \text{Some } (ts\_at \text{ rho } l') \wedge l \leq l' \wedge l' < i$   
 using L'\_def L\_nonempty assms  
 by (fastforce simp add: sup\_leadsto\_def split: if\_splits)  
 qed

**lemma** *sup\_leadsto\_SomeE*:  $i \leq j \implies \text{sup\_leadsto init step rho } i j q = \text{Some } ts \implies \exists l < i. \text{steps step rho init } (l, j) = q \wedge ts\_at \text{ rho } l = ts$

**proof** –

assume *assms*:  $i \leq j \text{sup\_leadsto init step rho } i j q = \text{Some } ts$   
 define *L'* **where**  $L' = \{l. l < i \wedge \text{steps step rho init } (l, j) = q\}$   
 have *fin\_L'*: *finite* *L'*  
 unfolding L'\_def by auto  
 moreover have *L\_nonempty*:  $L' \neq \{\}$   
 using *assms*(2) unfolding L'\_def  
 by (auto simp add: sup\_leadsto\_def split: if\_splits)  
 ultimately have  $\text{Max } L' \in L'$   
 using Max\_in by auto  
 then show  $\exists l < i. \text{steps step rho init } (l, j) = q \wedge ts\_at \text{ rho } l = ts$   
 using *assms*(2) L'\_def  
 apply (auto simp add: sup\_leadsto\_def split: if\_splits)  
 using  $\langle \text{Max } L' \in L' \rangle$  by blast

qed

**lemma** *Mapping\_keys\_dest*:  $x \in \text{mmap\_keys } f \implies \exists y. \text{mmap\_lookup } f x = \text{Some } y$   
 by (auto simp add: mmap\_keys\_def mmap\_lookup\_def weak\_map\_of\_SomeI)

**lemma** *Mapping\_keys\_intro*:  $\text{mmap\_lookup } f x \neq \text{None} \implies x \in \text{mmap\_keys } f$   
 by (auto simp add: mmap\_keys\_def mmap\_lookup\_def)  
 (metis *map\_of\_eq\_None\_iff option.distinct*(1))

**lemma** *Mapping\_not\_keys\_intro*:  $\text{mmap\_lookup } f x = \text{None} \implies x \notin \text{mmap\_keys } f$   
 unfolding mmap\_lookup\_def mmap\_keys\_def  
 using weak\_map\_of\_SomeI by force

**lemma** *Mapping\_lookup\_None\_intro*:  $x \notin \text{mmap\_keys } f \implies \text{mmap\_lookup } f x = \text{None}$   
 unfolding mmap\_lookup\_def mmap\_keys\_def  
 by (simp add: map\_of\_eq\_None\_iff)

**primrec** *mmap\_combine* ::  $'key \Rightarrow 'val \Rightarrow ('val \Rightarrow 'val \Rightarrow 'val) \Rightarrow ('key \times 'val) \text{list} \Rightarrow ('key \times 'val) \text{list}$

**where**

$\text{mmap\_combine } k v c [] = [(k, v)]$   
 $|\ \text{mmap\_combine } k v c (p \# ps) = (\text{case } p \text{ of } (k', v') \Rightarrow$   
 $\quad \text{if } k = k' \text{ then } (k, c v' v) \# ps \text{ else } p \# \text{mmap\_combine } k v c ps)$

**lemma** *mmap\_combine\_distinct\_set*:  $\text{distinct } (\text{map fst } r) \implies \text{distinct } (\text{map fst } (\text{mmap\_combine } k v c r)) \wedge \text{set } (\text{map fst } (\text{mmap\_combine } k v c r)) = \text{set } (\text{map fst } r) \cup \{k\}$   
 by (induction *r*) force+

**lemma** *mmap\_combine\_lookup*:  $\text{distinct } (\text{map fst } r) \implies \text{mmap\_lookup } (\text{mmap\_combine } k v c r) z =$   
 $(\text{if } k = z \text{ then } (\text{case } \text{mmap\_lookup } r k \text{ of } \text{None} \Rightarrow \text{Some } v \mid \text{Some } v' \Rightarrow \text{Some } (c v' v))$   
 $\text{else } \text{mmap\_lookup } r z)$   
 using *eq\_key\_imp\_eq\_value*  
 by (induction *r*) (fastforce simp: mmap\_lookup\_def split: option.splits)+

**definition** `mmap_fold` :: ('c, 'd) mmap ⇒ (('c × 'd) ⇒ ('c × 'd)) ⇒ ('d ⇒ 'd ⇒ 'd) ⇒ ('c, 'd) mmap ⇒ ('c, 'd) mmap **where**  
`mmap_fold m f c r = foldl (λr p. case f p of (k, v) ⇒ mmap_combine k v c r) r m`

**definition** `mmap_fold'` :: ('c, 'd) mmap ⇒ 'e ⇒ (('c × 'd) × 'e ⇒ ('c × 'd) × 'e) ⇒ ('d ⇒ 'd ⇒ 'd) ⇒ ('c, 'd) mmap ⇒ ('c, 'd) mmap × 'e **where**  
`mmap_fold' m e f c r = foldl (λ(r, e) p. case f (p, e) of ((k, v), e') ⇒ (mmap_combine k v c r, e')) (r, e) m`

**lemma** `mmap_fold'_eq`: `mmap_fold' m e f' c r = (m', e') ⇒ P e ⇒ (∧p e p' e'. P e ⇒ f' (p, e) = (p', e') ⇒ p' = f p ∧ P e') ⇒ m' = mmap_fold m f c r ∧ P e'`

**proof** (induction m arbitrary: e r m' e')

**case** (Cons p m)

**obtain** k v e'' **where** kv\_def: f' (p, e) = ((k, v), e'') P e''

**using** Cons

**by** (cases f' (p, e)) fastforce

**have** `mmap_fold`: `mmap_fold m f c (mmap_combine k v c r) = mmap_fold (p # m) f c r`

**using** Cons(1)[OF kv\_def(2), **where** ?r=mmap\_combine k v c r] Cons(2,3,4)

**by** (simp add: mmap\_fold\_def mmap\_fold'\_def kv\_def(1))

**have** `mmap_fold'`: `mmap_fold' m e'' f' c (mmap_combine k v c r) = (m', e')`

**using** Cons(2)

**by** (auto simp: mmap\_fold'\_def kv\_def)

**show** ?case

**using** Cons(1)[OF mmap\_fold' kv\_def(2) Cons(4)]

**unfolding** `mmap_fold`

**by** auto

**qed** (auto simp: mmap\_fold\_def mmap\_fold'\_def)

**lemma** `foldl_mmap_combine_distinct_set`: `distinct (map fst r) ⇒`

`distinct (map fst (mmap_fold m f c r)) ∧`

`set (map fst (mmap_fold m f c r)) = set (map fst r) ∪ set (map (fst ∘ f) m)`

**apply** (induction m arbitrary: r)

**using** `mmap_combine_distinct_set`

**apply** (auto simp: mmap\_fold\_def split: prod.splits)

**apply** force

**apply** (smt Un\_iff fst\_conv imageI insert\_iff)

**using** `mk_disjoint_insert`

**apply** fastforce+

**done**

**lemma** `mmap_fold_lookup_rec`: `distinct (map fst r) ⇒ mmap_lookup (mmap_fold m f c r) z =`

`(case map (snd ∘ f) (filter (λ(k, v). fst (f (k, v)) = z) m) of [] ⇒ mmap_lookup r z`

`| v # vs ⇒ (case mmap_lookup r z of None ⇒ Some (foldl c v vs)`

`| Some w ⇒ Some (foldl c w (v # vs))))`

**proof** (induction m arbitrary: r)

**case** (Cons p ps)

**obtain** k v **where** kv\_def: f p = (k, v)

**by** fastforce

**have** `distinct`: `distinct (map fst (mmap_combine k v c r))`

**using** `mmap_combine_distinct_set`[OF Cons(2)]

**by** auto

**show** ?case

**using** Cons(1)[OF distinct, unfolded mmap\_combine\_lookup[OF Cons(2)]]

**by** (auto simp: mmap\_lookup\_def kv\_def mmap\_fold\_def split: list.splits option.splits)

**qed** (auto simp: mmap\_fold\_def)

**lemma** *mmap\_fold\_distinct*:  $\text{distinct} (\text{map fst } m) \implies \text{distinct} (\text{map fst} (\text{mmap\_fold } m \text{ f c } []))$   
**using** *foldl\_mmap\_combine\_distinct\_set*[of []]  
**by** *auto*

**lemma** *mmap\_fold\_set*:  $\text{distinct} (\text{map fst } m) \implies$   
 $\text{set} (\text{map fst} (\text{mmap\_fold } m \text{ f c } [])) = (\text{fst} \circ \text{f}) \text{ ` set } m$   
**using** *foldl\_mmap\_combine\_distinct\_set*[of []]  
**by** *force*

**lemma** *mmap\_lookup\_empty*:  $\text{mmap\_lookup } [] \text{ z} = \text{None}$   
**by** (*auto simp: mmap\_lookup\_def*)

**lemma** *mmap\_fold\_lookup*:  $\text{distinct} (\text{map fst } m) \implies \text{mmap\_lookup} (\text{mmap\_fold } m \text{ f c } []) \text{ z} =$   
 $(\text{case map (snd} \circ \text{f) (filter } (\lambda(k, v). \text{fst } (\text{f } (k, v)) = \text{z}) \text{ m) of } [] \Rightarrow \text{None}$   
 $| v \# \text{ vs} \Rightarrow \text{Some } (\text{foldl } c \text{ v vs}))$   
**using** *mmap\_fold\_lookup\_rec*[of [] \_ f c]  
**by** (*auto simp: mmap\_lookup\_empty split: list.splits*)

**definition** *fold\_sup* :: ('c, 'd :: timestamp) *mmap*  $\Rightarrow$  ('c  $\Rightarrow$  'c)  $\Rightarrow$  ('c, 'd) *mmap* **where**  
*fold\_sup* *m f* = *mmap\_fold* *m* ( $\lambda(x, y). (\text{f } x, y)$ ) *sup* []

**lemma** *mmap\_lookup\_distinct*:  $\text{distinct} (\text{map fst } m) \implies (k, v) \in \text{set } m \implies$   
 $\text{mmap\_lookup } m \text{ k} = \text{Some } v$   
**by** (*auto simp: mmap\_lookup\_def*)

**lemma** *fold\_sup\_distinct*:  $\text{distinct} (\text{map fst } m) \implies \text{distinct} (\text{map fst} (\text{fold\_sup } m \text{ f}))$   
**using** *mmap\_fold\_distinct*  
**by** (*auto simp: fold\_sup\_def*)

**lemma** *fold\_sup*:  
**fixes** *v* :: 'd :: timestamp  
**shows**  $\text{foldl } \text{sup } v \text{ vs} = \text{fold } \text{sup } \text{vs } v$   
**by** (*induction vs arbitrary: v*) (*auto simp: sup.commute*)

**lemma** *lookup\_fold\_sup*:  
**assumes** *distinct*:  $\text{distinct} (\text{map fst } m)$   
**shows**  $\text{mmap\_lookup} (\text{fold\_sup } m \text{ f}) \text{ z} =$   
 $(\text{let } Z = \{x \in \text{mmap\_keys } m. \text{f } x = \text{z}\} \text{ in}$   
 $\text{if } Z = \{\} \text{ then None else Some } (\text{Sup\_fin } ((\text{the} \circ \text{mmap\_lookup } m) \text{ ` } Z)))$   
**proof** (*cases*  $\{x \in \text{mmap\_keys } m. \text{f } x = \text{z}\} = \{\}$ )

**case** *True*  
**have**  $z \notin \text{mmap\_keys} (\text{mmap\_fold } m (\lambda(x, y). (\text{f } x, y)) \text{sup } [])$   
**using** *True*[*unfolded* *mmap\_keys\_def*] *mmap\_fold\_set*[*OF distinct*]  
**by** (*auto simp: mmap\_keys\_def*)  
**then have**  $\text{mmap\_lookup} (\text{fold\_sup } m \text{ f}) \text{ z} = \text{None}$   
**unfolding** *fold\_sup\_def*  
**by** (*meson Mapping\_keys\_intro*)  
**then show** *?thesis*  
**unfolding** *True*  
**by** *auto*

**next**  
**case** *False*  
**have**  $z \text{ in\_keys: } z \in \text{mmap\_keys} (\text{mmap\_fold } m (\lambda(x, y). (\text{f } x, y)) \text{sup } [])$   
**using** *False*[*unfolded* *mmap\_keys\_def*] *mmap\_fold\_set*[*OF distinct*]  
**by** (*force simp: mmap\_keys\_def*)  
**obtain** *v vs* **where** *vs\_def*:  $\text{mmap\_lookup} (\text{fold\_sup } m \text{ f}) \text{ z} = \text{Some } (\text{foldl } \text{sup } v \text{ vs})$   
 $v \# \text{ vs} = \text{map snd } (\text{filter } (\lambda(k, v). \text{f } k = \text{z}) \text{ m})$   
**using** *mmap\_fold\_lookup*[*OF distinct*, of  $(\lambda(x, y). (\text{f } x, y)) \text{sup } z$ ]

```

    Mapping_keys_dest[OF z_in_keys]
  by (force simp: fold_sup_def mmap_keys_def comp_def split: list.splits prod.splits)
have set (v # vs) = (the ∘ mmap_lookup m) ‘ {x ∈ mmap_keys m. f x = z}
proof (rule set_eqI, rule iffI)
  fix w
  assume w ∈ set (v # vs)
  then obtain x where x_def: x ∈ mmap_keys m f x = z (x, w) ∈ set m
    using vs_def(2)
  by (auto simp add: mmap_keys_def rev_image_eqI)
  show w ∈ (the ∘ mmap_lookup m) ‘ {x ∈ mmap_keys m. f x = z}
    using x_def(1,2) mmap_lookup_distinct[OF distinct x_def(3)]
  by force
next
fix w
assume w ∈ (the ∘ mmap_lookup m) ‘ {x ∈ mmap_keys m. f x = z}
then obtain x where x_def: x ∈ mmap_keys m f x = z (x, w) ∈ set m
  using mmap_lookup_distinct[OF distinct]
  by (auto simp add: Mapping_keys_intro distinct mmap_lookup_def dest: Mapping_keys_dest)
show w ∈ set (v # vs)
  using x_def
  by (force simp: vs_def(2))
qed
then have foldl sup v vs = Sup_fin ((the ∘ mmap_lookup m) ‘ {x ∈ mmap_keys m. f x = z})
  unfolding fold_sup
  by (metis Sup_fin.set_eq_fold)
then show ?thesis
  using False
  by (auto simp: vs_def(1))
qed

definition mmap_map :: ('a ⇒ 'b ⇒ 'c) ⇒ ('a, 'b) mmap ⇒ ('a, 'c) mmap where
  mmap_map f m = map (λ(k, v). (k, f k v)) m

lemma mmap_map_keys: mmap_keys (mmap_map f m) = mmap_keys m
  by (force simp: mmap_map_def mmap_keys_def)

lemma mmap_map_fst: map fst (mmap_map f m) = map fst m
  by (auto simp: mmap_map_def)

definition cstep :: ('c ⇒ 'b ⇒ 'c) ⇒ ('c × 'b, 'c) mapping ⇒
  'c ⇒ 'b ⇒ ('c × ('c × 'b, 'c) mapping) where
  cstep step st q bs = (case Mapping.lookup st (q, bs) of None ⇒ (let res = step q bs in
    (res, Mapping.update (q, bs) res st)) | Some v ⇒ (v, st))

definition cac :: ('c ⇒ bool) ⇒ ('c, bool) mapping ⇒ 'c ⇒ (bool × ('c, bool) mapping) where
  cac accept ac q = (case Mapping.lookup ac q of None ⇒ (let res = accept q in
    (res, Mapping.update q res ac)) | Some v ⇒ (v, ac))

fun mmap_fold_s :: ('c ⇒ 'b ⇒ 'c) ⇒ ('c × 'b, 'c) mapping ⇒
  ('c ⇒ bool) ⇒ ('c, bool) mapping ⇒
  'b ⇒ 'd ⇒ nat ⇒ ('c, 'c × ('d × nat) option) mmap ⇒
  (('c, 'c × ('d × nat) option) mmap × ('c × 'b, 'c) mapping × ('c, bool) mapping) where
  mmap_fold_s step st accept ac bs t j [] = ([], st, ac)
| mmap_fold_s step st accept ac bs t j ((q, (q', tstp)) # qbss) =
  (let (q'', st') = cstep step st q' bs;
    (β, ac') = cac accept ac q'';
    (qbss', st'', ac'') = mmap_fold_s step st' accept ac' bs t j qbss in
  ((q, (q'', if β then Some (t, j) else tstp)) # qbss', st'', ac''))

```

**lemma** *mmap\_fold\_s\_sound*:  $mmap\_fold\_s\ step\ st\ accept\ ac\ bs\ t\ j\ qbss = (qbss', st', ac') \implies$   
 $(\bigwedge q\ bs.\ case\ Mapping.lookup\ st\ (q,\ bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v) \implies$   
 $(\bigwedge q\ bs.\ case\ Mapping.lookup\ ac\ q\ of\ None \Rightarrow True \mid Some\ v \Rightarrow accept\ q = v) \implies$   
 $qbss' = mmap\_map\ (\lambda q\ (q',\ tstp).\ (step\ q'\ bs,\ if\ accept\ (step\ q'\ bs)\ then\ Some\ (t,\ j)\ else\ tstp))\ qbss \wedge$   
 $(\forall q\ bs.\ case\ Mapping.lookup\ st'\ (q,\ bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v) \wedge$   
 $(\forall q\ bs.\ case\ Mapping.lookup\ ac'\ q\ of\ None \Rightarrow True \mid Some\ v \Rightarrow accept\ q = v)$

**proof** (induction qbss arbitrary: st ac qbss')

**case** (Cons a qbss)

**obtain**  $q\ q'\ tstp$  **where**  $a\_def: a = (q,\ (q',\ tstp))$

**by** (cases a) auto

**obtain**  $q''\ st''$  **where**  $q''\_def: cstep\ step\ st\ q'\ bs = (q'',\ st'')$

$q'' = step\ q'\ bs$

**using** Cons(3)

**by** (cases cstep step st q' bs)

(auto simp: cstep\_def Let\_def option.case\_eq\_if split: option.splits if\_splits)

**obtain**  $b\ ac''$  **where**  $b\_def: cac\ accept\ ac\ q'' = (b,\ ac'')$

$b = accept\ q''$

**using** Cons(4)

**by** (cases cac accept ac q'')

(auto simp: cac\_def Let\_def option.case\_eq\_if split: option.splits if\_splits)

**obtain**  $qbss''$  **where**  $qbss''\_def: mmap\_fold\_s\ step\ st''\ accept\ ac''\ bs\ t\ j\ qbss =$

$(qbss'',\ st'',\ ac'')\ qbss' = (q,\ q'',\ if\ b\ then\ Some\ (t,\ j)\ else\ tstp)\ \#\ qbss''$

**using** Cons(2)[unfolded a\_def mmap\_fold\_s.simps q''\_def(1) b\_def(1)]

**unfolding** Let\_def

**by** (auto simp: b\_def(1) split: prod.splits)

**have**  $ih: \bigwedge q\ bs.\ case\ Mapping.lookup\ st''\ (q,\ bs)\ of\ None \Rightarrow True \mid Some\ a \Rightarrow step\ q\ bs = a$

$\bigwedge q\ bs.\ case\ Mapping.lookup\ ac''\ q\ of\ None \Rightarrow True \mid Some\ a \Rightarrow accept\ q = a$

**using** q''\_def b\_def Cons(3,4)

**by** (auto simp: cstep\_def cac\_def Let\_def Mapping.lookup\_update' option.case\_eq\_if split: option.splits if\_splits)

**show** ?case

**using** Cons(1)[OF qbss''\_def(1) ih]

**unfolding** a\_def q''\_def(2) b\_def(2) qbss''\_def(2)

**by** (auto simp: mmap\_map\_def)

**qed** (auto simp: mmap\_map\_def)

**definition** *adv\_end* ::  $(b,\ 'c,\ 'd :: timestamp,\ 't,\ 'e)\ args \Rightarrow$

$(b,\ 'c,\ 'd,\ 't,\ 'e)\ window \Rightarrow (b,\ 'c,\ 'd,\ 't,\ 'e)\ window\ option$  **where**

$adv\_end\ args\ w = (let\ step = w\_step\ args;\ accept = w\_accept\ args;$

$run\_t = w\_run\_t\ args;\ run\_sub = w\_run\_sub\ args;\ st = w\_st\ w;\ ac = w\_ac\ w;$

$j = w\_j\ w;\ tj = w\_tj\ w;\ sj = w\_sj\ w;\ s = w\_s\ w;\ e = w\_e\ w\ in$

$(case\ run\_t\ tj\ of\ None \Rightarrow None \mid Some\ (tj',\ t) \Rightarrow (case\ run\_sub\ sj\ of\ None \Rightarrow None \mid Some\ (sj',\ bs)$

$\Rightarrow$

$let\ (s',\ st',\ ac') = mmap\_fold\_s\ step\ st\ accept\ ac\ bs\ t\ j\ s;$

$(e',\ st'') = mmap\_fold'\ e\ st'\ (\lambda((x,\ y),\ st).\ let\ (q',\ st') = cstep\ step\ st\ x\ bs\ in\ ((q',\ y),\ st'))\ sup\ []\ in$

$Some\ (w(w\_st := st'',\ w\_ac := ac',\ w\_j := Suc\ j,\ w\_tj := tj',\ w\_sj := sj',\ w\_s := s',\ w\_e := e'))))$

**lemma** *map\_values\_lookup*:  $mmap\_lookup\ (mmap\_map\ f\ m)\ z = Some\ v' \implies$

$\exists v.\ mmap\_lookup\ m\ z = Some\ v \wedge v' = f\ z\ v$

**by** (induction m) (auto simp: mmap\_lookup\_def mmap\_map\_def)

**lemma** *acc\_app*:

**assumes**  $i \leq j$  *steps*  $\rho\ q\ (i,\ Suc\ j) = q'\ accept\ q'$

**shows**  $sup\_acc\ step\ accept\ \rho\ q\ i\ (Suc\ j) = Some\ (ts\_at\ \rho\ j,\ j)$

**proof** –

**define**  $L$  **where**  $L = \{l \in \{i..<Suc\ j\}.\ accept\ (steps\ step\ \rho\ q\ (i,\ Suc\ l))\}$

```

have nonempty: finite L L ≠ {}
  using assms unfolding L_def by auto
moreover have Max {l ∈ {i..<Suc j}. accept (steps step rho q (i, Suc l))} = j
  unfolding L_def[symmetric] Max_eq_iff[OF nonempty, of j]
  unfolding L_def using assms by auto
ultimately show ?thesis
  by (auto simp add: sup_acc_def acc_def L_def)
qed

```

```

lemma acc_app_idle:
  assumes i ≤ j steps step rho q (i, Suc j) = q' ¬accept q'
  shows sup_acc step accept rho q i (Suc j) = sup_acc step accept rho q i j
  using assms
  by (auto simp add: sup_acc_def Let_def acc_def elim: less_SucE) (metis less_Suc_eq)+

```

```

lemma sup_fin_closed: finite A ⇒ A ≠ {} ⇒
  (∧x y. x ∈ A ⇒ y ∈ A ⇒ sup x y ∈ {x, y}) ⇒ ⋂fin A ∈ A
  apply (induct A rule: finite.induct)
  using Sup_fin.insert
  by auto fastforce

```

```

lemma valid_adv_end:
  assumes valid_window args t0 sub rho w w_run_t args (w_tj w) = Some (tj', t)
    w_run_sub args (w_sj w) = Some (sj', bs)
    ∧t'. t' ∈ set (map fst rho) ⇒ t' ≤ t
  shows case adv_end args w of None ⇒ False | Some w' ⇒ valid_window args t0 sub (rho @ [(t, bs)])
  w'

```

**proof** –

```

define init where init = w_init args
define step where step = w_step args
define accept where accept = w_accept args
define run_t where run_t = w_run_t args
define run_sub where run_sub = w_run_sub args
define st where st = w_st w
define ac where ac = w_ac w
define i where i = w_i w
define ti where ti = w_ti w
define si where si = w_si w
define j where j = w_j w
define tj where tj = w_tj w
define sj where sj = w_sj w
define s where s = w_s w
define e where e = w_e w
have valid_before: reach_window args t0 sub rho (i, ti, si, j, tj, sj)
  ∧i j. i ≤ j ⇒ j < length rho ⇒ ts_at rho i ≤ ts_at rho j
  (∧q bs. case Mapping.lookup st (q, bs) of None ⇒ True | Some v ⇒ step q bs = v)
  (∧q bs. case Mapping.lookup ac q of None ⇒ True | Some v ⇒ accept q = v)
  ∀q. mmap_lookup e q = sup_leadsto init step rho i j q distinct (map fst e)
  valid_s init step st accept rho i i j s
  using assms(1)
  unfolding valid_window_def valid_s_def Let_def init_def step_def accept_def run_t_def
    run_sub_def st_def ac_def i_def ti_def si_def j_def tj_def sj_def s_def e_def
  by auto
have i_j: i ≤ j
  using valid_before(1)
  by auto
have distinct_before: distinct (map fst s) distinct (map fst e)
  using valid_before

```

```

by (auto simp: valid_s_def)
note run_tj = assms(2)[folded run_t_def tj_def]
note run_sj = assms(3)[folded run_sub_def sj_def]
define rho' where rho' = rho @ [(t, bs)]
have ts_at_mono:  $\bigwedge i j. i \leq j \implies j < \text{length } \rho' \implies \text{ts\_at } \rho' i \leq \text{ts\_at } \rho' j$ 
  using valid_before(2) assms(4)
  by (auto simp: rho'_def ts_at_def nth_append split: option.splits list.splits if_splits)
obtain s' st' ac' where s'_def:  $\text{mmap\_fold\_s } \text{step } \text{st } \text{accept } \text{ac } \text{bs } t j s = (s', st', ac')$ 
  apply (cases  $\text{mmap\_fold\_s } \text{step } \text{st } \text{accept } \text{ac } \text{bs } t j s$ )
  apply (auto)
done
have s'_mmap_map:  $s' = \text{mmap\_map } (\lambda q (q', tstp).$ 
  (step q' bs, if accept (step q' bs) then Some (t, j) else tstp)) s
  ( $\bigwedge q \text{ bs. case Mapping.lookup } st' (q, \text{bs}) \text{ of None } \implies \text{True} \mid \text{Some } v \implies \text{step } q \text{ bs} = v$ )
  ( $\bigwedge q \text{ bs. case Mapping.lookup } ac' q \text{ of None } \implies \text{True} \mid \text{Some } v \implies \text{accept } q = v$ )
  using  $\text{mmap\_fold\_s\_sound}[OF s'_def valid_before(3,4)]$ 
  by auto
obtain e' st'' where e'_def:  $\text{mmap\_fold}' e st' (\lambda((x, y), st).$ 
  let (q', st') = cstep step st x bs in ((q', y), st')) sup [] = (e', st'')
  by (metis old.prod.exhaust)
define inv where  $\text{inv} \equiv \lambda st'. \forall q \text{ bs. case Mapping.lookup } st' (q, \text{bs}) \text{ of None } \implies \text{True}$ 
  |  $\text{Some } v \implies \text{step } q \text{ bs} = v$ 
have inv_st':  $\text{inv } st'$ 
  using s'_mmap_map(2)
  by (auto simp: inv_def)
have  $\bigwedge p e p' e'. \text{inv } e \implies (\text{case } (p, e) \text{ of } (x, xa) \implies (\text{case } x \text{ of } (x, y) \implies$ 
   $\lambda st. \text{let } (q', st') = \text{cstep } \text{step } \text{st } x \text{ bs in } ((q', y), st')) xa) = (p', e') \implies$ 
   $p' = (\text{case } p \text{ of } (x, y) \implies (\text{step } x \text{ bs}, y)) \wedge \text{inv } e'$ 
  by (auto simp: inv_def cstep_def Let_def Mapping.lookup_update' split: option.splits if_splits)
then have e'_fold_sup_st'':  $e' = \text{fold\_sup } e (\lambda q. \text{step } q \text{ bs})$ 
  ( $\bigwedge q \text{ bs. case Mapping.lookup } st'' (q, \text{bs}) \text{ of None } \implies \text{True} \mid \text{Some } v \implies \text{step } q \text{ bs} = v$ )
  using  $\text{mmap\_fold}'\_eq[OF e'_def, of inv \lambda(x, y). (\text{step } x \text{ bs}, y), OF inv\_st']$ 
  by (fastforce simp: fold_sup_def inv_def)+
have adv_end:  $\text{adv\_end } \text{args } w = \text{Some } (w \setminus w\_st := st'', w\_ac := ac',$ 
   $w\_j := \text{Suc } j, w\_tj := tj', w\_sj := sj', w\_s := s', w\_e := e')$ 
  using run_tj run_sj e'_def[unfolded st_def]
  unfolding adv_end_def init_def step_def accept_def run_t_def run_sub_def
  i_def ti_def si_def j_def tj_def sj_def s_def e_def s'_def e'_def
  by (auto simp: Let_def s'_def[unfolded step_def st_def accept_def ac_def j_def s_def])
have keys_s':  $\text{mmap\_keys } s' = \text{mmap\_keys } s$ 
  by (force simp: mmap_keys_def mmap_map_def s'_mmap_map(1))
have lookup_s':  $\bigwedge q q' tstp. \text{mmap\_lookup } s q = \text{Some } (q', tstp) \implies$ 
   $\text{steps } \text{step } \rho' q (i, j) = q' \wedge tstp = \text{sup\_acc } \text{step } \text{accept } \rho' q i j$ 
  using valid_before Mapping_keys_intro
  by (force simp add: Let_def rho'_def valid_s_def steps_app_cong sup_acc_app_cong
    split: option.splits)
have bs_at_rho'_j:  $\text{bs\_at } \rho' j = \text{bs}$ 
  using valid_before
  by (auto simp: rho'_def bs_at_def nth_append)
have ts_at_rho'_j:  $\text{ts\_at } \rho' j = t$ 
  using valid_before
  by (auto simp: rho'_def ts_at_def nth_append)
have lookup_s'':  $\bigwedge q q' tstp. \text{mmap\_lookup } s' q = \text{Some } (q', tstp) \implies$ 
   $\text{steps } \text{step } \rho' q (i, \text{Suc } j) = q' \wedge tstp = \text{sup\_acc } \text{step } \text{accept } \rho' q i (\text{Suc } j)$ 
proof -
  fix q q'' tstp'
  assume  $\text{assm: mmap\_lookup } s' q = \text{Some } (q'', tstp')$ 
  obtain q' tstp where  $\text{mmap\_lookup } s q = \text{Some } (q', tstp) \quad q'' = \text{step } q' \text{ bs}$ 

```

```

    tstp' = (if accept (step q' bs) then Some (t, j) else tstp)
  using map_values_lookup[OF assm[unfolded s'_mmap_map]] by auto
then show steps step rho' q (i, Suc j) = q'' ∧ tstp' = sup_acc step accept rho' q i (Suc j)
  using lookup_s
  apply (auto simp: bs_at_rho'_j ts_at_rho'_j)
    apply (metis Suc_eq_plus1 bs_at_rho'_j i_j steps_app)
    apply (metis acc_app bs_at_rho'_j i_j steps_appE ts_at_rho'_j)
    apply (metis Suc_eq_plus1 bs_at_rho'_j i_j steps_app)
    apply (metis (no_types, lifting) acc_app_idle bs_at_rho'_j i_j steps_appE)
  done
qed
have lookup_e: ∧q. mmap_lookup e q = sup_leadsto init step rho' i j q
  using valid_before sup_leadsto_app_cong[of _ _ rho init step]
  by (auto simp: rho'_def)
have keys_e_alt: mmap_keys e = {q. ∃ l < i. steps step rho' init (l, j) = q}
  using valid_before
  apply (auto simp add: sup_leadsto_def rho'_def)
  apply (metis (no_types, lifting) Mapping_keys_dest lookup_e rho'_def sup_leadsto_SomeE)
  apply (metis (no_types, lifting) Mapping_keys_intro option.simps(3) order_refl steps_app_cong)
  done
have finite_keys_e: finite (mmap_keys e)
  unfolding keys_e_alt
  by (rule finite_surj[of {l. l < i}]) auto
have reaches_on run_sub sub (map snd rho) sj
  using valid_before reaches_on_trans
  unfolding run_sub_def sub_def
  by fastforce
then have reaches_on': reaches_on run_sub sub (map snd rho @ [bs]) sj'
  using reaches_on_app run_sj
  by fast
have reaches_on run_t t0 (map fst rho) tj
  using valid_before reaches_on_trans
  unfolding run_t_def
  by fastforce
then have reach_t': reaches_on run_t t0 (map fst rho^') tj'
  using reaches_on_app run_tj
  unfolding rho'_def
  by fastforce
have lookup_e': ∧q. mmap_lookup e' q = sup_leadsto init step rho' i (Suc j) q
proof -
  fix q
  define Z where Z = {x ∈ mmap_keys e. step x bs = q}
  show mmap_lookup e' q = sup_leadsto init step rho' i (Suc j) q
  proof (cases Z = {})
    case True
    then have mmap_lookup e' q = None
      using Z_def lookup_fold_sup[OF distinct_before(2)]
      unfolding e'_fold_sup_st''
      by (auto simp: Let_def)
    moreover have sup_leadsto init step rho' i (Suc j) q = None
  proof (rule ccontr)
    assume assm: sup_leadsto init step rho' i (Suc j) q ≠ None
    obtain l where l_def: l < i steps step rho' init (l, Suc j) = q
      using i_j sup_leadsto_SomeE[of i Suc j] assm
      by force
    have l_j: l ≤ j
      using less_le_trans[OF l_def(1) i_j] by auto
    obtain q'' where q''_def: steps step rho' init (l, j) = q'' step q'' bs = q

```

```

    using steps_appE[OF l_def(2)] l_j
    by (auto simp: bs_at_rho'_j)
  then have q'' ∈ mmap_keys e
    using keys_e_alt l_def(1)
    by auto
  then show False
    using Z_def q''_def(2) True
    by auto
qed
ultimately show ?thesis
  by auto
next
case False
then have lookup_e': mmap_lookup e' q = Some (Sup_fin ((the ∘ mmap_lookup e) ' Z))
  using Z_def lookup_fold_sup[OF distinct_before(2)]
  unfolding e'_fold_sup_st''
  by (auto simp: Let_def)
define L where L = {l. l < i ∧ steps step rho' init (l, Suc j) = q}
have fin_L: finite L
  unfolding L_def by auto
have Z_alt: Z = {x. ∃ l < i. steps step rho' init (l, j) = x ∧ step x bs = q}
  using Z_def[unfolded keys_e_alt] by auto
have fin_Z: finite Z
  unfolding Z_alt by auto
have L_nonempty: L ≠ {}
  using L_def Z_alt False i_j steps_app[of _ _ step rho q]
  by (auto simp: bs_at_rho'_j)
  (smt Suc_eq_plus1 bs_at_rho'_j less_irrefl_nat less_le_trans nat_le_linear steps_app)
have sup_leadsto: sup_leadsto init step rho' i (Suc j) q = Some (ts_at rho' (Max L))
  using L_nonempty L_def
  by (auto simp add: sup_leadsto_def)
have j_lt_rho': j < length rho'
  using valid_before
  by (auto simp: rho'_def)
have Sup_fin ((the ∘ mmap_lookup e) ' Z) = ts_at rho' (Max L)
proof (rule antisym)
obtain z ts where zts_def: z ∈ Z (the ∘ mmap_lookup e) z = ts
  Sup_fin ((the ∘ mmap_lookup e) ' Z) = ts
proof -
  assume lassm:  $\bigwedge z ts. z \in Z \implies (the \circ mmap\_lookup \ e) \ z = ts \implies$ 
 $\bigsqcup_{fin} ((the \circ mmap\_lookup \ e) \ ' \ Z) = ts \implies thesis$ 
  define T where T = (the ∘ mmap_lookup e) ' Z
  have T_sub: T ⊆ ts_at rho' '{..j}
    using lookup_e keys_e_alt i_j
    by (auto simp add: T_def Z_def sup_leadsto_def)
  have finite T T ≠ {}
    using fin_Z False
    by (auto simp add: T_def)
  then have sup_in:  $\bigsqcup_{fin} T \in T$ 
  proof (rule sup_fin_closed)
    fix x y
    assume xy: x ∈ T y ∈ T
    then obtain a c where x = ts_at rho' a y = ts_at rho' c a ≤ j c ≤ j
      using T_sub
      by (meson atMost_iff imageE subsetD)
    then show sup x y ∈ {x, y}
      using ts_at_mono j_lt_rho'
      by (cases a ≤ c) (auto simp add: sup.absorb1 sup.absorb2)
  end
end

```

```

qed
then show ?thesis
  using lassm
  by (auto simp add: T_def)
qed
from zts_def(2) have lookup_e_z: mmap_lookup e z = Some ts
  using zts_def(1) Z_def by (auto dest: Mapping_keys_dest)
have sup_leadsto_init step rho' i j z = Some ts
  using lookup_e_z lookup_e
  by auto
then obtain l where l_def: l < i steps step rho' init (l, j) = z ts_at rho' l = ts
  using sup_leadsto_SomeE[OF i_j]
  by (fastforce simp: rho'_def ts_at_def nth_append)
have l_j: l ≤ j
  using less_le_trans[OF l_def(1) i_j] by auto
have l ∈ L
  unfolding L_def using l_def zts_def(1) Z_alt
  by auto (metis (no_types, lifting) Suc_eq_plus1 bs_at_rho'_j l_j steps_app)
then have l ≤ Max L Max L < i
  using L_nonempty fin_L
  by (auto simp add: L_def)
then show Sup_fin ((the ∘ mmap_lookup e) ' Z) ≤ ts_at rho' (Max L)
  unfolding zts_def(3) l_def(3)[symmetric]
  using ts_at_mono i_j j_lt_rho'
  by (auto simp: rho'_def)
next
obtain l where l_def: Max L = l l < i steps step rho' init (l, Suc j) = q
  using Max_in[OF fin_L L_nonempty] L_def by auto
obtain z where z_def: steps step rho' init (l, j) = z step z bs = q
  using l_def(2,3) i_j bs_at_rho'_j
  by (metis less_imp_le_nat less_le_trans steps_appE)
have z_in_Z: z ∈ Z
  unfolding Z_alt
  using l_def(2) z_def i_j
  by fastforce
have lookup_e_z: mmap_lookup e z = sup_leadsto_init step rho' i j z
  using lookup_e z_in_Z Z_alt
  by auto
obtain l' where l'_def: sup_leadsto_init step rho' i j z = Some (ts_at rho' l')
  l ≤ l' l' < i
  using sup_leadsto_SomeI[OF l_def(2) z_def(1)] by auto
have ts_at_rho' l' ∈ (the ∘ mmap_lookup e) ' Z
  using lookup_e_z l'_def(1) z_in_Z
  by force
then have ts_at_rho' l' ≤ Sup_fin ((the ∘ mmap_lookup e) ' Z)
  using Inf_fin_le_Sup_fin fin_Z z_in_Z
  by (simp add: Sup_fin.coboundedI)
then show ts_at_rho' (Max L) ≤ Sup_fin ((the ∘ mmap_lookup e) ' Z)
  unfolding l_def(1)
  using ts_at_mono l'_def(2,3) i_j j_lt_rho'
  by (fastforce simp: rho'_def)
qed
then show ?thesis
  unfolding lookup_e' sup_leadsto by auto
qed
qed
have distinct (map fst s') distinct (map fst e')
  using distinct_before mmap_fold_distinct

```

```

unfolding s'_mmap_map mmap_map fst e'_fold_sup_st'' fold_sup_def
by auto
moreover have mmap_keys s' = {q.  $\exists l \leq i. \text{steps step rho' init } (l, i) = q$ }
unfolding keys_s' rho'_def
using valid_before(1,7) valid_s_def[of init step st accept rho i i j s]
by (auto simp: steps_app_cong[of _ rho step])
moreover have reaches_on run_t ti (drop i (map fst rho')) tj'
reaches_on run_sub si (drop i (map snd rho')) sj'
using valid_before reaches_on_app run_tj run_sj
by (auto simp: rho'_def run_t_def run_sub_def)
ultimately show ?thesis
unfolding adv_end
using valid_before lookup_e' lookup_s' ts_at_mono s'_mmap_map(3) e'_fold_sup_st''(2)
by (fastforce simp: valid_window_def Let_def init_def step_def accept_def run_t_def
run_sub_def i_def ti_def si_def j_def tj_def sj_def s_def e'_def
rho'_def valid_s_def intro!: exI[of _ rho] split: option.splits)
qed

```

**lemma** *adv\_end\_bounds*:

```

assumes w_run_t args (w_tj w) = Some (tj', t)
w_run_sub args (w_sj w) = Some (sj', bs)
adv_end args w = Some w'
shows w_i w' = w_i w w_ti w' = w_ti w w_si w' = w_si w
w_j w' = Suc (w_j w) w_tj w' = tj' w_sj w' = sj'
using assms
by (auto simp: adv_end_def Let_def split: prod.splits)

```

**definition** *drop\_cur* ::  $\text{nat} \Rightarrow ('c \times ('d \times \text{nat}) \text{option}) \Rightarrow ('c \times ('d \times \text{nat}) \text{option})$  **where**  
*drop\_cur* i =  $\lambda(q', \text{tstp}). (q', \text{case } \text{tstp} \text{ of } \text{Some } (ts, tp) \Rightarrow$   
*if*  $tp = i$  *then*  $\text{None}$  *else*  $\text{tstp} \mid \text{None} \Rightarrow \text{tstp}$ )

**definition** *adv\_d* ::  $('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('c \times 'b, 'c) \text{ mapping} \Rightarrow \text{nat} \Rightarrow 'b \Rightarrow$   
 $('c, 'c \times ('d \times \text{nat}) \text{option}) \text{ mmap} \Rightarrow$   
 $((('c, 'c \times ('d \times \text{nat}) \text{option}) \text{ mmap} \times ('c \times 'b, 'c) \text{ mapping}) \text{ where}$   
*adv\_d* *step* *st* *i* *b* *s* =  $(\text{mmap\_fold}' s \text{ st } (\lambda((x, v), \text{st}). \text{case } \text{cstep } \text{step } \text{st } x \text{ b of } (x', \text{st}') \Rightarrow$   
 $((x', \text{drop\_cur } i \text{ v}), \text{st}')) (\lambda x y. x) \square$ )

**lemma** *adv\_d\_mmap\_fold*:

```

assumes inv:  $\bigwedge q \text{ bs. case } \text{Mapping.lookup } \text{st } (q, \text{bs}) \text{ of } \text{None} \Rightarrow \text{True} \mid \text{Some } v \Rightarrow \text{step } q \text{ bs} = v$ 
and fold':  $\text{mmap\_fold}' s \text{ st } (\lambda((x, v), \text{st}). \text{case } \text{cstep } \text{step } \text{st } x \text{ bs of } (x', \text{st}') \Rightarrow$   

 $((x', \text{drop\_cur } i \text{ v}), \text{st}')) (\lambda x y. x) r = (s', \text{st}')$ 
shows  $s' = \text{mmap\_fold } s (\lambda(x, v). (\text{step } x \text{ bs}, \text{drop\_cur } i \text{ v})) (\lambda x y. x) r \wedge$   

 $(\forall q \text{ bs. case } \text{Mapping.lookup } \text{st}' (q, \text{bs}) \text{ of } \text{None} \Rightarrow \text{True} \mid \text{Some } v \Rightarrow \text{step } q \text{ bs} = v)$ 

```

**proof** –

```

define inv where  $\text{inv} \equiv \lambda \text{st}. \forall q \text{ bs. case } \text{Mapping.lookup } \text{st } (q, \text{bs}) \text{ of } \text{None} \Rightarrow \text{True}$   

 $\mid \text{Some } v \Rightarrow \text{step } q \text{ bs} = v$ 
have inv_st:  $\text{inv } \text{st}$ 
using inv
by (auto simp: inv_def)
show ?thesis
by (rule mmap_fold'_eq[OF fold', of inv  $\lambda(x, v). (\text{step } x \text{ bs}, \text{drop\_cur } i \text{ v})$ ,  

OF inv_st, unfolded inv_def])
(auto simp: cstep_def Let_def Mapping.lookup_update'
split: prod.splits option.splits if_splits)

```

**qed**

**definition** *keys\_idem* ::  $('c \Rightarrow 'b \Rightarrow 'c) \Rightarrow \text{nat} \Rightarrow 'b \Rightarrow$   
 $('c, 'c \times ('d \times \text{nat}) \text{option}) \text{ mmap} \Rightarrow \text{bool}$  **where**

$keys\_idem\ step\ i\ b\ s = (\forall x \in mmap\_keys\ s. \forall x' \in mmap\_keys\ s.$   
 $step\ x\ b = step\ x'\ b \longrightarrow drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x)) =$   
 $drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x')))$

**lemma** *adv\_d\_keys*:

**assumes** *inv*:  $\bigwedge q\ bs. case\ Mapping.lookup\ st\ (q, bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v$   
**and** *distinct*:  $distinct\ (map\ fst\ s)$   
**and** *adv\_d*:  $adv\_d\ step\ st\ i\ bs\ s = (s', st')$   
**shows**  $mmap\_keys\ s' = (\lambda q. step\ q\ bs) \text{ ' } (mmap\_keys\ s)$   
**using** *adv\_d\_mmap\_fold*[*OF inv adv\_d[unfolded adv\_d\_def]*]  
*mmap\_fold\_set*[*OF distinct*]  
**unfolding** *mmap\_keys\_def*  
**by** *fastforce*

**lemma** *lookup\_adv\_d\_None*:

**assumes** *inv*:  $\bigwedge q\ bs. case\ Mapping.lookup\ st\ (q, bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v$   
**and** *distinct*:  $distinct\ (map\ fst\ s)$   
**and** *adv\_d*:  $adv\_d\ step\ st\ i\ bs\ s = (s', st')$   
**and** *Z\_empty*:  $\{x \in mmap\_keys\ s. step\ x\ bs = z\} = \{\}$   
**shows**  $mmap\_lookup\ s'\ z = None$

**proof** –

**have**  $z \notin mmap\_keys\ (mmap\_fold\ s\ (\lambda(x, v). (step\ x\ bs, drop\_cur\ i\ v))\ (\lambda x\ y. x)) \ []$   
**using** *Z\_empty*[*unfolded mmap\_keys\_def*] *mmap\_fold\_set*[*OF distinct*]  
**by** (*auto simp: mmap\_keys\_def*)

**then show** *?thesis*

**using** *adv\_d\_adv\_d\_mmap\_fold*[*OF inv adv\_d[unfolded adv\_d\_def]*]  
**unfolding** *adv\_d\_def*  
**by** (*simp add: Mapping\_lookup\_None\_intro*)

**qed**

**lemma** *lookup\_adv\_d\_Some*:

**assumes** *inv*:  $\bigwedge q\ bs. case\ Mapping.lookup\ st\ (q, bs)\ of\ None \Rightarrow True \mid Some\ v \Rightarrow step\ q\ bs = v$   
**and** *distinct*:  $distinct\ (map\ fst\ s)$  **and** *idem*:  $keys\_idem\ step\ i\ bs\ s$   
**and** *wit*:  $x \in mmap\_keys\ s\ step\ x\ bs = z$   
**and** *adv\_d*:  $adv\_d\ step\ st\ i\ bs\ s = (s', st')$   
**shows**  $mmap\_lookup\ s'\ z = Some\ (drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x)))$

**proof** –

**have**  $z\_in\_keys: z \in mmap\_keys\ (mmap\_fold\ s\ (\lambda(x, v). (step\ x\ bs, drop\_cur\ i\ v))\ (\lambda x\ y. x)) \ []$   
**using** *wit*(1,2)[*unfolded mmap\_keys\_def*] *mmap\_fold\_set*[*OF distinct*]  
**by** (*force simp: mmap\_keys\_def*)

**obtain** *v vs* **where** *vs\_def*:  $mmap\_lookup\ s'\ z = Some\ (foldl\ (\lambda x\ y. x)\ v\ vs)$

$v \# vs = map\ (\lambda(x, v). drop\_cur\ i\ v)\ (filter\ (\lambda(k, v). step\ k\ bs = z)\ s)$

**using** *adv\_d\_adv\_d\_mmap\_fold*[*OF inv adv\_d[unfolded adv\_d\_def]*]

**unfolding** *adv\_d\_def*

**using** *mmap\_fold\_lookup*[*OF distinct, of (\lambda(x, v). (step\ x\ bs, drop\\_cur\ i\ v)) \lambda x\ y. x\ z*]

*Mapping\_keys\_dest*[*OF z\_in\_keys*]

**by** (*force simp: adv\_d\_def mmap\_keys\_def split: list.splits*)

**have**  $set\ (v \# vs) = drop\_cur\ i \text{ ' } (the \circ mmap\_lookup\ s) \text{ ' } \{x \in mmap\_keys\ s. step\ x\ bs = z\}$

**proof** (*rule set\_eqI, rule iffI*)

**fix** *w*

**assume**  $w \in set\ (v \# vs)$

**then obtain** *x y* **where** *xy\_def*:  $x \in mmap\_keys\ s\ step\ x\ bs = z\ (x, y) \in set\ s$

$w = drop\_cur\ i\ y$

**using** *vs\_def*(2)

**by** (*auto simp add: mmap\_keys\_def rev\_image\_eqI*)

**show**  $w \in drop\_cur\ i \text{ ' } (the \circ mmap\_lookup\ s) \text{ ' } \{x \in mmap\_keys\ s. step\ x\ bs = z\}$

**using** *xy\_def*(1,2,4) *mmap\_lookup\_distinct*[*OF distinct xy\_def*(3)]

**by** *force*

```

next
  fix w
  assume w ∈ drop_cur i ‘ (the ∘ mmap_lookup s) ‘ {x ∈ mmap_keys s. step x bs = z}
  then obtain x y where xy_def: x ∈ mmap_keys s step x bs = z (x, y) ∈ set s
    w = drop_cur i y
    using mmap_lookup_distinct[OF distinct]
    by (auto simp add: Mapping_keys_intro distinct mmap_lookup_def dest: Mapping_keys_dest)
  show w ∈ set (v # vs)
    using xy_def
    by (force simp: vs_def(2))
qed
then have foldl (λx y. x) v vs = drop_cur i (the (mmap_lookup s x))
  using wit
  apply (induction vs arbitrary: v)
  apply (auto)
  apply (smt empty_is_image idem imageE insert_not_empty keys_idem_def mem_Collect_eq
    the_elem_eq the_elem_image_unique)
  apply (smt Collect_cong idem imageE insert_compr keys_idem_def mem_Collect_eq)
  done
then show ?thesis
  using wit
  by (auto simp: vs_def(1))
qed

```

**definition** *loop\_cond* j = (λ(st, ac, i, ti, si, q, s, tstp). i < j ∧ q ∉ mmap\_keys s)

**definition** *loop\_body* step accept run\_t run\_sub =  
 (λ(st, ac, i, ti, si, q, s, tstp). case run\_t ti of Some (ti', t) ⇒  
 case run\_sub si of Some (si', b) ⇒ case adv\_d step st i b s of (s', st') ⇒  
 case cstep step st' q b of (q', st'') ⇒ case cac accept ac q' of (β, ac') ⇒  
 (st'', ac', Suc i, ti', si', q', s', if β then Some (t, i) else tstp))

**definition** *loop\_inv* init step accept args t0 sub rho u j tj sj =  
 (λ(st, ac, i, ti, si, q, s, tstp). u + 1 ≤ i ∧  
 reach\_window args t0 sub rho (i, ti, si, j, tj, sj) ∧  
 steps step rho init (u + 1, i) = q ∧  
 (∀ q. case Mapping.lookup ac q of None ⇒ True | Some v ⇒ accept q = v) ∧  
 valid\_s init step st accept rho u i j s ∧ tstp = sup\_acc step accept rho init (u + 1) i)

**definition** *mmap\_update* :: 'a ⇒ 'b ⇒ ('a, 'b) mmap ⇒ ('a, 'b) mmap **where**  
*mmap\_update* = AList.update

**lemma** *mmap\_update\_distinct*: distinct (map fst m) ⇒ distinct (map fst (mmap\_update k v m))  
 by (auto simp: mmap\_update\_def distinct\_update)

**definition** *adv\_start* :: ('b, 'c, 'd :: timestamp, 't, 'e) args ⇒  
 ('b, 'c, 'd, 't, 'e) window ⇒ ('b, 'c, 'd, 't, 'e) window **where**  
*adv\_start* args w = (let init = w\_init args; step = w\_step args; accept = w\_accept args;  
 run\_t = w\_run\_t args; run\_sub = w\_run\_sub args; st = w\_st w; ac = w\_ac w;  
 i = w\_i w; ti = w\_ti w; si = w\_si w; j = w\_j w;  
 s = w\_s w; e = w\_e w in  
 (case run\_t ti of Some (ti', t) ⇒ (case run\_sub si of Some (si', bs) ⇒  
 let (s', st') = adv\_d step st i bs s;  
 e' = mmap\_update (fst (the (mmap\_lookup s init))) t e;  
 (st\_cur, ac\_cur, i\_cur, ti\_cur, si\_cur, q\_cur, s\_cur, tstp\_cur) =  
 while (loop\_cond j) (loop\_body step accept run\_t run\_sub)  
 (st', ac, Suc i, ti', si', init, s', None);  
 s'' = mmap\_update init (case mmap\_lookup s\_cur q\_cur of Some (q', tstp') ⇒  
 (case tstp' of Some (ts, tp) ⇒ (q', tstp') | None ⇒ (q', tstp\_cur))  
 | None ⇒ (q\_cur, tstp\_cur)) s' in

$w(w\_st := st\_cur, w\_ac := ac\_cur, w\_i := Suc\ i, w\_ti := ti', w\_si := si', w\_s := s'', w\_e := e'()))$

**lemma** *valid\_adv\_d*:

**assumes** *valid\_before*: *valid\_s init step st accept rho u i j s*  
**and** *u\_le\_i*:  $u \leq i$  **and** *i\_lt\_j*:  $i < j$  **and** *b\_def*:  $b = bs\_at\ rho\ i$   
**and** *adv\_d*: *adv\_d step st i b s = (s', st')*  
**shows** *valid\_s init step st' accept rho u (i + 1) j s'*

**proof** –

**have** *inv\_st*:  $\bigwedge q\ bs.$  *case Mapping.lookup st (q, bs) of None  $\Rightarrow$  True | Some v  $\Rightarrow$  step q bs = v*  
**using** *valid\_before* **by** (*auto simp add: valid\_s\_def*)  
**have** *keys\_s*:  $mmap\_keys\ s = \{q. (\exists l \leq u. steps\ step\ rho\ init\ (l, i) = q)\}$   
**using** *valid\_before* **by** (*auto simp add: valid\_s\_def*)  
**have** *fin\_keys\_s*: *finite (mmap\_keys s)*  
**using** *valid\_before* **by** (*auto simp add: valid\_s\_def*)  
**have** *lookup\_s*:  $\bigwedge q\ q'\ tstp.$  *mmap\_lookup s q = Some (q', tstp)  $\Longrightarrow$  steps step rho q (i, j) = q'  $\wedge$  tstp = sup\_acc step accept rho q i j*  
**using** *valid\_before Mapping\_keys\_intro*  
**by** (*auto simp add: valid\_s\_def*) (*smt case\_prodD option.simps(5)*)  
**have** *drop\_cur\_i*:  $\bigwedge x.$   $x \in mmap\_keys\ s \Longrightarrow drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x)) = (steps\ step\ rho\ (step\ x\ (bs\_at\ rho\ i))\ (i + 1, j), sup\_acc\ step\ accept\ rho\ (step\ x\ (bs\_at\ rho\ i))\ (i + 1)\ j)$

**proof** –

**fix** *x*

**assume** *assms*:  $x \in mmap\_keys\ s$

**obtain** *q tstp* **where** *q\_def*: *mmap\_lookup s x = Some (q, tstp)*

**using** *assms(1)* **by** (*auto dest: Mapping\_keys\_dest*)

**have** *q\_q'*:  $q = steps\ step\ rho\ (step\ x\ (bs\_at\ rho\ i))\ (i + 1, j)$

$tstp = sup\_acc\ step\ accept\ rho\ x\ i\ j$

**using** *lookup\_s[OF q\_def] steps\_split[OF i\_lt\_j] assms(1)* **by** *auto*

**show** *drop\_cur i (the (mmap\_lookup s x)) =*

*(steps step rho (step x (bs\_at rho i)) (i + 1, j),*

*sup\_acc step accept rho (step x (bs\_at rho i)) (i + 1) j)*

**using** *q\_def sup\_acc\_None[OF i\_lt\_j, of step accept rho]*

$sup\_acc\_i[OF\ i\_lt\_j, of\ step\ accept\ rho]\ sup\_acc\_l[OF\ i\_lt\_j, of\ \_ step\ accept\ rho]$

**unfolding** *q\_q'*

**by** (*auto simp add: drop\_cur\_def split: option.splits*)

**qed**

**have** *valid\_drop\_cur*:  $\bigwedge x\ x'. x \in mmap\_keys\ s \Longrightarrow x' \in mmap\_keys\ s \Longrightarrow$

$step\ x\ (bs\_at\ rho\ i) = step\ x'\ (bs\_at\ rho\ i) \Longrightarrow drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x)) =$

$drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x'))$

**using** *drop\_cur\_i* **by** *auto*

**then have** *keys\_idem*: *keys\_idem step i b s*

**unfolding** *keys\_idem\_def b\_def*

**by** *blast*

**have** *distinct*: *distinct (map fst s)*

**using** *valid\_before*

**by** (*auto simp: valid\_s\_def*)

**have**  $(\lambda q. step\ q\ (bs\_at\ rho\ i))\ \{q. \exists l \leq u. steps\ step\ rho\ init\ (l, i) = q\} =$

$\{q. \exists l \leq u. steps\ step\ rho\ init\ (l, i + 1) = q\}$

**using** *steps\_app[of \_ i step rho init] u\_le\_i*

**by** *auto*

**then have** *keys\_s'*:  $mmap\_keys\ s' = \{q. \exists l \leq u. steps\ step\ rho\ init\ (l, i + 1) = q\}$

**using** *adv\_d\_keys[OF \_ distinct adv\_d] inv\_st*

**unfolding** *keys\_s b\_def*

**by** *auto*

**have** *lookup\_s'*:  $\bigwedge q\ q'\ tstp.$  *mmap\_lookup s' q = Some (q', tstp)  $\Longrightarrow$*

$steps\ step\ rho\ q\ (i + 1, j) = q' \wedge tstp = sup\_acc\ step\ accept\ rho\ q\ (i + 1)\ j$

```

proof –
  fix  $q\ q'\ tstp$ 
  assume  $assm: mmap\_lookup\ s'\ q = Some\ (q',\ tstp)$ 
  obtain  $x$  where  $wit: x \in mmap\_keys\ s\ step\ x\ (bs\_at\ rho\ i) = q$ 
  using  $assm\ lookup\_adv\_d\_None[OF\ \_distinct\ adv\_d]\ inv\_st$ 
  by  $(fastforce\ simp: b\_def)$ 
  have  $lookup\_s'\_q: mmap\_lookup\ s'\ q = Some\ (drop\_cur\ i\ (the\ (mmap\_lookup\ s\ x)))$ 
  using  $lookup\_adv\_d\_Some[OF\ \_distinct\ keys\_idem\ wit[folded\ b\_def]\ adv\_d]\ inv\_st$ 
  by  $auto$ 
  then show  $steps\ step\ rho\ q\ (i + 1, j) = q' \wedge tstp = sup\_acc\ step\ accept\ rho\ q\ (i + 1)\ j$ 
  using  $assm$ 
  by  $(simp\ add: drop\_cur\_i\ wit)$ 
qed
have  $distinct\ (map\ fst\ s')$ 
  using  $mmap\_fold\_distinct[OF\ distinct]\ adv\_d\_mmap\_fold[OF\ inv\_st\ adv\_d[unfolded\ adv\_d\_def]]$ 
  unfolding  $adv\_d\_def\ mmap\_map\_fst$ 
  by  $auto$ 
then show  $valid\_s\ init\ step\ st'\ accept\ rho\ u\ (i + 1)\ j\ s'$ 
  unfolding  $valid\_s\_def$ 
  using  $keys\_s'\ lookup\_s'\ u\_le\_i\ inv\_st\ adv\_d[unfolded\ adv\_d\_def]$ 
   $adv\_d\_mmap\_fold[OF\ inv\_st\ adv\_d[unfolded\ adv\_d\_def]]$ 
  by  $(auto\ split: option.splits\ dest: Mapping\_keys\_dest)$ 
qed

lemma  $mmap\_lookup\_update'$ :
   $mmap\_lookup\ (mmap\_update\ k\ v\ kvs)\ z = (if\ k = z\ then\ Some\ v\ else\ mmap\_lookup\ kvs\ z)$ 
  unfolding  $mmap\_lookup\_def\ mmap\_update\_def$ 
  by  $(auto\ simp\ add: update\_conv')$ 

lemma  $mmap\_keys\_update$ :  $mmap\_keys\ (mmap\_update\ k\ v\ kvs) = mmap\_keys\ kvs \cup \{k\}$ 
  by  $(induction\ kvs)\ (auto\ simp: mmap\_keys\_def\ mmap\_update\_def)$ 

lemma  $valid\_adv\_start$ :
  assumes  $valid\_window\ args\ t0\ sub\ rho\ w\ w\_i\ w < w\_j\ w$ 
  shows  $valid\_window\ args\ t0\ sub\ rho\ (adv\_start\ args\ w)$ 
proof –
  define  $init$  where  $init = w\_init\ args$ 
  define  $step$  where  $step = w\_step\ args$ 
  define  $accept$  where  $accept = w\_accept\ args$ 
  define  $run\_t$  where  $run\_t = w\_run\_t\ args$ 
  define  $run\_sub$  where  $run\_sub = w\_run\_sub\ args$ 
  define  $st$  where  $st = w\_st\ w$ 
  define  $ac$  where  $ac = w\_ac\ w$ 
  define  $i$  where  $i = w\_i\ w$ 
  define  $ti$  where  $ti = w\_ti\ w$ 
  define  $si$  where  $si = w\_si\ w$ 
  define  $j$  where  $j = w\_j\ w$ 
  define  $tj$  where  $tj = w\_tj\ w$ 
  define  $sj$  where  $sj = w\_sj\ w$ 
  define  $s$  where  $s = w\_s\ w$ 
  define  $e$  where  $e = w\_e\ w$ 
  have  $valid\_before: reach\_window\ args\ t0\ sub\ rho\ (i, ti, si, j, tj, sj)$ 
   $\wedge i\ j. i \leq j \implies j < length\ rho \implies ts\_at\ rho\ i \leq ts\_at\ rho\ j$ 
   $(\wedge q\ bs. case\ Mapping.lookup\ st\ (q, bs)\ of\ None \implies True \mid Some\ v \implies step\ q\ bs = v)$ 
   $(\wedge q\ bs. case\ Mapping.lookup\ ac\ q\ of\ None \implies True \mid Some\ v \implies accept\ q = v)$ 
   $\forall q. mmap\_lookup\ e\ q = sup\_leadsto\ init\ step\ rho\ i\ j\ q\ distinct\ (map\ fst\ e)$ 
   $valid\_s\ init\ step\ st\ accept\ rho\ i\ i\ j\ s$ 
  using  $assms(1)$ 

```

```

unfolding valid_window_def valid_s_def Let_def init_def step_def accept_def run_t_def
  run_sub_def st_def ac_def i_def ti_def si_def j_def tj_def sj_def s_def e_def
by auto
have distinct_before: distinct (map fst s) distinct (map fst e)
using valid_before
by (auto simp: valid_s_def)
note i_lt_j = assms(2)[folded i_def j_def]
obtain ti' si' t b where tb_def: run_t ti = Some (ti', t)
  run_sub si = Some (si', b)
  reaches_on run_t ti' (drop (Suc i) (map fst rho)) tj
  reaches_on run_sub si' (drop (Suc i) (map snd rho)) sj
  t = ts_at rho i b = bs_at rho i
using valid_before i_lt_j
apply (auto simp: ts_at_def bs_at_def run_t_def[symmetric] run_sub_def[symmetric]
  elim!: reaches_on.cases[of run_t ti drop i (map fst rho) tj]
  reaches_on.cases[of run_sub si drop i (map snd rho) sj])
by (metis Cons_nth_drop_Suc length_map list.inject nth_map)
have reaches_on_si': reaches_on run_sub sub (take (Suc i) (map snd rho)) si'
using valid_before tb_def(2,3,4) i_lt_j reaches_on_app tb_def(1)
by (auto simp: run_sub_def sub_def bs_at_def take_Suc_conv_app_nth reaches_on_app tb_def(6))
have reaches_on_ti': reaches_on run_t t0 (take (Suc i) (map fst rho)) ti'
using valid_before tb_def(2,3,4) i_lt_j reaches_on_app tb_def(1)
by (auto simp: run_t_def ts_at_def take_Suc_conv_app_nth reaches_on_app tb_def(5))
define e' where e' = mmap_update (fst (the (mmap_lookup s init))) t e
obtain st' s' where s'_def: adv_d step st i b s = (s', st')
by (metis old.prod.exhaust)
obtain st_cur ac_cur i_cur ti_cur si_cur q_cur s_cur tstp_cur where loop_def:
  (st_cur, ac_cur, i_cur, ti_cur, si_cur, q_cur, s_cur, tstp_cur) =
  while (loop_cond j) (loop_body step accept run_t run_sub)
  (st', ac, Suc i, ti', si', init, s', None)
by (cases while (loop_cond j) (loop_body step accept run_t run_sub)
  (st', ac, Suc i, ti', si', init, s', None)) auto
define s'' where s'' = mmap_update init (case mmap_lookup s_cur q_cur of
  Some (q', tstp')  $\Rightarrow$  (case tstp' of Some (ts, tp)  $\Rightarrow$  (q', tstp') | None  $\Rightarrow$  (q', tstp_cur))
  | None  $\Rightarrow$  (q_cur, tstp_cur)) s'
have i_le_j: i  $\leq$  j
using i_lt_j by auto
have length_rho: length rho = j
using valid_before by auto
have lookup_s:  $\bigwedge q q' tstp. \text{mmap\_lookup } s \ q = \text{Some } (q', tstp) \Longrightarrow$ 
  steps step rho q (i, j) = q'  $\wedge$  tstp = sup_acc step accept rho q i j
using valid_before Mapping_keys_intro
by (auto simp: valid_s_def) (smt case_prodD option.simps(5))+
have init_in_keys_s: init  $\in$  mmap_keys s
using valid_before by (auto simp add: valid_s_def)
then have run_init_i_j: steps step rho init (i, j) = fst (the (mmap_lookup s init))
using lookup_s by (auto dest: Mapping_keys_dest)
have lookup_e:  $\bigwedge q. \text{mmap\_lookup } e \ q = \text{sup\_leadsto } \text{init } \text{step } \text{rho } i \ j \ q$ 
using valid_before by auto
have lookup_e':  $\bigwedge q. \text{mmap\_lookup } e' \ q = \text{sup\_leadsto } \text{init } \text{step } \text{rho } (i + 1) \ j \ q$ 
proof -
  fix q
  show mmap_lookup e' q = sup_leadsto init step rho (i + 1) j q
proof (cases steps step rho init (i, j) = q)
  case True
  have Max {l. l < Suc i  $\wedge$  steps step rho init (l, j) = steps step rho init (i, j)} = i
  by (rule iffD2[OF Max_eq_iff]) auto
  then have sup_leadsto init step rho (i + 1) j q = Some (ts_at rho i)

```

```

    by (auto simp add: sup_leadsto_def True)
  then show ?thesis
    unfolding e'_def using run_init_i_j tb_def
    by (auto simp add: mmap_lookup_update' True)
next
  case False
  show ?thesis
    using run_init_i_j sup_leadsto_idle[OF i_lt_j False] lookup_e[of q] False
    by (auto simp add: e'_def mmap_lookup_update')
qed
qed
have reach_split: {q.  $\exists l \leq i + 1. \text{steps step rho init } (l, i + 1) = q$ } =
  {q.  $\exists l \leq i. \text{steps step rho init } (l, i + 1) = q$ }  $\cup$  {init}
  using le_Suc_eq by auto
have valid_s_i: valid_s init step st accept rho i i j s
  using valid_before by auto
have valid_s'_Suc_i: valid_s init step st' accept rho i (i + 1) j s'
  using valid_adv_d[OF valid_s_i order.refl i_lt_j, OF tb_def(6) s'_def] unfolding s'_def .
have loop: loop_inv init step accept args t0 sub rho i j tj sj
  (st_cur, ac_cur, i_cur, ti_cur, si_cur, q_cur, s_cur, tstp_cur)  $\wedge$ 
   $\neg$ loop_cond j (st_cur, ac_cur, i_cur, ti_cur, si_cur, q_cur, s_cur, tstp_cur)
  unfolding loop_def
proof (rule while_rule_lemma[of loop_inv init step accept args t0 sub rho i j tj sj
  loop_cond j loop_body step accept run_t run_sub
   $\lambda s. \text{loop\_inv init step accept args t0 sub rho i j tj sj s} \wedge \neg \text{loop\_cond j s}$ ])
  show loop_inv init step accept args t0 sub rho i j tj sj
    (st', ac, Suc i, ti', si', init, s', None)
    unfolding loop_inv_def
    using i_lt_j valid_s'_Suc_i sup_acc_same[of step accept rho]
    length_rho_reaches_on_si' reaches_on_ti' tb_def(3,4) valid_before(4)
    by (auto simp: run_t_def run_sub_def split: prod.splits)
next
  have {(t, s). loop_inv init step accept args t0 sub rho i j tj sj s  $\wedge$ 
  loop_cond j s  $\wedge$  t = loop_body step accept run_t run_sub s}  $\subseteq$ 
  measure ( $\lambda(st, ac, i\_cur, ti, si, q, s, tstp). j - i\_cur$ )
  unfolding loop_inv_def loop_cond_def loop_body_def
  apply (auto simp: run_t_def run_sub_def split: option.splits)
  apply (metis drop_eq_Nil length_map not_less option.distinct(1) reaches_on.simps)
  apply (metis (no_types, lifting) drop_eq_Nil length_map not_less option.distinct(1)
    reaches_on.simps)
  apply (auto split: prod.splits)
  done
  then show wf {(t, s). loop_inv init step accept args t0 sub rho i j tj sj s  $\wedge$ 
  loop_cond j s  $\wedge$  t = loop_body step accept run_t run_sub s}
    using wf_measure wf_subset by auto
next
  fix state
  assume assms: loop_inv init step accept args t0 sub rho i j tj sj state
    loop_cond j state
  obtain st_cur ac_cur i_cur ti_cur si_cur q_cur s_cur tstp_cur
    where state_def: state = (st_cur, ac_cur, i_cur, ti_cur, si_cur, q_cur, s_cur, tstp_cur)
    by (cases state) auto
  obtain ti'_cur si'_cur t_cur b_cur where tb_cur_def: run_t ti_cur = Some (ti'_cur, t_cur)
    run_sub si_cur = Some (si'_cur, b_cur)
    reaches_on run_t ti'_cur (drop (Suc i_cur) (map fst rho)) tj
    reaches_on run_sub si'_cur (drop (Suc i_cur) (map snd rho)) sj
    t_cur = ts_at rho i_cur b_cur = bs_at rho i_cur
  using assms

```

```

unfolding loop_inv_def loop_cond_def state_def
apply (auto simp: ts_at_def bs_at_def run_t_def[symmetric] run_sub_def[symmetric]
  elim!: reaches_on.cases[of run_t ti_cur drop i_cur (map fst rho) tj]
  reaches_on.cases[of run_sub si_cur drop i_cur (map snd rho) sj])
by (metis Cons_nth_drop_Suc length_map list.inject nth_map)
obtain s'_cur st'_cur where s'_cur_def: adv_d step st_cur i_cur b_cur s_cur =
  (s'_cur, st'_cur)
by fastforce
have valid_s'_cur: valid_s init step st'_cur accept rho i (i_cur + 1) j s'_cur
using assms valid_adv_d[of init step st_cur accept rho] tb_cur_def(6) s'_cur_def
unfolding loop_inv_def loop_cond_def state_def
by auto
obtain q' st''_cur where q'_def: cstep step st'_cur q_cur b_cur = (q', st''_cur)
by fastforce
obtain  $\beta$  ac'_cur where b_def: cac accept ac_cur q' = ( $\beta$ , ac'_cur)
by fastforce
have step: q' = step q_cur b_cur  $\wedge$  q bs. case Mapping.lookup st''_cur (q, bs) of
  None  $\Rightarrow$  True | Some v  $\Rightarrow$  step q bs = v
using valid_s'_cur q'_def
unfolding valid_s_def
by (auto simp: cstep_def Let_def Mapping.lookup_update' split: option.splits if_splits)
have accept:  $\beta$  = accept q'  $\wedge$  q. case Mapping.lookup ac'_cur q of
  None  $\Rightarrow$  True | Some v  $\Rightarrow$  accept q = v
using assms b_def
unfolding loop_inv_def state_def
by (auto simp: cac_def Let_def Mapping.lookup_update' split: option.splits if_splits)
have steps_q': steps step rho init (i + 1, Suc i_cur) = q'
using assms
unfolding loop_inv_def state_def
by auto (metis local.step(1) steps_appE tb_cur_def(6))
have b_acc:  $\beta$  = acc step accept rho init (i + 1, Suc i_cur)
unfolding accept(1) acc_def steps_q'
by (auto simp: tb_cur_def)
have valid_s''_cur: valid_s init step st''_cur accept rho i (i_cur + 1) j s'_cur
using valid_s'_cur step(2)
unfolding valid_s_def
by auto
have reaches_on_si': reaches_on run_sub sub (take (Suc i_cur) (map snd rho)) si'_cur
using assms
unfolding loop_inv_def loop_cond_def state_def
by (auto simp: run_sub_def sub_def bs_at_def take_Suc_conv_app_nth reaches_on_app
  tb_cur_def(2,4,6))
  (metis bs_at_def reaches_on_app run_sub_def tb_cur_def(2) tb_cur_def(6))
have reaches_on_ti': reaches_on run_t t0 (take (Suc i_cur) (map fst rho)) ti'_cur
using assms
unfolding loop_inv_def loop_cond_def state_def
by (auto simp: run_t_def ts_at_def take_Suc_conv_app_nth reaches_on_app tb_cur_def(1,3,5))
  (metis reaches_on_app run_t_def tb_cur_def(1) tb_cur_def(5) ts_at_def)
have reach_window args t0 sub rho (Suc i_cur, ti'_cur, si'_cur, j, tj, sj)
using reaches_on_si' reaches_on_ti' tb_cur_def(3,4) length_rho assms(2)
unfolding loop_cond_def state_def
by (auto simp: run_t_def run_sub_def)
moreover have steps step rho init (i + 1, Suc i_cur) = q'
using assms steps_app
unfolding loop_inv_def state_def step(1)
by (auto simp: tb_cur_def(6))
ultimately show loop_inv init step accept args t0 sub rho i j tj sj
  (loop_body step accept run_t run_sub state)

```

```

using assms accept(2) valid_s''_cur sup_acc_ext[of __ step accept rho]
  sup_acc_ext_idle[of __ step accept rho]
unfolding loop_inv_def loop_body_def state_def
by (auto simp: tb_cur_def(1,2,5) s'_cur_def q'_def b_def b_acc
  split: option.splits prod.splits)
qed auto
have valid_stac_cur:  $\forall q$  bs. case Mapping.lookup st_cur (q, bs) of None  $\Rightarrow$  True
  | Some v  $\Rightarrow$  step q bs = v  $\forall q$ . case Mapping.lookup ac_cur q of None  $\Rightarrow$  True
  | Some v  $\Rightarrow$  accept q = v
using loop unfolding loop_inv_def valid_s_def
by auto
have valid_s'': valid_s init step st_cur accept rho (i + 1) (i + 1) j s''
proof (cases mmap_lookup s_cur q_cur)
  case None
  then have added: steps step rho init (i + 1, j) = q_cur
    tstp_cur = sup_acc step accept rho init (i + 1) j
    using loop unfolding loop_inv_def loop_cond_def
    by (auto dest: Mapping_keys_dest)
  have s''_case: s'' = mmap_update init (q_cur, tstp_cur) s'
    unfolding s''_def using None by auto
  show ?thesis
    using valid_s'_Suc_i reach_split added mmap_update_distinct valid_stac_cur
    unfolding s''_case valid_s_def mmap_keys_update
    by (auto simp add: mmap_lookup_update' split: option.splits)
next
  case (Some p)
  obtain q' tstp' where p_def: p = (q', tstp')
    by (cases p) auto
  note lookup_s_cur = Some[unfolded p_def]
  have i_cur_in: i + 1  $\leq$  i_cur i_cur  $\leq$  j
    using loop unfolding loop_inv_def by auto
  have q_cur_def: steps step rho init (i + 1, i_cur) = q_cur
    using loop unfolding loop_inv_def by auto
  have valid_s_cur: valid_s init step st_cur accept rho i i_cur j s_cur
    using loop unfolding loop_inv_def by auto
  have q'_steps: steps step rho q_cur (i_cur, j) = q'
    using Some valid_s_cur unfolding valid_s_def p_def
    by (auto intro: Mapping_keys_intro) (smt case_prodD option.simps(5))
  have tstp_cur: tstp_cur = sup_acc step accept rho init (i + 1) i_cur
    using loop unfolding loop_inv_def by auto
  have tstp': tstp' = sup_acc step accept rho q_cur i_cur j
    using loop Some unfolding loop_inv_def p_def valid_s_def
    by (auto intro: Mapping_keys_intro) (smt case_prodD option.simps(5))
  have added: steps step rho init (i + 1, j) = q'
    using steps_comp[OF i_cur_in q_cur_def q'_steps] .
  show ?thesis
proof (cases tstp')
  case None
  have s''_case: s'' = mmap_update init (q', tstp_cur) s'
    unfolding s''_def lookup_s_cur None by auto
  have tstp_cur_opt: tstp_cur = sup_acc step accept rho init (i + 1) j
    using sup_acc_comp_None[OF i_cur_in, of step accept rho init, unfolded q_cur_def,
    OF tstp'[unfolded None, symmetric]]
    unfolding tstp_cur by auto
  then show ?thesis
    using valid_s'_Suc_i reach_split added mmap_update_distinct valid_stac_cur
    unfolding s''_case valid_s_def mmap_keys_update
    by (auto simp add: mmap_lookup_update' split: option.splits)

```

```

next
  case (Some p')
  obtain ts tp where p'_def: p' = (ts, tp)
  by (cases p') auto
  have True: tp ≥ i_cur
  using sup_acc_SomeE[OF tstp'[unfolded Some p'_def, symmetric]] by auto
  have s''_case: s'' = mmap_update init (q', tstp') s'
  unfolding s''_def lookup_s_cur Some p'_def using True by auto
  have tstp'_opt: tstp' = sup_acc step accept rho init (i + 1) j
  using sup_acc_comp_Some_ge[OF i_cur_in True
    tstp'[unfolded Some p'_def q_cur_def[symmetric], symmetric]]
  unfolding tstp' by (auto simp: q_cur_def[symmetric])
  then show ?thesis
  using valid_s'_Suc_i reach_split added mmap_update_distinct valid_stac_cur
  unfolding s''_case valid_s_def mmap_keys_update
  by (auto simp add: mmap_lookup_update' split: option.splits)
qed
qed
have distinct (map fst e')
  using mmap_update_distinct[OF distinct_before(2), unfolded e'_def]
  unfolding e'_def .
then have valid_window args t0 sub rho
  (w(w_st := st_cur, w_ac := ac_cur, w_i := Suc i, w_ti := ti', w_si := si', w_s := s'', w_e :=
e'))
  using i_lt_j lookup_e' valid_s'' length_rho tb_def(3,4) reaches_on_si' reaches_on_ti'
  valid_before[unfolded step_def accept_def] valid_stac_cur(2)[unfolded accept_def]
  by (auto simp: valid_window_def Let_def init_def step_def accept_def run_t_def
    run_sub_def st_def ac_def i_def ti_def si_def j_def tj_def sj_def s_def e_def)
moreover have adv_start args w = w(w_st := st_cur, w_ac := ac_cur, w_i := Suc i,
w_ti := ti', w_si := si', w_s := s'', w_e := e')
  unfolding adv_start_def Let_def s''_def e'_def
  using tb_def(1,2) s'_def i_lt_j loop_def valid_before(3)
  by (auto simp: valid_window_def Let_def init_def step_def accept_def run_t_def
    run_sub_def st_def ac_def i_def ti_def si_def j_def tj_def sj_def s_def e_def
    split: prod.splits)
ultimately show ?thesis
  by auto
qed

lemma valid_adv_start_bounds:
  assumes valid_window args t0 sub rho w w_i w < w_j w
  shows w_i (adv_start args w) = Suc (w_i w) w_j (adv_start args w) = w_j w
  w_tj (adv_start args w) = w_tj w w_sj (adv_start args w) = w_sj w
  using assms
  by (auto simp: adv_start_def Let_def valid_window_def split: option.splits prod.splits
    elim: reaches_on.cases)

lemma valid_adv_start_bounds':
  assumes valid_window args t0 sub rho w w_run_t args (w_ti w) = Some (ti', t)
  w_run_sub args (w_si w) = Some (si', bs)
  shows w_ti (adv_start args w) = ti' w_si (adv_start args w) = si'
  using assms
  by (auto simp: adv_start_def Let_def valid_window_def split: option.splits prod.splits)
end
theory Temporal
  imports MDL NFA Window
begin

```

```

fun state_cnt :: ('a, 'b :: timestamp) regex  $\Rightarrow$  nat where
  state_cnt (Lookahead phi) = 1
| state_cnt (Symbol phi) = 2
| state_cnt (Plus r s) = 1 + state_cnt r + state_cnt s
| state_cnt (Times r s) = state_cnt r + state_cnt s
| state_cnt (Star r) = 1 + state_cnt r

```

```

lemma state_cnt_pos: state_cnt r > 0
by (induction r rule: state_cnt.induct) auto

```

```

fun collect_subfmlas :: ('a, 'b :: timestamp) regex  $\Rightarrow$  ('a, 'b) formula list  $\Rightarrow$ 
  ('a, 'b) formula list where
  collect_subfmlas (Lookahead phi)phis = (if phi  $\in$  set phis then phis else phis @ [phi])
| collect_subfmlas (Symbol phi)phis = (if phi  $\in$  set phis then phis else phis @ [phi])
| collect_subfmlas (Plus r s)phis = collect_subfmlas s (collect_subfmlas r phis)
| collect_subfmlas (Times r s)phis = collect_subfmlas s (collect_subfmlas r phis)
| collect_subfmlas (Star r)phis = collect_subfmlas r phis

```

```

lemma bf_collect_subfmlas: bounded_future_regex r  $\Longrightarrow$  phi  $\in$  set (collect_subfmlas r phis)  $\Longrightarrow$ 
  phi  $\in$  set phis  $\vee$  bounded_future_fmula phi
by (induction r phi rule: collect_subfmlas.induct) (auto split: if_splits)

```

```

lemma collect_subfmlas_atms: set (collect_subfmlas r phis) = set phis  $\cup$  atms r
by (induction r phi rule: collect_subfmlas.induct) (auto split: if_splits)

```

```

lemma collect_subfmlas_set: set (collect_subfmlas r phis) = set (collect_subfmlas r [])  $\cup$  set phis
proof (induction r arbitrary: phis)

```

```

  case (Plus r1 r2)
  show ?case
  using Plus(1)[of phis] Plus(2)[of collect_subfmlas r1 phis]
    Plus(2)[of collect_subfmlas r1 []]
  by auto
next
  case (Times r1 r2)
  show ?case
  using Times(1)[of phis] Times(2)[of collect_subfmlas r1 phis]
    Times(2)[of collect_subfmlas r1 []]
  by auto
next
  case (Star r)
  show ?case
  using Star[of phis]
  by auto
qed auto

```

```

lemma collect_subfmlas_size: x  $\in$  set (collect_subfmlas r [])  $\Longrightarrow$  size x < size r

```

```

proof (induction r)
  case (Plus r1 r2)
  then show ?case
  by (auto simp: collect_subfmlas_set[of r2 collect_subfmlas r1 []])
next
  case (Times r1 r2)
  then show ?case
  by (auto simp: collect_subfmlas_set[of r2 collect_subfmlas r1 []])
next
  case (Star r)
  then show ?case

```

**by** *fastforce*  
**qed** (*auto split: if\_splits*)

**lemma** *collect\_subfmlas\_app*:  $\exists \text{phis}'.$  *collect\_subfmlas* *r phis* = *phis* @ *phis'*  
**by** (*induction r phis rule: collect\_subfmlas.induct*) *auto*

**lemma** *length\_collect\_subfmlas*: *length (collect\_subfmlas r phis)*  $\geq$  *length phis*  
**by** (*induction r phis rule: collect\_subfmlas.induct*) *auto*

**fun** *pos* :: '*a*  $\Rightarrow$  '*a* list  $\Rightarrow$  nat option **where**  
*pos* *a* [] = None  
| *pos* *a* (*x* # *xs*) =  
(*if a = x then Some 0 else (case pos a xs of Some n  $\Rightarrow$  Some (Suc n) | \_  $\Rightarrow$  None)*)

**lemma** *pos\_sound*: *pos a xs* = *Some i*  $\implies$  *i* < *length xs*  $\wedge$  *xs* ! *i* = *a*  
**by** (*induction a xs arbitrary: i rule: pos.induct*) (*auto split: if\_splits option.splits*)

**lemma** *pos\_complete*: *pos a xs* = None  $\implies$  *a*  $\notin$  set *xs*  
**by** (*induction a xs rule: pos.induct*) (*auto split: if\_splits option.splits*)

**fun** *build\_nfa\_impl* :: ('*a*, '*b* :: timestamp) *rexex*  $\Rightarrow$  (state  $\times$  state  $\times$  ('*a*, '*b*) formula list)  $\Rightarrow$   
transition list **where**  
*build\_nfa\_impl* (*Lookahead*  $\varphi$ ) (*q0*, *qf*, *phis*) = (*case pos*  $\varphi$  *phis* of  
*Some n*  $\Rightarrow$  [*eps\_trans qf n*]  
| *None*  $\Rightarrow$  [*eps\_trans qf (length phis)*])  
| *build\_nfa\_impl* (*Symbol*  $\varphi$ ) (*q0*, *qf*, *phis*) = (*case pos*  $\varphi$  *phis* of  
*Some n*  $\Rightarrow$  [*eps\_trans (Suc q0) n, symb\_trans qf*]  
| *None*  $\Rightarrow$  [*eps\_trans (Suc q0) (length phis), symb\_trans qf*])  
| *build\_nfa\_impl* (*Plus* *r s*) (*q0*, *qf*, *phis*) = (  
*let ts\_r* = *build\_nfa\_impl r (q0 + 1, qf, phis)*;  
*ts\_s* = *build\_nfa\_impl s (q0 + 1 + state\_cnt r, qf, collect\_subfmlas r phis)* in  
*split\_trans (q0 + 1) (q0 + 1 + state\_cnt r) # ts\_r @ ts\_s*)  
| *build\_nfa\_impl* (*Times* *r s*) (*q0*, *qf*, *phis*) = (  
*let ts\_r* = *build\_nfa\_impl r (q0, q0 + state\_cnt r, phis)*;  
*ts\_s* = *build\_nfa\_impl s (q0 + state\_cnt r, qf, collect\_subfmlas r phis)* in  
*ts\_r @ ts\_s*)  
| *build\_nfa\_impl* (*Star* *r*) (*q0*, *qf*, *phis*) = (  
*let ts\_r* = *build\_nfa\_impl r (q0 + 1, q0, phis)* in  
*split\_trans (q0 + 1) qf # ts\_r*)

**lemma** *build\_nfa\_impl\_state\_cnt*: *length (build\_nfa\_impl r (q0, qf, phis))* = *state\_cnt r*  
**by** (*induction r (q0, qf, phis) arbitrary: q0 qf phis rule: build\_nfa\_impl.induct*)  
(*auto split: option.splits*)

**lemma** *build\_nfa\_impl\_not\_Nil*: *build\_nfa\_impl r (q0, qf, phis)*  $\neq$  []  
**by** (*induction r (q0, qf, phis) arbitrary: q0 qf phis rule: build\_nfa\_impl.induct*)  
(*auto split: option.splits*)

**lemma** *build\_nfa\_impl\_state\_set*: *t*  $\in$  set (*build\_nfa\_impl r (q0, qf, phis)*)  $\implies$   
state\_set *t*  $\subseteq$  {*q0*..*q0* + *length (build\_nfa\_impl r (q0, qf, phis))*}  $\cup$  {*qf*}  
**by** (*induction r (q0, qf, phis) arbitrary: q0 qf phis t rule: build\_nfa\_impl.induct*)  
(*fastforce simp add: build\_nfa\_impl\_state\_cnt state\_cnt\_pos build\_nfa\_impl\_not\_Nil*  
*split: option.splits*) +

**lemma** *build\_nfa\_impl\_fmula\_set*: *t*  $\in$  set (*build\_nfa\_impl r (q0, qf, phis)*)  $\implies$   
*n*  $\in$  *fmula\_set t*  $\implies$  *n* < *length (collect\_subfmlas r phis)*

**proof** (*induction r (q0, qf, phis) arbitrary: q0 qf phis t rule: build\_nfa\_impl.induct*)  
**case** (*1*  $\varphi$  *q0 qf phis*)

```

then show ?case
  using pos_sound pos_complete by (force split: option.splits)
next
  case (2  $\varphi$  q0 qf phis)
  then show ?case
    using pos_sound pos_complete by (force split: option.splits)
next
  case (3 r s q0 qf phis)
  then show ?case
    using length_collect_subfmlas dual_order.strict_trans1 by fastforce
next
  case (4 r s q0 qf phis)
  then show ?case
    using length_collect_subfmlas dual_order.strict_trans1 by fastforce
next
  case (5 r q0 qf phis)
  then show ?case
    using length_collect_subfmlas dual_order.strict_trans1 by fastforce
qed

```

```

context MDL
begin

```

```

definition IH r q0 qf phis transs bss bs i  $\equiv$ 
  let n = length (collect_subfmlas r phis) in
  transs = build_nfa_impl r (q0, qf, phis)  $\wedge$  ( $\forall$  cs  $\in$  set bss. length cs  $\geq$  n)  $\wedge$  length bs  $\geq$  n  $\wedge$ 
  qf  $\notin$  NFA.SQ q0 (build_nfa_impl r (q0, qf, phis))  $\wedge$ 
  ( $\forall$  k < n. (bs ! k  $\longleftrightarrow$  sat (collect_subfmlas r phis ! k) (i + length bss)))  $\wedge$ 
  ( $\forall$  j < length bss.  $\forall$  k < n. ((bss ! j) ! k  $\longleftrightarrow$  sat (collect_subfmlas r phis ! k) (i + j)))

```

```

lemma nfa_correct: IH r q0 qf phis transs bss bs i  $\implies$ 
  NFA.run_accept_eps q0 qf transs {q0} bss bs  $\longleftrightarrow$  (i, i + length bss)  $\in$  match r

```

```

proof (induct r arbitrary: q0 qf phis transs bss bs i rule: regex_induct)
case (Lookahead  $\varphi$ )

```

```

  have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 transs
    using Lookahead_unfolding IH_def by (auto simp: Let_def)
  have qf_not_q0_Suc_q0: qf  $\notin$  {q0}
    using Lookahead_unfolding IH_def
    by (auto simp: NFA.SQ_def split: option.splits)
  have transs_def: transs = build_nfa_impl (Lookahead  $\varphi$ ) (q0, qf, phis)
    using Lookahead(1)
    by (auto simp: Let_def IH_def)
  interpret base: nfa q0 qf transs
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding IH_def NFA.Q_def NFA.SQ_def transs_def
  by (auto split: option.splits)
  define n where n  $\equiv$  case pos  $\varphi$  phis of Some n  $\Rightarrow$  n | _  $\Rightarrow$  length phis
  then have collect: n < length (collect_subfmlas (Lookahead  $\varphi$ ) phis)
    (collect_subfmlas (Lookahead  $\varphi$ ) phis) ! n =  $\varphi$ 
    using pos_sound pos_complete by (force split: option.splits)+
  have  $\bigwedge$  cs q. base.step_eps cs q0 q  $\longleftrightarrow$  n < length cs  $\wedge$  cs ! n  $\wedge$  q = qf  $\wedge$  cs q.  $\neg$ base.step_eps cs qf q
    using base.q0_sub_SQ qf_not_in_SQ
    by (auto simp: NFA.step_eps_def transs_def n_def split: option.splits)
  then have base_eps: base.step_eps_closure_set {q0} cs = (if n < length cs  $\wedge$  cs ! n then {q0, qf}
  else {q0}) for cs
    using NFA.step_eps_closure_set_unfold[where ?X={qf}]
    using NFA.step_eps_closure_set_step_id[where ?R={q0}]

```

```

    using NFA.step_eps_closure_set_step_id[where ?R={qf}]
    by auto
  have base_delta: base.delta {q0} cs = {} for cs
    unfolding NFA.delta_def NFA.step_symb_set_def base_eps
    by (auto simp: NFA.step_symb_def NFA.SQ_def transs_def split: option.splits)
  show ?case
  proof (cases bss)
    case Nil
    have sat: n < length bs ∧ bs ! n ↔ sat φ i
      using Lookahead(1) collect
      by (auto simp: Let_def IH_def Nil)
    show ?thesis
      using qf_not_q0_Suc_q0
      unfolding NFA.run_accept_eps_def NFA.run_def NFA.accept_eps_def Nil
      by (auto simp: base_eps sat)
  next
    case bss_def: (Cons cs css)
    show ?thesis
      using NFA.run_accept_eps_empty
      unfolding NFA.run_accept_eps_def NFA.run_def NFA.accept_eps_def bss_def
      by (auto simp: bss_def base_delta)
  qed
next
case (Symbol φ)
have qf_not_in_SQ: qf ∉ NFA.SQ q0 transs
  using Symbol unfolding IH_def by (auto simp: Let_def)
have qf_not_q0_Suc_q0: qf ∉ {q0, Suc q0}
  using Symbol unfolding IH_def
  by (auto simp: NFA.SQ_def split: option.splits)
have transs_def: transs = build_nfa_impl (Symbol φ) (q0, qf, phis)
  using Symbol(1)
  by (auto simp: Let_def IH_def)
interpret base: nfa q0 qf transs
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding IH_def NFA.Q_def NFA.SQ_def transs_def
  by (auto split: option.splits)
define n where n ≡ case pos φ phis of Some n ⇒ n | _ ⇒ length phis
then have collect: n < length (collect_subfmlas (Symbol φ) phis)
  (collect_subfmlas (Symbol φ) phis) ! n = φ
  using pos_sound pos_complete by (force split: option.splits)+
have ∧cs q. base.step_eps cs q0 q ↔ n < length cs ∧ cs ! n ∧ q = Suc q0 ∧ cs q. ¬base.step_eps cs
(Suc q0) q
  using base.q0_sub_SQ
  by (auto simp: NFA.step_eps_def transs_def n_def split: option.splits)
then have base_eps: base.step_eps_closure_set {q0} cs = (if n < length cs ∧ cs ! n then {q0, Suc
q0} else {q0}) for cs
  using NFA.step_eps_closure_set_unfold[where ?X={Suc q0}]
  using NFA.step_eps_closure_set_step_id[where ?R={q0}]
  using NFA.step_eps_closure_set_step_id[where ?R={Suc q0}]
  by auto
have base_delta: base.delta {q0} cs = (if n < length cs ∧ cs ! n then {qf} else {}) for cs
  unfolding NFA.delta_def NFA.step_symb_set_def base_eps
  by (auto simp: NFA.step_symb_def NFA.SQ_def transs_def split: option.splits)
show ?case
proof (cases bss)
  case Nil
  show ?thesis

```

```

    using qf_not_q0_Suc_q0
    unfolding NFA.run_accept_eps_def NFA.run_def NFA.accept_eps_def Nil
    by (auto simp: base_eps)
next
case bss_def: (Cons cs css)
have sat:  $n < \text{length } cs \wedge cs ! n \longleftrightarrow \text{sat } \varphi i$ 
  using Symbol(1) collect
  by (auto simp: Let_def IH_def bss_def)
show ?thesis
proof (cases css)
case Nil
show ?thesis
  unfolding NFA.run_accept_eps_def NFA.run_def NFA.accept_eps_def bss_def Nil
  by (auto simp: base_delta sat NFA.step_eps_closure_set_def NFA.step_eps_closure_def)
next
case css_def: (Cons ds dss)
have base_delta { } ds = { } base_delta {qf} ds = { }
  using base.step_eps_closure_qf qf_not_in_SQ step_symb_dest
  by (fastforce simp: NFA.delta_def NFA.step_eps_closure_set_def NFA.step_symb_set_def)+
then show ?thesis
  using NFA.run_accept_eps_empty
  unfolding NFA.run_accept_eps_def NFA.run_def NFA.accept_eps_def bss_def css_def
  by (auto simp: base_delta)
qed
qed
next
case (Plus r s)
obtain phis' where collect: collect_subfmlas (Plus r s) phis =
  collect_subfmlas r phis @ phis'
  using collect_subfmlas_app by auto
have qf_not_in_SQ:  $qf \notin \text{NFA.SQ } q0$  (build_nfa_impl (Plus r s) (q0, qf, phis))
  using Plus unfolding IH_def by auto
interpret base: nfa q0 qf build_nfa_impl (Plus r s) (q0, qf, phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt by fast+
interpret left: nfa q0 + 1 qf build_nfa_impl r (q0 + 1, qf, phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
  by fastforce+
interpret right: nfa q0 + 1 + state_cnt r qf
  build_nfa_impl s (q0 + 1 + state_cnt r, qf, collect_subfmlas r phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
  by fastforce+
from Plus(3) have IH r (q0 + 1) qf phis (build_nfa_impl r (q0 + 1, qf, phis)) bss bs i
  unfolding Let_def IH_def collect
  using left.qf_not_in_SQ
  by (auto simp: nth_append)
then have left_IH:  $\text{left.run\_accept\_eps } \{q0 + 1\} \text{ bss bs } \longleftrightarrow$ 
   $(i, i + \text{length } \text{bss}) \in \text{match } r$ 
  using Plus(1) build_nfa_impl_state_cnt
  by auto
have IH s (q0 + 1 + state_cnt r) qf (collect_subfmlas r phis)
  (build_nfa_impl s (q0 + 1 + state_cnt r, qf, collect_subfmlas r phis)) bss bs i
  using right.qf_not_in_SQ IH_def Plus

```

```

  by (auto simp: Let_def)
then have right_IH: right.run_accept_eps {q0 + 1 + state_cnt r} bss bs  $\longleftrightarrow$ 
  (i, i + length bss)  $\in$  match s
  using Plus(2) build_nfa_impl_state_cnt
  by auto
interpret cong: nfa_cong Plus q0 q0 + 1 q0 + 1 + state_cnt r qf qf
  build_nfa_impl (Plus r s) (q0, qf, phis) build_nfa_impl r (q0 + 1, qf, phis)
  build_nfa_impl s (q0 + 1 + state_cnt r, qf, collect_subfmlas r phis)
apply unfold_locales
unfolding NFA.SQ_def build_nfa_impl_state_cnt
  NFA.step_eps_def NFA.step_symb_def
  by (auto simp add: nth_append build_nfa_impl_state_cnt)
show ?case
  using cong.run_accept_eps_cong left_IH right_IH Plus
  by (auto simp: Let_def IH_def)
next
case (Times r s)
obtain phis' where collect: collect_subfmlas (Times r s) phis =
  collect_subfmlas r phis @ phis'
  using collect_subfmlas_app by auto
have transs_def: transs = build_nfa_impl (Times r s) (q0, qf, phis)
  using Times unfolding IH_def by (auto simp: Let_def)
have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 (build_nfa_impl (Times r s) (q0, qf, phis))
  using Times unfolding IH_def by auto
interpret base: nfa q0 qf build_nfa_impl (Times r s) (q0, qf, phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt by fast+
interpret left: nfa q0 q0 + state_cnt r build_nfa_impl r (q0, q0 + state_cnt r, phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
  by fastforce+
interpret right: nfa q0 + state_cnt r qf
  build_nfa_impl s (q0 + state_cnt r, qf, collect_subfmlas r phis)
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
  by fastforce+
from Times(3) have left_IH: IH r q0 (q0 + state_cnt r) phis
  (build_nfa_impl r (q0, q0 + state_cnt r, phis)) bss bs i
  unfolding Let_def IH_def collect
  using left.qf_not_in_SQ
  by (auto simp: nth_append)
from Times(3) have left_IH_take:  $\bigwedge n. n < \text{length } bss \implies$ 
  IH r q0 (q0 + state_cnt r) phis
  (build_nfa_impl r (q0, q0 + state_cnt r, phis)) (take n bss) (hd (drop n bss)) i
  unfolding Let_def IH_def collect
  using left.qf_not_in_SQ
  apply (auto simp: nth_append min_absorb2 hd_drop_conv_nth)
  apply (meson in_set_takeD le_add1 le_trans)
  by (meson le_add1 le_trans nth_mem)
have left_IH_match: left.run_accept_eps {q0} bss bs  $\longleftrightarrow$ 
  (i, i + length bss)  $\in$  match r
  using Times(1) build_nfa_impl_state_cnt left_IH
  by auto
have left_IH_match_take:  $\bigwedge n. n < \text{length } bss \implies$ 
  left.run_accept_eps {q0} (take n bss) (hd (drop n bss))  $\longleftrightarrow$  (i, i + n)  $\in$  match r

```

```

using Times(1) build_nfa_impl_state_cnt left_IH_take
by (fastforce simp add: nth_append min_absorb2)
have IH s (q0 + state_cnt r) qf (collect_subfmlas r phis)
  (build_nfa_impl s (q0 + state_cnt r, qf, collect_subfmlas r phis)) bss bs i
using right.qf_not_in_SQ IH_def Times
by (auto simp: Let_def)
then have right_IH:  $\bigwedge n. n \leq \text{length } bss \implies$  IH s (q0 + state_cnt r) qf (collect_subfmlas r phis)
  (build_nfa_impl s (q0 + state_cnt r, qf, collect_subfmlas r phis)) (drop n bss) bs (i + n)
unfolding Let_def IH_def
by (auto simp: nth_append add.assoc) (meson in_set_dropD)
have right_IH_match:  $\bigwedge n. n \leq \text{length } bss \implies$ 
  right.run_accept_eps {q0 + state_cnt r} (drop n bss) bs  $\longleftrightarrow$  (i + n, i + length bss)  $\in$  match s
using Times(2)[OF right_IH] build_nfa_impl_state_cnt
by (auto simp: IH_def)
interpret cong: nfa_cong_Times q0 q0 + state_cnt r qf
  build_nfa_impl (Times r s) (q0, qf, phis)
  build_nfa_impl r (q0, q0 + state_cnt r, phis)
  build_nfa_impl s (q0 + state_cnt r, qf, collect_subfmlas r phis)
apply unfold_locales
using NFA.Q_def NFA.SQ_def NFA.step_eps_def NFA.step_symb_def build_nfa_impl_state_set
by (fastforce simp add: nth_append build_nfa_impl_state_cnt build_nfa_impl_not_Nil
  state_cnt_pos)+
have right_IH_Nil: right.run_accept_eps {q0 + state_cnt r} [] bs  $\longleftrightarrow$ 
  (i + length bss, i + length bss)  $\in$  match s
using right_IH_match
by fastforce
show ?case
unfolding match_Times transs_def cong.run_accept_eps_cong left_IH_match right_IH_Nil
using left_IH_match_take right_IH_match less_imp_le_nat le_eq_less_or_eq
by auto
next
case (Star r)
then show ?case
proof (induction length bss arbitrary: bss bs i rule: nat_less_induct)
case 1
have transs_def: transs = build_nfa_impl (Star r) (q0, qf, phis)
using 1 unfolding IH_def by (auto simp: Let_def)
have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 (build_nfa_impl (Star r) (q0, qf, phis))
using 1 unfolding IH_def by auto
interpret base: nfa q0 qf build_nfa_impl (Star r) (q0, qf, phis)
apply unfold_locales
using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
by fast+
interpret left: nfa q0 + 1 q0 build_nfa_impl r (q0 + 1, q0, phis)
apply unfold_locales
using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt
by fastforce+
from 1(3) have left_IH: IH r (q0 + 1) q0 phis (build_nfa_impl r (q0 + 1, q0, phis)) bss bs i
using left.qf_not_in_SQ
unfolding Let_def IH_def
by (auto simp add: nth_append)
from 1(3) have left_IH_take:  $\bigwedge n. n < \text{length } bss \implies$ 
  IH r (q0 + 1) q0 phis (build_nfa_impl r (q0 + 1, q0, phis)) (take n bss) (hd (drop n bss)) i
using left.qf_not_in_SQ
unfolding Let_def IH_def
by (auto simp add: nth_append min_absorb2 hd_drop_conv_nth) (meson in_set_takeD)

```

```

have left_IH_match: left.run_accept_eps {q0 + 1} bss bs  $\longleftrightarrow$ 
  (i, i + length bss)  $\in$  match r
using 1(2) left_IH
unfolding build_nfa_impl_state_cnt NFA.SQ_def
by auto
have left_IH_match_take:  $\bigwedge n. n < \text{length } bss \implies$ 
  left.run_accept_eps {q0 + 1} (take n bss) (hd (drop n bss))  $\longleftrightarrow$ 
  (i, i + n)  $\in$  match r
using 1(2) left_IH_take
unfolding build_nfa_impl_state_cnt NFA.SQ_def
by (fastforce simp add: nth_append min_absorb2)
interpret cong: nfa_cong_Star q0 q0 + 1 qf
  build_nfa_impl (Star r) (q0, qf, phis)
  build_nfa_impl r (q0 + 1, q0, phis)
apply unfold_locales
unfolding NFA.SQ_def build_nfa_impl_state_cnt NFA.step_eps_def NFA.step_symb_def
by (auto simp add: nth_append build_nfa_impl_state_cnt)
show ?case
using cong.run_accept_eps_Nil
proof (cases bss)
case Nil
show ?thesis
  unfolding transs_def Nil
  using cong.run_accept_eps_Nil run_Nil run_accept_eps_Nil
  by auto
next
case (Cons cs css)
have aux:  $\bigwedge n j x P. n < x \implies j < x - n \implies (\forall j < \text{Suc } x. P j) \implies P (\text{Suc } (n + j))$ 
by auto
from 1(3) have star_IH:  $\bigwedge n. n < \text{length } css \implies$ 
  IH (Star r) q0 qf phis transs (drop n css) bs (i + n + 1)
unfolding Cons_Let_def IH_def
using aux[of _ _ _ λj.  $\forall k < \text{length } (\text{collect\_subfmlas } r \text{ phis}).$ 
  (cs # css) ! j ! k = sat (collect_subfmlas r phis ! k) (i + j)]
by (auto simp add: nth_append add.assoc dest: in_set_dropD)
have IH_inst:  $\bigwedge xs i. \text{length } xs \leq \text{length } css \implies \text{IH } (Star r) q0 qf \text{ phis transs } xs \text{ bs } i \longrightarrow$ 
  (base.run_accept_eps {q0} xs bs  $\longleftrightarrow$  (i, i + length xs)  $\in$  match (Star r))
using 1
unfolding Cons
by (auto simp add: nth_append less_Suc_eq_le transs_def)
have  $\bigwedge n. n < \text{length } css \implies \text{base.run\_accept\_eps } \{q0\} (\text{drop } n \text{ css}) \text{ bs } \longleftrightarrow$ 
  (i + n + 1, i + length (cs # css))  $\in$  match (Star r)
proof -
fix n
assume assm: n < length css
then show base.run_accept_eps {q0} (drop n css) bs  $\longleftrightarrow$ 
  (i + n + 1, i + length (cs # css))  $\in$  match (Star r)
using IH_inst[of drop n css i + n + 1] star_IH
by (auto simp add: nth_append)
qed
then show ?thesis
using match_Star length_Cons Cons cong.run_accept_eps_cong_Cons
using cong.run_accept_eps_Nil left_IH_match left_IH_match_take
apply (auto simp add: Cons transs_def)
apply (metis Suc_less_eq add_Suc_right drop_Suc_Cons less_imp_le_nat take_Suc_Cons)
apply (metis Suc_less_eq add_Suc_right drop_Suc_Cons le_eq_less_or_eq lessThan_iff
  take_Suc_Cons)
done

```

qed  
 qed  
 qed

```

lemma step_eps_closure_set_empty_list:
  assumes wf_regex r IH r q0 qf phis transs bss bs i NFA.step_eps_closure q0 transs bs q qf
  shows NFA.step_eps_closure q0 transs [] q qf
  using assms
proof (induction r arbitrary: q0 qf phis transs q)
  case (Symbol  $\varphi$ )
  have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 transs
  using Symbol unfolding IH_def by (auto simp: Let_def)
  have qf_not_q0_Suc_q0: qf  $\notin$  {q0, Suc q0}
  using Symbol unfolding IH_def
  by (auto simp: NFA.SQ_def split: option.splits)
  have transs_def: transs = build_nfa_impl (Symbol  $\varphi$ ) (q0, qf, phis)
  using Symbol(2)
  by (auto simp: Let_def IH_def)
  interpret base: nfa q0 qf transs
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding IH_def NFA.Q_def NFA.SQ_def transs_def
  by (auto split: option.splits)
  define n where n  $\equiv$  case pos  $\varphi$  phis of Some n  $\Rightarrow$  n | _  $\Rightarrow$  length phis
  then have collect: n < length (collect_subfmlas (Symbol  $\varphi$ ) phis)
  (collect_subfmlas (Symbol  $\varphi$ ) phis) ! n =  $\varphi$ 
  using pos_sound pos_complete by (force split: option.splits)+
  have SQD: q  $\in$  NFA.SQ q0 transs  $\Longrightarrow$  q = q0  $\vee$  q = Suc q0 for q
  by (auto simp: NFA.SQ_def transs_def split: option.splits)
  have  $\neg$ base.step_eps cs q qf if q  $\in$  NFA.SQ q0 transs for cs q
  using SQD[OF that] qf_not_q0_Suc_q0
  by (auto simp: NFA.step_eps_def transs_def split: option.splits transition.splits)
  then show ?case
  using Symbol(3)
  by (auto simp: NFA.step_eps_closure_def) (metis rtranclp.simps step_eps_dest)
next
  case (Plus r s)
  have transs_def: transs = build_nfa_impl (Plus r s) (q0, qf, phis)
  using Plus(4)
  by (auto simp: IH_def Let_def)
  define ts_l where ts_l = build_nfa_impl r (q0 + 1, qf, phis)
  define ts_r where ts_r = build_nfa_impl s (q0 + 1 + state_cnt r, qf, collect_subfmlas r phis)
  have len_ts: length ts_l = state_cnt r length ts_r = state_cnt s length transs = Suc (state_cnt r +
  state_cnt s)
  by (auto simp: ts_l_def ts_r_def transs_def build_nfa_impl_state_cnt)
  have transs_eq: transs = split_trans (q0 + 1) (q0 + 1 + state_cnt r) # ts_l @ ts_r
  by (auto simp: transs_def ts_l_def ts_r_def)
  have ts_nonempty: ts_l = []  $\Longrightarrow$  False ts_r = []  $\Longrightarrow$  False
  by (auto simp: ts_l_def ts_r_def build_nfa_impl_not_Nil)
  obtain phis' where collect: collect_subfmlas (Plus r s) phis = collect_subfmlas r phis @ phis'
  using collect_subfmlas_app by auto
  have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 (build_nfa_impl (Plus r s) (q0, qf, phis))
  using Plus unfolding IH_def by auto
  interpret base: nfa q0 qf transs
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt transs_def by fast+
  interpret left: nfa Suc q0 qf ts_l

```

```

apply unfold_locales
using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt ts_l_def
by fastforce+
interpret right: nfa Suc (q0 + state_cnt r) qf ts_r
apply unfold_locales
using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt ts_r_def
by fastforce+
interpret cong: nfa_cong_Plus q0 Suc q0 Suc (q0 + state_cnt r) qf qf qf transs ts_l ts_r
apply unfold_locales
unfolding NFA.SQ_def build_nfa_impl_state_cnt
NFA.step_eps_def NFA.step_symb_def transs_def ts_l_def ts_r_def
by (auto simp add: nth_append build_nfa_impl_state_cnt)
have IH s (Suc (q0 + state_cnt r)) qf (collect_subfmlas r phis) ts_r bss bs i
using right.qf_not_in_SQ IH_def Plus
by (auto simp: Let_def ts_r_def)
then have case_right: base.step_eps_closure [] q qf if base.step_eps_closure bs q qf q ∈ right.Q for q
using cong.right.eps_nfa'_step_eps_closure[OF that] Plus(2,3) cong.right.nfa'_eps_step_eps_closure[OF _
that(2)]
by auto
from Plus(4) have IH r (Suc q0) qf phis ts_l bss bs i
using left.qf_not_in_SQ
unfolding Let_def IH_def collect ts_l_def
by (auto simp: nth_append)
then have case_left: base.step_eps_closure [] q qf if base.step_eps_closure bs q qf q ∈ left.Q for q
using cong.eps_nfa'_step_eps_closure[OF that] Plus(1,3) cong.nfa'_eps_step_eps_closure[OF _
that(2)]
by auto
have q = q0 ∨ q ∈ left.Q ∨ q ∈ right.Q
using Plus(5)
by (auto simp: NFA.Q_def NFA.SQ_def len_ts dest!: NFA.step_eps_closure_dest)
moreover have ?case if q = q0
proof –
have q0 ≠ qf
using qf_not_in_SQ
by (auto simp: NFA.SQ_def)
then obtain q' where q'_def: base.step_eps bs q q' base.step_eps_closure bs q' qf
using Plus(5)
by (auto simp: q = q0 NFA.step_eps_closure_def elim: converse_rtranclpE)
have fst_step_eps: base.step_eps [] q q'
using q'_def(1)
by (auto simp: q = q0 NFA.step_eps_def transs_eq)
have q' ∈ left.Q ∨ q' ∈ right.Q
using q'_def(1)
by (auto simp: NFA.step_eps_def NFA.Q_def NFA.SQ_def q = q0 transs_eq dest: ts_nonempty split:
transition.splits)
then show ?case
using fst_step_eps case_left[OF q'_def(2)] case_right[OF q'_def(2)]
by (auto simp: NFA.step_eps_closure_def)
qed
ultimately show ?case
using Plus(5) case_left case_right
by auto
next
case (Times r s)
obtain phis' where collect: collect_subfmlas (Times r s) phis =
collect_subfmlas r phis @ phis'

```

```

    using collect_subfmlas_app by auto
  have transs_def: transs = build_nfa_impl (Times r s) (q0, qf, phis)
    using Times unfolding IH_def by (auto simp: Let_def)
  define ts_l where ts_l = build_nfa_impl r (q0, q0 + state_cnt r, phis)
  define ts_r where ts_r = build_nfa_impl s (q0 + state_cnt r, qf, collect_subfmlas r phis)
  have len_ts: length ts_l = state_cnt r length ts_r = state_cnt s length transs = state_cnt r + state_cnt
s
  by (auto simp: ts_l_def ts_r_def transs_def build_nfa_impl_state_cnt)
  have transs_eq: transs = ts_l @ ts_r
    by (auto simp: transs_def ts_l_def ts_r_def)
  have ts_nonempty: ts_l = []  $\implies$  False ts_r = []  $\implies$  False
    by (auto simp: ts_l_def ts_r_def build_nfa_impl_not_Nil)
  have qf_not_in_SQ: qf  $\notin$  NFA.SQ q0 (build_nfa_impl (Times r s) (q0, qf, phis))
    using Times unfolding IH_def by auto
  interpret base: nfa q0 qf transs
    apply unfold_locales
    using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
    unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt transs_def by fast+
  interpret left: nfa q0 q0 + state_cnt r ts_l
    apply unfold_locales
    using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
    unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt ts_l_def
    by fastforce+
  interpret right: nfa q0 + state_cnt r qf ts_r
    apply unfold_locales
    using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
    unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt ts_r_def
    by fastforce+
  interpret cong: nfa_cong Times q0 q0 + state_cnt r qf transs ts_l ts_r
    apply unfold_locales
    using NFA.Q_def NFA.SQ_def NFA.step_eps_def NFA.step_symb_def build_nfa_impl_state_set
    by (auto simp add: nth_append build_nfa_impl_state_cnt build_nfa_impl_not_Nil
      state_cnt_pos len_ts transs_eq)
  have qf  $\notin$  base.SQ
    using Times(4)
    by (auto simp: IH_def Let_def)
  then have qf_left_Q: qf  $\in$  left.Q  $\implies$  False
    by (auto simp: NFA.Q_def NFA.SQ_def len_ts state_cnt_pos)
  have left_IH: IH r q0 (q0 + state_cnt r) phis ts_l bss bs i
    using left.qf_not_in_SQ Times
    unfolding Let_def IH_def collect
    by (auto simp: nth_append ts_l_def)
  have case_left: base.step_eps_closure [] q (q0 + state_cnt r) if left.step_eps_closure bs q (q0 +
state_cnt r) q  $\in$  left.Q and wf: wf_regex r for q
    using that(1) Times(1)[OF wf left_IH] cong.nfa'_step_eps_closure_cong[OF _ that(2)]
    by auto
  have left_IH: IH s (q0 + state_cnt r) qf (collect_subfmlas r phis) ts_r bss bs i
    using right.qf_not_in_SQ IH_def Times
    by (auto simp: Let_def ts_r_def)
  then have case_right: base.step_eps_closure [] q qf if base.step_eps_closure bs q qf q  $\in$  right.Q for q
    using cong.right.eps_nfa'_step_eps_closure[OF that] Times(2,3) cong.right.nfa'_eps_step_eps_closure[OF
that(2)]
    by auto
  have init_right: q0 + state_cnt r  $\in$  right.Q
    by (auto simp: NFA.Q_def NFA.SQ_def dest: ts_nonempty)
  {
  assume q_left_Q: q  $\in$  left.Q
    then have split: left.step_eps_closure bs q (q0 + state_cnt r) base.step_eps_closure bs (q0 +

```

```

state_cnt r) qf
  using cong.eps_nfa'_step_eps_closure_cong[OF Times(5)]
  by (auto dest: qf_left_Q)
have empty_IH: IH s (q0 + state_cnt r) qf (collect_subfmlas r phis) ts_r [] bs (i + length bss)
  using left_IH
  by (auto simp: IH_def Let_def ts_r_def)
have right.step_eps_closure bs (q0 + state_cnt r) qf
  using cong.right.eps_nfa'_step_eps_closure[OF split(2) init_right]
  by auto
then have right.run_accept_eps {q0 + state_cnt r} [] bs
  by (auto simp: NFA.run_accept_eps_def NFA.accept_eps_def NFA.step_eps_closure_set_def
NFA.run_def)
then have wf: wf_regex r
  using nfa_correct[OF empty_IH] Times(3) match_refl_eps
  by auto
have ?case
  using case_left[OF split(1) q_left_Q wf] case_right[OF split(2) init_right]
  by (auto simp: NFA.step_eps_closure_def)
}
moreover have q ∈ left.Q ∨ q ∈ right.Q
  using Times(5)
  by (auto simp: NFA.Q_def NFA.SQ_def transs_eq len_ts dest!: NFA.step_eps_closure_dest)
ultimately show ?case
  using case_right[OF Times(5)]
  by auto
next
case (Star r)
have transs_def: transs = build_nfa_impl (Star r) (q0, qf, phis)
  using Star_unfolding IH_def by (auto simp: Let_def)
obtain ts_r where ts_r: transs = split_trans (q0 + 1) qf # ts_r ts_r = build_nfa_impl r (Suc q0,
q0, phis)
  using Star(3)
  by (auto simp: Let_def IH_def)
have qf_not_in_SQ: qf ∉ NFA.SQ q0 transs
  using Star_unfolding IH_def transs_def by auto
interpret base: nfa q0 qf transs
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt transs_def
  by fast+
interpret left: nfa Suc q0 q0 ts_r
  apply unfold_locales
  using build_nfa_impl_state_set build_nfa_impl_not_Nil qf_not_in_SQ
  unfolding NFA.Q_def NFA.SQ_def build_nfa_impl_state_cnt ts_r(2)
  by fastforce+
interpret cong: nfa_cong_Star q0 Suc q0 qf transs ts_r
  apply unfold_locales
  unfolding NFA.SQ_def build_nfa_impl_state_cnt NFA.step_eps_def NFA.step_symb_def
  by (auto simp add: nth_append build_nfa_impl_state_cnt ts_r(1))
have IH: wf_regex r IH r (Suc q0) q0 phis ts_r bss bs i
  using Star(2,3)
  by (auto simp: Let_def IH_def NFA.SQ_def ts_r(2))
have step_eps_q'_qf: q' = q0 if base.step_eps bs q' qf for q'
proof (rule ccontr)
  assume q' ≠ q0
  then have q' ∈ left.SQ
    using that
    by (auto simp: NFA.step_eps_def NFA.SQ_def ts_r(1))

```

```

then have left.step_eps bs q' qf
  using cong.step_eps_cong_SQ that
  by simp
then show False
  using qf_not_in_SQ
  by (metis NFA.Q_def UnE base.q0_sub_SQ cong.SQ_sub left.step_eps_closed subset_eq)
qed
show ?case
proof (cases q = qf)
case False
then have base_q_q0: base.step_eps_closure bs q q0 base.step_eps bs q0 qf
  using Star(4) step_eps_q'_qf
  by (auto simp: NFA.step_eps_closure_def) (metis rtranclp.cases)+
have base_Nil_q0_qf: base.step_eps [] q0 qf
  by (auto simp: NFA.step_eps_def NFA.SQ_def ts_r(1))
have q_left_Q: q ∈ left.Q
  using base_q_q0
  by (auto simp: NFA.Q_def NFA.SQ_def ts_r(1) dest: step_eps_closure_dest)
have left.step_eps_closure [] q q0
  using cong.eps_nfa'_step_eps_closure_cong[OF base_q_q0(1) q_left_Q] Star(1)[OF IH]
  by auto
then show ?thesis
  using cong.nfa'_step_eps_closure_cong[OF _ q_left_Q] base_Nil_q0_qf
  by (auto simp: NFA.step_eps_closure_def) (meson rtranclp.rtrancl_into_rtrancl)
qed (auto simp: NFA.step_eps_closure_def)
qed auto

```

**lemma** *accept\_eps\_iff\_accept*:

```

assumes wf_regex r IH r q0 qf phis transs bss bs i
shows NFA.accept_eps q0 qf transs R bs = NFA.accept q0 qf transs R
using step_eps_closure_set_empty_list[OF assms] step_eps_closure_set_mono'
unfolding NFA.accept_eps_def NFA.accept_def
by (fastforce simp: NFA.accept_eps_def NFA.accept_def NFA.step_eps_closure_set_def)

```

**lemma** *run\_accept\_eps\_iff\_run\_accept*:

```

assumes wf_regex r IH r q0 qf phis transs bss bs i
shows NFA.run_accept_eps q0 qf transs {q0} bss bs ↔ NFA.run_accept q0 qf transs {q0} bss
unfolding NFA.run_accept_eps_def NFA.run_accept_def accept_eps_iff_accept[OF assms] ..

```

**end**

**definition** *pred\_option'* :: ('a ⇒ bool) ⇒ 'a option ⇒ bool **where**  
*pred\_option'* P z = (case z of Some z' ⇒ P z' | None ⇒ False)

**definition** *map\_option'* :: ('b ⇒ 'c option) ⇒ 'b option ⇒ 'c option **where**  
*map\_option'* f z = (case z of Some z' ⇒ f z' | None ⇒ None)

**definition** *while\_break* :: ('a ⇒ bool) ⇒ ('a ⇒ 'a option) ⇒ 'a ⇒ 'a option **where**  
*while\_break* P f x = while (pred\_option' P) (map\_option' f) (Some x)

**lemma** *wf\_while\_break*:

```

assumes wf {(t, s). P s ∧ b s ∧ Some t = c s}
shows wf {(t, s). pred_option P s ∧ pred_option' b s ∧ t = map_option' c s}
proof -
have sub: {(t, s). pred_option P s ∧ pred_option' b s ∧ t = map_option' c s} ⊆
  map_prod Some Some ' {(t, s). P s ∧ b s ∧ Some t = c s} ∪ ({None} × (Some ' UNIV))
  by (auto simp: pred_option'_def map_option'_def split: option.splits)
  (smt (z3) case_prodI map_prod_imageI mem_Collect_eq not_Some_eq)

```

```

show ?thesis
  apply (rule wf_subset[OF _ sub])
  apply (rule wf_union_compatible)
  apply (rule wf_map_prod_image)
  apply (fastforce simp: wf_def intro: assms)+
done
qed

lemma wf_while_break':
  assumes wf {(t, s). P s ∧ b s ∧ Some t = c s}
  shows wf {(t, s). pred_option' P s ∧ pred_option' b s ∧ t = map_option' c s}
  by (rule wf_subset[OF wf_while_break[OF assms]]) (auto simp: pred_option'_def split: option.splits)

lemma while_break_sound:
  assumes  $\bigwedge s s'. P s \implies b s \implies c s = \text{Some } s' \implies P s' \wedge s. P s \implies \neg b s \implies Q s$  wf {(t, s). P s ∧
  b s ∧ Some t = c s} P s
  shows pred_option Q (while_break b c s)
proof -
  have aux: P t  $\implies$  b t  $\implies$  pred_option P (c t) for t
    using assms(1)
    by (cases c t) auto
  show ?thesis
    using assms aux
    by (auto simp: while_break_def pred_option'_def map_option'_def split: option.splits
      intro!: while_rule_lemma[where ?P=pred_option P and ?Q=pred_option Q and ?b=pred_option'
      b and ?c=map_option' c, OF _ _ wf_while_break])
qed

lemma while_break_complete: ( $\bigwedge s. P s \implies b s \implies \text{pred\_option}' P (c s) \implies (\bigwedge s. P s \implies \neg b s \implies Q s) \implies$ 
  wf {(t, s). P s ∧ b s ∧ Some t = c s}  $\implies$  P s  $\implies$ 
  pred_option' Q (while_break b c s)
  unfolding while_break_def
  by (rule while_rule_lemma[where ?P=pred_option' P and ?Q=pred_option' Q and ?b=pred_option'
  b and ?c=map_option' c, OF _ _ wf_while_break])
  (force simp: pred_option'_def map_option'_def split: option.splits elim!: case_optionE)+

context
  fixes args :: (bool iarray, nat set, 'd :: timestamp, 't, 'e) args
begin

abbreviation reach_w  $\equiv$  reach_window args

qualified definition in_win = init_window args

definition valid_window_matchP :: 'd  $\mathcal{I} \Rightarrow$  't  $\Rightarrow$  'e  $\Rightarrow$ 
  ('d  $\times$  bool iarray) list  $\Rightarrow$  nat  $\Rightarrow$  (bool iarray, nat set, 'd, 't, 'e) window  $\Rightarrow$  bool where
  valid_window_matchP I t0 sub rho j w  $\longleftrightarrow$  j = w_j w  $\wedge$ 
  valid_window args t0 sub rho w  $\wedge$ 
  reach_w t0 sub rho (w_i w, w_ti w, w_si w, w_j w, w_tj w, w_sj w)  $\wedge$ 
  (case w_read_t args (w_tj w) of None  $\Rightarrow$  True
  | Some t  $\Rightarrow$  ( $\forall l < w_i w. \text{memL } (ts\_at \text{ rho } l) t I$ ))

lemma valid_window_matchP_reach_tj: valid_window_matchP I t0 sub rho i w  $\implies$ 
  reaches_on (w_run_t args) t0 (map fst rho) (w_tj w)
  using reach_window_run_tj
  by (fastforce simp: valid_window_matchP_def simp del: reach_window.simps)

lemma valid_window_matchP_reach_sj: valid_window_matchP I t0 sub rho i w  $\implies$ 

```

```

reaches_on (w_run_sub args) sub (map snd rho) (w_sj w)
using reach_window_run_sj
by (fastforce simp: valid_window_matchP_def simp del: reach_window.simps)

```

**lemma** *valid\_window\_matchP\_len\_rho*: *valid\_window\_matchP I t0 sub rho i w  $\implies$  length rho = i*  
**by** (auto simp: valid\_window\_matchP\_def)

**definition** *matchP\_loop\_cond* *I t* = ( $\lambda w. w_i w < w_j w \wedge \text{memL (the (w_read_t args (w_ti w))) t I}$ )

**definition** *matchP\_loop\_inv* *I t0 sub rho j0 tj0 sj0 t* =  
( $\lambda w. \text{valid\_window args t0 sub rho w} \wedge$   
 $w_j w = j0 \wedge w_{tj} w = tj0 \wedge w_{sj} w = sj0 \wedge (\forall l < w_i w. \text{memL (ts\_at rho l) t I})$ )

```

fun ex_key :: ('c, 'd) mmap  $\Rightarrow$  ('d  $\Rightarrow$  bool)  $\Rightarrow$ 
('c  $\Rightarrow$  bool)  $\Rightarrow$  ('c, bool) mapping  $\Rightarrow$  (bool  $\times$  ('c, bool) mapping) where
ex_key [] time accept ac = (False, ac)
| ex_key ((q, t) # qts) time accept ac = (if time t then
(case cac accept ac q of ( $\beta, ac'$ )  $\Rightarrow$ 
if  $\beta$  then (True,  $ac'$ ) else ex_key qts time accept  $ac'$ )
else ex_key qts time accept ac)

```

**lemma** *ex\_key\_sound*:

**assumes** *inv*:  $\bigwedge q. \text{Case Mapping.lookup ac q of None} \Rightarrow \text{True} \mid \text{Some } v \Rightarrow \text{accept } q = v$

**and** *distinct*: *distinct (map fst qts)*

**and** *eval*: *ex\_key qts time accept ac = (b,  $ac'$ )*

**shows**  $b = (\exists q \in \text{mmap\_keys } qts. \text{time (the (mmap\_lookup } qts \ q))} \wedge \text{accept } q) \wedge$   
 $(\forall q. \text{Case Mapping.lookup } ac' \ q \text{ of None} \Rightarrow \text{True} \mid \text{Some } v \Rightarrow \text{accept } q = v)$

**using** *assms*

**proof** (*induction qts arbitrary: ac*)

**case** (*Cons a qts*)

**obtain** *q t* **where** *qt\_def*:  $a = (q, t)$

**by** *fastforce*

**show** *?case*

**proof** (*cases time t*)

**case** *True*

**note** *time\_t* = *True*

**obtain**  $\beta$   $ac''$  **where** *ac''\_def*:  $\text{cac accept } ac \ q = (\beta, ac'')$

**by** *fastforce*

**have** *accept*:  $\beta = \text{accept } q \wedge q. \text{Case Mapping.lookup } ac'' \ q \text{ of None} \Rightarrow \text{True}$

$\mid \text{Some } v \Rightarrow \text{accept } q = v$

**using** *ac''\_def* *Cons(2)*

**by** (*fastforce simp: cac\_def Let\_def Mapping.lookup\_update' split: option.splits if\_splits*)

**show** *?thesis*

**proof** (*cases  $\beta$* )

**case** *True*

**then show** *?thesis*

**using** *accept(2)* *time\_t* *Cons(4)*

**by** (*auto simp: qt\_def mmap\_keys\_def accept(1) mmap\_lookup\_def ac''\_def*)

**next**

**case** *False*

**have** *ex\_key*:  $\text{ex\_key } qts \ \text{time } \text{accept } ac'' = (b, ac')$

**using** *Cons(4)* *time\_t* *False*

**by** (*auto simp: qt\_def ac''\_def*)

**show** *?thesis*

**using** *Cons(1)[OF accept(2) \_ ex\_key]* *False[unfolded accept(1)]* *Cons(3)*

**by** (*auto simp: mmap\_keys\_def mmap\_lookup\_def qt\_def*)

**qed**

**next**



```

    Some ((t, β), adv_start args w'))))

end

locale MDL_window = MDL σ
  for σ :: ('a, 'd :: timestamp) trace +
  fixes r :: ('a, 'd :: timestamp) regex
  and t0 :: 't
  and sub :: 'e
  and args :: (bool iarray, nat set, 'd, 't, 'e) args
  assumes init_def: w_init args = {0 :: nat}
  and step_def: w_step args =
    NFA.delta' (IArray (build_nfa_impl r (0, state_cnt r, []))) (state_cnt r)
  and accept_def: w_accept args = NFA.accept' (IArray (build_nfa_impl r (0, state_cnt r, [])))
(state_cnt r)
  and run_t_sound: reaches_on (w_run_t args) t0 ts t ==>
    w_run_t args t = Some (t', x) ==> x = τ σ (length ts)
  and run_sub_sound: reaches_on (w_run_sub args) sub bs s ==>
    w_run_sub args s = Some (s', b) ==>
    b = IArray (map (λphi. sat phi (length bs)) (collect_subfmlas r []))
  and run_t_read: w_run_t args t = Some (t', x) ==> w_read_t args t = Some x
  and read_t_run: w_read_t args t = Some x ==> ∃ t'. w_run_t args t = Some (t', x)
begin

definition qf = state_cnt r
definition transs = build_nfa_impl r (0, qf, [])

abbreviation init ≡ w_init args
abbreviation step ≡ w_step args
abbreviation accept ≡ w_accept args
abbreviation run ≡ NFA.run' (IArray transs) qf
abbreviation wacc ≡ Window.acc (w_step args) (w_accept args)
abbreviation rw ≡ reach_window args

abbreviation valid_matchP ≡ valid_window_matchP args
abbreviation eval_mP ≡ eval_matchP args
abbreviation matchP_inv ≡ matchP_loop_inv args
abbreviation matchP_cond ≡ matchP_loop_cond args

abbreviation valid_matchF ≡ valid_window_matchF args
abbreviation eval_mF ≡ eval_matchF args
abbreviation matchF_inv ≡ matchF_loop_inv args
abbreviation matchF_inv' ≡ matchF_loop_inv' args
abbreviation matchF_cond ≡ matchF_loop_cond args

lemma run_t_sound':
  assumes reaches_on (w_run_t args) t0 ts t i < length ts
  shows ts ! i = τ σ i
proof -
  obtain t' t'' where t'_def: reaches_on (w_run_t args) t0 (take i ts) t'
    w_run_t args t' = Some (t'', ts ! i)
  using reaches_on_split[OF assms]
  by auto
  show ?thesis
  using run_t_sound[OF t'_def] assms(2)
  by simp
qed

```

```

lemma run_sub_sound':
  assumes reaches_on (w_run_sub args) sub bs s i < length bs
  shows bs ! i = IArray (map (λphi. sat phi i) (collect_subfmlas r []))
proof -
  obtain s' s'' where s'_def: reaches_on (w_run_sub args) sub (take i bs) s'
    w_run_sub args s' = Some (s'', bs ! i)
    using reaches_on_split[OF assms]
    by auto
  show ?thesis
  using run_sub_sound[OF s'_def] assms(2)
  by simp
qed

lemma run_ts: reaches_on (w_run_t args) t ts t'  $\implies$  t = t0  $\implies$  chain_le ts
proof (induction t ts t' rule: reaches_on_rev_induct)
  case (2 s s' v vs s'')
  show ?case
proof (cases vs rule: rev_cases)
  case (snoc zs z)
  show ?thesis
  using 2(3)[OF 2(4)]
  using chain_le_app[OF _ τ_mono[of length zs Suc (length zs) σ]]
    run_t_sound'[OF reaches_on_app[OF 2(1,2), unfolded 2(4)], of length zs]
    run_t_sound'[OF reaches_on_app[OF 2(1,2), unfolded 2(4)], of Suc (length zs)]
  unfolding snoc
  by (auto simp: nth_append)
qed (auto intro: chain_le.intros)
qed (auto intro: chain_le.intros)

lemma ts_at_tau: reaches_on (w_run_t args) t0 (map fst rho) t  $\implies$  i < length rho  $\implies$ 
  ts_at rho i = τ σ i
  using run_t_sound'
  unfolding ts_at_def
  by fastforce

lemma length_bs_at: reaches_on (w_run_sub args) sub (map snd rho) s  $\implies$  i < length rho  $\implies$ 
  IArray.length (bs_at rho i) = length (collect_subfmlas r [])
  using run_sub_sound'
  unfolding bs_at_def
  by fastforce

lemma bs_at_nth: reaches_on (w_run_sub args) sub (map snd rho) s  $\implies$  i < length rho  $\implies$ 
  n < IArray.length (bs_at rho i)  $\implies$ 
  IArray.sub (bs_at rho i) n  $\longleftrightarrow$  sat (collect_subfmlas r [] ! n) i
  using run_sub_sound'
  unfolding bs_at_def
  by fastforce

lemma ts_at_mono:  $\bigwedge i j.$  reaches_on (w_run_t args) t0 (map fst rho) t  $\implies$ 
  i ≤ j  $\implies$  j < length rho  $\implies$  ts_at rho i ≤ ts_at rho j
  using ts_at_tau
  by fastforce

lemma steps_is_run: steps (w_step args) rho q ij = run q (sub_bs rho ij)
  unfolding NFA.run'_def steps_def step_def transs_def qf_def ..

lemma acc_is_accept: wacc rho q (i, j) = w_accept args (run q (sub_bs rho (i, j)))
  unfolding acc_def steps_is_run by auto

```

```

lemma iarray_list_of: IArray (IArray.list_of xs) = xs
  by (cases xs) auto

lemma map_iarray_list_of: map IArray (map IArray.list_of bss) = bss
  using iarray_list_of
  by (induction bss) auto

lemma acc_match:
  fixes ts :: 'd list
  assumes reaches_on (w_run_sub args) sub (map snd rho) s  $i \leq j \leq \text{length } rho$  wf_regex r
  shows wacc rho {0} (i, j)  $\longleftrightarrow (i, j) \in \text{match } r$ 
proof -
  have j_eq:  $j = i + \text{length } (\text{sub\_bs } rho \ (i, j))$ 
    using assms by auto
  define bs where bs = map ( $\lambda phi. \text{sat } phi \ j$ ) (collect_subfmlas r [])
  have IH: IH r 0 qf [] transs (map IArray.list_of (sub_bs rho (i, j))) bs i
    unfolding IH_def transs_def qf_def NFA.SQ_def build_nfa_impl_state_cnt bs_def
    using assms run_sub_sound bs_at_nth length_bs_at by fastforce
  interpret NFA_array: nfa_array transs IArray transs qf
  by unfold_locales (auto simp: qf_def transs_def build_nfa_impl_state_cnt)
  have run_run': NFA_array.run R (map IArray.list_of (sub_bs rho (i, j))) = NFA_array.run' R
(sub_bs rho (i, j)) for R
    using NFA_array.run'_eq[of sub_bs rho (i, j) map IArray.list_of (sub_bs rho (i, j))]
    unfolding map_iarray_list_of
    by auto
  show ?thesis
    using nfa_correct[OF IH, unfolded NFA.run_accept_def]
    unfolding run_accept_eps_iff_run_accept[OF assms(4) IH] acc_is_accept NFA.run_accept_def
run_run' NFA_array.accept'_eq
    by (simp add: j_eq[symmetric] accept_def assms(2) qf_def transs_def)
qed

lemma accept_match:
  fixes ts :: 'd list
  shows reaches_on (w_run_sub args) sub (map snd rho) s  $\implies i \leq j \implies j \leq \text{length } rho \implies wf\_regex$ 
r  $\implies$ 
  w_accept args (steps (w_step args) rho {0} (i, j))  $\longleftrightarrow (i, j) \in \text{match } r$ 
  using acc_match acc_is_accept steps_is_run
  by metis

lemma drop_take_drop:  $i \leq j \implies j \leq \text{length } rho \implies \text{drop } i \ (\text{take } j \ rho) \ @ \ \text{drop } j \ rho = \text{drop } i \ rho$ 
  apply (induction i arbitrary: j rho)
  by auto (metis append_take_drop_id diff_add drop_drop drop_take)

lemma take_Suc:  $\text{drop } n \ xs = y \ \# \ ys \implies \text{take } n \ xs \ @ \ [y] = \text{take } (\text{Suc } n) \ xs$ 
  by (metis drop_all list.distinct(1) list.sel(1) not_less take_hd_drop)

lemma valid_init_matchP: valid_matchP I t0 sub [] 0 (init_window args t0 sub)
  using valid_init_window
  by (fastforce simp: valid_window_matchP_def Let_def intro: reaches_on.intros split: option.splits)

lemma valid_init_matchF: valid_matchF I t0 sub [] 0 (init_window args t0 sub)
  using valid_init_window
  by (fastforce simp: valid_window_matchF_def Let_def intro: reaches_on.intros split: option.splits)

lemma valid_eval_matchP:
  assumes valid_before': valid_matchP I t0 sub rho j w

```

**and** *before\_end*:  $w\_run\_t$  args ( $w\_tj$   $w$ ) = *Some* ( $tj'''$ ,  $t$ )  $w\_run\_sub$  args ( $w\_sj$   $w$ ) = *Some* ( $sj'''$ ,  
b) **and** *wf*: *wf\_regex*  $r$   
**shows**  $\exists w'. eval\_mP$   $I$   $w$  = *Some* ( $(\tau \sigma j, sat (MatchP$   $I$   $r)$   $j)$ ,  $w'$ )  $\wedge$   
 $t = \tau \sigma j \wedge valid\_matchP$   $I$   $t0$  *sub* ( $\rho @ [(t, b)]$ ) (*Suc*  $j$ )  $w'$   
**proof** –  
**obtain**  $w'$  **where**  $w'\_def$ : *adv\_end* args  $w$  = *Some*  $w'$   
**using** *before\_end*  
**by** (*fastforce simp: adv\_end\_def Let\_def split: prod.splits*)  
**define**  $st$  **where**  $st = w\_st$   $w'$   
**define**  $i$  **where**  $i = w\_i$   $w'$   
**define**  $ti$  **where**  $ti = w\_ti$   $w'$   
**define**  $si$  **where**  $si = w\_si$   $w'$   
**define**  $tj$  **where**  $tj = w\_tj$   $w'$   
**define**  $sj$  **where**  $sj = w\_sj$   $w'$   
**define**  $s$  **where**  $s = w\_s$   $w'$   
**define**  $e$  **where**  $e = w\_e$   $w'$   
**define**  $\rho'$  **where**  $\rho' = \rho @ [(t, b)]$   
**have** *reaches\_on'*: *reaches\_on* ( $w\_run\_t$  args)  $t0$  (*map fst*  $\rho'$ )  $tj'''$   
**using** *valid\_before' reach\_window\_run\_tj[OF reach\_window\_app[OF \_ before\_end]]*  
**by** (*auto simp: valid\_window\_matchP\_def rho'\_def*)  
**have**  $\rho\_mono$ :  $\wedge t'. t' \in set (map fst \rho) \implies t' \leq t$   
**using** *ts\_at\_mono[OF reaches\_on'] nat\_less\_le*  
**by** (*fastforce simp: rho'\_def ts\_at\_def nth\_append in\_set\_conv\_nth split: list.splits*)  
**have** *valid\_adv\_end\_w*: *valid\_window* args  $t0$  *sub*  $\rho'$   $w'$   
**using** *valid\_before' valid\_adv\_end[OF \_ before\_end rho\_mono]*  
**by** (*auto simp: valid\_window\_matchP\_def rho'\_def w'\_def*)  
**have**  $w\_ij\_adv\_end$ :  $w\_i$   $w' = w\_i$   $w$   $w\_j$   $w' = Suc$   $j$   
**using** *valid\_before' w'\_def*  
**by** (*auto simp: valid\_window\_matchP\_def adv\_end\_def Let\_def before\_end split: prod.splits*)  
**have** *valid\_before*: *rw*  $t0$  *sub*  $\rho'$  ( $i, ti, si, Suc$   $j, tj, sj$ )  
 $\wedge i j. i \leq j \implies j < length \rho' \implies ts\_at \rho' i \leq ts\_at \rho' j$   
 $\forall q. mmap\_lookup$   $e$   $q = sup\_leadsto$  *init* *step*  $\rho' i$  (*Suc*  $j$ )  $q$   
 $valid\_s$  *init* *step*  $st$  *accept*  $\rho' i$  (*Suc*  $j$ )  $s$   
 $w\_j$   $w' = Suc$   $j$   $i \leq Suc$   $j$   
**using** *valid\_adv\_end\_w*  
**unfolding** *valid\_window\_def Let\_def ti\_def si\_def i\_def tj\_def sj\_def s\_def e\_def w\_ij\_adv\_end*  
*st\_def*  
**by** *auto*  
**note** *read\_t\_def* = *run\_t\_read*[*OF before\_end(1)*]  
**have**  $t\_props$ :  $\forall l < i. memL$  ( $ts\_at \rho' l$ )  $t$   $I$   
**using** *valid\_before'*  
**by** (*auto simp: valid\_window\_matchP\_def i\_def w\_ij\_adv\_end read\_t\_def rho'\_def ts\_at\_def*  
*nth\_append*)  
  
**note** *reaches\_on\_tj* = *reach\_window\_run\_tj*[*OF valid\_before(1)*]  
**note** *reaches\_on\_sj* = *reach\_window\_run\_sj*[*OF valid\_before(1)*]  
**have**  $length\_rho'$ :  $length \rho' = Suc$   $j$   $length \rho = j$   
**using** *valid\_before*  
**by** (*auto simp: rho'\_def*)  
**have**  $j\_len\_rho'$ :  $j < length \rho'$   
**by** (*auto simp: length\_rho'*)  
**have**  $tj\_eq$ :  $t = \tau \sigma j$   $t = ts\_at \rho' j$   
**using** *run\_t\_sound'[OF reaches\_on\_tj, of j]*  
**by** (*auto simp: rho'\_def length\_rho' nth\_append ts\_at\_def*)  
**have**  $bj\_def$ :  $b = bs\_at \rho' j$   
**using** *run\_sub\_sound'[OF reaches\_on\_sj, of j]*  
**by** (*auto simp: rho'\_def length\_rho' nth\_append bs\_at\_def*)

```

define w'' where loop_def: w'' = while (matchP_cond I t) (adv_start args) w'
have inv_before: matchP_inv I t0 sub rho' (Suc j) tj sj t w'
  using valid_adv_end_w valid_before t_props
  unfolding matchP_loop_inv_def
  by (auto simp: tj_def sj_def i_def)
have loop: matchP_inv I t0 sub rho' (Suc j) tj sj t w''  $\wedge$   $\neg$ matchP_cond I t w''
  unfolding loop_def
proof (rule while_rule_lemma[of matchP_inv I t0 sub rho' (Suc j) tj sj t])
  fix w_cur :: (bool iarray, nat set, 'd, 't, 'e) window
  assume assms: matchP_inv I t0 sub rho' (Suc j) tj sj t w_cur matchP_cond I t w_cur
  define st_cur where st_cur = w_st w_cur
  define i_cur where i_cur = w_i w_cur
  define ti_cur where ti_cur = w_ti w_cur
  define si_cur where si_cur = w_si w_cur
  define s_cur where s_cur = w_s w_cur
  define e_cur where e_cur = w_e w_cur
  have valid_loop: rw t0 sub rho' (i_cur, ti_cur, si_cur, Suc j, tj, sj)
 $\wedge$   $i.j. i \leq j \implies j < \text{length } \rho' \implies \text{ts\_at } \rho' i \leq \text{ts\_at } \rho' j$ 
 $\forall q. \text{mmap\_lookup } e\_cur q = \text{sup\_leadsto init step } \rho' i\_cur (Suc j) q$ 
  valid_s init step st_cur accept rho' i_cur i_cur (Suc j) s_cur
  w_j w_cur = Suc j
  using assms(1)[unfolded matchP_loop_inv_def valid_window_matchP_def] valid_before(6)
  ti_cur_def si_cur_def i_cur_def s_cur_def e_cur_def
  by (auto simp: valid_window_def Let_def init_def step_def st_cur_def accept_def
    split: option.splits)
obtain ti'_cur si'_cur t_cur b_cur where run_si_cur:
  w_run_t args ti_cur = Some (ti'_cur, t_cur) w_run_sub args si_cur = Some (si'_cur, b_cur)
  t_cur = ts_at rho' i_cur b_cur = bs_at rho' i_cur
  using assms(2) reach_window_run_si[OF valid_loop(1)] reach_window_run_ti[OF valid_loop(1)]
  unfolding matchP_loop_cond_def valid_loop(5) i_cur_def
  by auto
have  $\wedge l. l < i\_cur \implies \text{memL } (\text{ts\_at } \rho' l) t I$ 
  using assms(1)
  unfolding matchP_loop_inv_def i_cur_def
  by auto
then have  $\wedge l. l < \text{Suc } (i\_cur) \implies \text{memL } (\text{ts\_at } \rho' l) t I$ 
  using assms(2) run_t_read[OF run_si_cur(1), unfolded run_si_cur(3)]
  unfolding matchP_loop_cond_def i_cur_def ti_cur_def
  by (auto simp: less_Suc_eq)
then show matchP_inv I t0 sub rho' (Suc j) tj sj t (adv_start args w_cur)
  using assms i_cur_def valid_adv_start valid_adv_start_bounds
  unfolding matchP_loop_inv_def matchP_loop_cond_def
  by fastforce
next
{
  fix w1 w2
  assume lassms: matchP_inv I t0 sub rho' (Suc j) tj sj t w1 matchP_cond I t w1
  w2 = adv_start args w1
  define i_cur where i_cur = w_i w1
  define ti_cur where ti_cur = w_ti w1
  define si_cur where si_cur = w_si w1
  have valid_loop: rw t0 sub rho' (i_cur, ti_cur, si_cur, Suc j, tj, sj) w_j w1 = Suc j
  using lassms(1)[unfolded matchP_loop_inv_def valid_window_matchP_def] valid_before(6)
  ti_cur_def si_cur_def i_cur_def
  by (auto simp: valid_window_def Let_def)
obtain ti'_cur si'_cur t_cur b_cur where run_si_cur:
  w_run_t args ti_cur = Some (ti'_cur, t_cur)
  w_run_sub args si_cur = Some (si'_cur, b_cur)

```

```

using lassms(2) reach_window_run_si[OF valid_loop(1)] reach_window_run_ti[OF valid_loop(1)]
  unfolding matchP_loop_cond_def valid_loop_i_cur_def
  by auto
have w1_ij: w_i w1 < Suc j w_j w1 = Suc j
  using lassms
  unfolding matchP_loop_inv_def matchP_loop_cond_def
  by auto
have w2_ij: w_i w2 = Suc (w_i w1) w_j w2 = Suc j
  using w1_ij lassms(3) run_si_cur(1,2)
  unfolding ti_cur_def si_cur_def
  by (auto simp: adv_start_def Let_def split: option.splits prod.splits if_splits)
have w_j w2 - w_i w2 < w_j w1 - w_i w1
  using w1_ij w2_ij
  by auto
}
then have {(s', s). matchP_inv I t0 sub rho' (Suc j) tj sj t s ∧ matchP_cond I t s ∧
  s' = adv_start args s} ⊆ measure (λw. w_j w - w_i w)
  by auto
then show wf {(s', s). matchP_inv I t0 sub rho' (Suc j) tj sj t s ∧ matchP_cond I t s ∧
  s' = adv_start args s}
  using wf_measure wf_subset by auto
qed (auto simp: inv_before)
have valid_w': valid_window args t0 sub rho' w''
  using conjunct1[OF loop]
  unfolding matchP_loop_inv_def
  by auto
have w_tsj_w': w_tj w'' = tj w_sj w'' = sj w_j w'' = Suc j
  using loop
  by (auto simp: matchP_loop_inv_def)
define st' where st' = w_st w''
define ac where ac = w_ac w''
define i' where i' = w_i w''
define ti' where ti' = w_ti w''
define si' where si' = w_si w''
define s' where s' = w_s w''
define e' where e' = w_e w''
define tj' where tj' = w_tj w''
define sj' where sj' = w_sj w''
have i'_le_Suc_j: i' ≤ Suc j
  using loop
  unfolding matchP_loop_inv_def
  by (auto simp: valid_window_def Let_def i'_def)
have valid_after: rw t0 sub rho' (i', ti', si', Suc j, tj', sj')
  ∧ i j. i ≤ j ⇒ j < length rho' ⇒ ts_at rho' i ≤ ts_at rho' j
  distinct (map fst e')
  ∀ q. mmap_lookup e' q = sup_leadsto init step rho' i' (Suc j) q
  ∧ q. case Mapping.lookup ac q of None ⇒ True | Some v ⇒ accept q = v
  valid_s init step st' accept rho' i' i' (Suc j) s' i' ≤ Suc j Suc j ≤ length rho'
  using valid_w' i'_le_Suc_j
  unfolding valid_window_def Let_def i'_def ti'_def si'_def s'_def e'_def tj'_def sj'_def ac_def
  st'_def w_tsj_w'
  by auto
note lookup_e' = valid_after(3,4,5,6)
obtain β ac' where ac'_def: ex_key e' (λt'. memR t' t I)
  (w_accept args) ac = (β, ac')
  by fastforce
have β_def: β = (∃ q ∈ mmap_keys e'. memR (the (mmap_lookup e' q)) t I ∧ accept q)
  ∧ q. case Mapping.lookup ac' q of None ⇒ True | Some v ⇒ accept q = v

```

```

using ex_key_sound[OF valid_after(5) valid_after(3) ac'_def]
by auto
have i'_set:  $\bigwedge l. l < w_i w'' \implies \text{memL } (ts\_at \text{ rho}' l) (ts\_at \text{ rho}' j) I$ 
using loop_length_rho' i'_le_Suc_j
unfolding matchP_loop_inv_def
by (auto simp: ts_at_def rho'_def nth_append i'_def)
have b_alt:  $(\exists q \in \text{mmap\_keys } e'. \text{memR } (the (\text{mmap\_lookup } e' q)) t I \wedge \text{accept } q) \longleftrightarrow \text{sat } (\text{MatchP } I r) j$ 
proof (rule iffI)
assume  $\exists q \in \text{mmap\_keys } e'. \text{memR } (the (\text{mmap\_lookup } e' q)) t I \wedge \text{accept } q$ 
then obtain q where q_def:  $q \in \text{mmap\_keys } e'$ 
  memR (the (mmap_lookup e' q)) t I accept q
by auto
then obtain ts' where ts_def:  $\text{mmap\_lookup } e' q = \text{Some } ts'$ 
by (auto dest: Mapping_keys_dest)
have sup_leadsto_init_step_rho' i' (Suc j) q = Some ts'
using lookup_e' ts_def q_def valid_after(4,7,8)
by (auto simp: rho'_def sup_leadsto_app_cong)
then obtain l where l_def:  $l < i'$  steps step_rho' init (l, Suc j) = q
  ts_at_rho' l = ts'
using sup_leadsto_SomeE[OF i'_le_Suc_j]
unfolding i'_def
by fastforce
have l_le_j:  $l \leq j$  and l_le_Suc_j:  $l \leq \text{Suc } j$ 
using l_def(1) i'_le_Suc_j
by (auto simp: i'_def)
have tau_l:  $l < j \implies \text{fst } (\text{rho } ! l) = \tau \sigma l$ 
using run_t_sound'[OF reaches_on_tj, of l] length_rho'
by (auto simp: rho'_def nth_append)
have tau_l_left:  $\text{memL } ts' t I$ 
unfolding l_def(3)[symmetric] tj_eq(2)
using i'_set l_def(1)
by (auto simp: i'_def)
have (l, Suc j)  $\in \text{match } r$ 
using accept_match[OF reaches_on_sj l_le_Suc_j_wf] q_def(3) length_rho' init_def l_def(2)
  rho'_def
by auto
then show sat (MatchP I r) j
using l_le_j q_def(2) ts_at_tau[OF reaches_on_tj] tau_l_left
by (auto simp: mem_def tj_eq rho'_def ts_def l_def(3)[symmetric] tau_l tj_def ts_at_def
  nth_append length_rho' intro: exI[of _ l] split: if_splits)
next
assume sat (MatchP I r) j
then obtain l where l_def:  $l \leq j \leq \text{Suc } j$  mem  $(\tau \sigma l) (\tau \sigma j) I$  (l, Suc j)  $\in \text{match } r$ 
by auto
show  $(\exists q \in \text{mmap\_keys } e'. \text{memR } (the (\text{mmap\_lookup } e' q)) t I \wedge \text{accept } q)$ 
proof -
have l_lt_j:  $l < \text{Suc } j$ 
using l_def(1) by auto
then have ts_at_l_j:  $ts\_at \text{ rho}' l \leq ts\_at \text{ rho}' j$ 
using ts_at_mono[OF reaches_on'_j_len_rho']
by (auto simp: rho'_def length_rho')
have ts_j_l:  $\text{memL } (ts\_at \text{ rho}' l) (ts\_at \text{ rho}' j) I$ 
using l_def(3) ts_at_tau[OF reaches_on_tj] l_lt_j length_rho' tj_eq
unfolding rho'_def mem_def
by auto
have i' = Suc j  $\vee \neg \text{memL } (ts\_at \text{ rho}' i') (ts\_at \text{ rho}' j) I$ 
proof (rule Meson.disj_comm, rule disjCI)

```

```

assume  $i' \neq \text{Suc } j$ 
then have  $i'_j: i' < \text{Suc } j$ 
  using valid_after
  by auto
obtain  $t' b'$  where  $tbi\_cur\_def: w\_read\_t \text{ args } ti' = \text{Some } t'$ 
   $t' = ts\_at \text{ rho}' i' b' = bs\_at \text{ rho}' i'$ 
  using reach_window_run_ti[OF valid_after(1) i'_j]
  reach_window_run_si[OF valid_after(1) i'_j] run_t_read
  by auto
show  $\neg memL (ts\_at \text{ rho}' i') (ts\_at \text{ rho}' j) I$ 
  using loop tbi_cur_def(1) i'_j length_rho'
  unfolding matchP_loop_inv_def matchP_loop_cond_def tj_eq(2) ti'_def[symmetric]
  by (auto simp: i'_def tbi_cur_def)
qed
then have  $l\_lt\_i': l < i'$ 
proof (rule disjE)
  assume assm:  $\neg memL (ts\_at \text{ rho}' i') (ts\_at \text{ rho}' j) I$ 
  show  $l < i'$ 
  proof (rule ccontr)
    assume  $\neg l < i'$ 
    then have  $ts\_at \text{ rho}' i' l: ts\_at \text{ rho}' i' \leq ts\_at \text{ rho}' l$ 
      using ts_at_mono[OF reaches_on']  $l\_lt\_j$  length_rho'
      by (auto simp: rho'_def length_rho')
    show False
      using assm memL_mono[OF ts_j_l ts_at i'_l]
      by auto
  qed
qed (auto simp add: l_lt_j)
define  $q$  where q_def:  $q = \text{steps step rho}' \text{ init } (l, \text{Suc } j)$ 
then obtain  $l'$  where l'_def:  $\text{sup\_leadsto init step rho}' i' (\text{Suc } j) q = \text{Some } (ts\_at \text{ rho}' l')$ 
 $l \leq l' l' < i'$ 
  using sup_leadsto_SomeI[OF l_lt_i'] by fastforce
have  $ts\_j \text{ rho}' l': memR (ts\_at \text{ rho}' l') (ts\_at \text{ rho}' j) I$ 
proof -
  have  $ts\_at \text{ rho}' l' l': ts\_at \text{ rho}' l' \leq ts\_at \text{ rho}' l'$ 
    using ts_at_mono[OF reaches_on' l'_def(2)]  $l'_def(3)$  valid_after(4,7,8)
    by (auto simp add: rho'_def length_rho' dual_order.order_iff_strict)
  show ?thesis
    using  $l\_def(3)$  memR_mono[OF ts_at_l_l']
    ts_at_tau[OF reaches_on_tj]  $l'_def(2,3)$  valid_after(4,7,8)
    by (auto simp: mem_def rho'_def length_rho')
  qed
have lookup_e'_q:  $mmap\_lookup e' q = \text{Some } (ts\_at \text{ rho}' l')$ 
  using lookup_e' l'_def(1) valid_after(4,7,8)
  by (auto simp: rho'_def sup_leadsto_app_cong)
show ?thesis
  using accept_match[OF reaches_on_sj l_def(2) wf]  $l\_def(4)$   $ts\_j \text{ rho}' l'$  lookup_e'_q tj_eq(2)
  by (auto simp: bs_at_def nth_append init_def length_rho'(1) q_def intro!: bexI[of _ q] Map-
ping_keys_intro)
  qed
qed
have read_tj_Some:  $\bigwedge t' l. w\_read\_t \text{ args } tj = \text{Some } t' \implies l < i' \implies memL (ts\_at \text{ rho}' l) t' I$ 
proof -
  fix  $t' l$ 
  assume lassms:  $(w\_read\_t \text{ args } tj = \text{Some } t' l < i')$ 
  obtain  $tj''''$  where reaches_on_tj'''':
reaches_on (w_run_t args) t0 (map fst (rho' @ [(t', undefined)]))  $tj''''$ 
  using reaches_on_app[OF reaches_on_tj] read_t_run[OF lassms(1)]

```

```

    by auto
  have  $t \leq t'$ 
    using  $ts\_at\_mono[OF\ reaches\_on\_tj''',\ of\ length\ rho\ length\ rho']$ 
    by (auto simp:  $ts\_at\_def\ nth\_append\ rho'\_def$ )
  then show  $memL\ (ts\_at\ rho'\ l)\ t'\ I$ 
    using  $memL\_mono'\ lassms(2)\ loop$ 
    unfolding  $matchP\_loop\_inv\_def$ 
    by (fastforce simp:  $i'\_def$ )
qed
define  $w'''$  where  $w''' = w''(w\_ac := ac')$ 
have  $rw\ t0\ sub\ rho'\ (w\_i\ w''',\ w\_ti\ w''',\ w\_si\ w''',\ w\_j\ w''',\ w\_tj\ w''',\ w\_sj\ w''')$ 
  using  $valid\_after(1)$ 
  by (auto simp del:  $reach\_window.simps$  simp:  $w'''\_def\ i'\_def\ ti'\_def\ si'\_def\ tj'\_def\ sj'\_def\ w\_tsj\_w'$ )
moreover have  $valid\_window\ args\ t0\ sub\ rho'\ w'''$ 
  using  $valid\_w'$ 
  by (auto simp:  $w'''\_def\ valid\_window\_def\ Let\_def\ \beta\_def(2)$ )
ultimately have  $valid\_matchP\ I\ t0\ sub\ rho'\ (Suc\ j)\ w'''$ 
  using  $i'\_set\ read\_tj\_Some$ 
  by (auto simp:  $valid\_window\_matchP\_def\ w'''\_def\ w\_tsj\_w'\ i'\_def\ split: option.splits$ )
moreover have  $eval\_mP\ I\ w = Some\ ((t,\ sat\ (MatchP\ I\ r)\ j),\ w''')$ 
  by (simp add:  $read\_t\_def\ Let\_def\ loop\_def[symmetric]\ ac'\_def[unfolded\ e'\_def\ ac\_def]\ w'''\_def\ w'\_def\ trans[OF\ \beta\_def(1)\ b\_alt]$ )
ultimately show  $?thesis$ 
  by (auto simp:  $tj\_eq\ rho'\_def$ )
qed

```

lemma  $valid\_eval\_matchF\_Some$ :

```

  assumes  $valid\_before'$ :  $valid\_matchF\ I\ t0\ sub\ rho\ i\ w$ 
  and  $eval$ :  $eval\_mF\ I\ w = Some\ ((t,\ b),\ w'')$ 
  and  $bounded$ :  $right\ I \in\ tfin$ 
  shows  $\exists\ rho'\ tm. reaches\_on\ (w\_run\_t\ args)\ (w\_tj\ w)\ (map\ fst\ rho')\ (w\_tj\ w'') \wedge$ 
 $reaches\_on\ (w\_run\_sub\ args)\ (w\_sj\ w)\ (map\ snd\ rho')\ (w\_sj\ w'') \wedge$ 
 $(w\_read\_t\ args)\ (w\_ti\ w) = Some\ t \wedge$ 
 $(w\_read\_t\ args)\ (w\_tj\ w'') = Some\ tm \wedge$ 
 $\neg memR\ t\ tm\ I$ 

```

proof –

```

  define  $st$  where  $st = w\_st\ w$ 
  define  $ti$  where  $ti = w\_ti\ w$ 
  define  $si$  where  $si = w\_si\ w$ 
  define  $j$  where  $j = w\_j\ w$ 
  define  $tj$  where  $tj = w\_tj\ w$ 
  define  $sj$  where  $sj = w\_sj\ w$ 
  define  $s$  where  $s = w\_s\ w$ 
  define  $e$  where  $e = w\_e\ w$ 
  have  $valid\_before$ :  $rw\ t0\ sub\ rho\ (i,\ ti,\ si,\ j,\ tj,\ sj)$ 
 $\wedge i\ j. i \leq j \implies j < length\ rho \implies ts\_at\ rho\ i \leq ts\_at\ rho\ j$ 
 $\forall q. mmap\_lookup\ e\ q = sup\_leadsto\ init\ step\ rho\ i\ j\ q$ 
 $valid\_s\ init\ step\ st\ accept\ rho\ i\ i\ j\ s$ 
 $i = w\_i\ w\ i \leq j\ length\ rho = j$ 
  using  $valid\_before'[unfolded\ valid\_window\_matchF\_def]\ ti\_def$ 
 $si\_def\ j\_def\ tj\_def\ sj\_def\ s\_def\ e\_def$ 
  by (auto simp:  $valid\_window\_def\ Let\_def\ init\_def\ step\_def\ st\_def\ accept\_def$ )
  obtain  $ti'''$  where  $tbi\_def$ :  $w\_run\_t\ args\ ti = Some\ (ti''',\ t)$ 
  using  $eval\ read\_t\_run$ 
  by (fastforce simp:  $Let\_def\ ti\_def\ si\_def\ split: option.splits\ if\_splits$ )
  have  $t\_tau$ :  $t = \tau\ \sigma\ i$ 
  using  $run\_t\_sound[OF\ _\ tbi\_def]\ valid\_before(1)$ 
  by auto

```

```

note  $t\_def = run\_t\_read[OF\ tbi\_def(1)]$ 
obtain  $w'$  where  $loop\_def: while\_break (matchF\_cond\ I\ t) (adv\_end\ args) w = Some\ w'$ 
using  $eval$ 
by ( $auto\ simp: ti\_def[symmetric]\ t\_def\ split: option.splits$ )
have  $adv\_start\_last:$ 
 $adv\_start\ args\ w' = w''$ 
using  $eval\ loop\_def[symmetric]\ run\_t\_read[OF\ tbi\_def(1)]$ 
by ( $auto\ simp: ti\_def\ split: option.splits\ prod.splits\ if\_splits$ )
have  $inv\_before: matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w$ 
using  $valid\_before(1)\ valid\_before'$ 
unfolding  $matchF\_loop\_inv'\_def\ valid\_before(6)\ valid\_window\_matchF\_def$ 
by ( $auto\ simp\ add: ti\_def\ si\_def\ j\_def\ tj\_def\ sj\_def\ simp\ del: reach\_window.simps$ 
 $dest: reach\_window\_shift\_all\ intro!: exI[of\_ \_\_]$ )
have  $i\_j: i \leq j\ length\ rho = j$ 
using  $valid\_before\ by\ auto$ 
define  $j''$  where  $j'' = w\_j\ w''$ 
define  $tj''$  where  $tj'' = w\_tj\ w''$ 
define  $sj''$  where  $sj'' = w\_sj\ w''$ 
have  $loop: matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w' \wedge \neg matchF\_cond\ I\ t\ w'$ 
proof ( $rule\ while\_break\_sound[of\ matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ matchF\_cond\ I\ t\ adv\_end\ args$ 
 $\lambda w'. matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w' \wedge \neg matchF\_cond\ I\ t\ w'\ w, unfolded\ loop\_def, simplified]$ )
 $fix\ w\_cur\ w\_cur' :: (bool\ iarray, nat\ set, 'd, 't, 'e)\ window$ 
assume  $assms: matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w\_cur\ matchF\_cond\ I\ t\ w\_cur\ adv\_end\ args$ 
 $w\_cur = Some\ w\_cur'$ 
define  $j\_cur$  where  $j\_cur = w\_j\ w\_cur$ 
define  $tj\_cur$  where  $tj\_cur = w\_tj\ w\_cur$ 
define  $sj\_cur$  where  $sj\_cur = w\_sj\ w\_cur$ 
obtain  $rho'$  where  $rho'\_def: valid\_window\ args\ t0\ sub\ (rho\ @\ rho')\ w\_cur$ 
 $rw\ t0\ sub\ (rho\ @\ rho')\ (j, tj, sj, w\_j\ w\_cur, w\_tj\ w\_cur, w\_sj\ w\_cur)$ 
using  $assms(1)[unfolded\ matchF\_loop\_inv'\_def\ valid\_window\_matchF\_def]$ 
by  $auto$ 
obtain  $tj'\ x\ sj'\ y$  where  $append: w\_run\_t\ args\ tj\_cur = Some\ (tj', x)$ 
 $w\_run\_sub\ args\ sj\_cur = Some\ (sj', y)$ 
using  $assms(3)$ 
unfolding  $tj\_cur\_def\ sj\_cur\_def$ 
by ( $auto\ simp: adv\_end\_def\ Let\_def\ split: option.splits$ )
note  $append' = append[unfolded\ tj\_cur\_def\ sj\_cur\_def]$ 
define  $rho''$  where  $rho'' = rho\ @\ rho'$ 
have  $reach: reaches\_on\ (w\_run\_t\ args)\ t0\ (map\ fst\ (rho''\ @\ [(x, undefined)]))\ tj'$ 
using  $reaches\_on\_app[OF\ reach\_window\_run\_tj[OF\ rho'\_def(2)]\ append'(1)]$ 
by ( $auto\ simp: rho''\_def$ )
have  $mono: \bigwedge t'. t' \in set\ (map\ fst\ rho'') \implies t' \leq x$ 
using  $ts\_at\_mono[OF\ reach, of\_ \_ length\ rho'']\ nat\_less\_le$ 
by ( $fastforce\ simp: ts\_at\_def\ nth\_append\ in\_set\ conv\_nth\ split: list.splits$ )
show  $matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w\_cur'$ 
using  $assms(1,3)\ reach\_window\_app[OF\ rho'\_def(2)]\ append[unfolded\ tj\_cur\_def\ sj\_cur\_def]$ 
 $valid\_adv\_end[OF\ rho'\_def(1)]\ append'\ mono\ adv\_end\_bounds[OF\ append']$ 
unfolding  $matchF\_loop\_inv'\_def\ matchF\_loop\_cond\_def\ rho''\_def$ 
by  $auto$ 
next
obtain  $l$  where  $l\_def: \neg\tau\ \sigma\ l \leq t + right\ I$ 
unfolding  $t\_tau$ 
using  $ex\_lt\_tau[OF\ bounded]$ 
by  $auto$ 
{
fix  $w1\ w2$ 
assume  $lassms: matchF\_inv'\ t0\ sub\ rho\ i\ ti\ si\ j\ tj\ sj\ w1\ matchF\_cond\ I\ t\ w1$ 
 $Some\ w2 = adv\_end\ args\ w1$ 

```

```

define  $j\_cur$  where  $j\_cur = w\_j w1$ 
define  $tj\_cur$  where  $tj\_cur = w\_tj w1$ 
define  $sj\_cur$  where  $sj\_cur = w\_sj w1$ 
obtain  $\rho'$  where  $\rho'\_def: valid\_window\ args\ t0\ sub\ (\rho\ @\ \rho')$   $w1$ 
 $rw\ t0\ sub\ (\rho\ @\ \rho')\ (j,\ tj,\ sj,\ w\_j\ w1,\ w\_tj\ w1,\ w\_sj\ w1)$ 
using  $lassms(1)[unfolding\ matchF\_loop\_inv'\_def\ valid\_window\_matchF\_def]$ 
by auto
obtain  $tj'$   $x\ sj'$   $y$  where  $append: w\_run\_t\ args\ tj\_cur = Some\ (tj',\ x)$ 
 $w\_run\_sub\ args\ sj\_cur = Some\ (sj',\ y)$ 
using  $lassms(3)$ 
unfolding  $tj\_cur\_def\ sj\_cur\_def$ 
by (auto simp: adv\_end\_def Let\_def split: option.splits)
note  $append' = append[unfolding\ tj\_cur\_def\ sj\_cur\_def]$ 
define  $\rho''$  where  $\rho'' = \rho\ @\ \rho'$ 
have  $reach: reaches\_on\ (w\_run\_t\ args)\ t0\ (map\ fst\ (\rho''\ @\ [(x,\ undefined)]))\ tj'$ 
using  $reaches\_on\_app[OF\ reach\_window\_run\_tj[OF\ \rho'\_def(2)]\ append'(1)]$ 
by (auto simp: \rho''\_def)
have  $mono: \bigwedge t'. t' \in set\ (map\ fst\ \rho'') \implies t' \leq x$ 
using  $ts\_at\_mono[OF\ reach,\ of\_length\ \rho'']\ nat\_less\_le$ 
by (fastforce simp: ts\_at\_def nth\_append\ in\_set\_conv\_nth\ split: list.splits)
have  $t\_cur\_tau: x = \tau\ \sigma\ j\_cur$ 
using  $ts\_at\_tau[OF\ reach,\ of\ length\ \rho'']\ \rho'\_def(2)$ 
by (auto simp: ts\_at\_def\ j\_cur\_def\ \rho''\_def)
have  $j\_cur < l$ 
using  $lassms(2)[unfolding\ matchF\_loop\_cond\_def]\ l\_def\ memR\_mono'[OF\ \_ \tau\_mono[of\ l\ j\_cur$ 
 $\sigma]]$ 
unfolding  $run\_t\_read[OF\ append(1),\ unfolding\ t\_cur\_tau\ tj\_cur\_def]$ 
by (fastforce dest: memR\_dest)
moreover have  $w\_j\ w2 = Suc\ j\_cur$ 
using  $adv\_end\_bounds[OF\ append']$ 
unfolding  $lassms(3)[symmetric]\ j\_cur\_def$ 
by auto
ultimately have  $l - w\_j\ w2 < l - w\_j\ w1$ 
unfolding  $j\_cur\_def$ 
by auto
}
then have  $\{(ta,\ s). matchF\_inv'\ t0\ sub\ \rho\ i\ ti\ si\ j\ tj\ sj\ s \wedge matchF\_cond\ I\ t\ s \wedge$ 
 $Some\ ta = adv\_end\ args\ s\} \subseteq measure\ (\lambda w. l - w\_j\ w)$ 
by auto
then show  $wf\ \{(ta,\ s). matchF\_inv'\ t0\ sub\ \rho\ i\ ti\ si\ j\ tj\ sj\ s \wedge matchF\_cond\ I\ t\ s \wedge$ 
 $Some\ ta = adv\_end\ args\ s\}$ 
using  $wf\_measure\ wf\_subset$ 
by auto
qed (auto simp: inv\_before)
define  $i'$  where  $i' = w\_i\ w'$ 
define  $ti'$  where  $ti' = w\_ti\ w'$ 
define  $si'$  where  $si' = w\_si\ w'$ 
define  $j'$  where  $j' = w\_j\ w'$ 
define  $tj'$  where  $tj' = w\_tj\ w'$ 
define  $sj'$  where  $sj' = w\_sj\ w'$ 
obtain  $\rho'$  where  $\rho'\_def: valid\_window\ args\ t0\ sub\ (\rho\ @\ \rho')$   $w'$ 
 $rw\ t0\ sub\ (\rho\ @\ \rho')\ (j,\ tj,\ sj,\ j',\ tj',\ sj')$ 
 $i = i' j \leq j'$ 
using loop
unfolding  $matchF\_loop\_inv'\_def\ i'\_def\ j'\_def\ tj'\_def\ sj'\_def$ 
by auto
obtain  $tje\ tm$  where  $tm\_def: w\_read\_t\ args\ tj' = Some\ tm\ w\_run\_t\ args\ tj' = Some\ (tje,\ tm)$ 
using  $eval\ read\_t\_run\ loop\_def\ t\_def\ ti\_def$ 

```

```

    by (auto simp: t_def Let_def tj'_def split: option.splits if_splits)
  have drop_j_rho: drop j (map fst (rho @ rho')) = map fst rho'
    using i_j
    by auto
  have reaches_on (w_run_t args) ti (drop i (map fst rho)) tj
    using valid_before(1)
    by auto
  then have reaches_on (w_run_t args) ti
    (drop i (map fst rho) @ (drop j (map fst (rho @ rho')))) tj'
    using rho'_def reaches_on_trans
    by fastforce
  then have reaches_on (w_run_t args) ti (drop i (map fst (rho @ rho'))) tj'
    unfolding drop_j_rho
    by (auto simp: i_j)
  then have reach_tm: reaches_on (w_run_t args) ti (drop i (map fst (rho @ rho'))) @ [tm] tje
    using reaches_on_app tm_def(2)
    by fastforce
  have run_tsi': w_run_t args ti' ≠ None
    using tbi_def loop
    by (auto simp: matchF_loop_inv'_def ti'_def si'_def)
  have memR_t_tm: ¬ memR t tm I
    using loop tm_def
    by (auto simp: tj'_def matchF_loop_cond_def)
  have i_le_rho: i ≤ length rho
    using valid_before
    by auto
  define rho'' where rho'' = rho @ rho'
  have t_tfin: t ∈ tfin
    using τ_fin
    by (auto simp: t_tau)
  have i'_lt_j': i' < j'
    using rho'_def(1,2,3)[folded rho''_def] i_j reach_tm[folded rho''_def] memR_t_tm tbi_def memR_tfin_refl[OF
t_tfin]
    by (cases i' = j') (auto dest!: reaches_on_NilD elim!: reaches_on.cases[of _ _ [tm]])
  have adv_last_bounds: j'' = j' tj'' = tj' sj'' = sj'
    using valid_adv_start_bounds[OF rho'_def(1) i'_lt_j'[unfolded i'_def j'_def]]
    unfolding adv_start_last j'_def j''_def tj'_def tj''_def sj'_def sj''_def
    by auto
  show ?thesis
    using eval rho'_def run_tsi' i_j(2) adv_last_bounds tj''_def tj_def sj''_def sj_def
    loop_def t_def ti_def tj'_def tm_def memR_t_tm
    by (auto simp: drop_map run_t_read[OF tbi_def(1)] Let_def
split: option.splits prod.splits if_splits intro!: exI[of _ rho'])
qed

lemma valid_eval_matchF_complete:
  assumes valid_before': valid_matchF I t0 sub rho i w
    and before_end: reaches_on (w_run_t args) (w_tj w) (map fst rho') tj'''
    reaches_on (w_run_sub args) (w_sj w) (map snd rho') sj'''
    w_read_t args (w_ti w) = Some t w_read_t args tj''' = Some tm ¬memR t tm I
    and wf: wf_regex r
  shows ∃ w'. eval_mF I w = Some ((τ σ i, sat (MatchF I r) i), w') ∧
    valid_matchF I t0 sub (take (w_j w') (rho @ rho')) (Suc i) w'
proof -
  define st where st = w_st w
  define ti where ti = w_ti w
  define si where si = w_si w
  define j where j = w_j w

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define tj where tj = w_tj w
define sj where sj = w_sj w
define s where s = w_s w
define e where e = w_e w
have valid_before: rw t0 sub rho (i, ti, si, j, tj, sj)
   $\bigwedge i j. i \leq j \implies j < \text{length } \rho \implies \text{ts\_at } \rho \ i \leq \text{ts\_at } \rho \ j$ 
   $\forall q. \text{mmap\_lookup } e \ q = \text{sup\_leadsto init step } \rho \ i \ j \ q$ 
  valid_s init step st accept rho i i j s
  i = w_i w i \leq j \text{ length } \rho = j
  using valid_before[unfolded valid_window_matchF_def] ti_def
    si_def j_def tj_def sj_def s_def e_def
  by (auto simp: valid_window_def Let_def init_def step_def st_def accept_def)
define rho'' where rho'' = rho @ rho'
have ij_le: i \leq j j = \text{length } \rho
  using valid_before
  by auto
have reach_tj: reaches_on (w_run_t args) t0 (take j (map fst rho'')) tj
  using valid_before(1) ij_le
  by (auto simp: take_map rho''_def simp del: reach_window.simps dest!: reach_window_run_tj)
have reach_ti: reaches_on (w_run_t args) t0 (take i (map fst rho'')) ti
  using valid_before(1) ij_le
  by (auto simp: take_map rho''_def)
have reach_si: reaches_on (w_run_sub args) sub (take i (map snd rho'')) si
  using valid_before(1) ij_le
  by (auto simp: take_map rho''_def)
have reach_sj: reaches_on (w_run_sub args) sub (take j (map snd rho'')) sj
  using valid_before(1) ij_le
  by (auto simp: take_map rho''_def simp del: reach_window.simps dest!: reach_window_run_sj)
have reach_tj''': reaches_on (w_run_t args) t0 (map fst rho'') tj'''
  using reaches_on_trans[OF reach_tj before_end(1)[folded tj_def]] ij_le(2)
  by (auto simp del: map_append simp: rho''_def take_map drop_map map_append[symmetric])
have rho''_mono:  $\bigwedge i j. i \leq j \implies j < \text{length } \rho'' \implies \text{ts\_at } \rho'' \ i \leq \text{ts\_at } \rho'' \ j$ 
  using ts_at_mono[OF reach_tj'''] .
obtain tm' where reach_tm: reaches_on (w_run_t args) t0
  (map fst (rho'' @ [(tm, undefined)])) tm'
  using reaches_on_app[OF reach_tj'''] read_t_run[OF before_end(4)]
  by auto
have tj'''_eq:  $\bigwedge tj\_cur. \text{reaches\_on } (w\_run\_t \text{ args}) \ t0 \ (map \text{fst } \rho'') \ tj\_cur \implies$ 
  tj\_cur = tj'''
  using reaches_on_inj[OF reach_tj''']
  by blast
have reach_sj''': reaches_on (w_run_sub args) sub (map snd rho'') sj'''
  using reaches_on_trans[OF reach_sj before_end(2)[folded sj_def]] ij_le(2)
  by (auto simp del: map_append simp: rho''_def take_map drop_map map_append[symmetric])
have sj'''_eq:  $\bigwedge sj\_cur. \text{reaches\_on } (w\_run\_sub \text{ args}) \ sub \ (map \text{snd } \rho'') \ sj\_cur \implies$ 
  sj\_cur = sj'''
  using reaches_on_inj[OF reach_sj''']
  by blast
have reach_window_i: rw t0 sub rho'' (i, ti, si, length rho'', tj''', sj''')
  using reach_windowI[OF reach_ti reach_si reach_tj''' reach_sj''' _ refl] ij_le
  by (auto simp: rho''_def)
have reach_window_j: rw t0 sub rho'' (j, tj, sj, length rho'', tj''', sj''')
  using reach_windowI[OF reach_tj reach_sj reach_tj''' reach_sj''' _ refl] ij_le
  by (auto simp: rho''_def)
have t_def: t = \tau \sigma i
  using valid_before(6) read_t_run[OF before_end(3)] reaches_on_app[OF reach_ti]
  ts_at_tau[where ?rho=take i rho'' @ [(t, undefined)]]
  by (fastforce simp: ti_def rho''_def valid_before(5,7) take_map ts_at_def nth_append)

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have t_tfin: t ∈ tfin
  using τ_tfin
  by (auto simp: t_def)
have i_lt_rho'': i < length rho''
  using ij_le before_end(3,4,5) reach_window_i memR_tfin_refl[OF t_tfin]
  by (cases i = length rho'') (auto simp: rho''_def ti_def dest!: reaches_on_NilD)
obtain ti''' si''' b where tbi_def: w_run_t args ti = Some (ti''', t)
  w_run_sub args si = Some (si''', b) t = ts_at rho'' i b = bs_at rho'' i
  using reach_window_run_ti[OF reach_window_i i_lt_rho'']
  reach_window_run_si[OF reach_window_i i_lt_rho'']
  read_t_run[OF before_end(3), folded ti_def]
  by auto
note before_end' = before_end(5)
have read_ti: w_read_t args ti = Some t
  using run_t_read[OF tbi_def(1)] .
have inv_before: matchF_inv I t0 sub rho'' i ti si tj''' sj''' w
  using valid_before' before_end(1,2,3) reach_window_j ij_le ti_def si_def j_def tj_def sj_def
  unfolding matchF_loop_inv_def valid_window_matchF_def
  by (auto simp: rho''_def ts_at_def nth_append)
have i_j: i ≤ j
  using valid_before by auto
have loop: pred_option' (λw'. matchF_inv I t0 sub rho'' i ti si tj''' sj''' w' ∧ ¬ matchF_cond I t w')
(while_break (matchF_cond I t) (adv_end args) w)
proof (rule while_break_complete[of matchF_inv I t0 sub rho'' i ti si tj''' sj''', OF ___ inv_before])
  fix w_cur :: (bool iarray, nat set, 'd, 't, 'e) window
  assume assms: matchF_inv I t0 sub rho'' i ti si tj''' sj''' w_cur matchF_cond I t w_cur
  define j_cur where j_cur = w_j w_cur
  define tj_cur where tj_cur = w_tj w_cur
  define sj_cur where sj_cur = w_sj w_cur
  define s_cur where s_cur = w_s w_cur
  define e_cur where e_cur = w_e w_cur
  have loop: valid_window_args t0 sub (take (w_j w_cur) rho'') w_cur
    rw t0 sub rho'' (j_cur, tj_cur, sj_cur, length rho'', tj''', sj''')
    ∧ l. l ∈ {w_i w_cur..<w_j w_cur} ⇒ memR (ts_at rho'' i) (ts_at rho'' l) I
  using j_cur_def tj_cur_def sj_cur_def s_cur_def e_cur_def
    assms(1)[unfolded matchF_loop_inv_def]
  by auto
  have j_cur_lt_rho'': j_cur < length rho''
    using assms tj'''_eq before_end(4,5)
    unfolding matchF_loop_inv_def matchF_loop_cond_def
    by (cases j_cur = length rho'') (auto simp: j_cur_def split: option.splits)
  obtain tj_cur' sj_cur' x b_cur where tbi_cur_def: w_run_t args tj_cur = Some (tj_cur', x)
    w_run_sub args sj_cur = Some (sj_cur', b_cur)
    x = ts_at rho'' j_cur b_cur = bs_at rho'' j_cur
    using reach_window_run_ti[OF loop(2) j_cur_lt_rho'']
    reach_window_run_si[OF loop(2) j_cur_lt_rho'']
    by auto
  note reach_window_j'_cur = reach_window_shift[OF loop(2) j_cur_lt_rho'' tbi_cur_def(1,2)]
  note tbi_cur_def' = tbi_cur_def(1,2)[unfolded tj_cur_def sj_cur_def]
  have mono: ∧ t'. t' ∈ set (map fst (take (w_j w_cur) rho'')) ⇒ t' ≤ x
    using rho''_mono[of _ j_cur] j_cur_lt_rho'' nat_less_le
    by (fastforce simp: tbi_cur_def(3) j_cur_def ts_at_def nth_append in_set_conv_nth
      split: list.splits)
  have take_unfold: take (w_j w_cur) rho'' @ [(x, b_cur)] = (take (Suc (w_j w_cur)) rho'')
    using j_cur_lt_rho''
    unfolding tbi_cur_def(3,4)
    by (auto simp: ts_at_def bs_at_def j_cur_def take_Suc_conv_app_nth)
  obtain w_cur' where w_cur'_def: adv_end args w_cur = Some w_cur'

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by (fastforce simp: adv_end_def Let_def tj_cur_def[symmetric] sj_cur_def[symmetric] tbi_cur_def(1,2)
split: prod.splits)
have  $\bigwedge l. l \in \{w_i w_{cur'} .. < w_j w_{cur'}\} \implies$ 
  memR (ts_at rho'' i) (ts_at rho'' l) I
  using loop(3) assms(2) w_cur'_def
  unfolding adv_end_bounds[OF tbi_cur_def' w_cur'_def] matchF_loop_cond_def
  run_t_read[OF tbi_cur_def(1)[unfolded tj_cur_def]] tbi_cur_def(3) tbi_def(3)
  by (auto simp: j_cur_def elim: less_SucE)
then show pred_option' (matchF_inv I t0 sub rho'' i ti si tj''' sj''') (adv_end args w_cur)
  using assms(1) reach_window_j'_cur valid_adv_end[OF loop(1) tbi_cur_def' mono]
  w_cur'_def adv_end_bounds[OF tbi_def' w_cur'_def]
  unfolding matchF_loop_inv_def j_cur_def take_unfold
  by (auto simp: pred_option'_def)
next
{
  fix w1 w2
  assume lassms: matchF_inv I t0 sub rho'' i ti si tj''' sj''' w1 matchF_cond I t w1
  Some w2 = adv_end args w1
  define j_cur where j_cur = w_j w1
  define tj_cur where tj_cur = w_tj w1
  define sj_cur where sj_cur = w_sj w1
  define s_cur where s_cur = w_s w1
  define e_cur where e_cur = w_e w1
  have loop: valid_window args t0 sub (take (w_j w1) rho'') w1
    rw t0 sub rho'' (j_cur, tj_cur, sj_cur, length rho'', tj''', sj''')
     $\bigwedge l. l \in \{w_i w1 .. < w_j w1\} \implies$  memR (ts_at rho'' i) (ts_at rho'' l) I
    using j_cur_def tj_cur_def sj_cur_def s_cur_def e_cur_def
    lassms(1)[unfolded matchF_loop_inv_def]
    by auto
  have j_cur_lt_rho'': j_cur < length rho''
    using lassms tj'''_eq ij_le before_end(4,5)
  unfolding matchF_loop_inv_def matchF_loop_cond_def
  by (cases j_cur = length rho'') (auto simp: j_cur_def split: option.splits)
  obtain tj_cur' sj_cur' x b_cur where tbi_cur_def: w_run_t args tj_cur = Some (tj_cur', x)
  w_run_sub args sj_cur = Some (sj_cur', b_cur)
  x = ts_at rho'' j_cur b_cur = bs_at rho'' j_cur
  using reach_window_run_ti[OF loop(2) j_cur_lt_rho'']
  reach_window_run_si[OF loop(2) j_cur_lt_rho'']
  by auto
  note tbi_cur_def' = tbi_cur_def(1,2)[unfolded tj_cur_def sj_cur_def]
  have length rho'' - w_j w2 < length rho'' - w_j w1
    using j_cur_lt_rho'' adv_end_bounds[OF tbi_cur_def', folded lassms(3)]
  unfolding j_cur_def
  by auto
}
then have  $\{(ta, s). \text{matchF\_inv } I \ t0 \ \text{sub } \rho'' \ i \ ti \ si \ tj''' \ sj''' \ s \ \wedge \ \text{matchF\_cond } I \ t \ s \ \wedge$ 
  Some ta = adv_end args s $\} \subseteq$  measure  $(\lambda w. \text{length } \rho'' - w_j \ w)$ 
  by auto
then show wf  $\{(ta, s). \text{matchF\_inv } I \ t0 \ \text{sub } \rho'' \ i \ ti \ si \ tj''' \ sj''' \ s \ \wedge \ \text{matchF\_cond } I \ t \ s \ \wedge$ 
  Some ta = adv_end args s $\}$ 
  using wf_measure wf_subset
  by auto
qed (auto simp add: inv_before)
obtain w' where w'_def: while_break (matchF_cond I t) (adv_end args) w = Some w'
  using loop
  by (auto simp: pred_option'_def split: option.splits)
define w'' where adv_start_last: w'' = adv_start args w'
define st' where st' = w_st w'

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define i' where i' = w_i w'
define ti' where ti' = w_ti w'
define si' where si' = w_si w'
define j' where j' = w_j w'
define tj' where tj' = w_tj w'
define sj' where sj' = w_sj w'
define s' where s' = w_s w'
define e' where e' = w_e w'
have valid_after: valid_window args t0 sub (take (w_j w') rho'') w'
  rw t0 sub rho'' (j', tj', sj', length rho'', tj''', sj''')
   $\wedge l. l \in \{i..<j'\} \implies \text{memR} (ts\_at\ rho''\ i) (ts\_at\ rho''\ l) I$ 
  i' = i ti' = ti si' = si
  using loop
  unfolding matchF_loop_inv_def w'_def i'_def ti'_def si'_def j'_def tj'_def sj'_def
  by (auto simp: pred_option'_def)
define i'' where i'' = w_i w''
define j'' where j'' = w_j w''
define tj'' where tj'' = w_tj w''
define sj'' where sj'' = w_sj w''
have j'_le_rho'': j' ≤ length rho''
  using loop
  unfolding matchF_loop_inv_def valid_window_matchF_def w'_def j'_def
  by (auto simp: pred_option'_def)
obtain te where tbj'_def: w_read_t args tj' = Some te
  te = ts_at (rho'' @ [(tm, undefined)]) j'
proof (cases j' < length rho'')
  case True
  show ?thesis
  using reach_window_run_ti[OF valid_after(2) True] that True
  by (auto simp: ts_at_def nth_append dest!: run_t_read)
next
  case False
  then have tj' = tj''' j' = length rho''
    using valid_after(2) j'_le_rho'' tj'''_eq
    by auto
  then show ?thesis
    using that before_end(4)
    by (auto simp: ts_at_def nth_append)
qed
have not_ets_te: ¬memR (ts_at rho'' i) te I
  using loop
  unfolding w'_def
  by (auto simp: pred_option'_def matchF_loop_cond_def tj'_def[symmetric] tbj'_def(1) tbi_def(3)
  split: option.splits)
have i'_set:  $\wedge l. l \in \{i..<j'\} \implies \text{memR} (ts\_at\ rho''\ i) (ts\_at\ rho''\ l) I$ 
  ¬memR (ts_at rho'' i) (ts_at (rho'' @ [(tm, undefined)]) j') I
  using loop tbj'_def not_ets_te valid_after atLeastLessThan_iff
  unfolding matchF_loop_inv_def matchF_loop_cond_def tbi_def(3)
  by (auto simp: tbi_def tj'_def split: option.splits)
have i_le_j': i ≤ j'
  using valid_after(1)
  unfolding valid_after(4)[symmetric]
  by (auto simp: valid_window_def Let_def i'_def j'_def)
have i_lt_j': i < j'
  using i_le_j' i'_set(2) i_lt_rho''
  using memR_tfn_refl[OF  $\tau\_fin$ ] ts_at_tau[OF reach_tj''', of j']
  by (cases i = j') (auto simp: ts_at_def nth_append)
then have i'_lt_j': i' < j'

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unfolding valid_after
by auto
have adv_last_bounds:  $i'' = \text{Suc } i' \ w\_ti \ w'' = ti''' \ w\_si \ w'' = si''' \ j'' = j'$ 
 $tj'' = tj' \ sj'' = sj'$ 
using valid_adv_start_bounds[OF valid_after(1)  $i'_lt\_j'$ [unfolded  $i'_def \ j'_def$ ]]
  valid_adv_start_bounds'[OF valid_after(1)  $tbi\_def(1,2)$ [folded valid_after(5,6),
  unfolded  $ti'_def \ si'_def$ ]]
unfolding adv_start_last  $i'_def \ i''_def \ j'_def \ j''_def \ tj'_def \ tj''_def \ sj'_def \ sj''_def$ 
by auto
have  $i''_i$ :  $i'' = i + 1$ 
using valid_after_adv_last_bounds by auto
have  $i\_le\_j'$ :  $i \leq j'$ 
using valid_after  $i'_lt\_j'$ 
by auto
then have  $i\_le\_rho$ :  $i \leq \text{length } rho''$ 
using valid_after(2)
by auto
have valid_s_init_step  $st'$  accept (take  $j' \ rho''$ )  $i \ i \ j' \ s'$ 
using valid_after(1,4)  $i'_def$ 
by (auto simp: valid_window_def Let_def init_def step_def  $st'_def$   $accept\_def \ j'_def \ s'_def$ )
note valid_s' = this[unfolded valid_s_def]
have  $q0\_in\_keys$ :  $\{0\} \in \text{mmap\_keys } s'$ 
using valid_s' init_def steps_refl by auto
then obtain  $q' \ tstp$  where  $\text{lookup\_s}'$ :  $\text{mmap\_lookup } s' \ \{0\} = \text{Some } (q', \ tstp)$ 
by (auto dest: Mapping_keys_dest)
have  $\text{lookup\_sup\_acc}$ :  $\text{snd } (\text{the } (\text{mmap\_lookup } s' \ \{0\})) =$ 
 $\text{sup\_acc } \text{step } \text{accept } (\text{take } j' \ rho'') \ \{0\} \ i \ j'$ 
using conjunct2[OF valid_s']  $\text{lookup\_s}'$ 
by auto (smt case_prodD option.simps(5))
have  $b\_alt$ : (case  $\text{snd } (\text{the } (\text{mmap\_lookup } s' \ \{0\}))$  of None  $\Rightarrow$  False
|  $\text{Some } \text{tstp} \Rightarrow \text{memL } t \ (\text{fst } \text{tstp}) \ I \longleftrightarrow \text{sat } (\text{MatchF } I \ r) \ i$ 
proof (rule iffI)
assume  $\text{asm}$ : case  $\text{snd } (\text{the } (\text{mmap\_lookup } s' \ \{0\}))$  of None  $\Rightarrow$  False
|  $\text{Some } \text{tstp} \Rightarrow \text{memL } t \ (\text{fst } \text{tstp}) \ I$ 
then obtain  $ts \ tp$  where  $\text{tstp\_def}$ :
 $\text{sup\_acc } \text{step } \text{accept } (\text{take } j' \ rho'') \ \{0\} \ i \ j' = \text{Some } (ts, tp)$ 
 $\text{memL } (ts\_at \ rho'' \ i) \ ts \ I$ 
unfolding  $\text{lookup\_sup\_acc}$ 
by (auto simp:  $tbi\_def$  split: option.splits)
then have  $\text{sup\_acc\_rho''}$ :  $\text{sup\_acc } \text{step } \text{accept } \rho'' \ \{0\} \ i \ j' = \text{Some } (ts, tp)$ 
using  $\text{sup\_acc\_concat\_cong}$ [of  $j' \ \text{take } j' \ rho'' \ \text{step } \text{accept } \text{drop } j' \ rho''$ ]  $j'_le\_rho''$ 
by auto
have  $tp\_props$ :  $tp \in \{i..<j'\} \ \text{acc } \text{step } \text{accept } \rho'' \ \{0\} \ (i, \ \text{Suc } tp)$ 
using  $\text{sup\_acc\_SomeE}$ [OF  $\text{sup\_acc\_rho''}$ ] by auto
have  $ts\_ts\_at$ :  $ts = ts\_at \ rho'' \ tp$ 
using  $\text{sup\_acc\_Some\_ts}$ [OF  $\text{sup\_acc\_rho''}$ ] .
have  $i\_le\_tp$ :  $i \leq \text{Suc } tp$ 
using  $tp\_props$  by auto
have  $\text{memR } (ts\_at \ rho'' \ i) \ (ts\_at \ rho'' \ tp) \ I$ 
using  $i'_set(1)$ [OF  $tp\_props(1)$ ] .
then have  $\text{mem } (ts\_at \ rho'' \ i) \ (ts\_at \ rho'' \ tp) \ I$ 
using  $\text{tstp\_def}(2)$  unfolding  $ts\_ts\_at$   $\text{mem\_def}$  by auto
then show  $\text{sat } (\text{MatchF } I \ r) \ i$ 
using  $i\_le\_tp \ \text{acc\_match}$ [OF  $\text{reach\_sj''' } \ i\_le\_tp \ \_ \ \text{wf}$ ]  $tp\_props(2) \ ts\_at\_tau$ [OF  $\text{reach\_tj'''}$ ]
 $tp\_props(1) \ j'_le\_rho''$ 
by auto
next
assume  $\text{sat } (\text{MatchF } I \ r) \ i$ 

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then obtain  $l$  where  $l\_def: l \geq i \text{ mem } (\tau \sigma i) (\tau \sigma l) I (i, \text{Suc } l) \in \text{match } r$ 
  by auto
have  $l\_lt\_rho: l < \text{length } rho''$ 
proof (rule ccontr)
  assume  $contr: \neg l < \text{length } rho''$ 
  have  $tm = ts\_at (rho'' @ [(tm, \text{undefined}]]) (\text{length } rho'')$ 
    using  $i\_le\_rho$ 
    by (auto simp add: ts\_at\_def rho''\_def)
  moreover have  $\dots \leq \tau \sigma l$ 
    using  $\tau\_mono \ ts\_at\_tau[OF \text{reach\_tm}] \ i\_le\_rho \ contr$ 
    by (metis One\_nat\_def Suc\_eq\_plus1 length\_append lessI list.size(3)
       $list.size(4) \ not\_le\_imp\_less$ )
  moreover have  $memR (\tau \sigma i) (\tau \sigma l) I$ 
    using  $l\_def(2)$ 
    unfolding  $mem\_def$ 
    by auto
  ultimately have  $memR (\tau \sigma i) tm I$ 
    using  $memR\_mono'$ 
    by auto
  then show False
    using  $\text{before\_end}' \ ts\_at\_tau[OF \text{reach\_tj}'''] \ i\_lt\_rho'$   $tbi\_def(3)$ 
    by (auto simp: rho''\_def)
qed
have  $l\_lt\_j': l < j'$ 
proof (rule ccontr)
  assume  $contr: \neg l < j'$ 
  then have  $ts\_at\_j'\_l: ts\_at \ rho'' \ j' \leq ts\_at \ rho'' \ l$ 
    using  $ts\_at\_mono[OF \text{reach\_tj}'''] \ l\_lt\_rho$ 
    by (auto simp add: order.not\_eq\_order\_implies\_strict)
  have  $ts\_at\_l\_iu: memR (ts\_at \ rho'' \ i) (ts\_at \ rho'' \ l) I$ 
    using  $l\_def(2) \ ts\_at\_tau[OF \text{reach\_tj}'''] \ l\_lt\_rho \ ts\_at\_tau[OF \text{reach\_tj}'''] \ i\_lt\_rho'$ 
    unfolding  $mem\_def$ 
    by auto
  show False
    using  $i'\_set(2) \ ts\_at\_j'\_l \ ts\_at\_l\_iu \ contr \ l\_lt\_rho \ memR\_mono'$ 
    by (auto simp: ts\_at\_def nth\_append split: if\_splits)
qed
have  $i\_le\_Suc\_l: i \leq \text{Suc } l$ 
  using  $l\_def(1)$ 
  by auto
obtain  $tp$  where  $tstp\_def: sup\_acc \ step \ accept \ rho'' \ \{0\} \ i \ j' = \text{Some} (ts\_at \ rho'' \ tp, tp)$ 
   $l \leq tp \ tp < j'$ 
  using  $l\_def(1,3) \ l\_lt\_j' \ l\_lt\_rho$ 
  by (meson accept\_match[OF reach\_sj'''] i\_le\_Suc\_l\_wf, unfolded steps\_is\_run] sup\_acc\_SomeI[unfolded
   $acc\_is\_accept, \ of \ step \ accept] \ acc\_is\_accept \ atLeastLessThan\_iff \ less\_eq\_Suc\_le$ )
  have  $memL (ts\_at \ rho'' \ i) (ts\_at \ rho'' \ l) I$ 
    using  $l\_def(2)$ 
    unfolding  $ts\_at\_tau[OF \text{reach\_tj}'''] \ i\_lt\_rho'', \ symmetric$ 
       $ts\_at\_tau[OF \text{reach\_tj}'''] \ l\_lt\_rho, \ symmetric] \ mem\_def$ 
    by auto
  then have  $memL (ts\_at \ rho'' \ i) (ts\_at \ rho'' \ tp) I$ 
    using  $ts\_at\_mono[OF \text{reach\_tj}'''] \ tstp\_def(2) \ tstp\_def(3) \ j'\_le\_rho'' \ memL\_mono'$ 
    by auto
  then show  $\text{case } snd \ (the \ (mmap\_lookup \ s' \ \{0\})) \ \text{of } None \Rightarrow \text{False}$ 
     $| \ \text{Some } tstp \Rightarrow memL \ t \ (fst \ tstp) \ I$ 
    using  $lookup\_sup\_acc \ tstp\_def \ j'\_le\_rho''$ 
       $sup\_acc\_concat\_cong[of \ j' \ take \ j' \ rho'' \ step \ accept \ drop \ j' \ rho'']$ 
    by (auto simp: tbi\_def split: option.splits)

```

```

qed
have valid_matchF I t0 sub (take j'' rho'') i'' (adv_start args w')
proof -
  have  $\forall l \in \{i'..<j'\}$ . memR (ts_at rho'' i') (ts_at rho'' l) I
    using loop i'_def j'_def valid_after
    unfolding matchF_loop_inv_def
    by auto
  then have  $\forall l \in \{i''..<j''\}$ . memR (ts_at rho'' i'') (ts_at rho'' l) I
    unfolding i''_i valid_after adv_last_bounds
    apply safe
    subgoal for l
      apply (drule ballE[of _ _ l])
      using ts_at_mono[OF reach_tj''', of i Suc i] j'_le_rho'' memR_mono
      apply auto
    done
  done
  moreover have rw t0 sub (take j'' rho'') (i'', ti''', si''', j'', tj'', sj'')
  proof -
    have rw: rw t0 sub (take j' rho'') (i', ti', si', j', tj', sj')
      using valid_after(1)
      by (auto simp: valid_window_def Let_def i'_def ti'_def si'_def j'_def tj'_def sj'_def)
    show ?thesis
      using reach_window_shift[OF rw i'_lt_j'
        tbi_def(1,2)[unfolded valid_after(5,6)[symmetric]]] adv_last_bounds
      by auto
  qed
  moreover have valid_window args t0 sub (take j' rho'') w''
    using valid_adv_start[OF valid_after(1) i'_lt_j'[unfolded i'_def j'_def]]
    unfolding adv_start_last j'_def .
  ultimately show valid_matchF I t0 sub (take j'' rho'') i'' (adv_start args w')
    using j'_le_rho''
  unfolding valid_window_matchF_def adv_last_bounds adv_start_last[symmetric] i''_def[symmetric]
    j'_def j''_def[symmetric] tj'_def tj''_def[symmetric] sj'_def sj''_def[symmetric]
    by (auto simp: ts_at_def)
qed
moreover have eval_mF I w = Some (( $\tau$   $\sigma$  i, sat (MatchF I r) i), w'')
  unfolding j''_def adv_start_last[symmetric] adv_last_bounds valid_after rho''_def
    eval_matchF.simps run_t_read[OF tbi_def(1)[unfolded ti_def]]
  using tbj'_def[unfolded tj'_def] not_ets_te[folded tbi_def(3)]
    b_alt[unfolded s'_def] t_def adv_start_last w'_def
  by (auto simp only: Let_def split: option.splits if_splits)
ultimately show ?thesis
  unfolding j''_def adv_start_last[symmetric] adv_last_bounds valid_after rho''_def
  by auto
qed

lemma valid_eval_matchF_sound:
  assumes valid_before: valid_matchF I t0 sub rho i w
  and eval: eval_mF I w = Some ((t, b), w'')
  and bounded: right I  $\in$  tfin
  and wf: wf_regex r
  shows t =  $\tau$   $\sigma$  i  $\wedge$  b = sat (MatchF I r) i  $\wedge$  ( $\exists$  rho'. valid_matchF I t0 sub rho' (Suc i) w'')
  proof -
    obtain rho' t tm where rho'_def: reaches_on (w_run_t args) (w_tj w) (map fst rho') (w_tj w'')
      reaches_on (w_run_sub args) (w_sj w) (map snd rho') (w_sj w'')
      w_read_t args (w_ti w) = Some t
      w_read_t args (w_tj w'') = Some tm
    -memR t tm I

```

```

using valid_eval_matchF_Some[OF assms(1-3)]
by auto
show ?thesis
using valid_eval_matchF_complete[OF assms(1) rho'_def wf]
unfolding eval
by blast
qed

thm valid_eval_matchP
thm valid_eval_matchF_sound
thm valid_eval_matchF_complete

end

end

theory Monitor
imports MDL Temporal
begin

type_synonym ('h, 't) time = ('h × 't) option

datatype (dead 'a, dead 't :: timestamp, dead 'h) vydra_aux =
  VYDRA_None
| VYDRA_Bool bool 'h
| VYDRA_Atom 'a 'h
| VYDRA_Neg ('a, 't, 'h) vydra_aux
| VYDRA_Bin bool ⇒ bool ⇒ bool ('a, 't, 'h) vydra_aux ('a, 't, 'h) vydra_aux
| VYDRA_Prev 't  $\mathcal{I}$  ('a, 't, 'h) vydra_aux 'h ('t × bool) option
| VYDRA_Next 't  $\mathcal{I}$  ('a, 't, 'h) vydra_aux 'h 't option
| VYDRA_Since 't  $\mathcal{I}$  ('a, 't, 'h) vydra_aux ('a, 't, 'h) vydra_aux ('h, 't) time nat nat nat option 't
option
| VYDRA_Until 't  $\mathcal{I}$  ('h, 't) time ('a, 't, 'h) vydra_aux ('a, 't, 'h) vydra_aux ('h, 't) time nat ('t ×
bool × bool) option
| VYDRA_MatchP 't  $\mathcal{I}$  transition iarray nat
  (bool iarray, nat set, 't, ('h, 't) time, ('a, 't, 'h) vydra_aux list) window
| VYDRA_MatchF 't  $\mathcal{I}$  transition iarray nat
  (bool iarray, nat set, 't, ('h, 't) time, ('a, 't, 'h) vydra_aux list) window

type_synonym ('a, 't, 'h) vydra = nat × ('a, 't, 'h) vydra_aux

fun msize_vydra :: nat ⇒ ('a, 't :: timestamp, 'h) vydra_aux ⇒ nat where
  msize_vydra n VYDRA_None = 0
| msize_vydra n (VYDRA_Bool b e) = 0
| msize_vydra n (VYDRA_Atom a e) = 0
| msize_vydra (Suc n) (VYDRA_Bin f v1 v2) = msize_vydra n v1 + msize_vydra n v2 + 1
| msize_vydra (Suc n) (VYDRA_Neg v) = msize_vydra n v + 1
| msize_vydra (Suc n) (VYDRA_Prev I v e tb) = msize_vydra n v + 1
| msize_vydra (Suc n) (VYDRA_Next I v e to) = msize_vydra n v + 1
| msize_vydra (Suc n) (VYDRA_Since I vphi vpsi e cphi cpsi cpsi tpsi) = msize_vydra n vphi +
msize_vydra n vpsi + 1
| msize_vydra (Suc n) (VYDRA_Until I e vphi vpsi epsi c zo) = msize_vydra n vphi + msize_vydra n
vpsi + 1
| msize_vydra (Suc n) (VYDRA_MatchP I transs qf w) = size_list (msize_vydra n) (w_si w) + size_list
(msize_vydra n) (w_sj w) + 1
| msize_vydra (Suc n) (VYDRA_MatchF I transs qf w) = size_list (msize_vydra n) (w_si w) + size_list
(msize_vydra n) (w_sj w) + 1
| msize_vydra _ _ = 0

```

```

fun next_vydra :: ('a, 't :: timestamp, 'h) vydra_aux  $\Rightarrow$  nat where
  next_vydra (VYDRA_Next I v e None) = 1
| next_vydra _ = 0

context
  fixes init_hd :: 'h
  and run_hd :: 'h  $\Rightarrow$  ('h  $\times$  ('t :: timestamp  $\times$  'a set)) option
begin

definition t0 :: ('h, 't) time where
  t0 = (case run_hd init_hd of None  $\Rightarrow$  None | Some (e', (t, X))  $\Rightarrow$  Some (e', t))

fun run_t :: ('h, 't) time  $\Rightarrow$  (('h, 't) time  $\times$  't) option where
  run_t None = None
| run_t (Some (e, t)) = (case run_hd e of None  $\Rightarrow$  Some (None, t)
| Some (e', (t', X))  $\Rightarrow$  Some (Some (e', t'), t))

fun read_t :: ('h, 't) time  $\Rightarrow$  't option where
  read_t None = None
| read_t (Some (e, t)) = Some t

lemma run_t_read: run_t x = Some (x', t)  $\Longrightarrow$  read_t x = Some t
  by (cases x) (auto split: option.splits)

lemma read_t_run: read_t x = Some t  $\Longrightarrow$   $\exists$  x'. run_t x = Some (x', t)
  by (cases x) (auto split: option.splits)

lemma reach_event_t: reaches_on run_hd e vs e''  $\Longrightarrow$  run_hd e = Some (e', (t, X))  $\Longrightarrow$ 
  run_hd e'' = Some (e''', (t', X'))  $\Longrightarrow$ 
  reaches_on run_t (Some (e', t)) (map fst vs) (Some (e''', t'))
proof (induction e vs e'' arbitrary: t' X' e''' rule: reaches_on_rev_induct)
case (2 s s' v vs s')
  obtain v_t v_X where v_def: v = (v_t, v_X)
  by (cases v) auto
  have run_t s'': run_t (Some (s'', v_t)) = Some (Some (e''', t'), v_t)
  by (auto simp: 2(5))
  show ?case
  using reaches_on_app[OF 2(3)][OF 2(4) 2(2)[unfolded v_def]] run_t_s''
  by (auto simp: v_def)
qed (auto intro: reaches_on.intros)

lemma reach_event_t0_t:
  assumes reaches_on run_hd init_hd vs e'' run_hd e'' = Some (e''', (t', X'))
  shows reaches_on run_t t0 (map fst vs) (Some (e''', t'))
proof -
  have t0_not_None: t0  $\neq$  None
  apply (rule reaches_on.cases[OF assms(1)])
  using assms(2)
  by (auto simp: t0_def split: option.splits prod.splits)
  then show ?thesis
  using reach_event_t[OF assms(1) _ assms(2)]
  by (auto simp: t0_def split: option.splits)
qed

lemma reaches_on_run_hd_t:
  assumes reaches_on run_hd init_hd vs e
  shows  $\exists$  x. reaches_on run_t t0 (map fst vs) x
proof (cases vs rule: rev_cases)

```

```

case (snoc ys y)
show ?thesis
  using assms
  apply (cases y)
  apply (auto simp: snoc dest!: reaches_on_split_last)
  apply (meson reaches_on_app[OF reach_event_t0_t] read_t.simps(2) read_t_run)
  done
qed (auto intro: reaches_on.intros)

```

**definition** run\_subs run = ( $\lambda vs$ . let vs' = map run vs in  
 (if  $(\exists x \in \text{set } vs'. \text{Option.is\_none } x)$  then None  
 else Some (map (fst  $\circ$  the) vs', iarray\_of\_list (map (snd  $\circ$  snd  $\circ$  the) vs'))))

**lemma** run\_subs\_ID: run\_subs run vs = Some (vs', bs)  $\implies$   
 length vs' = length vs  $\wedge$  IArray.length bs = length vs  
**by** (auto simp: run\_subs\_def Let\_def iarray\_of\_list\_def split: option.splits if\_splits)

**lemma** run\_subs\_vD: run\_subs run vs = Some (vs', bs)  $\implies j < \text{length } vs \implies$   
 $\exists vj' \ tj \ bj. \text{run } (vs \ ! \ j) = \text{Some } (vj', (tj, bj)) \wedge vs' \ ! \ j = vj' \wedge \text{IArray.sub } bs \ j = bj$   
**apply** (cases run (vs ! j))  
**apply** (auto simp: Option.is\_none\_def run\_subs\_def Let\_def iarray\_of\_list\_def  
 split: option.splits if\_splits)  
**by** (metis image\_eqI nth\_mem)

**fun** msize\_fmula :: ('a, 'b :: timestamp) formula  $\Rightarrow$  nat  
**and** msize\_regeX :: ('a, 'b) regeX  $\Rightarrow$  nat **where**  
 msize\_fmula (Bool b) = 0  
| msize\_fmula (Atom a) = 0  
| msize\_fmula (Neg phi) = Suc (msize\_fmula phi)  
| msize\_fmula (Bin f phi psi) = Suc (msize\_fmula phi + msize\_fmula psi)  
| msize\_fmula (Prev I phi) = Suc (msize\_fmula phi)  
| msize\_fmula (Next I phi) = Suc (msize\_fmula phi)  
| msize\_fmula (Since phi I psi) = Suc (max (msize\_fmula phi) (msize\_fmula psi))  
| msize\_fmula (Until phi I psi) = Suc (max (msize\_fmula phi) (msize\_fmula psi))  
| msize\_fmula (MatchP I r) = Suc (msize\_regeX r)  
| msize\_fmula (MatchF I r) = Suc (msize\_regeX r)  
| msize\_regeX (Lookahead phi) = msize\_fmula phi  
| msize\_regeX (Symbol phi) = msize\_fmula phi  
| msize\_regeX (Plus r s) = max (msize\_regeX r) (msize\_regeX s)  
| msize\_regeX (Times r s) = max (msize\_regeX r) (msize\_regeX s)  
| msize\_regeX (Star r) = msize\_regeX r

**lemma** collect\_subfmlas\_msize:  $x \in \text{set } (\text{collect\_subfmlas } r \ []) \implies$   
 msize\_fmula x  $\leq$  msize\_regeX r

**proof** (induction r)  
**case** (Lookahead phi)  
**then show** ?case  
**by** (auto split: if\_splits)  
**next**  
**case** (Symbol phi)  
**then show** ?case  
**by** (auto split: if\_splits)  
**next**  
**case** (Plus r1 r2)  
**then show** ?case  
**by** (auto simp: collect\_subfmlas\_set[of r2 collect\_subfmlas r1 []])  
**next**  
**case** (Times r1 r2)

```

then show ?case
  by (auto simp: collect_subfmlas_set[of r2 collect_subfmlas r1 []])
next
  case (Star r)
  then show ?case
    by fastforce
qed

```

**definition** *until\_ready*  $I t c zo = (\text{case } (c, zo) \text{ of } (\text{Suc } \_, \text{Some } (t', b1, b2)) \Rightarrow (b2 \wedge \text{memL } t t' I) \vee \neg b1 \mid \_ \Rightarrow \text{False})$

**definition** *while\_since\_cond*  $I t = (\lambda(vpsi, e, cpsi :: \text{nat}, cpsi, tpsi). cpsi > 0 \wedge \text{memL } (\text{the } (\text{read\_t } e)) t I)$

**definition** *while\_since\_body*  $\text{run} = (\lambda(vpsi, e, cpsi :: \text{nat}, cpsi, tpsi). \text{case run vpsi of Some } (vpsi', (t', b')) \Rightarrow \text{Some } (vpsi', \text{fst } (\text{the } (\text{run\_t } e))), cpsi - 1, \text{if } b' \text{ then Some } cpsi \text{ else cpsi, if } b' \text{ then Some } t' \text{ else tpsi}) \mid \_ \Rightarrow \text{None})$

**definition** *while\_until\_cond*  $I t = (\lambda(vphi, vpsi, epsi, c, zo). \neg \text{until\_ready } I t c zo \wedge (\text{case read\_t epsi of Some } t' \Rightarrow \text{memR } t t' I \mid \text{None} \Rightarrow \text{False}))$

**definition** *while\_until\_body*  $\text{run} = (\lambda(vphi, vpsi, epsi, c, zo). \text{case run\_t epsi of Some } (epsi', t') \Rightarrow (\text{case run vphi of Some } (vphi', (\_, b1)) \Rightarrow (\text{case run vpsi of Some } (vpsi', (\_, b2)) \Rightarrow \text{Some } (vphi', vpsi', epsi', \text{Suc } c, \text{Some } (t', b1, b2)) \mid \_ \Rightarrow \text{None}) \mid \_ \Rightarrow \text{None}))$

**function** (sequential) *run* ::  $\text{nat} \Rightarrow ('a, 't, 'h) \text{vydra\_aux} \Rightarrow (('a, 't, 'h) \text{vydra\_aux} \times ('t \times \text{bool})) \text{option}$   
**where**

```

run n (VYDRA_None) = None
| run n (VYDRA_Bool b e) = (case run_hd e of None => None
  | Some (e', (t, _)) => Some (VYDRA_Bool b e', (t, b)))
| run n (VYDRA_Atom a e) = (case run_hd e of None => None
  | Some (e', (t, X)) => Some (VYDRA_Atom a e', (t, a ∈ X)))
| run (Suc n) (VYDRA_Neg v) = (case run n v of None => None
  | Some (v', (t, b)) => Some (VYDRA_Neg v', (t, ¬b)))
| run (Suc n) (VYDRA_Bin f vl vr) = (case run n vl of None => None
  | Some (vl', (t, bl)) => (case run n vr of None => None
  | Some (vr', (_, br)) => Some (VYDRA_Bin f vl' vr', (t, f bl br))))
| run (Suc n) (VYDRA_Prev I v e tb) = (case run_hd e of Some (e', (t, _)) =>
  (let β = (case tb of Some (t', b') => b' ∧ mem t' t I | None => False) in
  case run n v of Some (v', _, b') => Some (VYDRA_Prev I v' e' (Some (t, b')), (t, β))
  | None => Some (VYDRA_None, (t, β)))
  | None => None)
| run (Suc n) (VYDRA_Next I v e to) = (case run_hd e of Some (e', (t, _)) =>
  (case to of None =>
  (case run n v of Some (v', _, _) => run (Suc n) (VYDRA_Next I v' e' (Some t))
  | None => None)
  | Some t' =>
  (case run n v of Some (v', _, b) => Some (VYDRA_Next I v' e' (Some t), (t', b ∧ mem t' t I))
  | None => if mem t' t I then None else Some (VYDRA_None, (t', False))))
  | None => None)
| run (Suc n) (VYDRA_Since I vphi vpsi e cphi cpsi cpsi tpsi) = (case run n vphi of
  Some (vphi', (t, b1)) =>
  let cphi = (if b1 then Suc cphi else 0) in

```

```

    let cpsi = Suc cpsi in
    let cpsi = map_option Suc cpsi in
    (case while_break (while_since_cond I t) (while_since_body (run n)) (vpsi, e, cpsi, cpsi, tpsi) of
    Some (vpsi', e', cpsi', cpsi', tpsi') ⇒
      (let β = (case cpsi' of Some k ⇒ k - 1 ≤ cphi ∧ memR (the tpsi') t I | _ ⇒ False) in
      Some (VYDRA_Since I vphi' vpsi' e' cphi cpsi' cpsi' tpsi', (t, β)))
    | _ ⇒ None)
  | _ ⇒ None)
| run (Suc n) (VYDRA_Until I e vphi vpsi epsi c zo) = (case run_t e of Some (e', t) ⇒
  (case while_break (while_until_cond I t) (while_until_body (run n)) (vphi, vpsi, epsi, c, zo) of Some
  (vphi', vpsi', epsi', c', zo') ⇒
    if c' = 0 then None else
    (case zo' of Some (t', b1, b2) ⇒
      (if b2 ∧ memL t t' I then Some (VYDRA_Until I e' vphi' vpsi' epsi' (c' - 1) zo', (t, True))
      else if ¬b1 then Some (VYDRA_Until I e' vphi' vpsi' epsi' (c' - 1) zo', (t, False))
      else (case read_t epsi' of Some t' ⇒ Some (VYDRA_Until I e' vphi' vpsi' epsi' (c' - 1) zo', (t,
      False)) | _ ⇒ None))
    | _ ⇒ None)
  | _ ⇒ None)
  | _ ⇒ None)
| run (Suc n) (VYDRA_MatchP I transs qf w) =
  (case eval_matchP (init_args {0}, NFA.delta' transs qf, NFA.accept' transs qf)
  (run_t, read_t) (run_subs (run n))) I w of None ⇒ None
  | Some ((t, b), w') ⇒ Some (VYDRA_MatchP I transs qf w', (t, b)))
| run (Suc n) (VYDRA_MatchF I transs qf w) =
  (case eval_matchF (init_args {0}, NFA.delta' transs qf, NFA.accept' transs qf)
  (run_t, read_t) (run_subs (run n))) I w of None ⇒ None
  | Some ((t, b), w') ⇒ Some (VYDRA_MatchF I transs qf w', (t, b)))
| run _ _ = undefined
by pat_completeness auto

```

**termination**

```

by (relation (λp. size (fst p)) <*mlex*> (λp. next_vydra (snd p)) <*mlex*> (λp. msize_vydra (fst p)
(snd p)) <*mlex*> {}) (auto simp: mlex_prod_def)

```

**lemma** *wf\_since*:  $wf \{(t, s). \text{while\_since\_cond } I \ t \ t \ s \wedge \text{Some } t = \text{while\_since\_body } (\text{run } n) \ s\}$

**proof** –

```

let ?X = {(t, s). while_since_cond I t t s ∧ Some t = while_since_body (run n) s}
have sub: ?X ⊆ measure (λ(vpsi, e, cpsi, cpsi, tpsi). cpsi)
  by (auto simp: while_since_cond_def while_since_body_def Let_def split: option.splits)
then show ?thesis
  using wf_subset[OF wf_measure]
  by auto

```

**qed**

**definition** *run\_vydra* ::  $('a, 't, 'h) \text{vydra} \Rightarrow (('a, 't, 'h) \text{vydra} \times ('t \times \text{bool})) \text{option}$  **where**  
 $\text{run\_vydra } v = (\text{case } v \text{ of } (n, w) \Rightarrow \text{map\_option } (\text{apfst } (\text{Pair } n)) (\text{run } n \ w))$

**fun** *sub* ::  $\text{nat} \Rightarrow ('a, 't) \text{formula} \Rightarrow ('a, 't, 'h) \text{vydra\_aux}$  **where**

```

sub n (Bool b) = VYDRA_Bool b init_hd
| sub n (Atom a) = VYDRA_Atom a init_hd
| sub (Suc n) (Neg phi) = VYDRA_Neg (sub n phi)
| sub (Suc n) (Bin f phi psi) = VYDRA_Bin f (sub n phi) (sub n psi)
| sub (Suc n) (Prev I phi) = VYDRA_Prev I (sub n phi) init_hd None
| sub (Suc n) (Next I phi) = VYDRA_Next I (sub n phi) init_hd None
| sub (Suc n) (Since phi I psi) = VYDRA_Since I (sub n phi) (sub n psi) t0 0 0 None None
| sub (Suc n) (Until phi I psi) = VYDRA_Until I t0 (sub n phi) (sub n psi) t0 0 None
| sub (Suc n) (MatchP I r) = (let qf = state_cnt r;
  transs = iarray_of_list (build_nfa_impl r (0, qf, [])) in

```

```

  VYDRA_MatchP I transs qf (init_window (init_args
    ({0}, NFA.delta' transs qf, NFA.accept' transs qf)
    (run_t, read_t) (run_subs (run n)))
    t0 (map (sub n) (collect_subfmlas r [])))
| sub (Suc n) (MatchF I r) = (let qf = state_cnt r;
  transs = iarray_of_list (build_nfa_impl r (0, qf, [])) in
  VYDRA_MatchF I transs qf (init_window (init_args
    ({0}, NFA.delta' transs qf, NFA.accept' transs qf)
    (run_t, read_t) (run_subs (run n)))
    t0 (map (sub n) (collect_subfmlas r [])))
| sub _ _ = undefined

```

**definition** *init\_vydra* :: ('a, 't) formula  $\Rightarrow$  ('a, 't, 'h) vydra **where**  
*init\_vydra*  $\varphi = (\text{let } n = \text{msize\_fmla } \varphi \text{ in } (n, \text{sub } n \varphi))$

**end**

**locale** *VYDRA\_MDL* = *MDL*  $\sigma$

**for**  $\sigma :: ('a, 't :: \text{timestamp}) \text{ trace} +$

**fixes** *init\_hd* :: 'h

**and** *run\_hd* :: 'h  $\Rightarrow$  ('h  $\times$  ('t  $\times$  'a set)) option

**assumes** *run\_hd\_sound*: reaches *run\_hd* *init\_hd*  $n \ s \Longrightarrow \text{run\_hd } s = \text{Some } (s', (t, X)) \Longrightarrow (t, X) = (\tau \sigma \ n, \Gamma \sigma \ n)$

**begin**

**lemma** *reaches\_on\_run\_hd*: reaches\_on *run\_hd* *init\_hd*  $es \ s \Longrightarrow \text{run\_hd } s = \text{Some } (s', (t, X)) \Longrightarrow t = \tau \sigma (\text{length } es) \wedge X = \Gamma \sigma (\text{length } es)$

**using** *run\_hd\_sound*

**by** (auto dest: reaches\_on\_n)

**abbreviation** *ru\_t*  $\equiv \text{run\_t } \text{run\_hd}$

**abbreviation** *l\_t0*  $\equiv t0 \ \text{init\_hd } \text{run\_hd}$

**abbreviation** *ru*  $\equiv \text{run } \text{run\_hd}$

**abbreviation** *su*  $\equiv \text{sub } \text{init\_hd } \text{run\_hd}$

**lemma** *ru\_t\_event*: reaches\_on *ru\_t*  $t \ ts \ t' \Longrightarrow t = l\_t0 \Longrightarrow \text{ru\_t } t' = \text{Some } (t'', x) \Longrightarrow$

$\exists \rho \ e \ tt. t' = \text{Some } (e, tt) \wedge \text{reaches\_on } \text{run\_hd } \text{init\_hd } \rho \ e \wedge \text{length } \rho = \text{Suc } (\text{length } ts) \wedge$

$x = \tau \sigma (\text{length } ts)$

**proof** (induction  $t \ ts \ t'$  arbitrary:  $t'' \ x$  rule: reaches\_on\_rev\_induct)

**case** (1  $s$ )

**show** ?case

**using** 1 reaches\_on\_run\_hd[OF reaches\_on.intros(1)]

**by** (auto simp: t0\_def split: option.splits intro!: reaches\_on.intros)

**next**

**case** (2  $s \ s' \ v \ vs \ s''$ )

**obtain**  $\rho \ e \ tt$  **where**  $\rho\_def: s' = \text{Some } (e, tt) \text{ reaches\_on } \text{run\_hd } \text{init\_hd } \rho \ e$

$\text{length } \rho = \text{Suc } (\text{length } vs)$

**using** 2(3)[OF 2(4,2)]

**by** auto

**then show** ?case

**using** 2(2,5) reaches\_on\_app[OF  $\rho\_def$ (2)] reaches\_on\_run\_hd[OF  $\rho\_def$ (2)]

**by** (fastforce split: option.splits)

**qed**

**lemma** *ru\_t\_tau*: reaches\_on *ru\_t*  $l\_t0 \ ts \ t' \Longrightarrow \text{ru\_t } t' = \text{Some } (t'', x) \Longrightarrow x = \tau \sigma (\text{length } ts)$

**using** *ru\_t\_event*

**by** fastforce

```

lemma ru_t_Some_tau:
  assumes reaches_on ru_t l_t0 ts (Some (e, t))
  shows  $t = \tau \sigma$  (length ts)
proof -
  obtain z where z_def: ru_t (Some (e, t)) = Some (z, t)
    by (cases run_hd e) auto
  show ?thesis
    by (rule ru_t_tau[OF assms z_def])
qed

lemma ru_t_tau_in:
  assumes reaches_on ru_t l_t0 ts t j < length ts
  shows  $ts ! j = \tau \sigma j$ 
proof -
  obtain t' where t'_def: reaches_on ru_t l_t0 (take j ts) t' reaches_on ru_t t' (drop j ts) t
    using reaches_on_split'[OF assms(1), where ?i=j] assms(2)
    by auto
  have drop: drop j ts = ts ! j # tl (drop j ts)
    using assms(2)
    by (cases drop j ts) (auto simp add: nth_via_drop)
  obtain t'' where t''_def: ru_t t' = Some (t'', ts ! j)
    using t'_def(2) assms(2) drop
    by (auto elim: reaches_on.cases)
  show ?thesis
    using ru_t_event[OF t'_def(1) refl t''_def] assms(2)
    by auto
qed

lemmas run_hd_tau_in = ru_t_tau_in[OF reach_event_t0_t, simplified]

fun last_before :: (nat  $\Rightarrow$  bool)  $\Rightarrow$  nat  $\Rightarrow$  nat option where
  last_before P 0 = None
| last_before P (Suc n) = (if P n then Some n else last_before P n)

lemma last_before_None: last_before P n = None  $\implies$  m < n  $\implies$   $\neg$ P m
proof (induction P n rule: last_before.induct)
  case ( $\emptyset P n$ )
  then show ?case
    by (cases m = n) (auto split: if_splits)
qed (auto split: if_splits)

lemma last_before_Some: last_before P n = Some m  $\implies$  m < n  $\wedge$  P m  $\wedge$  ( $\forall k \in \{m <.. <n\}. \neg P k$ )
apply (induction P n rule: last_before.induct)
apply (auto split: if_splits)
apply (metis greaterThanLessThan_iff less_antisym)
done

inductive wf_vydra :: ('a, 't :: timestamp) formula  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  ('a, 't, 'h) vydra_aux  $\Rightarrow$  bool where
  wf_vydra phi i n w  $\implies$  ru n w = None  $\implies$  wf_vydra (Prev I phi) (Suc i) (Suc n) VYDRA_None
| wf_vydra phi i n w  $\implies$  ru n w = None  $\implies$  wf_vydra (Next I phi) i (Suc n) VYDRA_None
| reaches_on run_hd init_hd es sub'  $\implies$  length es = i  $\implies$  wf_vydra (Bool b) i n (VYDRA_Bool b sub')
| reaches_on run_hd init_hd es sub'  $\implies$  length es = i  $\implies$  wf_vydra (Atom a) i n (VYDRA_Atom a sub')
| wf_vydra phi i n v  $\implies$  wf_vydra (Neg phi) i (Suc n) (VYDRA_Neg v)
| wf_vydra phi i n v  $\implies$  wf_vydra psi i n v'  $\implies$  wf_vydra (Bin f phi psi) i (Suc n) (VYDRA_Bin f v v')
| wf_vydra phi i n v  $\implies$  reaches_on run_hd init_hd es sub'  $\implies$  length es = i  $\implies$ 
  wf_vydra (Prev I phi) i (Suc n) (VYDRA_Prev I v sub' (case i of 0  $\Rightarrow$  None | Suc j  $\Rightarrow$  Some ( $\tau \sigma j$ ),

```

```

sat phi j)))
| wf_vydra phi i n v  $\implies$  reaches_on run_hd init_hd es sub'  $\implies$  length es = i  $\implies$ 
  wf_vydra (Next I phi) (i - 1) (Suc n) (VYDRA_Next I v sub' (case i of 0  $\implies$  None | Suc j  $\implies$  Some
  (tau sigma j)))
| wf_vydra phi i n vphi  $\implies$  wf_vydra psi j n vpsi  $\implies$  j  $\leq$  i  $\implies$ 
  reaches_on ru_t l_t0 es sub'  $\implies$  length es = j  $\implies$  ( $\bigwedge$ t. t  $\in$  set es  $\implies$  memL t (tau sigma i) I)  $\implies$ 
  cphi = i - (case last_before (lambda k.  $\neg$ sat phi k) i of None  $\implies$  0 | Some k  $\implies$  Suc k)  $\implies$  cpsi = i - j  $\implies$ 
  cppsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some (i - k))  $\implies$ 
  tppsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some (tau sigma k))  $\implies$ 
  wf_vydra (Since phi I psi) i (Suc n) (VYDRA_Since I vphi vpsi sub' cphi cpsi cppsi tppsi)
| wf_vydra phi j n vphi  $\implies$  wf_vydra psi j n vpsi  $\implies$  i  $\leq$  j  $\implies$ 
  reaches_on ru_t l_t0 es back  $\implies$  length es = i  $\implies$ 
  reaches_on ru_t l_t0 es' front  $\implies$  length es' = j  $\implies$  ( $\bigwedge$ t. t  $\in$  set es'  $\implies$  memR (tau sigma i) t I)  $\implies$ 
  c = j - i  $\implies$  z = (case j of 0  $\implies$  None | Suc k  $\implies$  Some (tau sigma k, sat phi k, sat psi k))  $\implies$ 
  ( $\bigwedge$ k. k  $\in$  {i..<j-1}  $\implies$  sat phi k  $\wedge$  (memL (tau sigma i) (tau sigma k) I  $\longrightarrow$   $\neg$ sat psi k))  $\implies$ 
  wf_vydra (Until phi I psi) i (Suc n) (VYDRA_Until I back vphi vpsi front c z)
| valid_window_matchP args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w  $\implies$ 
  n  $\geq$  msize_regex r  $\implies$  qf = state_cnt r  $\implies$ 
  transs = iarray_of_list (build_nfa_impl r (0, qf, []))  $\implies$ 
  args = init_args ({0}, NFA.delta' transs qf, NFA.accept' transs qf)
  (ru_t, read_t) (run_subs (ru n))  $\implies$ 
  wf_vydra (MatchP I r) i (Suc n) (VYDRA_MatchP I transs qf w)
| valid_window_matchF args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w  $\implies$ 
  n  $\geq$  msize_regex r  $\implies$  qf = state_cnt r  $\implies$ 
  transs = iarray_of_list (build_nfa_impl r (0, qf, []))  $\implies$ 
  args = init_args ({0}, NFA.delta' transs qf, NFA.accept' transs qf)
  (ru_t, read_t) (run_subs (ru n))  $\implies$ 
  wf_vydra (MatchF I r) i (Suc n) (VYDRA_MatchF I transs qf w)

```

**lemma reach\_run\_subs\_len:**

```

assumes reaches_ons: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) rho vs
shows length vs = length (collect_subfmlas r [])
using reaches_ons run_subs_ID
by (induction map (su n) (collect_subfmlas r [])) rho vs rule: reaches_on_rev_induct) fastforce+

```

**lemma reach\_run\_subs\_run:**

```

assumes reaches_ons: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) rho vs
and subfmla: j < length (collect_subfmlas r []) phi = collect_subfmlas r [] ! j
shows  $\exists$  rho'. reaches_on (ru n) (su n phi) rho' (vs ! j)  $\wedge$  length rho' = length rho
using reaches_ons subfmla

```

**proof** (induction map (su n) (collect\_subfmlas r [])) rho vs rule: reaches\_on\_rev\_induct)

**case 1**

**then show** ?case

**by** (auto intro: reaches\_on.intros)

**next**

**case** (2 s' v vs' s')

**note** len\_s'\_vs = reach\_run\_subs\_len[OF 2(1)]

**obtain** rho' **where** reach\_s'\_vs: reaches\_on (ru n) (su n phi) rho' (s' ! j)

length rho' = length vs'

**using** 2(2)[OF 2(4,5)]

**by** auto

**note** run\_subslD = run\_subs\_ID[OF 2(3)]

**note** run\_subsvD = run\_subs\_vD[OF 2(3) 2(4)[unfolded len\_s'\_vs[symmetric]]]

**obtain** vj' tj bj **where** vj'\_def: ru n (s' ! j) = Some (vj', tj, bj)

s'' ! j = vj' IArray.sub v j = bj

**using** run\_subsvD **by** auto

**obtain** rho'' **where** rho''\_def: reaches\_on (ru n) (su n phi) rho'' (s'' ! j)

length rho'' = Suc (length vs')

```

using reaches_on_app[OF reach_s'_vs(1) vj'_def(1)] vj'_def(2) reach_s'_vs(2)
by auto
then show ?case
using conjunct1[OF run_subslD, unfolded len_s'_vs[symmetric]]
by auto
qed

lemma IArray_nth_equalityI: IArray.length xs = length ys  $\implies$ 
( $\bigwedge i. i < IArray.length xs \implies IArray.sub xs i = ys ! i$ )  $\implies xs = IArray ys$ 
by (induction xs arbitrary: ys) (auto intro: nth_equalityI)

lemma bs_sat:
assumes IH:  $\bigwedge phi i v v' b. phi \in set (collect\_subfmlas r []) \implies wf\_vydra phi i n v \implies ru n v = Some (v', b) \implies snd b = sat phi i$ 
and reaches_ons:  $\bigwedge j. j < length (collect\_subfmlas r []) \implies wf\_vydra (collect\_subfmlas r [] ! j) i n (vs ! j)$ 
and run_subs:  $run\_subs (ru n) vs = Some (vs', bs) \wedge length vs = length (collect\_subfmlas r [])$ 
shows  $bs = iarray\_of\_list (map (\lambda phi. sat phi i) (collect\_subfmlas r []))$ 
proof -
have  $\bigwedge j. j < length (collect\_subfmlas r []) \implies IArray.sub bs j = sat (collect\_subfmlas r [] ! j) i$ 
proof -
fix j
assume lassm:  $j < length (collect\_subfmlas r [])$ 
define phi where  $phi = collect\_subfmlas r [] ! j$ 
have phi_in_set:  $phi \in set (collect\_subfmlas r [])$ 
using lassm
by (auto simp: phi_def)
have wf:  $wf\_vydra phi i n (vs ! j)$ 
using reaches_ons lassm phi_def
by metis
show  $IArray.sub bs j = sat (collect\_subfmlas r [] ! j) i$ 
using IH(1)[OF phi_in_set wf] run_subs_vD[OF run_subs(1) lassm[folded run_subs(2)]]
unfolding phi_def[symmetric]
by auto
qed
moreover have  $length (IArray.list\_of bs) = length vs$ 
using run_subs(1)
by (auto simp: run_subs_def Let_def iarray_of_list_def split: if_splits)
ultimately show ?thesis
using run_subs(2)
by (auto simp: iarray_of_list_def intro!: IArray_nth_equalityI)
qed

lemma run_induct[case_names Bool Atom Neg Bin Prev Next Since Until MatchP MatchF, consumes 1]:
fixes phi :: ('a, 't) formula
assumes msize_fm1a phi  $\leq n$  ( $\bigwedge b n. P n (Bool b)$ ) ( $\bigwedge a n. P n (Atom a)$ )
( $\bigwedge n phi. msize_fm1a phi \leq n \implies P n phi \implies P (Suc n) (Neg phi)$ )
( $\bigwedge n f phi psi. msize_fm1a (Bin f phi psi) \leq Suc n \implies P n phi \implies P n psi \implies P (Suc n) (Bin f phi psi)$ )
( $\bigwedge n I phi. msize_fm1a phi \leq n \implies P n phi \implies P (Suc n) (Prev I phi)$ )
( $\bigwedge n I phi. msize_fm1a phi \leq n \implies P n phi \implies P (Suc n) (Next I phi)$ )
( $\bigwedge n I phi psi. msize_fm1a phi \leq n \implies msize_fm1a psi \leq n \implies P n phi \implies P n psi \implies P (Suc n) (Since phi I psi)$ )
( $\bigwedge n I phi psi. msize_fm1a phi \leq n \implies msize_fm1a psi \leq n \implies P n phi \implies P n psi \implies P (Suc n) (Until phi I psi)$ )
( $\bigwedge n I r. msize_fm1a (MatchP I r) \leq Suc n \implies (\bigwedge x. msize_fm1a x \leq n \implies P n x) \implies$ 

```

```

    P (Suc n) (MatchP I r)
  (∧ n I r. msize_fmula (MatchF I r) ≤ Suc n ⇒ (∧ x. msize_fmula x ≤ n ⇒ P n x) ⇒
    P (Suc n) (MatchF I r))
  shows P n phi
  using assms(1)
proof (induction n arbitrary: phi rule: nat_less_induct)
  case (1 n)
  show ?case
  proof (cases n)
    case 0
    show ?thesis
      using 1 assms(2-)
      by (cases phi) (auto simp: 0)
  next
  case (Suc m)
  show ?thesis
    using 1 assms(2-)
    by (cases phi) (auto simp: Suc)
  qed
qed

lemma wf_vydra_sub: msize_fmula φ ≤ n ⇒ wf_vydra φ 0 n (su n φ)
proof (induction n φ rule: run_induct)
  case (Prev n I phi)
  then show ?case
    using wf_vydra.intros(7)[where ?i=0, OF _ reaches_on.intros(1)]
    by auto
  next
  case (Next n I phi)
  then show ?case
    using wf_vydra.intros(8)[where ?i=0, OF _ reaches_on.intros(1)]
    by auto
  next
  case (MatchP n I r)
  let ?qf = state_cnt r
  let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
  let ?args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t, read_t) (run_subs
(ru n))
  show ?case
    using MatchP valid_init_window[of ?args l_t0 map (su n) (collect_subfmlas r []), simplified]
    by (auto simp: Let_def valid_window_matchP_def split: option.splits intro: reaches_on.intros
intro!: wf_vydra.intros(11)[where ?xs=[], OF _ _ refl refl refl])
  next
  case (MatchF n I r)
  let ?qf = state_cnt r
  let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
  let ?args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t, read_t) (run_subs
(ru n))
  show ?case
    using MatchF valid_init_window[of ?args l_t0 map (su n) (collect_subfmlas r []), simplified]
    by (auto simp: Let_def valid_window_matchF_def split: option.splits intro: reaches_on.intros
intro!: wf_vydra.intros(12)[where ?xs=[], OF _ _ refl refl refl])
  qed (auto simp: Let_def intro: wf_vydra.intros reaches_on.intros)

lemma ru_t_Some: ∃ e' et. ru_t e = Some (e', et) if reaches_Suc_i: reaches_on run_hd init_hd fs f
length fs = Suc i
  and aux: reaches_on ru_t l_t0 es e length es ≤ i for es e
proof -

```

```

obtain  $fs' ft$  where  $ft\_def: reaches\_on\ ru\_t\ l\_t0\ (map\ fst\ (fs' :: ('t \times 'a\ set)\ list))\ (Some\ (f,\ ft))$ 
   $map\ fst\ fs = map\ fst\ fs' @ [ft]\ length\ fs' = i$ 
  using  $reaches\_Suc\_i$ 
  by  $(cases\ fs\ rule: rev\_cases)\ (auto\ dest!: reaches\_on\_split\_last\ reach\_event\_t0\_t)$ 
show  $?thesis$ 
proof  $(cases\ length\ es = i)$ 
  case  $True$ 
  have  $e\_def: e = Some\ (f,\ ft)$ 
    using  $reaches\_on\_inj[OF\ aux(1)\ ft\_def(1)]$ 
    by  $(auto\ simp: True\ ft\_def(3))$ 
  then show  $?thesis$ 
    by  $(cases\ run\_hd\ f)\ (auto\ simp: e\_def)$ 
next
  case  $False$ 
  obtain  $s' s''$  where  $split: reaches\_on\ ru\_t\ l\_t0\ (take\ (length\ es)\ (map\ fst\ fs'))\ s'$ 
     $ru\_t\ s' = Some\ (s'',\ map\ fst\ fs' ! (length\ es))$ 
    using  $reaches\_on\_split[OF\ ft\_def(1),\ where\ ?i=length\ es]\ False\ aux(2)$ 
    by  $(auto\ simp: ft\_def(3))$ 
  show  $?thesis$ 
    using  $reaches\_on\_inj[OF\ aux(1)\ split(1)]\ aux(2)$ 
    by  $(auto\ simp: ft\_def(3)\ split(2))$ 
qed
qed

lemma  $vydra\_sound\_aux:$ 
  assumes  $msize\_fmla\ \varphi \leq n\ wf\_vydra\ \varphi\ i\ n\ v\ ru\ n\ v = Some\ (v',\ t,\ b)\ bounded\_future\_fmla\ \varphi\ wf\_fmla$ 
   $\varphi$ 
  shows  $wf\_vydra\ \varphi\ (Suc\ i)\ n\ v' \wedge (\exists\ es\ e.\ reaches\_on\ run\_hd\ init\_hd\ es\ e \wedge length\ es = Suc\ i) \wedge t =$ 
 $\tau\ \sigma\ i \wedge b = sat\ \varphi\ i$ 
  using  $assms$ 
proof  $(induction\ n\ \varphi\ arbitrary: i\ v\ v'\ t\ b\ rule: run\_induct)$ 
  case  $(Bool\ \beta\ n)$ 
  then show  $?case$ 
    using  $reaches\_on\_run\_hd\ reaches\_on\_app\ wf\_vydra.intros(3)[OF\ reaches\_on\_app\ refl]$ 
    by  $(fastforce\ elim!: wf\_vydra.cases[of\ \_\_\_\ v]\ split: option.splits)$ 
next
  case  $(Atom\ a\ n)$ 
  then show  $?case$ 
    using  $reaches\_on\_run\_hd\ reaches\_on\_app\ wf\_vydra.intros(4)[OF\ reaches\_on\_app\ refl]$ 
    by  $(fastforce\ elim!: wf\_vydra.cases[of\ \_\_\_\ v]\ split: option.splits)$ 
next
  case  $(Neg\ n\ x)$ 
  have  $IH: wf\_vydra\ x\ i\ n\ v \implies ru\ n\ v = Some\ (v',\ t,\ b) \implies wf\_vydra\ x\ (Suc\ i)\ n\ v' \wedge (\exists\ es\ e.$ 
 $reaches\_on\ run\_hd\ init\_hd\ es\ e \wedge length\ es = Suc\ i) \wedge t = \tau\ \sigma\ i \wedge b = sat\ x\ i$  for  $v\ v'\ t\ b$ 
    using  $Neg(2,5,6)$ 
    by  $auto$ 
  show  $?case$ 
    apply  $(rule\ wf\_vydra.cases[OF\ Neg(3)])$ 
    using  $Neg(4)\ IH\ wf\_vydra.intros(5)$ 
    by  $(fastforce\ split: option.splits)+$ 
next
  case  $(Bin\ n\ f\ x1\ x2)$ 
  have  $IH1: wf\_vydra\ x1\ i\ n\ v \implies ru\ n\ v = Some\ (v',\ t,\ b) \implies wf\_vydra\ x1\ (Suc\ i)\ n\ v' \wedge (\exists\ es\ e.$ 
 $reaches\_on\ run\_hd\ init\_hd\ es\ e \wedge length\ es = Suc\ i) \wedge t = \tau\ \sigma\ i \wedge b = sat\ x1\ i$  for  $v\ v'\ t\ b$ 
    using  $Bin(2,6,7)$ 
    by  $auto$ 
  have  $IH2: wf\_vydra\ x2\ i\ n\ v \implies ru\ n\ v = Some\ (v',\ t,\ b) \implies wf\_vydra\ x2\ (Suc\ i)\ n\ v' \wedge t = \tau\ \sigma\ i$ 
 $\wedge b = sat\ x2\ i$  for  $v\ v'\ t\ b$ 

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```

    using Bin(3,6,7)
    by auto
  show ?case
    apply (rule wf_vydra.cases[OF Bin(4)])
    using Bin(5) IH1 IH2 wf_vydra.intros(6)
    by (fastforce split: option.splits)+
next
case (Prev n I phi)
show ?case
proof (cases i)
  case 0
  then show ?thesis
    using Prev run_hd_sound[OF reaches.intros(1)] wf_vydra.intros(7)[OF _ reaches_on.intros(2)[OF
    _ reaches_on.intros(1)], where ?i=Suc 0, simplified]
    by (fastforce split: nat.splits option.splits dest!: reaches_on_NilD elim!: wf_vydra.cases[of _ _ _ v]
    intro: wf_vydra.intros(1) reaches_on.intros(2)[OF _ reaches_on.intros(1)])
  next
  case (Suc j)
  obtain vphi es sub where v_def: v = VYDRA_Prev I vphi sub (Some (τ σ j, sat phi j))
    wf_vydra phi i n vphi reaches_on run_hd init_hd es sub length es = i
    using Prev(3,4)
    by (auto simp: Suc elim!: wf_vydra.cases[of _ _ _ v])
  obtain sub' X where run_sub: run_hd sub = Some (sub', (t, X))
    using Prev(4)
    by (auto simp: v_def(1) Let_def split: option.splits)
  note reaches_sub' = reaches_on_app[OF v_def(3) run_sub]
  have t_def: t = τ σ (Suc j)
    using reaches_on_run_hd[OF v_def(3) run_sub]
    by (auto simp: Suc v_def(2,4))
  show ?thesis
  proof (cases v' = VYDRA_None)
    case v'_def: True
    show ?thesis
      using Prev(4) v_def(2) reaches_sub'
      by (auto simp: Suc Let_def v_def(1,4) v'_def run_sub t_def split: option.splits intro: wf_vydra.intros(1))
  next
  case False
  obtain vphi' where ru_vphi: ru n vphi = Some (vphi', (τ σ i, sat phi i))
    using Prev(2)[OF v_def(2)] Prev(4,5,6) False
    by (auto simp: v_def(1) Let_def split: option.splits)
  have wf': wf_vydra phi (Suc (Suc j)) n vphi'
    using Prev(2)[OF v_def(2) ru_vphi] Prev(5,6)
    by (auto simp: Suc)
  show ?thesis
    using Prev(4) wf_vydra.intros(7)[OF wf' reaches_sub'] reaches_sub'
    by (auto simp: Let_def Suc t_def v_def(1,4) run_sub ru_vphi)
qed
qed
next
case (Next n I phi)
obtain w sub to es where v_def: v = VYDRA_Next I w sub to wf_vydra phi (length es) n w
  reaches_on run_hd init_hd es sub length es = (case to of None ⇒ 0 | _ ⇒ Suc i)
  case to of None ⇒ i = 0 | Some told ⇒ told = τ σ i
  using Next(3,4)
  by (auto elim!: wf_vydra.cases[of _ _ _ v] split: option.splits nat.splits)
obtain sub' tnew X where run_sub: run_hd sub = Some (sub', (tnew, X))
  using Next(4)
  by (auto simp: v_def(1) split: option.splits)

```

```

have tnew_def: tnew =  $\tau \sigma$  (length es)
  using reaches_on_run_hd[OF v_def(3) run_sub]
  by auto
have aux: ?case if aux_assms: wf_vydra phi (Suc i) n w
  ru (Suc n) (VYDRA_Next I w sub (Some t0)) = Some (v', t, b)
  reaches_on_run_hd init_hd es sub length es = Suc i t0 =  $\tau \sigma$  i for w sub t0 es
  using aux_assms(1,2,5) wf_vydra.intros(2)[OF aux_assms(1)]
  Next(2)[where ?i=Suc i and ?v=w] Next(5,6) reaches_on_run_hd[OF aux_assms(3)]
  wf_vydra.intros(8)[OF _ reaches_on_app[OF aux_assms(3)], where ?phi=phi and ?i=Suc (Suc
i) and ?n=n] aux_assms(3)
  by (auto simp: run_sub aux_assms(4,5) split: option.splits if_splits)
show ?case
proof (cases to)
case None
obtain w' z where w_def: ru (Suc n) v = ru (Suc n) (VYDRA_Next I w' sub' (Some tnew))
  ru n w = Some (w', z)
  using Next(4)
  by (cases ru n w) (auto simp: v_def(1) run_sub None split: option.splits)
have wf: wf_vydra phi (Suc i) n w'
  using v_def w_def(2) Next(2,5,6)
  by (cases z) (auto simp: None intro: wf_vydra.intros(1))
show ?thesis
  using aux[OF wf Next(4)[unfolded w_def(1)] reaches_on_app[OF v_def(3) run_sub]] v_def(4,5)
tnew_def
  by (auto simp: None)
next
case (Some z)
show ?thesis
  using aux[OF _ _ v_def(3), where ?w=w] v_def(2,4,5) Next(4)
  by (auto simp: v_def(1) Some simp del: run.simps)
qed
next
case (Since n I phi psi)
obtain vphi vpsi e cphi cpsi cpsi tpsi j es where v_def:
  v = VYDRA_Since I vphi vpsi e cphi cpsi cpsi tpsi
  wf_vydra phi i n vphi wf_vydra psi j n vpsi j  $\leq$  i
  reaches_on ru t l t0 es e length es = j  $\wedge$  t. t  $\in$  set es  $\implies$  memL t ( $\tau \sigma$  i) I
  cphi = i - (case last_before ( $\lambda k. \neg$ sat phi k) i of None  $\implies$  0 | Some k  $\implies$  Suc k) cpsi = i - j
  cpsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some (i - k))
  tpsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some ( $\tau \sigma$  k))
  using Since(5)
  by (auto elim: wf_vydra.cases)
obtain vphi' b1 where run_vphi: ru n vphi = Some (vphi', t, b1)
  using Since(6)
  by (auto simp: v_def(1) Let_def split: option.splits)
obtain fs f where wf_vphi': wf_vydra phi (Suc i) n vphi'
  and reaches_Suc_i: reaches_on run_hd init_hd fs f length fs = Suc i
  and t_def: t =  $\tau \sigma$  i and b1_def: b1 = sat phi i
  using Since(3)[OF v_def(2) run_vphi] Since(7,8)
  by auto
note ru_t_Some = ru_t_Some[OF reaches_Suc_i]
define loop_inv where loop_inv = ( $\lambda$ (vpsi, e, cpsi :: nat, cpsi, tpsi).
  let j = Suc i - cpsi in cpsi  $\leq$  Suc i  $\wedge$ 
  wf_vydra psi j n vpsi  $\wedge$  ( $\exists$  es. reaches_on ru_t l t0 es e  $\wedge$  length es = j  $\wedge$  ( $\forall$  t  $\in$  set es. memL t ( $\tau \sigma$  i) I))  $\wedge$ 
  cpsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some (Suc i - k))  $\wedge$ 
  tpsi = (case last_before (sat psi) j of None  $\implies$  None | Some k  $\implies$  Some ( $\tau \sigma$  k)))
define loop_init where loop_init = (vpsi, e, Suc cpsi, map_option Suc cpsi, tpsi)

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obtain  $vpsi' e' cpsi' cpsi' tpsi'$  where  $loop\_def: while\_break (while\_since\_cond I t) (while\_since\_body$ 
 $run\_hd (ru n)) loop\_init =$ 
   $Some (vpsi', e', cpsi', cpsi', tpsi')$ 
  using  $Since(6)$ 
  by  $(auto simp: v\_def(1) run\_vphi loop\_init\_def Let\_def split: option.splits)$ 
have  $j\_def: j = i - cpsi$ 
  using  $v\_def(4,9)$ 
  by  $auto$ 
have  $cpsi \leq i$ 
  using  $v\_def(9)$ 
  by  $auto$ 
then have  $loop\_inv\_init: loop\_inv loop\_init$ 
  using  $v\_def(3,5,6,7,10,11) last\_before\_Some$ 
  by  $(fastforce simp: loop\_inv\_def loop\_init\_def Let\_def j\_def split: option.splits)$ 
have  $wf\_loop: wf \{(s', s). loop\_inv s \wedge while\_since\_cond I t s \wedge Some s' = while\_since\_body run\_hd$ 
 $(ru n) s\}$ 
  by  $(auto intro: wf\_subset[OF wf\_since])$ 
have  $step\_loop: loop\_inv s'$  if  $loop\_assms: loop\_inv s while\_since\_cond I t s while\_since\_body run\_hd$ 
 $(ru n) s = Some s'$  for  $s s'$ 
proof -
  obtain  $vpsi e cpsi cpsi tpsi$  where  $s\_def: s = (vpsi, e, cpsi, cpsi, tpsi)$ 
  by  $(cases s) auto$ 
  define  $j$  where  $j = Suc i - cpsi$ 
  obtain  $es$  where  $loop\_before: cpsi \leq Suc i wf\_vydra psi j n vpsi$ 
   $reaches\_on ru\_t l\_t0 es e length es = j \wedge t. t \in set es \implies memL t (\tau \sigma i)$ 
   $cpsi = (case last\_before (sat psi) j of None \implies None | Some k \implies Some (Suc i - k))$ 
   $tpsi = (case last\_before (sat psi) j of None \implies None | Some k \implies Some (\tau \sigma k))$ 
  using  $loop\_assms(1)$ 
  by  $(auto simp: s\_def j\_def loop\_inv\_def Let\_def)$ 
  obtain  $tt h$  where  $tt\_def: read\_t e = Some tt memL tt t I e = Some (h, tt)$ 
  using  $ru\_t\_Some[OF loop\_before(3)] loop\_before(4) loop\_assms(2)$ 
  by  $(cases e) (fastforce simp: while\_since\_cond\_def s\_def j\_def split: option.splits) +$ 
obtain  $e'$  where  $e'\_def: reaches\_on ru\_t l\_t0 (es @ [tt]) e' ru\_t e = Some (e', tt)$ 
  using  $reaches\_on\_app[OF loop\_before(3)] tt\_def(1)$ 
  by  $(cases run\_hd h) (auto simp: tt\_def(3))$ 
  obtain  $vpsi' t' b'$  where  $run\_vpsi: ru n vpsi = Some (vpsi', (t', b'))$ 
  using  $loop\_assms(3)$ 
  by  $(auto simp: while\_since\_body\_def s\_def Let\_def split: option.splits)$ 
  have  $wf\_psi': wf\_vydra psi (Suc j) n vpsi'$  and  $t'\_def: t' = \tau \sigma j$  and  $b'\_def: b' = sat psi j$ 
  using  $Since(4)[OF loop\_before(2) run\_vpsi] Since(7,8)$ 
  by  $auto$ 
  define  $j'$  where  $j'\_def: j' = Suc i - (cpsi - Suc 0)$ 
  have  $cpsi\_pos: cpsi > 0$ 
  using  $loop\_assms(2)$ 
  by  $(auto simp: while\_since\_cond\_def s\_def)$ 
  have  $j'\_j: j' = Suc j$ 
  using  $loop\_before(1) cpsi\_pos$ 
  by  $(auto simp: j'\_def j\_def)$ 
  define  $cpsi'$  where  $cpsi' = (if b' then Some cpsi else cpsi)$ 
  define  $tpsi'$  where  $tpsi' = (if b' then Some t' else tpsi)$ 
  have  $cpsi': cpsi' = (case last\_before (sat psi) j' of None \implies None | Some k \implies Some (Suc i - k))$ 
  using  $cpsi\_pos loop\_before(1)$ 
  by  $(auto simp: cpsi'\_def b'\_def j'\_j loop\_before(6) j\_def)$ 
  have  $tpsi': tpsi' = (case last\_before (sat psi) j' of None \implies None | Some k \implies Some (\tau \sigma k))$ 
  by  $(auto simp: tpsi'\_def t'\_def b'\_def j'\_j loop\_before(7) split: option.splits)$ 
  have  $s'\_def: s' = (vpsi', fst (the (ru\_t e)), cpsi - Suc 0, cpsi', tpsi')$ 
  using  $loop\_assms(3)$ 
  by  $(auto simp: while\_since\_body\_def s\_def run\_vpsi cpsi'\_def tpsi'\_def)$ 

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show ?thesis
  using loop_before(1,4,5) tt_def(2) wf_psi'[folded j'_j] cpsi' tpsi' e'_def(1)
  by (fastforce simp: loop_inv_def s'_def j'_def[symmetric] e'_def(2) j'_j t_def)
qed
have loop: loop_inv (vpsi', e', cpsi', cpsi', tpsi')  $\neg$ while_since_cond I t (vpsi', e', cpsi', cpsi', tpsi')
  using while_break_sound[where ?P=loop_inv and ?Q= $\lambda$ s. loop_inv s  $\wedge$   $\neg$ while_since_cond I t s,
  OF step_loop _ wf_loop loop_inv_init]
  by (auto simp: loop_def)
define cphi' where cphi' = (if b1 then Suc cphi else 0)
have v'_def: v' = VYDRA_Since I vphi' vpsi' e' cphi' cpsi' cpsi' tpsi'
  and b_def: b = (case cpsi' of None  $\Rightarrow$  False | Some k  $\Rightarrow$  k - 1  $\leq$  cphi'  $\wedge$  memR (the tpsi') t I)
  using Since(6)
  by (auto simp: v_def(1) run_vphi loop_init_def[symmetric] loop_def cphi'_def Let_def split: option.splits)
have read_t_e': cpsi' > 0  $\implies$  read_t e' = None  $\implies$  False
  using loop(1) ru_t_Some[where ?e=e'] run_t_read
  by (fastforce simp: loop_inv_def Let_def)
define j' where j' = Suc i - cpsi'
have wf_vpsi': wf_vydra psi j' n vpsi' and cpsi'_le_Suc_i: cpsi'  $\leq$  Suc i
  and cpsi'_def: cpsi' = (case last_before (sat psi) j' of None  $\Rightarrow$  None | Some k  $\Rightarrow$  Some (Suc i -
  k))
  and tpsi'_def: tpsi' = (case last_before (sat psi) j' of None  $\Rightarrow$  None | Some k  $\Rightarrow$  Some ( $\tau$   $\sigma$  k))
  using loop(1)
  by (auto simp: loop_inv_def j'_def[symmetric])
obtain es' where es'_def: reaches_on ru_t l_t0 es' e' length es' = j'  $\wedge$  t. t  $\in$  set es'  $\implies$  memL t ( $\tau$ 
 $\sigma$  i) I
  using loop(1)
  by (auto simp: loop_inv_def j'_def[symmetric])
have wf_v': wf_vydra (Since phi I psi) (Suc i) (Suc n) v'
  and cphi'_sat: cphi' = Suc i - (case last_before ( $\lambda$ k.  $\neg$ sat phi k) (Suc i) of None  $\Rightarrow$  0 | Some k  $\Rightarrow$ 
  Suc k)
  using cpsi'_le_Suc_i last_before_Some es'_def(3) memL_mono'[OF  $\tau$ _mono[of i Suc i  $\sigma$ ]]
  by (force simp: v'_def cpsi'_def tpsi'_def j'_def cphi'_def b1_def v_def(8) split: option.splits
  intro!: wf_vydra.intros(9)[OF wf_vphi' wf_vpsi' _ es'_def(1-2)])+
have j' = Suc i  $\vee$   $\neg$ memL ( $\tau$   $\sigma$  j') ( $\tau$   $\sigma$  i) I
  using loop(2) j'_def read_t_e' ru_t_tau[OF es'_def(1)] read_t_run[where ?run_hd=run_hd]
  by (fastforce simp: while_since_cond_def es'_def(2) t_def split: option.splits)
then have tau_k_j': k  $\leq$  i  $\implies$  memL ( $\tau$   $\sigma$  k) ( $\tau$   $\sigma$  i) I  $\longleftrightarrow$  k < j' for k
  using ru_t_tau_in[OF es'_def(1)] es'_def(3)  $\tau$ _mono[of j' k  $\sigma$ ] memL_mono
  by (fastforce simp: es'_def(2) in_set_conv_nth)
have b_sat: b = sat (Since phi I psi) i
proof (rule iffI)
  assume b: b
obtain m where m_def: last_before (sat psi) j' = Some m i - m  $\leq$  cphi' memR ( $\tau$   $\sigma$  m) ( $\tau$   $\sigma$  i) I
  using b
  by (auto simp: b_def t_def cpsi'_def tpsi'_def split: option.splits)
note aux = last_before_Some[OF m_def(1)]
have mem: mem ( $\tau$   $\sigma$  m) ( $\tau$   $\sigma$  i) I
  using m_def(3) tau_k_j' aux
  by (auto simp: mem_def j'_def)
have sat_phi: sat phi x if m < x  $\leq$  i for x
  using m_def(2) that le_neq_implies_less
  by (fastforce simp: cphi'_sat dest: last_before_None last_before_Some split: option.splits if_splits)
show sat (Since phi I psi) i
  using aux mem sat_phi
  by (auto simp: j'_def intro!: exI[of _ m])
next
assume sat: sat (Since phi I psi) i

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then obtain k where k_def: k ≤ i mem (τ σ k) (τ σ i) I sat psi k ∧ k'. k < k' ∧ k' ≤ i ⇒ sat phi
k'
  by auto
have k_j': k < j'
  using tau_k_j'[OF k_def(1)] k_def(2)
  by (auto simp: mem_def)
obtain m where m_def: last_before (sat psi) j' = Some m
  using last_before_None[where ?P=sat psi and ?n=j' and ?m=k] k_def(3) k_j'
  by (cases last_before (sat psi) j') auto
have cppsi'_Some: cppsi' = Some (Suc i - m)
  by (auto simp: cppsi'_def m_def)
have tppsi'_Some: tppsi' = Some (τ σ m)
  by (auto simp: tppsi'_def m_def)
have m_k: k ≤ m
  using last_before_Some[OF m_def] k_def(3) k_j'
  by auto
have tau_i_m: memR (τ σ m) (τ σ i) I
  using τ_mono[OF m_k, where ?s=σ] memR_mono k_def(2)
  by (auto simp: mem_def)
have i - m ≤ cphi'
  using k_def(1) k_def(4) m_k
  apply (cases k = i)
  apply (auto simp: cphi'_sat b1_def dest!: last_before_Some split: option.splits)
  apply (metis diff_le_mono2 le_neq_implies_less le_trans less_imp_le_nat nat_le_linear)
  done
then show b
  using tau_i_m
  by (auto simp: b_def t_def cppsi'_Some tppsi'_Some)
qed
show ?case
  using wf_v' reaches_Suc_i
  by (auto simp: t_def b_sat)
next
case (Until n I phi psi)
obtain back vphi vpsi front c z es es' j where v_def:
  v = VYDRA_Until I back vphi vpsi front c z
  wf_vydra phi j n vphi wf_vydra psi j n vpsi i ≤ j
  reaches_on ru_t l_t0 es back length es = i
  reaches_on ru_t l_t0 es' front length es' = j ∧ t. t ∈ set es' ⇒ memR (τ σ i) t I
  c = j - i z = (case j of 0 ⇒ None | Suc k ⇒ Some (τ σ k, sat phi k, sat psi k))
  ∧ k. k ∈ {i..<j - 1} ⇒ sat phi k ∧ (memL (τ σ i) (τ σ k) I → ¬sat psi k)
  using Until(5)
  by (auto elim: wf_vydra.cases)
define loop_init where loop_init = (vphi, vpsi, front, c, z)
obtain back' vphi' vpsi' epsi' c' zo' zt zb1 zb2 where run_back: ru_t back = Some (back', t)
  and loop_def: while_break (while_until_cond I t) (while_until_body run_hd (ru n)) loop_init =
Some (vphi', vpsi', epsi', c', zo')
  and v'_def: v' = VYDRA_Until I back' vphi' vpsi' epsi' (c' - 1) zo'
  and c'_pos: ¬c' = 0
  and zo'_Some: zo' = Some (zt, (zb1, zb2))
  and b_def: b = (zb2 ∧ memL t zt I)
  using Until(6)
  apply (auto simp: v_def(1) Let_def loop_init_def[symmetric] split: option.splits nat.splits if_splits)
  done
define j' where j' = i + c'
have j_eq: j = i + c
  using v_def(4)
  by (auto simp: v_def(10))

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have  $t\_def$ :  $t = \tau \sigma i$ 
  using  $ru\_t\_tau$ [ $OF$   $v\_def(5)$   $run\_back$ ]
  by ( $auto$   $simp$ :  $v\_def(6)$ )
define  $loop\_inv$  where  $loop\_inv = (\lambda(vphi, vpsi, epsi, c, zo).$ 
   $let$   $j = i + c$   $in$ 
   $wf\_vydra$   $phi$   $j$   $n$   $vphi$   $\wedge$   $wf\_vydra$   $psi$   $j$   $n$   $vpsi$   $\wedge$ 
   $(\exists gs. reaches\_on$   $ru\_t$   $l\_t0$   $gs$   $epsi$   $\wedge$   $length$   $gs = j$   $\wedge$   $(\forall t. t \in set$   $gs \longrightarrow memR$   $(\tau \sigma i)$   $t$   $I))$   $\wedge$ 
   $zo = (case$   $j$   $of$   $0 \Rightarrow None$   $|$   $Suc$   $k \Rightarrow Some$   $(\tau \sigma k, sat$   $phi$   $k, sat$   $psi$   $k))$   $\wedge$ 
   $(\forall k. k \in \{i..<j - 1\} \longrightarrow sat$   $phi$   $k$   $\wedge$   $(memL$   $(\tau \sigma i)$   $(\tau \sigma k)$   $I \longrightarrow \neg sat$   $psi$   $k)))$ 
have  $loop\_inv\_init$ :  $loop\_inv$   $loop\_init$ 
  using  $v\_def(2,3,7,9,12)$ 
  by ( $auto$   $simp$ :  $loop\_inv\_def$   $loop\_init\_def$   $j\_eq$ [ $symmetric$ ]  $v\_def(8,11)$ )
have  $loop\_step$ :  $loop\_inv$   $s'$  if  $loop\_assms$ :  $loop\_inv$   $s$   $while\_until\_cond$   $I$   $t$   $s$   $while\_until\_body$   $run\_hd$ 
 $(ru$   $n)$   $s = Some$   $s'$  for  $s$   $s'$ 
proof -
  obtain  $vphi\_cur$   $vpsi\_cur$   $epsi\_cur$   $c\_cur$   $zo\_cur$  where  $s\_def$ :  $s = (vphi\_cur, vpsi\_cur, epsi\_cur,$ 
 $c\_cur, zo\_cur)$ 
  by ( $cases$   $s$ )  $auto$ 
  define  $j\_cur$  where  $j\_cur = i + c\_cur$ 
  obtain  $epsi'_cur$   $t'_cur$   $vphi'_cur$   $tphi\_cur$   $bphi\_cur$   $vpsi'_cur$   $tpsi\_cur$   $bpsi\_cur$  where
   $run\_epsi$ :  $ru\_t$   $epsi\_cur = Some$   $(epsi'_cur, t'_cur)$ 
  and  $run\_vphi$ :  $ru$   $n$   $vphi\_cur = Some$   $(vphi'_cur, (tphi\_cur, bphi\_cur))$ 
  and  $run\_vpsi$ :  $ru$   $n$   $vpsi\_cur = Some$   $(vpsi'_cur, (tpsi\_cur, bpsi\_cur))$ 
  using  $loop\_assms(2,3)$ 
  apply ( $auto$   $simp$ :  $while\_until\_cond\_def$   $while\_until\_body\_def$   $s\_def$   $split$ :  $option.splits$   $dest$ :
 $read\_t\_run$ [where  $?run\_hd=run\_hd$ ])
  done
  have  $wf$ :  $wf\_vydra$   $phi$   $j\_cur$   $n$   $vphi\_cur$   $wf\_vydra$   $psi$   $j\_cur$   $n$   $vpsi\_cur$ 
  and  $zo\_cur\_def$ :  $zo\_cur = (case$   $j\_cur$   $of$   $0 \Rightarrow None$   $|$   $Suc$   $k \Rightarrow Some$   $(\tau \sigma k, sat$   $phi$   $k, sat$   $psi$   $k))$ 
  using  $loop\_assms(1)$ 
  by ( $auto$   $simp$ :  $loop\_inv\_def$   $s\_def$   $j\_cur\_def$ [ $symmetric$ ])
  have  $wf'$ :  $wf\_vydra$   $phi$   $(Suc$   $j\_cur)$   $n$   $vphi'_cur$   $tphi\_cur = \tau \sigma j\_cur$   $bphi\_cur = sat$   $phi$   $j\_cur$ 
 $wf\_vydra$   $psi$   $(Suc$   $j\_cur)$   $n$   $vpsi'_cur$   $tpsi\_cur = \tau \sigma j\_cur$   $bpsi\_cur = sat$   $psi$   $j\_cur$ 
  using  $Until(3)$ [ $OF$   $wf(1)$   $run\_vphi$ ]  $Until(4)$ [ $OF$   $wf(2)$   $run\_vpsi$ ]  $Until(7,8)$ 
  apply ( $auto$ )
  done
have  $s'_def$ :  $s' = (vphi'_cur, vpsi'_cur, epsi'_cur, Suc$   $c\_cur, Some$   $(t'_cur, (bphi\_cur, bpsi\_cur)))$ 
  using  $loop\_assms(3)$ 
  by ( $auto$   $simp$ :  $while\_until\_body\_def$   $s\_def$   $run\_epsi$   $run\_vphi$   $run\_vpsi$ )
  obtain  $gs\_cur$  where  $gs\_cur\_def$ :  $reaches\_on$   $ru\_t$   $l\_t0$   $gs\_cur$   $epsi\_cur$ 
 $length$   $gs\_cur = j\_cur$   $\wedge t. t \in set$   $gs\_cur \implies memR$   $(\tau \sigma i)$   $t$   $I$ 
  using  $loop\_assms(1)$ 
  by ( $auto$   $simp$ :  $loop\_inv\_def$   $s\_def$   $j\_cur\_def$ [ $symmetric$ ])
  have  $t'_cur\_def$ :  $t'_cur = \tau \sigma j\_cur$ 
  using  $ru\_t\_tau$ [ $OF$   $gs\_cur\_def(1)$   $run\_epsi$ ]
  by ( $auto$   $simp$ :  $gs\_cur\_def(2)$ )
  have  $t'_cur\_right\_I$ :  $memR$   $t$   $t'_cur$   $I$ 
  using  $loop\_assms(2)$   $run\_t\_read$ [ $OF$   $run\_epsi$ ]
  by ( $auto$   $simp$ :  $while\_until\_cond\_def$   $s\_def$ )
  have  $c\_cur\_zero$ :  $c\_cur = 0 \implies j\_cur = i$ 
  by ( $auto$   $simp$ :  $j\_cur\_def$ )
  have  $k \in \{i..<Suc$   $j\_cur - 1\} \implies sat$   $phi$   $k$   $\wedge$   $(memL$   $(\tau \sigma i)$   $(\tau \sigma k)$   $I \longrightarrow \neg sat$   $psi$   $k)$  for  $k$ 
  using  $loop\_assms(1,2)$ 
  by ( $cases$   $k = j\_cur - Suc$   $0$ ) ( $auto$   $simp$ :  $while\_until\_cond\_def$   $loop\_inv\_def$   $j\_cur\_def$ [ $symmetric$ ]
 $zo\_cur\_def$   $s\_def$   $until\_ready\_def$   $t\_def$   $split$ :  $nat.splits$   $dest$ :  $c\_cur\_zero$ )
  then show  $?thesis$ 
  using  $wf'$   $t'_cur\_right\_I$ 
  using  $reaches\_on\_app$ [ $OF$   $gs\_cur\_def(1)$   $run\_epsi$ ]  $gs\_cur\_def(2,3)$ 

```

```

    by (auto simp: loop_inv_def s'_def j_cur_def[symmetric] t_def t'_cur_def intro!: exI[of _ gs_cur
@ [t'_cur]])
  qed
  have wf_loop: wf {(s', s). loop_inv s ∧ while_until_cond I t s ∧ Some s' = while_until_body run_hd
(ru n) s}
  proof -
    obtain m where m_def: ¬τ σ m ≤ τ σ i + right I
      using ex_lt_τ[where ?x=right I and ?s=σ] Until(7)
      by auto
    define X where X = {(s', s). loop_inv s ∧ while_until_cond I t s ∧ Some s' = while_until_body
run_hd (ru n) s}
    have memR t (τ σ (i + c)) I ⇒ i + c < m for c
      using m_def order_trans[OF τ_mono[where ?i=m and ?j=i + c and ?s=σ]]
      by (fastforce simp: t_def dest!: memR_dest)
    then have X ⊆ measure (λ(vphi, vpsi, epsi, c, zo). m - c)
      by (fastforce simp: X_def while_until_cond_def while_until_body_def loop_inv_def Let_def split:
option.splits
      dest!: read_t_run[where ?run_hd=run_hd] dest: ru_t_tau)
    then show ?thesis
      using wf_subset
      by (auto simp: X_def[symmetric])
  qed
  have loop: loop_inv (vphi', vpsi', epsi', c', zo') ¬while_until_cond I t (vphi', vpsi', epsi', c', zo')
    using while_break_sound[where ?Q=λs. loop_inv s ∧ ¬while_until_cond I t s, OF _ _ wf_loop
loop_inv_init] loop_step
    by (auto simp: loop_def)
  have tau_right_I: k < j' ⇒ memR (τ σ i) (τ σ k) I for k
    using loop(1) ru_t_tau_in
    by (auto simp: loop_inv_def j'_def[symmetric] in_set_conv_nth)
  have read_t_epsi': read_t_epsi' = Some et ⇒ et = τ σ j' for et
    using loop(1) ru_t_tau
    by (fastforce simp: loop_inv_def Let_def j'_def dest!: read_t_run[where ?run_hd=run_hd])
  have end_cond: until_ready I t c' zo' ∨ ¬(memR (τ σ i) (τ σ j') I)
    unfolding t_def[symmetric]
    using Until(6) c'_pos loop(2)[unfolded while_until_cond_def]
    by (auto simp: until_ready_def v_def(1) run_back loop_init_def[symmetric] loop_def zo'_Some
split: if_splits option.splits nat.splits dest: read_t_epsi')
  have Suc_i_le_j': Suc i ≤ j' and c'_j': c' - Suc 0 = j' - Suc i
    using end_cond c'_pos
    by (auto simp: until_ready_def j'_def split: nat.splits)
  have zo'_def: zo' = (case j' of 0 ⇒ None | Suc k ⇒ Some (τ σ k, sat phi k, sat psi k))
    and sat_phi: k ∈ {i..<j' - 1} ⇒ sat phi k
    and not_sat_psi: k ∈ {i..<j' - 1} ⇒ memL (τ σ i) (τ σ k) I ⇒ ¬sat psi k for k
    using loop(1)
    by (auto simp: loop_inv_def j'_def[symmetric])
  have b_sat: b = sat (Until phi I psi) i
  proof (rule iffI)
    assume b: b
    have mem: mem (τ σ i) (τ σ (j' - Suc 0)) I
      using b zo'_def tau_right_I[where ?k=j' - 1]
      by (auto simp: mem_def b_def t_def zo'_Some split: nat.splits)
    have sat_psi: sat psi (j' - 1)
      using b zo'_def
      by (auto simp: b_def zo'_Some split: nat.splits)
    show sat (Until phi I psi) i
      using Suc_i_le_j' mem sat_psi sat_phi
      by (auto intro!: exI[of _ j' - 1])
  next

```

```

assume sat (Until phi I psi) i
then obtain k where k_def: i ≤ k mem (τ σ i) (τ σ k) I sat psi k ∀ m ∈ {i..<k}. sat phi m
  by auto
define X where X = {m ∈ {i..k}. memL (τ σ i) (τ σ m) I ∧ sat psi m}
have fin_X: finite X and X_nonempty: X ≠ {} and k_X: k ∈ X
  using k_def
  by (auto simp: X_def mem_def)
define km where km = Min X
note aux = Min_in[OF fin_X X_nonempty, folded km_def]
have km_def: i ≤ km km ≤ k memL (τ σ i) (τ σ km) I sat psi km ∀ m ∈ {i..<km}. sat phi m
  ∀ m ∈ {i..<km}. memL (τ σ i) (τ σ m) I → ¬sat psi m
  using aux Min_le[OF fin_X, folded km_def] k_def(4)
  by (fastforce simp: X_def)+
have j'_le_km: j' - 1 ≤ km
  using not_sat_psi[OF km_def(3)] km_def(1,4)
  by fastforce
show b
proof (cases j' - 1 < km)
  case True
  have until_ready I t c' zo'
    using end_cond True km_def(2) k_def(2) memR_mono'[OF τ_mono[where ?i=j' and ?j=k
and ?s=σ]]
    by (auto simp: mem_def)
  then show ?thesis
    using c'_pos zo'_def km_def(5) Suc_i_le_j' True
    by (auto simp: until_ready_def zo'_Some b_def split: nat.splits)
  next
  case False
  have km_j': km = j' - 1
    using j'_le_km False
    by auto
  show ?thesis
    using c'_pos zo'_def km_def(3,4)
    by (auto simp: zo'_Some b_def km_j' t_def split: nat.splits)
  qed
qed
obtain gs where gs_def: reaches_on ru_t l_t0 gs epsi' length gs = j'
  ∧ t. t ∈ set gs ⇒ memR (τ σ i) t I
  using loop(1)
  by (auto simp: loop_inv_def j'_def[symmetric])
note aux = τ_mono[where ?s=σ and ?i=i and ?j=Suc i]
have wf_v': wf_vydra (Until phi I psi) (Suc i) (Suc n) v'
  unfolding v'_def
  apply (rule wf_vydra.intros(10)[where ?j=j' and ?es'=gs])
  using loop(1) reaches_on_app[OF v_def(5) run_back] Suc_i_le_j' c'_j' memL_mono[OF aux]
  memR_mono[OF aux] gs_def
  by (auto simp: loop_inv_def j'_def[symmetric] v_def(6,8))
show ?case
  using wf_v' ru_t_event[OF v_def(5) refl run_back]
  by (auto simp: b_sat v_def(6))
next
case (MatchP n I r)
have IH: x ∈ set (collect_subfmlas r []) ⇒ wf_vydra x j n v ⇒ ru n v = Some (v', t, b) ⇒ wf_vydra
  x (Suc j) n v' ∧ t = τ σ j ∧ b = sat x j for x j v v' t b
  using MatchP(2,1,5,6) le_trans[OF collect_subfmlas_msize]
  using bf_collect_subfmlas[where ?r=r and ?phis=[]]
  by (fastforce simp: collect_subfmlas_atms[where ?phis=[], simplified, symmetric])
have reaches_on (ru n) (su n phi) vs v ⇒ wf_vydra phi (length vs) n v if phi: phi ∈ set (collect_subfmlas

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r [] for phi vs v
  apply (induction vs arbitrary: v rule: rev_induct)
  using MatchP(1) wf_vydra_sub collect_subfmlas_msize[OF phi]
  apply (auto elim!: reaches_on.cases)[1]
  using IH[OF phi]
  apply (fastforce dest!: reaches_on_split_last)
  done
  then have wf: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) bs s  $\implies$  j < length
  (collect_subfmlas r [])  $\implies$ 
    wf_vydra (collect_subfmlas r [] ! j) (length bs) n (s ! j) for bs s j
    using reach_run_subs_run
    by (fastforce simp: in_set_conv_nth)
  let ?qf = state_cnt r
  let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
  define args where args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t,
  read_t) (run_subs (ru n))
  interpret MDL_window  $\sigma$  r l_t0 map (su n) (collect_subfmlas r []) args
  using bs_sat[where ?r=r and ?n=n, OF wf_reach_run_subs_len[where ?n=n]] IH run_t_read[of
  run_hd]
  read_t_run[of __ run_hd] ru_t_tau MatchP(5) collect_subfmlas_atms[where ?phis=[]]
  unfolding args_def iarray_of_list_def
  by unfold_locales auto
  obtain w xs where w_def: v = VYDRA_MatchP I ?transs ?qf w
  valid_window_matchP args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w
  using MatchP(3,4)
  by (auto simp: args_def elim!: wf_vydra.cases[of __ v])
  obtain tj' t' sj' bs where somes: w_run_t args (w_tj w) = Some (tj', t')
  w_run_sub args (w_sj w) = Some (sj', bs)
  using MatchP(4)
  by (auto simp: w_def(1) adv_end_def args_def Let_def split: option.splits prod.splits)
  obtain w' where w'_def: eval_mP I w = Some (( $\tau$   $\sigma$  i, sat (MatchP I r) i), w')
  t' =  $\tau$   $\sigma$  i valid_matchP I l_t0 (map (su n) (collect_subfmlas r [])) (xs @ [(t', bs)]) (Suc i) w'
  using valid_eval_matchP[OF w'_def(2) somes] MatchP(6)
  by auto
  have v'_def: v' = VYDRA_MatchP I ?transs ?qf w' (t, b) = ( $\tau$   $\sigma$  i, sat (MatchP I r) i)
  using MatchP(4)
  by (auto simp: w_def(1) args_def[symmetric] w'_def(1) simp del: eval_matchP.simps init_args.simps)
  have len_xs: length xs = i
  using w'_def(3)
  by (auto simp: valid_window_matchP_def)
  have  $\exists$  es e. reaches_on run_hd init_hd es e  $\wedge$  length es = Suc i
  using ru_t_event valid_window_matchP_reach_tj[OF w'_def(2)] somes(1) len_xs
  by (fastforce simp: args_def)
  then show ?case
  using MatchP(1) v'_def(2) w'_def(3)
  by (auto simp: v'_def(1) args_def[symmetric] simp del: init_args.simps intro!: wf_vydra.intros(11))
next
case (MatchF n I r)
  have IH:  $x \in \text{set } (\text{collect\_subfmlas } r \ []) \implies \text{wf\_vydra } x j n v \implies \text{ru } n v = \text{Some } (v', t, b) \implies \text{wf\_vydra}$ 
   $x (\text{Suc } j) n v' \wedge t = \tau \sigma j \wedge b = \text{sat } x j$  for  $x j v v' t b$ 
  using MatchF(2,1,5,6) le_trans[OF collect_subfmlas_msize]
  using bf_collect_subfmlas[where ?r=r and ?phis=[]]
  by (fastforce simp: collect_subfmlas_atms[where ?phis=[], simplified, symmetric])
  have reaches_on (ru n) (su n phi) vs v  $\implies$  wf_vydra phi (length vs) n v if phi: phi  $\in$  set (collect_subfmlas
  r []) for phi vs v
  apply (induction vs arbitrary: v rule: rev_induct)
  using MatchF(1) wf_vydra_sub collect_subfmlas_msize[OF phi]
  apply (auto elim!: reaches_on.cases)[1]

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```

using IH[OF phi]
apply (fastforce dest!: reaches_on_split_last)
done
then have wf: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) bs s  $\implies$  j < length
(collect_subfmlas r [])  $\implies$ 
  wf_vydra (collect_subfmlas r [] ! j) (length bs) n (s ! j) for bs s j
  using reach_run_subs_run
  by (fastforce simp: in_set_conv_nth)
let ?qf = state_cnt r
let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
define args where args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t,
read_t) (run_subs (ru n))
interpret MDL_window  $\sigma$  r l_t0 map (su n) (collect_subfmlas r []) args
  using bs_sat[where ?r=r and ?n=n, OF_wf_reach_run_subs_len[where ?n=n]] IH run_t_read[of
run_hd]
  read_t_run[of __ run_hd] ru_t_tau MatchF(5) collect_subfmlas_atms[where ?phis=[]]
  unfolding args_def iarray_of_list_def
  by unfold_locales auto
obtain w xs where w_def: v = VYDRA_MatchF I ?transs ?qf w
  valid_window_matchF args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w
  using MatchF(3,4)
  by (auto simp: args_def elim!: wf_vydra.cases[of __ __ v])
obtain w' rho' where w'_def: eval_mF I w = Some ((t, b), w') valid_matchF I l_t0 (map (su n)
(collect_subfmlas r [])) rho' (Suc i) w' t =  $\tau$   $\sigma$  i b = sat (MatchF I r) i
  using valid_eval_matchF_sound[OF w_def(2)] MatchF(4,5,6)
  by (fastforce simp: w_def(1) args_def[symmetric] simp del: eval_matchF.simps init_args.simps split:
option.splits)
have v'_def: v' = VYDRA_MatchF I ?transs ?qf w'
  using MatchF(4)
  by (auto simp: w_def(1) args_def[symmetric] w'_def(1) simp del: eval_matchF.simps init_args.simps)
obtain w_ti' ti where ru_w_ti: ru_t (w_ti w) = Some (w_ti', ti)
  using MatchF(4) read_t_run
  by (auto simp: w_def(1) args_def split: option.splits)
have  $\exists$  es e. reaches_on run_hd init_hd es e  $\wedge$  length es = Suc i
  using w_def(2) ru_t_event[OF_refl ru_w_ti, where ?ts=take (w_i w) (map fst xs)]
  by (auto simp: valid_window_matchF_def args_def)
then show ?case
  using MatchF(1) w'_def(2)
  by (auto simp: v'_def(1) args_def[symmetric] w'_def(3,4) simp del: init_args.simps intro!: wf_vydra.intros(12))
qed

```

```

lemma reaches_ons_run_lD: reaches_on (run_subs (ru n)) vs ws vs'  $\implies$ 
length vs = length vs'
by (induction vs ws vs' rule: reaches_on_rev_induct)
  (auto simp: run_subs_def Let_def split: option.splits if_splits)

```

```

lemma reaches_ons_run_vD: reaches_on (run_subs (ru n)) vs ws vs'  $\implies$ 
i < length vs  $\implies$  ( $\exists$  ys. reaches_on (ru n) (vs ! i) ys (vs' ! i)  $\wedge$  length ys = length ws)

```

```

proof (induction vs ws vs' rule: reaches_on_rev_induct)

```

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case (2 s s' bs bss s'')

```

```

obtain ys where ys_def: reaches_on (ru n) (s ! i) ys (s' ! i)

```

```

length s = length s' length ys = length bss

```

```

using reaches_ons_run_lD[OF 2(1)] 2(3)[OF 2(4)]

```

```

by auto

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obtain tb where tb_def: ru n (s' ! i) = Some (s'' ! i, tb)

```

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using run_subs_vD[OF 2(2)] 2(4)[unfolded ys_def(2)]

```

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by auto

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```

show ?case

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    using reaches_on_app[OF ys_def(1) tb_def(1)] ys_def(2,3) tb_def
    by auto
qed (auto intro: reaches_on.intros)

lemma reaches_on_runI:
  assumes  $\bigwedge \phi. \phi \in \text{set} (\text{collect\_subfmls } r \ []) \implies \exists ws\ v. \text{reaches\_on } (ru\ n) (su\ n\ \phi) ws\ v \wedge \text{length } ws = i$ 
  shows  $\exists ws\ v. \text{reaches\_on } (\text{run\_subs } (ru\ n)) (\text{map } (su\ n) (\text{collect\_subfmls } r \ [])) ws\ v \wedge \text{length } ws = i$ 
  using assms
proof (induction i)
  case 0
  show ?case
  by (fastforce intro: reaches_on.intros)
next
  case (Suc i)
  have IH':  $\bigwedge \phi. \phi \in \text{set} (\text{collect\_subfmls } r \ []) \implies \exists ws\ v. \text{reaches\_on } (ru\ n) (su\ n\ \phi) ws\ v \wedge \text{length } ws = i \wedge ru\ n\ v \neq \text{None}$ 
  proof -
    fix phi
    assume lassm:  $\phi \in \text{set} (\text{collect\_subfmls } r \ [])$ 
    obtain ws v where ws_def:  $\text{reaches\_on } (ru\ n) (su\ n\ \phi) ws\ v$ 
      length ws = Suc i
      using Suc(2)[OF lassm]
      by auto
    obtain y ys where ws_snoc:  $ws = ys @ [y]$ 
      using ws_def(2)
      by (cases ws rule: rev_cases) auto
    show  $\exists ws\ v. \text{reaches\_on } (ru\ n) (su\ n\ \phi) ws\ v \wedge \text{length } ws = i \wedge ru\ n\ v \neq \text{None}$ 
      using reaches_on_split_last[OF ws_def(1)[unfolded ws_snoc]] ws_def(2) ws_snoc
      by fastforce
  qed
  obtain ws v where ws_def:  $\text{reaches\_on } (\text{run\_subs } (ru\ n)) (\text{map } (su\ n) (\text{collect\_subfmls } r \ [])) ws\ v$ 
    length ws = i
    using Suc(1) IH'
    by blast
  have  $x \in \text{set } v \implies \text{Option.is\_none } (ru\ n\ x) \implies \text{False}$  for x
    using ws_def IH'
    by (auto simp: in_set_conv_nth) (metis is_none_code(2) reach_run_subs_len reach_run_subs_run reaches_on_inj)
  then obtain v' tb where v'_def:  $\text{run\_subs } (ru\ n) v = \text{Some } (v', tb)$ 
    by (fastforce simp: run_subs_def Let_def)
  show ?case
    using reaches_on_app[OF ws_def(1) v'_def] ws_def(2)
    by fastforce
qed

lemma reaches_on_takeWhile:  $\text{reaches\_on } r\ s\ vs\ s' \implies r\ s' = \text{Some } (s'', v) \implies \neg f\ v \implies vs' = \text{takeWhile } f\ vs \implies \exists t'\ t''\ v'. \text{reaches\_on } r\ s\ vs'\ t' \wedge r\ t' = \text{Some } (t'', v') \wedge \neg f\ v' \wedge \text{reaches\_on } r\ t' (\text{drop } (\text{length } vs')\ vs)\ s'$ 
  by (induction s vs s' arbitrary: vs' rule: reaches_on.induct) (auto intro: reaches_on.intros)

lemma reaches_on_suffix:
  assumes  $\text{reaches\_on } r\ s\ vs\ s' \text{ reaches\_on } r\ s\ vs'\ s'' \text{ length } vs' \leq \text{length } vs$ 
  shows  $\exists vs''. \text{reaches\_on } r\ s''\ vs''\ s' \wedge vs = vs' @ vs''$ 
  using reaches_on_split'[where ?i=length vs', OF assms(1,3)] assms(3) reaches_on_inj[OF assms(2)]
  by (metis add_diff_cancel_right' append_take_drop_id diff_diff_cancel length_append length_drop)

```

**lemma** *vydra\_wf\_reaches\_on*:  
**assumes**  $\bigwedge j v. j < i \implies wf\_vydra\ \varphi\ j\ n\ v \implies ru\ n\ v = None \implies False$  *bounded\_future\_fmula*  $\varphi$   
*wf\_fmula*  $\varphi$  *msize\_fmula*  $\varphi \leq n$   
**shows**  $\exists vs v. reaches\_on\ (ru\ n)\ (su\ n\ \varphi)\ vs\ v \wedge wf\_vydra\ \varphi\ i\ n\ v \wedge length\ vs = i$   
**using** *assms(1)*  
**proof** (*induction i*)  
**case** (*Suc i*)  
**obtain** *vs v* **where** *IH: reaches\_on (ru n) (su n  $\varphi$ ) vs v wf\_vydra  $\varphi$  i n v length vs = i*  
**using** *Suc(1) Suc(2)[OF less\_SucI]*  
**by** *auto*  
**show** *?case*  
**using** *reaches\_on\_app[OF IH(1)] Suc(2)[OF \_ IH(2)] vydra\_sound\_aux[OF assms(4) IH(2) \_ assms(2,3)] IH(3)*  
**by** *fastforce*  
**qed** (*auto intro: reaches\_on.intros wf\_vydra\_sub[OF assms(4)]*)

**lemma** *reaches\_on\_Some*:  
**assumes** *reaches\_on r s vs s' reaches\_on r s vs' s'' length vs < length vs'*  
**shows**  $\exists s''' x. r\ s' = Some\ (s''', x)$   
**using** *reaches\_on\_split[OF assms(2,3)] reaches\_on\_inj[OF assms(1)] assms(3)*  
**by** *auto*

**lemma** *reaches\_on\_progress*:  
**assumes** *reaches\_on run\_hd init\_hd vs e*  
**shows** *progress phi (map fst vs)  $\leq$  length vs*  
**using** *progress\_le\_ts[of map fst vs phi] reaches\_on\_run\_hd  $\tau\_fin$*   
**by** (*fastforce dest!: reaches\_on\_setD[OF assms] reaches\_on\_split\_last*)

**lemma** *vydra\_complete\_aux*:  
**assumes** *prefix: reaches\_on run\_hd init\_hd vs e*  
**and** *run: wf\_vydra  $\varphi$  i n v ru n v = None i < progress  $\varphi$  (map fst vs) bounded\_future\_fmula  $\varphi$  wf\_fmula*  
 $\varphi$   
**and** *msize: msize\_fmula  $\varphi \leq n$*   
**shows** *False*  
**using** *msize run*  
**proof** (*induction n  $\varphi$  arbitrary; i v rule: run\_induct*)  
**case** (*Bool b n*)  
**have** *False if v\_def: v = VYDRA\_Bool b g for g*  
**proof** –  
**obtain** *es* **where** *g\_def: reaches\_on run\_hd init\_hd es g length es = i*  
**using** *Bool(1)*  
**by** (*auto simp: v\_def elim: wf\_vydra.cases*)  
**show** *False*  
**using** *Bool(2) reaches\_on\_Some[OF g\_def(1) prefix] Bool(3)*  
**by** (*auto simp: v\_def g\_def(2) split: option.splits*)  
**qed**  
**then show** *False*  
**using** *Bool(1)*  
**by** (*auto elim!: wf\_vydra.cases[of \_ \_ \_ v]*)  
**next**  
**case** (*Atom a n*)  
**have** *False if v\_def: v = VYDRA\_Atom a g for g*  
**proof** –  
**obtain** *es* **where** *g\_def: reaches\_on run\_hd init\_hd es g length es = i*  
**using** *Atom(1)*  
**by** (*auto simp: v\_def elim: wf\_vydra.cases*)  
**show** *False*  
**using** *Atom(2) reaches\_on\_Some[OF g\_def(1) prefix] Atom(3)*

```

      by (auto simp: v_def g_def(2) split: option.splits)
qed
then show False
  using Atom(1)
  by (auto elim!: wf_vydra.cases[of _ _ _ v])
next
case (Neg n phi)
show ?case
  apply (rule wf_vydra.cases[OF Neg(3)])
  using Neg
  by (fastforce split: option.splits)+
next
case (Bin n f phi psi)
show ?case
  apply (rule wf_vydra.cases[OF Bin(4)])
  using Bin
  by (fastforce split: option.splits)+
next
case (Prev n I phi)
show ?case
proof (cases i)
  case 0
  obtain vphi where v_def: v = VYDRA_Prev I vphi init_hd None
    using Prev(3)
  by (auto simp: 0 dest: reaches_on_NilD elim!: wf_vydra.cases[of Prev I phi])
  show ?thesis
    using Prev(4,5) prefix
    by (auto simp: 0 v_def elim: reaches_on.cases split: option.splits)
next
case (Suc j)
show ?thesis
proof (cases v = VYDRA_None)
  case v_def: True
  obtain w where w_def: wf_vydra phi j n w ru n w = None
    using Prev(3)
  by (auto simp: Suc v_def elim!: wf_vydra.cases[of Prev I phi])
  show ?thesis
    using Prev(2)[OF w_def(1,2)] Prev(5,6,7)
    by (auto simp: Suc)
next
case False
  obtain vphi tphi bphi es g where v_def: v = VYDRA_Prev I vphi g (Some (tphi, bphi))
    wf_vydra phi (Suc j) n vphi reaches_on_run_hd init_hd es g length es = Suc j
    using Prev(3) False
  by (auto simp: Suc elim!: wf_vydra.cases[of Prev I phi])
  show ?thesis
    using Prev(4,5) reaches_on_Some[OF v_def(3) prefix]
    by (auto simp: Let_def Suc v_def(1,4) split: option.splits)
qed
qed
next
case (Next n I phi)
show ?case
proof (cases v = VYDRA_None)
  case True
  obtain w where w_def: wf_vydra phi i n w ru n w = None
    using Next(3)
  by (auto simp: True elim: wf_vydra.cases)

```

```

show ?thesis
  using Next(2)[OF w_def] Next(5,6,7)
  by (auto split: nat.splits)
next
case False
obtain w sub to es where v_def: v = VYDRA_Next I w sub to wf_vydra phi (length es) n w
  reaches_on run_hd init_hd es sub length es = (case to of None  $\Rightarrow$  0 | _  $\Rightarrow$  Suc i)
  case to of None  $\Rightarrow$  i = 0 | Some told  $\Rightarrow$  told =  $\tau$   $\sigma$  i
  using Next(3) False
  by (auto elim!: wf_vydra.cases[of _ _ _ v] split: option.splits nat.splits)
show ?thesis
proof (cases to)
  case None
  obtain w' tw' bw' where w'_def: ru n w = Some (w', (tw', bw'))
  using Next(2)[OF v_def(2)] Next(5,6,7)
  by (fastforce simp: v_def(4) None split: nat.splits)
  have wf: wf_vydra phi (Suc (length es)) n w'
  using v_def(4,5) vydra_sound_aux[OF Next(1) v_def(2) w'_def] Next(6,7)
  by (auto simp: None)
  show ?thesis
  using Next(2)[OF wf] Next(4,5,6,7) reaches_on_Some[OF v_def(3) prefix]
    reaches_on_Some[OF reaches_on_app[OF v_def(3)] prefix] reaches_on_progress[OF prefix,
where ?phi=phi]
  by (cases vs) (fastforce simp: v_def(1,4) w'_def None split: option.splits nat.splits if_splits)+
  next
  case (Some z)
  show ?thesis
  using Next(2)[OF v_def(2)] Next(4,5,6,7) reaches_on_Some[OF v_def(3) prefix] reaches_on_progress[OF
prefix, where ?phi=phi]
  by (auto simp: v_def(1,4) Some split: option.splits nat.splits)
  qed
qed
next
case (Since n I phi psi)
obtain vphi vpsi g cphi cpsi cpsi tpsi j gs where v_def:
  v = VYDRA_Since I vphi vpsi g cphi cpsi cpsi tpsi
  wf_vydra phi i n vphi wf_vydra psi j n vpsi j  $\leq$  i
  reaches_on ru_t l_t0 gs g length gs = j cpsi = i - j
  using Since(5)
  by (auto elim: wf_vydra.cases)
obtain vphi' t1 b1 where run_vphi: ru n vphi = Some (vphi', t1, b1)
  using Since(3)[OF v_def(2)] Since(7,8,9)
  by fastforce
obtain cs c where wf_vphi': wf_vydra phi (Suc i) n vphi'
  and reaches_Suc_i: reaches_on run_hd init_hd cs c length cs = Suc i
  and t1_def: t1 =  $\tau$   $\sigma$  i
  using vydra_sound_aux[OF Since(1) v_def(2) run_vphi] Since(8,9)
  by auto
note ru_t_Some = ru_t_Some[OF reaches_Suc_i]
define loop_inv where loop_inv = ( $\lambda$ (vpsi, e, cpsi :: nat, cpsi :: nat option, tpsi :: 't option).
  let j = Suc i - cpsi in cpsi  $\leq$  Suc i  $\wedge$  wf_vydra psi j n vpsi  $\wedge$  ( $\exists$  es. reaches_on ru_t l_t0 es e  $\wedge$ 
length es = j))
define loop_init where loop_init = (vpsi, g, Suc cpsi, map_option Suc cpsi, tpsi)
have j_def: j = i - cpsi and cpsi_i: cpsi  $\leq$  i
  using v_def(4,7)
  by auto
then have loop_inv_init: loop_inv loop_init
  using v_def(3,5,6,7) last_before_Some

```

```

    by (fastforce simp: loop_inv_def loop_init_def Let_def j_def split: option.splits)
  have wf_loop: wf {(s', s). loop_inv s ∧ while_since_cond I t1 s ∧ Some s' = while_since_body run_hd
(ru n) s}
    by (auto intro: wf_subset[OF wf_since])
  have step_loop: pred_option' loop_inv (while_since_body run_hd (ru n) s)
    if loop_assms: loop_inv s while_since_cond I t1 s for s
  proof -
    obtain vpsi d cpsi cpsi tpsi where s_def: s = (vpsi, d, cpsi, cpsi, tpsi)
      by (cases s) auto
    have cpsi_pos: cpsi > 0
      using loop_assms(2)
    by (auto simp: while_since_cond_def s_def)
    define j where j = Suc i - cpsi
    have j_i: j ≤ i
      using cpsi_pos
    by (auto simp: j_def)
    obtain ds where loop_before: cpsi ≤ Suc i wf_vydra psi j n vpsi reaches_on ru_t l_t0 ds d length
ds = j
      using loop_assms(1)
    by (auto simp: s_def j_def loop_inv_def Let_def)
    obtain h tt where tt_def: read_t d = Some tt d = Some (h, tt)
      using ru_t_Some[OF loop_before(3)] loop_before(4) loop_assms(2)
    by (cases d) (fastforce simp: while_since_cond_def s_def j_def split: option.splits)+
    obtain d' where d'_def: reaches_on ru_t l_t0 (ds @ [tt]) d' ru_t d = Some (d', tt)
      using reaches_on_app[OF loop_before(3)] tt_def(1)
    by (cases run_hd h) (auto simp: tt_def(2))
    obtain vpsi' tpsi' bpsi' where run_vpsi: ru n vpsi = Some (vpsi', (tpsi', bpsi'))
      using Since(4) j_i Since(7,8,9) loop_before(2)
    by fastforce
    have wf_psi': wf_vydra psi (Suc j) n vpsi' and t'_def: tpsi' = τ σ j and b'_def: bpsi' = sat psi j
      using vydra_sound_aux[OF Since(2) loop_before(2) run_vpsi] Since(8,9)
    by auto
    define j' where j'_def: j' = Suc i - (cpsi - Suc 0)
    have j'_j: j' = Suc j
      using loop_before(1) cpsi_pos
    by (auto simp: j'_def j_def)
    show ?thesis
      using wf_psi'(1) loop_before(1,4) d'_def(1)
    by (fastforce simp: while_since_body_def s_def run_vpsi pred_option'_def loop_inv_def j'_def[symmetric]
j'_j d'_def(2) t1_def)
    qed
    show ?case
      using while_break_complete[OF step_loop _ wf_loop loop_inv_init, where ?Q=λ_. True] Since(6)
    by (auto simp: pred_option'_def v_def(1) run_vphi Let_def loop_init_def split: option.splits)
  next
  case (Until n I phi psi)
  obtain back vphi vpsi front c z es es' j where v_def:
    v = VYDRA_Until I back vphi vpsi front c z
    wf_vydra phi j n vphi wf_vydra psi j n vpsi i ≤ j
    reaches_on ru_t l_t0 es back length es = i
    reaches_on ru_t l_t0 es' front length es' = j ∧ t. t ∈ set es' ⇒ memR (τ σ i) t I
    c = j - i z = (case j of 0 ⇒ None | Suc k ⇒ Some (τ σ k, sat phi k, sat psi k))
    ∧ k. k ∈ {i..<j - 1} ⇒ sat phi k ∧ (memL (τ σ i) (τ σ k) I → ¬sat psi k)
      using Until(5)
    by (auto simp: elim: wf_vydra.cases)
  have ru_t_Some: reaches_on ru_t l_t0 gs g ⇒ length gs < length vs ⇒ ∃ g' gt. ru_t g = Some (g',
gt) for gs g
      using reaches_on_Some reaches_on_run_hd_t[OF prefix]

```

```

  by fastforce
have vs_tau: map fst vs ! k = τ σ k if k_vs: k < length vs for k
  using reaches_on_split[OF prefix k_vs] run_hd_sound k_vs
  apply (cases vs ! k)
  apply (auto)
  apply (metis fst_conv length_map nth_map prefix reaches_on_run_hd_t ru_t_tau_in)
done
define m where m = min (length (map fst vs) - 1) (min (progress phi (map fst vs)) (progress psi
(map fst vs)))
have m_vs: m < length vs
  using Until(7)
  by (cases vs) (auto simp: m_def split: if_splits)
define A where A = {j. 0 ≤ j ∧ j ≤ m ∧ memR (map fst vs ! j) (map fst vs ! m) I}
have m_A: m ∈ A
  using memR_tfin_refl[OF τ_fin] vs_tau[OF m_vs]
  by (fastforce simp: A_def)
then have A_nonempty: A ≠ {}
  by auto
have A_finite: finite A
  by (auto simp: A_def)
have p: progress (Until phi I psi) (map fst vs) = Min A
  using Until(7)
  unfolding progress.simps m_def[symmetric] Let_def A_def[symmetric]
  by auto
have i_A: i ∉ A
  using Until(7) A_finite A_nonempty
  by (auto simp del: progress.simps simp: p)
have i_m: i < m
  using Until(7) m_A A_finite A_nonempty
  by (auto simp del: progress.simps simp: p)
have memR_i_m: ¬memR (map fst vs ! i) (map fst vs ! m) I
  using i_A i_m
  by (auto simp: A_def)
have i_vs: i < length vs
  using i_m m_vs
  by auto
have j_m: j ≤ m
  using ru_t_tau_in[OF v_def(7), of m] v_def(9)[of τ σ m] memR_i_m
  unfolding vs_tau[OF i_vs] vs_tau[OF m_vs]
  by (force simp: in_set_conv_nth v_def(8))
have j_vs: j < length vs
  using j_m m_vs
  by auto
obtain back' t where run_back: ru_t back = Some (back', t) and t_def: t = τ σ i
  using ru_t_Some[OF v_def(5)] v_def(4) j_vs ru_t_tau[OF v_def(5)]
  by (fastforce simp: v_def(6))
define loop_inv where loop_inv = (λ(vphi, vpsi, front, c, z :: ('t × bool × bool) option).
  let j = i + c in wf_vydra phi j n vphi ∧ wf_vydra psi j n vpsi ∧ j < length vs ∧
  (∃ es'. reaches_on ru_t l_t0 es' front ∧ length es' = j) ∧
  (z = None → j = 0))
define loop_init where loop_init = (vphi, vpsi, front, c, z)
have j_eq: j = i + c
  using v_def(4)
  by (auto simp: v_def(10))
have j = 0 ⇒ c = 0
  by (auto simp: j_eq)
then have loop_inv_init: loop_inv loop_init
  using v_def(2,3,4,7,8,9,11) j_vs

```

```

    by (auto simp: loop_inv_def loop_init_def j_eq[symmetric] split: nat.splits)
  have loop_step: pred_option' loop_inv (while_until_body run_hd (ru n) s) if loop_assms: loop_inv s
  while_until_cond I t s for s
  proof -
    obtain vphi_cur vpsi_cur epsi_cur c_cur zo_cur where s_def: s = (vphi_cur, vpsi_cur, epsi_cur,
  c_cur, zo_cur)
    by (cases s) auto
    define j_cur where j_cur = i + c_cur
    obtain gs where wf: wf_vydra phi j_cur n vphi_cur wf_vydra psi j_cur n vpsi_cur
    and gs_def: reaches_on ru_t l_t0 gs epsi_cur length gs = j_cur
    and j_cur_vs: j_cur < length vs
    using loop_assms(1)
    by (auto simp: loop_inv_def s_def j_cur_def[symmetric])
    obtain epsi'_cur t'_cur where run_epsi: ru_t epsi_cur = Some (epsi'_cur, t'_cur)
    and t'_cur_def: t'_cur =  $\tau$   $\sigma$  (length gs)
    using ru_t_Some[OF gs_def(1)] ru_t_event[OF gs_def(1) refl] j_cur_vs
    by (auto simp: gs_def(2))
    have j_m: j_cur < m
    using loop_assms(2) memR_i_m memR_mono'[OF  $\tau$ _mono, of _ _ _ _ m]
    unfolding vs_tau[OF i_vs] vs_tau[OF m_vs]
    by (fastforce simp: gs_def(2) while_until_cond_def s_def run_t_read[OF run_epsi] t_def t'_cur_def)
    have j_cur_prog_phi: j_cur < progress phi (map fst vs)
    using j_m
    by (auto simp: m_def)
    have j_cur_prog_psi: j_cur < progress psi (map fst vs)
    using j_m
    by (auto simp: m_def)
    obtain vphi'_cur tphi_cur bphi_cur where run_vphi: ru n vphi_cur = Some (vphi'_cur, (tphi_cur,
  bphi_cur))
    using Until(3)[OF wf(1) _ j_cur_prog_phi] Until(8,9)
    by fastforce
    obtain vpsi'_cur tpsi_cur bpsi_cur where run_vpsi: ru n vpsi_cur = Some (vpsi'_cur, (tpsi_cur,
  bpsi_cur))
    using Until(4)[OF wf(2) _ j_cur_prog_psi] Until(8,9)
    by fastforce
    have wf': wf_vydra phi (Suc j_cur) n vphi'_cur wf_vydra psi (Suc j_cur) n vpsi'_cur
    using vydra_sound_aux[OF Until(1) wf(1) run_vphi] vydra_sound_aux[OF Until(2) wf(2)
  run_vpsi] Until(8,9)
    by auto
    show ?thesis
    using wf' reaches_on_app[OF gs_def(1) run_epsi] gs_def(2) j_m m_vs
    by (auto simp: pred_option'_def while_until_body_def s_def run_epsi run_vphi run_vpsi loop_inv_def
  j_cur_def[symmetric])
  qed
  have wf_loop: wf {(s', s). loop_inv s  $\wedge$  while_until_cond I t s  $\wedge$  Some s' = while_until_body run_hd
  (ru n) s}
  proof -
    obtain m where m_def:  $\neg \tau$   $\sigma$  m  $\leq$   $\tau$   $\sigma$  i + right I
    using ex_lt_tau[where ?x=right I and ?s= $\sigma$ ] Until(8)
    by auto
    define X where X = {(s', s). loop_inv s  $\wedge$  while_until_cond I t s  $\wedge$  Some s' = while_until_body
  run_hd (ru n) s}
    have memR t ( $\tau$   $\sigma$  (i + c)) I  $\implies$  i + c < m for c
    using m_def order_trans[OF  $\tau$ _mono[where ?i=m and ?j=i + c and ?s= $\sigma$ ]]
    by (fastforce simp: t_def dest!: memR_dest)
    then have X  $\subseteq$  measure ( $\lambda$ (vphi, vpsi, epsi, c, zo). m - c)
    by (fastforce simp: X_def while_until_cond_def while_until_body_def loop_inv_def Let_def split:
  option.splits)

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```

    dest!: read_t_run[where ?run_hd=run_hd] dest: ru_t_tau)
then show ?thesis
  using wf_subset
  by (auto simp: X_def[symmetric])
qed
obtain vphi' vpsi' front' c' z' where loop:
  while_break (while_until_cond I t) (while_until_body run_hd (ru n)) loop_init = Some (vphi', vpsi',
front', c', z')
  loop_inv (vphi', vpsi', front', c', z')  $\neg$ while_until_cond I t (vphi', vpsi', front', c', z')
  using while_break_complete[where ?P=loop_inv, OF loop_step_wf_loop loop_inv_init]
  by (cases while_break (while_until_cond I t) (while_until_body run_hd (ru n)) loop_init) (force
simp: pred_option'_def)+
  define j' where j' = i + c'
  obtain tf where read_front': read_t front' = Some tf
  using loop(2)
  by (auto simp: loop_inv_def j'_def[symmetric] dest!: ru_t_Some run_t_read[where ?run_hd=run_hd])
  have tf_fin: tf  $\in$  tfin
  using loop(2) ru_t_Some[where ?g=front'] ru_t_tau[where ?t'=front'] read_t_run[OF read_front']
 $\tau$ _fin[where ? $\sigma$ = $\sigma$ ]
  by (fastforce simp: loop_inv_def j'_def[symmetric])
  have c'_pos: c' = 0  $\implies$  False
  using loop(2,3) ru_t_tau ru_t_tau[OF reaches_on.intros(1)] read_t_run[OF read_front']
  memR_tfin_refl[OF tf_fin]
  by (fastforce simp: loop_inv_def while_until_cond_def until_ready_def read_front'_t_def dest!:
reaches_on_NilD)
  have z'_Some: z' = None  $\implies$  False
  using loop(2) c'_pos
  by (auto simp: loop_inv_def j'_def[symmetric])
  show ?case
  using Until(6) c'_pos z'_Some
  by (auto simp: v_def(1) run_back loop_init_def[symmetric] loop(1) read_front'_split: if_splits op-
tion.splits)
next
  case (MatchP n I r)
  have msize_sub:  $\bigwedge x. x \in \text{set}(\text{collect\_subfmlas } r \ []) \implies \text{msize\_fmla } x \leq n$ 
  using le_trans[OF collect_subfmlas_msize] MatchP(1)
  by auto
  have sound:  $x \in \text{set}(\text{collect\_subfmlas } r \ []) \implies \text{wf\_vydra } x \ j \ n \ v \implies \text{ru } n \ v = \text{Some}(v', t, b) \implies$ 
wf_vydra x (Suc j) n v'  $\wedge$  t =  $\tau$   $\sigma$  j  $\wedge$  b = sat x j for x j v v' t b
  using MatchP vydra_sound_aux[OF msize_sub] le_trans[OF collect_subfmlas_msize]
  using bf_collect_subfmlas[where ?r=r and ?phis=[]]
  by (fastforce simp: collect_subfmlas_atms[where ?phis=[], simplified, symmetric])
  have reaches_on (ru n) (su n phi) vs v  $\implies$  wf_vydra phi (length vs) n v if phi: phi  $\in$  set (collect_subfmlas
r []) for phi vs v
  apply (induction vs arbitrary: v rule: rev_induct)
  using MatchP(1) wf_vydra_sub collect_subfmlas_msize[OF phi]
  apply (auto elim!: reaches_on.cases)[1]
  using sound[OF phi]
  apply (fastforce dest!: reaches_on_split_last)
  done
  then have wf: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) bs s  $\implies$  j < length
(collect_subfmlas r [])  $\implies$ 
  wf_vydra (collect_subfmlas r [] ! j) (length bs) n (s ! j) for bs s j
  using reach_run_subs_run
  by (fastforce simp: in_set_conv_nth)
  let ?qf = state_cnt r
  let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
  define args where args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t,

```

```

read_t) (run_subs (ru n))
  interpret MDL_window  $\sigma$  r l_t0 map (su n) (collect_subfmlas r []) args
    using bs_sat[where ?r=r and ?n=n, OF _ wf _ reach_run_subs_len[where ?n=n]] sound
run_t_read[of run_hd]
  read_t_run[of _ _ run_hd] ru_t_tau MatchP(5) collect_subfmlas_atms[where ?phis=[]]
  unfolding args_def iarray_of_list_def
  by unfold_locales auto
obtain w xs where w_def: v = VYDRA_MatchP I ?transs ?qf w
  valid_window_matchP args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w
  using MatchP(3)
  by (auto simp: args_def elim!: wf_vydra.cases)
note args' = args_def[unfolded init_args.simps, symmetric]
have run_args: w_run_t args = ru_t w_run_sub args = run_subs (ru n)
  by (auto simp: args_def)
have len_xs: length xs = i
  using w_def(2)
  by (auto simp: valid_window_matchP_def)
obtain ej tj where w_tj: ru_t (w_tj w) = Some (ej, tj)
  using reaches_on_Some[OF valid_window_matchP_reach_tj[OF w_def(2)]] reaches_on_run_hd_t[OF
prefix]
  MatchP(5) reaches_on_progress[OF prefix, where ?phi=MatchP I r]
  by (fastforce simp: run_args len_xs simp del: progress.simps)
have run_subs (ru n) (w_sj w) = None
  using valid_eval_matchP[OF w_def(2), unfolded run_args] w_tj MatchP(4,7)
  by (cases run_subs (ru n) (w_sj w)) (auto simp: w_def(1) args' simp del: eval_matchP.simps split:
option.splits)
then obtain j where j_def: j < length (w_sj w) ru n (w_sj w ! j) = None
  by (auto simp: run_subs_def Let_def in_set_conv_nth Option.is_none_def split: if_splits)
have len_w_sj: length (w_sj w) = length (collect_subfmlas r [])
  using valid_window_matchP_reach_sj[OF w_def(2)] reach_run_subs_len
  by (auto simp: run_args)
define phi where phi = collect_subfmlas r [] ! j
have phi_in_set: phi  $\in$  set (collect_subfmlas r [])
  using j_def(1)
  by (auto simp: phi_def in_set_conv_nth len_w_sj)
have wf: wf_vydra phi i n (w_sj w ! j)
  using valid_window_matchP_reach_sj[OF w_def(2)] wf[folded len_w_sj, OF _ j_def(1)] len_xs
  by (fastforce simp: run_args phi_def)
have i < progress phi (map fst vs)
  using MatchP(5) phi_in_set atms_nonempty[of r] atms_finite[of r] collect_subfmlas_atms[of r []]
  by auto
then show ?case
  using MatchP(2)[OF msize_sub[OF phi_in_set] wf j_def(2)] MatchP(6,7) phi_in_set
  bf_collect_subfmlas[where ?r=r and ?phis=[]]
  by (auto simp: collect_subfmlas_atms)
next
case (MatchF n I r)
have subfmla: msize_fmfa x  $\leq$  n bounded_future_fmfa x wf_fmfa x if x  $\in$  set (collect_subfmlas r [])
for x
  using that MatchF(1,6,7) le_trans[OF collect_subfmlas_msize] bf_collect_subfmlas[where ?r=r
and ?phis=[] and ?phi=x]
  collect_subfmlas_atms[where ?phis=[] and ?r=r]
  by auto
have sound: x  $\in$  set (collect_subfmlas r [])  $\implies$  wf_vydra x j n v  $\implies$  ru n v = Some (v', t, b)  $\implies$ 
wf_vydra x (Suc j) n v'  $\wedge$  t =  $\tau$   $\sigma$  j  $\wedge$  b = sat x j for x j v v' t b
  using MatchF vydra_sound_aux subfmla
  by fastforce
have reaches_on (ru n) (su n phi) vs v  $\implies$  wf_vydra phi (length vs) n v if phi: phi  $\in$  set (collect_subfmlas

```

```

r []) for phi vs v
  apply (induction vs arbitrary: v rule: rev_induct)
  using MatchF(1) wf_vydra_sub subfmla(1)[OF phi] sound[OF phi]
  apply (auto elim!: reaches_on.cases)[1]
  using sound[OF phi]
  apply (fastforce dest!: reaches_on_split_last)
  done
then have wf: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) bs s  $\implies$  j < length
(collect_subfmlas r [])  $\implies$ 
  wf_vydra (collect_subfmlas r [] ! j) (length bs) n (s ! j) for bs s j
  using reach_run_subs_run
  by (fastforce simp: in_set_conv_nth)
let ?qf = state_cnt r
let ?transs = iarray_of_list (build_nfa_impl r (0, ?qf, []))
define args where args = init_args ({0}, NFA.delta' ?transs ?qf, NFA.accept' ?transs ?qf) (ru_t,
read_t) (run_subs (ru n))
interpret MDL_window  $\sigma$  r l_t0 map (su n) (collect_subfmlas r []) args
  using bs_sat[where ?r=r and ?n=n, OF _ wf _ reach_run_subs_len[where ?n=n]] sound
run_t_read[of run_hd]
  read_t_run[of _ _ run_hd] ru_t_tau MatchF(5) subfmla
  unfolding args_def iarray_of_list_def
  by unfold_locales auto
obtain w xs where w_def: v = VYDRA_MatchF I ?transs ?qf w
  valid_window_matchF args I l_t0 (map (su n) (collect_subfmlas r [])) xs i w
  using MatchF(3)
  by (auto simp: args_def elim!: wf_vydra.cases)
note args' = args_def[unfolded init_args.simps, symmetric]
have run_args: w_run_t args = ru_t w_read_t args = read_t w_run_sub args = run_subs (ru n)
  by (auto simp: args_def)
have vs_tau: map fst vs ! k =  $\tau$   $\sigma$  k if k_vs: k < length vs for k
  using reaches_on_split[OF prefix k_vs] run_hd_sound k_vs
  apply (cases vs ! k)
  apply (auto)
  apply (metis fst_conv length_map_nth_map prefix_reaches_on_run_hd_t ru_t_tau_in)
  done
define m where m = min (length (map fst vs) - 1) (Min (( $\lambda$ f. progress f (map fst vs)) 'atms r))
have m_vs: m < length vs
  using MatchF(5)
  by (cases vs) (auto simp: m_def split: if_splits)
define A where A = {j. 0  $\leq$  j  $\wedge$  j  $\leq$  m  $\wedge$  memR (map fst vs ! j) (map fst vs ! m) I}
have m_A: m  $\in$  A
  using memR_tfin_refl[OF  $\tau$ _fin] vs_tau[OF m_vs]
  by (fastforce simp: A_def)
then have A_nonempty: A  $\neq$  {}
  by auto
have A_finite: finite A
  by (auto simp: A_def)
have p: progress (MatchF I r) (map fst vs) = Min A
  using MatchF(5)
  unfolding progress.simps m_def[symmetric] Let_def A_def[symmetric]
  by auto
have i_A: i  $\notin$  A
  using MatchF(5) A_finite A_nonempty
  by (auto simp del: progress.simps simp: p)
have i_m: i < m
  using MatchF(5) m_A A_finite A_nonempty
  by (auto simp del: progress.simps simp: p)
have memR_i_m:  $\neg$ memR (map fst vs ! i) (map fst vs ! m) I

```

```

using i_A i_m
by (auto simp: A_def)
have i_vs: i < length vs
using i_m m_vs
by auto
obtain ti where read_ti: w_read_t args (w_ti w) = Some ti
using w_def(2) reaches_on_Some[where ?r=ru_t and ?s=l_t0 and ?s'=w_ti w]
reaches_on_run_hd_t[OF prefix] i_vs run_t_read[where ?t=w_ti w]
by (fastforce simp: valid_window_matchF_def run_args)
have ti_def: ti =  $\tau$   $\sigma$  i
using w_def(2) ru_t_tau read_t_run[OF read_ti]
by (fastforce simp: valid_window_matchF_def run_args)
note reach_tj = valid_window_matchF_reach_tj[OF w_def(2), unfolded run_args]
note reach_sj = valid_window_matchF_reach_sj[OF w_def(2), unfolded run_args]
have len_xs: length xs = w_j w and memR_xs:  $\bigwedge l. l \in \{w_i w..<w_j w\} \implies \text{memR} (ts\_at\ xs\ i) (ts\_at\ xs\ l)\ I$ 
and i_def: w_i w = i
using w_def(2)
unfolding valid_window_matchF_def
by (auto simp: valid_window_matchF_def run_args)
have j_m: w_j w  $\leq$  m
using ru_t_tau_in[OF reach_tj, of i] ru_t_tau_in[OF reach_tj, of m] memR_i_m i_m memR_xs[of m]
unfolding vs_tau[OF i_vs] vs_tau[OF m_vs]
by (force simp: in_set_conv_nth len_xs ts_at_def i_def)
obtain tm tm' where tm_def: reaches_on ru_t l_t0 (take m (map fst vs)) tm
ru_t tm = Some (tm', fst (vs ! m))
using reaches_on_split[where ?i=m] reaches_on_run_hd_t[OF prefix] m_vs
by fastforce
have reach_tj_m: reaches_on (w_run_t args) (w_tj w) (drop (w_j w) (take m (map fst vs))) tm
using reaches_on_split'[OF tm_def(1), where ?i=w_j w] reaches_on_inj[OF reach_tj] m_vs j_m
by (auto simp: len_xs run_args)
have vs_m: fst (vs ! m) =  $\tau$   $\sigma$  m
using vs_tau[OF m_vs] m_vs
by auto
have memR_ti_m:  $\neg \text{memR} ti (\tau \sigma m) I$ 
using memR_i_m
unfolding vs_tau[OF i_vs] vs_tau[OF m_vs]
by (auto simp: ti_def)
have m_prog: m  $\leq$  progress phi (map fst vs) if phi  $\in$  set (collect_subfmlas r []) for phi
using collect_subfmlas_atms[where ?r=r and ?phis=[]] that atms_finite[of r]
by (auto simp: m_def min.coboundedI2)
obtain ws s where ws_def: reaches_on (run_subs (ru n)) (map (su n) (collect_subfmlas r [])) ws s
length ws = m
using reaches_ons_runI[where ?r=r and ?n=n and ?i=m]
vydra_wf_reaches_on[where ?i=m and ?n=n] subfmula
MatchF(2) m_prog
by fastforce
have reach_sj_m: reaches_on (run_subs (ru n)) (w_sj w) (drop (w_j w) ws) s
using reaches_on_split'[OF ws_def(1), where ?i=w_j w] reaches_on_inj[OF reach_sj] i_m j_m
by (auto simp: ws_def(2) len_xs)
define rho where rho = zip (drop (w_j w) (take m (map fst vs))) (drop (w_j w) ws)
have map_fst_rho: map fst rho = drop (w_j w) (take m (map fst vs))
and map_snd_rho: map snd rho = drop (w_j w) ws
using ws_def(2) j_m m_vs
by (auto simp: rho_def)
show False
using valid_eval_matchF_complete[OF w_def(2) reach_tj_m[folded map_fst_rho] reach_sj_m[folded

```

*map\_snd\_rho run\_args*] *read\_ti run\_t\_read*[*OF tm\_def(2)*][*folded run\_args*], *unfolded vs\_m*] *memR\_ti\_m*] *MatchF(4,7)*  
**by** (*auto simp: w\_def(1) args\_def simp del: eval\_matchF.simps*)  
**qed**

**definition** *ru'*  $\varphi = ru$  (*msize\_fm**la*  $\varphi$ )  
**definition** *su'*  $\varphi = su$  (*msize\_fm**la*  $\varphi$ )  $\varphi$

**lemma** *vydra\_wf*:  
**assumes** *reaches* (*ru n*) (*su n*  $\varphi$ ) *i v* *bounded\_future\_fm**la*  $\varphi$  *wf\_fm**la*  $\varphi$  *msize\_fm**la*  $\varphi \leq n$   
**shows** *wf\_vydra*  $\varphi$  *i n v*  
**using** *assms(1)*  
**proof** (*induction i arbitrary: v*)  
**case** 0  
**then show** ?*case*  
**using** *wf\_vydra\_sub*[*OF assms(4)*]  
**by** (*auto elim: reaches.cases*)  
**next**  
**case** (*Suc i*)  
**show** ?*case*  
**using** *reaches\_Suc\_split\_last*[*OF Suc(2)*] *Suc(1) vydra\_sound\_aux*[*OF assms(4) \_\_ assms(2,3)*]  
**by** *fastforce*  
**qed**

**lemma** *vydra\_sound'*:  
**assumes** *reaches* (*ru'*  $\varphi$ ) (*su'*  $\varphi$ ) *n v* *ru'*  $\varphi$  *v = Some* (*v'*, (*t*, *b*)) *bounded\_future\_fm**la*  $\varphi$  *wf\_fm**la*  $\varphi$   
**shows** (*t*, *b*) = ( $\tau$   $\sigma$  *n*, *sat*  $\varphi$  *n*)  
**using** *vydra\_sound\_aux*[*OF order.refl vydra\_wf*][*OF assms(1)*][*unfolded ru'\_def su'\_def*] *assms(3,4)*  
*order.refl*] *assms(2)*[*unfolded ru'\_def*] *assms(3,4)*]  
**by** *auto*

**lemma** *vydra\_complete'*:  
**assumes** *prefix: reaches\_on run\_hd init\_hd vs e*  
**and** *prog: n < progress*  $\varphi$  (*map fst vs*) *bounded\_future\_fm**la*  $\varphi$  *wf\_fm**la*  $\varphi$   
**shows**  $\exists v v'$ . *reaches* (*ru'*  $\varphi$ ) (*su'*  $\varphi$ ) *n v*  $\wedge$  *ru'*  $\varphi$  *v = Some* (*v'*, ( $\tau$   $\sigma$  *n*, *sat*  $\varphi$  *n*))  
**proof** –  
**have** *aux: False* **if** *aux\_assms: j*  $\leq$  *n* *wf\_vydra*  $\varphi$  *j* (*msize\_fm**la*  $\varphi$ ) *v* *ru* (*msize\_fm**la*  $\varphi$ ) *v = None* **for**  
*j v*  
**using** *vydra\_complete\_aux*[*OF prefix aux\_assms(2,3) \_\_ prog(2-)*] *aux\_assms(1) prog(1)*  
**by** *auto*  
**obtain** *ws v* **where** *ws\_def: reaches\_on* (*ru'*  $\varphi$ ) (*su'*  $\varphi$ ) *ws v* *wf\_vydra*  $\varphi$  *n* (*msize\_fm**la*  $\varphi$ ) *v* *length*  
*ws = n*  
**using** *vydra\_wf\_reaches\_on*[*OF \_\_ prog(2,3)*] *aux*[*OF less\_imp\_le\_nat*]  
**unfolding** *ru'\_def su'\_def*  
**by** *blast*  
**have** *ru\_Some: ru'*  $\varphi$  *v*  $\neq$  *None*  
**using** *aux*[*OF order.refl ws\_def(2)*]  
**by** (*fastforce simp: ru'\_def*)  
**obtain** *v' t b* **where** *tb\_def: ru'*  $\varphi$  *v = Some* (*v'*, (*t*, *b*))  
**using** *ru\_Some*  
**by** *auto*  
**show** ?*thesis*  
**using** *reaches\_on\_n*[*OF ws\_def(1)*] *tb\_def vydra\_sound'*[*OF reaches\_on\_n*][*OF ws\_def(1)*] *tb\_def*  
*prog(2,3)*]  
**by** (*auto simp: ws\_def(3)*)  
**qed**

**lemma** *map\_option\_apfst\_idle*: *map\_option* (*apfst snd*) (*map\_option* (*apfst* (*Pair n*)) *x*) = *x*

by (cases x) auto

**lemma** *vydra\_sound*:

**assumes** *reaches* (*run\_vydra* *run\_hd*) (*init\_vydra* *init\_hd* *run\_hd*  $\varphi$ ) *n* *v* *run\_vydra* *run\_hd* *v* = *Some* (*v'*, (*t*, *b*)) *bounded\_future\_fm*  $\varphi$  *wf\_fm*  $\varphi$   
**shows** (*t*, *b*) = ( $\tau$   $\sigma$  *n*, *sat*  $\varphi$  *n*)  
**proof** –  
**have** *fst\_v*: *fst* *v* = *msize\_fm*  $\varphi$   
**by** (*rule* *reaches\_invar*[*OF* *assms*(1)]) (*auto simp*: *init\_vydra\_def* *run\_vydra\_def* *Let\_def*)  
**have** *reaches* (*ru'*  $\varphi$ ) (*su'*  $\varphi$ ) *n* (*snd* *v*)  
**using** *reaches\_cong*[*OF* *assms*(1), **where**  $?P = \lambda(m, w). m = \text{msize\_fm } \varphi$  **and**  $?g = \text{snd}$ ]  
**by** (*auto simp*: *init\_vydra\_def* *run\_vydra\_def* *ru'\_def* *su'\_def* *map\_option\_apfst\_idle* *Let\_def* *simp*  
*del*: *sub.simps*)  
**then show** *?thesis*  
**using** *vydra\_sound'*[*OF* \_\_ *assms*(3,4)] *assms*(2) *fst\_v*  
**by** (*auto simp*: *run\_vydra\_def* *ru'\_def* *split*: *prod.splits*)  
**qed**

**lemma** *vydra\_complete*:

**assumes** *prefix*: *reaches\_on* *run\_hd* *init\_hd* *vs* *e*  
**and** *prog*: *n* < *progress*  $\varphi$  (*map* *fst* *vs*) *bounded\_future\_fm*  $\varphi$  *wf\_fm*  $\varphi$   
**shows**  $\exists v v'. \text{reaches}(\text{run\_vydra } \text{run\_hd}) (\text{init\_vydra } \text{init\_hd } \text{run\_hd } \varphi) n v \wedge \text{run\_vydra } \text{run\_hd } v = \text{Some } (v', (\tau \sigma n, \text{sat } \varphi n))$   
**proof** –  
**obtain** *v v'* **where** *wits*: *reaches* (*ru'*  $\varphi$ ) (*su'*  $\varphi$ ) *n* *v* *ru'*  $\varphi$  *v* = *Some* (*v'*,  $\tau$   $\sigma$  *n*, *sat*  $\varphi$  *n*)  
**using** *vydra\_complete'*[*OF* *assms*]  
**by** *auto*  
**have** *reach*: *reaches* (*run\_vydra* *run\_hd*) (*init\_vydra* *init\_hd* *run\_hd*  $\varphi$ ) *n* (*msize\_fm*  $\varphi$ , *v*)  
**using** *reaches\_cong*[*OF* *wits*(1), **where**  $?P = \lambda x. \text{True}$  **and**  $?f' = \text{run\_vydra } \text{run\_hd}$  **and**  $?g = \text{Pair}$   
(*msize\_fm*  $\varphi$ )]  
**by** (*auto simp*: *init\_vydra\_def* *run\_vydra\_def* *ru'\_def* *su'\_def* *Let\_def* *simp* *del*: *sub.simps*)  
**show** *?thesis*  
**apply** (*rule* *exI*[*of* \_\_ (*msize\_fm*  $\varphi$ , *v*)])  
**apply** (*rule* *exI*[*of* \_\_ (*msize\_fm*  $\varphi$ , *v'*)])  
**apply** (*auto simp*: *run\_vydra\_def* *wits*(2)[*unfolded* *ru'\_def*] *intro*: *reach*)  
**done**  
**qed**

**end**

**context** *MDL*

**begin**

**lemma** *reach\_elem*:

**assumes** *reaches* ( $\lambda i. \text{if } P \ i \ \text{then } \text{Some } (\text{Suc } i, (\tau \sigma i, \Gamma \sigma i)) \ \text{else } \text{None}$ ) *s* *n* *s'* *s* = 0  
**shows** *s'* = *n*  
**proof** –  
**obtain** *vs* **where** *vs\_def*: *reaches\_on* ( $\lambda i. \text{if } P \ i \ \text{then } \text{Some } (\text{Suc } i, (\tau \sigma i, \Gamma \sigma i)) \ \text{else } \text{None}$ ) *s* *vs* *s'*  
*length* *vs* = *n*  
**using** *reaches\_on*[*OF* *assms*(1)]  
**by** *auto*  
**have** *s'* = *length* *vs*  
**using** *vs\_def*(1) *assms*(2)  
**by** (*induction* *s* *vs* *s'* *rule*: *reaches\_on\_rev\_induct*) (*auto split*: *if\_splits*)  
**then show** *?thesis*  
**using** *vs\_def*(2)  
**by** *auto*  
**qed**

**interpretation** *default\_vydra*:  $VYDRA\_MDL \sigma 0 \lambda i. \text{Some} (Suc\ i, (\tau\ \sigma\ i, \Gamma\ \sigma\ i))$   
**using** *reach\_elem*[**where**  $?P = \lambda \_ . \text{True}$ ]  
**by** *unfold\_locales auto*

**end**

**lemma** *reaches\_inj*:  $\text{reaches}\ r\ s\ i\ t \implies \text{reaches}\ r\ s\ i\ t' \implies t = t'$   
**using** *reaches\_on\_inj reaches\_on*  
**by** *metis*

**lemma** *progress\_sound*:

**assumes**

$\bigwedge n. n < \text{length}\ ts \implies ts\ !\ n = \tau\ \sigma\ n$

$\bigwedge n. n < \text{length}\ ts \implies \tau\ \sigma\ n = \tau\ \sigma'\ n$

$\bigwedge n. n < \text{length}\ ts \implies \Gamma\ \sigma\ n = \Gamma\ \sigma'\ n$

$n < \text{progress}\ \phi\ i\ ts$

*bounded\_future\_fm*  $\phi$

*wf\_fm*  $\phi$

**shows**  $MDL.\text{sat}\ \sigma\ \phi\ n \iff MDL.\text{sat}\ \sigma'\ \phi\ n$

**proof** –

**define** *run\_hd* **where**  $\text{run\_hd} = (\lambda i. \text{if}\ i < \text{length}\ ts \text{ then } \text{Some}\ (Suc\ i, (\tau\ \sigma\ i, \Gamma\ \sigma\ i)) \text{ else } \text{None})$

**interpret** *vydra*:  $VYDRA\_MDL \sigma 0 \text{run\_hd}$

**using** *MDL.reach\_elem*[**where**  $?P = \lambda i. i < \text{length}\ ts$ ]

**by** *unfold\_locales (auto simp: run\_hd\_def split: if\_splits)*

**define** *run\_hd'* **where**  $\text{run\_hd}' = (\lambda i. \text{if}\ i < \text{length}\ ts \text{ then } \text{Some}\ (Suc\ i, (\tau\ \sigma'\ i, \Gamma\ \sigma'\ i)) \text{ else } \text{None})$

**interpret** *vydra'*:  $VYDRA\_MDL \sigma' 0 \text{run\_hd}'$

**using** *MDL.reach\_elem*[**where**  $?P = \lambda i. i < \text{length}\ ts$ ]

**by** *unfold\_locales (auto simp: run\_hd'\_def split: if\_splits)*

**have** *run\_hd\_hd'*:  $\text{run\_hd} = \text{run\_hd}'$

**using** *assms(1-3)*

**by** *(auto simp: run\_hd\_def run\_hd'\_def)*

**have** *reaches\_run\_hd*:  $n \leq \text{length}\ ts \implies \text{reaches\_on}\ \text{run\_hd}\ 0\ (\text{map}\ (\lambda i. (\tau\ \sigma\ i, \Gamma\ \sigma\ i))\ [0..<n])\ n$   
**for**  $n$

**by** *(induction n) (auto simp: run\_hd\_def intro: reaches\_on.intros(1) intro!: reaches\_on\_app)*

**have** *ts\_map*:  $ts = \text{map}\ \text{fst}\ (\text{map}\ (\lambda i. (\tau\ \sigma\ i, \Gamma\ \sigma\ i))\ [0..<\text{length}\ ts])$

**by** *(subst map\_nth[symmetric]) (auto simp: assms(1))*

**obtain**  $v\ v'$  **where** *vv\_def*:  $\text{reaches}\ (\text{run\_vydra}\ \text{run\_hd})\ (\text{init\_vydra}\ 0\ \text{run\_hd}\ \phi)\ n\ v\ \text{run\_vydra}\ \text{run\_hd}\ v = \text{Some}\ (v', \tau\ \sigma\ n, \text{vydra}.\text{sat}\ \phi\ n)$

**using** *vydra.vydra\_complete[OF reaches\_run\_hd[OF order.refl] \_ assms(5,6)] assms(4)*

**unfolding** *ts\_map*[*symmetric*]

**by** *blast*

**have** *reaches\_run\_hd'*:  $n \leq \text{length}\ ts \implies \text{reaches\_on}\ \text{run\_hd}'\ 0\ (\text{map}\ (\lambda i. (\tau\ \sigma'\ i, \Gamma\ \sigma'\ i))\ [0..<n])\ n$   
**for**  $n$

**by** *(induction n) (auto simp: run\_hd'\_def intro: reaches\_on.intros(1) intro!: reaches\_on\_app)*

**have** *ts'\_map*:  $ts = \text{map}\ \text{fst}\ (\text{map}\ (\lambda i. (\tau\ \sigma'\ i, \Gamma\ \sigma'\ i))\ [0..<\text{length}\ ts])$

**by** *(subst map\_nth[symmetric]) (auto simp: assms(1,2))*

**obtain**  $w\ w'$  **where** *ww\_def*:  $\text{reaches}\ (\text{run\_vydra}\ \text{run\_hd}')\ (\text{init\_vydra}\ 0\ \text{run\_hd}'\ \phi)\ n\ w\ \text{run\_vydra}\ \text{run\_hd}'\ w = \text{Some}\ (w', \tau\ \sigma'\ n, \text{vydra}'.\text{sat}\ \phi\ n)$

**using** *vydra'.vydra\_complete[OF reaches\_run\_hd'[OF order.refl] \_ assms(5,6)] assms(4)*

**unfolding** *ts'\_map*[*symmetric*]

**by** *blast*

**note**  $v\ w = \text{reaches\_inj}[OF\ \text{vv\_def}(1)\ \text{ww\_def}(1)][\text{folded}\ \text{run\_hd\_hd}']$

**show** *?thesis*

**using** *vv\_def(2) ww\_def(2)*

**by** *(auto simp: run\_hd\_hd' v\_w)*

**qed**

```

end
theory Preliminaries
  imports MDL
begin

```

## 4 Formulas and Satisfiability

```

declare [[typedef_overloaded]]

```

```

context
begin

```

```

qualified datatype ('a, 't :: timestamp) formula = Bool bool | Atom 'a | Neg ('a, 't) formula |
  Bin bool  $\Rightarrow$  bool  $\Rightarrow$  bool ('a, 't) formula ('a, 't) formula |
  Prev 't  $\mathcal{I}$  ('a, 't) formula | Next 't  $\mathcal{I}$  ('a, 't) formula |
  Since ('a, 't) formula 't  $\mathcal{I}$  ('a, 't) formula |
  Until ('a, 't) formula 't  $\mathcal{I}$  ('a, 't) formula |
  MatchP 't  $\mathcal{I}$  ('a, 't) regex | MatchF 't  $\mathcal{I}$  ('a, 't) regex
and ('a, 't) regex = Test ('a, 't) formula | Wild |
  Plus ('a, 't) regex ('a, 't) regex | Times ('a, 't) regex ('a, 't) regex |
  Star ('a, 't) regex

```

```

end

```

```

fun mdl2mdl :: ('a, 't :: timestamp) Preliminaries.formula  $\Rightarrow$  ('a, 't) formula
and embed :: ('a, 't) Preliminaries.regex  $\Rightarrow$  ('a, 't) regex where
  mdl2mdl (Preliminaries.Bool b) = Bool b
| mdl2mdl (Preliminaries.Atom a) = Atom a
| mdl2mdl (Preliminaries.Neg phi) = Neg (mdl2mdl phi)
| mdl2mdl (Preliminaries.Bin f phi psi) = Bin f (mdl2mdl phi) (mdl2mdl psi)
| mdl2mdl (Preliminaries.Prev I phi) = Prev I (mdl2mdl phi)
| mdl2mdl (Preliminaries.Next I phi) = Next I (mdl2mdl phi)
| mdl2mdl (Preliminaries.Since phi I psi) = Since (mdl2mdl phi) I (mdl2mdl psi)
| mdl2mdl (Preliminaries.Until phi I psi) = Until (mdl2mdl phi) I (mdl2mdl psi)
| mdl2mdl (Preliminaries.MatchP I r) = MatchP I (Times (embed r) (Symbol (Bool True)))
| mdl2mdl (Preliminaries.MatchF I r) = MatchF I (Times (embed r) (Symbol (Bool True)))
| embed (Preliminaries.Test phi) = Lookahead (mdl2mdl phi)
| embed Preliminaries.Wild = Symbol (Bool True)
| embed (Preliminaries.Plus r s) = Plus (embed r) (embed s)
| embed (Preliminaries.Times r s) = Times (embed r) (embed s)
| embed (Preliminaries.Star r) = Star (embed r)

```

```

lemma mdl2mdl_wf:

```

```

  fixes phi :: ('a, 't :: timestamp) Preliminaries.formula
  shows wf_fm1a (mdl2mdl phi)
  by (induction phi rule: mdl2mdl_embed.induct(1)[where ?Q= $\lambda$ r. wf_regex (Times (embed r) (Symbol (Bool True)))]  $\wedge$  ( $\forall$  phi  $\in$  atms (embed r). wf_fm1a phi)]) auto

```

```

fun embed' :: (('a, 't :: timestamp) formula  $\Rightarrow$  ('a, 't) Preliminaries.formula)  $\Rightarrow$  ('a, 't) regex  $\Rightarrow$  ('a, 't)
Preliminaries.regex where

```

```

  embed' f (Lookahead phi) = Preliminaries.Test (f phi)
| embed' f (Symbol phi) = Preliminaries.Times (Preliminaries.Test (f phi)) Preliminaries.Wild
| embed' f (Plus r s) = Preliminaries.Plus (embed' f r) (embed' f s)
| embed' f (Times r s) = Preliminaries.Times (embed' f r) (embed' f s)
| embed' f (Star r) = Preliminaries.Star (embed' f r)

```

```

lemma embed'_cong[fundef_cong]: ( $\wedge$  phi. phi  $\in$  atms r  $\implies$  f phi = f' phi)  $\implies$  embed' f r = embed' f'
r

```

by (induction r) auto

```

fun mdl2mdl' :: ('a, 't :: timestamp) formula  $\Rightarrow$  ('a, 't) Preliminaries.formula where
  mdl2mdl' (Bool b) = Preliminaries.Bool b
| mdl2mdl' (Atom a) = Preliminaries.Atom a
| mdl2mdl' (Neg phi) = Preliminaries.Neg (mdl2mdl' phi)
| mdl2mdl' (Bin f phi psi) = Preliminaries.Bin f (mdl2mdl' phi) (mdl2mdl' psi)
| mdl2mdl' (Prev I phi) = Preliminaries.Prev I (mdl2mdl' phi)
| mdl2mdl' (Next I phi) = Preliminaries.Next I (mdl2mdl' phi)
| mdl2mdl' (Since phi I psi) = Preliminaries.Since (mdl2mdl' phi) I (mdl2mdl' psi)
| mdl2mdl' (Until phi I psi) = Preliminaries.Until (mdl2mdl' phi) I (mdl2mdl' psi)
| mdl2mdl' (MatchP I r) = Preliminaries.MatchP I (embed' mdl2mdl' (rderive r))
| mdl2mdl' (MatchF I r) = Preliminaries.MatchF I (embed' mdl2mdl' (rderive r))

```

**context** MDL

**begin**

```

fun rvsat :: ('a, 't) Preliminaries.formula  $\Rightarrow$  nat  $\Rightarrow$  bool
  and rvmatch :: ('a, 't) Preliminaries.regex  $\Rightarrow$  (nat  $\times$  nat) set where
  rvsat (Preliminaries.Bool b) i = b
| rvsat (Preliminaries.Atom a) i = (a  $\in$   $\Gamma$   $\sigma$  i)
| rvsat (Preliminaries.Neg  $\varphi$ ) i = ( $\neg$  rvsat  $\varphi$  i)
| rvsat (Preliminaries.Bin f  $\varphi$   $\psi$ ) i = (f (rvsat  $\varphi$  i) (rvsat  $\psi$  i))
| rvsat (Preliminaries.Prev I  $\varphi$ ) i = (case i of 0  $\Rightarrow$  False | Suc j  $\Rightarrow$  mem ( $\tau$   $\sigma$  j) ( $\tau$   $\sigma$  i) I  $\wedge$  rvsat  $\varphi$  j)
| rvsat (Preliminaries.Next I  $\varphi$ ) i = (mem ( $\tau$   $\sigma$  i) ( $\tau$   $\sigma$  (Suc i)) I  $\wedge$  rvsat  $\varphi$  (Suc i))
| rvsat (Preliminaries.Since  $\varphi$  I  $\psi$ ) i = ( $\exists j \leq i$ . mem ( $\tau$   $\sigma$  j) ( $\tau$   $\sigma$  i) I  $\wedge$  rvsat  $\psi$  j  $\wedge$  ( $\forall k \in \{j..i\}$ . rvsat  $\varphi$  k))
| rvsat (Preliminaries.Until  $\varphi$  I  $\psi$ ) i = ( $\exists j \geq i$ . mem ( $\tau$   $\sigma$  i) ( $\tau$   $\sigma$  j) I  $\wedge$  rvsat  $\psi$  j  $\wedge$  ( $\forall k \in \{i..j\}$ . rvsat  $\varphi$  k))
| rvsat (Preliminaries.MatchP I r) i = ( $\exists j \leq i$ . mem ( $\tau$   $\sigma$  j) ( $\tau$   $\sigma$  i) I  $\wedge$  (j, i)  $\in$  rvmatch r)
| rvsat (Preliminaries.MatchF I r) i = ( $\exists j \geq i$ . mem ( $\tau$   $\sigma$  i) ( $\tau$   $\sigma$  j) I  $\wedge$  (i, j)  $\in$  rvmatch r)
| rvmatch (Preliminaries.Test  $\varphi$ ) = {(i, i) | i. rvsat  $\varphi$  i}
| rvmatch Preliminaries.Wild = {(i, i + 1) | i. True}
| rvmatch (Preliminaries.Plus r s) = rvmatch r  $\cup$  rvmatch s
| rvmatch (Preliminaries.Times r s) = rvmatch r  $O$  rvmatch s
| rvmatch (Preliminaries.Star r) = rtrancl (rvmatch r)

```

**lemma** mdl2mdl\_equivalent:

```

  fixes phi :: ('a, 't :: timestamp) Preliminaries.formula
  shows  $\bigwedge i$ . sat (mdl2mdl phi) i  $\longleftrightarrow$  rvsat phi i
  by (induction phi rule: mdl2mdl_embed.induct(1)[where ?Q= $\lambda r$ . match (embed r) = rvmatch r]) (auto split: nat.splits)

```

**lemma** mdlstar2mdl:

```

  fixes phi :: ('a, 't :: timestamp) Preliminaries.formula
  shows wf_fm1a (mdl2mdl phi)  $\wedge$   $\bigwedge i$ . sat (mdl2mdl phi) i  $\longleftrightarrow$  rvsat phi i
  apply (rule mdl2mdl_wf)
  apply (rule mdl2mdl_equivalent)
  done

```

**lemma** rvmatch\_embed':

```

  assumes  $\bigwedge phi$  i. phi  $\in$  atms r  $\Longrightarrow$  rvsat (mdl2mdl' phi) i  $\longleftrightarrow$  sat phi i
  shows rvmatch (embed' mdl2mdl' r) = match r
  using assms
  by (induction r) auto

```

**lemma** mdl2mdlstar:

```

  fixes phi :: ('a, 't :: timestamp) formula

```

```

assumes wf_fm1a phi
shows  $\bigwedge i. \text{rvsat} (\text{mdl2mdl}' \text{ phi}) i \longleftrightarrow \text{sat phi } i$ 
using assms
apply (induction phi rule: mdl2mdl'.induct)
      apply (auto split: nat.splits)[8]
subgoal for I r i
  by auto (metis atms_rderive match_rderive rvmatch_embed' wf_fm1a.simps(1))+
subgoal for I r i
  by auto (metis atms_rderive match_rderive rvmatch_embed' wf_fm1a.simps(1))+
done

end

end
theory Monitor_Code
  imports HOL-Library.Code_Target_Nat Containers.Containers Monitor Preliminaries
begin

derive (eq) ceq enat

instantiation enat :: ccompare begin

definition ccompare_enat :: enat comparator option where
  ccompare_enat = Some ( $\lambda x y. \text{if } x = y \text{ then order.Eq else if } x < y \text{ then order.Lt else order.Gt}$ )

instance by intro_classes
  (auto simp: ccompare_enat_def split: if_splits intro!: comparator.intro)

end

code_printing
  code_module IArray  $\rightarrow$  (OCaml)
  <module IArray : sig
    val length' : 'a array  $\rightarrow$  Z.t
    val sub' : 'a array * Z.t  $\rightarrow$  'a
  end = struct

  let length' xs = Z.of_int (Array.length xs);;

  let sub' (xs, i) = Array.get xs (Z.to_int i);;

  end> for type_constructor iarray constant IArray.length' IArray.sub'

code_reserved OCaml IArray

code_printing
  type_constructor iarray  $\rightarrow$  (OCaml) _ array
  | constant iarray_of_list  $\rightarrow$  (OCaml) Array.of'_list
  | constant IArray.list_of  $\rightarrow$  (OCaml) Array.to'_list
  | constant IArray.length'  $\rightarrow$  (OCaml) IArray.length'
  | constant IArray.sub'  $\rightarrow$  (OCaml) IArray.sub'

lemma iarray_list_of_inj: IArray.list_of xs = IArray.list_of ys  $\implies$  xs = ys
  by (cases xs; cases ys) auto

instantiation iarray :: (ccompare) ccompare
begin

```

**definition** *ccompare\_iarray* :: ('a iarray ⇒ 'a iarray ⇒ order) option **where**  
*ccompare\_iarray* = (case ID CCOMPARE('a list) of None ⇒ None  
| Some c ⇒ Some (λxs ys. c (IArray.list\_of xs) (IArray.list\_of ys)))

**instance**

**apply** standard  
**apply** unfold\_locales  
**using** comparator.sym[OF ID\_ccompare'] comparator.weak\_eq[OF ID\_ccompare']  
comparator.comp\_trans[OF ID\_ccompare'] iarray\_list\_of\_inj  
**apply** (auto simp: ccompare\_iarray\_def split: option.splits)  
**apply** blast+  
**done**

**end**

**derive** (rbt) mapping\_impl iarray

**definition** *mk\_db* :: String.literal list ⇒ String.literal set **where** *mk\_db* = set

**definition** *init\_vydra\_string\_enat* :: \_ ⇒ \_ ⇒ \_ ⇒ (String.literal, enat, 'e) vydra **where**  
*init\_vydra\_string\_enat* = *init\_vydra*

**definition** *run\_vydra\_string\_enat* :: \_ ⇒ (String.literal, enat, 'e) vydra ⇒ \_ **where**  
*run\_vydra\_string\_enat* = *run\_vydra*

**definition** *progress\_enat* :: (String.literal, enat) formula ⇒ enat list ⇒ nat **where**  
*progress\_enat* = *progress*

**definition** *bounded\_future\_fmula\_enat* :: (String.literal, enat) formula ⇒ bool **where**  
*bounded\_future\_fmula\_enat* = *bounded\_future\_fmula*

**definition** *wf\_fmula\_enat* :: (String.literal, enat) formula ⇒ bool **where**  
*wf\_fmula\_enat* = *wf\_fmula*

**definition** *mdl2mdl'\_enat* :: (String.literal, enat) formula ⇒ (String.literal, enat) Preliminaries.formula **where**  
*mdl2mdl'\_enat* = *mdl2mdl'*

**definition** *interval\_enat* :: enat ⇒ enat ⇒ bool ⇒ bool ⇒ enat  $\mathcal{I}$  **where**  
*interval\_enat* = *interval*

**definition** *rep\_interval\_enat* :: enat  $\mathcal{I}$  ⇒ enat × enat × bool × bool **where**  
*rep\_interval\_enat* = *Rep\_* $\mathcal{I}$

**definition** *init\_vydra\_string\_ereal* :: \_ ⇒ \_ ⇒ \_ ⇒ (String.literal, ereal, 'e) vydra **where**  
*init\_vydra\_string\_ereal* = *init\_vydra*

**definition** *run\_vydra\_string\_ereal* :: \_ ⇒ (String.literal, ereal, 'e) vydra ⇒ \_ **where**  
*run\_vydra\_string\_ereal* = *run\_vydra*

**definition** *progress\_ereal* :: (String.literal, ereal) formula ⇒ ereal list ⇒ real **where**  
*progress\_ereal* = *progress*

**definition** *bounded\_future\_fmula\_ereal* :: (String.literal, ereal) formula ⇒ bool **where**  
*bounded\_future\_fmula\_ereal* = *bounded\_future\_fmula*

**definition** *wf\_fmula\_ereal* :: (String.literal, ereal) formula ⇒ bool **where**  
*wf\_fmula\_ereal* = *wf\_fmula*

**definition** *mdl2mdl'\_ereal* :: (String.literal, ereal) formula ⇒ (String.literal, ereal) Preliminaries.formula **where**  
*mdl2mdl'\_ereal* = *mdl2mdl'*

**definition** *interval\_ereal* :: ereal ⇒ ereal ⇒ bool ⇒ bool ⇒ ereal  $\mathcal{I}$  **where**  
*interval\_ereal* = *interval*

**definition** *rep\_interval\_ereal* :: ereal  $\mathcal{I}$  ⇒ ereal × ereal × bool × bool **where**  
*rep\_interval\_ereal* = *Rep\_* $\mathcal{I}$

**lemma** *tfin\_enat\_code*[code]: (tfin :: enat set) = Collect\_set (λx. x ≠ ∞)  
**by** (auto simp: tfin\_enat\_def)

```

lemma tfin_ereal_code[code]: (tfin :: ereal set) = Collect_set ( $\lambda x. x \neq -\infty \wedge x \neq \infty$ )
  by (auto simp: tfin_ereal_def)

lemma Ball_atms[code_unfold]: Ball (atms r) P = list_all P (collect_subfmlas r [])
  using collect_subfmlas_atms[where ?phis=[]]
  by (auto simp: list_all_def)

lemma MIN_fold: (MIN  $x \in \text{set } (z \# \text{zs}). f x$ ) = fold min (map f zs) (f z)
  by (metis Min.set_eq_fold list.set_map list.simps(9))

declare progress.simps(1-8)[code]

lemma progress_matchP_code[code]:
  progress (MatchP I r) ts = (case collect_subfmlas r [] of  $x \# xs \Rightarrow \text{fold min } (\text{map } (\lambda f. \text{progress } f \text{ ts}))$ 
  xs) (progress x ts))
  using collect_subfmlas_atms[where ?r=r and ?phis=[]] atms_nonempty[of r]
  by (auto split: list.splits) (smt (z3) MIN_fold[where ?f= $\lambda f. \text{progress } f \text{ ts}$ ] list.simps(15))

lemma progress_matchF_code[code]:
  progress (MatchF I r) ts = (if length ts = 0 then 0 else
  (let k = min (length ts - 1) (case collect_subfmlas r [] of  $x \# xs \Rightarrow \text{fold min } (\text{map } (\lambda f. \text{progress } f \text{ ts}))$ 
  xs) (progress x ts)) in
  Min {j  $\in \{..k\}. \text{memR } (ts ! j) (ts ! k) I$ }))
  by (auto simp: progress_matchP_code[unfolded progress.simps])

export_code init_vydra_string_enat run_vydra_string_enat progress_enat bounded_future_fmla_enat
  wf_fmla_enat mdl2mdl'_enat
  Bool Preliminaries.Bool enat interval_enat rep_interval_enat nat_of_integer integer_of_nat mk_db
  in OCaml module_name VYDRA file_prefix verified

end
theory Timestamp_Lex
  imports Timestamp
begin

instantiation prod :: (timestamp_total_strict, timestamp_total_strict) timestamp_total_strict
begin

definition tfin_prod :: ('a  $\times$  'b) set where
  tfin_prod = tfin  $\times$  UNIV

definition  $\iota$ _prod :: nat  $\Rightarrow$  'a  $\times$  'b where
   $\iota$ _prod n = ( $\iota$  n,  $\iota$  n)

fun sup_prod :: 'a  $\times$  'b  $\Rightarrow$  'a  $\times$  'b  $\Rightarrow$  'a  $\times$  'b where
  sup_prod (a, b) (c, d) = (if a < c then (c, d) else if c < a then (a, b) else (a, sup b d))

fun less_eq_prod :: 'a  $\times$  'b  $\Rightarrow$  'a  $\times$  'b  $\Rightarrow$  bool where
  less_eq_prod (a, b) (c, d)  $\longleftrightarrow$  a < c  $\vee$  (a = c  $\wedge$  b  $\leq$  d)

definition less_prod :: 'a  $\times$  'b  $\Rightarrow$  'a  $\times$  'b  $\Rightarrow$  bool where
  less_prod x y  $\longleftrightarrow$  x  $\leq$  y  $\wedge$  x  $\neq$  y

instance
  apply standard
  apply (auto simp: zero_prod_def less_prod_def)[2]
subgoal for x y z
  using order.strict_trans

```

```

    by (cases x; cases y; cases z) auto
  subgoal for x y
    by (cases x; cases y) auto
  subgoal for x y
    by (cases x; cases y) (auto simp add: sup commute sup.strict_order_iff)
  subgoal for x y
    apply (cases x; cases y)
    apply (auto simp add: sup commute sup.strict_order_iff)
    apply (metis sup.absorb_iff2 sup.order_iff timestamp_total)
    done
  subgoal for y x z
    by (cases x; cases y; cases z) auto
  subgoal for i j
    using  $\iota$ _mono less_le
    apply (auto simp:  $\iota$ _prod_def less_prod_def)
    by (simp add:  $\iota$ _mono)
  subgoal for i
    by (auto simp:  $\iota$ _prod_def tfin_prod_def intro:  $\iota$ _tfin)
  subgoal for x i
    apply (cases x)
    apply (auto simp:  $\iota$ _prod_def tfin_prod_def)
    apply (metis  $\iota$ _progressing dual_order.refl order_less_le)
    done
    apply (auto simp: tfin_prod_def tfin_closed)[1]
    apply (auto simp: zero_prod_def tfin_prod_def intro: zero_tfin)[1]
  subgoal for c d
    by (cases c; cases d) (auto simp: tfin_prod_def intro: tfin_closed)
  subgoal for c d a
    by (cases c; cases d; cases a) (auto simp: add_mono add_mono_strict)
  subgoal for a c
    apply (cases a; cases c)
    apply (auto simp: tfin_prod_def zero_prod_def)
    apply (metis less_eq_prod.simps add.right_neutral add_mono_strict less_prod_def order_le_less
order_less_le prod.inject)
    done
  subgoal for c d a
    apply (cases c; cases d; cases a)
    apply (auto simp add: add_mono_strict less_prod_def order.strict_implies_order)
    apply (metis add_mono_strict sup.strict_order_iff)
    apply (metis add_mono_strict sup.strict_order_iff)
    by (metis add_mono_strict dual_order.order_iff_strict less_le_not_le)
  subgoal for a b
    apply (cases a; cases b)
    apply (auto)
    apply (metis antisym_conv1 timestamp_total)
    apply (metis antisym_conv1 timestamp_total)
    apply (metis antisym_conv1 timestamp_total)
    apply (metis timestamp_total)
    done
  subgoal for a b
    apply (cases a; cases b)
    apply (auto simp: zero_prod_def tfin_prod_def order_less_le timestamp_tfin_le_not_tfin)
    done
done
end
end

```

```

theory Timestamp_Prod
  imports Timestamp
begin

instantiation prod :: (timestamp, timestamp) timestamp
begin

definition tfin_prod :: ('a × 'b) set where
  tfin_prod = tfin × tfin

definition ι_prod :: nat ⇒ 'a × 'b where
  ι_prod n = (ι n, ι n)

fun sup_prod :: 'a × 'b ⇒ 'a × 'b ⇒ 'a × 'b where
  sup_prod (a, b) (c, d) = (sup a c, sup b d)

fun less_eq_prod :: 'a × 'b ⇒ 'a × 'b ⇒ bool where
  less_eq_prod (a, b) (c, d) ⇔ a ≤ c ∧ b ≤ d

definition less_prod :: 'a × 'b ⇒ 'a × 'b ⇒ bool where
  less_prod x y ⇔ x ≤ y ∧ x ≠ y

instance
  apply standard
    apply (auto simp: add commute zero_prod_def less_prod_def)[7]
  subgoal for i j
    using ι_mono ι_mono less_le
    by (fastforce simp: ι_prod_def less_prod_def)
  subgoal for i
    by (auto simp: ι_prod_def tfin_prod_def intro: ι_tfin)
  subgoal for x i
    apply (cases x)
    using ι_progressing
    by (auto simp: tfin_prod_def ι_prod_def)
  apply (auto simp: zero_prod_def tfin_prod_def intro: zero_tfin)[1]
  subgoal for c d
    by (cases c; cases d) (auto simp: tfin_prod_def intro: tfin_closed)
  subgoal for c d a
    by (cases c; cases d; cases a) (auto simp add: add_mono)
  subgoal for a c
    apply (cases a; cases c)
    apply (auto simp: tfin_prod_def zero_prod_def)
    apply (metis add.right_neutral add_pos less_eq_prod.simps less_prod_def order_less_le prod.inject
timestamp_class.add_mono)
    done
  done

end

end

```

## References

- [1] R. Koymans. Specifying real-time properties with metric temporal logic. *Real Time Syst.*, 2(4):255–299, 1990.
- [2] M. Raszyk, D. A. Basin, and D. Traytel. Multi-head monitoring of metric dynamic logic. In

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