

# Tycon: Type Constructor Classes and Monad Transformers

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## Abstract

These theories contain a formalization of first class type constructors and axiomatic constructor classes for HOLCF. This work is described in detail in the ICFP 2012 paper “Formal Verification of Monad Transformers” by the author [1]. The formalization is a revised and updated version of earlier joint work with Matthews and White [3].

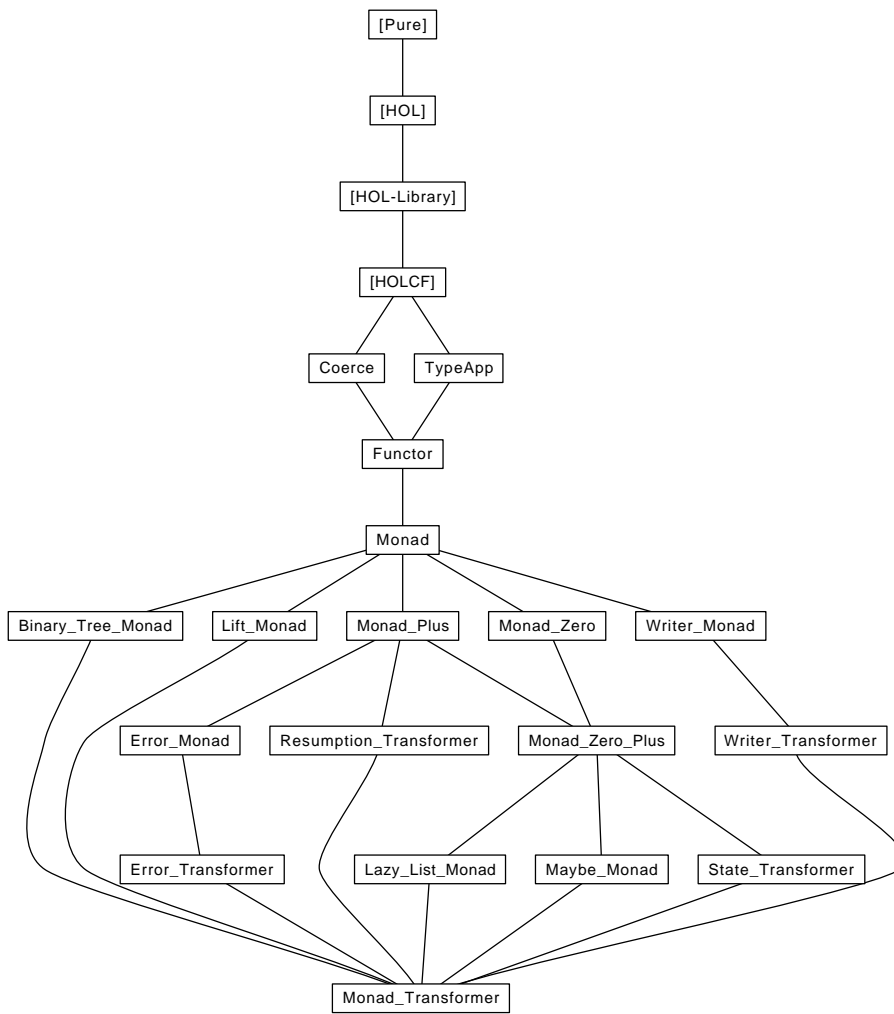
Based on the hierarchy of type classes in Haskell, we define classes for functors, monads, monad-plus, etc. Each one includes all the standard laws as axioms. We also provide a new user command, *tycondef*, for defining new type constructors in HOLCF. Using *tycondef*, we instantiate the type class hierarchy with various monads and monad transformers.

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# 1 Type Application

```
theory TypeApp
imports HOLCF
begin
```

## 1.1 Class of type constructors

In HOLCF, the type *udom defl* consists of deflations over the universal domain—each value of type *udom defl* represents a bifinite domain. In turn, values of the continuous function type *udom defl*  $\rightarrow$  *udom defl* represent functions from domains to domains, i.e. type constructors.

Class *tycon*, defined below, will be populated with dummy types: For example, if the type *foo* is an instance of class *tycon*, then users will never deal with any values *x::foo* in practice. Such types are only used with the overloaded constant *tc*, which associates each type *'a::tycon* with a value of type *udom defl*  $\rightarrow$  *udom defl*.

```
class tycon =
  fixes tc :: ('a::type) itself  $\Rightarrow$  udom defl  $\rightarrow$  udom defl
```

Type *'a itself* is defined in Isabelle's meta-logic; it is inhabited by a single value, written *TYPE('a)*. We define the syntax *TC('a)* to abbreviate *tc TYPE('a)*.

```
syntax -TC :: type  $\Rightarrow$  logic ((1TC/(1'(-))))
```

```
translations TC('a)  $\equiv$  CONST tc TYPE('a)
```

## 1.2 Type constructor for type application

We now define a binary type constructor that models type application: Type *('a, 't) app* is the result of applying the type constructor *'t* (from class *tycon*) to the type argument *'a* (from class *domain*).

We define type *('a, 't) app* using *domaindef*, a low-level type-definition command provided by HOLCF (similar to *typedef* in Isabelle/HOL) that defines a new domain type represented by the given deflation. Note that in HOLCF, *DEFL('a)* is an abbreviation for *defl TYPE('a)*, where *defl* :: ('a::domain) *itself*  $\Rightarrow$  *udom defl* is an overloaded function from the *domain* type class that yields the deflation representing the given type.

```
domaindef ('a,'t) app = TC('t::tycon)·DEFL('a::domain)
```

We define the infix syntax *'a·'t* for the type *('a,'t) app*. Note that for consistency with Isabelle's existing type syntax, we have used postfix order

for type application: type argument on the left, type constructor on the right.

**type-notation** *app* ((*--*) [999,1000] 999)

The *domaindef* command generates the theorem *DEFL-app*:  $DEFL(?'a \cdot ?'t) = TC(?'t) \cdot DEFL(?'a)$ , which we can use to derive other useful lemmas.

**lemma** *TC-DEFL*:  $TC('t::tycon) \cdot DEFL('a) = DEFL('a \cdot 't)$   
**by** (*rule DEFL-app* [*symmetric*])

**lemma** *DEFL-app-mono* [*simp, intro*]:  
 $DEFL('a) \sqsubseteq DEFL('b) \implies DEFL('a \cdot 't::tycon) \sqsubseteq DEFL('b \cdot 't)$   
**apply** (*simp add: DEFL-app*)  
**apply** (*erule monofun-cfun-arg*)  
**done**

**end**

## 2 Coercion Operator

**theory** *Coerce*  
**imports** *HOLCF*  
**begin**

### 2.1 Coerce

The *domain* type class, which is the default type class in HOLCF, fixes two overloaded functions:  $emb::'a \rightarrow udom$  and  $prj::udom \rightarrow 'a$ . By composing the *prj* and *emb* functions together, we can coerce values between any two types in class *domain*.

**definition** *coerce* ::  $'a \rightarrow 'b$   
**where** *coerce*  $\equiv prj \circ emb$

When working with proofs involving *emb*, *prj*, and *coerce*, it is often difficult to tell at which types those constants are being used. To alleviate this problem, we define special input and output syntax to indicate the types.

**syntax**  
 $-emb :: type \Rightarrow logic ((1EMB/(1'(-))))$   
 $-prj :: type \Rightarrow logic ((1PRJ/(1'(-))))$   
 $-coerce :: type \Rightarrow type \Rightarrow logic ((1COERCE/(1'(-, -))))$

**translations**  
 $EMB('a) \rightarrow CONST emb :: 'a \rightarrow udom$   
 $PRJ('a) \rightarrow CONST prj :: udom \rightarrow 'a$   
 $COERCE('a, 'b) \rightarrow CONST coerce :: 'a \rightarrow 'b$

```

typed-print-translation ⟨⟨
  let
    fun emb-tr' (ctxt : Proof.context) (Type(-, [T, -])) [] =
      Syntax.const @{{syntax-const -emb}} $ Syntax-Phases.term-of-typ ctxt T
    fun prj-tr' ctxt (Type(-, [-, T])) [] =
      Syntax.const @{{syntax-const -prj}} $ Syntax-Phases.term-of-typ ctxt T
    fun coerce-tr' ctxt (Type(-, [T, U])) [] =
      Syntax.const @{{syntax-const -coerce}} $
        Syntax-Phases.term-of-typ ctxt T $ Syntax-Phases.term-of-typ ctxt U
  in
    [(@{{const-syntax emb}}, emb-tr'),
     (@{{const-syntax prj}}, prj-tr'),
     (@{{const-syntax coerce}}, coerce-tr')]
  end
  ⟩⟩

```

**lemma** *beta-coerce*:  $coerce.x = prj.(emb.x)$   
**by** (*simp add: coerce-def*)

**lemma** *prj-emb*:  $prj.(emb.x) = coerce.x$   
**by** (*simp add: coerce-def*)

**lemma** *coerce-strict* [*simp*]:  $coerce.\perp = \perp$   
**by** (*simp add: coerce-def*)

Certain type instances of *coerce* may reduce to the identity function, *emb*, or *prj*.

**lemma** *coerce-eq-ID* [*simp*]:  $COERCE('a, 'a) = ID$   
**by** (*rule cfun-eqI, simp add: beta-coerce*)

**lemma** *coerce-eq-emb* [*simp*]:  $COERCE('a, udom) = EMB('a)$   
**by** (*rule cfun-eqI, simp add: beta-coerce*)

**lemma** *coerce-eq-prj* [*simp*]:  $COERCE(udom, 'a) = PRJ('a)$   
**by** (*rule cfun-eqI, simp add: beta-coerce*)

Cancellation rules

**lemma** *emb-coerce*:  
 $DEFL('a) \sqsubseteq DEFL('b)$   
 $\implies EMB('b).(COERCE('a, 'b).x) = EMB('a).x$   
**by** (*simp add: beta-coerce emb-prj-emb*)

**lemma** *coerce-prj*:  
 $DEFL('a) \sqsubseteq DEFL('b)$   
 $\implies COERCE('b, 'a).(PRJ('b).x) = PRJ('a).x$   
**by** (*simp add: beta-coerce prj-emb-prj*)

**lemma** *coerce-idem* [*simp*]:  
 $DEFL('a) \sqsubseteq DEFL('b)$   
 $\implies COERCE('b, 'c) \cdot (COERCE('a, 'b) \cdot x) = COERCE('a, 'c) \cdot x$   
**by** (*simp add: beta-coerce emb-prj-emb*)

## 2.2 More lemmas about emb and prj

**lemma** *prj-cast-DEFL* [*simp*]:  $PRJ('a) \cdot (cast \cdot DEFL('a) \cdot x) = PRJ('a) \cdot x$   
**by** (*simp add: cast-DEFL*)

**lemma** *cast-DEFL-emb* [*simp*]:  $cast \cdot DEFL('a) \cdot (EMB('a) \cdot x) = EMB('a) \cdot x$   
**by** (*simp add: cast-DEFL*)

$DEFL(udom)$

**lemma** *below-DEFL-udom* [*simp*]:  $A \sqsubseteq DEFL(udom)$   
**apply** (*rule cast-below-imp-below*)  
**apply** (*rule cast.belowI*)  
**apply** (*simp add: cast-DEFL*)  
**done**

## 2.3 Coercing various datatypes

Coercing from the strict product type  $'a \otimes 'b$  to another strict product type  $'c \otimes 'd$  is equivalent to mapping the *coerce* function separately over each component using *sprod-map*  $:: ('a \rightarrow 'c) \rightarrow ('b \rightarrow 'd) \rightarrow 'a \otimes 'b \rightarrow 'c \otimes 'd$ . Each of the several type constructors defined in HOLCF satisfies a similar property, with respect to its own map combinator.

**lemma** *coerce-u*:  $coerce = u\text{-map} \cdot coerce$   
**apply** (*rule cfun-eqI, simp add: coerce-def*)  
**apply** (*simp add: emb-u-def prj-u-def liftemb-eq liftprj-eq*)  
**apply** (*subst ep-pair.e-inverse [OF ep-pair-u]*)  
**apply** (*simp add: u-map-map ccomp1*)  
**done**

**lemma** *coerce-sfun*:  $coerce = sfun\text{-map} \cdot coerce \cdot coerce$   
**apply** (*rule cfun-eqI, simp add: coerce-def*)  
**apply** (*simp add: emb-sfun-def prj-sfun-def*)  
**apply** (*subst ep-pair.e-inverse [OF ep-pair-sfun]*)  
**apply** (*simp add: sfun-map-map ccomp1*)  
**done**

**lemma** *coerce-cfun'*:  $coerce = cfun\text{-map} \cdot coerce \cdot coerce$   
**apply** (*rule cfun-eqI, simp add: prj-emb [symmetric]*)  
**apply** (*simp add: emb-cfun-def prj-cfun-def*)  
**apply** (*simp add: prj-emb coerce-sfun coerce-u*)  
**apply** (*simp add: encode-cfun-map [symmetric]*)  
**done**



```

lemma coerce-ssum: coerce = ssum-map·coerce·coerce
apply (rule cfun-eqI, simp add: coerce-def)
apply (simp add: emb-ssum-def prj-ssum-def)
apply (subst ep-pair.e-inverse [OF ep-pair-ssum])
apply (simp add: ssum-map-map cfcomp1)
done

```

```

lemma coerce-sprod: coerce = sprod-map·coerce·coerce
apply (rule cfun-eqI, simp add: coerce-def)
apply (simp add: emb-sprod-def prj-sprod-def)
apply (subst ep-pair.e-inverse [OF ep-pair-sprod])
apply (simp add: sprod-map-map cfcomp1)
done

```

```

lemma coerce-prod: coerce = prod-map·coerce·coerce
apply (rule cfun-eqI, simp add: coerce-def)
apply (simp add: emb-prod-def prj-prod-def)
apply (subst ep-pair.e-inverse [OF ep-pair-prod])
apply (simp add: prod-map-map cfcomp1)
done

```

## 2.4 Simplifying coercions

When simplifying applications of the *coerce* function, rewrite rules are always oriented to replace *coerce* at complex types with other applications of *coerce* at simpler types.

The safest rewrite rules for *coerce* are given the [*simp*] attribute. For other rules that do not belong in the global simpset, we use dynamic theorem list called *coerce-simp*, which will collect additional rules for simplifying coercions.

**named-theorems** *coerce-simp* rule for simplifying coercions

The *coerce* function commutes with data constructors for various HOLCF datatypes.

```

lemma coerce-up [simp]: coerce·(up·x) = up·(coerce·x)
by (simp add: coerce-u)

```

```

lemma coerce-sinl [simp]: coerce·(sinl·x) = sinl·(coerce·x)
by (simp add: coerce-ssum ssum-map-sinl')

```

```

lemma coerce-sinr [simp]: coerce·(sinr·x) = sinr·(coerce·x)
by (simp add: coerce-ssum ssum-map-sinr')

```

```

lemma coerce-spair [simp]: coerce·(:x, y) = (:coerce·x, coerce·y)
by (simp add: coerce-sprod sprod-map-spair')

```

**lemma** *coerce-Pair* [*simp*]:  $\text{coerce} \cdot (x, y) = (\text{coerce} \cdot x, \text{coerce} \cdot y)$   
**by** (*simp add: coerce-prod*)

**lemma** *beta-coerce-cfun* [*simp*]:  $\text{coerce} \cdot f \cdot x = \text{coerce} \cdot (f \cdot (\text{coerce} \cdot x))$   
**by** (*simp add: coerce-cfun'*)

**lemma** *coerce-cfun*:  $\text{coerce} \cdot f = \text{coerce} \circ f \circ \text{coerce}$   
**by** (*simp add: cfun-eqI*)

**lemma** *coerce-cfun-app* [*coerce-simp*]:  
 $\text{coerce} \cdot f = (\lambda x. \text{coerce} \cdot (f \cdot (\text{coerce} \cdot x)))$   
**by** (*simp add: cfun-eqI*)

**end**

### 3 Functor Class

**theory** *Functor*  
**imports** *TypeApp Coerce*  
**keywords** *tycondef* :: *thy-decl* **and** ·  
**begin**

#### 3.1 Class definition

Here we define the *functor* class, which models the Haskell class `Functor`. For technical reasons, we split the definition of *functor* into two separate classes: First, we introduce *prefunctor*, which only requires *fmap* to preserve the identity function, and not function composition.

The Haskell class `Functor f` fixes a polymorphic function  $\text{fmap} :: (a \rightarrow b) \rightarrow f\ a \rightarrow f\ b$ . Since functions in Isabelle type classes can only mention one type variable, we have the *prefunctor* class fix a function *fmapU* that fixes both of the polymorphic types to be the universal domain. We will use the coercion operator to recover a polymorphic *fmap*.

The single axiom of the *prefunctor* class is stated in terms of the HOLCF constant *isodefl*, which relates a function  $f :: 'a \rightarrow 'a$  with a deflation  $t :: \text{udom defl}$ :  $\text{isodefl } f\ t = (\text{cast} \cdot t = \text{EMB}('a) \circ f \circ \text{PRJ}('a))$ .

**class** *prefunctor* = *tycon* +  
**fixes** *fmapU* ::  $(\text{udom} \rightarrow \text{udom}) \rightarrow \text{udom} \cdot 'a \rightarrow \text{udom} \cdot 'a :: \text{tycon}$   
**assumes** *isodefl-fmapU*:  
 $\text{isodefl } (fmapU \cdot (\text{cast} \cdot t))\ (TC('a :: \text{tycon}) \cdot t)$

The *functor* class extends *prefunctor* with an axiom stating that *fmapU* preserves composition.

**class** *functor* = *prefunctor* +

**assumes**  $fmapU$ - $fmapU$  [*coerce-simp*]:  
 $\bigwedge f g (xs :: udom \cdot 'a :: tycon).$   
 $fmapU \cdot f \cdot (fmapU \cdot g \cdot xs) = fmapU \cdot (\lambda x. f \cdot (g \cdot x)) \cdot xs$

We define the polymorphic  $fmap$  by coercion from  $fmapU$ , then we proceed to derive the polymorphic versions of the functor laws.

**definition**  $fmap :: ('a \rightarrow 'b) \rightarrow 'a \cdot 'f \rightarrow 'b \cdot 'f :: functor$   
**where**  $fmap = coerce \cdot (fmapU :: - \rightarrow udom \cdot 'f \rightarrow udom \cdot 'f)$

### 3.2 Polymorphic functor laws

**lemma**  $fmapU$ - $eq$ - $fmap$ :  $fmapU = fmap$   
**by** (*simp add: fmap-def eta-cfun*)

**lemma**  $fmap$ - $eq$ - $fmapU$ :  $fmap = fmapU$   
**by** (*simp only: fmapU-eq-fmap*)

**lemma**  $cast$ - $TC$ :  
 $cast \cdot (TC('f) \cdot t) = emb \circ fmapU \cdot (cast \cdot t) \circ PRJ(udom \cdot 'f :: prefunctor)$   
**by** (*rule isodeft-fmapU [unfolded isodeft-def]*)

**lemma**  $isodeft$ - $cast$ :  $isodeft (cast \cdot t) t$   
**by** (*simp add: isodeft-def*)

**lemma**  $cast$ - $cast$ - $below1$ :  $A \sqsubseteq B \implies cast \cdot A \cdot (cast \cdot B \cdot x) = cast \cdot A \cdot x$   
**by** (*intro deflation-below-comp1 deflation-cast monofun-cfun-arg*)

**lemma**  $cast$ - $cast$ - $below2$ :  $A \sqsubseteq B \implies cast \cdot B \cdot (cast \cdot A \cdot x) = cast \cdot A \cdot x$   
**by** (*intro deflation-below-comp2 deflation-cast monofun-cfun-arg*)

**lemma**  $isodeft$ - $fmap$ :  
**assumes**  $isodeft d t$   
**shows**  $isodeft (fmap \cdot d :: 'a \cdot 'f \rightarrow -) (TC('f :: functor) \cdot t)$   
**proof** –  
**have**  $deflation$ - $d$ :  $deflation d$   
**using** *assms* **by** (*rule isodeft-imp-deflation*)  
**have**  $cast$ - $t$ :  $cast \cdot t = emb \circ d \circ prj$   
**using** *assms* **unfolding**  $isodeft$ - $def$  .  
**have**  $t$ - $below$ :  $t \sqsubseteq DEFL('a)$   
**apply** (*rule cast-below-imp-below*)  
**apply** (*simp only: cast-t cast-DEFL*)  
**apply** (*simp add: cfun-below-iff deflation.below [OF deflation-d]*)  
**done**  
**have**  $fmap$ - $eq$ :  $fmap \cdot d = PRJ('a \cdot 'f) \circ cast \cdot (TC('f) \cdot t) \circ emb$   
**by** (*simp add: fmap-def coerce-cfun cast-TC cast-t prj-emb ccomp1*)  
**show** *?thesis*  
**apply** (*simp add: fmap-eq isodeft-def cfun-eq-iff emb-prj*)  
**apply** (*simp add: DEFL-app*)  
**apply** (*simp add: cast-cast-below1 monofun-cfun t-below*)

```

  apply (simp add: cast-cast-below2 monofun-cfun t-below)
done
qed

```

```

lemma fmap-ID [simp]: fmap·ID = ID
apply (rule isodeft-DEFL-imp-ID)
apply (subst DEFL-app)
apply (rule isodeft-fmap)
apply (rule isodeft-ID-DEFL)
done

```

```

lemma fmap-ident [simp]: fmap·(λ x. x) = ID
by (simp add: ID-def [symmetric])

```

```

lemma coerce-coerce-eq-fmapU-cast [coerce-simp]:
  fixes xs :: udom.'f::functor
  shows COERCE('a.'f, udom.'f)·(COERCE(udom.'f, 'a.'f)·xs) =
    fmapU·(cast·DEFL('a))·xs
by (simp add: coerce-def emb-prj DEFL-app cast-TC)

```

```

lemma fmap-fmap:
  fixes xs :: 'a.'f::functor and g :: 'a → 'b and f :: 'b → 'c
  shows fmap·f·(fmap·g·xs) = fmap·(λ x. f·(g·x))·xs
unfolding fmap-def
by (simp add: coerce-simp)

```

```

lemma fmap-cfcomp: fmap·(f oo g) = fmap·f oo fmap·g
by (simp add: cfcomp1 fmap-fmap eta-cfun)

```

### 3.3 Derived properties of *fmap*

Other theorems about *fmap* can be derived using only the abstract functor laws.

```

lemma deflation-fmap:
  deflation d ⇒ deflation (fmap·d)
apply (rule deflation.intro)
apply (simp add: fmap-fmap deflation.idem eta-cfun)
apply (subgoal-tac fmap·d·x ⊆ fmap·ID·x, simp)
apply (rule monofun-cfun-fun, rule monofun-cfun-arg)
apply (erule deflation.below-ID)
done

```

```

lemma ep-pair-fmap:
  ep-pair e p ⇒ ep-pair (fmap·e) (fmap·p)
apply (rule ep-pair.intro)
apply (simp add: fmap-fmap ep-pair.e-inverse)
apply (simp add: fmap-fmap)
apply (rule-tac y=fmap·ID·y in below-trans)

```

```

apply (rule monofun-cfun-fun)
apply (rule monofun-cfun-arg)
apply (rule cfun-belowI, simp)
apply (erule ep-pair.e-p-below)
apply simp
done

```

```

lemma fmap-strict:
  fixes f :: 'a → 'b
  assumes f·⊥ = ⊥ shows fmap·f·⊥ = (⊥::'b.'f::functor)
proof (rule bottomI)
  have fmap·f·(⊥::'a.'f) ⊆ fmap·f·(fmap·⊥·(⊥::'b.'f))
    by (simp add: monofun-cfun)
  also have ... = fmap·(λ x. f·(⊥·x))·(⊥::'b.'f)
    by (simp add: fmap-fmap)
  also have ... ⊆ fmap·ID·⊥
    by (simp add: monofun-cfun assms del: fmap-ID)
  also have ... = ⊥
    by simp
  finally show fmap·f·⊥ ⊆ (⊥::'b.'f::functor) .
qed

```

### 3.4 Proving that $fmap \cdot coerce = coerce$

```

lemma fmapU-cast-eq:
  fmapU·(cast·A) =
    PRJ(udom.'f) oo cast·(TC('f::functor)·A) oo emb
by (subst cast-TC, rule cfun-eqI, simp)

```

```

lemma fmapU-cast-DEFL:
  fmapU·(cast·DEFL('a)) =
    PRJ(udom.'f) oo cast·DEFL('a.'f::functor) oo emb
by (simp add: fmapU-cast-eq DEFL-app)

```

```

lemma coerce-functor: COERCE('a.'f, 'b.'f::functor) = fmap·coerce
apply (rule cfun-eqI, rename-tac xs)
apply (simp add: fmap-def coerce-cfun)
apply (simp add: coerce-def)
apply (simp add: cfcomp1)
apply (simp only: emb-prj)
apply (subst fmapU-fmapU [symmetric])
apply (simp add: fmapU-cast-DEFL)
apply (simp add: emb-prj)
apply (simp add: cast-cast-below1 cast-cast-below2)
done

```

### 3.5 Lemmas for reasoning about coercion

```

lemma fmapU-cast-coerce [coerce-simp]:
  fixes m :: 'a.'f::functor

```

```

shows fmapU·(cast·DEFL('a))·(COERCE('a·'f, udom·'f)·m) =
  COERCE('a·'f, udom·'f)·m
by (simp add: coerce-functor cast-DEFL fmapU-eq-fmap fmap-fmap eta-cfun)

```

```

lemma coerce-fmap [coerce-simp]:
  fixes xs :: 'a·'f::functor and f :: 'a → 'b
  shows COERCE('b·'f, 'c·'f)·(fmap·f·xs) = fmap·(λ x. COERCE('b,'c)·(f·x))·xs
by (simp add: coerce-functor fmap-fmap)

```

```

lemma fmap-coerce [coerce-simp]:
  fixes xs :: 'a·'f::functor and f :: 'b → 'c
  shows fmap·f·(COERCE('a·'f, 'b·'f)·xs) = fmap·(λ x. f·(COERCE('a,'b)·x))·xs
by (simp add: coerce-functor fmap-fmap)

```

### 3.6 Configuration of Domain package

We make various theorem declarations to enable Domain package definitions that involve *tycon* application.

```

setup << Domain-Take-Proofs.add-rec-type (@{type-name app}, [true, false]) >>

```

```

declare DEFL-app [domain-defl-simps]
declare fmap-ID [domain-map-ID]
declare deflation-fmap [domain-deflation]
declare isodefl-fmap [domain-isodefl]

```

### 3.7 Configuration of the Tycon package

We now set up a new type definition command, which is used for defining new *tycon* instances. The *tycondef* command is implemented using much of the same code as the Domain package, and supports a similar input syntax. It automatically generates a *prefunctor* instance for each new type. (The user must provide a proof of the composition law to obtain a *functor* class instance.)

```

ML-file tycondef.ML

```

```

end

```

## 4 Monad Class

```

theory Monad
imports Functor
begin

```

### 4.1 Class definition

In Haskell, class *Monad* is defined as follows:

```

class Monad m where
  return :: a -> m a
  (>>=) :: m a -> (a -> m b) -> m b

```

We formalize class *monad* in a manner similar to the *functor* class: We fix monomorphic versions of the class constants, replacing type variables with *udom*, and assume monomorphic versions of the class axioms.

Because the monad laws imply the composition rule for *fmap*, we declare *prefunctor* as the superclass, and separately prove a subclass relationship with *functor*.

```

class monad = prefunctor +
  fixes returnU :: udom -> udom.'a::tycon
  fixes bindU :: udom.'a -> (udom -> udom.'a) -> udom.'a
  assumes fmapU-eq-bindU:
     $\bigwedge xs. \text{fmapU} \cdot f \cdot xs = \text{bindU} \cdot xs \cdot (\lambda x. \text{returnU} \cdot (f \cdot x))$ 
  assumes bindU-returnU:
     $\bigwedge f x. \text{bindU} \cdot (\text{returnU} \cdot x) \cdot f = f \cdot x$ 
  assumes bindU-bindU:
     $\bigwedge xs f g. \text{bindU} \cdot (\text{bindU} \cdot xs \cdot f) \cdot g = \text{bindU} \cdot xs \cdot (\lambda x. \text{bindU} \cdot (f \cdot x) \cdot g)$ 

```

```
instance monad  $\subseteq$  functor
```

```
proof
```

```

  fix f g :: udom -> udom and xs :: udom.'a
  show fmapU f (fmapU g xs) = fmapU (lambda x. f (g x)) xs
  by (simp add: fmapU-eq-bindU bindU-bindU bindU-returnU)
qed

```

As with *fmap*, we define the polymorphic *return* and *bind* by coercion from the monomorphic *returnU* and *bindU*.

```

definition return :: 'a -> 'a.'m::monad
  where return = coerce (returnU :: udom -> udom.'m)

```

```

definition bind :: 'a.'m::monad -> ('a -> 'b.'m) -> 'b.'m
  where bind = coerce (bindU :: udom.'m -> -)

```

```

abbreviation bind-syn :: 'a.'m::monad => ('a -> 'b.'m) => 'b.'m (infixl  $\gg=$  55)
  where m  $\gg=$  f  $\equiv$  bind . m . f

```

## 4.2 Naturality of bind and return

The three class axioms imply naturality properties of *returnU* and *bindU*, i.e., that both commute with *fmapU*.

```

lemma fmapU-returnU [coerce-simp]:
  fmapU f (returnU x) = returnU (f x)
by (simp add: fmapU-eq-bindU bindU-returnU)

```

**lemma** *fmapU-bindU* [*coerce-simp*]:  
 $fmapU \cdot f \cdot (bindU \cdot m \cdot k) = bindU \cdot m \cdot (\Lambda x. fmapU \cdot f \cdot (k \cdot x))$   
**by** (*simp add: fmapU-eq-bindU bindU-bindU*)

**lemma** *bindU-fmapU*:  
 $bindU \cdot (fmapU \cdot f \cdot xs) \cdot k = bindU \cdot xs \cdot (\Lambda x. k \cdot (f \cdot x))$   
**by** (*simp add: fmapU-eq-bindU bindU-returnU bindU-bindU*)

### 4.3 Polymorphic versions of class assumptions

**lemma** *monad-fmap*:  
**fixes**  $xs :: 'a \cdot 'm :: monad$  **and**  $f :: 'a \rightarrow 'b$   
**shows**  $fmap \cdot f \cdot xs = xs \gg (\Lambda x. return \cdot (f \cdot x))$   
**unfolding** *bind-def return-def fmap-def*  
**by** (*simp add: coerce-simp fmapU-eq-bindU bindU-returnU*)

**lemma** *monad-left-unit* [*simp*]:  $(return \cdot x \gg f) = (f \cdot x)$   
**unfolding** *bind-def return-def*  
**by** (*simp add: coerce-simp bindU-returnU*)

**lemma** *bind-bind*:  
**fixes**  $m :: 'a \cdot 'm :: monad$   
**shows**  $((m \gg f) \gg g) = (m \gg (\Lambda x. f \cdot x \gg g))$   
**unfolding** *bind-def*  
**by** (*simp add: coerce-simp bindU-bindU*)

### 4.4 Derived rules

The following properties can be derived using only the abstract monad laws.

**lemma** *monad-right-unit* [*simp*]:  $(m \gg return) = m$   
**apply** (*subgoal-tac fmap.ID \cdot m = m*)  
**apply** (*simp only: monad-fmap*)  
**apply** (*simp add: eta-cfun*)  
**apply** *simp*  
**done**

**lemma** *fmap-return*:  $fmap \cdot f \cdot (return \cdot x) = return \cdot (f \cdot x)$   
**by** (*simp add: monad-fmap*)

**lemma** *fmap-bind*:  $fmap \cdot f \cdot (bind \cdot xs \cdot k) = bind \cdot xs \cdot (\Lambda x. fmap \cdot f \cdot (k \cdot x))$   
**by** (*simp add: monad-fmap bind-bind*)

**lemma** *bind-fmap*:  $bind \cdot (fmap \cdot f \cdot xs) \cdot k = bind \cdot xs \cdot (\Lambda x. k \cdot (f \cdot x))$   
**by** (*simp add: monad-fmap bind-bind*)

Bind is strict in its first argument, if its second argument is a strict function.

**lemma** *bind-strict*:  
**assumes**  $k \cdot \perp = \perp$  **shows**  $\perp \gg k = \perp$



**proof** –  
**have**  $\perp \gg k \sqsubseteq \text{return}.\perp \gg k$   
**by** (*intro monofun-cfun below-refl minimal*)  
**thus**  $\perp \gg k = \perp$   
**by** (*simp add: assms*)  
**qed**

**lemma congruent-bind:**  
 $(\forall m. m \gg k1 = m \gg k2) = (k1 = k2)$   
**apply** (*safe, rule cfun-eqI*)  
**apply** (*drule-tac x=return.x in spec, simp*)  
**done**

## 4.5 Laws for join

**definition** *join* ::  $('a.'m) \rightarrow 'a.'m::\text{monad}$   
**where** *join*  $\equiv \Lambda m. m \gg (\Lambda x. x)$

**lemma join-fmap-fmap:**  $\text{join} \cdot (\text{fmap} \cdot (\text{fmap} \cdot f) \cdot xss) = \text{fmap} \cdot f \cdot (\text{join} \cdot xss)$   
**by** (*simp add: join-def monad-fmap bind-bind*)

**lemma join-return:**  $\text{join} \cdot (\text{return} \cdot xs) = xs$   
**by** (*simp add: join-def*)

**lemma join-fmap-return:**  $\text{join} \cdot (\text{fmap} \cdot \text{return} \cdot xs) = xs$   
**by** (*simp add: join-def monad-fmap eta-cfun bind-bind*)

**lemma join-fmap-join:**  $\text{join} \cdot (\text{fmap} \cdot \text{join} \cdot xsss) = \text{join} \cdot (\text{join} \cdot xsss)$   
**by** (*simp add: join-def monad-fmap bind-bind*)

**lemma bind-def2:**  $m \gg k = \text{join} \cdot (\text{fmap} \cdot k \cdot m)$   
**by** (*simp add: join-def monad-fmap eta-cfun bind-bind*)

## 4.6 Equivalence of monad laws and fmap/join laws

**lemma**  $(\text{return} \cdot x \gg f) = (f \cdot x)$   
**by** (*simp only: bind-def2 fmap-return join-return*)

**lemma**  $(m \gg \text{return}) = m$   
**by** (*simp only: bind-def2 join-fmap-return*)

**lemma**  $((m \gg f) \gg g) = (m \gg (\Lambda x. f \cdot x \gg g))$   
**apply** (*simp only: bind-def2*)  
**apply** (*subgoal-tac join \cdot (\text{fmap} \cdot g \cdot (\text{join} \cdot (\text{fmap} \cdot f \cdot m))) =*  
 $\text{join} \cdot (\text{fmap} \cdot \text{join} \cdot (\text{fmap} \cdot (\text{fmap} \cdot g) \cdot (\text{fmap} \cdot f \cdot m)))$ )  
**apply** (*simp add: fmap-fmap*)  
**apply** (*simp add: join-fmap-join join-fmap-fmap*)  
**done**

## 4.7 Simplification of coercions

We configure rewrite rules that push coercions inwards, and reduce them to coercions on simpler types.

**lemma** *coerce-return* [*coerce-simp*]:

$COERCE('a.'m, 'b.'m::monad).(return.x) = return.(COERCE('a, 'b).x)$   
by (*simp add: coerce-functor fmap-return*)

**lemma** *coerce-bind* [*coerce-simp*]:

**fixes**  $m :: 'a.'m::monad$  **and**  $k :: 'a \rightarrow 'b.'m$   
**shows**  $COERCE('b.'m, 'c.'m).(m \gg= k) = m \gg= (\Lambda x. COERCE('b.'m, 'c.'m).(k.x))$   
by (*simp add: coerce-functor fmap-bind*)

**lemma** *bind-coerce* [*coerce-simp*]:

**fixes**  $m :: 'a.'m::monad$  **and**  $k :: 'b \rightarrow 'c.'m$   
**shows**  $COERCE('a.'m, 'b.'m).m \gg= k = m \gg= (\Lambda x. k.(COERCE('a, 'b).x))$   
by (*simp add: coerce-functor bind-fmap*)

**end**

## 5 Monad-Zero Class

**theory** *Monad-Zero*  
**imports** *Monad*  
**begin**

**class** *zeroU* = *tycon* +  
**fixes**  $zeroU :: udom.'a::tycon$

**class** *functor-zero* = *zeroU* + *functor* +  
**assumes** *fmapU-zeroU* [*coerce-simp*]:  
 $fmapU.f.zeroU = zeroU$

**class** *monad-zero* = *zeroU* + *monad* +  
**assumes** *bindU-zeroU*:  
 $bindU.zeroU.f = zeroU$

**instance** *monad-zero*  $\subseteq$  *functor-zero*

**proof**

**fix**  $f$  **show**  $fmapU.f.zeroU = (zeroU :: udom.'a)$   
**unfolding** *fmapU-eq-bindU*  
**by** (*rule bindU-zeroU*)

**qed**

**definition**  $fzero :: 'a.'f::functor-zero$   
**where**  $fzero = coerce.(zeroU :: udom.'f)$

**lemma** *fmap-fzero*:

$fmap \cdot f \cdot (fzero :: 'a \cdot 'f :: functor\text{-}zero) = (fzero :: 'b \cdot 'f)$   
**unfolding** *fmap-def fzero-def*  
**by** (*simp add: coerce-simp*)

**abbreviation**  $mzero :: 'a \cdot 'm :: monad\text{-}zero$   
**where**  $mzero \equiv fzero$

**lemmas**  $mzero\text{-}def = fzero\text{-}def$  [**where**  $'f = 'm :: monad\text{-}zero$ ] **for**  $f$   
**lemmas**  $fmap\text{-}mzero = fmap\text{-}fzero$  [**where**  $'f = 'm :: monad\text{-}zero$ ] **for**  $f$

**lemma**  $bindU\text{-}eq\text{-}bind$ :  $bindU = bind$   
**unfolding** *bind-def* **by** *simp*

**lemma**  $bind\text{-}mzero$ :  
 $bind \cdot (fzero :: 'a \cdot 'm :: monad\text{-}zero) \cdot k = (mzero :: 'b \cdot 'm)$   
**unfolding** *bind-def mzero-def*  
**by** (*simp add: coerce-simp bindU-zeroU*)

**end**

## 6 Monad-Plus Class

**theory** *Monad-Plus*  
**imports** *Monad*  
**begin**

**hide-const** (**open**) *Fixrec.mplus*

**class**  $plusU = tycon +$   
**fixes**  $plusU :: udom \cdot 'a \rightarrow udom \cdot 'a \rightarrow udom \cdot 'a :: tycon$

**class**  $functor\text{-}plus = plusU + functor +$   
**assumes**  $fmapU\text{-}plusU$  [*coerce-simp*]:  
 $fmapU \cdot f \cdot (plusU \cdot a \cdot b) = plusU \cdot (fmapU \cdot f \cdot a) \cdot (fmapU \cdot f \cdot b)$   
**assumes**  $plusU\text{-}assoc$ :  
 $plusU \cdot (plusU \cdot a \cdot b) \cdot c = plusU \cdot a \cdot (plusU \cdot b \cdot c)$

**class**  $monad\text{-}plus = plusU + monad +$   
**assumes**  $bindU\text{-}plusU$ :  
 $bindU \cdot (plusU \cdot xs \cdot ys) \cdot k = plusU \cdot (bindU \cdot xs \cdot k) \cdot (bindU \cdot ys \cdot k)$   
**assumes**  $plusU\text{-}assoc'$ :  
 $plusU \cdot (plusU \cdot a \cdot b) \cdot c = plusU \cdot a \cdot (plusU \cdot b \cdot c)$

**instance**  $monad\text{-}plus \subseteq functor\text{-}plus$   
**by** *standard (simp-all only: fmapU-eq-bindU bindU-plusU plusU-assoc')*

**definition**  $fplus :: 'a \cdot 'f :: functor\text{-}plus \rightarrow 'a \cdot 'f \rightarrow 'a \cdot 'f$   
**where**  $fplus = coerce \cdot (plusU :: udom \cdot 'f \rightarrow -)$

```

lemma fmap-fplus:
  fixes f :: 'a → 'b and a b :: 'a.'f::functor-plus
  shows fmap·f·(fplus·a·b) = fplus·(fmap·f·a)·(fmap·f·b)
unfolding fmap-def fplus-def
by (simp add: coerce-simp)

lemma fplus-assoc:
  fixes a b c :: 'a.'f::functor-plus
  shows fplus·(fplus·a·b)·c = fplus·a·(fplus·b·c)
unfolding fplus-def
by (simp add: coerce-simp plusU-assoc)

abbreviation mplus :: 'a.'m::monad-plus → 'a.'m → 'a.'m
  where mplus ≡ fplus

lemmas mplus-def = fplus-def [where 'f='m::monad-plus for f]
lemmas fmap-mplus = fmap-fplus [where 'f='m::monad-plus for f]
lemmas mplus-assoc = fplus-assoc [where 'f='m::monad-plus for f]

lemma bind-mplus:
  fixes a b :: 'a.'m::monad-plus
  shows bind·(mplus·a·b)·k = mplus·(bind·a·k)·(bind·b·k)
unfolding bind-def mplus-def
by (simp add: coerce-simp bindU-plusU)

lemma join-mplus:
  fixes xss yss :: ('a.'m)·'m::monad-plus
  shows join·(mplus·xss·yss) = mplus·(join·xss)·(join·yss)
by (simp add: join-def bind-mplus)

end

```

## 7 Monad-Zero-Plus Class

```

theory Monad-Zero-Plus
imports Monad-Zero Monad-Plus
begin

hide-const (open) Fixrec.mplus

class functor-zero-plus = functor-zero + functor-plus +
  assumes plusU-zeroU-left:
    plusU·zeroU·m = m
  assumes plusU-zeroU-right:
    plusU·m·zeroU = m

class monad-zero-plus = monad-zero + monad-plus + functor-zero-plus

lemma fplus-fzero-left:

```

```

fixes m :: 'a.'f::functor-zero-plus
shows fplus.fzero.m = m
unfolding fplus-def fzero-def
by (simp add: coerce-simp plusU-zeroU-left)

```

```

lemma fplus-fzero-right:
  fixes m :: 'a.'f::functor-zero-plus
  shows fplus.m.fzero = m
unfolding fplus-def fzero-def
by (simp add: coerce-simp plusU-zeroU-right)

```

```

lemmas mplus-mzero-left =
  fplus-fzero-left [where 'f='m::monad-zero-plus] for f

```

```

lemmas mplus-mzero-right =
  fplus-fzero-right [where 'f='m::monad-zero-plus] for f

```

```

end

```

## 8 Lazy list monad

```

theory Lazy-List-Monad
imports Monad-Zero-Plus
begin

```

To illustrate the general process of defining a new type constructor, we formalize the datatype of lazy lists. Below are the Haskell datatype definition and class instances.

```

data List a = Nil | Cons a (List a)

instance Functor List where
  fmap f Nil = Nil
  fmap f (Cons x xs) = Cons (f x) (fmap f xs)

instance Monad List where
  return x          = Cons x Nil
  Nil              >>= k = Nil
  Cons x xs >>= k = mplus (k x) (xs >>= k)

instance MonadZero List where
  mzero = Nil

instance MonadPlus List where
  mplus Nil          ys = ys
  mplus (Cons x xs) ys = Cons x (mplus xs ys)

```

## 8.1 Type definition

The first step is to register the datatype definition with *tycondef*.

```
tycondef 'a·llist = LNil | LCons (lazy 'a) (lazy 'a·llist)
```

The *tycondef* command generates lots of theorems automatically, but there are a few more involving *coerce* and *fmapU* that we still need to prove manually. These proofs could be automated in a later version of *tycondef*.

```
lemma coerce-llist-abs [simp]: coerce·(llist-abs·x) = llist-abs·(coerce·x)  
apply (simp add: llist-abs-def coerce-def)  
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-llist)  
done
```

```
lemma coerce-LNil [simp]: coerce·LNil = LNil  
unfolding LNil-def by simp
```

```
lemma coerce-LCons [simp]: coerce·(LCons·x·xs) = LCons·(coerce·x)·(coerce·xs)  
unfolding LCons-def by simp
```

```
lemma fmapU-llist-simps [simp]:  
  fmapU·f·( $\perp$ ::udom·llist) =  $\perp$   
  fmapU·f·LNil = LNil  
  fmapU·f·(LCons·x·xs) = LCons·(f·x)·(fmapU·f·xs)  
unfolding fmapU-llist-def llist-map-def  
apply (subst fix-eq, simp)  
apply (subst fix-eq, simp add: LNil-def)  
apply (subst fix-eq, simp add: LCons-def)  
done
```

## 8.2 Class instances

The *tycondef* command defines *fmapU* for us and proves a *prefunctor* class instance automatically. For the *functor* instance we only need to prove the composition law, which we can do by induction.

```
instance llist :: functor  
proof  
  fix f g and xs :: udom·llist  
  show fmapU·f·(fmapU·g·xs) = fmapU·( $\Lambda$  x. f·(g·x))·xs  
  by (induct xs rule: llist.induct) simp-all  
qed
```

For the other class instances, we need to provide definitions for a few constants: *returnU*, *bindU*, *zeroU*, and *plusU*. We can use ordinary commands like *definition* and *fixrec* for this purpose. Finally we prove the class axioms, along with a few helper lemmas, using ordinary proof procedures like *induction*.

**instantiation** *llist* :: *monad-zero-plus*  
**begin**

**fixrec** *plusU-llist* :: *udom-llist* → *udom-llist* → *udom-llist*  
**where** *plusU-llist*·*LNil*·*ys* = *ys*  
| *plusU-llist*·(*LCons*·*x*·*xs*)·*ys* = *LCons*·*x*·(*plusU-llist*·*xs*·*ys*)

**lemma** *plusU-llist-strict* [*simp*]: *plusU*·⊥·*ys* = (⊥::*udom-llist*)  
**by** *fixrec-simp*

**fixrec** *bindU-llist* :: *udom-llist* → (*udom* → *udom-llist*) → *udom-llist*  
**where** *bindU-llist*·*LNil*·*k* = *LNil*  
| *bindU-llist*·(*LCons*·*x*·*xs*)·*k* = *plusU*·(*k*·*x*)·(*bindU-llist*·*xs*·*k*)

**lemma** *bindU-llist-strict* [*simp*]: *bindU*·⊥·*k* = (⊥::*udom-llist*)  
**by** *fixrec-simp*

**definition** *zeroU-llist-def*:  
*zeroU* = *LNil*

**definition** *returnU-llist-def*:  
*returnU* = (Λ *x*. *LCons*·*x*·*LNil*)

**lemma** *plusU-LNil-right*: *plusU*·*xs*·*LNil* = *xs*  
**by** (*induct xs rule: llist.induct*) *simp-all*

**lemma** *plusU-llist-assoc*:  
**fixes** *xs ys zs* :: *udom-llist*  
**shows** *plusU*·(*plusU*·*xs*·*ys*)·*zs* = *plusU*·*xs*·(*plusU*·*ys*·*zs*)  
**by** (*induct xs rule: llist.induct*) *simp-all*

**lemma** *bindU-plusU-llist*:  
**fixes** *xs ys* :: *udom-llist* **shows**  
*bindU*·(*plusU*·*xs*·*ys*)·*f* = *plusU*·(*bindU*·*xs*·*f*)·(*bindU*·*ys*·*f*)  
**by** (*induct xs rule: llist.induct*) (*simp-all add: plusU-llist-assoc*)

**instance proof**  
**fix** *x* :: *udom*  
**fix** *f* :: *udom* → *udom*  
**fix** *h k* :: *udom* → *udom-llist*  
**fix** *xs ys zs* :: *udom-llist*  
**show** *fmapU*·*f*·*xs* = *bindU*·*xs*·(Λ *x*. *returnU*·(*f*·*x*))  
**by** (*induct xs rule: llist.induct*, *simp-all add: returnU-llist-def*)  
**show** *bindU*·(*returnU*·*x*)·*k* = *k*·*x*  
**by** (*simp add: returnU-llist-def plusU-LNil-right*)  
**show** *bindU*·(*bindU*·*xs*·*h*)·*k* = *bindU*·*xs*·(Λ *x*. *bindU*·(*h*·*x*)·*k*)  
**by** (*induct xs rule: llist.induct*)  
(*simp-all add: bindU-plusU-llist*)  
**show** *bindU*·(*plusU*·*xs*·*ys*)·*k* = *plusU*·(*bindU*·*xs*·*k*)·(*bindU*·*ys*·*k*)

```

    by (induct xs rule: llist.induct)
      (simp-all add: plusU-llist-assoc)
  show plusU·(plusU·xs·ys)·zs = plusU·xs·(plusU·ys·zs)
    by (rule plusU-llist-assoc)
  show bindU·zeroU·k = zeroU
    by (simp add: zeroU-llist-def)
  show fmapU·f·(plusU·xs·ys) = plusU·(fmapU·f·xs)·(fmapU·f·ys)
    by (induct xs rule: llist.induct) simp-all
  show fmapU·f·zeroU = (zeroU :: udom·llist)
    by (simp add: zeroU-llist-def)
  show plusU·zeroU·xs = xs
    by (simp add: zeroU-llist-def)
  show plusU·xs·zeroU = xs
    by (simp add: zeroU-llist-def plusU-LNil-right)
qed

end

```

### 8.3 Transfer properties to polymorphic versions

After proving the class instances, there is still one more step: We must transfer all the list-specific lemmas about the monomorphic constants (e.g.,  $fmapU$  and  $bindU$ ) to the corresponding polymorphic constants ( $fmap$  and  $bind$ ). These lemmas primarily consist of the defining equations for each constant. The polymorphic constants are defined using *coerce*, so the proofs proceed by unfolding the definitions and simplifying with the *coerce-simp* rules.

```

lemma fmap-llist-simps [simp]:
  fmap·f·(⊥::'a·llist) = ⊥
  fmap·f·LNil = LNil
  fmap·f·(LCons·x·xs) = LCons·(f·x)·(fmap·f·xs)
unfolding fmap-def by simp-all

```

```

lemma mplus-llist-simps [simp]:
  mplus·(⊥::'a·llist)·ys = ⊥
  mplus·LNil·ys = ys
  mplus·(LCons·x·xs)·ys = LCons·x·(mplus·xs·ys)
unfolding mplus-def by simp-all

```

```

lemma bind-llist-simps [simp]:
  bind·(⊥::'a·llist)·f = ⊥
  bind·LNil·f = LNil
  bind·(LCons·x·xs)·f = mplus·(f·x)·(bind·xs·f)
unfolding bind-def mplus-def
by (simp-all add: coerce-simp)

```

```

lemma return-llist-def:
  return = (Λ x. LCons·x·LNil)

```



**unfolding** *return-def returnU-llist-def*  
**by** (*simp add: coerce-simp*)

**lemma** *mzero-llist-def*:  
 $mzero = LNil$   
**unfolding** *mzero-def zeroU-llist-def*  
**by** *simp*

**lemma** *join-llist-simps* [*simp*]:  
 $join.(⊥::'a·llist·llist) = ⊥$   
 $join.LNil = LNil$   
 $join.(LCons·xs·xss) = mplus·xs·(join·xss)$   
**unfolding** *join-def* **by** *simp-all*

**end**

## 9 Maybe monad

**theory** *Maybe-Monad*  
**imports** *Monad-Zero-Plus*  
**begin**

### 9.1 Type definition

**tycondef** *'a·maybe = Nothing | Just (lazy 'a)*

**lemma** *coerce-maybe-abs* [*simp*]:  $coerce.(maybe-abs·x) = maybe-abs.(coerce·x)$   
**apply** (*simp add: maybe-abs-def coerce-def*)  
**apply** (*simp add: emb-prj-emb prj-emb-prj DEFL-eq-maybe*)  
**done**

**lemma** *coerce-Nothing* [*simp*]:  $coerce.Nothing = Nothing$   
**unfolding** *Nothing-def* **by** *simp*

**lemma** *coerce-Just* [*simp*]:  $coerce.(Just·x) = Just.(coerce·x)$   
**unfolding** *Just-def* **by** *simp*

**lemma** *fmapU-maybe-simps* [*simp*]:  
 $fmapU·f.(⊥::udom·maybe) = ⊥$   
 $fmapU·f.Nothing = Nothing$   
 $fmapU·f.(Just·x) = Just.(f·x)$   
**unfolding** *fmapU-maybe-def maybe-map-def fix-const*  
**apply** *simp*  
**apply** (*simp add: Nothing-def*)  
**apply** (*simp add: Just-def*)  
**done**

## 9.2 Class instance proofs

**instance** *maybe* :: *functor*

**apply** *standard*

**apply** (*induct-tac xs rule: maybe.induct, simp-all*)

**done**

**instantiation** *maybe* :: {*functor-zero-plus, monad-zero*}

**begin**

**fixrec** *plusU-maybe* :: *udom·maybe* → *udom·maybe* → *udom·maybe*

**where** *plusU-maybe.Nothing·ys* = *ys*

| *plusU-maybe.(Just·x)·ys* = *Just·x*

**lemma** *plusU-maybe-strict* [*simp*]: *plusU·⊥·ys* = ( $\perp$ ::*udom·maybe*)

**by** *fixrec-simp*

**fixrec** *bindU-maybe* :: *udom·maybe* → (*udom* → *udom·maybe*) → *udom·maybe*

**where** *bindU-maybe.Nothing·k* = *Nothing*

| *bindU-maybe.(Just·x)·k* = *k·x*

**lemma** *bindU-maybe-strict* [*simp*]: *bindU·⊥·k* = ( $\perp$ ::*udom·maybe*)

**by** *fixrec-simp*

**definition** *zeroU-maybe-def*:

*zeroU* = *Nothing*

**definition** *returnU-maybe-def*:

*returnU* = *Just*

**lemma** *plusU-Nothing-right*: *plusU·xs·Nothing* = *xs*

**by** (*induct xs rule: maybe.induct*) *simp-all*

**lemma** *bindU-plusU-maybe*:

**fixes** *xs ys* :: *udom·maybe* **shows**

*bindU·(plusU·xs·ys)·f* = *plusU·(bindU·xs·f)·(bindU·ys·f)*

**apply** (*induct xs rule: maybe.induct*)

**apply** *simp-all*

**oops**

**instance proof**

**fix** *x* :: *udom*

**fix** *f* :: *udom* → *udom*

**fix** *h k* :: *udom* → *udom·maybe*

**fix** *xs ys zs* :: *udom·maybe*

**show** *fmapU·f·xs* = *bindU·xs·(λ x. returnU·(f·x))*

**by** (*induct xs rule: maybe.induct, simp-all add: returnU-maybe-def*)

**show** *bindU·(returnU·x)·k* = *k·x*

**by** (*simp add: returnU-maybe-def plusU-Nothing-right*)

**show** *bindU·(bindU·xs·h)·k* = *bindU·xs·(λ x. bindU·(h·x)·k)*

by (*induct xs rule: maybe.induct simp-all*)  
**show**  $plusU \cdot (plusU \cdot xs \cdot ys) \cdot zs = plusU \cdot xs \cdot (plusU \cdot ys \cdot zs)$   
 by (*induct xs rule: maybe.induct simp-all*)  
**show**  $bindU \cdot zeroU \cdot k = zeroU$   
 by (*simp add: zeroU-maybe-def*)  
**show**  $fmapU \cdot f \cdot (plusU \cdot xs \cdot ys) = plusU \cdot (fmapU \cdot f \cdot xs) \cdot (fmapU \cdot f \cdot ys)$   
 by (*induct xs rule: maybe.induct simp-all*)  
**show**  $fmapU \cdot f \cdot zeroU = (zeroU :: udom \cdot maybe)$   
 by (*simp add: zeroU-maybe-def*)  
**show**  $plusU \cdot zeroU \cdot xs = xs$   
 by (*simp add: zeroU-maybe-def*)  
**show**  $plusU \cdot xs \cdot zeroU = xs$   
 by (*simp add: zeroU-maybe-def plusU-Nothing-right*)  
**qed**

**end**

### 9.3 Transfer properties to polymorphic versions

**lemma** *fmap-maybe-simps* [*simp*]:

$fmap \cdot f \cdot (\perp :: 'a \cdot maybe) = \perp$   
 $fmap \cdot f \cdot Nothing = Nothing$   
 $fmap \cdot f \cdot (Just \cdot x) = Just \cdot (f \cdot x)$

**unfolding** *fmap-def* **by** *simp-all*

**lemma** *fplus-maybe-simps* [*simp*]:

$fplus \cdot (\perp :: 'a \cdot maybe) \cdot ys = \perp$   
 $fplus \cdot Nothing \cdot ys = ys$   
 $fplus \cdot (Just \cdot x) \cdot ys = Just \cdot x$

**unfolding** *fplus-def* **by** *simp-all*

**lemma** *fplus-Nothing-right* [*simp*]:

$fplus \cdot m \cdot Nothing = m$

**by** (*simp add: fplus-def plusU-Nothing-right*)

**lemma** *bind-maybe-simps* [*simp*]:

$bind \cdot (\perp :: 'a \cdot maybe) \cdot f = \perp$   
 $bind \cdot Nothing \cdot f = Nothing$   
 $bind \cdot (Just \cdot x) \cdot f = f \cdot x$

**unfolding** *bind-def fplus-def* **by** *simp-all*

**lemma** *return-maybe-def*:  $return = Just$

**unfolding** *return-def returnU-maybe-def*

**by** (*simp add: coerce-cfun cfcomp1 eta-cfun*)

**lemma** *mzero-maybe-def*:  $mzero = Nothing$

**unfolding** *mzero-def zeroU-maybe-def*

**by** *simp*

**lemma** *join-maybe-simps* [simp]:  
 $join.(⊥::'a.maybe.maybe) = ⊥$   
 $join.Nothing = Nothing$   
 $join.(Just.xs) = xs$   
**unfolding** *join-def* **by** *simp-all*

## 9.4 Maybe is not in *monad-plus*

The *maybe* type does not satisfy the law *bind-mplus*.

**lemma** *maybe-counterexample1*:  
 $[[a = Just.x; b = ⊥; k.x = Nothing]]$   
 $⇒ fplus.a.b ≫ k ≠ fplus.(a ≫ k).(b ≫ k)$   
**by** *simp*

**lemma** *maybe-counterexample2*:  
 $[[a = Just.x; b = Just.y; k.x = Nothing; k.y = Just.z]]$   
 $⇒ fplus.a.b ≫ k ≠ fplus.(a ≫ k).(b ≫ k)$   
**by** *simp*

**end**

## 10 Error monad

**theory** *Error-Monad*  
**imports** *Monad-Plus*  
**begin**

### 10.1 Type definition

**tycondef** *'a.'e error* = *Err* (lazy 'e) | *Ok* (lazy 'a)

**lemma** *coerce-error-abs* [simp]:  $coerce.(error-abs.x) = error-abs.(coerce.x)$   
**apply** (*simp add: error-abs-def coerce-def*)  
**apply** (*simp add: emb-prj-emb prj-emb-prj DEFL-eq-error*)  
**done**

**lemma** *coerce-Err* [simp]:  $coerce.(Err.x) = Err.(coerce.x)$   
**unfolding** *Err-def* **by** *simp*

**lemma** *coerce-Ok* [simp]:  $coerce.(Ok.m) = Ok.(coerce.m)$   
**unfolding** *Ok-def* **by** *simp*

**lemma** *fmapU-error-simps* [simp]:  
 $fmapU.f.(⊥::udom.'a error) = ⊥$   
 $fmapU.f.(Err.e) = Err.e$   
 $fmapU.f.(Ok.x) = Ok.(f.x)$   
**unfolding** *fmapU-error-def error-map-def fix-const*  
**apply** *simp*

```

apply (simp add: Err-def)
apply (simp add: Ok-def)
done

```

## 10.2 Monad class instance

```

instantiation error :: (domain) {monad, functor-plus}
begin

```

**definition**

```

  returnU = Ok

```

```

fixrec bindU-error :: udom.'a error → (udom → udom.'a error) → udom.'a error
  where bindU-error.(Err.e).f = Err.e
  | bindU-error.(Ok.x).f = f.x

```

```

lemma bindU-error-strict [simp]: bindU.⊥.k = (⊥::udom.'a error)
by fixrec-simp

```

```

fixrec plusU-error :: udom.'a error → udom.'a error → udom.'a error
  where plusU-error.(Err.e).f = f
  | plusU-error.(Ok.x).f = Ok.x

```

```

lemma plusU-error-strict [simp]: plusU.(⊥ :: udom.'a error) = ⊥
by fixrec-simp

```

**instance proof**

```

  fix f g :: udom → udom and r :: udom.'a error
  show fmapU.f.(fmapU.g.r) = fmapU.(λ x. f.(g.x)).r
  by (induct r rule: error.induct) simp-all

```

**next**

```

  fix f :: udom → udom and r :: udom.'a error
  show fmapU.f.r = bindU.r.(λ x. returnU.(f.x))
  by (induct r rule: error.induct)
  (simp-all add: returnU-error-def)

```

**next**

```

  fix f :: udom → udom.'a error and x :: udom
  show bindU.(returnU.x).f = f.x
  by (simp add: returnU-error-def)

```

**next**

```

  fix r :: udom.'a error and f g :: udom → udom.'a error
  show bindU.(bindU.r.f).g = bindU.r.(λ x. bindU.(f.x).g)
  by (induct r rule: error.induct)
  simp-all

```

**next**

```

  fix f :: udom → udom and a b :: udom.'a error
  show fmapU.f.(plusU.a.b) = plusU.(fmapU.f.a).(fmapU.f.b)
  by (induct a rule: error.induct) simp-all

```

**next**

```

fix a b c :: udom.'a error
show plusU.(plusU.a.b).c = plusU.a.(plusU.b.c)
  by (induct a rule: error.induct) simp-all
qed

end

```

### 10.3 Transfer properties to polymorphic versions

```

lemma fmap-error-simps [simp]:
  fmap.f.(⊥ :: 'a.'e error) = ⊥
  fmap.f.(Err.e :: 'a.'e error) = Err.e
  fmap.f.(Ok.x :: 'a.'e error) = Ok.(f.x)
unfolding fmap-def [where 'f='e error]
by (simp-all add: coerce-simp)

```

```

lemma return-error-def: return = Ok
unfolding return-def returnU-error-def
by (simp add: coerce-simp eta-cfun)

```

```

lemma bind-error-simps [simp]:
  bind.(⊥ :: 'a.'e error).f = ⊥
  bind.(Err.e :: 'a.'e error).f = Err.e
  bind.(Ok.x :: 'a.'e error).f = f.x
unfolding bind-def
by (simp-all add: coerce-simp)

```

```

lemma join-error-simps [simp]:
  join.⊥ = (⊥ :: 'a.'e error)
  join.(Err.e) = Err.e
  join.(Ok.x) = x
unfolding join-def by simp-all

```

```

lemma fplus-error-simps [simp]:
  fplus.⊥.r = (⊥ :: 'a.'e error)
  fplus.(Err.e).r = r
  fplus.(Ok.x).r = Ok.x
unfolding fplus-def
by (simp-all add: coerce-simp)

```

**end**

## 11 Writer monad

```

theory Writer-Monad
imports Monad
begin

```

### 11.1 Monoid class

```
class monoid = domain +
  fixes mempty :: 'a
  fixes mappend :: 'a → 'a → 'a
  assumes mempty-left:  $\bigwedge ys. mappend \cdot mempty \cdot ys = ys$ 
  assumes mempty-right:  $\bigwedge xs. mappend \cdot xs \cdot mempty = xs$ 
  assumes mappend-assoc:
     $\bigwedge xs \ ys \ zs. mappend \cdot (mappend \cdot xs \cdot ys) \cdot zs = mappend \cdot xs \cdot (mappend \cdot ys \cdot zs)$ 
```

### 11.2 Writer monad type

Below is the standard Haskell definition of a writer monad type; it is an isomorphic copy of the lazy pair type `(a, w)`.

```
newtype Writer w a = Writer { runWriter :: (a, w) }
```

Since HOLCF does not have a pre-defined lazy pair type, we will base this formalization on an equivalent, more direct definition:

```
data Writer w a = Writer w a
```

We can directly translate the above Haskell type definition using *tycondef*.

```
tycondef 'a.'w writer = Writer (lazy 'w) (lazy 'a)
```

```
lemma coerce-writer-abs [simp]: coerce.(writer-abs.x) = writer-abs.(coerce.x)
apply (simp add: writer-abs-def coerce-def)
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-writer)
done
```

```
lemma coerce-Writer [simp]:
  coerce.(Writer.w.x) = Writer.(coerce.w).(coerce.x)
unfolding Writer-def by simp
```

```
lemma fmapU-writer-simps [simp]:
  fmapU.f.( $\perp :: \text{udom} \cdot 'w \text{ writer}$ ) =  $\perp$ 
  fmapU.f.(Writer.w.x) = Writer.w.(f.x)
unfolding fmapU-writer-def writer-map-def fix-const
apply simp
apply (simp add: Writer-def)
done
```

### 11.3 Class instance proofs

```
instance writer :: (domain) functor
proof
```

```
  fix f g :: udom → udom and xs :: udom.'a writer
```

```

show fmapU.f.(fmapU.g.xs) = fmapU.( $\Lambda$  x. f.(g.x)).xs
  by (induct xs rule: writer.induct) simp-all
qed

instantiation writer :: (monoid) monad
begin

fixrec bindU-writer ::
  udom.'a writer  $\rightarrow$  (udom  $\rightarrow$  udom.'a writer)  $\rightarrow$  udom.'a writer
  where bindU-writer.(Writer.w.x).f =
    (case f.x of Writer.w'.y  $\Rightarrow$  Writer.(mappend.w.w').y)

lemma bindU-writer-strict [simp]: bindU. $\perp$ .k = ( $\perp$ ::udom.'a writer)
by fixrec-simp

definition
  returnU = Writer.empty

instance proof
  fix f :: udom  $\rightarrow$  udom and m :: udom.'a writer
  show fmapU.f.m = bindU.m.( $\Lambda$  x. returnU.(f.x))
  by (induct m rule: writer.induct)
    (simp-all add: returnU-writer-def mempty-right)
next
  fix f :: udom  $\rightarrow$  udom.'a writer and x :: udom
  show bindU.(returnU.x).f = f.x
  by (cases f.x rule: writer.exhaust)
    (simp-all add: returnU-writer-def mempty-left)
next
  fix m :: udom.'a writer and f g :: udom  $\rightarrow$  udom.'a writer
  show bindU.(bindU.m.f).g = bindU.m.( $\Lambda$  x. bindU.(f.x).g)
  apply (induct m rule: writer.induct, simp)
  apply (case-tac f.a rule: writer.exhaust, simp)
  apply (case-tac g.aa rule: writer.exhaust, simp)
  apply (simp add: mappend-assoc)
  done
qed

end

```

## 11.4 Transfer properties to polymorphic versions

```

lemma fmap-writer-simps [simp]:
  fmap.f.( $\perp$ ::'a.'w writer) =  $\perp$ 
  fmap.f.(Writer.w.x :: 'a.'w writer) = Writer.w.(f.x)
unfolding fmap-def [where 'f='w writer]
by (simp-all add: coerce-simp)

```

```

lemma return-writer-def: return = Writer.empty

```



**unfolding** *return-def returnU-writer-def*  
**by** (*simp add: coerce-simp eta-cfun*)

**lemma** *bind-writer-simps* [*simp*]:  
 $bind.(⊥ :: 'a.'w::monoid\ writer).f = ⊥$   
 $bind.(Writer.w.x :: 'a.'w::monoid\ writer).k =$   
 $(case\ k.x\ of\ Writer.w'.y \Rightarrow\ Writer.(mappend.w.w').y)$   
**unfolding** *bind-def*  
**apply** (*simp add: coerce-simp*)  
**apply** (*cases k.x rule: writer.exhaust*)  
**apply** (*simp-all add: coerce-simp*)  
**done**

**lemma** *join-writer-simps* [*simp*]:  
 $join.⊥ = (⊥ :: 'a.'w::monoid\ writer)$   
 $join.(Writer.w.(Writer.w'.x)) = Writer.(mappend.w.w').x$   
**unfolding** *join-def* **by** *simp-all*

## 11.5 Extra operations

**definition** *tell* ::  $'w \rightarrow unit.( 'w::monoid\ writer)$   
**where** *tell* =  $(\Lambda\ w.\ Writer.w.())$

**end**

## 12 Binary tree monad

**theory** *Binary-Tree-Monad*  
**imports** *Monad*  
**begin**

### 12.1 Type definition

**tycondef**  $'a.btree =$   
 $Leaf\ (lazy\ 'a) \mid Node\ (lazy\ 'a.btree)\ (lazy\ 'a.btree)$

**lemma** *coerce-btree-abs* [*simp*]:  $coerce.(btree-abs.x) = btree-abs.(coerce.x)$   
**apply** (*simp add: btree-abs-def coerce-def*)  
**apply** (*simp add: emb-prj-emb prj-emb-prj DEFL-eq-btree*)  
**done**

**lemma** *coerce-Leaf* [*simp*]:  $coerce.(Leaf.x) = Leaf.(coerce.x)$   
**unfolding** *Leaf-def* **by** *simp*

**lemma** *coerce-Node* [*simp*]:  $coerce.(Node.xs.ys) = Node.(coerce.xs).(coerce.ys)$   
**unfolding** *Node-def* **by** *simp*

**lemma** *fmapU-btree-simps* [*simp*]:  
 $fmapU.f.(⊥::udom.btree) = ⊥$

```

  fmapU·f·(Leaf·x) = Leaf·(f·x)
  fmapU·f·(Node·xs·ys) = Node·(fmapU·f·xs)·(fmapU·f·ys)
unfolding fmapU-btree-def btree-map-def
apply (subst fix-eq, simp)
apply (subst fix-eq, simp add: Leaf-def)
apply (subst fix-eq, simp add: Node-def)
done

```

## 12.2 Class instance proofs

```

instance btree :: functor
apply standard
apply (induct-tac xs rule: btree.induct, simp-all)
done

```

```

instantiation btree :: monad
begin

```

**definition**

```

  returnU = Leaf

```

```

fixrec bindU-btree :: udom·btree → (udom → udom·btree) → udom·btree
  where bindU-btree·(Leaf·x)·k = k·x
  | bindU-btree·(Node·xs·ys)·k =
    Node·(bindU-btree·xs·k)·(bindU-btree·ys·k)

```

```

lemma bindU-btree-strict [simp]: bindU·⊥·k = (⊥::udom·btree)
by fixrec-simp

```

**instance proof**

```

  fix x :: udom
  fix f :: udom → udom
  fix h k :: udom → udom·btree
  fix xs :: udom·btree
  show fmapU·f·xs = bindU·xs·(λ x. returnU·(f·x))
    by (induct xs rule: btree.induct, simp-all add: returnU-btree-def)
  show bindU·(returnU·x)·k = k·x
    by (simp add: returnU-btree-def)
  show bindU·(bindU·xs·h)·k = bindU·xs·(λ x. bindU·(h·x)·k)
    by (induct xs rule: btree.induct) simp-all
qed

```

**end**

## 12.3 Transfer properties to polymorphic versions

```

lemma fmap-btree-simps [simp]:
  fmap·f·(⊥::'a·btree) = ⊥
  fmap·f·(Leaf·x) = Leaf·(f·x)
  fmap·f·(Node·xs·ys) = Node·(fmap·f·xs)·(fmap·f·ys)

```

**unfolding** *fmap-def* **by** *simp-all*

**lemma** *bind-btree-simps* [*simp*]:

$bind.(⊥::'a.btree).k = ⊥$

$bind.(Leaf.x).k = k.x$

$bind.(Node.xs.ys).k = Node.(bind.xs.k).(bind.ys.k)$

**unfolding** *bind-def*

**by** (*simp-all add: coerce-simp*)

**lemma** *return-btree-def*:

$return = Leaf$

**unfolding** *return-def returnU-btree-def*

**by** (*simp add: coerce-simp eta-cfun*)

**lemma** *join-btree-simps* [*simp*]:

$join.(⊥::'a.btree.btree) = ⊥$

$join.(Leaf.xs) = xs$

$join.(Node.xss.yss) = Node.(join.xss).(join.yss)$

**unfolding** *join-def* **by** *simp-all*

**end**

## 13 Lift monad

**theory** *Lift-Monad*

**imports** *Monad*

**begin**

### 13.1 Type definition

**tycondef** *'a.lifted = Lifted (lazy 'a)*

**lemma** *coerce-lifted-abs* [*simp*]:  $coerce.(lifted-abs.x) = lifted-abs.(coerce.x)$

**apply** (*simp add: lifted-abs-def coerce-def*)

**apply** (*simp add: emb-prj-emb prj-emb-prj DEFL-eq-lifted*)

**done**

**lemma** *coerce-Lifted* [*simp*]:  $coerce.(Lifted.x) = Lifted.(coerce.x)$

**unfolding** *Lifted-def* **by** *simp*

**lemma** *fmapU-lifted-simps* [*simp*]:

$fmapU.f.(⊥::udom.lifted) = ⊥$

$fmapU.f.(Lifted.x) = Lifted.(f.x)$

**unfolding** *fmapU-lifted-def lifted-map-def fix-const*

**apply** *simp*

**apply** (*simp add: Lifted-def*)

**done**

## 13.2 Class instance proofs

```

instance lifted :: functor
  by standard (induct-tac xs rule: lifted.induct, simp-all)

instantiation lifted :: monad
begin

fixrec bindU-lifted :: udom-lifted → (udom → udom-lifted) → udom-lifted
  where bindU-lifted·(Lifted·x)·k = k·x

lemma bindU-lifted-strict [simp]: bindU·⊥·k = (⊥::udom-lifted)
by fixrec-simp

definition returnU-lifted-def:
  returnU = Lifted

instance proof
  fix x :: udom
  fix f :: udom → udom
  fix h k :: udom → udom-lifted
  fix xs :: udom-lifted
  show fmapU·f·xs = bindU·xs·(λ x. returnU·(f·x))
    by (induct xs rule: lifted.induct, simp-all add: returnU-lifted-def)
  show bindU·(returnU·x)·k = k·x
    by (simp add: returnU-lifted-def)
  show bindU·(bindU·xs·h)·k = bindU·xs·(λ x. bindU·(h·x)·k)
    by (induct xs rule: lifted.induct) simp-all
qed

end

```

## 13.3 Transfer properties to polymorphic versions

```

lemma fmap-lifted-simps [simp]:
  fmap·f·(⊥::'a·lifted) = ⊥
  fmap·f·(Lifted·x) = Lifted·(f·x)
unfolding fmap-def by simp-all

lemma bind-lifted-simps [simp]:
  bind·(⊥::'a·lifted)·f = ⊥
  bind·(Lifted·x)·f = f·x
unfolding bind-def by simp-all

lemma return-lifted-def: return = Lifted
unfolding return-def returnU-lifted-def
by (simp add: coerce-cfun ccomp1 eta-cfun)

lemma join-lifted-simps [simp]:
  join·(⊥::'a·lifted·lifted) = ⊥

```

```

    join.(Lifted.xs) = xs
  unfolding join-def by simp-all

```

```
end
```

## 14 Resumption monad transformer

```

theory Resumption-Transformer
imports Monad-Plus
begin

```

### 14.1 Type definition

The standard Haskell libraries do not include a resumption monad transformer type; below is the Haskell definition for the one we will use here.

```
data ResT m a = Done a | More (m (ResT m a))
```

The above datatype definition can be translated directly into HOLCF using *tycondef*.

```

tycondef 'a.(f::functor) resT =
  Done (lazy 'a) | More (lazy ('a.'f resT).'f)

```

```

lemma coerce-resT-abs [simp]: coerce.(resT-abs.x) = resT-abs.(coerce.x)
apply (simp add: resT-abs-def coerce-def)
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-resT)
done

```

```

lemma coerce-Done [simp]: coerce.(Done.x) = Done.(coerce.x)
unfolding Done-def by simp

```

```

lemma coerce-More [simp]: coerce.(More.m) = More.(coerce.m)
unfolding More-def by simp

```

```

lemma resT-induct [case-names adm bottom Done More]:
  fixes P :: 'a.'f::functor resT => bool
  assumes adm: adm P
  assumes bottom: P ⊥
  assumes Done:  $\bigwedge x. P (Done.x)$ 
  assumes More:  $\bigwedge m f. (\bigwedge (r::'a.'f resT). P (f.r)) \implies P (More.(fmap.f.m))$ 
  shows P r

```

```

proof (induct r rule: resT.take-induct [OF adm])
  fix n show P (resT.take n.r)
  apply (induct n arbitrary: r)
  apply (simp add: bottom)
  apply (case-tac r rule: resT.exhaust)
  apply (simp add: bottom)

```

```

    apply (simp add: Done)
    apply (simp add: More)
  done
qed

```

## 14.2 Class instance proofs

```

lemma fmapU-resT-simps [simp]:
  fmapU.f.( $\perp$ ::udom.f::functor resT) =  $\perp$ 
  fmapU.f.(Done.x) = Done.(f.x)
  fmapU.f.(More.m) = More.(fmap.(fmapU.f).m)
unfolding fmapU-resT-def resT-map-def
apply (subst fix-eq, simp)
apply (subst fix-eq, simp add: Done-def)
apply (subst fix-eq, simp add: More-def)
done

```

```

instance resT :: (functor) functor
proof
  fix f g :: udom  $\rightarrow$  udom and xs :: udom.'a resT
  show fmapU.f.(fmapU.g.xs) = fmapU.( $\Lambda$  x. f.(g.x)).xs
  by (induct xs rule: resT-induct, simp-all add: fmap-fmap)
qed

```

```

instantiation resT :: (functor) monad
begin

```

```

fixrec bindU-resT :: udom.'a resT  $\rightarrow$  (udom  $\rightarrow$  udom.'a resT)  $\rightarrow$  udom.'a resT
  where bindU-resT.(Done.x).f = f.x
  | bindU-resT.(More.m).f = More.(fmap.( $\Lambda$  r. bindU-resT.r.f).m)

```

```

lemma bindU-resT-strict [simp]: bindU. $\perp$ .k = ( $\perp$ ::udom.'a resT)
by fixrec-simp

```

**definition**

```

returnU = Done

```

**instance proof**

```

  fix f :: udom  $\rightarrow$  udom and xs :: udom.'a resT
  show fmapU.f.xs = bindU.xs.( $\Lambda$  x. returnU.(f.x))
  by (induct xs rule: resT-induct)
  (simp-all add: fmap-fmap returnU-resT-def)
next
  fix f :: udom  $\rightarrow$  udom.'a resT and x :: udom
  show bindU.(returnU.x).f = f.x
  by (simp add: returnU-resT-def)
next
  fix xs :: udom.'a resT and h k :: udom  $\rightarrow$  udom.'a resT
  show bindU.(bindU.xs.h).k = bindU.xs.( $\Lambda$  x. bindU.(h.x).k)

```

by (*induct xs rule: resT-induct*)  
 (*simp-all add: fmap-fmap*)

qed

end

### 14.3 Transfer properties to polymorphic versions

**lemma** *fmap-resT-simps* [*simp*]:

$fmap.f \cdot (\perp :: 'a.'f::functor\ resT) = \perp$

$fmap.f \cdot (Done.x :: 'a.'f::functor\ resT) = Done.(f.x)$

$fmap.f \cdot (More.m :: 'a.'f::functor\ resT) = More.(fmap.(fmap.f).m)$

**unfolding** *fmap-def* [**where**  $'f = 'f\ resT$ ]

by (*simp-all add: coerce-simp*)

**lemma** *return-resT-def*:  $return = Done$

**unfolding** *return-def returnU-resT-def*

by (*simp add: coerce-simp eta-cfun*)

**lemma** *bind-resT-simps* [*simp*]:

$bind.(\perp :: 'a.'f::functor\ resT).f = \perp$

$bind.(Done.x :: 'a.'f::functor\ resT).f = f.x$

$bind.(More.m :: 'a.'f::functor\ resT).f = More.(fmap.(\lambda r. bind.r.f).m)$

**unfolding** *bind-def*

by (*simp-all add: coerce-simp*)

**lemma** *join-resT-simps* [*simp*]:

$join.\perp = (\perp :: 'a.'f::functor\ resT)$

$join.(Done.x) = x$

$join.(More.m) = More.(fmap.join.m)$

**unfolding** *join-def* by *simp-all*

### 14.4 Nondeterministic interleaving

In this section we present a more general formalization of the nondeterministic interleaving operation presented in Chapter 7 of the author's PhD thesis [2]. If both arguments are *Done*, then *zipRT* combines the results with the function  $f$  and terminates. While either argument is *More*, *zipRT* nondeterministically chooses one such argument, runs it for one step, and then calls itself recursively.

**fixrec** *zipRT* ::

$('a \rightarrow 'b \rightarrow 'c) \rightarrow 'a.( 'm::functor-plus)\ resT \rightarrow 'b.'m\ resT \rightarrow 'c.'m\ resT$

**where** *zipRT-Done-Done*:

$zipRT.f.(Done.x).(Done.y) = Done.(f.x.y)$

| *zipRT-Done-More*:

$zipRT.f.(Done.x).(More.b) =$

$More.(fmap.(\lambda r. zipRT.f.(Done.x).r).b)$

| *zipRT-More-Done*:

$$\begin{aligned} & \text{zipRT}\cdot f\cdot(\text{More}\cdot a)\cdot(\text{Done}\cdot y) = \\ & \quad \text{More}\cdot(\text{fmap}\cdot(\Lambda r. \text{zipRT}\cdot f\cdot r\cdot(\text{Done}\cdot y))\cdot a) \\ | \text{zipRT-More-More:} \\ & \text{zipRT}\cdot f\cdot(\text{More}\cdot a)\cdot(\text{More}\cdot b) = \\ & \quad \text{More}\cdot(\text{fplus}\cdot(\text{fmap}\cdot(\Lambda r. \text{zipRT}\cdot f\cdot(\text{More}\cdot a)\cdot r)\cdot b) \\ & \quad \cdot(\text{fmap}\cdot(\Lambda r. \text{zipRT}\cdot f\cdot r\cdot(\text{More}\cdot b))\cdot a)) \end{aligned}$$

**lemma** *zipRT-strict1* [*simp*]:  $\text{zipRT}\cdot f\cdot\perp\cdot r = \perp$   
**by** *fixrec-simp*

**lemma** *zipRT-strict2* [*simp*]:  $\text{zipRT}\cdot f\cdot r\cdot\perp = \perp$   
**by** (*fixrec-simp*, *cases r*, *simp-all*)

**abbreviation** *apR* (*infixl*  $\diamond$  70)  
**where**  $a \diamond b \equiv \text{zipRT}\cdot \text{ID}\cdot a\cdot b$

Proofs that *zipRT* satisfies the applicative functor laws:

**lemma** *zipRT-homomorphism*:  $\text{Done}\cdot f \diamond \text{Done}\cdot x = \text{Done}\cdot(f\cdot x)$   
**by** *simp*

**lemma** *zipRT-identity*:  $\text{Done}\cdot \text{ID} \diamond r = r$   
**by** (*induct r rule: resT-induct*, *simp-all add: fmap-fmap eta-cfun*)

**lemma** *zipRT-interchange*:  $r \diamond \text{Done}\cdot x = \text{Done}\cdot(\Lambda f. f\cdot x) \diamond r$   
**by** (*induct r rule: resT-induct*, *simp-all add: fmap-fmap*)

The associativity rule is the hard one!

**lemma** *zipRT-associativity*:  $\text{Done}\cdot \text{cfcomp} \diamond r1 \diamond r2 \diamond r3 = r1 \diamond (r2 \diamond r3)$

**proof** (*induct r1 arbitrary: r2 r3 rule: resT-induct*)

**case** (*Done x1*) **thus** *?case*

**proof** (*induct r2 arbitrary: r3 rule: resT-induct*)

**case** (*Done x2*) **thus** *?case*

**proof** (*induct r3 rule: resT-induct*)

**case** (*More p3 c3*) **thus** *?case*

**by** (*simp add: fmap-fmap*)

**qed** *simp-all*

**next**

**case** (*More p2 c2*) **thus** *?case*

**proof** (*induct r3 rule: resT-induct*)

**case** (*Done x3*) **thus** *?case*

**by** (*simp add: fmap-fmap*)

**next**

**case** (*More p3 c3*) **thus** *?case*

**by** (*simp add: fmap-fmap fmap-fplus*)

**qed** *simp-all*

**qed** *simp-all*

**next**

**case** (*More p1 c1*) **thus** *?case*

**proof** (*induct r2 arbitrary: r3 rule: resT-induct*)



```

    case (Done y) thus ?case
  proof (induct r3 rule: resT-induct)
    case (Done x3) thus ?case
      by (simp add: fmap-fmap)
    next
      case (More p3 c3) thus ?case
        by (simp add: fmap-fmap)
      qed simp-all
    next
      case (More p2 c2) thus ?case
    proof (induct r3 rule: resT-induct)
      case (Done x3) thus ?case
        by (simp add: fmap-fmap fmap-fplus)
      next
        case (More p3 c3) thus ?case
          by (simp add: fmap-fmap fmap-fplus fplus-assoc)
        qed simp-all
      qed simp-all
    qed simp-all
  end

```

## 15 State monad transformer

```

theory State-Transformer
imports Monad-Zero-Plus
begin

```

This version has non-lifted product, and a non-lifted function space.

```

tycondef 'a.(f::functor, 's) stateT =
  StateT (runStateT :: 's → ('a × 's).f)

```

```

lemma coerce-stateT-abs [simp]: coerce.(stateT-abs.x) = stateT-abs.(coerce.x)
apply (simp add: stateT-abs-def coerce-def)
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-stateT)
done

```

```

lemma coerce-StateT [simp]: coerce.(StateT.k) = StateT.(coerce.k)
unfolding StateT-def by simp

```

```

lemma stateT-cases [case-names StateT]:
  obtains k where y = StateT.k
proof
  show y = StateT.(runStateT.y)
    by (cases y, simp-all)
qed

```

```

lemma stateT-induct [case-names StateT]:
  fixes P :: 'a.(f::functor, 's) stateT ⇒ bool

```

**assumes**  $\bigwedge k. P (StateT \cdot k)$   
**shows**  $P y$   
**by** (*cases y rule: stateT-cases, simp add: assms*)

**lemma** *stateT-eqI*:  
 $(\bigwedge s. runStateT \cdot a \cdot s = runStateT \cdot b \cdot s) \implies a = b$   
**apply** (*cases a rule: stateT-cases*)  
**apply** (*cases b rule: stateT-cases*)  
**apply** (*simp add: cfun-eq-iff*)  
**done**

**lemma** *runStateT-coerce [simp]*:  
 $runStateT \cdot (coerce \cdot k) \cdot s = coerce \cdot (runStateT \cdot k \cdot s)$   
**by** (*induct k rule: stateT-induct, simp*)

## 15.1 Functor class instance

**lemma** *fmapU-StateT [simp]*:  
 $fmapU \cdot f \cdot (StateT \cdot k) =$   
 $StateT \cdot (\bigwedge s. fmap \cdot (\bigwedge (x, s'). (f \cdot x, s')) \cdot (k \cdot s))$   
**unfolding** *fmapU-stateT-def stateT-map-def StateT-def*  
**by** (*subst fix-eq, simp add: cfun-map-def csplit-def prod-map-def*)

**lemma** *runStateT-fmapU [simp]*:  
 $runStateT \cdot (fmapU \cdot f \cdot m) \cdot s =$   
 $fmap \cdot (\bigwedge (x, s'). (f \cdot x, s')) \cdot (runStateT \cdot m \cdot s)$   
**by** (*cases m rule: stateT-cases, simp*)

**instantiation** *stateT* :: (*functor, domain*) *functor*  
**begin**

**instance**  
**apply** *standard*  
**apply** (*induct-tac xs rule: stateT-induct*)  
**apply** (*simp-all add: fmap-fmap ID-def csplit-def*)  
**done**

**end**

## 15.2 Monad class instance

**instantiation** *stateT* :: (*monad, domain*) *monad*  
**begin**

**definition** *returnU-stateT-def*:  
 $returnU = (\bigwedge x. StateT \cdot (\bigwedge s. return \cdot (x, s)))$

**definition** *bindU-stateT-def*:  
 $bindU = (\bigwedge m k. StateT \cdot (\bigwedge s. runStateT \cdot m \cdot s \gg= (\bigwedge (x, s'). runStateT \cdot (k \cdot x) \cdot s'))$

**lemma** *bindU-stateT-StateT* [*simp*]:  
 $bindU \cdot (StateT \cdot f) \cdot k =$   
 $StateT \cdot (\Lambda s. f \cdot s \gg (\Lambda (x, s'). runStateT \cdot (k \cdot x) \cdot s'))$   
**unfolding** *bindU-stateT-def* **by** *simp*

**lemma** *runStateT-bindU* [*simp*]:  
 $runStateT \cdot (bindU \cdot m \cdot k) \cdot s = runStateT \cdot m \cdot s \gg (\Lambda (x, s'). runStateT \cdot (k \cdot x) \cdot s')$   
**unfolding** *bindU-stateT-def* **by** *simp*

**instance proof**

**fix**  $f :: udom \rightarrow udom$  **and**  $r :: udom \cdot ('a, 'b) \text{ stateT}$   
**show**  $fmapU \cdot f \cdot r = bindU \cdot r \cdot (\Lambda x. returnU \cdot (f \cdot x))$   
**by** (*rule stateT-eqI*)  
(*simp add: returnU-stateT-def monad-fmap prod-map-def csplit-def*)

**next**

**fix**  $f :: udom \rightarrow udom \cdot ('a, 'b) \text{ stateT}$  **and**  $x :: udom$   
**show**  $bindU \cdot (returnU \cdot x) \cdot f = f \cdot x$   
**by** (*rule stateT-eqI*)  
(*simp add: returnU-stateT-def eta-cfun*)

**next**

**fix**  $r :: udom \cdot ('a, 'b) \text{ stateT}$  **and**  $f g :: udom \rightarrow udom \cdot ('a, 'b) \text{ stateT}$   
**show**  $bindU \cdot (bindU \cdot r \cdot f) \cdot g = bindU \cdot r \cdot (\Lambda x. bindU \cdot (f \cdot x) \cdot g)$   
**by** (*rule stateT-eqI*)  
(*simp add: bind-bind csplit-def*)

**qed**

**end**

### 15.3 Monad zero instance

**instantiation** *stateT* :: (*monad-zero, domain*) *monad-zero*  
**begin**

**definition** *zeroU-stateT-def*:  
 $zeroU = StateT \cdot (\Lambda s. mzero)$

**lemma** *runStateT-zeroU* [*simp*]:  
 $runStateT \cdot zeroU \cdot s = mzero$   
**unfolding** *zeroU-stateT-def* **by** *simp*

**instance proof**

**fix**  $k :: udom \rightarrow udom \cdot ('a, 'b) \text{ stateT}$   
**show**  $bindU \cdot zeroU \cdot k = zeroU$   
**by** (*rule stateT-eqI, simp add: bind-mzero*)

**qed**

**end**

## 15.4 Monad plus instance

**instantiation**  $stateT :: (monad-plus, domain) monad-plus$   
**begin**

**definition**  $plusU-stateT-def$ :

$plusU = (\Lambda a b. StateT \cdot (\Lambda s. mplus \cdot (runStateT \cdot a \cdot s) \cdot (runStateT \cdot b \cdot s)))$

**lemma**  $runStateT-plusU$  [*simp*]:

$runStateT \cdot (plusU \cdot a \cdot b) \cdot s =$   
 $mplus \cdot (runStateT \cdot a \cdot s) \cdot (runStateT \cdot b \cdot s)$

**unfolding**  $plusU-stateT-def$  **by** *simp*

**instance proof**

**fix**  $a b :: udom \cdot ('a, 'b) stateT$  **and**  $k :: udom \rightarrow udom \cdot ('a, 'b) stateT$

**show**  $bindU \cdot (plusU \cdot a \cdot b) \cdot k = plusU \cdot (bindU \cdot a \cdot k) \cdot (bindU \cdot b \cdot k)$

**by** (*rule stateT-eqI, simp add: bind-mplus*)

**next**

**fix**  $a b c :: udom \cdot ('a, 'b) stateT$

**show**  $plusU \cdot (plusU \cdot a \cdot b) \cdot c = plusU \cdot a \cdot (plusU \cdot b \cdot c)$

**by** (*rule stateT-eqI, simp add: mplus-assoc*)

**qed**

**end**

## 15.5 Monad zero plus instance

**instance**  $stateT :: (monad-zero-plus, domain) monad-zero-plus$   
**proof**

**fix**  $m :: udom \cdot ('a, 'b) stateT$

**show**  $plusU \cdot zeroU \cdot m = m$

**by** (*rule stateT-eqI, simp add: mplus-mzero-left*)

**next**

**fix**  $m :: udom \cdot ('a, 'b) stateT$

**show**  $plusU \cdot m \cdot zeroU = m$

**by** (*rule stateT-eqI, simp add: mplus-mzero-right*)

**qed**

## 15.6 Transfer properties to polymorphic versions

**lemma**  $coerce-csplit$  [*coerce-simp*]:

**shows**  $coerce \cdot (csplit \cdot f \cdot p) = csplit \cdot (\Lambda x y. coerce \cdot (f \cdot x \cdot y)) \cdot p$

**unfolding**  $csplit-def$  **by** *simp*

**lemma**  $csplit-coerce$  [*coerce-simp*]:

**fixes**  $p :: 'a \times 'b$

**shows**  $csplit \cdot f \cdot (COERCE ('a \times 'b, 'c \times 'd) \cdot p) =$

$csplit \cdot (\Lambda x y. f \cdot (COERCE ('a, 'c) \cdot x) \cdot (COERCE ('b, 'd) \cdot y)) \cdot p$

**unfolding**  $coerce-prod csplit-def prod-map-def$  **by** *simp*

**lemma** *fmap-stateT-simps* [*simp*]:  
 $fmap \cdot f \cdot (StateT \cdot m :: 'a \cdot ('f :: functor, 's) \ stateT) =$   
 $StateT \cdot (\Lambda s. fmap \cdot (\Lambda (x, s'). (f \cdot x, s')) \cdot (m \cdot s))$   
**unfolding** *fmap-def* [**where** *'f = ('f, 's) stateT*]  
**by** (*simp add: coerce-simp eta-cfun*)

**lemma** *runStateT-fmap* [*simp*]:  
 $runStateT \cdot (fmap \cdot f \cdot m) \cdot s = fmap \cdot (\Lambda (x, s'). (f \cdot x, s')) \cdot (runStateT \cdot m \cdot s)$   
**by** (*induct m rule: stateT-induct, simp*)

**lemma** *return-stateT-def*:  
 $(return :: - \rightarrow 'a \cdot ('m :: monad, 's) \ stateT) =$   
 $(\Lambda x. StateT \cdot (\Lambda s. return \cdot (x, s)))$   
**unfolding** *return-def* [**where** *'m = ('m, 's) stateT*] *returnU-stateT-def*  
**by** (*simp add: coerce-simp*)

**lemma** *bind-stateT-def*:  
 $bind = (\Lambda m \ k. StateT \cdot (\Lambda s. runStateT \cdot m \cdot s \gg (\Lambda (x, s'). runStateT \cdot (k \cdot x) \cdot s')))$   
**apply** (*subst bind-def, subst bindU-stateT-def*)  
**apply** (*simp add: coerce-simp*)  
**apply** (*simp add: coerce-idem domain-defl-simps monofun-cfun*)  
**apply** (*simp add: eta-cfun*)  
**done**

TODO: add *coerce-idem* to *coerce-simps*, along with monotonicity rules for DEFL.

**lemma** *bind-stateT-simps* [*simp*]:  
 $bind \cdot (StateT \cdot m :: 'a \cdot ('m :: monad, 's) \ stateT) \cdot k =$   
 $StateT \cdot (\Lambda s. m \cdot s \gg (\Lambda (x, s'). runStateT \cdot (k \cdot x) \cdot s'))$   
**unfolding** *bind-stateT-def* **by** *simp*

**lemma** *runStateT-bind* [*simp*]:  
 $runStateT \cdot (m \gg k) \cdot s = runStateT \cdot m \cdot s \gg (\Lambda (x, s'). runStateT \cdot (k \cdot x) \cdot s')$   
**unfolding** *bind-stateT-def* **by** *simp*

**end**

## 16 Error monad transformer

**theory** *Error-Transformer*  
**imports** *Error-Monad*  
**begin**

### 16.1 Type definition

The error monad transformer is defined in Haskell by composing the given monad with a standard error monad:

```

data Error e a = Err e | Ok a
newtype ErrorT e m a = ErrorT { runErrorT :: m (Error e a) }

```

We can formalize this definition directly using *tycondef*.

```

tycondef 'a.(f::functor,'e::domain) errorT =
  ErrorT (runErrorT :: ('a.'e error).f)

```

```

lemma coerce-errorT-abs [simp]: coerce.(errorT-abs.x) = errorT-abs.(coerce.x)
apply (simp add: errorT-abs-def coerce-def)
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-errorT)
done

```

```

lemma coerce-ErrorT [simp]: coerce.(ErrorT.k) = ErrorT.(coerce.k)
unfolding ErrorT-def by simp

```

```

lemma errorT-cases [case-names ErrorT]:
  obtains k where y = ErrorT.k
proof
  show y = ErrorT.(runErrorT.y)
  by (cases y, simp-all)
qed

```

```

lemma ErrorT-runErrorT [simp]: ErrorT.(runErrorT.m) = m
by (cases m rule: errorT-cases, simp)

```

```

lemma errorT-induct [case-names ErrorT]:
  fixes P :: 'a.(f::functor,'e) errorT => bool
  assumes  $\bigwedge k. P (ErrorT.k)$ 
  shows P y
by (cases y rule: errorT-cases, simp add: assms)

```

```

lemma errorT-eq-iff:
  a = b  $\longleftrightarrow$  runErrorT.a = runErrorT.b
apply (cases a rule: errorT-cases)
apply (cases b rule: errorT-cases)
apply simp
done

```

```

lemma errorT-eqI:
  runErrorT.a = runErrorT.b  $\implies$  a = b
by (simp add: errorT-eq-iff)

```

```

lemma runErrorT-coerce [simp]:
  runErrorT.(coerce.k) = coerce.(runErrorT.k)
by (induct k rule: errorT-induct, simp)

```

## 16.2 Functor class instance

```

lemma fmap-error-def: fmap = error-map.ID

```

```

apply (rule cfun-eqI, rename-tac f)
apply (rule cfun-eqI, rename-tac x)
apply (case-tac x rule: error.exhaust, simp-all)
apply (simp add: error-map-def fix-const)
apply (simp add: error-map-def fix-const Err-def)
apply (simp add: error-map-def fix-const Ok-def)
done

```

```

lemma fmapU-ErrorT [simp]:
  fmapU.f.(ErrorT.m) = ErrorT.(fmap.(fmap.f).m)
unfolding fmapU-errorT-def errorT-map-def fmap-error-def fix-const ErrorT-def
by simp

```

```

lemma runErrorT-fmapU [simp]:
  runErrorT.(fmapU.f.m) = fmap.(fmap.f).(runErrorT.m)
by (induct m rule: errorT-induct) simp

```

```

instance errorT :: (functor, domain) functor
proof

```

```

  fix f g and xs :: udom.('a, 'b) errorT
  show fmapU.f.(fmapU.g.xs) = fmapU.( $\Lambda$  x. f.(g.x)).xs
  apply (induct xs rule: errorT-induct)
  apply (simp add: fmap-fmap eta-cfun)
  done

```

```

qed

```

### 16.3 Transfer properties to polymorphic versions

```

lemma fmap-ErrorT [simp]:
  fixes f :: 'a  $\rightarrow$  'b and m :: 'a.'e error.('m::functor)
  shows fmap.f.(ErrorT.m) = ErrorT.(fmap.(fmap.f).m)
unfolding fmap-def [where 'f=(('m,'e) errorT)]
by (simp-all add: coerce-simp eta-cfun)

```

```

lemma runErrorT-fmap [simp]:
  fixes f :: 'a  $\rightarrow$  'b and m :: 'a.('m::functor,'e) errorT
  shows runErrorT.(fmap.f.m) = fmap.(fmap.f).(runErrorT.m)
using fmap-ErrorT [of f runErrorT.m]
by simp

```

```

lemma errorT-fmap-strict [simp]:
  shows fmap.f.( $\perp$ ::'a.('m::monad,'e) errorT) =  $\perp$ 
by (simp add: errorT-eq-iff fmap-strict)

```

### 16.4 Monad operations

The error monad transformer does not yield a monad in the usual sense: We cannot prove a *monad* class instance, because type  $'a.('m,'e) errorT$  contains values that break the monad laws. However, it turns out that

such values are inaccessible: The monad laws are satisfied by all values constructible from the abstract operations.

To explore the properties of the error monad transformer operations, we define them all as non-overloaded functions.

**definition**  $unitET :: 'a \rightarrow 'a \cdot ('m :: monad, 'e) errorT$   
**where**  $unitET = (\Lambda x. ErrorT \cdot (return \cdot (Ok \cdot x)))$

**definition**  $bindET :: 'a \cdot ('m :: monad, 'e) errorT \rightarrow$   
 $('a \rightarrow 'b \cdot ('m, 'e) errorT) \rightarrow 'b \cdot ('m, 'e) errorT$   
**where**  $bindET = (\Lambda m k. ErrorT \cdot (bind \cdot (runErrorT \cdot m) \cdot$   
 $(\Lambda n. case\ n\ of\ Err \cdot e \Rightarrow return \cdot (Err \cdot e) \mid Ok \cdot x \Rightarrow runErrorT \cdot (k \cdot x))))$

**definition**  $liftET :: 'a \cdot 'm :: monad \rightarrow 'a \cdot ('m, 'e) errorT$   
**where**  $liftET = (\Lambda m. ErrorT \cdot (fmap \cdot Ok \cdot m))$

**definition**  $throwET :: 'e \rightarrow 'a \cdot ('m :: monad, 'e) errorT$   
**where**  $throwET = (\Lambda e. ErrorT \cdot (return \cdot (Err \cdot e)))$

**definition**  $catchET :: 'a \cdot ('m :: monad, 'e) errorT \rightarrow$   
 $('e \rightarrow 'a \cdot ('m, 'e) errorT) \rightarrow 'a \cdot ('m, 'e) errorT$   
**where**  $catchET = (\Lambda m h. ErrorT \cdot (bind \cdot (runErrorT \cdot m) \cdot (\Lambda n. case\ n\ of$   
 $Err \cdot e \Rightarrow runErrorT \cdot (h \cdot e) \mid Ok \cdot x \Rightarrow return \cdot (Ok \cdot x))))$

**definition**  $fmapET :: ('a \rightarrow 'b) \rightarrow$   
 $'a \cdot ('m :: monad, 'e) errorT \rightarrow 'b \cdot ('m, 'e) errorT$   
**where**  $fmapET = (\Lambda f m. bindET \cdot m \cdot (\Lambda x. unitET \cdot (f \cdot x)))$

**lemma**  $runErrorT \cdot unitET$  [simp]:  
 $runErrorT \cdot (unitET \cdot x) = return \cdot (Ok \cdot x)$   
**unfolding**  $unitET$ -def **by** simp

**lemma**  $runErrorT \cdot bindET$  [simp]:  
 $runErrorT \cdot (bindET \cdot m \cdot k) = bind \cdot (runErrorT \cdot m) \cdot$   
 $(\Lambda n. case\ n\ of\ Err \cdot e \Rightarrow return \cdot (Err \cdot e) \mid Ok \cdot x \Rightarrow runErrorT \cdot (k \cdot x))$   
**unfolding**  $bindET$ -def **by** simp

**lemma**  $runErrorT \cdot liftET$  [simp]:  
 $runErrorT \cdot (liftET \cdot m) = fmap \cdot Ok \cdot m$   
**unfolding**  $liftET$ -def **by** simp

**lemma**  $runErrorT \cdot throwET$  [simp]:  
 $runErrorT \cdot (throwET \cdot e) = return \cdot (Err \cdot e)$   
**unfolding**  $throwET$ -def **by** simp

**lemma**  $runErrorT \cdot catchET$  [simp]:  
 $runErrorT \cdot (catchET \cdot m \cdot h) =$   
 $bind \cdot (runErrorT \cdot m) \cdot (\Lambda n. case\ n\ of$



$Err.e \Rightarrow runErrorT.(h.e) \mid Ok.x \Rightarrow return.(Ok.x)$   
**unfolding** *catchET-def* **by** *simp*

**lemma** *runErrorT-fmapET* [*simp*]:  
 $runErrorT.(fmapET.f.m) =$   
 $bind.(runErrorT.m).(\Lambda n. case\ n\ of$   
 $Err.e \Rightarrow return.(Err.e) \mid Ok.x \Rightarrow return.(Ok.(f.x)))$   
**unfolding** *fmapET-def* **by** *simp*

## 16.5 Laws

**lemma** *bindET-unitET* [*simp*]:  
 $bindET.(unitET.x).k = k.x$   
**by** (*rule errorT-eqI*, *simp*)

**lemma** *catchET-unitET* [*simp*]:  
 $catchET.(unitET.x).h = unitET.x$   
**by** (*rule errorT-eqI*, *simp*)

**lemma** *catchET-throwET* [*simp*]:  
 $catchET.(throwET.e).h = h.e$   
**by** (*rule errorT-eqI*, *simp*)

**lemma** *liftET-return*:  
 $liftET.(return.x) = unitET.x$   
**by** (*rule errorT-eqI*, *simp add: fmap-return*)

**lemma** *liftET-bind*:  
 $liftET.(bind.m.k) = bindET.(liftET.m).(liftET\ oo\ k)$   
**by** (*rule errorT-eqI*, *simp add: fmap-bind bind-fmap*)

**lemma** *bindET-throwET*:  
 $bindET.(throwET.e).k = throwET.e$   
**by** (*rule errorT-eqI*, *simp*)

**lemma** *bindET-bindET*:  
 $bindET.(bindET.m.h).k = bindET.m.(\Lambda x. bindET.(h.x).k)$   
**apply** (*rule errorT-eqI*)  
**apply** *simp*  
**apply** (*simp add: bind-bind*)  
**apply** (*rule cfun-arg-cong*)  
**apply** (*rule cfun-eqI*, *simp*)  
**apply** (*case-tac x*)  
**apply** (*simp add: bind-strict*)  
**apply** *simp*  
**apply** *simp*  
**done**

**lemma** *fmapET-fmapET*:

$fmapET \cdot f \cdot (fmapET \cdot g \cdot m) = fmapET \cdot (\lambda x. f \cdot (g \cdot x)) \cdot m$   
**by** (*simp add: fmapET-def bindET-bindET*)

Right unit monad law is not satisfied in general.

**lemma** *bindET-unitET-right-counterexample*:  
**fixes**  $m :: 'a \cdot ('m :: monad, 'e) \text{ errorT}$   
**assumes**  $m = ErrorT \cdot (return \cdot \perp)$   
**assumes**  $return \cdot \perp \neq (\perp :: ('a \cdot 'e \text{ error}) \cdot 'm)$   
**shows**  $bindET \cdot m \cdot unitET \neq m$   
**by** (*simp add: errorT-eq-iff assms*)

Right unit is satisfied for inner monads with strict return.

**lemma** *bindET-unitET-right-restricted*:  
**fixes**  $m :: 'a \cdot ('m :: monad, 'e) \text{ errorT}$   
**assumes**  $return \cdot \perp = (\perp :: ('a \cdot 'e \text{ error}) \cdot 'm)$   
**shows**  $bindET \cdot m \cdot unitET = m$   
**unfolding** *errorT-eq-iff*  
**apply** *simp*  
**apply** (*rule trans [OF - monad-right-unit]*)  
**apply** (*rule cfun-arg-cong*)  
**apply** (*rule cfun-eqI*)  
**apply** (*case-tac x, simp-all add: assms*)  
**done**

## 16.6 Error monad transformer invariant

This inductively-defined invariant is supposed to represent the set of all values constructible using the standard *errorT* operations.

**inductive** *invar* ::  $'a \cdot ('m :: monad, 'e) \text{ errorT} \Rightarrow \text{bool}$   
**where** *invar-bottom*:  $invar \perp$   
| *invar-lub*:  $\bigwedge Y. \llbracket chain Y; \bigwedge i. invar (Y i) \rrbracket \Longrightarrow invar (\bigsqcup i. Y i)$   
| *invar-unitET*:  $\bigwedge x. invar (unitET \cdot x)$   
| *invar-bindET*:  $\bigwedge m k. \llbracket invar m; \bigwedge x. invar (k \cdot x) \rrbracket \Longrightarrow invar (bindET \cdot m \cdot k)$   
| *invar-throwET*:  $\bigwedge e. invar (throwET \cdot e)$   
| *invar-catchET*:  $\bigwedge m h. \llbracket invar m; \bigwedge e. invar (h \cdot e) \rrbracket \Longrightarrow invar (catchET \cdot m \cdot h)$   
| *invar-liftET*:  $\bigwedge m. invar (liftET \cdot m)$

Right unit is satisfied for arguments built from standard functions.

**lemma** *bindET-unitET-right-invar*:  
**assumes** *invar m*  
**shows**  $bindET \cdot m \cdot unitET = m$   
**using** *assms*  
**apply** (*induct set: invar*)  
**apply** (*rule errorT-eqI, simp add: bind-strict*)  
**apply** (*rule admD, simp, assumption, assumption*)  
**apply** (*rule errorT-eqI, simp*)  
**apply** (*simp add: errorT-eq-iff bind-bind*)

```

apply (rule cfun-arg-cong, rule cfun-eqI, simp)
apply (case-tac x, simp add: bind-strict, simp, simp)
apply (rule errorT-eqI, simp)
apply (simp add: errorT-eq-iff bind-bind)
apply (rule cfun-arg-cong, rule cfun-eqI, simp)
apply (case-tac x, simp add: bind-strict, simp, simp)
apply (rule errorT-eqI, simp add: monad-fmap bind-bind)
done

```

Monad-fmap is satisfied for arguments built from standard functions.

```

lemma errorT-monad-fmap-invar:
  fixes f :: 'a → 'b and m :: 'a.( 'm::monad, 'e) errorT
  assumes invar m
  shows fmap.f.m = bindET.m.( $\Lambda$  x. unitET.(f.x))
using assms
apply (induct set: invar)
apply (rule errorT-eqI, simp add: bind-strict fmap-strict)
apply (rule admD, simp, assumption, assumption)
apply (rule errorT-eqI, simp add: fmap-return)
apply (simp add: errorT-eq-iff bind-bind fmap-bind)
apply (rule cfun-arg-cong, rule cfun-eqI, simp)
apply (case-tac x)
apply (simp add: bind-strict fmap-strict)
apply (simp add: fmap-return)
apply simp
apply (rule errorT-eqI, simp add: fmap-return)
apply (simp add: errorT-eq-iff bind-bind fmap-bind)
apply (rule cfun-arg-cong, rule cfun-eqI, simp)
apply (case-tac x)
apply (simp add: bind-strict fmap-strict)
apply simp
apply (simp add: fmap-return)
apply (rule errorT-eqI, simp add: monad-fmap bind-bind return-error-def)
done

```

## 16.7 Invariant expressed as a deflation

We can also define an invariant in a more semantic way, as the set of fixed-points of a deflation.

```

definition invar' :: 'a.( 'm::monad, 'e) errorT ⇒ bool
  where invar' m  $\longleftrightarrow$  fmapET.ID.m = m

```

All standard operations preserve the invariant.

```

lemma invar'-unitET: invar' (unitET.x)
  unfolding invar'-def by (simp add: fmapET-def)

```

```

lemma invar'-fmapET: invar' m  $\implies$  invar' (fmapET.f.m)
  unfolding invar'-def

```

**by** (*erule subst, simp add: fmapET-def bindET-bindET eta-cfun*)

**lemma** *invar'-bindET*:  $\llbracket \text{invar}' m; \bigwedge x. \text{invar}' (k \cdot x) \rrbracket \implies \text{invar}' (\text{bindET} \cdot m \cdot k)$   
**unfolding** *invar'-def*  
**by** (*simp add: fmapET-def bindET-bindET eta-cfun*)

**lemma** *invar'-throwET*:  $\text{invar}' (\text{throwET} \cdot e)$   
**unfolding** *invar'-def* **by** (*simp add: fmapET-def bindET-throwET eta-cfun*)

**lemma** *invar'-catchET*:  $\llbracket \text{invar}' m; \bigwedge e. \text{invar}' (h \cdot e) \rrbracket \implies \text{invar}' (\text{catchET} \cdot m \cdot h)$   
**unfolding** *invar'-def*  
**apply** (*simp add: fmapET-def eta-cfun*)  
**apply** (*rule errorT-eqI*)  
**apply** (*simp add: bind-bind eta-cfun*)  
**apply** (*rule cfun-arg-cong*)  
**apply** (*rule cfun-eqI*)  
**apply** (*case-tac x*)  
**apply** (*simp add: bind-strict*)  
**apply** *simp*  
**apply** (*drule-tac x=e in meta-spec*)  
**apply** (*erule-tac t=h \cdot e in subst*) **back**  
**apply** (*simp add: eta-cfun*)  
**apply** *simp*  
**done**

**lemma** *invar'-liftET*:  $\text{invar}' (\text{liftET} \cdot m)$   
**unfolding** *invar'-def*  
**apply** (*simp add: fmapET-def errorT-eq-iff*)  
**apply** (*simp add: monad-fmap bind-bind*)  
**done**

**lemma** *invar'-bottom*:  $\text{invar}' \perp$   
**unfolding** *invar'-def fmapET-def*  
**by** (*simp add: errorT-eq-iff bind-strict*)

**lemma** *adm-invar'*:  $\text{adm } \text{invar}'$   
**unfolding** *invar'-def [abs-def]* **by** *simp*

All monad laws are preserved by values satisfying the invariant.

**lemma** *bindET-fmapET-unitET*:  
**shows**  $\text{bindET} \cdot (\text{fmapET} \cdot f \cdot m) \cdot \text{unitET} = \text{fmapET} \cdot f \cdot m$   
**by** (*simp add: fmapET-def bindET-bindET*)

**lemma** *invar'-right-unit*:  $\text{invar}' m \implies \text{bindET} \cdot m \cdot \text{unitET} = m$   
**unfolding** *invar'-def* **by** (*erule subst, rule bindET-fmapET-unitET*)

**lemma** *invar'-monad-fmap*:  
 $\text{invar}' m \implies \text{fmapET} \cdot f \cdot m = \text{bindET} \cdot m \cdot (\bigwedge x. \text{unitET} \cdot (f \cdot x))$   
**unfolding** *invar'-def* **by** (*erule subst, simp add: errorT-eq-iff*)

```

lemma invar'-bind-assoc:
  
$$\llbracket \text{invar}'\ m; \bigwedge x. \text{invar}'\ (f \cdot x); \bigwedge y. \text{invar}'\ (g \cdot y) \rrbracket$$

  
$$\implies \text{bindET} \cdot (\text{bindET} \cdot m \cdot f) \cdot g = \text{bindET} \cdot m \cdot (\bigwedge x. \text{bindET} \cdot (f \cdot x) \cdot g)$$

  by (rule bindET-bindET)

end

```

## 17 Writer monad transformer

```

theory Writer-Transformer
imports Writer-Monad
begin

```

### 17.1 Type definition

Below is the standard Haskell definition of a writer monad transformer:

```

newtype WriterT w m a = WriterT { runWriterT :: m (a, w) }

```

In this development, since a lazy pair type is not pre-defined in HOLCF, we will use an equivalent formulation in terms of our previous `Writer` type:

```

data Writer w a = Writer w a
newtype WriterT w m a = WriterT { runWriterT :: m (Writer w a) }

```

We can translate this definition directly into HOLCF using *tycondef*.

```

tycondef 'a.('m::functor,'w) writerT =
  WriterT (runWriterT :: ('a.'w writer)..'m)

```

```

lemma coerce-writerT-abs [simp]:
  
$$\text{coerce} \cdot (\text{writerT-abs} \cdot x) = \text{writerT-abs} \cdot (\text{coerce} \cdot x)$$

apply (simp add: writerT-abs-def coerce-def)
apply (simp add: emb-prj-emb prj-emb-prj DEFL-eq-writerT)
done

```

```

lemma coerce-WriterT [simp]:  $\text{coerce} \cdot (\text{WriterT} \cdot k) = \text{WriterT} \cdot (\text{coerce} \cdot k)$ 
unfolding WriterT-def by simp

```

```

lemma writerT-cases [case-names WriterT]:
  obtains k where  $y = \text{WriterT} \cdot k$ 
proof
  show  $y = \text{WriterT} \cdot (\text{runWriterT} \cdot y)$ 
  by (cases y, simp-all)
qed

```

**lemma** *WriterT-runWriterT* [*simp*]:  $WriterT.(runWriterT.m) = m$   
**by** (*cases m rule: writerT-cases, simp*)

**lemma** *writerT-induct* [*case-names WriterT*]:  
**fixes**  $P :: 'a.(f::functor,'e) writerT \Rightarrow bool$   
**assumes**  $\bigwedge k. P (WriterT.k)$   
**shows**  $P y$   
**by** (*cases y rule: writerT-cases, simp add: assms*)

**lemma** *writerT-eq-iff*:  
 $a = b \longleftrightarrow runWriterT.a = runWriterT.b$   
**apply** (*cases a rule: writerT-cases*)  
**apply** (*cases b rule: writerT-cases*)  
**apply** *simp*  
**done**

**lemma** *writerT-below-iff*:  
 $a \sqsubseteq b \longleftrightarrow runWriterT.a \sqsubseteq runWriterT.b$   
**apply** (*cases a rule: writerT-cases*)  
**apply** (*cases b rule: writerT-cases*)  
**apply** *simp*  
**done**

**lemma** *writerT-eqI*:  
 $runWriterT.a = runWriterT.b \Longrightarrow a = b$   
**by** (*simp add: writerT-eq-iff*)

**lemma** *writerT-belowI*:  
 $runWriterT.a \sqsubseteq runWriterT.b \Longrightarrow a \sqsubseteq b$   
**by** (*simp add: writerT-below-iff*)

**lemma** *runWriterT-coerce* [*simp*]:  
 $runWriterT.(coerce.k) = coerce.(runWriterT.k)$   
**by** (*induct k rule: writerT-induct, simp*)

## 17.2 Functor class instance

**lemma** *fmap-writer-def*:  $fmap = writer-map.ID$   
**apply** (*rule cfun-eqI, rename-tac f*)  
**apply** (*rule cfun-eqI, rename-tac x*)  
**apply** (*case-tac x rule: writer.exhaust, simp-all*)  
**apply** (*simp add: writer-map-def fix-const*)  
**apply** (*simp add: writer-map-def fix-const Writer-def*)  
**done**

**lemma** *fmapU-WriterT* [*simp*]:  
 $fmapU.f.(WriterT.m) = WriterT.(fmap.(fmap.f).m)$   
**unfolding** *fmapU-writerT-def writerT-map-def fmap-writer-def fix-const*  
*WriterT-def* **by** *simp*

**lemma** *runWriterT-fmapU* [simp]:  
 $runWriterT.(fmapU.f.m) = fmap.(fmap.f).(runWriterT.m)$   
**by** (induct m rule: writerT-induct) simp

**instance** *writerT* :: (functor, domain) functor

**proof**

**fix**  $f\ g :: udom \rightarrow udom$  **and**  $xs :: udom.(a,b)$  *writerT*  
**show**  $fmapU.f.(fmapU.g.xs) = fmapU.(\lambda x. f.(g.x)).xs$   
**apply** (induct xs rule: writerT-induct)  
**apply** (simp add: fmap-fmap eta-cfun)  
**done**

qed

### 17.3 Monad operations

The writer monad transformer does not yield a monad in the usual sense: We cannot prove a *monad* class instance, because type  $a.(m,w)$  *writerT* contains values that break the monad laws. However, it turns out that such values are inaccessible: The monad laws are satisfied by all values constructible from the abstract operations.

To explore the properties of the writer monad transformer operations, we define them all as non-overloaded functions.

**definition** *unitWT* ::  $a \rightarrow a.(m::monad,w::monoid)$  *writerT*  
**where**  $unitWT = (\lambda x. WriterT.(return.(Writer.empty.x)))$

**definition** *bindWT* ::  $a.(m::monad,w::monoid)$  *writerT*  $\rightarrow (a \rightarrow b.(m,w)$  *writerT*)  $\rightarrow b.(m,w)$  *writerT*  
**where**  $bindWT = (\lambda m\ k. WriterT.(bind.(runWriterT.m).(\lambda (Writer.w.x). bind.(runWriterT.(k.x)).(\lambda (Writer.w'y). return.(Writer.(mappend.w.w').y))))$

**definition** *liftWT* ::  $a.m \rightarrow a.(m::monad,w::monoid)$  *writerT*  
**where**  $liftWT = (\lambda m. WriterT.(fmap.(Writer.empty).m))$

**definition** *tellWT* ::  $a \rightarrow w \rightarrow a.(m::monad,w::monoid)$  *writerT*  
**where**  $tellWT = (\lambda x\ w. WriterT.(return.(Writer.w.x)))$

**definition** *fmapWT* ::  $(a \rightarrow b) \rightarrow a.(m::monad,w::monoid)$  *writerT*  $\rightarrow b.(m,w)$  *writerT*  
**where**  $fmapWT = (\lambda f\ m. bindWT.m.(\lambda x. unitWT.(f.x)))$

**lemma** *runWriterT-fmap* [simp]:  
 $runWriterT.(fmap.f.m) = fmap.(fmap.f).(runWriterT.m)$   
**by** (subst fmap-def, simp add: coerce-simp eta-cfun)

**lemma** *runWriterT-unitWT* [simp]:

$runWriterT.(unitWT.x) = return.(Writer.empty.x)$   
**unfolding** *unitWT-def* **by** *simp*

**lemma** *runWriterT-bindWT* [*simp*]:  
 $runWriterT.(bindWT.m.k) = bind.(runWriterT.m).(\Lambda(Writer.w.x). bind.(runWriterT.(k.x)).(\Lambda(Writer.w'.y). return.(Writer.(mappend.w.w').y)))$   
**unfolding** *bindWT-def* **by** *simp*

**lemma** *runWriterT-liftWT* [*simp*]:  
 $runWriterT.(liftWT.m) = fmap.(Writer.empty).m$   
**unfolding** *liftWT-def* **by** *simp*

**lemma** *runWriterT-tellWT* [*simp*]:  
 $runWriterT.(tellWT.x.w) = return.(Writer.w.x)$   
**unfolding** *tellWT-def* **by** *simp*

**lemma** *runWriterT-fmapWT* [*simp*]:  
 $runWriterT.(fmapWT.f.m) = runWriterT.m \gg (\Lambda(Writer.w.x). return.(Writer.w.(f.x)))$   
**by** (*simp add: fmapWT-def bindWT-def mempty-right*)

## 17.4 Laws

The *liftWT* function maps *return* and *bind* on the inner monad to *unitWT* and *bindWT*, as expected.

**lemma** *liftWT-return*:  
 $liftWT.(return.x) = unitWT.x$   
**by** (*rule writerT-eqI, simp add: fmap-return*)

**lemma** *liftWT-bind*:  
 $liftWT.(bind.m.k) = bindWT.(liftWT.m).(liftWT oo k)$   
**by** (*rule writerT-eqI*)  
(*simp add: monad-fmap bind-bind mempty-left*)

The composition rule holds unconditionally for *fmap*. The *fmap* function also interacts as expected with *unit* and *bind*.

**lemma** *fmapWT-fmapWT*:  
 $fmapWT.f.(fmapWT.g.m) = fmapWT.(\Lambda x. f.(g.x)).m$   
**apply** (*simp add: writerT-eq-iff bind-bind*)  
**apply** (*rule cfun-arg-cong, rule cfun-eqI, simp*)  
**apply** (*case-tac x, simp add: bind-strict, simp add: mempty-right*)  
**done**

**lemma** *fmapWT-unitWT*:  
 $fmapWT.f.(unitWT.x) = unitWT.(f.x)$   
**by** (*simp add: writerT-eq-iff mempty-right*)



**lemma** *fmapWT-bindWT*:  
 $fmapWT \cdot f \cdot (bindWT \cdot m \cdot k) = bindWT \cdot m \cdot (\Lambda x. fmapWT \cdot f \cdot (k \cdot x))$   
**apply** (*simp add: writerT-eq-iff bind-bind*)  
**apply** (*rule cfun-arg-cong, rule cfun-eqI, rename-tac x, simp*)  
**apply** (*case-tac x, simp add: bind-strict, simp add: bind-bind*)  
**apply** (*rule cfun-arg-cong, rule cfun-eqI, rename-tac y, simp*)  
**apply** (*case-tac y, simp add: bind-strict, simp add: mempty-right*)  
**done**

**lemma** *bindWT-fmapWT*:  
 $bindWT \cdot (fmapWT \cdot f \cdot m) \cdot k = bindWT \cdot m \cdot (\Lambda x. k \cdot (f \cdot x))$   
**apply** (*simp add: writerT-eq-iff bind-bind*)  
**apply** (*rule cfun-arg-cong, rule cfun-eqI, rename-tac x, simp*)  
**apply** (*case-tac x, simp add: bind-strict, simp add: mempty-right*)  
**done**

The left unit monad law is not satisfied in general.

**lemma** *bindWT-unitWT-counterexample*:  
**fixes**  $k :: 'a \rightarrow 'b \cdot ('m :: monad, 'w :: monoid) \text{ writerT}$   
**assumes**  $1: k \cdot x = \text{WriterT} \cdot (\text{return} \cdot \perp)$   
**assumes**  $2: \text{return} \cdot \perp \neq (\perp :: ('b \cdot 'w \text{ writer}) \cdot 'm :: monad)$   
**shows**  $bindWT \cdot (\text{unitWT} \cdot x) \cdot k \neq k \cdot x$   
**by** (*simp add: writerT-eq-iff mempty-left assms*)

However, left unit is satisfied for inner monads with a strict *return* function.

**lemma** *bindWT-unitWT-restricted*:  
**fixes**  $k :: 'a \rightarrow 'b \cdot ('m :: monad, 'w :: monoid) \text{ writerT}$   
**assumes**  $\text{return} \cdot \perp = (\perp :: ('b \cdot 'w \text{ writer}) \cdot 'm)$   
**shows**  $bindWT \cdot (\text{unitWT} \cdot x) \cdot k = k \cdot x$   
**unfolding** *writerT-eq-iff*  
**apply** (*simp add: mempty-left*)  
**apply** (*rule trans [OF - monad-right-unit]*)  
**apply** (*rule cfun-arg-cong*)  
**apply** (*rule cfun-eqI*)  
**apply** (*case-tac x, simp-all add: assms*)  
**done**

The associativity of *bindWT* holds unconditionally.

**lemma** *bindWT-bindWT*:  
 $bindWT \cdot (bindWT \cdot m \cdot h) \cdot k = bindWT \cdot m \cdot (\Lambda x. bindWT \cdot (h \cdot x) \cdot k)$   
**apply** (*rule writerT-eqI*)  
**apply** *simp*  
**apply** (*simp add: bind-bind*)  
**apply** (*rule cfun-arg-cong*)  
**apply** (*rule cfun-eqI, simp*)  
**apply** (*case-tac x*)

```

apply (simp add: bind-strict)
apply (simp add: bind-bind)
apply (rule cfun-arg-cong)
apply (rule cfun-eqI, simp, rename-tac y)
apply (case-tac y)
apply (simp add: bind-strict)
apply (simp add: bind-bind)
apply (rule cfun-arg-cong)
apply (rule cfun-eqI, simp, rename-tac z)
apply (case-tac z)
apply (simp add: bind-strict)
apply (simp add: mappend-assoc)
done

```

The right unit monad law is not satisfied in general.

```

lemma bindWT-unitWT-right-counterexample:
  fixes m :: 'a.('m::monad,'w::monoid) writerT
  assumes m = WriterT.(return.⊥)
  assumes return.⊥ ≠ (⊥ :: ('a.'w writer).m)
  shows bindWT.m.unitWT ≠ m
by (simp add: writerT-eq-iff assms)

```

Right unit is satisfied for inner monads with a strict *return* function.

```

lemma bindWT-unitWT-right-restricted:
  fixes m :: 'a.('m::monad,'w::monoid) writerT
  assumes return.⊥ = (⊥ :: ('a.'w writer).m)
  shows bindWT.m.unitWT = m
unfolding writerT-eq-iff
apply simp
apply (rule trans [OF - monad-right-unit])
apply (rule cfun-arg-cong)
apply (rule cfun-eqI)
apply (case-tac x, simp-all add: assms mempty-right)
done

```

## 17.5 Writer monad transformer invariant

We inductively define a predicate that includes all values that can be constructed from the standard *writerT* operations.

```

inductive invar :: 'a.('m::monad,'w::monoid) writerT ⇒ bool
  where invar-bottom: invar ⊥
  | invar-lub:  $\bigwedge Y. \llbracket \text{chain } Y; \bigwedge i. \text{invar } (Y\ i) \rrbracket \implies \text{invar } (\bigsqcup i. Y\ i)$ 
  | invar-unitWT:  $\bigwedge x. \text{invar } (\text{unitWT}\cdot x)$ 
  | invar-bindWT:  $\bigwedge m\ k. \llbracket \text{invar } m; \bigwedge x. \text{invar } (k\cdot x) \rrbracket \implies \text{invar } (\text{bindWT}\cdot m\cdot k)$ 
  | invar-tellWT:  $\bigwedge x\ w. \text{invar } (\text{tellWT}\cdot x\cdot w)$ 
  | invar-liftWT:  $\bigwedge m. \text{invar } (\text{liftWT}\cdot m)$ 

```

Right unit is satisfied for arguments built from standard functions.

```

lemma bindWT-unitWT-right-invar:
  fixes  $m :: 'a \cdot ('m :: \text{monad}, 'w :: \text{monoid}) \text{writer}T$ 
  assumes invar m
  shows  $\text{bind}WT \cdot m \cdot \text{unit}WT = m$ 
using assms proof (induct set: invar)
  case invar-bottom thus ?case
    by (rule writerT-eqI, simp add: bind-strict)
next
  case invar-lub thus ?case
    by – (rule admD, simp, assumption, assumption)
next
  case invar-unitWT thus ?case
    by (rule writerT-eqI, simp add: bind-bind mempty-left)
next
  case invar-bindWT thus ?case
    apply (simp add: writerT-eq-iff bind-bind)
    apply (rule cfun-arg-cong, rule cfun-eqI, simp)
    apply (case-tac x, simp add: bind-strict, simp add: bind-bind)
    apply (rule cfun-arg-cong, rule cfun-eqI, simp, rename-tac y)
    apply (case-tac y, simp add: bind-strict, simp add: mempty-right)
    done
next
  case invar-tellWT thus ?case
    by (simp add: writerT-eq-iff mempty-right)
next
  case invar-liftWT thus ?case
    by (rule writerT-eqI, simp add: monad-fmap bind-bind mempty-right)
qed

```

Left unit is also satisfied for arguments built from standard functions.

```

lemma writerT-left-unit-invar-lemma:
  assumes invar m
  shows  $\text{run}WriterT \cdot m \gg= (\Lambda (\text{Writer} \cdot w \cdot x). \text{return} \cdot (\text{Writer} \cdot w \cdot x)) = \text{run}WriterT \cdot m$ 
using assms proof (induct m set: invar)
  case invar-bottom thus ?case
    by (simp add: bind-strict)
next
  case invar-lub thus ?case
    by – (rule admD, simp, assumption, assumption)
next
  case invar-unitWT thus ?case
    by simp
next
  case invar-bindWT thus ?case
    apply (simp add: bind-bind)
    apply (rule cfun-arg-cong)
    apply (rule cfun-eqI, simp, rename-tac n)

```

```

  apply (case-tac n, simp add: bind-strict)
  apply (simp add: bind-bind)
  apply (rule cfun-arg-cong)
  apply (rule cfun-eqI, simp, rename-tac p)
  apply (case-tac p, simp add: bind-strict)
  apply simp
  done
next
  case invar-tellWT thus ?case
  by simp
next
  case invar-liftWT thus ?case
  by (simp add: monad-fmap bind-bind)
qed

```

```

lemma bindWT-unitWT-invar:
  assumes invar (k·x)
  shows bindWT·(unitWT·x)·k = k·x
  apply (simp add: writerT-eq-iff mempty-left)
  apply (rule writerT-left-unit-invar-lemma [OF assms])
  done

```

## 17.6 Invariant expressed as a deflation

**definition**  $invar' :: 'a \cdot ('m :: monad, 'w :: monoid) \text{writer}T \Rightarrow bool$   
 where  $invar' m \longleftrightarrow fmapWT \cdot ID \cdot m = m$

All standard operations preserve the invariant.

```

lemma invar'-bottom: invar'  $\perp$ 
  unfolding invar'-def by (simp add: writerT-eq-iff bind-strict)

```

```

lemma adm-invar': adm invar'
  unfolding invar'-def [abs-def] by simp

```

```

lemma invar'-unitWT: invar' (unitWT·x)
  unfolding invar'-def by (simp add: writerT-eq-iff)

```

```

lemma invar'-bindWT:  $\llbracket invar' m; \bigwedge x. invar' (k \cdot x) \rrbracket \implies invar' (bindWT \cdot m \cdot k)$ 
  unfolding invar'-def
  apply (erule subst)
  apply (simp add: writerT-eq-iff)
  apply (simp add: bind-bind)
  apply (rule cfun-arg-cong)
  apply (rule cfun-eqI, case-tac x)
  apply (simp add: bind-strict)
  apply simp
  apply (simp add: bind-bind)
  apply (rule cfun-arg-cong)
  apply (rule cfun-eqI, rename-tac x, case-tac x)

```

**apply** (*simp add: bind-strict*)  
**apply** *simp*  
**done**

**lemma** *invar'-tellWT: invar' (tellWT·x·w)*  
**unfolding** *invar'-def by (simp add: writerT-eq-iff)*

**lemma** *invar'-liftWT: invar' (liftWT·m)*  
**unfolding** *invar'-def by (simp add: writerT-eq-iff monad-fmap bind-bind)*

Left unit is satisfied for arguments built from fmap.

**lemma** *bindWT-unitWT-fmapWT:*  
 $bindWT·(unitWT·x)·(\Lambda x. fmapWT·f·(k·x))$   
 $= fmapWT·f·(k·x)$   
**apply** (*simp add: fmapWT-def writerT-eq-iff bind-bind*)  
**apply** (*rule cfun-arg-cong, rule cfun-eqI, simp*)  
**apply** (*case-tac x, simp-all add: bind-strict mempty-left*)  
**done**

Right unit is satisfied for arguments built from fmap.

**lemma** *bindWT-fmapWT-unitWT:*  
**shows**  $bindWT·(fmapWT·f·m)·unitWT = fmapWT·f·m$   
**apply** (*simp add: bindWT-fmapWT*)  
**apply** (*simp add: fmapWT-def*)  
**done**

All monad laws are preserved by values satisfying the invariant.

**lemma** *invar'-right-unit: invar' m  $\implies$  bindWT·m·unitWT = m*  
**unfolding** *invar'-def by (erule subst, rule bindWT-fmapWT-unitWT)*

**lemma** *invar'-monad-fmap:*  
 $invar' m \implies fmapWT·f·m = bindWT·m·(\Lambda x. unitWT·(f·x))$   
**unfolding** *invar'-def*  
**by** (*erule subst, simp add: writerT-eq-iff mempty-right*)

**lemma** *invar'-bind-assoc:*  
 $\llbracket invar' m; \Lambda x. invar' (f·x); \Lambda y. invar' (g·y) \rrbracket$   
 $\implies bindWT·(bindWT·m·f)·g = bindWT·m·(\Lambda x. bindWT·(f·x)·g)$   
**by** (*rule bindWT-bindWT*)

**end**

## References

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