Theta Functions

Manuel Eberl

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Abstract

This entry defines the Ramanujan theta function

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$

and derives from it the more commonly known Jacobi theta function on the unit disc

$$\vartheta_{00}(w,q) = \sum_{n=-\infty}^{\infty} w^{2n} q^{n^2},$$

its version in the complex plane

$$\vartheta_{00}(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi(2nz + n^2\tau))$$

as well as its half-period variants ϑ_{01}, ϑ_{10}, and ϑ_{11}.

Notable results formalised include the fact that ϑ_{00} is a solution to the one-dimensional heat equation $\frac{\partial^2}{\partial z^2} f(z,t) = 4i\pi \frac{\partial}{\partial t} f(z,t)$, and Jacobi's triple product

$$\prod_{n=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1}w^2)(1 + q^{2m-1}w^{-2}) = \sum_{k=-\infty}^{\infty} q^{k^2} w^{2k}$$

as well as its corollary, Euler's famous pentagonal number theorem:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}$$

Various important theta nullwert identities are also derived, e.g. $\vartheta_2(q)^4 + \vartheta_4(q)^4 = \vartheta_3(q)^4$ and $\vartheta_1'(q) = \vartheta_2(q)\vartheta_3(q)\vartheta_4(q)$.

Contents

1 Auxiliary material

theory *Theta_Functions_Library*
imports
  "HOL-Computational_Algebra.Computational_Algebra" "HOL-Analysis.Analysis"
begin

lemma *add_in_Ints_iff_left [simp]: "x ∈ ℤ ⟹ x + y ∈ ℤ ⟷ y ∈ ℤ"*
  ⟨*proof*⟩

lemma *add_in_Ints_iff_right [simp]: "y ∈ ℤ ⟹ x + y ∈ ℤ ⟷ x ∈ ℤ"*
  ⟨*proof*⟩

lemma *diff_in_Ints_iff_left [simp]: "x ∈ ℤ ⟹ x - y ∈ ℤ ⟷ y ∈ ℤ"*
  ⟨*proof*⟩

lemma *diff_in_Ints_iff_right [simp]: "y ∈ ℤ ⟹ x - y ∈ ℤ ⟷ x ∈ ℤ"*
  ⟨*proof*⟩

lemmas *[simp] = minus_in_Ints_iff*

lemma *of_int_div_of_int_in_Ints_iff:*
  "(of_int n / of_int m :: 'a :: field_char_0) ∈ ℤ ⟷ m = 0 ∨ m dvd n"
⟨*proof*⟩

lemma *fraction_numeral_not_in_Ints [simp]:*
  assumes *"¬(numeral b :: int) dvd numeral a"*
  shows *"numeral a / numeral b ∉ (ℤ :: 'a :: {division_ring, ring_char_0} set)"*
  ⟨*proof*⟩

lemma *fraction_numeral_not_in_Ints' [simp]:*
  assumes *"b ≠ Num.One"*
  shows *"1 / numeral b ∉ (ℤ :: 'a :: {division_ring, ring_char_0} set)"*
  ⟨*proof*⟩

lemmas *[simp] = not_in_Ints_imp_not_in_nonpos_Ints minus_in_Ints_iff*

lemma *of_int_div: "b dvd a ⟹ of_int (a div b) = (of_int a / of_int b :: 'a :: field_char_0)"*
  ⟨*proof*⟩

lemma *is_square_mult_prime_left_iff:*
  assumes *"prime p"*
  shows *"is_square (p * x) ⟷ p dvd x ∧ is_square (x div p)"*
⟨*proof*⟩

lemma *is_square_mult2_nat_iff:*
  *"is_square (2 * b :: nat) ⟷ even b ∧ is_square (b div 2)"*
  ⟨*proof*⟩

lemma *is_square_mult2_int_iff:*
  *"is_square (2 * b :: int) ⟷ even b ∧ is_square (b div 2)"*
  ⟨*proof*⟩

end

2 Conversion from the complex plane to the nome

theory *Nome*
  imports *"HOL-Complex_Analysis.Complex_Analysis" "HOL-Library.Going_To_Filter"*
begin

definition *to_nome :: "complex ⇒ complex"*
  where *"to_nome z = exp (i * of_real pi * z)"*

lemma *to_nome_nonzero [simp]: "to_nome z ≠ 0"*
  ⟨*proof*⟩

lemma *norm_to_nome: "norm (to_nome z) = exp (-pi * Im z)"*
  ⟨*proof*⟩

lemma *to_nome_add: "to_nome (z + w) = to_nome z * to_nome w"*
  ⟨*proof*⟩

lemma *to_nome_diff: "to_nome (z - w) = to_nome z / to_nome w"*
  ⟨*proof*⟩

lemma *to_nome_minus: "to_nome (-z) = inverse (to_nome z)"*
  ⟨*proof*⟩

lemma *to_nome_cnj: "to_nome (cnj z) = cnj (to_nome (-z))"*
  ⟨*proof*⟩

lemma *to_nome_power: "to_nome z ^ n = to_nome (of_nat n * z)"*
  ⟨*proof*⟩

lemma *to_nome_power_int: "to_nome z powi n = to_nome (of_int n * z)"*
  ⟨*proof*⟩

lemma *cis_conv_to_nome: "cis x = to_nome (of_real (x / pi))"*
  ⟨*proof*⟩

lemma *to_nome_powr:*
  assumes *"Re z ∈ {-1<..1}"*
  shows *"to_nome z powr w = to_nome (z * w)"*

⟨proof⟩

lemma *to_nome_0 [simp]: "to_nome 0 = 1"*
⟨proof⟩

lemma *to_nome_1 [simp]: "to_nome 1 = -1"*
and *to_nome_neg1 [simp]: "to_nome (-1) = -1"*
⟨proof⟩

lemma *to_nome_of_nat [simp]: "to_nome (of_nat n) = (-1) ^ n"*
⟨proof⟩

lemma *to_nome_of_int [simp]: "to_nome (of_int n) = (-1) powi n"*
⟨proof⟩

lemma *to_nome_one_half [simp]: "to_nome (1 / 2) = i"*
⟨proof⟩

lemma *to_nome_three_halves [simp]: "to_nome (3 / 2) = -i"*
⟨proof⟩

lemma *to_nome_eq_1_iff: "to_nome z = 1 ⟷ (∃n. even n ∧ z = of_int n)"*
⟨proof⟩

lemma *to_nome_eq_neg1_iff: "to_nome z = -1 ⟷ (∃n. odd n ∧ z = of_int n)"*
⟨proof⟩

lemma *to_nome_eq_1_iff': "to_nome z = 1 ⟷ (z / 2) ∈ ℤ"*
⟨proof⟩

lemma *to_nome_eq_neg1_iff':"to_nome z = -1 ⟷ ((z-1) / 2) ∈ ℤ"*
⟨proof⟩

lemma *to_nome_neg_one_half [simp]: "to_nome (-(1 / 2)) = -i"*
⟨proof⟩

lemma *to_nome_2 [simp]: "to_nome 2 = 1"*
⟨proof⟩

lemma *has_field_derivative_to_nome [derivative_intros]:*
assumes *"(f has_field_derivative f') (at x within A)"*
shows *"((λx. to_nome (f x)) has_field_derivative (i * pi * to_nome (f x) * f')) (at x within A)"*
⟨proof⟩

lemma *holomorphic_to_nome [holomorphic_intros]:*
  "f holomorphic_on A ⟹ (λz. to_nome (f z)) holomorphic_on A"
  ⟨*proof*⟩

lemma *analytic_to_nome [analytic_intros]:*
  "f analytic_on A ⟹ (λz. to_nome (f z)) analytic_on A"
  ⟨*proof*⟩

lemma *tendsto_to_nome [tendsto_intros]:*
  assumes *"(f ⟶ w) F"*
  shows *"((λz. to_nome (f z)) ⟶ to_nome w) F"*
  ⟨*proof*⟩

lemma *continuous_on_to_nome [continuous_intros]:*
  assumes *"continuous_on A f"*
  shows *"continuous_on A (λz. to_nome (f z))"*
  ⟨*proof*⟩

lemma *continuous_to_nome [continuous_intros]:*
  assumes *"continuous F f"*
  shows *"continuous F (λz. to_nome (f z))"*
  ⟨*proof*⟩

lemma *tendsto_0_to_nome:*
  assumes *"filterlim (λx. Im (f x)) at_top F"*
  shows *"filterlim (λx. to_nome (f x)) (nhds 0) F"*
⟨*proof*⟩

lemma *tendsto_0_to_nome': "(to_nome ⟶ 0) (Im going_to at_top)"*
  ⟨*proof*⟩

lemma *filterlim_at_0_to_nome:*
  assumes *"filterlim (λx. Im (f x)) at_top F"*
  shows *"filterlim (λx. to_nome (f x)) (at 0) F"*
  ⟨*proof*⟩

end

3 General theta functions

theory *Theta_Functions*
imports
  Nome
  "Combinatorial_Q_Analogues.Q_Binomial_Identities"
  Theta_Functions_Library
begin

3.1 The Ramanujan theta function

We define the other theta functions in terms of the Ramanujan theta function:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \qquad (\text{for } |ab| < 1)$$

This is, in some sense, more general than Jacobi's theta function: Jacobi's theta function can be expressed very easily in terms of Ramanujan's; the other direction is only straightforward in the real case. Due to the presence of square roots, the complex case becomes tedious due to branch cuts.

However, even in the complex case, results can be transferred from Jacobi's theta function to Ramanujan's by using the connection on the real line and then doing analytic continuation.

Some of the proofs below are loosely based on Ramanujan's lost notebook (as edited by Berndt [2]).

```
definition ramanujan_theta :: "'a :: {real_normed_field, banach} ⇒ 'a
⇒ 'a" where
  "ramanujan_theta a b =
     (if norm (a*b) < 1 then (∑∞n. a powi (n*(n+1) div 2) * b powi (n*(n-1)
div 2)) else 0)"

lemma ramanujan_theta_outside [simp]: "norm (a * b) ≥ 1 ⟹ ramanujan_theta
a b = 0"
  ⟨proof⟩

lemma uniform_limit_ramanujan_theta_aux:
  fixes A :: "('a × 'a :: {real_normed_field, banach}) set"
  assumes "compact A" "⋀a b. (a, b) ∈ A ⟹ norm (a * b) < 1"
  shows   "uniform_limit A (λX (a,b). ∑n∈X. a powi (n*(n+1) div 2) *
b powi (n*(n-1) div 2))
             (λ(a,b). ∑∞n. a powi (n*(n+1) div 2) * b powi (n*(n-1)
div 2))
             (finite_subsets_at_top UNIV)"
⟨proof⟩

lemma uniform_limit_ramanujan_theta:
  fixes A :: "('a × 'a :: {real_normed_field, banach}) set"
  assumes "compact A" "⋀a b. (a, b) ∈ A ⟹ norm (a * b) < 1"
  shows   "uniform_limit A (λX (a,b). ∑n∈X. a powi (n*(n+1) div 2) *
b powi (n*(n-1) div 2))
             (λ(a,b). ramanujan_theta a b)
             (finite_subsets_at_top UNIV)"
⟨proof⟩

lemma has_sum_ramanujan_theta:
  assumes "norm (a*b) < 1"
```

```
  shows   "((λn. a powi (n*(n+1) div 2) * b powi (n*(n-1) div 2)) has_sum
ramanujan_theta a b) UNIV"
  ⟨proof⟩

lemma ramanujan_theta_commute: "ramanujan_theta a b = ramanujan_theta
b a"
⟨proof⟩

lemma ramanujan_theta_0_left [simp]: "ramanujan_theta 0 b = 1 + b"
⟨proof⟩

lemma ramanujan_theta_0_right [simp]: "ramanujan_theta a 0 = 1 + a"
  ⟨proof⟩

lemma has_sum_ramanujan_theta1:
  assumes "norm (a*b) < 1" and [simp]: "a ≠ 0"
  shows   "((λn. a powi n * (a*b) powi (n*(n-1) div 2)) has_sum ramanujan_theta
a b) UNIV"
⟨proof⟩

lemma has_sum_ramanujan_theta2:
  assumes "norm (a * b) < 1"
  shows    "((λn. (a*b) powi (n*(n-1) div 2) * (a powi n + b powi n))
has_sum
               (ramanujan_theta a b - 1)) {1..}"
⟨proof⟩

lemma ramanujan_theta_of_real:
  "ramanujan_theta (of_real a) (of_real b) = of_real (ramanujan_theta
a b)"
⟨proof⟩

lemma ramanujan_theta_cnj:
  "ramanujan_theta (cnj a) (cnj b) = cnj (ramanujan_theta a b)"
⟨proof⟩

lemma ramanujan_theta_holomorphic [holomorphic_intros]:
  assumes "f holomorphic_on A" "g holomorphic_on A"
  assumes "⋀z. z ∈ A ⟹ norm (f z * g z) < 1"  "open A"
  shows   "(λz. ramanujan_theta (f z) (g z)) holomorphic_on A"
⟨proof⟩

lemma ramanujan_theta_analytic [analytic_intros]:
  assumes "f analytic_on A" "g analytic_on A" "⋀z. z ∈ A ⟹ norm (f
z * g z) < 1"
  shows   "(λz. ramanujan_theta (f z) (g z)) analytic_on A"
⟨proof⟩

lemma tendsto_ramanujan_theta [tendsto_intros]:
```

fixes *f g :: "'a ⇒ 'b :: {real_normed_field, banach, heine_borel}"*
assumes *"(f ⟶ a) F" "(g ⟶ b) F" "norm (a * b) < 1"*
shows *"((λz. ramanujan_theta (f z) (g z)) ⟶ ramanujan_theta a b) F"*
⟨*proof*⟩

lemma *continuous_on_ramanujan_theta [continuous_intros]:*
fixes *f g :: "'a :: topological_space ⇒ 'b :: {real_normed_field, banach, heine_borel}"*
assumes *"continuous_on A f" "continuous_on A g" "⋀z. z ∈ A ⟹ norm (f z * g z) < 1"*
shows *"continuous_on A (λz. ramanujan_theta (f z) (g z))"*
⟨*proof*⟩

lemma *continuous_ramanujan_theta [continuous_intros]:*
fixes *f g :: "'a :: t2_space ⇒ 'b :: {real_normed_field, banach, heine_borel}"*
assumes *"continuous F f" "continuous F g" "norm (f (netlimit F) * g (netlimit F)) < 1"*
shows *"continuous F (λz. ramanujan_theta (f z) (g z))"*
⟨*proof*⟩

lemma *ramanujan_theta_1_left:*
*"ramanujan_theta 1 a = 2 * ramanujan_theta a (a ^ 3)"*
⟨*proof*⟩

lemma *ramanujan_theta_1_right: "ramanujan_theta a 1 = 2 * ramanujan_theta a (a ^ 3)"*
⟨*proof*⟩

lemma *ramanujan_theta_neg1_left [simp]: "ramanujan_theta (-1) a = 0"*
⟨*proof*⟩

lemma *ramanujan_theta_neg1_right [simp]: "ramanujan_theta a (-1) = 0"*
⟨*proof*⟩

lemma *ramanujan_theta_mult_power_int:*
assumes *[simp]: "a ≠ 0" "b ≠ 0"*
shows *"ramanujan_theta a b =*
a powi (m(m+1) div 2) * b powi (m*(m-1) div 2) **
*ramanujan_theta (a * (a*b) powi m) (b * (a*b) powi (-m))"*
⟨*proof*⟩

lemma *ramanujan_theta_mult:*
assumes *[simp]: "a ≠ 0" "b ≠ 0"*
shows *"ramanujan_theta a b = a * ramanujan_theta (a^2 * b) (1 / a)"*
⟨*proof*⟩

lemma *ramanujan_theta_mult':*
assumes *[simp]: "a ≠ 0" "b ≠ 0"*

shows *"ramanujan_theta a b = b * ramanujan_theta (1 / b) (a * b^2)"*
⟨*proof*⟩

3.2 The Jacobi theta function in terms of the nome

Based on Ramanujan's ϑ function, we introduce a version of Jacobi's ϑ function:

$$\vartheta(w,q) = \sum_{n=-\infty}^{\infty} w^n q^{n^2} \qquad (\text{for } |q| < 1, w \neq 0)$$

Both parameters are still in terms of the nome rather than the complex plane. This has some advantages, and we can easily derive the other versions from it later.

definition *jacobi_theta_nome :: "'a :: {real_normed_field,banach} ⇒ 'a ⇒ 'a"* **where**
*"jacobi_theta_nome w q = (if w = 0 then 0 else ramanujan_theta (q*w) (q/w))"*

lemma *jacobi_theta_nome_0_left [simp]: "jacobi_theta_nome 0 q = 0"*
⟨*proof*⟩

lemma *jacobi_theta_nome_outside [simp]:*
assumes *"norm q ≥ 1"*
shows *"jacobi_theta_nome w q = 0"*
⟨*proof*⟩

lemma *has_sum_jacobi_theta_nome:*
assumes *"norm q < 1"* **and** *[simp]: "w ≠ 0"*
shows *"((λn. w powi n * q powi (n ^ 2)) has_sum jacobi_theta_nome w q) UNIV"*
⟨*proof*⟩

lemma *jacobi_theta_nome_same:*
*"q ≠ 0 ⟹ jacobi_theta_nome q q = 2 * jacobi_theta_nome (1 / q^2) (q^4)"*
⟨*proof*⟩

lemma *jacobi_theta_nome_minus_same: "q ≠ 0 ⟹ jacobi_theta_nome (-q) q = 0"*
⟨*proof*⟩

lemma *jacobi_theta_nome_minus_same': "q ≠ 0 ⟹ jacobi_theta_nome q (-q) = 0"*
⟨*proof*⟩

lemma *jacobi_theta_nome_0_right [simp]: "w ≠ 0 ⟹ jacobi_theta_nome w 0 = 1"*
⟨*proof*⟩

lemma *jacobi_theta_nome_of_real:*
"jacobi_theta_nome (of_real w) (of_real q) = of_real (jacobi_theta_nome w q)"
⟨*proof*⟩

lemma *jacobi_theta_nome_cnj:*
"jacobi_theta_nome (cnj w) (cnj q) = cnj (jacobi_theta_nome w q)"
⟨*proof*⟩

lemma *jacobi_theta_nome_minus_left:*
"jacobi_theta_nome (-w) q = jacobi_theta_nome w (-q)"
⟨*proof*⟩

lemma *jacobi_theta_nome_quasiperiod':*
assumes *[simp]: "w ≠ 0" "q ≠ 0"*
shows *"w * q * jacobi_theta_nome (q^2 * w) q = jacobi_theta_nome w q"*
⟨*proof*⟩

lemma *jacobi_theta_nome_ii_left: "jacobi_theta_nome* i *q = jacobi_theta_nome (-1) (q^4)"*
⟨*proof*⟩

lemma *jacobi_theta_nome_quasiperiod:*
assumes *[simp]: "w ≠ 0" "q ≠ 0"*
shows *"jacobi_theta_nome (q^2 * w) q = jacobi_theta_nome w q / (w * q)"*
⟨*proof*⟩

lemma *jacobi_theta_nome_holomorphic [holomorphic_intros]:*
assumes *"f holomorphic_on A" "g holomorphic_on A"*
assumes *"⋀z. z ∈ A ⟹ norm (f z) ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1" "open A"*
shows *"(λz. jacobi_theta_nome (f z) (g z)) holomorphic_on A"*
⟨*proof*⟩

lemma *jacobi_theta_nome_analytic [analytic_intros]:*
assumes *"f analytic_on A" "g analytic_on A"*
assumes *"⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1"*
shows *"(λz. jacobi_theta_nome (f z) (g z)) analytic_on A"*
⟨*proof*⟩

lemma *tendsto_jacobi_theta_nome [tendsto_intros]:*
fixes *f g :: "'a ⇒ 'b :: {real_normed_field, banach, heine_borel}"*
assumes *"(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1"*
shows *"((λz. jacobi_theta_nome (f z) (g z)) ⟶ jacobi_theta_nome w q) F"*
⟨*proof*⟩

```
lemma continuous_on_jacobi_theta_nome [continuous_intros]:
  fixes f g :: "'a :: topological_space ⇒ 'b :: {real_normed_field, banach,
heine_borel}"
  assumes "continuous_on A f" "continuous_on A g"
  assumes "⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1"
  shows   "continuous_on A (λz. jacobi_theta_nome (f z) (g z))"
⟨proof⟩

lemma continuous_jacobi_theta_nome [continuous_intros]:
  fixes f g :: "'a :: t2_space ⇒ 'b :: {real_normed_field, banach, heine_borel}"
  assumes "continuous F f" "continuous F g" "f (netlimit F) ≠ 0" "norm
(g (netlimit F)) < 1"
  shows   "continuous F (λz. jacobi_theta_nome (f z) (g z))"
  ⟨proof⟩
```

3.3 The Jacobi theta function in the upper half of the complex plane

We now define the more usual version of the Jacobi ϑ function, which takes two complex parameters z and t where z is arbitrary and t must lie in the upper half of the complex plane.

```
definition jacobi_theta_00 :: "complex ⇒ complex ⇒ complex" where
  "jacobi_theta_00 z t = jacobi_theta_nome (to_nome z ^ 2) (to_nome t)"

lemma jacobi_theta_00_outside: "Im t ≤ 0 ⟹ jacobi_theta_00 z t = 0"
  ⟨proof⟩

lemma has_sum_jacobi_theta_00:
  assumes "Im t > 0"
  shows   "((λn. to_nome (of_int n ^ 2 * t + 2 * of_int n * z)) has_sum
jacobi_theta_00 z t) UNIV"
  ⟨proof⟩

lemma sums_jacobi_theta_00:
  assumes "Im t > 0"
  shows   "((λn. if n = 0 then 1 else 2 * to_nome t ^ n² *
             cos (2 * of_nat n * of_real pi * z)) sums jacobi_theta_00
z t)"
⟨proof⟩

lemma jacobi_theta_00_holomorphic [holomorphic_intros]:
  assumes "f holomorphic_on A" "g holomorphic_on A" "⋀z. z ∈ A ⟹ Im
(g z) > 0" "open A"
  shows   "(λz. jacobi_theta_00 (f z) (g z)) holomorphic_on A"
  ⟨proof⟩

lemma jacobi_theta_00_analytic [analytic_intros]:
```

assumes *"f analytic_on A" "g analytic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"(λz. jacobi_theta_00 (f z) (g z)) analytic_on A"*
⟨*proof*⟩

lemma *jacobi_theta_00_plus_half_left:*
"jacobi_theta_00 (z + 1 / 2) t = jacobi_theta_00 z (t + 1)"
⟨*proof*⟩

lemma *jacobi_theta_00_plus_2_right: "jacobi_theta_00 z (t + 2) = jacobi_theta_00 z t"*
⟨*proof*⟩

interpretation *jacobi_theta_00_left: periodic_fun_simple' "λz. jacobi_theta_00 z t"*
⟨*proof*⟩

lemma *jacobi_theta_00_1_plus_left: "jacobi_theta_00 (1 + z) t = jacobi_theta_00 z t"*
⟨*proof*⟩

interpretation *jacobi_theta_00_right: periodic_fun_simple "λt. jacobi_theta_00 z t" 2*
⟨*proof*⟩

lemma *jacobi_theta_00_plus_quasiperiod:*
*"jacobi_theta_00 (z + t) t = jacobi_theta_00 z t / to_nome (t + 2 * z)"*
⟨*proof*⟩

lemma *jacobi_theta_00_quasiperiodic:*
*"jacobi_theta_00 (z + of_int m + of_int n * t) t =*
*jacobi_theta_00 z t / to_nome (of_int (n^2) * t + 2 * of_int n * z)"*
⟨*proof*⟩

lemma *jacobi_theta_00_onequarter_left:*
*"jacobi_theta_00 (1/4) t = jacobi_theta_00 (1/2) (4 * t)"*
⟨*proof*⟩

lemma *jacobi_theta_00_eq_0: "jacobi_theta_00 ((t + 1) / 2) t = 0"*
⟨*proof*⟩

lemma *jacobi_theta_00_eq_0': "jacobi_theta_00 ((of_int m + 1/2) + (of_int n + 1/2) * t) t = 0"*
⟨*proof*⟩

lemma *tendsto_jacobi_theta_00 [tendsto_intros]:*
assumes *"(f ⟶ w) F" "(g ⟶ q) F" "Im q > 0"*

shows *"((λz. jacobi_theta_00 (f z) (g z)) ⟶ jacobi_theta_00 w q) F"*
⟨*proof*⟩

lemma *continuous_on_jacobi_theta_00 [continuous_intros]:*
assumes *"continuous_on A f" "continuous_on A g"*
assumes *"⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"continuous_on A (λz. jacobi_theta_00 (f z) (g z))"*
⟨*proof*⟩

lemma *continuous_jacobi_theta_00 [continuous_intros]:*
assumes *"continuous F f" "continuous F g" "Im (g (netlimit F)) > 0"*
shows *"continuous F (λz. jacobi_theta_00 (f z) (g z))"*
⟨*proof*⟩

3.4 The auxiliary theta functions in terms of the nome

definition *jacobi_theta_nome_00 :: "'a :: {real_normed_field, banach} ⇒ 'a ⇒ 'a"* **where**
"jacobi_theta_nome_00 w q = jacobi_theta_nome (w^2) q"

definition *jacobi_theta_nome_01 :: "'a :: {real_normed_field, banach} ⇒ 'a ⇒ 'a"* **where**
"jacobi_theta_nome_01 w q = jacobi_theta_nome (-(w^2)) q"

definition *jacobi_theta_nome_10 :: "'a :: {real_normed_field, banach, ln} ⇒ 'a ⇒ 'a"* **where**
*"jacobi_theta_nome_10 w q = w * q powr (1/4) * jacobi_theta_nome (w ^ 2 * q) q"*

definition *jacobi_theta_nome_11 :: "complex ⇒ complex ⇒ complex"* **where**
*"jacobi_theta_nome_11 w q = i * w * q powr (1/4) * jacobi_theta_nome (-(w ^ 2) * q) q"*

lemmas *jacobi_theta_nome_xx_defs =*
jacobi_theta_nome_00_def jacobi_theta_nome_01_def
jacobi_theta_nome_10_def jacobi_theta_nome_11_def

lemma *jacobi_theta_nome_00_outside [simp]: "norm q ≥ 1 ⟹ jacobi_theta_nome_00 w q = 0"*
and *jacobi_theta_nome_01_outside [simp]: "norm q ≥ 1 ⟹ jacobi_theta_nome_01 w q = 0"*
and *jacobi_theta_nome_10_outside [simp]: "norm q' ≥ 1 ⟹ jacobi_theta_nome_10 w' q' = 0"*
and *jacobi_theta_nome_11_outside [simp]: "norm q'' ≥ 1 ⟹ jacobi_theta_nome_11 w'' q'' = 0"*
⟨*proof*⟩

lemma *jacobi_theta_nome_01_conv_00: "jacobi_theta_nome_01 w' q' = jacobi_theta_nome_00*

w' (-q')"
  and jacobi_theta_nome_11_conv_10: "jacobi_theta_nome_11 w q = jacobi_theta_nome_10 (i * w) q"
  ⟨proof⟩

lemma jacobi_theta_nome_00_0_right [simp]: "w ≠ 0 ⟹ jacobi_theta_nome_00 w 0 = 1"
  and jacobi_theta_nome_01_0_right [simp]: "w ≠ 0 ⟹ jacobi_theta_nome_01 w 0 = 1"
  and jacobi_theta_nome_10_0_right [simp]: "jacobi_theta_nome_10 w' 0 = 0"
  and jacobi_theta_nome_11_0_right [simp]: "jacobi_theta_nome_11 w'' 0 = 0"
  ⟨proof⟩

lemma jacobi_theta_nome_00_of_real:
"jacobi_theta_nome_00 (of_real w :: 'a :: {banach, real_normed_field}) (of_real q) =
of_real (jacobi_theta_nome_00 w q)"
  and jacobi_theta_nome_01_of_real:
"jacobi_theta_nome_01 (of_real w :: 'a) (of_real q) = of_real (jacobi_theta_nome_01 w q)"
  and jacobi_theta_nome_10_complex_of_real:
"q ≥ 0 ⟹ jacobi_theta_nome_10 (complex_of_real w) (of_real q) =
of_real (jacobi_theta_nome_10 w q)"
  ⟨proof⟩

lemma jacobi_theta_nome_00_cnj:
"jacobi_theta_nome_00 (cnj w) (cnj q) = cnj (jacobi_theta_nome_00 w q)"
  and jacobi_theta_nome_01_cnj:
"jacobi_theta_nome_01 (cnj w) (cnj q) = cnj (jacobi_theta_nome_01 w q)"
  and jacobi_theta_nome_10_cnj:
"(Im q = 0 ⟹ Re q ≥ 0) ⟹
jacobi_theta_nome_10 (cnj w) (cnj q) = cnj (jacobi_theta_nome_10 w q)"
  and jacobi_theta_nome_11_cnj:
"(Im q = 0 ⟹ Re q ≥ 0) ⟹
jacobi_theta_nome_11 (cnj w) (cnj q) = -cnj (jacobi_theta_nome_11 w q)"
  ⟨proof⟩

lemma has_sum_jacobi_theta_nome_00:
  assumes "norm q < 1" "w ≠ 0"
  shows "((λn. w powi (2*n) * q powi n^2) has_sum jacobi_theta_nome_00 w q) UNIV"

⟨proof⟩

lemma *has_sum_jacobi_theta_nome_01:*
assumes *"norm q < 1" "w ≠ 0"*
shows *"((λn. (-1) powi n * w powi (2*n) * q powi n^2) has_sum jacobi_theta_nome_01 w q) UNIV"*
⟨proof⟩

lemma *has_sum_jacobi_theta_nome_10':*
assumes *q: "norm q < 1"* **and** *[simp]: "w ≠ 0" "q ≠ 0"*
shows *"((λn. w powi (2*n+1) * q powi (n*(n+1))) has_sum (jacobi_theta_nome_10 w q / q powr (1/4))) UNIV"*
⟨proof⟩

lemma *has_sum_jacobi_theta_nome_10:*
fixes *q :: "'a :: {real_normed_field, banach, ln}"*
assumes *q: "norm q < 1"* **and** *[simp]: "w ≠ 0" "exp (ln q) = q"*
shows *"((λn. w powi (2*n+1) * q powr (of_int n + 1 / 2) ^ 2) has_sum (jacobi_theta_nome_10 w q)) UNIV"*
⟨proof⟩

lemma *has_sum_jacobi_theta_nome_11':*
assumes *q: "norm q < 1"* **and** *[simp]: "w ≠ 0" "q ≠ 0"*
shows *"((λn. (-1) powi n * w powi (2*n+1) * q powi (n*(n+1))) has_sum (jacobi_theta_nome_11 w q / (i * q powr (1/4)))) UNIV"*
⟨proof⟩

lemma *has_sum_jacobi_theta_nome_11:*
assumes *q: "norm q < 1"* **and** *[simp]: "w ≠ 0" "q ≠ 0"*
shows *"((λn. i * (-1) powi n * w powi (2*n+1) * q powr (of_int n + 1/2) ^ 2) has_sum (jacobi_theta_nome_11 w q)) UNIV"*
⟨proof⟩

lemma *jacobi_theta_nome_00_holomorphic [holomorphic_intros]:*
assumes *"f holomorphic_on A" "g holomorphic_on A"*
assumes *"⋀z. z ∈ A ⟹ norm (f z) ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1" "open A"*
shows *"(λz. jacobi_theta_nome_00 (f z) (g z)) holomorphic_on A"*
⟨proof⟩

lemma *jacobi_theta_nome_01_holomorphic [holomorphic_intros]:*
assumes *"f holomorphic_on A" "g holomorphic_on A"*
assumes *"⋀z. z ∈ A ⟹ norm (f z) ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1" "open A"*
shows *"(λz. jacobi_theta_nome_01 (f z) (g z)) holomorphic_on A"*
⟨proof⟩

lemma *jacobi_theta_nome_10_holomorphic [holomorphic_intros]:*
  assumes *"f holomorphic_on A" "g holomorphic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1 ∧ g z ∉ $\mathbb{R}_{\leq 0}$" "open A"*
  shows *"(λz. jacobi_theta_nome_10 (f z) (g z)) holomorphic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_11_holomorphic [holomorphic_intros]:*
  assumes *"f holomorphic_on A" "g holomorphic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1 ∧ g z ∉ $\mathbb{R}_{\leq 0}$" "open A"*
  shows *"(λz. jacobi_theta_nome_11 (f z) (g z)) holomorphic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_11_holomorphic':*
  assumes *"f holomorphic_on A" "$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0" "norm q < 1" "open A"*
  shows *"(λz. jacobi_theta_nome_11 (f z) q) holomorphic_on A"*
⟨*proof*⟩

lemma *jacobi_theta_nome_00_analytic [analytic_intros]:*
  assumes *"f analytic_on A" "g analytic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0" "$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1"*
  shows *"(λz. jacobi_theta_nome_00 (f z) (g z)) analytic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_01_analytic [analytic_intros]:*
  assumes *"f analytic_on A" "g analytic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0" "$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1"*
  shows *"(λz. jacobi_theta_nome_01 (f z) (g z)) analytic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_10_analytic [analytic_intros]:*
  assumes *"f analytic_on A" "g analytic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1 ∧ g z ∉ $\mathbb{R}_{\leq 0}$"*
  shows *"(λz. jacobi_theta_nome_10 (f z) (g z)) analytic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_11_analytic [analytic_intros]:*
  assumes *"f analytic_on A" "g analytic_on A"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (f z) ≠ 0"*
  assumes *"$\bigwedge$z. z ∈ A ⟹ norm (g z) < 1 ∧ g z ∉ $\mathbb{R}_{\leq 0}$"*
  shows *"(λz. jacobi_theta_nome_11 (f z) (g z)) analytic_on A"*
  ⟨*proof*⟩

lemma *jacobi_theta_nome_11_analytic':*

```
  assumes "f analytic_on A" "⋀z. z ∈ A ⟹ norm (f z) ≠ 0" "norm q
< 1"
  shows   "(λz. jacobi_theta_nome_11 (f z) q) analytic_on A"
⟨proof⟩

lemma tendsto_jacobi_theta_nome_00 [tendsto_intros]:
  fixes f g :: "'a ⇒ 'b :: {real_normed_field, banach, heine_borel}"
  assumes "(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1"
  shows   "((λz. jacobi_theta_nome_00 (f z) (g z)) ⟶ jacobi_theta_nome_00
w q) F"
  ⟨proof⟩

lemma continuous_on_jacobi_theta_nome_00 [continuous_intros]:
  fixes f g :: "'a :: topological_space ⇒ 'b :: {real_normed_field, banach,
heine_borel}"
  assumes "continuous_on A f" "continuous_on A g"
  assumes "⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1"
  shows   "continuous_on A (λz. jacobi_theta_nome_00 (f z) (g z))"
  ⟨proof⟩

lemma continuous_jacobi_theta_nome_00 [continuous_intros]:
  fixes f g :: "'a :: t2_space ⇒ 'b :: {real_normed_field, banach, heine_borel}"
  assumes "continuous F f" "continuous F g" "f (netlimit F) ≠ 0" "norm
(g (netlimit F)) < 1"
  shows   "continuous F (λz. jacobi_theta_nome_00 (f z) (g z))"
  ⟨proof⟩


lemma tendsto_jacobi_theta_nome_01 [tendsto_intros]:
  fixes f g :: "'a ⇒ 'b :: {real_normed_field, banach, heine_borel}"
  assumes "(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1"
  shows   "((λz. jacobi_theta_nome_01 (f z) (g z)) ⟶ jacobi_theta_nome_01
w q) F"
  ⟨proof⟩

lemma continuous_on_jacobi_theta_nome_01 [continuous_intros]:
  fixes f g :: "'a :: topological_space ⇒ 'b :: {real_normed_field, banach,
heine_borel}"
  assumes "continuous_on A f" "continuous_on A g"
  assumes "⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1"
  shows   "continuous_on A (λz. jacobi_theta_nome_01 (f z) (g z))"
  ⟨proof⟩

lemma continuous_jacobi_theta_nome_01 [continuous_intros]:
  fixes f g :: "'a :: t2_space ⇒ 'b :: {real_normed_field, banach, heine_borel}"
  assumes "continuous F f" "continuous F g" "f (netlimit F) ≠ 0" "norm
(g (netlimit F)) < 1"
  shows   "continuous F (λz. jacobi_theta_nome_01 (f z) (g z))"
  ⟨proof⟩
```

lemma *tendsto_jacobi_theta_nome_10_complex [tendsto_intros]:*
  fixes *f g :: "complex ⇒ complex"*
  assumes *"(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1" "q ∉ ℝ≤0"*
  shows *"((λz. jacobi_theta_nome_10 (f z) (g z)) ⟶ jacobi_theta_nome_10 w q) F"*
  ⟨*proof*⟩

lemma *continuous_on_jacobi_theta_nome_10_complex [continuous_intros]:*
  fixes *f g :: "complex ⇒ complex"*
  assumes *"continuous_on A f" "continuous_on A g"*
  assumes *"⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1 ∧ (Re (g z) > 0 ∨ Im (g z) ≠ 0)"*
  shows *"continuous_on A (λz. jacobi_theta_nome_10 (f z) (g z))"*
  ⟨*proof*⟩

lemma *continuous_jacobi_theta_nome_10_complex [continuous_intros]:*
  assumes *"continuous F f" "continuous F g" "f (netlimit F) ≠ 0"*
  assumes *"norm (g (netlimit F)) < 1" "Re (g (netlimit F)) > 0 ∨ Im (g (netlimit F)) ≠ 0"*
  shows *"continuous F (λz. jacobi_theta_nome_10 (f z) (g z))"*
  ⟨*proof*⟩

lemma *tendsto_jacobi_theta_nome_10_real [tendsto_intros]:*
  fixes *f g :: "real ⇒ real"*
  assumes *"(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1" "q > 0"*
  shows *"((λz. jacobi_theta_nome_10 (f z) (g z)) ⟶ jacobi_theta_nome_10 w q) F"*
  ⟨*proof*⟩

lemma *continuous_on_jacobi_theta_nome_10_real [continuous_intros]:*
  fixes *f g :: "real ⇒ real"*
  assumes *"continuous_on A f" "continuous_on A g"*
  assumes *"⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ g z ∈ {0<..<1}"*
  shows *"continuous_on A (λz. jacobi_theta_nome_10 (f z) (g z))"*
  ⟨*proof*⟩

lemma *continuous_jacobi_theta_nome_10_real [continuous_intros]:*
  fixes *f g :: "real ⇒ real"*
  assumes *"continuous F f" "continuous F g" "f (netlimit F) ≠ 0" "g (netlimit F) ∈ {0<..<1}"*
  shows *"continuous F (λz. jacobi_theta_nome_10 (f z) (g z))"*
  ⟨*proof*⟩

lemma *tendsto_jacobi_theta_nome_11_complex [tendsto_intros]:*

fixes f g :: "complex ⇒ complex"
assumes "(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1" "q ∉ $\mathbb{R}_{\leq 0}$"
shows "((λz. jacobi_theta_nome_11 (f z) (g z)) ⟶ jacobi_theta_nome_11 w q) F"
⟨*proof*⟩

lemma continuous_on_jacobi_theta_nome_11_complex [continuous_intros]:
fixes f g :: "complex ⇒ complex"
assumes "continuous_on A f" "continuous_on A g"
assumes "⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ norm (g z) < 1 ∧ (Re (g z) > 0 ∨ Im (g z) ≠ 0)"
shows "continuous_on A (λz. jacobi_theta_nome_11 (f z) (g z))"
⟨*proof*⟩

lemma continuous_jacobi_theta_nome_11_complex [continuous_intros]:
assumes "continuous F f" "continuous F g" "f (netlimit F) ≠ 0"
assumes "norm (g (netlimit F)) < 1" "Re (g (netlimit F)) > 0 ∨ Im (g (netlimit F)) ≠ 0"
shows "continuous F (λz. jacobi_theta_nome_11 (f z) (g z))"
⟨*proof*⟩

lemma tendsto_jacobi_theta_nome_11_real [tendsto_intros]:
fixes f g :: "real ⇒ real"
assumes "(f ⟶ w) F" "(g ⟶ q) F" "w ≠ 0" "norm q < 1" "q > 0"
shows "((λz. jacobi_theta_nome_11 (f z) (g z)) ⟶ jacobi_theta_nome_11 w q) F"
⟨*proof*⟩

lemma continuous_on_jacobi_theta_nome_11_real [continuous_intros]:
fixes f g :: "real ⇒ real"
assumes "continuous_on A f" "continuous_on A g"
assumes "⋀z. z ∈ A ⟹ f z ≠ 0" "⋀z. z ∈ A ⟹ g z ∈ {0<..<1}"
shows "continuous_on A (λz. jacobi_theta_nome_11 (f z) (g z))"
⟨*proof*⟩

lemma continuous_jacobi_theta_nome_11_real [continuous_intros]:
fixes f g :: "real ⇒ real"
assumes "continuous F f" "continuous F g" "f (netlimit F) ≠ 0" "g (netlimit F) ∈ {0<..<1}"
shows "continuous F (λz. jacobi_theta_nome_11 (f z) (g z))"
⟨*proof*⟩

3.5 The auxiliary theta functions in the complex plane

definition jacobi_theta_01 :: "complex ⇒ complex ⇒ complex" **where**
"jacobi_theta_01 z t = jacobi_theta_00 (z + 1/2) t"

definition jacobi_theta_10 :: "complex ⇒ complex ⇒ complex" **where**

```
  "jacobi_theta_10 z t = to_nome (z + t/4) * jacobi_theta_00 (z + t/2)
t"

definition jacobi_theta_11 :: "complex ⇒ complex ⇒ complex" where
  "jacobi_theta_11 z t = to_nome (z + t/4 + 1/2) * jacobi_theta_00 (z
+ t/2 + 1/2) t"

lemma jacobi_theta_00_conv_nome:
  "jacobi_theta_00 z t = jacobi_theta_nome_00 (to_nome z) (to_nome t)"
  ⟨proof⟩

lemma jacobi_theta_01_conv_nome:
  "jacobi_theta_01 z t = jacobi_theta_nome_01 (to_nome z) (to_nome t)"
  ⟨proof⟩

lemma jacobi_theta_10_conv_nome:
  assumes "Re t ∈ {-1<..1}"
  shows   "jacobi_theta_10 z t = jacobi_theta_nome_10 (to_nome z) (to_nome
t)"
  ⟨proof⟩

lemma jacobi_theta_11_conv_nome:
  assumes "Re t ∈ {-1<..1}"
  shows   "jacobi_theta_11 z t = jacobi_theta_nome_11 (to_nome z) (to_nome
t)"
  ⟨proof⟩


lemma has_sum_jacobi_theta_01:
  assumes "Im t > 0"
  shows   "((λn. (-1) powi n * to_nome (of_int n ^ 2 * t + 2 * of_int
n * z))
             has_sum jacobi_theta_01 z t) UNIV"
⟨proof⟩

lemma sums_jacobi_theta_01:
  assumes "Im t > 0"
  shows   "((λn. if n = 0 then 1 else 2 * (-1) ^ n *  to_nome t ^ n²
*
             cos (2 * of_nat n * of_real pi * z)) sums jacobi_theta_01
z t)"
⟨proof⟩


interpretation jacobi_theta_01_left: periodic_fun_simple' "λz. jacobi_theta_01
z t"
⟨proof⟩

interpretation jacobi_theta_01_right: periodic_fun_simple "λt. jacobi_theta_01
```

z t" 2
⟨*proof*⟩

lemma *jacobi_theta_10_plus1_left: "jacobi_theta_10 (z + 1) t = -jacobi_theta_10 z t"*
⟨*proof*⟩

lemma *jacobi_theta_11_plus1_left: "jacobi_theta_11 (z + 1) t = -jacobi_theta_11 z t"*
⟨*proof*⟩

lemma *jacobi_theta_11_minus1_left: "jacobi_theta_11 (z - 1) t = -jacobi_theta_11 z t"*
⟨*proof*⟩

lemma *jacobi_theta_10_plus2_right: "jacobi_theta_10 z (t + 2) =* i ** jacobi_theta_10 z t"*
⟨*proof*⟩

lemma *jacobi_theta_11_plus2_right: "jacobi_theta_11 z (t + 2) =* i ** jacobi_theta_11 z t"*
⟨*proof*⟩

lemma *jacobi_theta_00_plus_half_left': "jacobi_theta_00 (z + 1/2) t = jacobi_theta_01 z t"*
⟨*proof*⟩

lemma *jacobi_theta_01_plus_half_left: "jacobi_theta_01 (z + 1/2) t = jacobi_theta_00 z t"*
⟨*proof*⟩

lemma *jacobi_theta_10_plus_half_left': "jacobi_theta_10 (z + 1/2) t = jacobi_theta_11 z t"*
⟨*proof*⟩

lemma *jacobi_theta_11_plus_half_left': "jacobi_theta_11 (z + 1/2) t = -jacobi_theta_10 z t"*
⟨*proof*⟩

The quasiperiodicity identities for the ϑ_{xy}:

lemma *jacobi_theta_01_plus_quasiperiod:*
*"jacobi_theta_01 (z + t) t = -jacobi_theta_01 z t / to_nome (2 * z + t)"*
⟨*proof*⟩

lemma *jacobi_theta_10_plus_quasiperiod:*
*"jacobi_theta_10 (z + t) t = jacobi_theta_10 z t / to_nome (t + 2 **

z)"
⟨proof⟩

lemma *jacobi_theta_11_plus_quasiperiod:*
  *"jacobi_theta_11 (z + t) t = -jacobi_theta_11 z t / to_nome (t + 2 *
z)"*
⟨proof⟩

ϑ_{11} is odd, the other ϑ_{xy} are even:

lemma *jacobi_theta_00_minus [simp]: "jacobi_theta_00 (-z) t = jacobi_theta_00
z t"*
  ⟨proof⟩

lemma *jacobi_theta_01_minus [simp]: "jacobi_theta_01 (-z) t = jacobi_theta_01
z t"*
  ⟨proof⟩

lemma *jacobi_theta_11_minus [simp]: "jacobi_theta_11 (-z) t = -jacobi_theta_11
z t"*
⟨proof⟩

lemma *jacobi_theta_10_minus [simp]: "jacobi_theta_10 (-z) t = jacobi_theta_10
z t"*
⟨proof⟩

lemma *jacobi_theta_11_0_left [simp]: "jacobi_theta_11 0 t = 0"*
  ⟨proof⟩

lemma *tendsto_jacobi_theta_01 [tendsto_intros]:*
  assumes *"(f ⟶ w) F" "(g ⟶ q) F" "Im q > 0"*
  shows *"((λz. jacobi_theta_01 (f z) (g z)) ⟶ jacobi_theta_01 w
q) F"*
  ⟨proof⟩

lemma *continuous_on_jacobi_theta_01 [continuous_intros]:*
  assumes *"continuous_on A f" "continuous_on A g"*
  assumes *"⋀z. z ∈ A ⟹ Im (g z) > 0"*
  shows *"continuous_on A (λz. jacobi_theta_01 (f z) (g z))"*
  ⟨proof⟩

lemma *continuous_jacobi_theta_01 [continuous_intros]:*
  assumes *"continuous F f" "continuous F g" "Im (g (netlimit F)) > 0"*
  shows *"continuous F (λz. jacobi_theta_01 (f z) (g z))"*
  ⟨proof⟩

lemma *holomorphic_jacobi_theta_01 [holomorphic_intros]:*
  assumes *"f holomorphic_on A" "g holomorphic_on A" "⋀z. z ∈ A ⟹ Im
(g z) > 0" "open A"*

shows *"(λz. jacobi_theta_01 (f z) (g z)) holomorphic_on A"*
⟨*proof*⟩

lemma *analytic_jacobi_theta_01 [analytic_intros]:*
assumes *"f analytic_on A" "g analytic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"(λz. jacobi_theta_01 (f z) (g z)) analytic_on A"*
⟨*proof*⟩

lemma *tendsto_jacobi_theta_10 [tendsto_intros]:*
assumes *"(f ⟶ w) F" "(g ⟶ q) F" "Im q > 0"*
shows *"((λz. jacobi_theta_10 (f z) (g z)) ⟶ jacobi_theta_10 w q) F"*
⟨*proof*⟩

lemma *continuous_on_jacobi_theta_10 [continuous_intros]:*
assumes *"continuous_on A f" "continuous_on A g"*
assumes *"⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"continuous_on A (λz. jacobi_theta_10 (f z) (g z))"*
⟨*proof*⟩

lemma *continuous_jacobi_theta_10 [continuous_intros]:*
assumes *"continuous F f" "continuous F g" "Im (g (netlimit F)) > 0"*
shows *"continuous F (λz. jacobi_theta_10 (f z) (g z))"*
⟨*proof*⟩

lemma *holomorphic_jacobi_theta_10 [holomorphic_intros]:*
assumes *"f holomorphic_on A" "g holomorphic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0" "open A"*
shows *"(λz. jacobi_theta_10 (f z) (g z)) holomorphic_on A"*
⟨*proof*⟩

lemma *analytic_jacobi_theta_10 [analytic_intros]:*
assumes *"f analytic_on A" "g analytic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"(λz. jacobi_theta_10 (f z) (g z)) analytic_on A"*
⟨*proof*⟩

lemma *tendsto_jacobi_theta_11 [tendsto_intros]:*
assumes *"(f ⟶ w) F" "(g ⟶ q) F" "Im q > 0"*
shows *"((λz. jacobi_theta_11 (f z) (g z)) ⟶ jacobi_theta_11 w q) F"*
⟨*proof*⟩

lemma *continuous_on_jacobi_theta_11 [continuous_intros]:*
assumes *"continuous_on A f" "continuous_on A g"*
assumes *"⋀z. z ∈ A ⟹ Im (g z) > 0"*
shows *"continuous_on A (λz. jacobi_theta_11 (f z) (g z))"*

⟨proof⟩

lemma *continuous_jacobi_theta_11 [continuous_intros]:*
  assumes *"continuous F f" "continuous F g" "Im (g (netlimit F)) > 0"*
  shows *"continuous F (λz. jacobi_theta_11 (f z) (g z))"*
  ⟨proof⟩

lemma *holomorphic_jacobi_theta_11 [holomorphic_intros]:*
  assumes *"f holomorphic_on A" "g holomorphic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0" "open A"*
  shows *"(λz. jacobi_theta_11 (f z) (g z)) holomorphic_on A"*
  ⟨proof⟩

lemma *analytic_jacobi_theta_11 [analytic_intros]:*
  assumes *"f analytic_on A" "g analytic_on A" "⋀z. z ∈ A ⟹ Im (g z) > 0"*
  shows *"(λz. jacobi_theta_11 (f z) (g z)) analytic_on A"*
  ⟨proof⟩

The derivatives of ϑ_{00}, ϑ_{01}, and ϑ_{10} at $z = 0$ vanish since they are even.

lemma *deriv_jacobi_theta_00_minus_left:*
  fixes *t :: complex*
  defines *"f ≡ (λz. jacobi_theta_00 z t)"*
  assumes *"Im t > 0"*
  shows *"deriv f (-z) = -deriv f z"*
⟨proof⟩

lemma *deriv_jacobi_theta_01_minus_left:*
  fixes *t :: complex*
  defines *"f ≡ (λz. jacobi_theta_01 z t)"*
  assumes *"Im t > 0"*
  shows *"deriv f (-z) = -deriv f z"*
⟨proof⟩

lemma *deriv_jacobi_theta_10_minus_left:*
  fixes *t :: complex*
  defines *"f ≡ (λz. jacobi_theta_10 z t)"*
  assumes *"Im t > 0"*
  shows *"deriv f (-z) = -deriv f z"*
⟨proof⟩

lemma *summable_on_nat_imp_summable_on_int_symmetric:*
  fixes *h :: "nat ⇒ 'b::topological_ab_group_add"*
  assumes *"h summable_on UNIV"*
  shows *"(λn. h (nat |n|)) summable_on UNIV"*
⟨proof⟩

```
context
  fixes f f' :: "int set ⇒ complex × complex ⇒ complex"
  defines "f ≡ (λN (z, t). ∑n∈N. to_nome (of_int n ^ 2 * t + 2 * of_int
n * z))"
  defines "f' ≡ (λN (z, t). ∑n∈N. 2 * i * of_real pi * of_int n *
                             to_nome (of_int n ^ 2 * t + 2 * of_int
n * z))"
begin

lemma uniform_limit_jacobi_theta_00:
  fixes A B :: "complex set"
  assumes compact: "compact A" "compact B"
  assumes B_subset: "B ⊆ {t. Im t > 0}"
  shows "uniform_limit (A × B) f (λ(z, t). jacobi_theta_00 z t) finite_sets_at_top"
⟨proof⟩

lemma has_sum_deriv_jacobi_theta_00_left:
  assumes t: "Im t > 0"
  shows   "((λn. 2 * i * of_real pi * of_int n * to_nome (of_int n ^ 2
* t + 2 * of_int n * z))
             has_sum deriv (λz. jacobi_theta_00 z t) z) UNIV"
⟨proof⟩

lemma uniform_limit_deriv_jacobi_theta_00_left:
  fixes A B :: "complex set"
  assumes compact: "compact A" "compact B"
  assumes B_subset: "B ⊆ {t. Im t > 0}"
  shows "uniform_limit (A × B) f' (λ(z, t). deriv (λz. jacobi_theta_00
z t) z) finite_sets_at_top"
⟨proof⟩

lemma analytic_deriv_jacobi_theta_00_left [analytic_intros]:
  assumes "h analytic_on A" "g analytic_on A" "⋀x. x ∈ A ⟹ Im (h x)
> 0"
  shows   "(λt. deriv (λz. jacobi_theta_00 z (h t)) (g t)) analytic_on
A"
⟨proof⟩

lemma analytic_deriv_jacobi_theta_11_left:
  "(λt. deriv (λz. jacobi_theta_11 z t) z) analytic_on {t. Im t > 0}"
⟨proof⟩

end
```

3.6 The heat equation

The Jacobi ϑ function in the complex plane is a solution of the the partial differential equation

$$\frac{\partial^2}{\partial z^2}\vartheta(z,t) = 4i\pi\frac{\partial}{\partial t}\vartheta(z,t)$$

This is a one-dimensional heat equation.

```
theorem jacobi_theta_00_heat_equation:
  assumes "Im t > 0"
  shows   "(deriv ^^ 2) (λz. jacobi_theta_00 z t) z = 4*pi*i * deriv (λt.
jacobi_theta_00 z t) t"
⟨proof⟩

bundle jacobi_theta_notation
begin

notation jacobi_theta_00 ("(<notation=<mixfix jacobi_theta_00>>ϑ₀₀'(_
; _'))")
notation jacobi_theta_01 ("(<notation=<mixfix jacobi_theta_01>>ϑ₀₁'(_
; _'))")
notation jacobi_theta_10 ("(<notation=<mixfix jacobi_theta_10>>ϑ₁₀'(_
; _'))")
notation jacobi_theta_11 ("(<notation=<mixfix jacobi_theta_11>>ϑ₁₁'(_
; _'))")

end

end
```

4 The Jacobi Triple Product

```
theory Jacobi_Triple_Product
imports
  Theta_Functions
  "Lambert_Series.Lambert_Series_Library"
  "Combinatorial_Q_Analogues.Euler_Function"
begin
```

4.1 Versions for Jacobi's theta function

```
unbundle qpochhammer_inf_notation
```

The following follows the short proof given by Andrews [?], which makes use of two series expansions for $(a;b)_\infty$ and $1/(a;b)\infty$ due to Euler.

We prove it for Jacobi's theta function and derive a version for Ramanujan's later on. One could possibly also adapt the prove to work for Ramanujan's version directly, which makes the transfer to Jacobi's function a bit easier.

However, I chose to do it for Jacobi's version first in order to stay closer to the text by Andrews.

The proof is fairly straightforward; the only messy part is proving the absolute convergence of the double sum (which Andrews does not bother doing). This is necessary in order to allow the exchange of the order of summation.

theorem *jacobi_theta_nome_triple_product_complex:*
  fixes *w q :: complex*
  assumes *"w ≠ 0" "norm q < 1"*
  shows *"jacobi_theta_nome w q = (q^2 ; q^2)$_\infty$ * (-q*w ; q^2)$_\infty$ * (-q/w ; q^2)$_\infty$"*
⟨*proof*⟩

lemma *jacobi_theta_nome_triple_product_real:*
  fixes *w q :: real*
  assumes *"w ≠ 0" "|q| < 1"*
  shows *"jacobi_theta_nome w q = (q^2 ; q^2)$_\infty$ * (-q*w ; q^2)$_\infty$ * (-q/w ; q^2)$_\infty$"*
⟨*proof*⟩

4.2 Version of Ramanujan's theta function

The triple product for Ramanujan's theta function, which follows easily from the above one, has a particularly elegant form:

$$f(a,b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty$$

It follows directly from the version for Jacobi's theta function, although I again have to employ analytic continuation to avoid dealing with the branch cuts when converting Ramanujan's theta function to Jacobi's.

theorem *ramanujan_theta_triple_product_complex:*
  fixes *a b :: complex*
  assumes *"norm (a * b) < 1"*
  shows *"ramanujan_theta a b = (-a ; a*b)$_\infty$ * (-b ; a*b)$_\infty$ * (a*b ; a*b)$_\infty$"*
⟨*proof*⟩

lemma *ramanujan_theta_triple_product_real:*
  fixes *a b :: real*
  assumes *ab: "|a * b| < 1"*
  shows *"ramanujan_theta a b = (-a ; a * b)$_\infty$ * (-b ; a * b)$_\infty$ * (a * b ; a * b)$_\infty$"*
⟨*proof*⟩

4.3 A related identity for $\varphi(q)^3$

By instantiating the Jacobi triple product for $f(qz, 1/z)$ and differentiating, we obtain the following identity:

$$\varphi(q)^3 = \sum_{n=-\infty}^{\infty} (-1)^n n q^{n(n+1)/2} = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}$$

lemma *has_sum_euler_phi_cube_complex:*
  fixes *q :: complex*
  assumes *q: "norm q < 1"*
  shows *"((λn. (-1) powi n * of_int n * q powi (n * (n + 1) div 2)) has_sum euler_phi q ^ 3) UNIV"*
⟨*proof*⟩
  include *qpochhammer_inf_notation*
  ⟨*proof*⟩

lemma *has_sum_euler_phi_cube_real:*
  fixes *q :: real*
  assumes *q: "|q| < 1"*
  shows *"((λn. (-1) powi n * of_int n * q powi (n * (n + 1) div 2)) has_sum euler_phi q ^ 3) UNIV"*
⟨*proof*⟩

lemma *powser_euler_phi_cube_complex:*
  fixes *q :: complex*
  assumes *q: "norm q < 1"*
  shows *"(λn. (-1) ^ n * of_nat (2 * n + 1) * q ^ (n * (n + 1) div 2)) sums euler_phi q ^ 3"*
⟨*proof*⟩

lemma *powser_euler_phi_cube_real:*
  fixes *q :: real*
  assumes *q: "|q| < 1"*
  shows *"(λn. (-1) ^ n * real (2 * n + 1) * q ^ (n * (n + 1) div 2)) sums euler_phi q ^ 3"*
⟨*proof*⟩

4.4 (Non-)vanishing of theta functinos

A corollary of the Jacobi triple product: the Jacobi theta function has no zeros apart from the "obvious" ones, i.e. the ones at the center of the cells of the lattice generated by the period 1 and the quasiperiod t.

corollary *jacobi_theta_00_eq_0_iff_complex:*
  fixes *z t :: complex*
  assumes *"Im t > 0"*
  shows *"jacobi_theta_00 z t = 0 ⟷ (∃m n. z = (of_int m + 1/2) + (of_int n + 1/2) * t)"*

```
⟨proof⟩

lemma jacobi_theta_00_nonzero:
  assumes z: "Im t > 0" and "Im z / Im t - 1 / 2 ∉ ℤ"
  shows   "jacobi_theta_00 z t ≠ 0"
⟨proof⟩

lemma jacobi_theta_00_0_left_nonzero:
  assumes "Im t > 0"
  shows   "jacobi_theta_00 0 t ≠ 0"
  ⟨proof⟩

lemma jacobi_theta_nome_nonzero_complex:
  fixes q w :: complex
  assumes [simp]: "w ≠ 0" "norm q < 1"
  assumes "q = 0 ∨ (ln (norm w) / ln (norm q) - 1) / 2 ∉ ℤ"
  shows   "jacobi_theta_nome w q ≠ 0"
⟨proof⟩

lemma jacobi_theta_nome_q_q_nonzero_complex:
  assumes "norm (q::complex) < 1" "q ≠ 0"
  shows   "jacobi_theta_nome q q ≠ 0"
⟨proof⟩

lemma jacobi_theta_nome_nonzero_real:
  fixes q w :: real
  assumes [simp]: "w ≠ 0" "norm q < 1" and "(ln |w| / ln |q| - 1) / 2
∉ ℤ"
  shows   "jacobi_theta_nome w q ≠ 0"
⟨proof⟩

lemma jacobi_theta_nome_1_left_nonzero_complex:
  assumes "norm (q :: complex) < 1"
  shows   "jacobi_theta_nome 1 q ≠ 0"
  ⟨proof⟩

lemma jacobi_theta_nome_1_left_nonzero_real:
  assumes "|q::real| < 1"
  shows   "jacobi_theta_nome 1 q ≠ 0"
  ⟨proof⟩

unbundle no qpochhammer_inf_notation

end
```

5 The theta nullwert functions

```
theory Theta_Nullwert
  imports "Sum_Of_Squares_Count.Sum_Of_Squares_Count" Jacobi_Triple_Product
```

begin

The theta nullwert function (nullwert being German for "zero value") are the four functions $\vartheta_{xy}(z;\tau)$ with $z = 0$. However, they are very commonly denoted in terms of the nome instead, corresponding to $\vartheta_{xy}(w, q)$ with $w = 1$.

It is easy to see that $\vartheta_{11}(0;\tau) = \vartheta_{11}(1, q)$ is identically zero and therefore uninteresting. The remaining three functions $\vartheta_{10}(0, q)$, $\vartheta_{00}(0, q)$, and $\vartheta_{01}(0, q)$ are denoted $\vartheta_2(q)$, $\vartheta_3(q)$, and $\vartheta_4(q)$.

It is also not hard to see that in fact $\vartheta_4(q) = \vartheta_3(-q)$, but we still introduce separate notation for ϑ_4 since it is very commonly used in the literature.

```
lemma jacobi_theta_nome_11_1_left [simp]: "jacobi_theta_nome_11 1 q =
0"
  ⟨proof⟩

abbreviation jacobi_theta_nw_10 :: "'a :: {real_normed_field, banach,
ln} ⇒ 'a" where
  "jacobi_theta_nw_10 q ≡ jacobi_theta_nome_10 1 q"

abbreviation jacobi_theta_nw_00 :: "'a :: {real_normed_field, banach}
⇒ 'a" where
  "jacobi_theta_nw_00 q ≡ jacobi_theta_nome_00 1 q"

abbreviation jacobi_theta_nw_01 :: "'a :: {real_normed_field, banach}
⇒ 'a" where
  "jacobi_theta_nw_01 q ≡ jacobi_theta_nome_01 1 q"
```

The function $\vartheta_{11}(0, q)$ is identically zero and therefore uninteresting. However, the derivative of ϑ_{11} at $z = 0$ *is* interesting. We will call it ϑ_1'. Note that we have a normalization factor of $\frac{1}{i}$. This is a bit arbitrary, but it is the one that leads to the "right" identities for ϑ_1' on NIST [1, Section 20.4].

```
definition jacobi_theta_nw_11' :: "complex ⇒ complex" where
  "jacobi_theta_nw_11' q ≡ deriv (λz. jacobi_theta_nome_11 z q) 1 / i"


bundle jacobi_theta_nw_notation
begin
notation jacobi_theta_nw_10 ("ϑ₂")
notation jacobi_theta_nw_00 ("ϑ₃")
notation jacobi_theta_nw_01 ("ϑ₄")
notation jacobi_theta_nw_11' ("ϑ₁''")
end

unbundle jacobi_theta_nw_notation


lemma jacobi_theta_nw_10_0 [simp]: "ϑ₂ 0 = 0"
  and jacobi_theta_nw_00_0 [simp]: "ϑ₃ 0 = 1"
```

```
  and jacobi_theta_nw_01_0 [simp]: "ϑ₄ 0 = 1"
  ⟨proof⟩

lemma jacobi_theta_nw_01_conv_00: "ϑ₄ q = ϑ₃ (-q)"
  ⟨proof⟩


lemma jacobi_theta_nw_10_of_real:
        "y ≥ 0 ⟹ ϑ₂ (complex_of_real y) = of_real (ϑ₂ y)"
  and jacobi_theta_nw_00_of_real: "ϑ₃ (of_real x) = of_real (ϑ₃ x)"
  and jacobi_theta_nw_01_of_real: "ϑ₄ (of_real x) = of_real (ϑ₄ x)"
  ⟨proof⟩

lemma jacobi_theta_nw_10_cnj:
        "(Im q = 0 ⟹ Re q ≥ 0) ⟹ ϑ₂ (cnj q) = cnj (ϑ₂ q)"
  and jacobi_theta_nw_00_cnj: "ϑ₃ (cnj q) = cnj (ϑ₃ q)"
  and jacobi_theta_nw_01_cnj: "ϑ₄ (cnj q) = cnj (ϑ₄ q)"
  ⟨proof⟩
```

The nullwerte have the following definitions as infinite sums:

$$\vartheta_2(q) = \sum_{-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$$

$$\vartheta_3(q) = \sum_{-\infty}^{\infty} q^{n^2}$$

$$\vartheta_4(q) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2}$$

```
lemma has_sum_jacobi_theta_nw_10_complex:
  assumes "norm (q :: complex) < 1"
  shows   "((λn. q powr ((of_int n + 1 / 2) ^ 2)) has_sum ϑ₂ q) UNIV"
⟨proof⟩

lemma has_sum_jacobi_theta_nw_10_real:
  assumes "q ∈ {0<..<1::real}"
  shows   "((λn. q powr ((of_int n + 1 / 2) ^ 2)) has_sum ϑ₂ q) UNIV"
⟨proof⟩

lemma has_sum_jacobi_theta_nw_00:
  assumes "norm q < 1"
  shows   "((λn. q powi (n ^ 2)) has_sum ϑ₃ q) UNIV"
  ⟨proof⟩

lemma has_sum_jacobi_theta_nw_01:
  assumes "norm q < 1"
  shows   "((λn. (-1) powi n * q powi (n ^ 2)) has_sum ϑ₄ q) UNIV"
  ⟨proof⟩
```

The theta nullwert functions do not vanish (except for ϑ_2 *(0) = 0*).

lemma *jacobi_theta_00_nw_nonzero_complex: "norm (q::complex) < 1 ⟹ ϑ_3 q ≠ 0"*
⟨*proof*⟩

lemma *jacobi_theta_01_nw_nonzero_complex: "norm (q::complex) < 1 ⟹ ϑ_4 q ≠ 0"*
⟨*proof*⟩

lemma *jacobi_theta_10_nw_nonzero_complex:*
assumes *"q ≠ 0" "norm (q::complex) < 1"*
shows *"ϑ_2 q ≠ 0"*
⟨*proof*⟩

lemma *jacobi_theta_00_nw_nonzero_real: "|q::real| < 1 ⟹ ϑ_3 q ≠ 0"*
and *jacobi_theta_01_nw_nonzero_real: "|q::real| < 1 ⟹ ϑ_4 q ≠ 0"*
and *jacobi_theta_10_nw_nonzero_real: "q ∈ {0..<1} ⟹ q ≠ 0 ⟹ ϑ_2 q ≠ 0"*
⟨*proof*⟩

lemma *has_field_derivative_jacobi_theta_nw_11:*
assumes *"norm q < 1"*
shows *"((λz. jacobi_theta_nome_11 z q) has_field_derivative (i * ϑ_1' q)) (at 1 within A)"*
⟨*proof*⟩

lemma *jacobi_theta_nw_11'_0 [simp]: "ϑ_1' 0 = 0"*
⟨*proof*⟩

5.1 The Maclaurin series of ϑ_3 and ϑ_4

It is easy to see from the above infinite sums that $\vartheta_3(q)$ and $\vartheta_4(q)$ have the Maclaurin series

$$1 + 2\sum_{n=1}^{\infty}[\exists i.\ n = i^2]c^n q^n$$

for $c = 1$ and $c = -1$, respectively.

In other words, $\vartheta_3(q)$ is the generating function of the number $r_1(n)$ of ways to write an integer n as a square of an integer – 1 for $n = 0$, 2 if n is a non-zero perfect square, and 0 otherwise.

Consequently, $\vartheta_3(q)^k$ is the generating function of the number $r_k(n)$ of ways to write an integer n as a square of k integers. The function $r_k(n)$ is written as *count_sos k n* in Isabelle.

definition *fps_jacobi_theta_nw :: "'a :: field ⇒ 'a fps"* **where**
*"fps_jacobi_theta_nw c = Abs_fps (λn. if n = 0 then 1 else if is_square n then 2 * c ^ n else 0)"*

theorem *fps_jacobi_theta_power_eq:*
*"fps_jacobi_theta_nw c ^ k = Abs_fps (λn. of_nat (count_sos k n) * c ^ n)"*
⟨*proof*⟩

corollary *fps_jacobi_theta_altdef:*
*"fps_jacobi_theta_nw c = Abs_fps (λn. of_nat (count_sos 1 n) * c ^ n)"*
⟨*proof*⟩

lemma *norm_summable_fps_jacobi_theta:*
fixes *q :: "'a :: {real_normed_field, banach}"*
assumes *"norm (c * q) < 1"*
shows *"summable (λn. norm (fps_nth (fps_jacobi_theta_nw c) n * q ^ n))"*
⟨*proof*⟩

lemma *summable_fps_jacobi_theta:*
fixes *q :: "'a :: {real_normed_field, banach}"*
assumes *"norm (c * q) < 1"*
shows *"summable (λn. fps_nth (fps_jacobi_theta_nw c) n * q ^ n)"*
⟨*proof*⟩

lemma *summable_ints_symmetric:*
fixes *f :: "int ⇒ 'a :: {real_normed_vector, banach}"*
assumes *"summable (λn. norm (f (int n)))"*
assumes *"⋀n. f (-n) = f n"*
shows *"f abs_summable_on UNIV"* **and** *"summable (λn. norm ((if n = 0 then 1 else 2) $*_R$ f (int n)))"*
⟨*proof*⟩

lemma *has_sum_ints_symmetric_iff:*
fixes *f :: "int ⇒ 'a :: {banach, real_normed_vector}"*
assumes *"⋀n. f (-n) = f n"*
shows *"(f has_sum S) UNIV ⟷ ((λn. (if n = 0 then 1 else 2) $*_R$ f (int n)) has_sum S) UNIV"*
⟨*proof*⟩

lemma *sums_jacobi_theta_nw_00:*
assumes *"norm q < 1"*
shows *"(λn. fps_nth (fps_jacobi_theta_nw 1) n * q ^ n) sums ϑ_3 q"*
⟨*proof*⟩

lemma *sums_jacobi_theta_nw_01:*
assumes *"norm q < 1"*
shows *"(λn. fps_nth (fps_jacobi_theta_nw (-1)) n * q ^ n) sums ϑ_4 q"*
⟨*proof*⟩

lemma *has_fps_expansion_jacobi_theta_3 [fps_expansion_intros]:*

```
  "ϑ3 has_fps_expansion fps_jacobi_theta_nw 1"
⟨proof⟩

lemma has_fps_expansion_jacobi_theta_4 [fps_expansion_intros]:
  "ϑ4 has_fps_expansion fps_jacobi_theta_nw (-1)"
⟨proof⟩

lemma fps_conv_radius_jacobi_theta_nw [simp]:
  fixes c :: "'a :: {banach, real_normed_field}"
  shows "fps_conv_radius (fps_jacobi_theta_nw c) = 1 / ereal (norm c)"
⟨proof⟩
```

Recall that $\vartheta_2(q) = q^{1/4}\vartheta(q,q)$. Since the factor $q^{1/4}$ has a branch cut, it is somewhat unpleasant to deal with and we will concentrate only on the factor $\vartheta(q,q)$ for now. This is a holomorphic function on the unit disc except for a removable singularity at $q = 0$.

For $q \neq 0$ and $|q| < 1$, $\vartheta(q,q)$ has following the power series expansion:

$$\vartheta(q,q) = \sum_{n=-\infty}^{\infty} q^{n(n+1)} = \sum_{n=0}^{\infty} 2q^{n(n+1)}$$

Note that $n(n+1)$ is twice the triangular number $n(n+1)/2$, so we can also see this as a series expansion in terms of powers of q^2.

```
lemma has_sum_jacobi_theta_nw_10_aux:
  assumes q: "norm q < 1" "q ≠ 0"
  shows   "((λn. 2 * q ^ (n*(n+1))) has_sum jacobi_theta_nome q q) UNIV"
⟨proof⟩

lemma sums_jacobi_theta_nw_10_aux:
  assumes q: "norm q < 1" "q ≠ 0"
  shows   "(λn. if ∃i. n = i*(i+1) then 2 * q ^ n else 0) sums jacobi_theta_nome
q q"
⟨proof⟩

definition fps_jacobi_theta_nw_10 :: "'a :: field fps" where
  "fps_jacobi_theta_nw_10 = Abs_fps (λn. if ∃i. n = i*(i+1) then 2 else
0)"

lemma fps_conv_radius_jacobi_theta_2 [simp]: "fps_conv_radius fps_jacobi_theta_nw_10
= 1"
⟨proof⟩

lemma has_laurent_expansion_jacobi_theta_2 [laurent_expansion_intros]:
  "(λq. jacobi_theta_nome q q) has_laurent_expansion fps_to_fls fps_jacobi_theta_nw_10"
  ⟨proof⟩
```

For $\vartheta(q,q)^2$, we can find the following expansion into a double sum:

$$\vartheta(q,q)^2 = \sum_{n=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{i(i+1)+j(j+1)}$$

lemma *has_sum_jacobi_theta_nw_10_aux_square:*
  fixes *q :: complex*
  assumes *q: "norm q < 1" "q ≠ 0"*
  shows *"((λ(i,j). q powi (i*(i+1) + j*(j+1))) has_sum jacobi_theta_nome q q ^ 2) UNIV"*
⟨*proof*⟩

With some creative reindexing, we find the following power series expansion:

$$q\vartheta(q^2,q^2)^2 = \sum_{n=0}^{\infty} r_2(2n+1)q^{2n+1}$$

lemma *sums_q_times_jacobi_theta_nw_10_aux_square_square:*
  fixes *q :: complex*
  assumes *q: "q ≠ 0" "norm q < 1"*
  shows *"(λn. (if odd n then of_nat (count_sos 2 n) else 0) * q ^ n) sums*
*(q * jacobi_theta_nome (q^2) (q^2) ^ 2)"*
⟨*proof*⟩

lemma *has_laurent_expansion_q_times_jacobi_theta_nw_10_aux_square_square:*
  defines *"F ≡ Abs_fps (λn. if odd n then of_nat (count_sos 2 n) else 0)"*
  shows *"(λq. q * jacobi_theta_nome (q^2) (q^2) ^ 2) has_laurent_expansion fps_to_fls F"*
  ⟨*proof*⟩

5.2 The nullwert derivative of ϑ_{11}

From the definition of ϑ_{11}, we directly get the following series representation for ϑ_1':

$$\vartheta_1'(q) = q^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)}$$

lemma *has_sum_jacobi_theta_nw_11'_aux1:*
  fixes *q :: complex*
  defines *"c ≡ q powr (1/4)"*
  assumes *q: "norm q < 1"*
  shows *"((λn. c * (-1) powi n * (2 * of_int n + 1) * q powi (n * (n + 1))) has_sum ϑ_1' q) UNIV"*
⟨*proof*⟩

We can also derive the following slightly simpler series:

$$\vartheta_1'(q) = 2q^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n n q^{n(n+1)}$$

The two are equivalent since their difference is

$$q^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)}$$

which clearly vanishes due to its symmetry under $n \mapsto -n-1$.

```
lemma has_sum_jacobi_theta_nw_11'_aux1':
  fixes q :: complex
  defines "c ≡ 2 * q powr (1/4)"
  assumes q: "norm q < 1"
  shows "((λn. c * (-1) powi n * of_int n * q powi (n * (n + 1))) has_sum
ϑ₁' q) UNIV"
⟨proof⟩

lemma has_sum_jacobi_theta_nw_11'_aux2:
  fixes q :: complex
  defines "c ≡ q powr (1/4)"
  assumes q: "norm q < 1" "q ≠ 0"
  shows "((λn. (-1) powi n * (2 * of_int n + 1) * q powi (n * (n + 1)))
has_sum (ϑ₁' q / c)) UNIV"
⟨proof⟩

lemma has_sum_jacobi_theta_nw_11'_aux2':
  fixes q :: complex
  defines "c ≡ 2 * q powr (1/4)"
  assumes q: "norm q < 1" "q ≠ 0"
  shows "((λn. (-1) powi n * of_int n * q powi (n * (n + 1))) has_sum
(ϑ₁' q / c)) UNIV"
⟨proof⟩

lemma has_sum_jacobi_theta_nw_11':
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "((λn. (-1) powi n * (2 * of_int n + 1) * q powr ((of_int n +
1 / 2)²)) has_sum ϑ₁' q) UNIV"
⟨proof⟩
```

5.3 Identities

Lastly, we derive a variety of identities between the different theta functions.

Using the Jacobi triple product three times and simplifying the result a bit, we obtain the following identity:

$$\vartheta_2(q)\vartheta_3(q)\vartheta_4(q) = 2q^{1/4}\varphi(q^2)^3$$

lemma *jacobi_theta_nw_10_00_01_conv_euler_phi:*
  assumes *q: "norm (q :: complex) < 1"*
  shows *"ϑ_2 q * ϑ_3 q * ϑ_4 q = 2 * q powr (1/4) * euler_phi (q^2) ^ 3"*
⟨*proof*⟩
  include *qpochhammer_inf_notation*
  ⟨*proof*⟩

Combining this with Jacobi's identity for $\varphi(q)^3$ and comparing the result with the series expansion for ϑ_1', we obtain the following identity for ϑ_1':

$$\vartheta_1'(q) = \vartheta_2(q)\vartheta_3(q)\vartheta_4(q)$$

theorem *jacobi_theta_nw_10_00_01_conv_11':*
  assumes *q: "norm (q :: complex) < 1"*
  shows *"ϑ_1' q = ϑ_2 q * ϑ_3 q * ϑ_4 q"*
⟨*proof*⟩

An equivalent identity for $\frac{\mathrm{d}}{\mathrm{d}z}\vartheta_1 1(z;\tau)$ also follows from this. However, we need analytic continuation in order to deal with the branch cuts.

lemma *deriv_jacobi_theta_11_at_0:*
  assumes *t: "Im t > 0"*
  shows *"deriv (λz. jacobi_theta_11 z t) 0 =*
    *-of_real pi * jacobi_theta_10 0 t * jacobi_theta_00 0 t * jacobi_theta_01 0 t"*
⟨*proof*⟩
  include *jacobi_theta_nw_notation*
  ⟨*proof*⟩

Next, we focus on some identities between ϑ_2, ϑ_3, and ϑ_4.

theorem *jacobi_theta_nw_00_plus_01_complex: "ϑ_3 q + ϑ_4 q = 2 * ϑ_3 (q ^ 4 :: complex)"*
⟨*proof*⟩

lemma *jacobi_theta_nw_00_plus_01_real: "ϑ_3 q + ϑ_4 q = 2 * ϑ_3 (q ^ 4 :: real)"*
  ⟨*proof*⟩

theorem *jacobi_theta_nw_00_plus_01_square_complex:*
  *"ϑ_3 q ^ 2 + ϑ_4 q ^ 2 = 2 * ϑ_3 (q ^ 2 :: complex) ^ 2"*
⟨*proof*⟩

corollary *midpoint_jacobi_theta_nw_00_01_square_complex:*
  "midpoint (ϑ_3 q ^ 2) (ϑ_4 q ^ 2) = ϑ_3 (q ^ 2 :: complex) ^ 2"
  ⟨*proof*⟩

lemma *jacobi_theta_nw_00_plus_01_square_real: "ϑ_3 q ^ 2 + ϑ_4 q ^ 2 = 2 * ϑ_3 (q ^ 2 :: real) ^ 2"*

⟨proof⟩

theorem *jacobi_theta_nw_00_times_01_complex:* "ϑ_3 q * ϑ_4 q = (ϑ_4 (q ^ 2) ^ 2 :: complex)"
⟨proof⟩

lemma *jacobi_theta_nw_00_times_01_real:* "ϑ_3 q * ϑ_4 q = (ϑ_4 (q ^ 2) ^ 2 :: real)"
⟨proof⟩

lemma *jacobi_theta_nw_00_plus_10_square_square_aux:*
fixes q :: complex
shows "ϑ_3 q ^ 2 - ϑ_3 (q^2) ^ 2 = q * jacobi_theta_nome (q^2) (q^2) ^ 2"
⟨proof⟩

theorem *jacobi_theta_nw_00_plus_10_square_square_complex:*
fixes q :: complex
assumes "Re q ≥ 0 ∧ (Re q = 0 ⟶ Im q ≥ 0)"
shows "ϑ_3 (q^2) ^ 2 + ϑ_2 (q^2) ^ 2 = ϑ_3 q ^ 2"
⟨proof⟩

lemma *jacobi_theta_nw_00_plus_10_square_square_real:*
assumes "q ≥ (0::real)"
shows "ϑ_3 (q^2) ^ 2 + ϑ_2 (q^2) ^ 2 = ϑ_3 q ^ 2"
⟨proof⟩

theorem *jacobi_theta_nw_00_minus_10_square_square_complex:*
assumes "0 ≤ Re q ∧ (Re q = 0 ⟶ 0 ≤ Im q)"
shows "ϑ_3 (q^2) ^ 2 - ϑ_2 (q^2) ^ 2 = ϑ_4 (q :: complex) ^ 2"
⟨proof⟩

lemma *jacobi_theta_nw_00_minus_10_square_square_real:*
assumes "q ≥ (0::real)"
shows "ϑ_3 (q^2) ^ 2 - ϑ_2 (q^2) ^ 2 = ϑ_4 q ^ 2"
⟨proof⟩

The following shows that the theta nullwerte provide a parameterisation of the Fermat curve $X^4 + Y^4 = Z^4$:

theorem *jacobi_theta_nw_pow4_complex:* "ϑ_2 q ^ 4 + ϑ_4 q ^ 4 = (ϑ_3 q ^ 4 :: complex)"
⟨proof⟩

unbundle *jacobi_theta_notation*

lemma *jacobi_theta_xy_0_pow4_complex:*
assumes "Im t > 0"
shows "ϑ_{10}(0; t) ^ 4 + ϑ_{01}(0; t) ^ 4 = ϑ_{00}(0; t) ^ 4"
⟨proof⟩

lemma *jacobi_theta_nw_pow4_real: "q ≥ 0 ⟹ ϑ_2 q ^ 4 + ϑ_4 q ^ 4 = (ϑ_3 q ^ 4 :: real)"*
⟨*proof*⟩

5.4 Properties of the nullwert functions on the real line

lemma *has_field_derivative_jacobi_theta_nw_00:*
fixes *q :: "'a :: {real_normed_field,banach}"*
assumes *q: "norm q < 1"*
defines *"a ≡ (λn. 2 * (of_nat n + 1)^2 * q ^ (n * (n + 2)))"*
shows *"summable a" "(ϑ_3 has_field_derivative (∑n. a n)) (at q)"*
⟨*proof*⟩

lemma *jacobi_theta_nw_10_le_00:*
assumes *"q ≥ (0::real)"*
shows *"ϑ_2 q ≤ ϑ_3 q"*
⟨*proof*⟩

lemma *jacobi_theta_nw_00_pos:*
fixes *q :: real*
assumes *"q ∈ {-1<..<1}"*
shows *"ϑ_3 q > 0"*
⟨*proof*⟩

lemma *jacobi_theta_nw_01_pos: "q ∈ {-1<..<1} ⟹ ϑ_4 q > (0::real)"*
⟨*proof*⟩

lemma *jacobi_theta_nw_00_nonneg: "ϑ_3 q ≥ (0::real)"*
⟨*proof*⟩

lemma *jacobi_theta_nw_01_nonneg: "ϑ_4 q ≥ (0::real)"*
⟨*proof*⟩

lemma *strict_mono_jacobi_theta_nw_00: "strict_mono_on {-1<..<1::real} ϑ_3"*
⟨*proof*⟩
include *qpochhammer_inf_notation*
⟨*proof*⟩

lemma *strict_antimono_jacobi_theta_nw_01: "strict_antimono_on {-1<..<1::real} ϑ_4"*
⟨*proof*⟩

lemma *jacobi_theta_nw_10_nonneg:*
assumes *"x ≥ 0"*
shows *"ϑ_2 x ≥ (0::real)"*
⟨*proof*⟩

lemma *strict_mono_jacobi_theta_nw_10: "strict_mono_on {0::real..<1} ϑ_2"*
⟨*proof*⟩

lemma *jacobi_theta_nw_10_pos:*
assumes *"x ∈ {0<..<1}"*
shows *"ϑ_2 x > (0::real)"*
⟨*proof*⟩

unbundle *no jacobi_theta_notation*
unbundle *no jacobi_theta_nw_notation*

end

References

[1] *NIST Digital Library of Mathematical Functions.* https://dlmf.nist.gov/, Release 1.2.5 of 2025-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[2] B. C. Berndt. *Ramanujans Notebooks.* Springer New York, 1991.