

The Teichmüller–Tukey Lemma

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Abstract

This entry formalizes the Teichmüller–Tukey lemma in Isabelle/HOL: every nonempty family of sets of finite character has a member that is maximal under inclusion. Instead of deriving the result from Zorn’s lemma, which is already available in Isabelle/HOL, the development follows the direct choice-function construction of Sun and Yu, originally formulated in Morse–Kelley set theory and checked in Coq. The Isabelle proof makes explicit the choice argument that turns a fixed point of the construction into a maximal member.

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1 The Teichmüller–Tukey Lemma

```
theory Teichmuller-Tukey-Lemma
imports Main
begin
```

The Teichmüller–Tukey lemma states that every nonempty family of sets of finite character has a member that is maximal under inclusion. Although the result follows from Zorn’s lemma, the development below formalizes the direct choice-function construction presented by Sun and Yu [2]. Their

proof is formulated in Morse–Kelley set theory and checked in Coq; here it is adapted to polymorphic HOL sets.

This result was developed as infrastructure for a separate formalization of first-order logic following Shoenfield [1].

1.1 Families of finite character

definition *Maximal* :: 'a set $\Rightarrow$ 'a set set $\Rightarrow$ bool **where**
Maximal A J $\equiv A \in J \wedge (\forall B \in J. \neg A \subset B)$

definition *Nest* :: 'a set set $\Rightarrow$ bool **where**
Nest A $\equiv \forall x \in A. \forall y \in A. x \subseteq y \vee y \subseteq x$

definition *Finite-character* :: 'a set set $\Rightarrow$ bool **where**
Finite-character J $\equiv \forall \Delta. \Delta \in J \longleftrightarrow (\forall \delta \subseteq \Delta. \text{finite } \delta \longrightarrow \delta \in J)$

A family of finite character is downward closed. It is also closed under the union of a nest. Nonemptiness is needed for the empty nest: it ensures that the empty set belongs to the family.

proposition *property-fin-char*:

assumes *Finite-character* J **and** $J \neq \{\}$

shows $\forall A \in J. \forall B \subseteq A. B \in J$

and $\forall G. \text{Nest } G \wedge G \subseteq J \longrightarrow \bigcup G \in J$

proof –

show $\forall A \in J. \forall B \subseteq A. B \in J$

using *Finite-character-def* *assms*(1) **by** (*meson subset-trans*)

next

show $\forall G. \text{Nest } G \wedge G \subseteq J \longrightarrow \bigcup G \in J$

proof (*rule allI*, *rule impI*)

fix G

assume B: $\text{Nest } G \wedge G \subseteq J$

have C: $\bigwedge g. \text{set } g \subseteq \bigcup G \Longrightarrow g \neq [] \Longrightarrow \exists X \in G. \text{set } g \subseteq X$

proof –

fix g

show $\text{set } g \subseteq \bigcup G \Longrightarrow g \neq [] \Longrightarrow \exists X \in G. \text{set } g \subseteq X$

proof (*induction g rule: measure-induct[of length]*)

case (1 g)

then show ?case

proof *cases*

assume $g = [] \vee (\exists a. g = [a])$

then show ?case **using** 1.premis(1,2) **by** *fastforce*

next

assume $\neg (g = [] \vee (\exists a. g = [a]))$

then obtain a **and** xs **where** axs: $g = a \# xs \wedge xs \neq []$

by (*metis list.exhaust*)

from this 1(1–2) **have**

$(\exists X \in G. \{a\} \subseteq X) \wedge (\exists X \in G. \text{set } xs \subseteq X)$

by *auto*

```

      thus ?case
      using B Nest-def axs
      by (metis dual-order.trans insert-subset list.simps(15))
    qed
  qed
  qed
  thus  $\bigcup G \in J$ 
  using assms(1-2) B C Finite-character-def
  by (smt (z3) ex-in-conv finite-list set-empty subset-eq list.set(1,1,1,1)
      subsetI subset-antisym)
  qed
qed

```

1.2 The choice extension

```

fun enlarge :: 'a set  $\Rightarrow$  'a set set  $\Rightarrow$  'a set where
  enlarge F J = {x. x  $\in$   $\bigcup J \wedge F \cup \{x\} \in J$ }

```

```

proposition enlarge-subset: F  $\in J \implies F \subseteq$  enlarge F J
  using UnionI insert-absorb by force

```

```

fun  $\mathcal{X}$ -choice :: 'a set  $\Rightarrow$  'a set set  $\Rightarrow$  ('a set  $\Rightarrow$  'a)  $\Rightarrow$  'a set where
   $\mathcal{X}$ -choice F J c =
    (if enlarge F J - F = {} then F else F  $\cup \{c$  (enlarge F J - F)})

```

For a member F of J , $\text{enlarge } F \ J$ contains the elements that can be adjoined to F while staying in J . The operation $\mathcal{X}\text{-choice } F \ J \ c$ adjoins an element chosen outside F , unless no such element exists.

The fixed-point argument below makes explicit a detail that is only implicit in Proof 4.7 of the source article. From $\mathcal{X}\text{-choice } F \ J \ c = F$ one may conclude $\text{enlarge } F \ J = F$ only by using the choice property of c : if $\text{enlarge } F \ J - F$ were nonempty, its chosen element would lie both outside and inside F . This is the step that turns a fixed point of the choice extension into a maximal member.

lemma $\mathcal{X}$ -implies-TTL:

```

assumes Finite-character J
and  $\exists F \in J. \exists c. ((\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y) \wedge \mathcal{X}\text{-choice } F \ J \ c = F)$ 
shows  $\exists \Delta \in J. \text{Maximal } \Delta \ J$ 
proof -
from assms(2) obtain  $\Delta$  and  $c$  where
   $\Delta c$ :  $\Delta \in J \wedge (\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y) \wedge \mathcal{X}\text{-choice } \Delta \ J \ c = \Delta$ 
by blast
hence  $\text{ig: enlarge } \Delta \ J = \Delta$ 
by (metis DiffD2 Un-upper2  $\mathcal{X}$ -choice.elims diff-shunt enlarge-subset
    insert-subset order-antisym-conv)
have Maximal  $\Delta \ J$ 
proof (rule ccontr)
  assume  $\neg \text{Maximal } \Delta \ J$ 

```

```

from this  $\Delta c$  obtain  $B$  where  $B: B \in J \wedge \Delta \subset B$ 
  using Maximal-def by meson
then obtain  $d$  where  $d: d \in B \wedge d \notin \Delta$  by blast
hence  $d \in \text{enlarge } \Delta J$ 
  using assms(1) Finite-character-def  $B$  property-fin-char(1) by fastforce
thus False using ig  $d$  by blast
qed
thus  $\exists \Delta \in J. \text{Maximal } \Delta J$  using  $\Delta c$  by blast
qed

```

1.3 The least closed subclass

Following the notation of the source proof, a *t-subclass* contains the empty set and is closed under both choice extensions and unions of nests. Intersecting all t-subclasses gives the least such subclass.

definition *tSubclass* :: '*a set set* $\Rightarrow$ '*a set set* $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ *bool* **where**
 $tSubclass\ G\ J\ c \equiv$
 $G \subseteq J \wedge \{\} \in G \wedge (\forall F \in G. \mathcal{X}\text{-choice}\ F\ J\ c \in G) \wedge$
 $(\forall L \subseteq G. \text{Nest}\ L \longrightarrow \bigcup L \in G)$

definition *inter-subclass* :: '*a set set* $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ '*a set set* **where**
 $inter\text{-subclass}\ J\ c \equiv \bigcap \{G. tSubclass\ G\ J\ c\}$

definition μ -*function* :: '*a set* $\Rightarrow$ '*a set set* $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ '*a set set* **where**
 $\mu\text{-function}\ C\ J\ c \equiv$
 $\{A \in inter\text{-subclass}\ J\ c. A \subseteq C \vee C \subseteq A\}$

definition *total-subclass* :: '*a set set* $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ '*a set set* **where**
 $total\text{-subclass}\ J\ c \equiv$
 $\{C \in inter\text{-subclass}\ J\ c. \mu\text{-function}\ C\ J\ c = inter\text{-subclass}\ J\ c\}$

definition ν -*function* :: '*a set* $\Rightarrow$ '*a set set* $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ '*a set set* **where**
 $\nu\text{-function}\ D\ J\ c \equiv$
 $\{A \in inter\text{-subclass}\ J\ c. A \subseteq D \vee \mathcal{X}\text{-choice}\ D\ J\ c \subseteq A\}$

proposition *property- $\mathcal{X}$* : $F \in J \implies F \subseteq \mathcal{X}\text{-choice}\ F\ J\ c$
by *auto*

proposition *property-inter-subclass*:
assumes *Finite-character* J **and** $J \neq \{\}$
and $\forall Y. Y \neq \{\} \longrightarrow c\ Y \in Y$
shows $tSubclass\ (inter\text{-subclass}\ J\ c)\ J\ c$
and $\forall G. tSubclass\ G\ J\ c \longrightarrow inter\text{-subclass}\ J\ c \subseteq G$

proof –

```

have  $JSubJ: tSubclass\ J\ J\ c$ 
  using tSubclass-def Nest-def assms(1–3) property-fin-char(2)
  by (smt (verit, ccfv-threshold) Diff-iff Sup-empty  $\mathcal{X}\text{-choice.simps}$ 
    dual-order.refl empty-iff empty-subsetI enlarge.simps mem-Collect-eq)
show  $tSubclass\ (inter\text{-subclass}\ J\ c)\ J\ c$ 

```

```

proof (auto simp only: tSubclass-def[of inter-subclass J c J c])
  show  $\{\} \in \text{inter-subclass } J \ c$ 
    using inter-subclass-def tSubclass-def
    by (metis (no-types, lifting) Inter-iff mem-Collect-eq)
next
  have  $\forall G \ F. \text{tSubclass } G \ J \ c \longrightarrow F \in G \longrightarrow \mathcal{X}\text{-choice } F \ J \ c \in G$ 
    using tSubclass-def by blast
  thus  $\bigwedge F. F \in \text{inter-subclass } J \ c \implies \mathcal{X}\text{-choice } F \ J \ c \in \text{inter-subclass } J \ c$ 
    using inter-subclass-def by blast
next
  have  $\forall G \ L. \text{tSubclass } G \ J \ c \longrightarrow L \subseteq G \longrightarrow \text{Nest } L \longrightarrow \bigcup L \in G$ 
    using tSubclass-def by metis
  thus  $\bigwedge L. L \subseteq \text{inter-subclass } J \ c \implies \text{Nest } L \implies \bigcup L \in \text{inter-subclass } J \ c$ 
    using inter-subclass-def[of J c] by blast
qed (auto simp add: JSubJ inter-subclass-def)
qed (auto simp add: inter-subclass-def)

```

1.4 The least t-subclass is a Nest

The next three lemmas correspond to Steps I–III of Section 4.1 in the source. The auxiliary subclasses μ -function $C \ J \ c$ and ν -function $D \ J \ c$ isolate the members comparable with a given set and allow comparability to be propagated through the choice extension.

lemma *step1*:

```

assumes Finite-character J and  $J \neq \{\}$ 
  and  $\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y$ 
  and  $D \in \text{total-subclass } J \ c$ 
shows tSubclass ( $\nu$ -function  $D \ J \ c$ ) J c
proof –
  show ?thesis
proof (auto simp only: tSubclass-def[of  $\nu$ -function  $D \ J \ c \ J \ c$ ])
  have  $\nu\text{-function } D \ J \ c \subseteq \text{inter-subclass } J \ c$ 
    using  $\nu$ -function-def by blast
  thus  $\bigwedge x. x \in \nu\text{-function } D \ J \ c \implies x \in J$ 
    using tSubclass-def property-inter-subclass(1) assms by (metis in-mono)
next
  fix F
  assume  $F \in \nu\text{-function } D \ J \ c$ 
  then have  $F: F \in \text{inter-subclass } J \ c \wedge$ 
     $(F \subseteq D \vee \mathcal{X}\text{-choice } D \ J \ c \subseteq F)$ 
    using  $\nu$ -function-def by blast
  show  $\mathcal{X}\text{-choice } F \ J \ c \in \nu\text{-function } D \ J \ c$ 
proof cases
  assume  $c1: \mathcal{X}\text{-choice } D \ J \ c \subseteq F$ 
  then show ?thesis
    using F  $\nu$ -function-def assms(1–3) property- $\mathcal{X}$ 
      property-inter-subclass(1) tSubclass-def
    by (smt (verit) in-mono mem-Collect-eq subset-trans)
next

```

```

assume  $\neg \mathcal{X}\text{-choice } D \ J \ c \subseteq F$ 
hence  $b: \mathcal{X}\text{-choice } F \ J \ c \subseteq D \vee D \subseteq \mathcal{X}\text{-choice } F \ J \ c$ 
  using assms  $F \ \mu\text{-function-def } t\text{Subclass-def } \text{property-inter-subclass}(1)$ 
    total-subclass-def
  by (smt (verit, del-insts) mem-Collect-eq)
show ?thesis
proof (rule ccontr)
  assume hyp:  $\mathcal{X}\text{-choice } F \ J \ c \notin \nu\text{-function } D \ J \ c$ 
  then have  $\neg \mathcal{X}\text{-choice } F \ J \ c \subseteq D$ 
    using assms( $1-3$ )  $\nu\text{-function-def } F \ t\text{Subclass-def}$ 
      property-inter-subclass( $1$ )
    by (metis (no-types, lifting) mem-Collect-eq)
  from this b have  $D \subset \mathcal{X}\text{-choice } F \ J \ c$  by blast
  hence  $D \subset F \vee D \subset F \cup \{c \ (\text{enlarge } F \ J - F)\}$ 
    by (metis  $\mathcal{X}\text{-choice.simps}$ )
  hence  $\mathcal{X}\text{-choice } D \ J \ c \subseteq \mathcal{X}\text{-choice } F \ J \ c$ 
    using  $F \ \langle \neg \mathcal{X}\text{-choice } D \ J \ c \subseteq F \rangle$ 
    by (metis Un-insert-right insert-subsetI leD order-class.order-eq-iff
      psubset-insert-iff sup-bot-right)
  thus False
    using hyp  $\nu\text{-function-def } F \ \text{assms}(1-3) \ \text{property-inter-subclass}(1)$ 
      tSubclass-def
    by (metis (no-types, lifting) mem-Collect-eq)
qed
qed
next
fix L
assume hyp1:  $L \subseteq \nu\text{-function } D \ J \ c$  and hyp2: Nest L
then have  $UL: \bigcup L \in \text{inter-subclass } J \ c$ 
  using  $\nu\text{-function-def}[\text{of } D \ J \ c] \ \text{property-inter-subclass}(1) \ \text{assms}(1-3)$ 
    tSubclass-def
  by (smt (verit) mem-Collect-eq subset-eq)
have  $\bigcup L \subseteq D \vee \mathcal{X}\text{-choice } D \ J \ c \subseteq \bigcup L$ 
proof cases
  assume  $\forall l \in L. l \subseteq D$ 
  then show ?thesis using UL  $\nu\text{-function-def}$  by blast
next
  assume  $\neg (\forall l \in L. l \subseteq D)$ 
  from this hyp1 show ?thesis using  $\nu\text{-function-def}[\text{of } D \ J \ c]$  by blast
qed
thus  $\bigcup L \in \nu\text{-function } D \ J \ c$ 
  using  $\nu\text{-function-def}[\text{of } D \ J \ c] \ UL$  by blast
qed (simp add:  $\nu\text{-function-def}$  inter-subclass-def tSubclass-def)
qed

lemma step2:
assumes Finite-character J and  $J \neq \{\}$ 
and  $\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y$ 
and  $D \in \text{total-subclass } J \ c$ 

```

shows $\mathcal{X}\text{-choice } D \ J \ c \in \text{total-subclass } J \ c$
proof –
 from *assms* have $\text{inter-subclass } J \ c \subseteq \nu\text{-function } D \ J \ c$
 using *step1* *property-inter-subclass*(2) **by** *metis*
 moreover have $\nu\text{-function } D \ J \ c \subseteq \mu\text{-function } (\mathcal{X}\text{-choice } D \ J \ c) \ J \ c$
 using $\mu\text{-function-def}$ $\nu\text{-function-def}$
 by (*smt* (*verit*, *best*) *Un-upper1* $\mathcal{X}\text{-choice.simps}$ *mem-Collect-eq* *subsetD* *subsetI*)
 moreover have $\dots \subseteq \text{inter-subclass } J \ c$ using $\mu\text{-function-def}$ **by** *blast*
 ultimately have
 $\mu\text{-function } (\mathcal{X}\text{-choice } D \ J \ c) \ J \ c = \text{inter-subclass } J \ c$
 by *order*
 thus ?thesis
 using *assms* *total-subclass-def*[of $J \ c$] *property-inter-subclass*[of $J \ c$]
tSubclass-def[of $\text{inter-subclass } J \ c \ J \ c$] **by** *simp*
qed

lemma *step3*:
 assumes *Finite-character* J and $J \neq \{\}$
 and $\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y$
 shows *Nest* ($\text{inter-subclass } J \ c$)
proof –
 have *Nest* ($\text{total-subclass } J \ c$)
 using *Nest-def*[of $\text{total-subclass } J \ c$] *total-subclass-def*[of $J \ c$]
 $\mu\text{-function-def}$ [of $J \ c$] **by** *blast*
 moreover have $a: \text{total-subclass } J \ c \subseteq \text{inter-subclass } J \ c$
 using *total-subclass-def* **by** *auto*
 moreover have *tSubclass* ($\text{total-subclass } J \ c$) $J \ c$
proof (*auto simp only: tSubclass-def*[of $\text{total-subclass } J \ c \ J \ c$])
 show $\bigwedge x. x \in \text{total-subclass } J \ c \implies x \in J$
 using *a* *assms* *property-inter-subclass*[of $J \ c$] *tSubclass-def* **by** (*meson subsetD*)
 show $\{\} \in \text{total-subclass } J \ c$
 using *assms* *property-inter-subclass*[of $J \ c$] $\mu\text{-function-def}$
tSubclass-def[of $\text{inter-subclass } J \ c$] *total-subclass-def*[of $J \ c$]
 by *fastforce*
 show $\bigwedge F. F \in \text{total-subclass } J \ c \implies$
 $\mathcal{X}\text{-choice } F \ J \ c \in \text{total-subclass } J \ c$
 using *assms* *step2* **by** *blast*
 show $\bigwedge L. L \subseteq \text{total-subclass } J \ c \implies \text{Nest } L \implies$
 $\bigcup L \in \text{total-subclass } J \ c$
proof –
 fix L
 assume *hyp1*: $L \subseteq \text{total-subclass } J \ c$ and *Nest* L
 then have $\bigcup L \in \text{inter-subclass } J \ c$
 using *property-inter-subclass*(1) *assms* *tSubclass-def* *a*
 by (*metis subset-trans*)
 moreover from *hyp1* have
 $\forall A \in \text{inter-subclass } J \ c. A \subseteq \bigcup L \vee \bigcup L \subseteq A$
 using *total-subclass-def*[of $J \ c$] $\mu\text{-function-def}$ [of $J \ c$] **by** *blast*

```

ultimately show  $\bigcup L \in \text{total-subclass } J \ c$ 
  using total-subclass-def[of  $J \ c$ ]  $\mu$ -function-def[of  $\bigcup L \ J \ c$ ]
  by auto
qed
qed
hence inter-subclass  $J \ c \subseteq \text{total-subclass } J \ c$ 
  using inter-subclass-def by blast
ultimately show ?thesis by simp
qed

```

1.5 The fixed point and the main theorem

Since the least t-subclass is a nest, its union belongs to it. Closure under the choice extension then places the extension in the same nest, while the definition of union bounds it from above. Hence the union is a fixed point.

```

lemma step4:
  assumes Finite-character  $J$  and  $J \neq \{\}$ 
    and  $\forall Y. Y \neq \{\} \longrightarrow c \ Y \in Y$ 
  shows  $\mathcal{X}\text{-choice } (\bigcup (\text{inter-subclass } J \ c)) \ J \ c = \bigcup (\text{inter-subclass } J \ c)$ 
  using assms property- $\mathcal{X}$  property-fin-char(2) property-inter-subclass step3
    tSubclass-def
  by (metis Union-upper subset-antisym)

```

```

theorem teichmueller-tuckey-lemma:
  fixes  $J :: 'a \ \text{set} \ \text{set}$ 
  assumes Finite-character  $J$  and  $J \neq \{\}$ 
  shows  $\exists \Delta \in J. \text{Maximal } \Delta \ J$ 

```

```

proof –
  let  $?c = \lambda Y :: 'a \ \text{set}. \text{SOME } y. y \in Y$ 
  have choice:  $\forall Y :: 'a \ \text{set}. Y \neq \{\} \longrightarrow ?c \ Y \in Y$ 
    by (simp add: some-in-eq)
  have fixed:
     $\mathcal{X}\text{-choice } (\bigcup (\text{inter-subclass } J \ ?c)) \ J \ ?c =$ 
       $\bigcup (\text{inter-subclass } J \ ?c)$ 
    using assms choice step4 by metis
  have member:  $\bigcup (\text{inter-subclass } J \ ?c) \in J$ 
    using assms choice property-fin-char(2) property-inter-subclass(1)
      step3 tSubclass-def by metis
  show ?thesis
    using assms(1) choice fixed member  $\mathcal{X}\text{-implies-TTL}$ [of  $J$ ]
    by metis
qed

```

end

References

- [1] J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

- [2] T. Sun and W. Yu. Formalization of the axiom of choice and its equivalent theorems. arXiv preprint arXiv:1906.03930v1, 2019. <https://arxiv.org/abs/1906.03930v1>.