Suppes' Theorem For Probability Logic

Matthew Doty

March 24, 2023

Abstract

We develop finitely additive probability logic and prove a theorem of Patrick Suppes that asserts that $\Psi \vdash \phi$ in classical propositional logic if and only if $(\sum \psi \leftarrow \Psi.\ 1 - \mathcal{P}\psi) \geq 1 - \mathcal{P}\phi$ holds for all probabilities $\mathcal{P}$. We also provide a novel *dual* form of Suppes' Theorem, which holds that $(\sum \phi \leftarrow \Phi.\ \mathcal{P}\phi) \leq \mathcal{P}\psi$ for all probabilities $\mathcal{P}$ if and only $(\bigvee \Phi) \vdash \psi$ and all of the formulae in Φ are logically exclusive from one another. Our proofs use *Maximally Consistent Sets*, and as a consequence, we obtain two *collapse* theorems. In particular, we show $(\sum \phi \leftarrow \Phi.\ \mathcal{P}\phi) \geq \mathcal{P}\psi$ holds for all probabilities $\mathcal{P}$ if and only if $(\sum \phi \leftarrow \Phi.\ \delta\ \phi) \geq \delta\ \psi$ holds for all binary-valued probabilities δ, along with the dual assertion that $(\sum \phi \leftarrow \Phi.\ \mathcal{P}\phi) \leq \mathcal{P}\psi$ holds for all probabilities $\mathcal{P}$ if and only if $(\sum \phi \leftarrow \Phi.\ \delta\ \phi) \leq \delta\ \psi$ holds for all binary-valued probabilities δ.

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no-notation *FuncSet.funcset* (**infixr** $\rightarrow$ *60*)

1.1 Definition of Probability Logic

Probability logic is defined in terms of an operator over classical logic obeying certain postulates. Scholars often credit George Boole for first conceiving this kind of formulation [1]. Theodore Hailperin in particular has written extensively on this subject [6, 7, 8].

The presentation below roughly follows Kolmogorov's axiomatization [10]. A key difference is that we only require *finite additivity*, rather than *countable additivity*. Finite additivity is also defined in terms of implication ($\rightarrow$).

class *probability-logic* = *classical-logic* +
fixes $\mathcal{P}$ :: $'a \Rightarrow real$
assumes *probability-non-negative*: $\mathcal{P}\ \varphi \geq 0$
assumes *probability-unity*: $\vdash \varphi \Longrightarrow \mathcal{P}\ \varphi = 1$
assumes *probability-implicational-additivity*:
$\vdash \varphi \rightarrow \psi \rightarrow \bot \Longrightarrow \mathcal{P}\ ((\varphi \rightarrow \bot) \rightarrow \psi) = \mathcal{P}\ \varphi + \mathcal{P}\ \psi$

A similar axiomatization may be credited to Rescher [11, pg. 185]. However, our formulation has fewer axioms. While Rescher assumes $\vdash \varphi \leftrightarrow \psi \Longrightarrow \mathcal{P}\ \varphi = \mathcal{P}\ \psi$, we show this is a lemma in §1.4.

1.2 Why Finite Additivity?

In this section we touch on why we have chosen to employ finite additivity in our axiomatization of *probability-logic* and deviate from conventional probability theory.

Conventional probability obeys an axiom known as *countable additivity*. Traditionally it states if S is a countable set of sets which are pairwise disjoint, then the limit $\sum s \in S.\ \mathcal{P}\ s$ exists and $\mathcal{P}\ (\bigcup S) = (\sum s \in S.\ \mathcal{P}\ s)$. This is more powerful than our finite additivity axiom $\vdash \varphi \to \psi \to \bot \Longrightarrow \mathcal{P}\ ((\varphi \to \bot) \to \psi) = \mathcal{P}\ \varphi + \mathcal{P}\ \psi$.

However, we argue that demanding countable additivity is not practical.

Historically, the statisticians Bruno de Finetti and Leonard Savage gave the most well known critiques. In [2] de Finetti shows various properties which are true for countably additive probability measures may not hold for finitely additive measures. Savage [12], on the other hand, develops probability based on choices prizes in lotteries.

We instead argue that if we demand countable additivity, then certain properties of real world software would no longer be formally verifiable as we demonstrate here. In particular, it prohibits conventional recursive data structures for defining propositions. Our argument is derivative of one given by Giangiacomo Gerla [5, Section 3].

By taking equivalence classes modulo $\lambda\varphi\ \psi.\ \vdash \varphi \leftrightarrow \psi$, any classical logic instance gives rise to a Boolean algebra known as a *Lindenbaum Algebra*. In the case of *′a classical-propositional-formula* this Boolean algebra algebra is both countable and *atomless*. A theorem of Horn and Tarski [9, Theorem 3.2] asserts there can be no countably additive *Pr* for a countable atomless Boolean algebra.

The above argument is not intended as a blanket refutation of conventional probability theory. It is simply an impossibility result with respect to software implementations of probability logic. Plenty of classic results in probability rely on countable additivity. A nice example, formalized in Isabelle/HOL, is Bouffon's needle [3].

1.3 Basic Properties of Probability Logic

lemma (**in** *probability-logic*) *probability-additivity*:
  assumes $\vdash \sim (\varphi \sqcap \psi)$
  shows $\mathcal{P}\ (\varphi \sqcup \psi) = \mathcal{P}\ \varphi + \mathcal{P}\ \psi$
  ⟨*proof*⟩

lemma (**in** *probability-logic*) *probability-alternate-additivity*:
  assumes $\vdash \varphi \to \psi \to \bot$
  shows $\mathcal{P}\ (\varphi \sqcup \psi) = \mathcal{P}\ \varphi + \mathcal{P}\ \psi$
  ⟨*proof*⟩

lemma (**in** *probability-logic*) *complementation*:
  $\mathcal{P}\ (\sim \varphi) = \mathit{1} - \mathcal{P}\ \varphi$
  ⟨*proof*⟩

lemma (**in** *probability-logic*) *unity-upper-bound*:
  $\mathcal{P}\ \varphi \leq \mathit{1}$
  ⟨*proof*⟩

1.4 Alternate Definition of Probability Logic

There is an alternate axiomatization of probability logic, due to Brian Gaines [4, pg. 159, postulates P7, P8, and P8] and independently formulated by Brian Weatherson [14]. As Weatherson notes, this axiomatization is suited to formulating *intuitionistic* probability logic. In the case where the underlying logic is classical the Gaines/Weatherson axiomatization is equivalent to the traditional Kolmogorov axiomatization from §1.1.

class *gaines-weatherson-probability* = *classical-logic* +
  fixes $\mathcal{P}$ :: $'a \Rightarrow$ *real*
  assumes *gaines-weatherson-thesis*:
    $\mathcal{P}\ \top = \mathit{1}$
  assumes *gaines-weatherson-antithesis*:
    $\mathcal{P}\ \bot = \mathit{0}$
  assumes *gaines-weatherson-monotonicity*:
    $\vdash \varphi \to \psi \Longrightarrow \mathcal{P}\ \varphi \leq \mathcal{P}\ \psi$
  assumes *gaines-weatherson-sum-rule*:
    $\mathcal{P}\ \varphi + \mathcal{P}\ \psi = \mathcal{P}\ (\varphi \sqcap \psi) + \mathcal{P}\ (\varphi \sqcup \psi)$

sublocale *gaines-weatherson-probability* $\subseteq$ *probability-logic*
⟨*proof*⟩

lemma (**in** *probability-logic*) *monotonicity*:
  $\vdash \varphi \to \psi \Longrightarrow \mathcal{P}\ \varphi \leq \mathcal{P}\ \psi$
⟨*proof*⟩

lemma (**in** *probability-logic*) *biconditional-equivalence*:
  $\vdash \varphi \leftrightarrow \psi \Longrightarrow \mathcal{P}\ \varphi = \mathcal{P}\ \psi$
  ⟨*proof*⟩

lemma (**in** *probability-logic*) *sum-rule*:
  $\mathcal{P}\ (\varphi \sqcup \psi) + \mathcal{P}\ (\varphi \sqcap \psi) = \mathcal{P}\ \varphi + \mathcal{P}\ \psi$
⟨*proof*⟩

sublocale *probability-logic* $\subseteq$ *gaines-weatherson-probability*
⟨*proof*⟩

sublocale *probability-logic* $\subseteq$ *consistent-classical-logic*
⟨*proof*⟩

lemma (**in** *probability-logic*) *subtraction-identity*:
$\mathcal{P}\ (\varphi \setminus \psi) = \mathcal{P}\ \varphi - \mathcal{P}\ (\varphi \sqcap \psi)$
⟨*proof*⟩

1.5 Basic Probability Logic Inequality Results

lemma (**in** *probability-logic*) *disjunction-sum-inequality*:
$\mathcal{P}\ (\varphi \sqcup \psi) \leq \mathcal{P}\ \varphi + \mathcal{P}\ \psi$
⟨*proof*⟩

lemma (**in** *probability-logic*)
arbitrary-disjunction-list-summation-inequality:
$\mathcal{P}\ (\bigsqcup\ \Phi) \leq (\sum \varphi \leftarrow \Phi.\ \mathcal{P}\ \varphi)$
⟨*proof*⟩

lemma (**in** *probability-logic*) *implication-list-summation-inequality*:
assumes $\vdash \varphi \rightarrow \bigsqcup\ \Psi$
shows $\mathcal{P}\ \varphi \leq (\sum \psi \leftarrow \Psi.\ \mathcal{P}\ \psi)$
⟨*proof*⟩

lemma (**in** *probability-logic*) *arbitrary-disjunction-set-summation-inequality*:
$\mathcal{P}\ (\bigsqcup\ \Phi) \leq (\sum \varphi \in set\ \Phi.\ \mathcal{P}\ \varphi)$
⟨*proof*⟩

lemma (**in** *probability-logic*) *implication-set-summation-inequality*:
assumes $\vdash \varphi \rightarrow \bigsqcup\ \Psi$
shows $\mathcal{P}\ \varphi \leq (\sum \psi \in set\ \Psi.\ \mathcal{P}\ \psi)$
⟨*proof*⟩

1.6 Dirac Measures

Before presenting *Dirac measures* in probability logic, we first give the set of all functions satisfying probability logic.

definition (**in** *classical-logic*) *probabilities* :: $('a \Rightarrow real)$ *set*
where *probabilities* =
$\{\mathcal{P}.\ class.probability\text{-}logic\ (\lambda\ \varphi.\ \vdash \varphi)\ (\rightarrow)\ \bot\ \mathcal{P}\ \}$

Traditionally, a Dirac measure is a function δ_x where $\delta_x\ S = 1$ if $x \in S$ and $\delta_x\ S = 0$ otherwise. This means that Dirac measures correspond to special ultrafilters on their underlying σ-algebra which are closed under countable unions.

Probability logic, as discussed in §1.2, may not have countable joins in its underlying logic. In the setting of probability logic, Dirac measures are simple probability functions that are either 0 or 1.

definition (**in** *classical-logic*) *dirac-measures* :: ($'a \Rightarrow$ *real*) *set*
where *dirac-measures* =
$\{\ \mathcal{P}.\ class.probability\text{-}logic\ (\lambda\ \varphi.\ \vdash \varphi)\ (\rightarrow)\ \bot\ \mathcal{P}$
$\wedge\ (\forall x.\ \mathcal{P}\ x = 0 \vee \mathcal{P}\ x = 1)\ \}$

lemma (**in** *classical-logic*) *dirac-measures-subset*:
dirac-measures ⊆ *probabilities*
⟨*proof*⟩

Maximally consistent sets correspond to Dirac measures. One direction of this correspondence is established below.

lemma (**in** *classical-logic*) *MCS-dirac-measure*:
assumes *MCS* Ω
shows ($\lambda\ \chi.$ *if* $\chi \in \Omega$ *then* (*1* :: *real*) *else* *0*) ∈ *dirac-measures*
(**is** *?*$\mathcal{P}$ ∈ *dirac-measures*)
⟨*proof*⟩

notation *FuncSet.funcset* (**infixr** → *60*)

end

Chapter 2

Suppes' Theorem

theory *Suppes-Theorem*
  imports *Probability-Logic*
begin

no-notation *FuncSet.funcset* (**infixr** → *60*)

An elementary completeness theorem for inequalities for probability logic is due to Patrick Suppes [13].

A consequence of this Suppes' theorem is an elementary form of *collapse*, which asserts that inequalities for probabilities are logically equivalent to the more restricted class of *Dirac measures* as defined in §1.6.

2.1 Suppes' List Theorem

We first establish Suppes' theorem for lists of propositions. This is done by establishing our first completeness theorem using *Dirac measures.*

First, we use the result from §1.5 that shows $\vdash \varphi \to \bigsqcup\ \Psi$ implies $\mathcal{P}\ \varphi \leq (\sum \psi{\leftarrow}\Psi.\ \mathcal{P}\ \psi)$. This can be understood as a *soundness* result.

To show completeness, assume $\neg \vdash \varphi \to \bigsqcup\ \Psi$. From this obtain a maximally consistent Ω such that $\varphi \to \bigsqcup\ \Psi \notin \Omega$. We then define $\delta\ \chi = (\mathit{if}\ \chi \in \Omega\ \mathit{then}\ \mathit{1}{::}'a\ \mathit{else}\ (\mathit{0}{::}'a))$ and show δ is a *Dirac measure* such that $\delta\ \varphi \leq (\sum \psi{\leftarrow}\Psi.\ \delta\ \psi)$.

lemma (**in** *classical-logic*) *dirac-list-summation-completeness*:
  $(\forall\ \delta \in \mathit{dirac\text{-}measures}.\ \delta\ \varphi \leq (\sum \psi{\leftarrow}\Psi.\ \delta\ \psi)) = \vdash \varphi \to \bigsqcup\ \Psi$
⟨*proof*⟩

theorem (**in** *classical-logic*) *list-summation-completeness*:
  $(\forall\ \mathcal{P} \in \mathit{probabilities}.\ \mathcal{P}\ \varphi \leq (\sum \psi{\leftarrow}\Psi.\ \mathcal{P}\ \psi)) = \vdash \varphi \to \bigsqcup\ \Psi$
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The collapse theorem asserts that to prove an inequalities for all probabilities in probability logic, one only needs to consider the case of functions which take on values of 0 or 1.

lemma (**in** *classical-logic*) *suppes-collapse*:
$(\forall\ \mathcal{P} \in \mathit{probabilities}.\ \mathcal{P}\ \varphi \leq (\sum \psi \leftarrow \Psi.\ \mathcal{P}\ \psi))$
$= (\forall\ \delta \in \mathit{dirac\text{-}measures}.\ \delta\ \varphi \leq (\sum \psi \leftarrow \Psi.\ \delta\ \psi))$
⟨*proof*⟩

lemma (**in** *classical-logic*) *probability-member-neg*:
fixes $\mathcal{P}$
assumes $\mathcal{P} \in \mathit{probabilities}$
shows $\mathcal{P}\ (\sim \varphi) = 1 - \mathcal{P}\ \varphi$
⟨*proof*⟩

Suppes' theorem has a philosophical interpretation. It asserts that if $\Psi :\vdash \varphi$, then our *uncertainty* in φ is bounded above by our uncertainty in Ψ. Here the uncertainty in the proposition φ is $1 - \mathcal{P}\ \varphi$. Our uncertainty in Ψ, on the other hand, is $\sum \psi \leftarrow \Psi.\ 1 - \mathcal{P}\ \psi$.

theorem (**in** *classical-logic*) *suppes-list-theorem*:
$\Psi :\vdash \varphi = (\forall\ \mathcal{P} \in \mathit{probabilities}.\ (\sum \psi \leftarrow \Psi.\ 1 - \mathcal{P}\ \psi) \geq 1 - \mathcal{P}\ \varphi)$
⟨*proof*⟩

2.2 Suppes' Set Theorem

Suppes theorem also obtains for *sets*.

lemma (**in** *classical-logic*) *dirac-set-summation-completeness*:
$(\forall\ \delta \in \mathit{dirac\text{-}measures}.\ \delta\ \varphi \leq (\sum \psi \in \mathit{set}\ \Psi.\ \delta\ \psi)) = \vdash \varphi \rightarrow \bigsqcup\ \Psi$
⟨*proof*⟩

theorem (**in** *classical-logic*) *set-summation-completeness*:
$(\forall\ \delta \in \mathit{probabilities}.\ \delta\ \varphi \leq (\sum \psi \in \mathit{set}\ \Psi.\ \delta\ \psi)) = \vdash \varphi \rightarrow \bigsqcup\ \Psi$
⟨*proof*⟩

lemma (**in** *classical-logic*) *suppes-set-collapse*:
$(\forall\ \mathcal{P} \in \mathit{probabilities}.\ \mathcal{P}\ \varphi \leq (\sum \psi \in \mathit{set}\ \Psi.\ \mathcal{P}\ \psi))$
$= (\forall\ \delta \in \mathit{dirac\text{-}measures}.\ \delta\ \varphi \leq (\sum \psi \in \mathit{set}\ \Psi.\ \delta\ \psi))$
⟨*proof*⟩

In our formulation of logic, there is not reason that $\sim a = \sim b$ while $a \neq b$. As a consequence the Suppes theorem for sets presented below is different than the one given in §2.1.

theorem (**in** *classical-logic*) *suppes-set-theorem*:
$\Psi :\vdash \varphi$
$= (\forall\ \mathcal{P} \in \mathit{probabilities}.\ (\sum \psi \in \mathit{set}\ (\boldsymbol{\sim}\ \Psi).\ \mathcal{P}\ \psi) \geq 1 - \mathcal{P}\ \varphi)$
⟨*proof*⟩

2.3 Converse Suppes' Theorem

A formulation of the converse of Suppes' theorem obtains for lists/sets of *logically disjoint* propositions.

lemma (**in** *probability-logic*) *exclusive-sum-list-identity*:
  assumes $\vdash \coprod\ \Phi$
  shows $\mathcal{P}\ (\bigsqcup\ \Phi) = (\sum \varphi \leftarrow \Phi.\ \mathcal{P}\ \varphi)$
  ⟨*proof*⟩

lemma *sum-list-monotone*:
  fixes $f :: {}'a \Rightarrow real$
  assumes $\forall\ x.\ f\ x \geq 0$
    and $set\ \Phi \subseteq set\ \Psi$
    and $distinct\ \Phi$
  shows $(\sum \varphi \leftarrow \Phi.\ f\ \varphi) \leq (\sum \psi \leftarrow \Psi.\ f\ \psi)$
  ⟨*proof*⟩

lemma *count-remove-all-sum-list*:
  fixes $f :: {}'a \Rightarrow real$
  shows $real\ (count\text{-}list\ xs\ x) * f\ x + (\sum x' \leftarrow (removeAll\ x\ xs).\ f\ x')$
    $= (\sum x \leftarrow xs.\ f\ x)$
  ⟨*proof*⟩

lemma (**in** *classical-logic*) *dirac-exclusive-implication-completeness*:
  $(\forall\ \delta \in dirac\text{-}measures.\ (\sum \varphi \leftarrow \Phi.\ \delta\ \varphi) \leq \delta\ \psi) = (\vdash \coprod\ \Phi \wedge\ \vdash \bigsqcup\ \Phi \rightarrow \psi)$
⟨*proof*⟩

theorem (**in** *classical-logic*) *exclusive-implication-completeness*:
  $(\forall\ \mathcal{P} \in probabilities.\ (\sum \varphi \leftarrow \Phi.\ \mathcal{P}\ \varphi) \leq \mathcal{P}\ \psi) = (\vdash \coprod\ \Phi \wedge\ \vdash \bigsqcup\ \Phi \rightarrow \psi)$
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma (**in** *classical-logic*) *dirac-inequality-completeness*:
  $(\forall\ \delta \in dirac\text{-}measures.\ \delta\ \varphi \leq \delta\ \psi) = \vdash \varphi \rightarrow \psi$
⟨*proof*⟩

2.4 Implication Inequality Completeness

The following theorem establishes the converse of $\vdash \varphi \rightarrow \psi \Longrightarrow \mathcal{P}\ \varphi \leq \mathcal{P}\ \psi$, which was proved in §1.4.

theorem (**in** *classical-logic*) *implication-inequality-completeness*:
  $(\forall\ \mathcal{P} \in probabilities.\ \mathcal{P}\ \varphi \leq \mathcal{P}\ \psi) = \vdash \varphi \rightarrow \psi$
⟨*proof*⟩

2.5 Characterizing Logical Exclusiveness In Probability Logic

Finally, we can say that $\mathcal{P}$ ($\bigsqcup$ Φ) = ($\sum\varphi\leftarrow\Phi$. $\mathcal{P}$ φ) if and only if the propositions in Φ are mutually exclusive (i.e. $\vdash$ $\coprod$ Φ). This result also obtains for sets.

lemma (**in** *classical-logic*) *dirac-exclusive-list-summation-completeness*:
  ($\forall$ δ $\in$ *dirac-measures*. δ ($\bigsqcup$ Φ) = ($\sum\varphi\leftarrow\Phi$. δ φ)) = $\vdash$ $\coprod$ Φ
  ⟨*proof*⟩

theorem (**in** *classical-logic*) *exclusive-list-summation-completeness*:
  ($\forall$ $\mathcal{P}$ $\in$ *probabilities*. $\mathcal{P}$ ($\bigsqcup$ Φ) = ($\sum\varphi\leftarrow\Phi$. $\mathcal{P}$ φ)) = $\vdash$ $\coprod$ Φ
  ⟨*proof*⟩

lemma (**in** *classical-logic*) *dirac-exclusive-set-summation-completeness*:
  ($\forall$ δ $\in$ *dirac-measures*. δ ($\bigsqcup$ Φ) = ($\sum\varphi$ $\in$ *set* Φ. δ φ))
    = $\vdash$ $\coprod$ (*remdups* Φ)
  ⟨*proof*⟩

theorem (**in** *classical-logic*) *exclusive-set-summation-completeness*:
  ($\forall$ $\mathcal{P}$ $\in$ *probabilities*.
      $\mathcal{P}$ ($\bigsqcup$ Φ) = ($\sum\varphi$ $\in$ *set* Φ. $\mathcal{P}$ φ)) = $\vdash$ $\coprod$ (*remdups* Φ)
  ⟨*proof*⟩

lemma (**in** *probability-logic*) *exclusive-list-set-inequality*:
  assumes $\vdash$ $\coprod$ Φ
  shows ($\sum\varphi\leftarrow\Phi$. $\mathcal{P}$ φ) = ($\sum\varphi\in$*set* Φ. $\mathcal{P}$ φ)
⟨*proof*⟩

notation *FuncSet.funcset* (**infixr** $\rightarrow$ *60*)

end

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