Formally Verified Suffix Array Construction

Louis Cheung and Christine Rizkallah

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Abstract

A suffix array [2] is a data structure that is extensively used in text retrieval and data compression applications, including query suggestion mechanisms in web search, and in bioinformatics tools for DNA sequencing and matching. This wide applicability means that algorithms for constructing suffix arrays are of great practical importance. The Suffix Array by Induced Sorting (SA-IS) algorithm [3] is a conceptually complex yet highly efficient suffix array construction technique, based on an earlier algorithm [1].

As part of this formalization, we have developed the SA-IS algorithm in Isabelle/HOL and formally verified that it is equivalent to a mathematical functional specification of suffix arrays. This required verifying a wide range of underlying properties of lists and suffixes, that could be reused in other contexts. We also used Isabelle's code extraction facilities to extract an executable Haskell implementation of SAIS. In particular, this entry includes the following: an axiomatic characterisation of suffix array construction; a formally verified encoding of a straightforward but inefficient suffix array construction algorithm (validating the specification); and a formally verified encoding of the linear time SA-IS algorithm.

Contents

1 HOL **9**

2 Natural Number Arithmetic **9**

3 Monotonic Functions **10**

4 Sets **11**
4.1 From AutoCorres 14

5 General Lists **14**

6 Find **16**

7 Filter **17**

8 Upt **22**

9 Lemmas about bijections **22**

10 Lemmas about monotone functions **26**

11 Sorting **28**
11.1 General sorting 28
11.2 Sorting on linear orders 29
11.3 Sorting on orders 31

12 Mapping elements to natural numbers **31**

13 Repeat Function At Most N Times **33**
13.1 Step and early termination lemmas 33

14 Repeat Function N Times **37**

15 Continuous Intervals **38**

16 List Slices **41**

17 Sorted List Slice **47**

18 General Non-standard Lexicographical Comparison **51**
18.1 Intro and Elimination 51
18.2 Simplification 52
18.3 Recursive version 53
18.4 Properties 54
18.5 Monotonicity 59
18.6 Other 61

19 Order definitions on lists of linorder elements **63**

20 Helper list comparison theorems **64**

21 *list-less-ns* helpers **66**

22 Lists of linorder elements are linorders with a bottom element **67**

23 Recursive Definition **68**

24 *list-less-ns-ex* helpers **69**

25 Valid List **70**

26 Order Equivalence **73**

27 Classical Lexicographical Order **74**

28 Non-standard Lexicographical Ordering **76**

29 Suffix **77**

30 Valid Lists and Suffixes **79**

31 Prefixes and Suffixes **79**

32 Suffix Comparisons **81**
32.1 Lexicographical Ordering 81
32.2 Non-standard List Ordering 82

33 List Slice **82**

34 Sorting **83**

35 Prefix Definition **87**

36 Axiomatic Suffix Array Specification **88**

37 Wrapper for Natural Number String only Algorithm **88**

38 General Suffix Array Properties **89**

39 Properties of Suffix Arrays on Valid Lists **90**

40 Equivalence **94**

41 Small and Large List Types **99**

42 Suffix Type **104**
42.1 General Suffix Type Simplifications 105
42.2 S-Type Simplifications 106
42.3 L-Type Simplifications 107
42.4 General Suffix Type Theories 110
42.5 S/L-Type Ordering 111
42.6 Implementation of Suffix Type Computation 113

43 SAIS Sublist Order **115**

44 Sorting **115**

45 LMS-Types **118**
45.1 LMS-Type Simplifications . . . 118
45.2 LMS-Type Sets and Subsets . . . 120
45.3 Implementation of LMS-Types Computation . . . 120
45.3.1 Properties . . . 121
45.4 Cardinality LMS-Types . . . 121
45.5 General Properties about LMS-types . . . 123

46 Buckets **129**
46.1 Entire Bucket . . . 129
46.2 L-types . . . 133
46.3 LMS-types . . . 135
46.4 S-types . . . 135

47 Continuous Buckets **143**

48 Bucket Initialisation **144**

49 Bucket Range **145**

50 Helpers **145**

51 LMS Slice **146**
51.1 Find the next LMS position . . . 146
51.2 LMS Prefix . . . 150
51.3 LMS Slice . . . 151
51.4 LMS Substring butlast . . . 153
51.5 Suffix Types . . . 154

52 Ordering LMS-substrings **165**

53 Mapping from suffix to lists of LMS-Substrings **174**
53.1 LMS Sequence . . . 175
53.2 LMS-Substring Sequence . . . 182
53.3 LMS Map . . . 190

54 Induce Sorting **198**
54.1 Bucket Insert . . . 198
54.2 Induce L-types . . . 198
54.3 Induce S-types . . . 199
54.4 Induce Sorting . . . 200

55 Rename Mapping **201**

56 Rename String **201**

57 Order LMS **201**

58 Extract LMS **201**

59 SAIS Definition **202**

60 Bucket Insert with Ghost State **203**

61 Simple Properties **203**

62 Invariants **203**
62.1 Defintions and Simple Helper Lemmas . . . 203
62.1.1 Distinctness . . . 203
62.1.2 LMS Bucket Ptr . . . 204
62.1.3 Unknowns . . . 205
62.1.4 Locations . . . 205
62.1.5 Unchanged . . . 205
62.1.6 Inserted . . . 206
62.1.7 Sorted . . . 206
62.2 Combined Invariant . . . 207
62.3 Helpers . . . 208
62.4 Establishment and Maintenance Steps . . . 219
62.4.1 Distinctness . . . 219
62.4.2 Bucket Ptr . . . 223
62.4.3 Unknowns . . . 227
62.4.4 Locations . . . 229
62.4.5 Unchanged . . . 232
62.4.6 Inserted . . . 234
62.4.7 Sorted . . . 235
62.5 Combined Establishment and Maintenance . . . 242

63 Exhaustiveness **244**

64 Postconditions **245**

65 Abstract Induce L-types Simple Properties **253**

66 Precondition Definitions **255**

67 Invariant Definitions **261**
67.1 Distinctness . . . 261
67.2 Predecessor . . . 261
67.3 L Bucket Ptr . . . 262
67.4 Unknowns . . . 262
67.5 Indexes . . . 263

67.6 Unchanged . . . 263
67.7 L Locations . . . 263
67.8 Seen . . . 264
67.9 Sortedness . . . 265
67.10Permutation . . . 266

68 Invariant Helpers **267**
68.1 Distinctness of New Insert . . . 267
68.2 Bucket Ranges . . . 269
68.3 No Overwrite . . . 271
68.4 Bucket Values . . . 275
68.5 Seen . . . 280

69 Distinctness **281**
69.1 Establishment . . . 281
69.2 Maintenance . . . 283

70 Unknowns **286**
70.1 Establishment . . . 286
70.2 Maintenance . . . 286

71 Number of L-types **289**
71.1 Establishment . . . 289
71.2 Maintenance . . . 291

72 L Locations **295**
72.1 Establishment . . . 295
72.2 Maintenance . . . 295

73 Unchanged **298**
73.1 Establishment . . . 298
73.2 Maintenance . . . 298

74 Invariant about the Current Index **301**
74.1 Establishment . . . 301
74.2 Maintenance . . . 302

75 Predecessor Invariant **305**
75.1 Establishment . . . 305
75.2 Maintenance . . . 305

76 Seen Invariant **308**
76.1 Establishment . . . 308
76.2 Maintenance . . . 308

77 Permutation **310**
77.1 Establishment . . . 310
77.2 Maintenance . . . 310

78 Sorted **313**

79 L-type Exhaustiveness **333**
79.1 Case 1 . . . 333
79.2 Case 2 . . . 334
79.3 Exhaustiveness Proof . . . 336

80 Correctness and Exhaustiveness **338**

81 Abstract Induce S Simple Properties **342**

82 Preconditions **344**

83 Invariants **346**
83.1 Definitions . . . 346
83.1.1 Distinctness . . . 346
83.1.2 S Bucket Ptr . . . 346
83.1.3 Locations . . . 346
83.1.4 Unchanged . . . 347
83.1.5 Seen . . . 348
83.1.6 Predecessor . . . 348
83.1.7 Successor . . . 348
83.1.8 Combined Permutation Invariant . . . 349
83.1.9 Sorted . . . 350
83.2 Helpers . . . 351
83.3 Establishment and Maintenance Steps . . . 363
83.3.1 Distinctness . . . 363
83.3.2 Bucket Pointer . . . 366
83.3.3 Locations . . . 369
83.3.4 Unchanged . . . 373
83.3.5 Seen . . . 375
83.3.6 Predecessor . . . 405
83.3.7 Successor . . . 410
83.3.8 Combined Permutation Invariant . . . 418
83.3.9 Sorted . . . 426

84 Induce S Correctness Theorems **458**

85 Bucket Initialisation Properties **459**

86 Bucket Insert Precondition **460**

87 Induce L Precondition 460

88 Induce S Precondition 461

89 Permutation 462

90 Sorting 463

91 Extract LMS types Proofs 466

92 Order LMS-types Proofs 467

93 Rename Mapping Proofs 468

94 Rename String Proofs 472

95 SAIS General Helpers 472

96 SAIS cases simplifications 473

97 SAIS returns a permutation 474

98 SAIS Sorted Helpers 477

99 SAIS sorts suffixes 477

100 Verification of a SAIS construction algorithm 480

101 Final Theorem: Verification of a generalised SAIS construction algorithm 480

102 Bucket Insert 481

103 Suffix Types 481

104 LMS types 482

105 Extracting LMS types 482

106 LMS Substrings 482

107 Rename Mapping 483

108 Induce L Refinement 483

109 Induce S Refinement 485

110 Induce 487

111 SAIS 489

112 Bucket Insert 491

113 Induce L Refinement 493

114 Induce S Refinement 495

115 Induce 500

116 Suffix Types 503

117 LMS types 506

118 Extracting LMS types 506

119 LMS Substrings 506

120 Rename Mapping 507

121 SAIS 508

122 Correctness 513

theory *Nat-Util*
imports *Main*
begin

1 HOL

lemma *duplicate-assms*:
$([\![P;\ P]\!] \Longrightarrow Q) \equiv (P \Longrightarrow Q)$
by *simp*

2 Natural Number Arithmetic

lemma *div-2-eq-Suc*:
$[\![x\ div\ 2 = y\ div\ 2;\ x \neq y]\!] \Longrightarrow (y = Suc\ x) \vee (x = Suc\ y)$
by *linarith*

lemma *Suc-m-sub-n-div-2*:
$Suc\ ((m - n)\ div\ 2) > (m - Suc\ n)\ div\ 2$
by (*simp add*: *div-le-mono le-Suc-eq*)

lemma *Suc-div-2-less-Suc*:
$Suc\ x\ div\ 2 < Suc\ x$
by *simp*

lemma *nat-x-less-y-le-Suc-x*:

$[\![x < y;\ y \leq Suc\ x]\!] \Longrightarrow y = Suc\ x$
by *simp*

lemma *nat-sub-eq-add*:
$[\![(a :: nat) - b = c - d;\ b < a]\!] \Longrightarrow a + d = c + b$
by *simp*

end
theory *Fun-Util*
imports *Main*
begin

3 Monotonic Functions

lemma *strict-mono-leD*: *strict-mono r* $\Longrightarrow m \leq n \Longrightarrow$ *r m* $\leq$ *r n*
by (*erule* (*1*) *monoD* [*OF strict-mono-mono*])

definition *map-to-nat* :: ('*a* :: *linorder list*) $\Rightarrow$ ('*a* $\Rightarrow$ *nat*)
where
map-to-nat xs = (λx. *card* $\{y|y.\ y \in set\ xs \wedge y < x\}$)

lemma *map-to-nat-strict-mono-on*:
strict-mono-on (*set xs*) (*map-to-nat xs*)
unfolding *strict-mono-on-def map-to-nat-def*
proof *safe*
fix *x y* :: '*a*
assume *x* $<$ *y x* $\in$ *set xs y* $\in$ *set xs*
have *finite* $\{a\ |a.\ a \in set\ xs \wedge a < y\}$
by *auto*
moreover
have $\{a\ |a.\ a \in set\ xs \wedge a < x\} \subset \{a\ |a.\ a \in set\ xs \wedge a < y\}$
proof (*intro psubsetI subsetI notI*)
fix *k*
assume $k \in \{a\ |a.\ a \in set\ xs \wedge a < x\}$
hence *k* $<$ *x k* $\in$ *set xs*
by *simp-all*
hence *k* $<$ *y k* $\in$ *set xs*
using ‹*x* $<$ *y*› **by** *auto*
then show $k \in \{a\ |a.\ a \in set\ xs \wedge a < y\}$
by *simp*
next
assume $\{a\ |a.\ a \in set\ xs \wedge a < x\} = \{a\ |a.\ a \in set\ xs \wedge a < y\}$
moreover
have $x \in \{a\ |a.\ a \in set\ xs \wedge a < y\}$
using ‹*x* $<$ *y*› ‹*x* $\in$ *set xs*› **by** *auto*
moreover
have $x \notin \{a\ |a.\ a \in set\ xs \wedge a < x\}$
by *auto*
ultimately show *False*

by *simp*
qed
ultimately show *card* $\{a\ |a.\ a \in set\ xs \wedge a < x\} < card\ \{a\ |a.\ a \in set\ xs \wedge a < y\}$
using *psubset-card-mono*[*of* $\{a\ |a.\ a \in set\ xs \wedge a < y\}$ $\{a\ |a.\ a \in set\ xs \wedge a < x\}$]
by *blast*
qed

lemma *strict-mono-on-map-set-ex*:
$\exists (f :: ('a :: linorder \Rightarrow nat)).$ *strict-mono-on* (*set xs*) *f*
using *map-to-nat-strict-mono-on* **by** *blast*

locale *Linorder-to-Nat-List* =
fixes *map-to-nat* :: $'a$ :: *linorder list* $\Rightarrow$ $'a \Rightarrow$ *nat*
and *xs* :: $'a$ :: *linorder list*
assumes *map-to-nat-strict-mono-on*: *strict-mono-on* (*set xs*) (*map-to-nat xs*)

context *Linorder-to-Nat-List* **begin**

lemma *strict-mono-on-Suc-map-to-nat*:
strict-mono-on (*set xs*) ($\lambda x.$ *Suc* (*map-to-nat xs x*))
by (*metis* (*mono-tags*, *lifting*) *Suc-mono ord.strict-mono-on-def map-to-nat-strict-mono-on*)

end

lemma *Linorder-to-Nat-List-ex*:
$\exists \alpha.$ *Linorder-to-Nat-List* α *xs*
by (*meson Linorder-to-Nat-List.intro strict-mono-on-map-set-ex*)

end
theory *Set-Util*
imports *Main*
begin

4 Sets

lemma *pigeonhole-principle-advanced*:
assumes *finite A*
and *finite B*
and $A \cap B = \{\}$
and *card A* $>$ *card B*
and *bij-betw f* $(A \cup B)$ $(A \cup B)$
shows $\exists\, a{\in}A.\ f\ a \in A$
proof (*rule ccontr*)
assume $\neg(\exists\, a{\in}A.\ f\ a \in A)$
hence $\forall\, a \in A.\ f\ a \notin A$
by *blast*

```
  hence ∀ a∈A. f a ∈ B
    using assms(5) bij-betw-apply by fastforce
  hence f ‘ A ⊆ B
    by blast

  have inj-on f A
    by (meson assms(5) bij-betw-def inj-on-Un)
  hence card (f ‘ A) = card A
    using card-image by blast
  hence f ‘ A = B
    by (metis ‹f ‘ A ⊆ B› assms(2,4) card-mono leD)
  hence card B = card A
    using ‹card (f ‘ A) = card A› by blast
  then show False
    using assms(4) by linarith
qed

lemma Suc-mod-n-bij-betw:
  bij-betw (λx. Suc x mod n) {0..<n} {0..<n}
proof (intro bij-betwI')
  fix x y
  assume x ∈ {0..<n} y ∈ {0..<n}
  then show (Suc x mod n = Suc y mod n) = (x = y)
    by (simp add: mod-Suc)
next
  fix x
  assume x ∈ {0..<n}
  then show Suc x mod n ∈ {0..<n}
    by clarsimp
next
  fix y
  assume y ∈ {0..<n}
  then show ∃ x∈{0..<n}. y = Suc x mod n
   by (metis atLeastLessThan-iff bot-nat-0.extremum less-nat-zero-code mod-Suc-eq
mod-less
              mod-less-divisor mod-self not-gr-zero old.nat.exhaust)
qed



lemma subset-upt-no-Suc:
  assumes A ⊆ {1..<n}
  and     ∀ x∈A. Suc x ∉ A
  shows card A ≤ n div 2
proof (rule ccontr)
  assume ¬ card A ≤ n div 2
  hence n div 2 < card A
    by auto
```

```
  have ∃ a∈A. Suc a mod n ∈ A
  proof (rule pigeonhole-principle-advanced[of A {0..<n} − A (λx. Suc x mod n),
simplified])
    show finite A
      using assms(1) finite-subset by blast
  next
    from card-Diff-subset
    have card ({0..<n} − A) = card {0..<n} − card A
      by (metis Diff-subset assms(1) dual-order.trans ivl-diff less-eq-nat.simps(1)
            subset-eq-atLeast0-lessThan-finite)
    moreover
    have card {0..<n} − card A < card A
      using ‹¬ card A ≤ n div 2› assms by simp
    ultimately show card ({0..<n} − A) < card A
      by simp
  next
    have A ∪ {0..<n} = {0..<n}
      using assms(1) dual-order.trans by auto
    with Suc-mod-n-bij-betw[of n]
    show bij-betw (λx. Suc x mod n) (A ∪ {0..<n}) (A ∪ {0..<n})
      by simp
  qed
  then obtain x where
    x ∈ A
    Suc x mod n ∈ A
    by blast

  show False
  proof (cases n)
    case 0
    then show ?thesis
      using ‹Suc x mod n ∈ A› ‹x ∈ A› assms(2) by force
  next
    case (Suc m)
    assume n = Suc m

    have x = m ∨ x < m
      using Suc ‹x ∈ A› assms(1) by auto
    then show ?thesis
    proof
      assume x = m
      then show False
        using Suc ‹Suc x mod n ∈ A› assms(1) by auto
    next
      assume x < m
      with mod-less[of Suc x Suc m]
      show False
        using Suc ‹Suc x mod n ∈ A› ‹x ∈ A› assms(2) by force
    qed
```

qed
qed

lemma *in-set-mapD*:
x ∈ *set* (*map f xs*) ⟹ ∃ *y* ∈ *set xs*. *x* = *f y*
by (*simp add*: *image-iff*)

4.1 From AutoCorres

lemma *disjointI'*:
assumes ⋀*x y*. ⟦ *x* ∈ *A*; *y* ∈ *B* ⟧ ⟹ *x* ≠ *y*
shows *A* ∩ *B* = {}
using *assms* **by** *fast*

lemma *disjoint-subset2*:
assumes *B'* ⊆ *B* **and** *A* ∩ *B* = {}
shows *A* ∩ *B'* = {}
using *assms* **by** *fast*

end
theory *List-Util*
imports *Main*
begin

5 General Lists

lemma *list-cases-3*:
T = [] ∨ (∃ *x*. *T* = [*x*]) ∨ (∃ *a b xs*. *T* = *a* # *b* # *xs*)
proof (*cases T*)
case *Nil*
then show *?thesis* **by** *simp*
next
case (*Cons a list*)
then show *?thesis*
proof (*cases list*)
case *Nil*
with ‹*T* = *a* # *list*›
show *?thesis*
by *simp*
next
case (*Cons a' list'*)
with ‹*T* = *a* # *list*›
show *?thesis*
by *simp*
qed
qed

lemma *length-cons-cons*:
$T = a \# b \# xs \Longrightarrow \exists n.\ length\ T = Suc\ (Suc\ n)$
by *simp*

lemma *length-Suc-Suc*:
$length\ T = Suc\ (Suc\ n) \Longrightarrow \exists a\ b\ xs.\ T = a \# b \# xs$
by (*metis length-Suc-conv*)

lemma *length-Suc-0*:
$length\ xs = Suc\ 0 \Longrightarrow \exists x.\ xs = [x]$
by (*simp add*: *length-Suc-conv*)

lemma *map-eq-replicate*:
$\forall x \in set\ xs.\ f\ x = k \Longrightarrow map\ f\ xs = replicate\ (length\ xs)\ k$
by (*metis map-eq-conv map-replicate-const*)

lemma *map-upt-eq-replicate*:
$\forall x \in set\ [i..<j].\ f\ x = k \Longrightarrow map\ f\ [i..<j] = replicate\ (j - i)\ k$
by (*metis length-upt map-eq-replicate*)

lemma *in-set-list-update*:
$[\![x \in set\ xs;\ xs\ !\ k \neq x]\!] \Longrightarrow x \in set\ (xs[k := y])$
by (*metis in-set-conv-nth length-list-update nth-list-update-neq*)

lemma *Max-greD*:
$i < length\ s \Longrightarrow Max\ (set\ s) \geq s\ !\ i$
by *clarsimp*

lemma *list-neq-rc1*:
$(\exists z\ zs.\ xs = ys\ @\ z \# zs) \Longrightarrow xs \neq ys$
by *fastforce*

lemma *list-neq-rc2*:
$(\exists z\ zs.\ ys = xs\ @\ z \# zs) \Longrightarrow xs \neq ys$
by *fastforce*

lemma *list-neq-rc3*:
$(\exists x\ y\ as\ bs\ cs.\ xs = as\ @\ x \# bs \wedge ys = as\ @\ y \# cs \wedge x \neq y) \Longrightarrow xs \neq ys$
by *fastforce*

lemma *list-neq-rc*:
$(\exists z\ zs.\ xs = ys\ @\ z \# zs) \vee$
$(\exists z\ zs.\ ys = xs\ @\ z \# zs) \vee$
$(\exists x\ y\ as\ bs\ cs.\ xs = as\ @\ x \# bs \wedge ys = as\ @\ y \# cs \wedge x \neq y) \Longrightarrow$
$xs \neq ys$
by (*elim disjE conjE list-neq-rc1 list-neq-rc2 list-neq-rc3*)

lemma *list-neq-fc*:

```
  xs ≠ ys ⟹
   (∃ z zs. xs = ys @ z # zs) ∨
   (∃ z zs. ys = xs @ z # zs) ∨
   (∃ x y as bs cs. xs = as @ x # bs ∧ ys = as @ y # cs ∧ x ≠ y)
proof (induct xs arbitrary: ys)
  case Nil
  then show ?case
    by (metis append-Nil list.exhaust)
next
  case (Cons a xs ys)
  note IH = this
  then show ?case
  proof (cases ys)
    case Nil
    then show ?thesis
      by simp
  next
    case (Cons b ys')
    assume ys = b # ys'
    show ?thesis
    proof (cases a = b)
      assume a ≠ b
      with ‹ys = b # ys'› 
      show ?thesis
        by blast
    next
      assume a = b
      with IH(2) ‹ys = b # ys'›
      have xs ≠ ys'
        by simp
      with IH(1)[of ys']
      show ?thesis
        by (metis Cons-eq-appendI ‹a = b› local.Cons)
    qed
  qed
qed

lemma list-neq-cases:
  xs ≠ ys ⟷
   (∃ z zs. xs = ys @ z # zs) ∨
   (∃ z zs. ys = xs @ z # zs) ∨
   (∃ x y as bs cs. xs = as @ x # bs ∧ ys = as @ y # cs ∧ x ≠ y)
  using list-neq-fc list-neq-rc by blast
```

6 Find

```
lemma findSomeD:
  find P xs = Some x ⟹ P x ∧ x ∈ set xs
  by (induct xs) (auto split: if-split-asm)
```

lemma *findNoneD*:
find P xs = None $\Longrightarrow \forall x \in$ *set xs*. $\neg$*P x*
by (*induct xs*) (*auto split*: *if-split-asm*)

7 Filter

lemma *filter-update-nth-success*:
$[\![$*P v*; *i* < *length xs*$]\!] \Longrightarrow$
filter P (*xs*[*i* := *v*]) = (*filter P* (*take i xs*)) @ [*v*] @ (*filter P* (*drop* (*Suc i*) *xs*))
by (*simp add*: *upd-conv-take-nth-drop*)

lemma *filter-update-nth-fail*:
$[\![\neg$*P v*; *i* < *length xs*$]\!] \Longrightarrow$
filter P (*xs*[*i* := *v*]) = (*filter P* (*take i xs*)) @ (*filter P* (*drop* (*Suc i*) *xs*))
by (*simp add*: *upd-conv-take-nth-drop*)

lemma *filter-take-nth-drop-success*:
$[\![$*i* < *length xs*; *P* (*xs* ! *i*)$]\!] \Longrightarrow$
filter P xs = (*filter P* (*take i xs*)) @ [*xs* ! *i*] @ (*filter P* (*drop* (*Suc i*) *xs*))
by (*metis filter-update-nth-success list-update-id*)

lemma *filter-take-nth-drop-fail*:
$[\![$*i* < *length xs*; $\neg$*P* (*xs* ! *i*)$]\!] \Longrightarrow$
filter P xs = (*filter P* (*take i xs*)) @ (*filter P* (*drop* (*Suc i*) *xs*))
by (*metis filter-update-nth-fail list-update-id*)

lemma *filter-nth-1*:
$[\![$*i* < *length xs*; *P* (*xs* ! *i*)$]\!] \Longrightarrow$
$\exists$ *i'*. *i'* < *length* (*filter P xs*) $\wedge$ (*filter P xs*) ! *i'* = *xs* ! *i*
proof (*induct xs arbitrary*: *i*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons a xs i*)
note *IH* = *this*
show *?case*
proof (*cases i*)
case *0*
with *IH*(*3*)
show *?thesis*
by *fastforce*
next
case (*Suc n*)
with *IH*(*1*)[*of n*] *IH*(*2*,*3*)
have $\exists$ *i'*<*length* (*filter P xs*). *filter P xs* ! *i'* = *xs* ! *n*
by *fastforce*
then show *?thesis*

```
    using Suc by auto
  qed
qed

lemma filter-nth-2:
  ⟦i < length (filter P xs)⟧ ⟹
   ∃ i'. i' < length xs ∧ (filter P xs) ! i = xs ! i'
proof (induct xs arbitrary: i)
  case Nil
  then show ?case
    by simp
next
  case (Cons a xs i)
  note IH = this
  then show ?case
  proof (cases i)
    case 0
    then show ?thesis
      using Cons.hyps Cons.prems by force
  next
    case (Suc n)
    with IH(1)[of n] IH(2)
    have ∃ i'<length xs. filter P xs ! n = xs ! i'
      by (metis filter.simps(2) length-Cons not-less-eq not-less-iff-gr-or-eq)
    then show ?thesis
      by (metis Cons.hyps Cons.prems Suc filter.simps(2) length-Cons not-less-eq
nth-Cons-Suc)
  qed
qed

lemma filter-nth-relative-1:
  ⟦i < length xs; P (xs ! i); j < i; P (xs ! j)⟧ ⟹
   ∃ i' j'. i' < length (filter P xs) ∧ j' < i' ∧ (filter P xs) ! i' = xs ! i ∧
    (filter P xs) ! j' = xs ! j
proof (induct xs arbitrary: i j)
  case Nil
  then show ?case
    by simp
next
  case (Cons a xs i j)
  note IH = this
  show ?case
  proof (cases i)
    case 0
    then show ?thesis
      using IH(4) by blast
  next
    case (Suc n)
    assume i = Suc n
```

```
    show ?thesis
    proof (cases j)
      case 0
      with filter-nth-1[of n xs P] IH(2,3) ‹i = Suc n› IH(4−)
      show ?thesis
        by (metis filter.simps(2) length-Cons not-less-eq nth-Cons-0 nth-Cons-Suc
zero-less-Suc)
    next
      case (Suc m)
      with IH(1)[of n m] IH(2−) ‹i = Suc n›
      show ?thesis
        by fastforce
    qed
  qed
qed

lemma filter-nth-relative-neq-1:
  assumes i < length xs P (xs ! i) j < length xs P (xs ! j) i ≠ j
  shows ∃ i' j'. i' < length (filter P xs) ∧ j' < length (filter P xs) ∧ (filter P xs) !
i' = xs ! i ∧
                (filter P xs) ! j' = xs ! j ∧ i' ≠ j'
proof (cases i < j)
  assume i < j
  with filter-nth-relative-1[where P = P, OF assms(3,4) - assms(2)]
  show ?thesis
    by (metis (no-types, lifting) nat-neq-iff order-less-trans)
next
  assume ¬ i < j
  with assms(5)
  have j < i
    by auto
  with filter-nth-relative-1[where P = P, OF assms(1,2) - assms(4)]
  show ?thesis
    using order-less-trans by blast
qed

lemma filter-nth-relative-2:
  ⟦i < length (filter P xs); j < i⟧ ⟹
   ∃ i' j'. i' < length xs ∧ j' < i' ∧ (filter P xs) ! i = xs ! i' ∧ (filter P xs) ! j = xs
! j'
proof (induct xs arbitrary: i j)
  case Nil
  then show ?case
    by simp
next
  case (Cons a xs i j)
  note IH = this

  let ?goal = ∃ i' j'. i' < length (a # xs) ∧ j' < i' ∧ filter P (a # xs) ! i = (a #
```

```
xs) ! i' ∧
                    filter P (a # xs) ! j = (a # xs) ! j'
  show ?case
  proof (cases i)
    case 0
    then show ?goal
      using IH(3) by blast
  next
    case (Suc n)
    assume i = Suc n
    show ?thesis
    proof (cases j)
      case 0
      assume j = 0
      from filter-nth-2[of n P xs] IH(2)
      have ∃ i'<length xs. filter P xs ! n = xs ! i'
        using Suc order-less-trans by fastforce

      show ?goal
      proof (cases P a)
        assume P a
        then show ?goal
          by (metis 0 Suc ‹∃ i'<length xs. filter P xs ! n = xs ! i'› filter.simps(2)
                length-Cons not-less-eq nth-Cons-0 nth-Cons-Suc zero-less-Suc)
      next
        assume ¬ P a
        then show ?goal
          using Cons.hyps IH(2,3) Suc-less-eq by fastforce
        qed
    next
      case (Suc m)
      with IH(1)[of n m] IH(2−) ‹i = Suc n›
      have ∃ i' j'. i' < length xs ∧ j' < i' ∧ filter P xs ! n = xs ! i' ∧
                  filter P xs ! m = xs ! j'
        by (metis filter.simps(2) length-Cons not-less-eq not-less-iff-gr-or-eq)
      then obtain i' j' where
        A: i' < length xs j' < i' filter P xs ! n = xs ! i' filter P xs ! m = xs ! j'
        by blast
      show ?goal
      proof (cases P a)
        assume P a
        then show ?goal
          using A Suc ‹i = Suc n› by force
      next
        assume ¬ P a
        then show ?goal
          using Cons.hyps IH(2,3) by fastforce
      qed
    qed
```

```
  qed
qed

lemma filter-nth-relative-neq-2:
  assumes i < length (filter P xs) j < length (filter P xs) i ≠ j
  shows ∃ i' j'. i' < length xs ∧ j' < length xs ∧ xs ! i' = (filter P xs) ! i ∧
                 xs ! j' = (filter P xs) ! j ∧ i' ≠ j'
proof (cases i < j)
  assume i < j
  with filter-nth-relative-2[OF assms(2), of i]
  show ?thesis
    by (metis dual-order.strict-trans exists-least-iff)
next
  assume ¬ i < j
  with assms(3)
  have j < i
    by auto
  with filter-nth-relative-2[OF assms(1), of j]
  show ?thesis
    by (metis nat-neq-iff order-less-trans)
qed

lemma filter-find:
  filter P xs ≠ [] ⟹ find P xs = Some ((filter P xs) ! 0)
  by (induct xs; auto)

lemma filter-nth-update-subset:
  set (filter P (xs[i := v])) ⊆ {v} ∪ set (filter P xs)
proof
  fix x
  assume x ∈ set (filter P (xs[i := v]))
  then have x ∈ set (xs[i := v]) P x
    by auto
  hence ∃ k < length (xs[i := v]). (xs[i := v]) ! k = x
    by (simp add: in-set-conv-nth)
  then obtain k where
    k < length (xs[i := v])
    (xs[i := v]) ! k = x
    by blast
  hence k < length xs
    by simp

  from ‹(xs[i := v]) ! k = x› ‹k < length (xs[i := v])›
  have k = i ⟹ x = v
    by simp
  moreover
  have k ≠ i ⟹ x ∈ set (filter P xs)
    using ‹P x› ‹k < length xs› ‹xs[i := v] ! k = x› by auto
  ultimately
```

show $x \in \{v\} \cup set\ (filter\ P\ xs)$
by *blast*
qed

8 Upt

lemma *card-upt*:
$card\ \{0..<n\} = n$
by *simp*

lemma *bounded-distinct-subset-upt-length*:
$[\![distinct\ xs;\ \forall\ i<length\ xs.\ xs\ !\ i < length\ xs]\!] \Longrightarrow set\ xs \subseteq \{0..<length\ xs\}$
by (*intro subsetI*; *clarsimp simp*: *in-set-conv-nth*)

lemma *bounded-distinct-eq-upt-length*:
assumes *distinct xs*
assumes $\forall\ i < length\ xs.\ xs\ !\ i < length\ xs$
shows $set\ xs = \{0..<length\ xs\}$
proof (*intro card-subset-eq finite-atLeastLessThan*
bounded-distinct-subset-upt-length[*OF assms*])
from *distinct-card*[*OF assms*(*1*)] *card-upt*[*of length xs*]
show $card\ (set\ xs) = card\ \{0..<length\ xs\}$
by *simp*
qed

lemma *set-map-nth-subset*:
assumes $n \leq length\ xs$
shows $set\ (map\ (nth\ xs)\ [0..<n]) \subseteq set\ xs$
using *assms* **by** *clarsimp*

lemma *set-map-nth-eq*:
$set\ (map\ (nth\ xs)\ [0..<length\ xs]) = set\ xs$
by (*intro equalityI set-map-nth-subset subsetI*; *simp add*: *map-nth*)

lemma *distinct-map-nth*:
assumes *distinct xs*
assumes $n \leq length\ xs$
shows $distinct\ (map\ (nth\ xs)\ [0..<n])$
using *assms* **by** (*simp add*: *distinct-conv-nth*)
end
theory *Sorting-Util*
imports *Main*
begin

9 Lemmas about bijections

A convenient definition of an inverses between two sets

definition

inverses-on ::
$('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where
inverses-on f g A B $\longleftrightarrow$
$(\forall x \in A.\ g\ (f\ x) = x) \wedge$
$(\forall x \in B.\ f\ (g\ x) = x)$

lemmas *inverses-onD1* = *inverses-on-def*[*THEN iffD1*, *THEN conjunct1*]
lemmas *inverses-onD2* = *inverses-on-def*[*THEN iffD1*, *THEN conjunct2*]

The inverses relation over maps

lemma *inverses-on-mapD*:
assumes *inverses-on* (*map f*) (*map g*) {*xs*. *set xs* $\subseteq A$} {*xs*. *set xs* $\subseteq B$}
shows *inverses-on f g A B*
proof −
from *inverses-onD1*[*OF assms*, *THEN bspec*, *of* [-], *simplified*]
have $\forall x \in A.\ g\ (f\ x) = x$
by *blast*
moreover
from *inverses-onD2*[*OF assms*, *THEN bspec*, *of* [-], *simplified*]
have $\forall x \in B.\ f\ (g\ x) = x$
by *blast*
ultimately show *?thesis*
by (*simp add*: *inverses-on-def*)
qed

lemma *inverses-on-map*:
assumes *inverses-on f g A B*
shows *inverses-on* (*map f*) (*map g*) {*xs*. *set xs* $\subseteq A$} {*xs*. *set xs* $\subseteq B$}
proof −
have $\forall x \in$ {*xs*. *set xs* $\subseteq A$}. *map g* (*map f x*) = *x*
using *list-eq-iff-nth-eq inverses-onD1 assms subsetD* **by** *fastforce*
moreover
have $\forall x \in$ {*xs*. *set xs* $\subseteq B$}. *map f* (*map g x*) = *x*
using *list-eq-iff-nth-eq inverses-onD2 assms subsetD* **by** *fastforce*
ultimately show *?thesis*
by (*simp add*: *inverses-on-def*)
qed

Inverses are symmetric

lemma *inverses-on-sym*:
inverses-on f g A B = *inverses-on g f B A*
apply (*simp add*: *inverses-on-def*)
apply (*subst conj.commute*)
apply *simp*
done

Convenient theorem to obtain the inverse of a bijection between two sets

lemma *bij-betw-inv-alt*:

assumes *bij-betw f A B*
shows $\exists g.$ *bij-betw g B A* $\wedge$ *inverses-on f g A B*
by (*meson assms bij-betw-imp-inj-on bij-betw-inv-into inv-into-f-f*
bij-betw-inv-into-right inverses-on-def[*THEN iffD2*])

Bijections over maps

lemma *bij-betw-map*:
assumes *bij-betw f A B*
shows *bij-betw* (*map f*) {*xs*. *set xs* $\subseteq$ *A*} {*xs*. *set xs* $\subseteq$ *B*}
proof (*rule bij-betwI'*)
fix *x y*
assume *x* $\in$ {*xs*. *set xs* $\subseteq$ *A*} *y* $\in$ {*xs*. *set xs* $\subseteq$ *A*}
hence *P*: *set x* $\subseteq$ *A set y* $\subseteq$ *A*
by *blast*+
note *inj* = *inj-on-eq-iff*[*THEN iffD1*, *OF bij-betw-imp-inj-on*[*OF assms*]]
from *list.inj-map-strong*[*OF inj*, *simplified*]
show (*map f x* = *map f y*) = (*x* = *y*)
using *P*(*1*) *P*(*2*) **by** *blast*
next
fix *x*
assume *x* $\in$ {*xs*. *set xs* $\subseteq$ *A*}
hence *set x* $\subseteq$ *A*
by *blast*
then show *map f x* $\in$ {*xs*. *set xs* $\subseteq$ *B*}
using *assms bij-betw-imp-surj-on* **by** *fastforce*
next
fix *y*
assume *y* $\in$ {*xs*. *set xs* $\subseteq$ *B*}
hence *set y* $\subseteq$ *B*
by *blast*

from *bij-betw-inv-alt*[*OF assms*, *simplified inverses-on-def*]
obtain *g* **where**
bij-betw g B A
$\forall x{\in}A.$ *g* (*f x*) = *x*
$\forall x{\in}B.$ *f* (*g x*) = *x*
by *blast*

have *set* (*map g y*) $\subseteq$ *A*
using ‹*bij-betw g B A*› ‹*set y* $\subseteq$ *B*› *bij-betw-imp-surj-on*
by *fastforce*
moreover
have *y* = *map f* (*map g y*)
proof −
have *length y* = *length* (*map f* (*map g y*))
by *simp*
moreover
have $\forall i <$ *length y*. *y* ! *i* = *map f* (*map g y*) ! *i*
using ‹$\forall x{\in}B.$ *f* (*g x*) = *x*› ‹*set y* $\subseteq$ *B*› *subsetD* **by** *fastforce*

```
    ultimately show ?thesis
      using list-eq-iff-nth-eq by blast
  qed
  ultimately have ∃x. set x ⊆ A ∧ y = map f x
    by blast
  then show ∃x∈{xs. set xs ⊆ A}. y = map f x
    by blast
qed
```

Eliminating the map from a bijection relation

```
lemma bij-betw-mapD:
  assumes bij-betw (map f) {xs. set xs ⊆ A} {xs. set xs ⊆ B}
  shows bij-betw f A B
proof (intro bij-betwI' iffI)
  fix x y
  assume x ∈ A y ∈ A f x = f y
  with inj-onD[OF bij-betw-imp-inj-on[OF assms], of [x] [y], simplified]
  show x = y
    by blast
next
  fix x
  assume x ∈ A
  with bij-betwE[OF assms, THEN bspec, of [x], simplified]
  show f x ∈ B
    by blast
next
  fix y
  assume y ∈ B
  with bij-betw-imp-surj-on[OF assms, simplified]
  have [y] ∈ map f ‘ {xs. set xs ⊆ A}
    by auto
  with image-iff[THEN iffD1, of [y] map f {xs. set xs ⊆ A}]
  obtain x' where
    x' ∈ {xs. set xs ⊆ A}
    [y] = map f x'
    by meson
  then show ∃x∈A. y = f x
    by auto
next
qed(clarsimp)
```

Obtaining the inverse over map

```
lemma bij-betw-inv-map:
  assumes bij-betw f A B
  shows ∃g. bij-betw (map g) {xs. set xs ⊆ B} {xs. set xs ⊆ A} ∧
            inverses-on (map f) (map g) {xs. set xs ⊆ A} {xs. set xs ⊆ B}
proof −
  from bij-betw-inv-alt[OF assms, simplified inverses-on-def]
  obtain g where
```

```
    bij-betw g B A
    ∀x∈A. g (f x) = x
    ∀x∈B. f (g x) = x
    by blast

  note bij-betw-map[OF ‹bij-betw g B A›]
  moreover
  {
    have ∀x. set x ⊆ A ⟶ map g (map f x) = x
    proof safe
      fix x
      assume set x ⊆ A
      then show map g (map f x) = x
        by (clarsimp simp: list-eq-iff-nth-eq ‹∀x∈A. g (f x) = x› subsetD)
    qed
    moreover
    have ∀x. set x ⊆ B ⟶ map f (map g x) = x
    proof safe
      fix x
      assume set x ⊆ B
      then show map f (map g x) = x
        by (clarsimp simp: list-eq-iff-nth-eq ‹∀x∈B. f (g x) = x› subsetD)
    qed
    ultimately have
      inverses-on (map f) (map g) {xs. set xs ⊆ A} {xs. set xs ⊆ B}
      by (simp add: inverses-on-def)
  }
  ultimately show ?thesis
    by blast
qed
```

10 Lemmas about monotone functions

Note that the base version of monotone is used as the sorts cause some issues with the types

Essentially a general version of *strict-mono-on UNIV ?f ⟹ (?f ?x < ?f ?y) = (?x < ?y)*

```
lemma monotone-on-iff:
  assumes monotone-on A orda ordb f
  and     asymp-on A orda
  and     totalp-on A orda
  and     asymp-on (f ‘ A) ordb
  and     totalp-on (f ‘ A) ordb
  and     x ∈ A
  and     y ∈ A
shows orda x y ⟷ ordb (f x) (f y)
proof (safe)
```

show *orda x y* $\Longrightarrow$ *ordb* (*f x*) (*f y*)
by (*meson assms monotone-onD*)
next
show *ordb* (*f x*) (*f y*) $\Longrightarrow$ *orda x y*
by (*metis* (*full-types*) *assms*(*1*,*3*,*4*,*6*,*7*)
asymp-onD monotone-onD totalp-onD imageI)
qed

The inverse of a monotonic function is also monotonic

lemma *monotone-on-bij-betw-inv*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*
and *asymp-on B ordb*
and *totalp-on B ordb*
and *bij-betw f A B*
and *bij-betw g B A*
and *inverses-on f g A B*
shows *monotone-on B ordb orda g*
proof (*rule monotone-onI*)
fix *x y*
assume $x \in B$ $y \in B$ *ordb x y*
moreover
have $g\ x \in A$
using ‹*bij-betw g B A*› *bij-betwE calculation*(*1*) **by** *auto*
moreover
have *f* (*g x*) = *x*
using *assms*(*8*) *calculation*(*1*) *inverses-onD2* **by** *blast*
moreover
have $g\ y \in A$
using ‹*bij-betw g B A*› *bij-betwE calculation*(*2*) **by** *auto*
moreover
have *f* (*g y*) = *y*
using *assms*(*8*) *calculation*(*2*) *inverses-onD2* **by** *blast*
ultimately show *orda* (*g x*) (*g y*)
using *assms bij-betw-imp-surj-on monotone-on-iff* **by** *fastforce*
qed

lemma *monotone-on-bij-betw*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*
and *asymp-on B ordb*
and *totalp-on B ordb*
and *bij-betw f A B*
shows $\exists g.$ *bij-betw g B A* $\wedge$ *inverses-on f g A B* $\wedge$ *monotone-on B ordb orda g*
using *assms bij-betw-inv-alt monotone-on-bij-betw-inv* **by** *fastforce*

11 Sorting

11.1 General sorting

Intro for *sorted-wrt*

lemmas *sorted-wrtI* = *sorted-wrt-iff-nth-less*[*THEN iffD2*, *OF allI*, *OF allI*, *OF impI*, *OF impI*]

lemma *sorted-wrt-mapI*:
(⋀*i j*. ⟦*i* < *j*; *j* < *length xs*⟧ ⟹ *P* (*f* (*xs* ! *i*)) (*f* (*xs* ! *j*))) ⟹
sorted-wrt P (*map f xs*)
by (*metis* (*mono-tags*, *lifting*) *length-map nth-map order-less-trans sorted-wrtI*)

lemma *sorted-wrt-mapD*:
(⋀*i j*. ⟦*sorted-wrt P* (*map f xs*); *i* < *j*; *j* < *length xs*⟧ ⟹ *P* (*f* (*xs* ! *i*)) (*f* (*xs* ! *j*)))
by (*metis* (*mono-tags*, *lifting*) *length-map nth-map order-less-trans sorted-wrt-iff-nth-less*)

lemma *monotone-on-sorted-wrt-map*:
assumes *monotone-on A orda ordb f*
and *sorted-wrt orda xs*
and *set xs* ⊆ *A*
shows *sorted-wrt ordb* (*map f xs*)
proof (*rule sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-nth-less*[*OF assms*(*2*)]
have *orda* (*xs* ! *i*) (*xs* ! *j*)
by *blast*
with *monotone-onD*[*OF assms*(*1*), *of xs* ! *i xs* ! *j*] *assms*(*3*)
show *ordb* (*f* (*xs* ! *i*)) (*f* (*xs* ! *j*))
by (*meson* ‹*i* < *j*› ‹*j* < *length xs*› *nth-mem order-less-trans subsetD*)
qed

lemma *monotone-on-map-sorted-wrt*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*
and *asymp-on* (*f* ‘ *A*) *ordb*
and *totalp-on* (*f* ‘ *A*) *ordb*
and *sorted-wrt ordb* (*map f xs*)
and *set xs* ⊆ *A*
shows *sorted-wrt orda xs*
proof (*rule sorted-wrtI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-nth-less*[*OF assms*(*6*)]
have *ordb* (*f* (*xs* ! *i*)) (*f* (*xs* ! *j*))
by *force*

with *monotone-on-iff*[*OF assms*(1−3), *of xs* ! *i xs* ! *j*]
show *orda* (*xs* ! *i*) (*xs* ! *j*)
using ‹*i* < *j*› ‹*j* < *length xs*› *assms*(4,5,7)
nth-mem order-less-trans **by** *blast*
qed

11.2 Sorting on linear orders

context *linorder* **begin**

abbreviation *strict-sorted xs* ≡ *sorted-wrt* (<) *xs*

lemma *sorted-nth-less-mono*:
⟦*sorted xs*; *i* < *length xs*; *j* < *length xs*; *i* ≠ *j*; *xs* ! *i* < *xs* ! *j*⟧ ⟹ *i* < *j*
by (*meson linorder-neqE-nat not-less sorted-iff-nth-mono-less*)

lemma *strict-sorted-nth-less-mono*:
⟦*strict-sorted xs*; *i* < *length xs*; *j* < *length xs*; *i* ≠ *j*; *xs* ! *i* < *xs* ! *j*⟧ ⟹ *i* < *j*
using *strict-sorted-iff sorted-nth-less-mono* **by** *blast*

lemma *strict-sorted-Min*:
⟦*strict-sorted xs*; *xs* ≠ []⟧ ⟹ *xs* ! *0* = *Min* (*set xs*)
by (*metis finite-set list.simps*(15) *Min-insert2 strict-sorted-imp-sorted*
nth-Cons-0 sorted-wrt.elims(2))

lemma *strict-sorted-take*:
assumes *strict-sorted xs*
and *i* < *length xs*
shows *set* (*take i xs*) = {*x*. *x* ∈ *set xs* ∧ *x* < *xs* ! *i*}
proof (*safe*)
fix *x*
assume *x* ∈ *set* (*take i xs*)
then show *x* ∈ *set xs*
by (*meson in-set-takeD*)
next
fix *x*
assume *x* ∈ *set* (*take i xs*)
then show *x* < *xs* ! *i*
by (*metis assms id-take-nth-drop list.set-intros*(1) *sorted-wrt-append*)
next
fix *x*
assume *x* ∈ *set xs x* < *xs* ! *i*
hence ∃*j* < *length xs*. *xs* ! *j* = *x*
by (*simp add*: *in-set-conv-nth*)
then obtain *j* **where**
j < *length xs xs* ! *j* = *x*
by *blast*
with *strict-sorted-nth-less-mono*[*OF assms*(1) - *assms*(2), *of j*] ‹*x* < *xs* ! *i*›
have *j* < *i*

by *blast*
then show $x \in set\ (take\ i\ xs)$
using ‹$j < length\ xs$› ‹$xs\ !\ j = x$› *in-set-conv-nth* **by** *fastforce*
qed

lemma *strict-sorted-card-idx*:
⟦*strict-sorted xs*; $i < length\ xs$⟧ ⟹ $card\ \{x.\ x \in set\ xs \wedge x < xs\ !\ i\} = i$
by (*metis* (*mono-tags*, *lifting*) *distinct-card distinct-take length-take strict-sorted-iff*
ord-class.min-def order-class.leD strict-sorted-take)

lemmas *strict-sorted-distinct-set-unique* =
sorted-distinct-set-unique[*OF strict-sorted-imp-sorted - strict-sorted-imp-sorted*]

lemma *sorted-and-distinct-imp-strict-sorted*:
⟦*sorted xs*; *distinct xs*⟧ ⟹ *strict-sorted xs*
using *strict-sorted-iff*
by *blast*

lemma *filter-sorted*:
sorted xs ⟹ *sorted* (*filter P xs*)
using *sorted-wrt-filter* **by** *blast*

lemma *sorted-nth-eq*:
assumes *sorted xs*
and $j < length\ xs$
and $xs\ !\ i = xs\ !\ j$
and $i \leq k$
and $k \leq j$
shows $xs\ !\ k = xs\ !\ i$
by (*metis assms sorted-iff-nth-mono preorder-class.le-less-trans nle-le*)

lemma *sorted-find-Min*:
sorted xs ⟹ $\exists\, x \in set\ xs.\ P\ x$ ⟹ *List.find P xs* = *Some* (*Min* $\{x \in set\ xs.\ P\ x\}$)
proof (*induct xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **show** *?case*
proof (*cases P x*)
case *True*
with *Cons* **show** *?thesis* **by** (*auto intro*: *Min-eqI* [*symmetric*])
next
case *False* **then have** $\{y.\ (y = x \vee y \in set\ xs) \wedge P\ y\} = \{y \in set\ xs.\ P\ y\}$
by *auto*
with *Cons False* **show** *?thesis* **by** (*simp-all*)
qed
qed

lemma *sorted-cons-nil*:
$xs = []$ ⟹ *sorted* (x # xs)

by *simp*

lemma *sorted-consI*:
⟦*xs* ≠ []; *sorted xs*; *x* ≤ *xs* ! *0*⟧ ⟹ *sorted* (*x* # *xs*)
by (*metis list.exhaust sorted2-simps*(*2*) *nth-Cons-0*)

end

11.3 Sorting on orders

context *order* **begin**

lemma *strict-mono-strict-sorted-map-1*:
assumes *strict-mono* α
and *strict-sorted xs*
shows *strict-sorted* (*map* α *xs*)
using *assms*(*1*) *assms*(*2*) *monotone-on-sorted-wrt-map* **by** *blast*

lemma *strict-mono-sorted-map-2*:
assumes *strict-mono* α
and *strict-sorted* (*map* α *xs*)
shows *strict-sorted xs*
using *assms*(*1*) *assms*(*2*) *monotone-on-map-sorted-wrt* **by** *fastforce*

end

12 Mapping elements to natural numbers

This section contains a mapping from elements to natural numbers that maintains ordering.

definition *elm-rank* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a set* ⇒ '*a* ⇒ *nat*
where
elm-rank ord A x = *card* {*y*. *y* ∈ *A* ∧ *ord y x*}

lemma *monotone-on-elm-rank*:
assumes *finite A*
and *transp-on A ord*
and *irreflp-on A ord*
shows *monotone-on A ord* (<) (*elm-rank ord A*)
proof (*rule monotone-onI*)
fix *a b*
assume *a* ∈ *A b* ∈ *A ord a b*

have {*y*. *y* ∈ *A* ∧ *ord y a*} ⊆ {*y*. *y* ∈ *A* ∧ *ord y b*}
proof *safe*
fix *x*
assume *x* ∈ *A ord x a*
then show *ord x b*

```
    by (meson ‹a ∈ A› ‹b ∈ A› ‹ord a b› assms(2) transp-onD)
  qed
  moreover
  have a ∈ {y. y ∈ A ∧ ord y b}
    by (simp add: ‹a ∈ A› ‹ord a b›)
  moreover
  have a ∉ {y. y ∈ A ∧ ord y a}
    using assms(3) irreflp-onD by fastforce
  ultimately have {y. y ∈ A ∧ ord y a} ⊂ {y. y ∈ A ∧ ord y b}
    by blast
  then show elm-rank ord A a < elm-rank ord A b
    by (simp add: elm-rank-def assms(1) psubset-card-mono)
qed

lemma elm-rank-insert-min:
  assumes finite A
  and     x ∉ A
  and     ∀ y ∈ A. ord x y
  and     z ∈ A
shows elm-rank ord (insert x A) z = Suc (elm-rank ord A z)
  unfolding elm-rank-def
proof −
  have ord x z
    by (simp add: assms(3,4))
  hence {y ∈ insert x A. ord y z} = insert x {y ∈ A. ord y z}
    by safe
  moreover
  have x ∉ {y ∈ A. ord y z}
    using assms(2) by auto
  ultimately show card {y ∈ insert x A. ord y z} = Suc (card {y ∈ A. ord y z})
    by (simp add: assms(1))
qed

definition (in order) elem-rank :: 'a set ⇒ 'a ⇒ nat
  where
elem-rank = elm-rank (<)

lemma (in order) strict-mono-on-elem-rank:
  assumes finite A
  shows strict-mono-on A (elem-rank A)
  by (simp add: assms elem-rank-def monotone-on-elm-rank)

lemma (in linorder) bij-betw-elem-rank-upt:
  assumes finite A
  shows bij-betw (elem-rank A) A {0..<card A}
proof −
  have ⋀x. x ∈ A ⟹ {y. y ∈ A ∧ y < x} ⊂ A
    by blast
  hence ⋀x. x ∈ A ⟹ card {y. y ∈ A ∧ y < x} < card A
```

by (*meson assms psubset-card-mono*)
hence $\bigwedge x.\ x \in A \Longrightarrow$ *elem-rank A x* $\in \{0..<$*card A*$\}$
by (*simp add*: *elm-rank-def elem-rank-def*)
moreover
have $\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow$ (*elem-rank A x* = *elem-rank A y*) = (*x* = *y*)
by (*metis assms less-irrefl-nat antisym-conv3 ord.strict-mono-onD strict-mono-on-elem-rank*)
moreover
{
let *?xs* = *sorted-list-of-set A*
have *strict-sorted ?xs*
by *simp*
moreover
have $\bigwedge x.$ *elem-rank A x* = *elem-rank* (*set ?xs*) *x*
using *assms* **by** *force*
moreover
have $\bigwedge y.\ y <$ *length ?xs* $\Longrightarrow$ *y* = *elem-rank* (*set ?xs*) (*?xs* ! *y*)
by (*metis* (*no-types*, *lifting*) *Collect-cong calculation*(*1*) *elem-rank-def elm-rank-def strict-sorted-card-idx*)
ultimately have $\bigwedge y.\ y \in \{0..<$*card A*$\} \Longrightarrow \exists\, x{\in}A.\ y =$ *elem-rank A x*
by (*metis assms length-sorted-list-of-set set-sorted-list-of-set nth-mem ord-class.atLeastLessThan-iff*)
}
ultimately show *?thesis*
using *bij-betwI'* **by** *blast*
qed

lemma (**in** *order*) *elem-rank-insert-min*:
$[\![$*finite A*; *x* $\notin$ *A*; $\forall\, y{\in}A.\ x < y$; *z* $\in$ *A*$]\!] \Longrightarrow$ *elem-rank* (*insert x A*) *z* = *Suc* (*elem-rank A z*)
by (*simp add*: *elm-rank-insert-min elem-rank-def*)
end
theory *Repeat*
imports *Main*
begin

13 Repeat Function At Most N Times

fun *repeatatm* :: *nat* $\Rightarrow$ (*'a* $\Rightarrow$ *'b* $\Rightarrow$ *bool*) $\Rightarrow$ (*'a* $\Rightarrow$ *'b* $\Rightarrow$ *'a*) $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'a*
where
repeatatm 0 - - acc - = acc |
repeatatm (*Suc n*) *f g acc obsv* = (*if f acc obsv then acc else repeatatm n f g* (*g acc obsv*) *obsv*)

declare *repeatatm.simps*[*simp del*]

13.1 Step and early termination lemmas

lemma *repeatatm-step-stop-Suc*:
f (*repeatatm n f g a b*) *b*

```
    ⟹ repeatatm (Suc n) f g a b = repeatatm n f g a b
proof (induct n arbitrary: a)
  case 0
  then show ?case
    by (simp add: repeatatm.simps)
next
  case (Suc n a)
  note IH = this
  show ?case
  proof (cases f a b)
    assume f a b
    then show ?case
      by (simp add: repeatatm.simps)
  next
    assume ¬ f a b
    with IH(2)
    have f (repeatatm n f g (g a b) b) b
      by (simp add: repeatatm.simps)
    with IH(1)
    have repeatatm (Suc n) f g (g a b) b = repeatatm n f g (g a b) b
      by blast
    then show ?case
      by (simp add: repeatatm.simps)
  qed
qed

lemma repeatatm-step:
  ¬f (repeatatm n f g a b) b
    ⟹ repeatatm (Suc n) f g a b = g (repeatatm n f g a b) b
proof (induct n arbitrary: a)
  case 0
  then show ?case
    by (simp add: repeatatm.simps)
next
  case (Suc n a)
  note IH = this
  show ?case
  proof (cases f a b)
    assume f a b
    with IH(2)
    show ?case
      by (simp add: repeatatm.simps)
  next
    assume ¬ f a b
    with IH(2)
    have ¬ f (repeatatm n f g (g a b) b) b
      by (simp add: repeatatm.simps)
    with IH(1)
    have repeatatm (Suc n) f g (g a b) b = g (repeatatm n f g (g a b) b) b
```

```
    by blast
  then show ?case
    by (simp add: repeatatm.simps ‹¬ f a b›)
 qed
qed

lemma repeatatm-step-forward:
 ¬f a b ⟹ repeatatm (Suc n) f g a b = repeatatm n f g (g a b) b
 by (induct n arbitrary: a; simp add: repeatatm.simps)

lemma repeatatm-stop-Suc:
 ⟦f (repeatatm n f g a b) b⟧ ⟹ f (repeatatm (Suc n) f g a b) b
proof (induct n arbitrary: a)
 case 0
 then show ?case
  by (simp add: repeatatm.simps)
next
 case (Suc n a)
 note IH = this
 show ?case
 proof (cases f a b)
  assume f a b
  then show ?case
   by (simp add: repeatatm.simps)
 next
  assume ¬ f a b
  with IH(2)
  have f (repeatatm n f g (g a b) b) b
   by (simp add: repeatatm.simps)
  with IH(1)
  have f (repeatatm (Suc n) f g (g a b) b) b
   by blast
  then show ?case
   by (simp add: repeatatm.simps)
 qed
qed

lemma repeatatm-stop:
 ⟦f (repeatatm n f g a b) b; n ≤ m⟧ ⟹ f (repeatatm m f g a b) b
proof (induct m arbitrary: a n)
 case 0
 then show ?case
  by simp
next
 case (Suc m a n)
 note IH = this
 show ?case
 proof (cases n ≤ m)
  assume n ≤ m
```

```
    with repeatatm-stop-Suc[of f m g a b] IH(1)[OF IH(2)]
    show ?case
      by simp
  next
    assume ¬ n ≤ m
    with IH(2,3)
    show ?case
      using le-Suc-eq by blast
  qed
qed


lemma repeatatm-step-stop:
  ⟦f (repeatatm n f g a b) b; n ≤ m⟧ ⟹ repeatatm m f g a b = repeatatm n f g a b
proof (induct m arbitrary: a n)
  case 0
  then show ?case
    by simp
next
  case (Suc m a n)
  note IH = this
  show ?case
  proof (cases n ≤ m)
    assume n ≤ m
    with repeatatm-step-stop-Suc[of f m g a b] IH(2) IH(1)[OF IH(2)]
    show ?case
      by simp
  next
    assume ¬ n ≤ m
    then show ?case
      using Suc.prems(2) le-SucE by blast
  qed
qed

lemma repeatatm-not-stop-Suc:
  ¬f (repeatatm (Suc n) f g a b) b ⟹ ¬f (repeatatm n f g a b) b
  apply (rule contrapos-nn[where Q = f (repeatatm (Suc n) f g a b) b]; simp)
  using repeatatm-stop-Suc[of f n g a b] by simp

lemma repeatatm-maintain-inv:
  assumes ⋀a. P a ⟹ P (g a b)
  shows P a ⟹ P (repeatatm n f g a b)
proof (induct n arbitrary: a)
  case 0
  then show ?case
    by (simp add: repeatatm.simps)
next
  case (Suc n a)
  note IH = this
```

from *IH*(*1*)[*OF IH*(*2*)]
have *P* (*repeatatm n f g a b*)
by *assumption*

with ‹⋀*a*. *P a* ⟹ *P* (*g a b*)›
show *?case*
by (*metis repeatatm-step repeatatm-step-stop-Suc*)
qed

14 Repeat Function N Times

definition *repeat* :: *nat* ⇒ (*'a* ⇒ *'b* ⇒ *'a*) ⇒ *'a* ⇒ *'b* ⇒ *'a*
where
repeat n f a b = *repeatatm n* (λ*x y*. *False*) *f a b*

lemma *repeat-0*:
repeat 0 f a b = *a*
by (*simp add*: *repeat-def repeatatm.simps*(*1*))

lemma *repeat-step*:
repeat (*Suc n*) *f a b* = *f* (*repeat n f a b*) *b*
unfolding *repeat-def*
by (*simp add*: *repeatatm-step*)

lemma *repeat-step-forward*:
repeat (*Suc n*) *f a b* = *repeat n f* (*f a b*) *b*
unfolding *repeat-def*
by (*simp add*: *repeatatm-step-forward*)

lemma *repeat-maintain-inv*:
assumes ⋀*a*. *P a* ⟹ *P* (*f a b*)
shows *P a* ⟹ *P* (*repeat n f a b*)
by (*simp add*: *assms repeat-def repeatatm-maintain-inv*)

lemma *repeat-eq-fold*:
repeat n f a b = *fold* (λ- *a*. *f a b*) [*0*..<*n*] *a*
apply (*induct n*)
apply (*simp add*: *repeat-def repeatatm.simps*)
apply (*subst repeat-step*)
apply *simp*
done

end
theory *Continuous-Interval*
imports *Main*
begin

15 Continuous Intervals

definition
continuous-list :: (*nat* × *nat*) *list* ⇒ *bool*
where
continuous-list xs =
(∀ *i*. *Suc i* < *length xs* ⟶ *fst* (*xs* ! *Suc i*) = *snd* (*xs* ! *i*))

lemma *continuous-list-nil*:
continuous-list []
by (*simp add*: *continuous-list-def*)

lemma *continuous-list-singleton*:
continuous-list [*x*]
by (*simp add*: *continuous-list-def*)

lemma *continuous-list-cons*:
continuous-list (*x* # *xs*) ⟹ *continuous-list xs*
by (*simp add*: *continuous-list-def*)

lemma *continuous-list-app*:
continuous-list (*xs* @ *ys*) ⟹ *continuous-list xs* ∧ *continuous-list ys*
proof (*induct xs*)
case *Nil*
then show *?case*
by (*clarsimp simp*: *continuous-list-nil continuous-list-cons*)
next
case (*Cons x1 xs*)
note *IH1* = *this*(*1*) **and**
IH2 = *this*(*2*)

from *continuous-list-cons IH2*
have *continuous-list* (*xs* @ *ys*)
by *simp*
with *IH1*
have *continuous-list ys*
by *simp*

have *xs* = [] ∨ *xs* ≠ []
by *simp*
then show *?case*
proof
assume *xs* = []
with *IH2 continuous-list-singleton*
have *continuous-list* (*x1* # *xs*)
by *blast*
with ‹*continuous-list ys*›
show *?thesis*
by *simp*

```
  next
    assume xs ≠ []
    hence ∃x xs'. xs = x # xs'
      using neq-Nil-conv by blast
    then obtain x xs' where
      xs = x # xs'
      by blast

    have continuous-list (x1 # (x # xs'))
      unfolding continuous-list-def
    proof(intro allI impI)
      fix i
      assume Suc i < length (x1 # (x # xs'))
      have i = 0 ∨ i ≠ 0
        by blast
      then show fst ((x1 # x # xs') ! Suc i) = snd ((x1 # x # xs') ! i)
      proof
        assume i = 0
        then show ?thesis
          using IH2 ‹xs = x # xs'› continuous-list-def by auto
      next
        assume i ≠ 0
        then show ?thesis
          using IH1 ‹Suc i < length (x1 # x # xs')› ‹continuous-list (xs @ ys)› ‹xs
= x # xs'›
                continuous-list-def
          by auto
      qed
    qed
    with ‹continuous-list ys› ‹xs = x # xs'›
    show ?thesis
      by simp
  qed
qed

lemma continuous-list-interval-1:
  assumes continuous-list xs
  and     xs ≠ []
  and     fst (hd xs) ≤ i
  and     i < snd (last xs)
  shows ∃j < length xs. fst (xs ! j) ≤ i ∧ i < snd (xs ! j)
  using assms
proof (induct xs)
  case Nil
  then show ?case
    by simp
next
  case (Cons x1 xs)
  note IH = this
```

```
have xs = [] ⟶ ?case
proof
  assume xs = []
  with IH(4,5)
  show ?case
    by simp
qed

have xs ≠ [] ⟶ ?case
proof
  assume xs ≠ []
  hence ∃ a b xs'. xs = (a, b) # xs'
    by (metis (full-types) list.exhaust surj-pair)
  then obtain a b xs' where
    xs = (a, b) # xs'
    by blast

  from IH(2)
  have continuous-list xs
    using continuous-list-cons by blast

  from IH(5) ‹xs ≠ []›
  have i < snd (last xs)
    by simp

  have a ≤ i ∨ i < a
    by linarith
  then show ?case
  proof
    assume a ≤ i
    with IH(1)
         [OF ‹continuous-list xs›
             ‹xs ≠ []› -
             ‹i < snd (last xs)›]
        ‹xs = (a, b) # xs'›
    show ?thesis
      by auto
  next
    assume i < a
    with IH(2) ‹xs = (a, b) # xs'›
    have i < snd x1
      using continuous-list-def by auto
    with IH(4)
    show ?thesis
      by auto
  qed
qed
```

```
  with ‹xs = [] ⟶ ?case›
  show ?case
    by blast
qed

lemma continuous-list-interval-2:
  assumes continuous-list xs
  and     length xs = Suc n
  and     fst (xs ! 0) ≤ i
  and     i < snd (xs ! n)
  shows ∃j < length xs. fst (xs ! j) ≤ i ∧ i < snd (xs ! j)
proof −
  from ‹length xs = Suc n›
  have xs ≠ []
    by auto

  from ‹fst (xs ! 0) ≤ i› ‹xs ≠ []›
  have fst (hd xs) ≤ i
    by (simp add: hd-conv-nth)

  from ‹i < snd (xs ! n)›
       ‹length xs = Suc n› ‹xs ≠ []›
  have i < snd (last xs)
    by (simp add: last-conv-nth)

  from continuous-list-interval-1
         [OF ‹continuous-list xs›
             ‹xs ≠ []›
             ‹fst (hd xs) ≤ i›
             ‹i < snd (last xs)›]
  show ?thesis
    by assumption
qed

end
theory List-Slice
  imports Main
begin
```

16 List Slices

```
fun list-slice ::
  'a list ⇒ nat ⇒ nat ⇒ 'a list
where
  list-slice xs i j = drop i (take j xs)

lemma length-list-slice[simp add]:
  length (list-slice xs i j) = (min j (length xs)) − i
  by simp
```

```
lemma list-slice-cons:
  fixes i j :: nat
  assumes i ≤ j
  assumes i > 0
  shows list-slice (x # xs) i j = list-slice xs (i − 1) (j − 1)
  using assms gr0-implies-Suc[OF order.strict-trans2]
  by (fastforce dest: gr0-implies-Suc)

lemma list-slice-append:
  fixes i j k :: nat
  assumes i ≤ j
  assumes j ≤ k
  shows list-slice xs i k = list-slice xs i j @ list-slice xs j k
using assms
proof (induct xs arbitrary: i j k)
  case (Cons a xs i j k)
  note IH = this
  show ?case
  proof (cases i > 0)
    assume ¬ i > 0
    hence i = 0
      by simp
    then show ?case
      unfolding list-slice.simps
      by (metis IH(3) append-take-drop-id drop0 min-def take-take)
  next
    assume i > 0
    with list-slice-cons[of i k a xs]
    have list-slice (a # xs) i k = list-slice xs (i − 1) (k − 1)
      using IH(2) IH(3) dual-order.trans by blast
    moreover
    from list-slice-cons[of i j a xs]
    have list-slice (a # xs) i j = list-slice xs (i − 1) (j − 1)
      using IH(2) ‹i > 0› by blast
    moreover
    from list-slice-cons[of j k a xs]
    have list-slice (a # xs) j k = list-slice xs (j − 1) (k − 1)
      using IH(2) IH(3) ‹0 < i› by blast
    moreover
    from IH(1)[of i−1 j−1 k−1]
    have list-slice xs (i − 1) (k − 1) =
          list-slice xs (i − 1) (j − 1) @ list-slice xs (j − 1) (k − 1)
      using IH(2) IH(3) diff-le-mono by blast
    ultimately show ?case
      by presburger
  qed
qed simp
```

lemma *list-slice-0-length*:
fixes *xs* :: *'a list*
fixes *n* :: *nat*
assumes *length xs* ≤ *n*
shows *list-slice xs 0 n* = *xs*
using *assms* **by** *simp*

lemma *list-slice-n-n*[*simp add*]:
fixes *xs* :: *'a list*
fixes *n* :: *nat*
shows *list-slice xs n n* = []
by *simp*

lemma *list-slice-nth*:
fixes *i s e* :: *nat*
fixes *xs* :: *'a list*
assumes *i* < *length xs*
assumes *s* ≤ *i*
assumes *i* < *e*
shows (*list-slice xs s e*) ! (*i* − *s*) = *xs* ! *i*
using *assms* **by** *simp*

lemma *list-slice-start-gre-length*:
fixes *xs* :: *'a list*
fixes *s* :: *nat*
assumes *length xs* ≤ *s*
shows *list-slice xs s e* = []
using *assms* **by** *simp*

lemma *list-slice-end-gre-length*:
fixes *xs* :: *'a list*
fixes *e* :: *nat*
assumes *length xs* ≤ *e*
shows *list-slice xs s e* = *list-slice xs s* (*length xs*)
using *assms* **by** *simp*

lemma *fold-list-slice*:
fixes *i j* :: *nat*
fixes *B* :: *nat list*
assumes *i* ≤ *j*
and *j* < *length B*
and *sorted B*
fixes *T zs* :: *'a list*
shows
fold (λ*x xs*. *xs* @ *list-slice T* (*B* ! *x*) (*B* ! *Suc x*)) [*i*..<*j*] *zs*
= *zs* @ (*list-slice T* (*B* ! *i*) (*B* ! *j*))
using *assms*
proof (*induct j arbitrary*: *i*)
case *0*

```
  then show ?case
    by (simp del: list-slice.simps)
next
  case (Suc j i)
  note IH = this
  show ?case
  proof (cases i ≤ j)
    assume i-leq-j: i ≤ j

    have fold (λx xs. xs @ list-slice T (B ! x) (B ! Suc x)) [i..<Suc j] zs =
          fold (λx xs. xs @ list-slice T (B ! x) (B ! Suc x)) [i..<j] zs @
            list-slice T (B ! j) (B ! Suc j)
      using ‹i ≤ j› by force
    moreover
    from IH(1)[OF ‹i ≤ j› - IH(4)]
    have fold (λx xs. xs @ list-slice T (B ! x) (B ! Suc x)) [i..<j] zs =
          zs @ list-slice T (B ! i) (B ! j)
      using Suc.prems(2) Suc-lessD by blast
    moreover
    have list-slice T (B ! i) (B ! Suc j) = list-slice T (B ! i) (B ! j) @
          list-slice T (B ! j) (B ! Suc j)
      by (meson Suc.prems(2,3) Suc-lessD i-leq-j list-slice-append
                sorted-iff-nth-Suc sorted-nth-mono)
    ultimately show ?case
      by (metis append.assoc)
  next
    assume ¬ i ≤ j
    then show ?case
      using Suc.prems(1) le-SucE by fastforce
  qed
qed

lemma nth-list-slice:
  fixes i s e :: nat
  fixes xs :: 'a list
  assumes i < length (list-slice xs s e)
  shows (list-slice xs s e) ! i = xs ! (s + i)
  using assms less-diff-conv by auto

lemma list-slice-nth-eq-iff-index-eq:
  fixes i s e j :: nat
  fixes xs :: 'a list
  assumes distinct (list-slice xs s e)
  assumes e ≤ length xs
  assumes s ≤ i and i < e
  and     s ≤ j and j < e
  shows   (xs ! i = xs ! j) ⟷ (i = j)
  using assms
  by (fastforce
```

dest: *nth-eq-iff-index-eq*[**where** $i = i - s$ **and** $j = j - s$])

lemma *distinct-list-slice*:
fixes $i\ j$ :: *nat*
fixes *xs* :: *'a list*
assumes *distinct xs*
shows *distinct* (*list-slice xs i j*)
using *assms* **by** *simp*

lemma *list-slice-nth-mem*:
fixes *e* :: *nat*
fixes *xs* :: *'a list*
fixes *s i* :: *nat*
assumes $s \leq i$ **and** $i < e$
assumes $e \leq$ *length xs*
shows *xs* ! $i \in$ *set* (*list-slice xs s e*)
by (*metis* (*no-types*, *opaque-lifting*) *assms nat-le-iff-add*
add-diff-cancel-left' diff-less-mono nth-mem
length-list-slice min-def nth-list-slice)

lemma *nth-mem-list-slice*:
fixes *x* :: *'a*
fixes *xs* :: *'a list*
fixes *s e* :: *nat*
assumes *x* ∈ *set* (*list-slice xs s e*)
shows $\exists\, i <$ *length xs*.
$s \leq i \wedge$
$i < e \wedge$
xs ! $i = x$
proof −
from *in-set-conv-nth*[*THEN iffD1*, *OF* ‹- ∈ -›]
obtain *i* **where**
$i <$ *length* (*list-slice xs s e*)
(*list-slice xs s e*) ! $i = x$
by *auto*
with *nth-list-slice*[*of i xs s e*]
have *xs* ! $(s + i) = x$
by *auto*
moreover
have $s \leq s + i$
by *linarith*
moreover
have $s + i <$ *length xs*
using ‹$i <$ *length* (*list-slice xs s e*)› **by** *auto*
moreover
have $s + i < e$
using ‹$i <$ *length* (*list-slice xs s e*)› **by** *auto*
ultimately show *?thesis*
by *blast*

qed

lemma *list-slice-subset*:
fixes *i j* :: *nat*
fixes *xs* :: *'a list*
shows *set* (*list-slice xs i j*) ⊆ *set xs*
using *set-drop-subset set-take-subset* **by** *force*

lemma *list-slice-Suc*:
fixes *i j* :: *nat*
fixes *xs* :: *'a list*
assumes *i* < *length xs*
assumes *i* < *j*
shows *list-slice xs i j* = *xs* ! *i* # *list-slice xs* (*Suc i*) *j*
by (*metis assms Cons-nth-drop-Suc Suc-diff-Suc*
list-slice.simps take-Suc-Cons drop-take)

lemma *list-slice-update-unchanged-1*:
fixes *xs* :: *'a list*
fixes *i j k* :: *nat*
assumes *i* < *j*
shows *list-slice* (*xs*[*i* := *x*]) *j k* = *list-slice xs j k*
by (*simp add*: *assms drop-take*)

lemma *list-slice-update-unchanged-2*:
fixes *i j k* :: *nat*
fixes *xs* :: *'a list*
assumes *k* ≤ *i*
shows *list-slice* (*xs*[*i* := *x*]) *j k* = *list-slice xs j k*
by (*metis assms list-slice.simps take-update-cancel*)

lemma *list-slice-update-changed*:
assumes *i* < *length xs*
assumes *j* ≤ *i*
assumes *i* < *k*
shows *list-slice* (*xs*[*i* := *x*]) *j k* = (*list-slice xs j k*)[*i* − *j* := *x*]
using *assms*
by (*fastforce*
intro: *list-eq-iff-nth-eq*[*THEN iffD2*]
simp: *nth-list-slice nth-list-update*)

lemma *list-slice-map-nth-upt*:
assumes *j* < *length xs*
shows *list-slice xs i j* = *map* (*nth xs*) [*i*..<*j*]
using *assms*
by (*fastforce intro*: *list-eq-iff-nth-eq*[*THEN iffD2*])

lemma *map-list-slice*:

map f (list-slice xs i j) = list-slice (map f xs) i j
by (*simp add*: *drop-map take-map*)

17 Sorted List Slice

lemma (**in** *linorder*) *sorted-list-slice*:
assumes *sorted xs*
shows *sorted* (*list-slice xs i j*)
by (*simp add*: *assms sorted-wrt-drop sorted-wrt-take*)

lemma (**in** *linorder*) *sorted-map-list-slice*:
assumes *sorted* (*map f xs*)
shows *sorted* (*map f* (*list-slice xs i j*))
by (*metis assms drop-map list-slice.simps local.sorted-list-slice take-map*)

lemma (**in** *linorder*) *sorted-map-filter-list-slice*:
assumes *sorted* (*map f* (*filter P xs*))
shows *sorted* (*map f* (*filter P* (*list-slice xs i j*)))
proof −
have $i \leq j \vee j < i$
using *linorder-class.le-less-linear* **by** *blast*
moreover
have $j < i \Longrightarrow$ *?thesis*
proof −
assume $j < i$
hence *list-slice xs i j* = []
by (*simp add*: *drop-take*)
then show *?thesis*
by *simp*
qed
moreover
have $i \leq j \Longrightarrow$ *?thesis*
proof −
let *?as* = *list-slice xs 0 i* **and**
?bs = *list-slice xs i j* **and**
?cs = *list-slice xs j* (*length xs*)
assume $i \leq j$
hence *xs* = *?as* @ *?bs* @ *?cs*
by (*metis le0 linorder-class.linear list-slice-0-length list-slice-append*
list-slice-start-gre-length)
hence *filter P xs* = *filter P ?as* @ *filter P ?bs* @ *filter P ?cs*
by (*metis filter-append*)
hence *map f* (*filter P xs*)
= (*map f* (*filter P ?as*)) @ (*map f* (*filter P ?bs*)) @ (*map f* (*filter P ?cs*))
by *simp*
with ‹*sorted* (*map f* (*filter P xs*))›
show *?thesis*
by (*simp add*: *local.sorted-append*)
qed

ultimately show *?thesis*
by *blast*
qed

lemma (**in** *linorder*) *list-slice-sorted-nth-mono*:
assumes *sorted* (*list-slice xs s e*)
and $s \leq i$
and $i \leq j$
and $j < e$
and $j < length\ xs$
shows $xs\ !\ i \leq xs\ !\ j$
proof −
have $\exists i'.\ i = s + i'$
using *assms*(*2*) *nat-le-iff-add* **by** *blast*
then obtain i' **where**
$i = s + i'$
by *blast*

have $\exists j'.\ j = s + j'$
using *assms*(*2*) *assms*(*3*) *nat-le-iff-add* **by** *auto*
then obtain j' **where**
$j = s + j'$
by *blast*

have $i' \leq j'$
using ‹$i = s + i'$› ‹$j = s + j'$› *assms*(*3*) **by** *auto*

have $j' < length\ (list\text{-}slice\ xs\ s\ e)$
using ‹$j = s + j'$› *assms*(*4*) *assms*(*5*) **by** *auto*
with *sorted-nth-mono*[*OF assms*(*1*) ‹$i' \leq j'$›]
have *list-slice xs s e* ! $i' \leq$ *list-slice xs s e* ! j' **.**
moreover
have $xs\ !\ i =$ *list-slice xs s e* ! i'
using ‹$i = s + i'$› *assms*(*3*−*5*) **by** *force*
moreover
have $xs\ !\ j =$ *list-slice xs s e* ! j'
using ‹$j = s + j'$› ‹$j' < length\ (list\text{-}slice\ xs\ s\ e)$› *nth-list-slice* **by** *force*
ultimately show *?thesis*
by *simp*
qed
end
theory *List-Lexorder-Util*
imports
HOL−*Library.List-Lexorder*
begin

lemma *same-equiv-def*:
$(\forall j{<}n.\ s\ !\ (i + j) = s\ !\ Suc\ (i + j)) = (\forall j{\leq}n.\ s\ !\ (i + j) = s\ !\ i)$
proof *safe*

```
  fix j
  assume ∀j<n. s ! (i + j) = s ! Suc (i + j) j ≤ n
  then show s ! (i + j) = s ! i
  proof (induct n arbitrary: j)
    case 0
    then show ?case
      by simp
  next
    case (Suc n j)
    note IH = this
    show ?case
    proof (cases j)
      case 0
      then show ?thesis
        by simp
    next
      case (Suc m)
      with IH(1)[of m] IH(2,3)
      have s ! (i + m) = s ! i
        by (meson Suc-le-mono less-Suc-eq)
      then show ?thesis
        using Suc Suc.prems(1) Suc.prems(2) by force
    qed
  qed
next
  fix j
  assume ∀j≤n. s ! (i + j) = s ! i j < n
  then show s ! (i + j) = s ! Suc (i + j)
    using less-eq-Suc-le by fastforce
qed

lemma list-less-ex:
  xs < ys ⟷
   (∃ b c as bs cs. xs = as @ b # bs ∧ ys = as @ c # cs ∧ b < c) ∨
   (∃ c cs. ys = xs @ c # cs)
  by (clarsimp simp: List-Lexorder.list-less-def lexord-def; blast)

end
theory List-Permutation-Util
  imports HOL−Combinatorics.List-Permutation ../util/List-Util
begin

lemma perm-distinct-set-of-upt-iff:
  xs <~~> [0..<n] ⟷ distinct xs ∧ set xs = {0..<n}
 by (metis atLeastLessThan-upt distinct-upt perm-distinct-iff set-eq-iff-mset-eq-distinct)

lemma distinct-set-of-upto-length:
  ⟦distinct xs; set xs = {0..<n}⟧ ⟹ length xs = n
  apply (drule (1) iffD2[OF perm-distinct-set-of-upt-iff conjI])
```

apply (*drule perm-length*; *simp*)
done

lemma *set-perm-upt*:
$xs <^{\sim\sim}> [0..<n] \Longrightarrow set\ xs = \{0..<n\}$
using *perm-distinct-set-of-upt-iff* **by** *blast*

lemma *perm-upt-length*:
$xs <^{\sim\sim}> [0..<n] \Longrightarrow length\ xs = n$
using *distinct-set-of-upto-length perm-distinct-set-of-upt-iff* **by** *blast*

lemma *perm-nth-ex*:
$[\![xs <^{\sim\sim}> [0..<n];\ i < n]\!] \Longrightarrow \exists\, k < n.\ xs\ !\ i = k$
using *perm-upt-length set-perm-upt* **by** *fastforce*

lemma *ex-perm-nth*:
$[\![xs <^{\sim\sim}> [0..<n];\ k < n]\!] \Longrightarrow \exists\, i < n.\ xs\ !\ i = k$
by (*metis atLeast-upt distinct-Ex1 distinct-upt lessThan-iff perm-distinct-iff perm-set-eq perm-upt-length*)

lemma *set-map-nth-perm-subset*:
$[\![ys <^{\sim\sim}> [0..<n];\ n \leq length\ xs]\!] \Longrightarrow set\ (map\ (nth\ xs)\ ys) \subseteq set\ xs$
by (*simp add*: *nth-image set-perm-upt set-take-subset*)

lemma *set-map-nth-perm-eq*:
$ys <^{\sim\sim}> [0..<length\ xs] \Longrightarrow set\ (map\ (nth\ xs)\ ys) = set\ xs$
by (*metis perm-set-eq set-map set-map-nth-eq*)

lemma *distinct-map-nth-perm*:
$[\![distinct\ xs;\ n \leq length\ xs;\ ys <^{\sim\sim}> [0..<n]]\!] \Longrightarrow distinct\ (map\ (nth\ xs)\ ys)$
by (*metis distinct-map distinct-map-nth perm-distinct-iff perm-set-eq*)

theorem *distinct-set-imp-perm*:
assumes *distinct xs*
and *distinct ys*
and *set xs = set ys*
shows $xs <^{\sim\sim}> ys$
proof −
from *set-eq-iff-mset-eq-distinct*[*OF assms*(*1*,*2*), *THEN iffD1*, *OF assms*(*3*)]
show *?thesis*
by *simp*
qed

theorem *perm-nth*:
assumes $xs <^{\sim\sim}> ys$
and $i < length\ xs$
shows $\exists j < length\ ys.\ ys\ !\ j = xs\ !\ i$
by (*metis assms*(*1*) *assms*(*2*) *in-set-conv-nth perm-set-eq*)

lemma *sort-perm*:
xs <~~> *sort xs*
by *simp*

end
theory *List-Lexorder-NS*
imports
../util/Sorting-Util
../util/List-Slice
../order/List-Permutation-Util

begin

18 General Non-standard Lexicographical Comparison

This section is based on the *lexord* classical lexicographical definition in the the List library but accounts for a variant of lexicographic order defined below that we rely on for verifying sais. The main difference is that this ordering preferences the original string over its prefix. For example, "aaa" is less than "aa", which in turn is less than "a".

definition *nslexord* :: ('*a* × '*a*) *set* ⇒ ('*a list* × '*a list*) *set* **where**
nslexord r = {(*x*,*y*). (∃ *a v*. *x* = *y* @ *a* # *v*) ∨
(∃ *u a b v w*. (*a*, *b*) ∈ *r* ∧ *x* = *u* @ *a* # *v* ∧ *y* = *u* @ *b* # *w*)}

definition *nslexordp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool*
where
nslexordp cmp xs ys ⟷
(∃ *b c as bs cs*. *xs* = *as* @ *b* # *bs* ∧ *ys* = *as* @ *c* # *cs* ∧ *cmp b c*) ∨
(∃ *c cs*. *xs* = *ys* @ *c* # *cs*)

lemma *nslexord-eq-nslexordp*:
(*xs*, *ys*) ∈ *nslexord* {(*x*, *y*). *cmp x y*} ⟷ *nslexordp cmp xs ys*
(*xs*, *ys*) ∈ *nslexord r* ⟷ *nslexordp* (λ*x y*. (*x*, *y*) ∈ *r*) *xs ys*
by (*clarsimp simp*: *nslexord-def nslexordp-def*; *blast*)+

definition *nslexordeqp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool*
where
nslexordeqp cmp xs ys ⟷ *nslexordp cmp xs ys* ∨ (*xs* = *ys*)

18.1 Intro and Elimination

lemma *nslexordpI1*:
∃ *b c as bs cs*. *xs* = *as* @ *b* # *bs* ∧ *ys* = *as* @ *c* # *cs* ∧ *cmp b c* ⟹ *nslexordp cmp xs ys*
by (*simp add*: *nslexordp-def*)

lemma *nslexordpI2*:
 $\exists\, c\ cs.\ xs = ys\ @\ c\ \#\ cs \Longrightarrow nslexordp\ cmp\ xs\ ys$
 by (*simp add*: *nslexordp-def*)

lemma *nslexordpE*:
 nslexordp cmp xs ys $\Longrightarrow$
 $(\exists\, b\ c\ as\ bs\ cs.\ xs = as\ @\ b\ \#\ bs \wedge ys = as\ @\ c\ \#\ cs \wedge cmp\ b\ c) \vee$
 $(\exists\, c\ cs.\ xs = ys\ @\ c\ \#\ cs)$
 by (*simp add*: *nslexordp-def*)

lemma *nslexordp-imp-eq*:
 nslexordp cmp xs ys $\Longrightarrow$ *nslexordeqp cmp xs ys*
 by (*simp add*: *nslexordeqp-def*)

lemma *nslexordeqp-imp-eq-or-less*:
 nslexordeqp cmp xs ys $\Longrightarrow xs = ys \vee$ *nslexordp cmp xs ys*
 using *nslexordeqp-def* **by** *auto*

18.2 Simplification

lemma *nslexord-Nil-left*[*simp*]: $([],\ y) \notin$ *nslexord r*
 by (*unfold nslexord-def*, *induct y*, *auto*)

lemma *nslexord-Nil-right*[*simp*]: $(y,\ []) \in$ *nslexord r* $= (\exists\, a\ x.\ y = a\ \#\ x)$
 by (*unfold nslexord-def*, *induct y*, *auto*)

lemma *nslexord-cons-cons*[*simp*]:
 $(a\ \#\ x,\ b\ \#\ y) \in$ *nslexord r* $\longleftrightarrow (a,\ b) \in r \vee (a = b \wedge (x,\ y) \in$ *nslexord r*$)$ (**is** *?lhs = ?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  apply (*simp add*: *nslexord-def*)
  apply (*metis hd-append list.sel*(*1*) *list.sel*(*3*) *tl-append2*)
  done
qed (*auto simp add*: *nslexord-def*; (*blast* | *meson Cons-eq-appendI*))

lemma *nslexordp-cons-cons*[*simp*]:
 nslexordp r $(a\ \#\ x)\ (b\ \#\ y) \longleftrightarrow r\ a\ b \vee (a = b \wedge$ *nslexordp r x y*$)$
 by (*metis mem-Collect-eq nslexord-cons-cons nslexord-eq-nslexordp*(*1*) *prod.simps*(*2*))

lemmas *nslexord-simps* = *nslexord-Nil-left nslexord-Nil-right nslexord-cons-cons*

lemma *nslexord-same-pref-iff*:
 $(xs\ @\ ys,\ xs\ @\ zs) \in$ *nslexord r* $\longleftrightarrow (\exists\, x \in set\ xs.\ (x,\ x) \in r) \vee (ys,\ zs) \in$ *nslexord r*
 by(*induction xs*) *auto*

lemma *nslexord-same-pref-if-irrefl*[*simp*]:

irrefl r $\Longrightarrow$ (*xs* @ *ys*, *xs* @ *zs*) $\in$ *nslexord r* $\longleftrightarrow$ (*ys*, *zs*) $\in$ *nslexord r*
by (*simp add*: *irrefl-def nslexord-same-pref-iff*)

lemma *nslexord-append-leftI*:
$\exists$ *b z*. *y* = *b* # *z* $\Longrightarrow$ (*x* @ *y*, *x*) $\in$ *nslexord r*
by (*simp add*: *nslexordpI2 nslexord-eq-nslexordp*(*2*))

lemma *nslexord-append-left-rightI*:
(*a* ,*b*) $\in$ *r* $\Longrightarrow$ (*u* @ *a* # *x*, *u* @ *b* # *y*) $\in$ *nslexord r*
by (*simp add*: *nslexord-same-pref-iff*)

lemma *nslexord-append-rightI*:
(*u*, *v*) $\in$ *nslexord r* $\Longrightarrow$ (*x* @ *u*, *x* @ *v*) $\in$ *nslexord r*
by (*simp add*: *nslexord-same-pref-iff*)

lemma *nslexord-append-rightD*:
$[\![$(*x* @ *u*, *x* @ *v*) $\in$ *nslexord r*; ($\forall$ *a*. (*a*,*a*) $\notin$ *r*) $]\!]$ $\Longrightarrow$ (*u*,*v*) $\in$ *nslexord r*
by (*simp add*: *nslexord-same-pref-iff*)

— nslexord is extension of partial ordering List.lex
lemma *nslexord-lex*:
(*x*,*y*) $\in$ *lex r* = ((*x*,*y*) $\in$ *nslexord r* $\wedge$ *length x* = *length y*)
proof (*induction x arbitrary*: *y*)
case (*Cons a x y*) **then show** *?case*
by (*cases y*) (*force*+)
qed *auto*

18.3 Recursive version

fun *nslexordrec* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *'a list* $\Rightarrow$ *'a list* $\Rightarrow$ *bool*
where
nslexordrec P [] - = *False* |
nslexordrec P - [] = *True* |
nslexordrec P (*x*#*xs*) (*y*#*ys*) = (*if P x y then True else if x* = *y then nslexordrec P xs ys else False*)

lemma *nslexordp-eq-nslexordrec*:
nslexordp cmp xs ys $\longleftrightarrow$ *nslexordrec cmp xs ys*
proof (*induct xs arbitrary*: *ys*)
case *Nil*
then show *?case*
by (*simp add*: *nslexordp-def*)
next
case (*Cons a xs*)
note *IH* = *this*

then show *?case*
proof (*cases ys*)
case *Nil*

```
    then show ?thesis
      by (simp add: nslexordp-def)
  next
    case (Cons b zs)
    note P = IH[of zs]
    have cmp a b ⟹ ?thesis
      by (simp add: Cons)
    moreover
    have ⟦¬cmp a b; a = b⟧ ⟹ ?thesis
      by (simp add: P Cons)
    moreover
    have ⟦¬cmp a b; a ≠ b⟧ ⟹ ?thesis
      by (simp add: local.Cons)
    ultimately show ?thesis
      by blast
  qed
qed

lemmas nslexordp-induct = nslexordrec.induct
```

18.4 Properties

Useful properties for proving things about relations, such as what type of order is satisfied

```
lemma nslexord-total-on:
  assumes total-on A R
  shows total-on {xs. set xs ⊆ A} (nslexord R)
proof (intro total-onI)
  fix x y
  assume x ∈ {xs. set xs ⊆ A} y ∈ {xs. set xs ⊆ A} x ≠ y
  hence  set x ⊆ A set y ⊆ A x ≠ y
    by blast+
  then show (x, y) ∈ nslexord R ∨ (y, x) ∈ nslexord R
  proof (induct x arbitrary: y)
    case Nil
    then show ?case
      by (metis list.exhaust nslexord-Nil-right)
  next
    case (Cons a x)
    note IH = this
    then show ?case
    proof (cases y)
      case Nil
      then show ?thesis
        by auto
    next
      case (Cons b z)
      then show ?thesis
        by (metis Cons.hyps IH(2−4) assms list.set-intros(1,2) nslexord-cons-cons
```

subset-code(*1*)
total-on-def)
qed
qed
qed

lemma *total-on-totalp-on-eq*:
total-on A {(*x*, *y*). *R x y*} = *totalp-on A R*
by (*simp add*: *total-on-def totalp-on-def*)

lemmas *nslexordp-totalp-on* =
nslexord-total-on[*OF total-on-totalp-on-eq*[*THEN iffD2*],
simplified nslexord-eq-nslexordp(*1*) *totalp-on-total-on-eq*[*symmetric*]]

lemma *nslexord-total*:
total r ⟹ *total* (*nslexord r*)
using *nslexord-total-on* **by** *fastforce*

lemma *nslexordp-totalp*:
totalp r ⟹ *totalp* (*nslexordp r*)
using *nslexordp-totalp-on* **by** *fastforce*

corollary *nslexord-linear*:
(∀ *a b*. (*a*,*b*) ∈ *r* ∨ *a* = *b* ∨ (*b*,*a*) ∈ *r*) ⟹ (*x*,*y*) ∈ *nslexord r* ∨ *x* = *y* ∨ (*y*,*x*) ∈ *nslexord r*
using *nslexord-total* **by** (*metis UNIV-I total-on-def*)

lemma *nslexord-irrefl-on*:
assumes *irrefl-on A R*
shows *irrefl-on* {*xs*. *set xs* ⊆ *A*} (*nslexord R*)
proof (*intro irrefl-onI*)
fix *x*
assume *x* ∈ {*xs*. *set xs* ⊆ *A*}
hence *set x* ⊆ *A*
by *blast*
then show (*x*, *x*) ∉ *nslexord R*
proof (*induct x*)
case *Nil*
then show *?case*
by *auto*
next
case (*Cons a x*)
then show *?case*
by (*meson assms irrefl-onD list.set-intros*(*1*) *nslexord-cons-cons set-subset-Cons*
subset-code(*1*))
qed
qed

lemma *irrefl-on-irreflp-on-eq*:

irefl-on A {(x, y). R x y} = ireflp-on A R
by (*simp add*: *irefl-on-def ireflp-on-def*)

lemmas *nslexordp-ireflp-on =*
nslexord-irefl-on[*OF irefl-on-ireflp-on-eq*[*THEN iffD2*],
simplified nslexord-eq-nslexordp(*1*) *ireflp-on-irefl-on-eq*[*symmetric*]]

lemma *nslexord-irreflexive*:
$\forall x.\ (x,x) \notin r \Longrightarrow (xs,xs) \notin$ *nslexord r*
by (*metis lex-take-index nslexord-lex*)

lemma *nslexord-irefl*:
irefl R $\Longrightarrow$ *irefl* (*nslexord R*)
by (*simp add*: *irefl-def nslexord-irreflexive*)

lemma *nslexordp-ireflp*:
assumes *ireflp R*
shows *ireflp* (*nslexordp R*)
using *assms nslexordp-ireflp-on* **by** *force*

lemma *asym-on-asymp-on-eq*:
asym-on A {(x, y). R x y} = asymp-on A R
by (*simp add*: *asym-on-def asymp-on-def*)

lemma *nslexord-asym-on*:
assumes *asym-on A R*
shows *asym-on* {*xs. set xs* $\subseteq$ *A*} (*nslexord R*)
proof (*intro asym-onI*)
fix *x y*
assume *x* $\in$ {*xs. set xs* $\subseteq$ *A*} *y* $\in$ {*xs. set xs* $\subseteq$ *A*} (*x, y*) $\in$ *nslexord R*
hence *set x* $\subseteq$ *A set y* $\subseteq$ *A* (*x, y*) $\in$ *nslexord R*
by *blast+*
then show (*y, x*) $\notin$ *nslexord R*
proof (*induct x arbitrary*: *y*)
case *Nil*
then show *?case*
by *force*
next
case (*Cons a x*)
note *IH = this*
then show *?case*
proof (*cases y*)
case *Nil*
then show *?thesis*
by *simp*
next
case (*Cons b z*)
hence (*a # x, b # z*) $\in$ *nslexord R*
using *IH*(*4*) **by** *blast*

```
    with nslexord-cons-cons[of a x b z R]
    have (a, b) ∈ R ∨ a = b ∧ (x, z) ∈ nslexord R
      by blast
    moreover
    have (a, b) ∈ R ⟹ ?thesis
      by (metis IH(2,3) assms asym-onD list.set-intros(1) local.Cons nslex-
ord-cons-cons
          subset-code(1))
    moreover
    have a = b ∧ (x, z) ∈ nslexord R ⟹ ?thesis
      using Cons.hyps IH(2,3) calculation(2) local.Cons by auto
    ultimately show ?thesis
      by blast
  qed
 qed
qed

lemmas nslexordp-asymp-on =
  nslexord-asym-on[OF asym-on-asymp-on-eq[THEN iffD2],
          simplified nslexord-eq-nslexordp(1) asymp-on-asym-on-eq[symmetric]]

lemma nslexord-asym:
 assumes asym R
 shows asym (nslexord R)
 using assms nslexord-asym-on by force

lemma nslexordp-asymp:
 assumes asymp R
 shows asymp (nslexordp R)
 using assms nslexordp-asymp-on by force

lemma nslexord-asymmetric:
 assumes asym R (a, b) ∈ nslexord R
 shows (b, a) ∉ nslexord R
 by (simp add: assms asymD nslexord-asym)

lemma trans-on-transp-on-eq:
 trans-on A {(x, y). R x y} = transp-on A R
 by (simp add: trans-on-def transp-on-def)

lemma nslexord-trans-on:
 assumes trans-on A R
 shows trans-on {xs. set xs ⊆ A} (nslexord R)
proof (intro trans-onI)
 fix x y z
 assume x ∈ {xs. set xs ⊆ A} y ∈ {xs. set xs ⊆ A} z ∈ {xs. set xs ⊆ A}
      (x, y) ∈ nslexord R (y, z) ∈ nslexord R
 hence set x ⊆ A set y ⊆ A set z ⊆ A (x, y) ∈ nslexord R (y, z) ∈ nslexord R
   by blast+
```

then show $(x, z) \in nslexord\ R$
proof (*induct x arbitrary*: $y\ z$)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons a x*)
note $IH = this$
then show *?case*
proof (*cases y*)
case *Nil*
then show *?thesis*
using $IH(6)$ **by** *auto*
next
case (*Cons b y'*)
hence $(a \# x, b \# y') \in nslexord\ R$
using $IH(5)$ **by** *blast*
with *nslexord-cons-cons*[*of a x b y' R*]
have P: $(a, b) \in R \vee a = b \wedge (x, y') \in nslexord\ R$
by *blast*
then show *?thesis*
proof (*cases z*)
case *Nil*
then show *?thesis*
by *simp*
next
case (*Cons c z'*)
hence $(b \# y', c \# z') \in nslexord\ R$
using $IH(6)$ ‹$y = b \# y'$› **by** *auto*
with *nslexord-cons-cons*[*of b y' c z' R*]
have $(b, c) \in R \vee b = c \wedge (y', z') \in nslexord\ R$
by *blast*
moreover
have $a \in A$ *set* $x \subseteq A$
using $IH(2)$ **by** *auto*
moreover
have $b \in A$ *set* $y' \subseteq A$
using $IH(3)$ ‹$y = b \# y'$› **by** *force+*
moreover
have $c \in A$ *set* $z' \subseteq A$
using $IH(4)$ ‹$z = c \# z'$› **by** *force+*
moreover
from P
have $(b, c) \in R \Longrightarrow$ *?thesis*
by (*metis assms calculation*(2,4,6) *local.Cons nslexord-cons-cons trans-onD*)
moreover
from P
have $b = c \wedge (y', z') \in nslexord\ R \Longrightarrow$ *?thesis*
by (*metis Cons.hyps calculation*(3,5,7) *local.Cons nslexord-cons-cons*)

```
      ultimately show ?thesis
        by blast
    qed
  qed
 qed
qed

lemmas nslexordp-transp-on =
  nslexord-trans-on[OF trans-on-transp-on-eq[THEN iffD2],
            simplified nslexord-eq-nslexordp(1) transp-on-trans-on-eq[symmetric]]

lemma nslexord-trans:
 assumes trans R
 shows trans (nslexord R)
 using assms nslexord-trans-on by force

lemma nslexordp-transp:
 assumes transp R
 shows transp (nslexordp R)
 using assms nslexordp-transp-on by force
```

18.5 Monotonicity

Properties about monotonicity

```
lemma monotone-on-nslexordp:
 assumes monotone-on A orda ordb f
 shows monotone-on {xs. set xs ⊆ A} (nslexordp orda) (nslexordp ordb) (map f)
proof (rule monotone-onI)
 fix x y
 assume x ∈ {xs. set xs ⊆ A} y ∈ {xs. set xs ⊆ A} nslexordp orda x y
 hence set x ⊆ A set y ⊆ A
   by blast+

 let ?c1 = ∃ b c as bs cs. x = as @ b # bs ∧ y = as @ c # cs ∧ orda b c
 and ?c2 = ∃ c cs. x = y @ c # cs

 let ?g = nslexordp ordb (map f x) (map f y)
 from nslexordpE[OF ‹nslexordp orda x y›]
 have ?c1 ∨ ?c2 .
 moreover
 have ?c2 ⟹ ?g
   using nslexordpI2 by fastforce
 moreover
 have ?c1 ⟹ ?g
 proof (rule nslexordpI1)
   assume ?c1
   then obtain b c as bs cs where
     x = as @ b # bs
     y = as @ c # cs
```

orda b c
by *blast*
moreover
have *map f x = map f as @ f b # map f bs*
by (*simp add*: *calculation*(*1*))
moreover
have *map f y = map f as @ f c # map f cs*
by (*simp add*: *calculation*(*2*))
moreover
have *ordb* (*f b*) (*f c*)
by (*metis ‹set x ⊆ A› ‹set y ⊆ A› assms calculation*(*1−3*) *in-set-conv-decomp monotone-on-def*
subset-code(*1*))
ultimately show ∃ *b c as bs cs. map f x = as @ b # bs ∧ map f y = as @ c # cs ∧ ordb b c*
by *blast*
qed
ultimately show *nslexordp ordb* (*map f x*) (*map f y*)
by *blast*
qed

lemma *monotone-on-bij-betw-inv-nslexordp*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*
and *asymp-on B ordb*
and *totalp-on B ordb*
and *bij-betw f A B*
and *bij-betw g B A*
and *inverses-on f g A B*
shows *monotone-on* {*xs. set xs ⊆ B*} (*nslexordp ordb*) (*nslexordp orda*) (*map g*)
by (*metis assms monotone-on-bij-betw-inv monotone-on-nslexordp*)

lemma *monotone-on-bij-betw-nslexordp*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*
and *asymp-on B ordb*
and *totalp-on B ordb*
and *bij-betw f A B*
shows ∃ *g. bij-betw* (*map g*) {*xs. set xs ⊆ B*} {*xs. set xs ⊆ A*} ∧
inverses-on (*map f*) (*map g*) {*xs. set xs ⊆ A*} {*xs. set xs ⊆ B*} ∧
monotone-on {*xs. set xs ⊆ B*} (*nslexordp ordb*) (*nslexordp orda*) (*map g*)
by (*metis assms bij-betw-inv-alt bij-betw-map inverses-on-map monotone-on-bij-betw-inv-nslexordp*)

lemma *monotone-on-iff-nslexordp*:
assumes *monotone-on A orda ordb f*
and *asymp-on A orda*
and *totalp-on A orda*

and *asymp-on B ordb*
and *totalp-on B ordb*
and *bij-betw f A B*
and *set xs* $\subseteq$ *A*
and *set ys* $\subseteq$ *A*
shows *nslexordp orda xs ys* $\longleftrightarrow$ *nslexordp ordb* (*map f xs*) (*map f ys*)
proof
assume *nslexordp orda xs ys*
with *monotone-onD*[*OF monotone-on-nslexordp*[*OF assms*(*1*)], *simplified*, *OF assms*(*7*,*8*)]
show *nslexordp ordb* (*map f xs*) (*map f ys*)
by *blast*
next
assume *A*: *nslexordp ordb* (*map f xs*) (*map f ys*)

from *monotone-on-bij-betw-nslexordp*[*OF assms*(*1*−*6*)]
obtain *g* **where** *P*:
bij-betw (*map g*) {*xs*. *set xs* $\subseteq$ *B*} {*xs*. *set xs* $\subseteq$ *A*}
inverses-on (*map f*) (*map g*) {*xs*. *set xs* $\subseteq$ *A*} {*xs*. *set xs* $\subseteq$ *B*}
monotone-on {*xs*. *set xs* $\subseteq$ *B*} (*nslexordp ordb*) (*nslexordp orda*) (*map g*)
by *blast*

have *Q*: *set* (*map f xs*) $\subseteq$ *B*
using *assms*(*6*,*7*) *bij-betw-imp-surj-on* **by** *fastforce*

have *R*: *set* (*map f ys*) $\subseteq$ *B*
using *assms*(*6*,*8*) *bij-betw-imp-surj-on* **by** *fastforce*

from *monotone-onD*[*OF P*(*3*), *simplified*, *OF Q R A*]
show *nslexordp orda xs ys*
by (*metis P*(*2*) *assms*(*7*,*8*) *inverses-on-def mem-Collect-eq*)
qed

18.6 Other

lemma *nslexordp-cons1-exE*:
assumes *nslexordp cmp xs* (*x* # *xs*)
shows $\exists$ *a as bs*. *x* # *xs* = *as* @ *x* # *a* # *bs* $\wedge$ *cmp a x* $\wedge$ ($\forall$ *b* $\in$ *set as*. *b* = *x*)
using *assms*
proof (*induct xs arbitrary*: *x*)
case *Nil*
then show *?case*
using *nslexord-Nil-left nslexord-eq-nslexordp*(*1*) **by** *blast*
next
case (*Cons a xs*)
note *IH* = *this*

have *cmp a x* $\Longrightarrow$ *?case*
by *fastforce*

```
  moreover
  have ⟦¬cmp a x; a ≠ x⟧ ⟹ ?case
    using IH(2) by force
  moreover
  have ⟦¬cmp a x; a = x⟧ ⟹ ?case
  proof −
    assume ¬cmp a x a = x
    with IH(2)
    have nslexordp cmp xs (x # xs)
      by simp
    with IH(1)[OF -] ‹a = x›
    have ∃ k as bs. a # xs = as @ x # k # bs ∧ cmp k x ∧ (∀ b∈set as. b = x)
      by simp
    then obtain k as bs where P:
      a # xs = as @ x # k # bs  cmp k x ∀ b∈set as. b = x
      by blast
    then show ?case
      by (metis Cons-eq-appendI set-ConsD)
  qed
  ultimately show ?case
    by blast
qed

lemma nslexordp-cons2-exE:
  assumes nslexordp cmp (x # xs) xs
  shows (∀ k ∈ set xs. k = x) ∨ (∃ a as bs. x # xs = as @ x # a # bs ∧ cmp x a
∧ (∀ b ∈ set as. b = x))
  using assms
proof (induct xs arbitrary: x)
  case Nil
  then show ?case
    by simp
next
  case (Cons a xs)
  note IH = this

  have cmp x a ⟹ ?case
    by (metis append-Nil empty-iff empty-set)
  moreover
  have ⟦¬cmp x a; a = x⟧ ⟹ ?case
  proof −
    assume ¬cmp x a a = x
    with IH(2)
    have nslexordp cmp (x # xs) xs
      by simp
    with IH(1)[of x] ‹a = x›
    have (∀ k∈set xs. k = x) ∨
          (∃ k as bs. a # xs = as @ x # k # bs ∧ cmp x k ∧ (∀ b∈set as. b = x))
      by simp
```

then show *?thesis*
proof
assume $\forall\, k{\in}set\ xs.\ k = x$
then show *?thesis*
by (*simp add*: ‹$a = x$›)
next
assume $\exists\, k\ as\ bs.\ a \# xs = as\ @\ x \# k \# bs \wedge cmp\ x\ k \wedge (\forall\, b{\in}set\ as.\ b = x)$
then obtain *k as bs* **where** *P*:
$a \# xs = as\ @\ x \# k \# bs\ cmp\ x\ k\ \forall\, b{\in}set\ as.\ b = x$
by *blast*
then show *?thesis*
by (*metis Cons-eq-appendI set-ConsD*)
qed
qed
moreover
have $[\![\neg cmp\ x\ a;\ a \neq x]\!] \Longrightarrow$ *?case*
using *Cons.prems* **by** *auto*

ultimately show *?case*
by *blast*
qed

19 Order definitions on lists of linorder elements

definition *list-less-ns* :: ('*a* :: *linorder*) *list* ⇒ '*a list* ⇒ *bool*
where
list-less-ns xs ys =
$(\exists\, n.\ n \leq length\ xs \wedge n \leq length\ ys \wedge$
$(\forall\, i < n.\ xs\ !\ i = ys\ !\ i) \wedge$
$(length\ ys = n \longrightarrow n < length\ xs) \wedge$
$(length\ ys \neq n \longrightarrow length\ xs \neq n \wedge xs\ !\ n < ys\ !\ n))$

definition *list-less-eq-ns* :: ('*a* :: *linorder*) *list* ⇒ '*a list* ⇒ *bool*
where
list-less-eq-ns xs ys =
$(\exists\, n.\ n \leq length\ xs \wedge n \leq length\ ys \wedge$
$(\forall\, i < n.\ xs\ !\ i = ys\ !\ i) \wedge$
$(length\ ys \neq n \longrightarrow length\ xs \neq n \wedge xs\ !\ n < ys\ !\ n))$

— Alternative definition

definition *list-less-ns-ex* :: ('*a* :: *linorder*) *list* ⇒ ('*a* :: *linorder*) *list* ⇒ *bool*
where
list-less-ns-ex xs ys ⟷
$(\exists\, b\ c\ as\ bs\ cs.\ xs = as\ @\ b \# bs \wedge ys = as\ @\ c \# cs \wedge b < c) \vee$
$(\exists\, c\ cs.\ xs = ys\ @\ c \# cs)$

20 Helper list comparison theorems

lemma *list-less-ns-alt-def*:
list-less-ns xs ys = list-less-ns-ex xs ys
proof
assume *list-less-ns xs ys*
with *list-less-ns-def*[*THEN iffD1*, *of xs ys*]
obtain *n* **where** *P*:
n ≤ *length xs n* ≤ *length ys* ∀ *i*<*n*. *xs* ! *i* = *ys* ! *i length ys* = *n* ⟶ *n* < *length xs*
length ys ≠ *n* ⟶ *length xs* ≠ *n* ∧ *xs* ! *n* < *ys* ! *n*
by *blast*
show *list-less-ns-ex xs ys*
proof (*cases length ys* = *n*)
assume *length ys* = *n*
then show *?thesis*
by (*metis P*(*1*,*2*,*3*,*4*) *id-take-nth-drop list-less-ns-ex-def nth-take-lemma take-all*)
next
assume *length ys* ≠ *n*
then show *?thesis*
by (*metis P*(*1*,*2*,*3*,*5*) *id-take-nth-drop le-neq-implies-less list-less-ns-ex-def nth-take-lemma*)
qed
next
assume *list-less-ns-ex xs ys*
hence (∃ *b c as bs cs*. *xs* = *as* @ *b* # *bs* ∧ *ys* = *as* @ *c* # *cs* ∧ *b* < *c*) ∨ (∃ *c cs*. *xs* = *ys* @ *c* # *cs*)
using *list-less-ns-ex-def* **by** *blast*
then show *list-less-ns xs ys*
proof
assume ∃ *b c as bs cs*. *xs* = *as* @ *b* # *bs* ∧ *ys* = *as* @ *c* # *cs* ∧ *b* < *c*
then obtain *b c as bs cs* **where** *P*:
xs = *as* @ *b* # *bs ys* = *as* @ *c* # *cs b* < *c*
by *blast*
hence *length as* < *length xs*
by *simp*
moreover
have *length as* < *length ys*
by (*simp add*: *P*(*2*))
moreover
have ∀ *i* < *length as*. *xs* ! *i* = *ys* ! *i*
by (*simp add*: *P*(*1*) *P*(*2*) *nth-append*)
moreover
have *xs* ! *length as* < *ys* ! *length as*
by (*simp add*: *P*(*1*) *P*(*2*) *P*(*3*))
ultimately show *list-less-ns xs ys*
using *list-less-ns-def*[*THEN iffD2*, *OF exI*, *of length as xs ys*] **by** *simp*
next

```
    assume ∃ c cs. xs = ys @ c # cs
    then obtain c cs where
      xs = ys @ c # cs
      by blast
    hence length ys < length xs
      by simp
    then show list-less-ns xs ys
      using list-less-ns-def[THEN iffD2, OF exI, of length ys xs ys]
      by (simp add: ‹xs = ys @ c # cs› nth-append)
  qed
qed

lemma nslexordp-eq-list-less-ns-ex:
  nslexordp (<) = list-less-ns-ex
  by (clarsimp simp: fun-eq-iff nslexordp-def list-less-ns-ex-def)

lemma nslexordp-eq-list-less-ns-ex-apply:
  nslexordp (<) x y = list-less-ns-ex x y
  by (simp add: nslexordp-eq-list-less-ns-ex)

lemma nslexordp-eq-list-less-ns:
  nslexordp (<) = list-less-ns
  by (clarsimp simp: fun-eq-iff list-less-ns-alt-def nslexordp-eq-list-less-ns-ex)

lemma nslexordp-eq-list-less-ns-app:
  nslexordp (<) x y = list-less-ns x y
  by (simp add: nslexordp-eq-list-less-ns)

lemma nslexordeqp-eq-list-less-eq-ns-apply:
  nslexordeqp (<) x y = list-less-eq-ns x y
proof (cases x = y)
  assume x = y
  then show ?thesis
    by (simp add: list-less-eq-ns-def nslexordeqp-def)
next
  assume x ≠ y
  hence nslexordeqp (<) x y = nslexordp (<) x y
    by (simp add: nslexordeqp-def)
  moreover
  have nslexordp (<) x y = list-less-ns x y
    by (simp add: nslexordp-eq-list-less-ns)
  moreover
  have list-less-eq-ns x y = list-less-ns x y
    unfolding list-less-eq-ns-def list-less-ns-def
  proof (intro iffI; elim exE conjE)
    fix n
    assume n ≤ length x n ≤ length y ∀ i<n. x ! i = y ! i
           length y ≠ n ⟶ length x ≠ n ∧ x ! n < y ! n
    then show ∃ n≤length x. n ≤ length y ∧ (∀ i<n. x ! i = y ! i) ∧
```

(*length y* = *n* ⟶ *n* < *length x*) ∧
(*length y* ≠ *n* ⟶ *length x* ≠ *n* ∧ *x* ! *n* < *y* ! *n*)
by (*metis* ‹*x* ≠ *y*› *le-eq-less-or-eq nth-equalityI*)
next
fix *n*
assume *n* ≤ *length x n* ≤ *length y* ∀ *i*<*n*. *x* ! *i* = *y* ! *i length y* = *n* ⟶ *n* < *length x*
length y ≠ *n* ⟶ *length x* ≠ *n* ∧ *x* ! *n* < *y* ! *n*
then show ∃ *n*≤*length x*. *n* ≤ *length y* ∧ (∀ *i*<*n*. *x* ! *i* = *y* ! *i*) ∧
(*length y* ≠ *n* ⟶ *length x* ≠ *n* ∧ *x* ! *n* < *y* ! *n*)
by *blast*
qed
ultimately show *?thesis*
by *blast*
qed

21 *list-less-ns* helpers

lemma *list-less-ns-cons-same*:
list-less-ns (*a* # *xs*) (*a* # *ys*) = *list-less-ns xs ys*
by (*metis nslexordp-cons-cons nslexordp-eq-list-less-ns order-less-irrefl*)

lemma *list-less-ns-cons-diff*:
a < *b* ⟹ *list-less-ns* (*a* # *xs*) (*b* # *ys*)
using *list-less-ns-def* **by** *fastforce*

lemma *list-less-ns-cons*:
list-less-ns (*a* # *xs*) (*b* # *ys*) = (*a* ≤ *b* ∧ (*a* = *b* ⟶ *list-less-ns xs ys*))
by (*metis length-Cons list-less-ns-cons-diff list-less-ns-cons-same list-less-ns-def nat.simps*(*3*)
not-less-iff-gr-or-eq not-less-zero nth-Cons-0 order.strict-iff-order order-class.order-eq-iff)

lemma *list-less-eq-ns-cons-same*:
list-less-eq-ns (*a* # *xs*) (*a* # *ys*) = *list-less-eq-ns xs ys*
by (*metis list.inject list-less-ns-cons-same nslexordeqp-def nslexordeqp-eq-list-less-eq-ns-apply*
nslexordp-eq-list-less-ns-app)

lemma *list-less-eq-ns-cons*:
list-less-eq-ns (*a* # *xs*) (*b* # *ys*) = (*a* ≤ *b* ∧ (*a* = *b* ⟶ *list-less-eq-ns xs ys*))
by (*metis list.inject list-less-ns-cons nle-le nslexordeqp-def*
nslexordeqp-eq-list-less-eq-ns-apply nslexordp-eq-list-less-ns)

lemma *list-less-ns-hd-same*:
⟦*hd xs* = *hd ys*; *xs* ≠ []; *ys* ≠ []⟧ ⟹ *list-less-ns xs ys* = *list-less-ns* (*tl xs*) (*tl ys*)
by (*metis list.collapse list-less-ns-cons-same*)

lemma *list-less-ns-recurse*:

$[\![xs \neq []; ys \neq []]\!] \Longrightarrow$
$(hd\ xs = hd\ ys \longrightarrow list\text{-}less\text{-}ns\ xs\ ys = list\text{-}less\text{-}ns\ (tl\ xs)\ (tl\ ys)) \wedge$
$(hd\ xs \neq hd\ ys \longrightarrow list\text{-}less\text{-}ns\ xs\ ys = (hd\ xs < hd\ ys))$
by (*metis list.collapse list-less-ns-cons list-less-ns-hd-same nless-le*)

lemma *list-less-ns-nil*:
$xs \neq [] \Longrightarrow list\text{-}less\text{-}ns\ xs\ []$
using *list-less-ns-def* **by** *auto*

lemma *list-less-ns-app*:
$bs \neq [] \Longrightarrow list\text{-}less\text{-}ns\ (as\ @\ bs)\ as$
by (*metis list.collapse nslexordpI2 nslexordp-eq-list-less-ns*)

22 Lists of linorder elements are linorders with a bottom element

lemma *list-less-ns-imp-less-eq-not-less-eq*:
$list\text{-}less\text{-}ns\ x\ y \Longrightarrow (list\text{-}less\text{-}eq\text{-}ns\ x\ y \wedge \neg\ list\text{-}less\text{-}eq\text{-}ns\ y\ x)$
apply (*clarsimp simp*: *nslexordp-eq-list-less-ns*[*symmetric*]
nslexordeqp-eq-list-less-eq-ns-apply[*symmetric*]
nslexordeqp-def
nslexord-eq-nslexordp(*1*)[*symmetric*])
apply (*rule conjI*)
apply (*erule nslexord-asymmetric*[*rotated*], *fastforce*)
by (*metis Product-Type.Collect-case-prodD fst-conv nslexord-irreflexive order-less-irrefl snd-conv*)

lemma *list-less-eq-ns-not-less-eq-imp-less*:
$list\text{-}less\text{-}eq\text{-}ns\ x\ y \wedge \neg\ list\text{-}less\text{-}eq\text{-}ns\ y\ x \Longrightarrow list\text{-}less\text{-}ns\ x\ y$
by (*metis nslexordeqp-eq-list-less-eq-ns-apply nslexordeqp-imp-eq-or-less nslexordp-eq-list-less-ns*)

lemma *list-less-eq-ns-trans*:
$[\![list\text{-}less\text{-}eq\text{-}ns\ x\ y; list\text{-}less\text{-}eq\text{-}ns\ y\ z]\!] \Longrightarrow list\text{-}less\text{-}eq\text{-}ns\ x\ z$
apply (*clarsimp simp*: *nslexordp-eq-list-less-ns*[*symmetric*]
nslexordeqp-eq-list-less-eq-ns-apply[*symmetric*]
nslexordeqp-def
nslexord-eq-nslexordp(*1*)[*symmetric*])
apply *safe*
apply (*erule* (*1*) *transD*[*OF nslexord-trans*, *rotated*])
by (*metis order-less-trans transp-def transp-trans*)

lemma *list-less-eq-ns-anti-sym*:
$[\![list\text{-}less\text{-}eq\text{-}ns\ x\ y; list\text{-}less\text{-}eq\text{-}ns\ y\ x]\!] \Longrightarrow x = y$
by (*metis list-less-ns-imp-less-eq-not-less-eq nslexordeqp-eq-list-less-eq-ns-apply nslexordeqp-imp-eq-or-less nslexordp-eq-list-less-ns*)

lemma *list-less-eq-ns-linear*:
list-less-eq-ns x y ∨ *list-less-eq-ns y x*
apply (*simp add*: *nslexordp-eq-list-less-ns*[*symmetric*]
nslexordeqp-eq-list-less-eq-ns-apply[*symmetric*]
nslexordeqp-def
nslexord-eq-nslexordp(*1*)[*symmetric*])
by (*metis case-prodI linorder-neqE mem-Collect-eq nslexord-linear*)

interpretation *ordlistns*: *linorder list-less-eq-ns list-less-ns*
proof
fix *x y z* :: *'a list*
show *list-less-ns x y* = (*list-less-eq-ns x y* ∧ ¬ *list-less-eq-ns y x*)
using *list-less-ns-imp-less-eq-not-less-eq list-less-eq-ns-not-less-eq-imp-less*
by *blast*
show *list-less-eq-ns x x*
unfolding *list-less-eq-ns-def*
by *simp*
show ⟦*list-less-eq-ns x y*; *list-less-eq-ns y z*⟧ ⟹ *list-less-eq-ns x z*
by (*rule list-less-eq-ns-trans*)
show ⟦*list-less-eq-ns x y*; *list-less-eq-ns y x*⟧ ⟹ *x* = *y*
by (*rule list-less-eq-ns-anti-sym*)
show *list-less-eq-ns x y* ∨ *list-less-eq-ns y x*
by (*rule list-less-eq-ns-linear*)
qed

interpretation *ordlistns*: *order-top list-less-eq-ns list-less-ns* []
proof
fix *a* :: *'a list*
show *list-less-eq-ns a* []
unfolding *list-less-eq-ns-def*
by *auto*
qed

23 Recursive Definition

fun *lt-ns* :: (*'a* :: *linorder*) *list* ⇒ *'a list* ⇒ *bool*
where
lt-ns [] [] = *False* |
lt-ns [] - = *False* |
lt-ns - [] = *True* |
lt-ns (*a* # *as*) (*b* # *bs*) =
(*if a* < *b then True*
else if a > *b then False*
else lt-ns as bs)

lemma *list-less-ns-lt-ns*:
list-less-ns xs ys = *lt-ns xs ys*
apply (*induct rule*: *lt-ns.induct*)
apply *simp*

apply *simp*
apply (*simp add*: *list-less-ns-nil*)
apply (*simp add*: *list-less-ns-cons*)
apply (*safe*; *simp*)
done

24 *list-less-ns-ex* helpers

lemma *list-less-ns-exI1*:
$\exists\, b\ c\ as\ bs\ cs.\ xs = as\ @\ b\ \#\ bs \wedge ys = as\ @\ c\ \#\ cs \wedge b < c \Longrightarrow$ *list-less-ns-ex xs ys*
by (*simp add*: *list-less-ns-ex-def*)

lemma *list-less-ns-exI2*:
$\exists\, c\ cs.\ xs = ys\ @\ c\ \#\ cs \Longrightarrow$ *list-less-ns-ex xs ys*
by (*simp add*: *list-less-ns-ex-def*)

lemma *list-less-ns-exE*:
list-less-ns-ex xs ys $\Longrightarrow$
$(\exists\, b\ c\ as\ bs\ cs.\ xs = as\ @\ b\ \#\ bs \wedge ys = as\ @\ c\ \#\ cs \wedge b < c) \vee$
$(\exists\, c\ cs.\ xs = ys\ @\ c\ \#\ cs)$
by (*simp add*: *list-less-ns-ex-def*)

lemma *list-less-ns-app-same*:
list-less-ns (*as* @ *xs*) (*as* @ *ys*) = *list-less-ns xs ys*
apply (*induct as arbitrary*: *xs ys*)
apply *simp*
apply (*simp add*: *list-less-ns-cons-same*)
done

lemma *list-less-eq-ns-app-same*:
list-less-eq-ns (*as* @ *xs*) (*as* @ *ys*) = *list-less-eq-ns xs ys*
apply (*induct as arbitrary*: *xs ys*)
apply *simp*
apply (*simp add*: *list-less-eq-ns-cons-same*)
done

lemma *list-less-ns-cons1-exE*:
assumes *list-less-ns xs* (*x* # *xs*)
shows $\exists\, a\ as\ bs.\ x\ \#\ xs = as\ @\ x\ \#\ a\ \#\ bs \wedge x > a \wedge (\forall\, b \in set\ as.\ b = x)$
by (*metis assms nslexordp-cons1-exE nslexordp-eq-list-less-ns*)

lemma *list-less-ns-cons1-exI*:
assumes $\exists\, a\ as\ bs.\ x\ \#\ xs = as\ @\ x\ \#\ a\ \#\ bs \wedge x > a \wedge (\forall\, b \in set\ as.\ b = x)$
shows *list-less-ns-ex xs* (*x* # *xs*)
proof −
from *assms*
obtain *a as bs* **where**
x # *xs* = *as* @ *x* # *a* # *bs*

```
    a < x
    ∀ b ∈ set as. b = x
    by blast

  have as = [] ⟹ ?thesis
   using ‹a < x› ‹x # xs = as @ x # a # bs› list-less-ns-alt-def list-less-ns-cons-diff
    by fastforce
  moreover
  have ∃ c cs. as = cs @ [c] ⟹ ?thesis
  proof −
    assume ∃ c cs. as = cs @ [c]
    then obtain c cs where
      as = cs @ [c]
      by blast
    with ‹∀ b ∈ set as. b = x›
    have c = x
      by auto
    hence x # xs = cs @ x # x # a # bs
      by (simp add: ‹as = cs @ [c]› ‹x # xs = as @ x # a # bs›)

    have ∀ b ∈ set cs. b = x
      by (simp add: ‹∀ b∈set as. b = x› ‹as = cs @ [c]›)
    hence ∃ n. cs = replicate n x
      by (meson replicate-eqI)
    then show ?thesis
       by (metis ‹a < x› ‹x # xs = cs @ x # x # a # bs› list-less-ns-alt-def
list-less-ns-app-same
              list-less-ns-cons-diff list-less-ns-cons-same replicate-app-Cons-same)
  qed
  ultimately show ?thesis
    by (meson rev-exhaust)
qed

lemma list-less-ns-cons2-ex:
  assumes list-less-ns (x # xs) xs
  shows (∀ k ∈ set xs. k = x) ∨ (∃ a as bs. x # xs = as @ x # a # bs ∧ x < a ∧
(∀ b ∈ set as. b = x))
  by (metis assms nslexordp-cons2-exE nslexordp-eq-list-less-ns)

end
theory Valid-List
  imports Main ../util/List-Util
begin
```

25 Valid List

```
definition
  valid-list :: ('a :: {linorder, order-bot}) list ⇒ bool
where
```

valid-list s = (*length s* > *0* ∧ (∀ *i* < *length s* − *1*. *s* ! *i* ≠ *bot*) ∧ *last s* = *bot*)

lemma *valid-list-ex-def*:
fixes *s* ::(*'a* :: {*linorder*, *order-bot*}) *list*
shows (*valid-list s*) =
(∃ *xs*. *s* = *xs* @ [*bot*] ∧
(∀ *i* < *length xs*. *xs* ! *i* ≠ *bot*))
unfolding *valid-list-def*
proof *safe*
assume
s ≠ []
∀ *i*<*length s* − *1*. *s* ! *i* ≠ *bot*
last s = *bot*
then
show ∃ *xs*. *s* = *xs* @ [*bot*] ∧
(∀ *i*<*length xs*. *xs* ! *i* ≠ *bot*)
by (*metis append-butlast-last-id length-butlast nth-butlast*)
qed (*simp add*: *nth-append*)+

lemma *valid-list-iff-butlast-app-last*:
fixes *s* :: (*'a* :: {*linorder*, *order-bot*}) *list*
shows *valid-list s* ⟷
s ≠ [] ∧
(∀ *x* ∈ *set* (*butlast s*). *x* ≠ *bot*) ∧
last s = *bot*
by (*metis append-butlast-last-id butlast-snoc in-set-conv-nth*
valid-list-def valid-list-ex-def length-greater-0-conv)

lemma *valid-list-consI*:
fixes *s* :: (*'a* :: {*linorder*, *order-bot*}) *list*
fixes *a* :: *'a*
assumes *valid-list s*
and *a* ≠ *bot*
shows *valid-list* (*a* # *s*)
using *assms*
by (*simp add*: *valid-list-iff-butlast-app-last*)

lemma *valid-list-consD*:
fixes *s* :: (*'a* :: {*linorder*, *order-bot*}) *list*
fixes *a* :: *'a*
assumes *valid-list* (*a* # *s*)
assumes *s* ≠ []
shows *valid-list s*
using *assms*
by (*simp add*: *valid-list-iff-butlast-app-last*)

lemma *Min-valid-list*:
fixes *s* :: (*'a* :: {*linorder*, *order-bot*}) *list*
assumes *valid-list s*

shows *Min* (*set s*) = *bot*
by (*metis assms List.finite-set Min.in-idem last-in-set min-bot valid-list-iff-butlast-app-last*)

lemma *valid-list-length*:
fixes *s* :: ('*a* :: {*linorder*, *order-bot*}) *list*
assumes *valid-list s*
shows *length s* > *0*
using *assms*
by (*clarsimp simp*: *valid-list-ex-def*)

lemma *valid-list-length-ex*:
fixes *s* :: ('*a* :: {*linorder*, *order-bot*}) *list*
assumes *valid-list s*
shows ∃ *n*. *length s* = *Suc n*
using *assms*
by (*clarsimp simp*: *valid-list-ex-def*)

lemma *valid-list-not-nil*:
fixes *s* :: ('*a* :: {*linorder*, *order-bot*}) *list*
assumes *valid-list s*
shows *s* ≠ []
using *assms* **by** (*simp add*: *valid-list-def*)

lemma *valid-list-Suc-mapping*:
fixes *f* :: '*a* ⇒ *nat*
fixes *s* :: '*a list*
shows *valid-list* ((*map* (λ*x*. *Suc* (*f x*)) *s*) @ [*bot*])
proof (*induct s*)
case (*Cons a s*)
note *IH* = *this*
have *map* (λ*x*. *Suc* (*f x*)) (*a* # *s*) = (*Suc* (*f a*)) # *map* (λ*x*. *Suc* (*f x*)) *s*
by *simp*
moreover
have *Suc* (*f a*) ≠ *bot*
by (*simp add*: *bot-nat-def*)
hence *valid-list* ((*Suc* (*f a*)) # (*map* (λ*x*. *Suc* (*f x*)) *s*) @ [*bot*])
by (*simp add*: *IH valid-list-consI*)
ultimately
show *?case*
by *simp*
qed (*simp add*: *valid-list-ex-def*)

lemma *valid-list-app*:
assumes *valid-list* (*xs* @ *y* # *ys*)
shows *valid-list* (*y* # *ys*)
using *assms*
by (*induct xs*) (*simp add*: *valid-list-consD*)+

lemma *not-valid-list-app*:

assumes *valid-list* (*xs* @ *y* # *ys*)
shows ¬*valid-list xs*
using *assms*
proof (*induct xs*)
case *Nil*
then show *?case*
using *valid-list-not-nil* **by** *auto*
next
case (*Cons a xs*)
then show *?case*
by (*metis Groups.add-ac*(2) *Nil-is-append-conv One-nat-def add-diff-cancel-left' append-Cons*
last-ConsL list.discI list.size(4) *nth-Cons-0 valid-list-consD valid-list-def*)
qed

lemma *valid-list-neqE*:
assumes *valid-list xs valid-list ys xs* ≠ *ys*
shows ∃ *x y as bs cs*. *xs* = *as* @ *x* # *bs* ∧ *ys* = *as* @ *y* # *cs* ∧ *x* ≠ *y*
proof −
note *cases* = *list-neq-fc*[*OF assms*(3)]
moreover
have ¬(∃ *z zs*. *xs* = *ys* @ *z* # *zs*)
using *assms*(1) *assms*(2) *not-valid-list-app* **by** *blast*
moreover
have ¬(∃ *z zs*. *ys* = *xs* @ *z* # *zs*)
using *assms*(1) *assms*(2) *not-valid-list-app* **by** *blast*
ultimately show *?thesis*
by *blast*
qed

end
theory *Valid-List-Util*
imports *List-Lexorder-Util List-Lexorder-NS Valid-List*
begin

26 Order Equivalence

lemma *valid-list-list-less-equiv-list-less-ns*:
assumes *valid-list s1*
and *valid-list s2*
shows *s1* < *s2* = *list-less-ns s1 s2*
proof
assume *s1* < *s2*
hence *s1* ≠ *s2*
by *simp*
with *valid-list-neqE*[*OF assms*(1,2)]
obtain *x y as bs cs* **where**
s1 = *as* @ *x* # *bs s2* = *as* @ *y* # *cs x* ≠ *y*
by *blast*

hence $x < y$
by (*metis* ‹*s1* < *s2*› *linorder-less-linear list-less-ex order-less-imp-not-less*)
then have *list-less-ns-ex s1 s2*
using ‹*s1* = *as* @ *x* # *bs*› ‹*s2* = *as* @ *y* # *cs*› *list-less-ns-ex-def* **by** *fastforce*
then show *list-less-ns s1 s2*
by (*simp add*: *list-less-ns-alt-def*)
next
assume *list-less-ns s1 s2*
hence *s1* ≠ *s2*
by *fastforce*
with *valid-list-neqE*[*OF assms*(*1*,*2*)]
obtain *x y as bs cs* **where**
s1 = *as* @ *x* # *bs s2* = *as* @ *y* # *cs x* ≠ *y*
by *blast*
hence *x* < *y*
by (*metis* ‹*list-less-ns s1 s2*› *list.distinct*(*1*) *list.sel*(*1*) *list-less-ns-app-same*
list-less-ns-recurse)
then show *s1* < *s2*
using ‹*s1* = *as* @ *x* # *bs*› ‹*s2* = *as* @ *y* # *cs*› *list-less-ex* **by** *fastforce*
qed

lemma *valid-list-list-less-eq-equiv-list-less-eq-ns*:
assumes *valid-list s1*
and *valid-list s2*
shows *s1* ≤ *s2* = *list-less-eq-ns s1 s2*
by (*simp add*: *assms order-le-less ordlistns.le-less valid-list-list-less-equiv-list-less-ns*)

27 Classical Lexicographical Order

lemma *valid-list-list-less-imp*:
assumes *valid-list* (*xs* @ [*bot*])
and *valid-list* (*ys* @ [*bot*])
and (*xs* @ [*bot*]) < (*ys* @ [*bot*])
shows *xs* < *ys*
proof −
from *assms*(*3*)
have *xs* @ [*bot*] ≠ *ys* @ [*bot*]
by *fastforce*
with *valid-list-neqE*[*OF assms*(*1*,*2*)]
obtain *x y as bs cs* **where**
xs @ [*bot*] = *as* @ *x* # *bs ys* @ [*bot*] = *as* @ *y* # *cs x* ≠ *y*
by *blast*
hence *x* < *y*
by (*metis assms*(*3*) *linorder-less-linear list-less-ex order-less-imp-not-less*)
then show *?thesis*
by (*metis* ‹*xs* @ [*bot*] = *as* @ *x* # *bs*› ‹*ys* @ [*bot*] = *as* @ *y* # *cs*› *append-self-conv*
bot.extremum-strict butlast.simps(*2*) *butlast-append last-snoc list-less-ex*
list-neq-rc1)
qed

lemma *strict-mono-on-list-less-map*:
fixes $\alpha :: 'a :: preorder \Rightarrow 'b :: ord$
assumes *strict-mono-on A* α
and *set xs* $\subseteq A$
and *set ys* $\subseteq A$
and *xs* $<$ *ys*
shows (*map* α *xs*) $<$ (*map* α *ys*)
using *assms*(*2*−*4*)
proof (*induct xs arbitrary*: *ys*)
case *Nil*
then show *?case*
using *list-le-def* **by** *fastforce*
next
case (*Cons a xs*)
note *IH* = *this*

have $\exists z\ zs.\ a \leq z \wedge ys = z \# zs$
by (*metis Cons-less-Cons IH*(*4*) *dual-order.refl dual-order.strict-iff-not neq-Nil-conv*
not-less-Nil)
then obtain *z zs* **where**
a $\leq$ *z ys* = *z* # *zs*
by *blast*
then show *?case*
using *IH assms*(*1*) *strict-mono-onD* **by** *fastforce*
qed

lemma *strict-mono-list-less-map*:
assumes *strict-mono* α
and *xs* $<$ *ys*
shows *map* α *xs* $<$ *map* α *ys*
using *assms*(*1*) *assms*(*2*) *strict-mono-on-list-less-map* **by** *blast*

lemma *strict-mono-on-map-list-less*:
fixes $\alpha :: 'a :: linorder \Rightarrow 'b :: order$
assumes *strict-mono-on A* α
and *set xs* $\subseteq A$
and *set ys* $\subseteq A$
and (*map* α *xs*) $<$ (*map* α *ys*)
shows *xs* $<$ *ys*
using *assms*(*2*−*4*)
proof (*induct xs arbitrary*: *ys*)
case *Nil*
then show *?case*
using *list-le-def* **by** *fastforce*
next
case (*Cons a xs*)
note *IH* = *this*

have *ys* = [] ∨ (∃ *b zs*. *ys* = *b* # *zs*)
using *neq-Nil-conv* **by** *blast*
moreover
have *ys* = [] ⟹ *?case*
using *Cons.prems* **by** *auto*
moreover
have ∃ *b zs*. *ys* = *b* # *zs* ⟹ *?case*
by (*metis IH*(2−4) *assms*(1) *linorder-neq-iff order-less-asym strict-mono-on-list-less-map*)
ultimately show *?case*
by *blast*
qed

lemma *strict-mono-map-list-less*:
fixes α :: 'a :: *linorder* ⇒ 'b :: *order*
assumes *strict-mono* α
and (*map* α *xs*) < (*map* α *ys*)
shows *xs* < *ys*
using *assms*(1) *assms*(2) *strict-mono-on-map-list-less* **by** *blast*

28 Non-standard Lexicographical Ordering

lemma *sorted-list-less-ns*:
assumes *sorted* (*a* # *bs* @ [*c*])
and *c* < *d*
shows *list-less-ns* (*a* # *bs* @ [*c*, *d*] @ *xs*) (*bs* @ [*c*, *d*] @ *ys*)
using *assms*
proof (*induct bs arbitrary*: *a*)
case *Nil*
then show *?case*
by (*metis append-Cons append-Nil le-less list-less-ns-cons-diff list-less-ns-cons-same sorted2*)
next
case (*Cons a bs x*)
note *IH* = *this*

from *IH*(2)
have *sorted* (*a* # *bs* @ [*c*])
by *simp*
with *IH*(1)[*OF - assms*(2)]
have *list-less-ns* (*a* # *bs* @ [*c*, *d*] @ *xs*) (*bs* @ [*c*, *d*] @ *ys*) .
with *sorted-nth-mono*[*OF IH*(2), *of 0 Suc 0*, *simplified*]
show *?case*
by (*simp add*: *list-less-ns-cons*)
qed

lemma *rev-sorted-list-less-ns*:
assumes *sorted* (*rev* (*a* # *bs* @ [*c*]))
and *c* > *d*
shows *list-less-ns* (*bs* @ [*c*, *d*] @ *xs*) (*a* # *bs* @ [*c*, *d*] @ *ys*)

```
  using assms
proof (induct bs arbitrary: a)
  case Nil
  then show ?case
    using list-less-ns-cons list-less-ns-cons-diff by fastforce
next
  case (Cons a bs x)
  note IH = this

  from IH(2)
  have sorted (rev (a # bs @ [c]))
    by (simp add: sorted-append)
  with IH(1)[OF - assms(2)]
  have list-less-ns (bs @ [c, d] @ xs) (a # bs @ [c, d] @ ys) .
  with sorted-rev-nth-mono[OF IH(2), of 0 Suc 0, simplified]
  show ?case
    using list-less-ns-cons by auto
qed

lemma sorted-cons-list-less-ns:
  assumes sorted (a # bs)
  shows list-less-ns (a # bs) bs
  using assms
proof (induct bs arbitrary: a)
case Nil
  then show ?case
    by (simp add: list-less-ns-nil)
next
  case (Cons a bs x)
  note IH = this

  from IH(2)
  have sorted (a # bs)
    by simp
  with IH(1)
  have list-less-ns (a # bs) bs .
  with sorted-nth-mono[OF IH(2), of 0 Suc 0, simplified]
  show ?case
    by (simp add: list-less-ns-cons)
qed

end
theory Suffix
  imports Main
begin
```

29 Suffix

```
abbreviation suffix :: 'a list ⇒ nat ⇒ 'a list
```

where
suffix xs i ≡ *drop i xs*

lemma *suffixes-neq*:
⟦*i* < *length s*; *j* < *length s*; *i* ≠ *j*⟧ ⟹ *suffix s i* ≠ *suffix s j*
by (*metis diff-diff-cancel length-drop less-or-eq-imp-le*)

lemma *distinct-suffixes*:
⟦*distinct xs*; ∀ *x* ∈ *set xs*. *x* < *length s*⟧ ⟹ *distinct* (*map* (*suffix s*) *xs*)
by (*simp add*: *distinct-conv-nth suffixes-neq*)

lemma *suffix-eq-index*:
⟦*i* < *length xs*; *j* < *length xs*; *suffix xs i* = *suffix xs j*⟧ ⟹ *i* = *j*
by (*metis diff-diff-cancel le-eq-less-or-eq length-drop*)

lemma *suffix-neq-nil*:
i < *length s* ⟹ *suffix s i* ≠ []
by *simp*

lemma *suffix-map*:
suffix (*map f xs*) *i* = *map f* (*suffix xs i*)
by (*simp add*: *drop-map*)

lemma *set-suffix-subset*:
set (*suffix s i*) ⊆ *set s*
by (*simp add*: *set-drop-subset*)
lemma *suffix-cons-suc*:
suffix (*a* # *xs*) (*Suc i*) = *suffix xs i*
by *simp*

lemma *suffix-app*:
i < *length xs* ⟹ *suffix* (*xs* @ *ys*) *i* = *suffix xs i* @ *ys*
by *simp*

lemma *suffix-cons-ex*:
i < *length T* ⟹ ∃ *x xs*. *suffix T i* = *x* # *xs* ∧ *x* = *T* ! *i*
by (*metis Cons-nth-drop-Suc*)

lemma *suffix-cons-Suc*:
i < *length T* ⟹ *suffix T i* = *T* ! *i* # *suffix T* (*Suc i*)
by (*metis Cons-nth-drop-Suc*)

lemma *suffix-cons-app*:
suffix T i = *as* @ *bs* ⟹ *suffix T* (*i* + *length as*) = *bs*
by (*metis add.commute append-eq-conv-conj drop-drop*)

lemma *suffix-0*:
suffix T 0 = *T*
by *simp*

end
theory *Suffix-Util*
imports
../util/List-Slice
Suffix
Valid-List
Valid-List-Util

begin

30 Valid Lists and Suffixes

lemma *valid-suffix*:
⟦*valid-list s*; *i* < *length s*⟧ ⟹ *valid-list* (*suffix s i*)
by (*clarsimp simp*: *valid-list-ex-def*)

lemma *last-suffix-index*:
assumes *valid-list s*
and *i* < *length s*
shows *hd* (*suffix s i*) = *bot* ⟷ *i* = *length s* − *1*
proof −
from *iffD1*[*OF valid-list-ex-def* ‹*valid-list s*›]
obtain *xs* **where**
s = *xs* @ [*bot*]
∀ *i* < *length xs*. *xs* ! *i* ≠ *bot*
by *blast*
show *?thesis*
proof
from ‹*s* = *xs* @ [*bot*]› ‹∀ *i* < *length xs*. *xs* ! *i* ≠ *bot*›
show *i* = *length s* − *1* ⟹ *hd* (*suffix s i*) = *bot*
by *simp*
next
from ‹*s* = *xs* @ [*bot*]› ‹∀ *i* < *length xs*. *xs* ! *i* ≠ *bot*› ‹*i* < *length s*›
show *hd* (*suffix s i*) = *bot* ⟹ *i* = *length s* − *1*
by (*clarsimp simp*: *hd-append hd-drop-conv-nth split*: *if-splits*)
qed
qed

31 Prefixes and Suffixes

lemma *suffix-has-no-prefix-suffix*:
assumes *valid-list*: *valid-list s*
and *i-less-len-s*: *i* < *length s*
and *j-less-len-s*: *j* < *length s*
and *i-neq-j*: *i* ≠ *j*
shows ¬ (∃ *s′*. *suffix s i* = (*suffix s j*) @ *s′*)
proof

assume $\exists\, s'.$ *suffix s i* = *suffix s j* @ s'
then obtain s' **where**
pref: *suffix s i* = *suffix s j* @ s'
by *blast*
with *i-neq-j i-less-len-s j-less-len-s*
have $i < j$
by (*metis diff-less-mono2 length-append length-drop less-Suc-eq not-add-less1 not-less-eq*)
with *pref i-less-len-s j-less-len-s*
have *s-not-nil*: $s' \neq []$
by (*metis append-Nil2 diff-less-mono2 length-drop less-irrefl-nat*)

from *valid-list i-less-len-s valid-suffix*
have *valid-suf-i*: *valid-list* (*suffix s i*)
by *force*

from *valid-list j-less-len-s valid-suffix*
have *valid-list* (*suffix s j*)
by *force*
with *pref valid-list-ex-def*
have $\exists\, xs.$ *suffix s i* = *xs* @ [*bot*] @ s'
using *append-assoc* **by** *auto*
then obtain *xs* **where**
suf-i: *suffix s i* = *xs* @ [*bot*] @ s'
by *blast*

from *valid-suf-i valid-list-ex-def*
have $\exists\, ys.$ *suffix s i* = *ys* @ [*bot*] $\wedge$ ($\forall\, k <$ *length ys*. *ys* ! $k \neq$ *bot*)
by *blast*
then obtain *ys* **where**
suffix s i = *ys* @ [*bot*] **and**
all-ys-not-0: $\forall\, k <$ *length ys*. *ys* ! $k \neq$ *bot*
by *blast*
with *suf-i*
have *suf-i-eq*: *xs* @ [*bot*] @ s' = *ys* @ [*bot*]
by *force*
with *s-not-nil*
have *length xs* < *length ys*
by (*metis* (*no-types*, *lifting*) *append-assoc append-eq-append-conv length-append length-append-singleton less-trans-Suc linorder-neqE-nat not-add-less1 self-append-conv*)
with *suf-i-eq all-ys-not-0*
show *False*
by (*metis append-Cons butlast-snoc nth-append-length nth-butlast*)
qed

32 Suffix Comparisons

32.1 Lexicographical Ordering

lemma *suffix-less-ex*:
fixes $s :: ('a :: \{linorder, order\text{-}bot\})\ list$
assumes *valid-list s*
and $i < length\ s$
and $j < length\ s$
and $suffix\ s\ i < suffix\ s\ j$
shows $\exists\, b\ c\ as\ bs\ cs.\ suffix\ s\ i = as\ @\ b\ \#\ bs \wedge$
$suffix\ s\ j = as\ @\ c\ \#\ cs \wedge b < c$
proof −
have *valid-list* (*suffix s i*)
using *assms*(*1*) *assms*(*2*) *valid-suffix* **by** *blast*
moreover
have *valid-list* (*suffix s j*)
using *assms*(*1*) *assms*(*3*) *valid-suffix* **by** *blast*
moreover
have $suffix\ s\ i \neq suffix\ s\ j$
using *assms*(*4*) *nless-le* **by** *blast*
ultimately have
$\exists\, b\ c\ as\ bs\ cs.\ suffix\ s\ i = as\ @\ b\ \#\ bs \wedge suffix\ s\ j = as\ @\ c\ \#\ cs \wedge b \neq c$
using *valid-list-neqE* **by** *blast*
then obtain *b c as bs cs* **where**
$suffix\ s\ i = as\ @\ b\ \#\ bs\ suffix\ s\ j = as\ @\ c\ \#\ cs\ b \neq c$
by *blast*
hence $b < c$
by (*metis assms*(*4*) *linorder-less-linear list-less-ex order-less-imp-not-less*)
then show *?thesis*
using ‹$suffix\ s\ i = as\ @\ b\ \#\ bs$› ‹$suffix\ s\ j = as\ @\ c\ \#\ cs$› **by** *blast*
qed

lemma *suffix-less-nth*:
assumes *valid-list s*
and $i < length\ s$
and $j < length\ s$
and $suffix\ s\ i < suffix\ s\ j$
shows
$\exists\, n.\ n < length\ (suffix\ s\ i) \wedge$
$n < length\ (suffix\ s\ j) \wedge$
$(\forall\, k < n.\ (suffix\ s\ i)\ !\ k = (suffix\ s\ j)\ !\ k) \wedge$
$(suffix\ s\ i)\ !\ n < (suffix\ s\ j)\ !\ n$
proof −
from *suffix-less-ex*[*OF assms*]
obtain *b c as bs cs* **where**
suf-i: $suffix\ s\ i = as\ @\ b\ \#\ bs$ **and**
suf-j: $suffix\ s\ j = as\ @\ c\ \#\ cs$ **and**
b-less-c: $b < c$
by *blast*

hence *length as* < *length* (*suffix s i*) **and**
length as < *length* (*suffix s j*) **and**
(*suffix s i*) ! *length as* = *b* **and**
(*suffix s j*) ! *length as* = *c*
by *fastforce*+
with *b-less-c suf-i suf-j*
show *?thesis*
by (*metis nth-append*)
qed

lemma *suffix-less-butlast*:
assumes *valid-list s*
and *i* < *length s*
and *j* < *length s*
and *suffix s i* < *suffix s j*
shows *butlast* (*suffix s i*) < *butlast* (*suffix s j*)
by (*metis append-butlast-last-id assms suffix-neq-nil valid-list-def valid-list-list-less-imp valid-suffix*)

32.2 Non-standard List Ordering

lemma *suffix-less-ns-ex*:
assumes *valid-list s*
and *i* < *length s*
and *j* < *length s*
and *list-less-ns* (*suffix s i*) (*suffix s j*)
shows ∃ *b c as bs cs*.
suffix s i = *as* @ *b* # *bs* ∧
suffix s j = *as* @ *c* # *cs* ∧ *b* < *c*
by (*meson assms suffix-less-ex valid-suffix valid-list-list-less-equiv-list-less-ns*)

lemma *suffix-less-ns-nth*:
assumes *valid-list s*
and *i* < *length s*
and *j* < *length s*
and *list-less-ns* (*suffix s i*) (*suffix s j*)
shows
∃ *n*. *n* < *length* (*suffix s i*) ∧
n < *length* (*suffix s j*) ∧
(∀ *k* < *n*. (*suffix s i*) ! *k* = (*suffix s j*) ! *k*) ∧
(*suffix s i*) ! *n* < (*suffix s j*) ! *n*
by (*meson assms suffix-less-nth valid-list-list-less-equiv-list-less-ns valid-suffix*)

33 List Slice

declare *list-slice.simps*[*simp del*]

lemma *list-slice-to-suffix*:
list-slice T i j = take (j − i) (suffix T i)
by (*simp add*: *list-slice.simps drop-take*)

lemma *suffix-eq-list-slice*:
suffix T i = list-slice T i (length T)
by (*simp add*: *list-slice.simps*)

lemma *list-slice-suffix*:
list-slice T i j = list-slice (suffix T i) 0 (j − i)
by (*metis drop0 drop-take list-slice.simps*)
lemma *suffix-to-list-slice-app*:
i ≤ j ⟹ suffix T i = (list-slice T i j) @ (list-slice T j (length T))
apply (*cases j ≤ length T*)
apply (*subst list-slice-append*[*symmetric*]; *simp?*)
apply (*clarsimp simp*: *list-slice.simps*)
apply (*clarsimp simp*: *not-le*)
apply (*subst list-slice-end-gre-length*, *arith*)
apply (*simp add*: *list-slice-start-gre-length list-slice.simps*)
done

34 Sorting

lemma *ordlist-strict-mono-strict-sorted-1*:
assumes *strict-mono α*
and *strict-sorted (map (suffix (map α s)) xs)*
shows *strict-sorted (map (suffix s) xs)*
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i < j j < length xs*
with *sorted-wrt-nth-less*[*OF assms*(*2*)]
have *suffix (map α s) (xs ! i) < suffix (map α s) (xs ! j)*
by *auto*
then show *suffix s (xs ! i) < suffix s (xs ! j)*
by (*metis assms*(*1*) *strict-mono-map-list-less suffix-map*)
qed

lemma *ordlist-strict-mono-on-strict-sorted-1*:
assumes *strict-mono-on A α*
and *set s ⊆ A*
and *strict-sorted (map (suffix (map α s)) xs)*
shows *strict-sorted (map (suffix s) xs)*
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i < j j < length xs*
with *sorted-wrt-nth-less*[*OF assms*(*3*)]
have *suffix (map α s) (xs ! i) < suffix (map α s) (xs ! j)*
by *auto*
hence *map α (suffix s (xs ! i)) < map α (suffix s (xs ! j))*

by (*metis suffix-map*)
then show *suffix s* (*xs* ! *i*) < *suffix s* (*xs* ! *j*)
by (*meson assms*(*1*,*2*) *dual-order.trans set-suffix-subset*
strict-mono-on-map-list-less)
qed

lemma *ordlist-strict-mono-strict-sorted-2*:
assumes *strict-mono* α
and *strict-sorted* (*map* (*suffix s*) *xs*)
shows *strict-sorted* (*map* (*suffix* (*map* α *s*)) *xs*)
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-nth-less*[*OF assms*(*2*)]
have *suffix s* (*xs* ! *i*) < *suffix s* (*xs* ! *j*)
by *auto*
then show *suffix* (*map* α *s*) (*xs* ! *i*) < *suffix* (*map* α *s*) (*xs* ! *j*)
by (*metis assms*(*1*) *strict-mono-list-less-map suffix-map*)
qed

lemma *ordlist-strict-mono-on-strict-sorted-2*:
assumes *strict-mono-on A* α
and *set s* ⊆ *A*
and *strict-sorted* (*map* (*suffix s*) *xs*)
shows *strict-sorted* (*map* (*suffix* (*map* α *s*)) *xs*)
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-nth-less*[*OF assms*(*3*)]
have *suffix s* (*xs* ! *i*) < *suffix s* (*xs* ! *j*)
by *auto*
moreover
have *set* (*suffix s* (*xs* ! *i*)) ⊆ *A*
by (*meson assms*(*2*) *dual-order.trans set-suffix-subset*)
moreover
have *set* (*suffix s* (*xs* ! *j*)) ⊆ *A*
by (*meson assms*(*2*) *dual-order.trans set-suffix-subset*)
ultimately show *suffix* (*map* α *s*) (*xs* ! *i*) < *suffix* (*map* α *s*) (*xs* ! *j*)
using *strict-mono-on-list-less-map*[*OF assms*(*1*)]
by (*metis suffix-map*)
qed

lemma *valid-list-ordlist-ordlistns-strict-sorted-eq*:
assumes *valid-list T*
and *set xs* ⊆ {*0*..<*length T*}
shows *ordlistns.strict-sorted* (*map* (*suffix T*) *xs*) ⟷
strict-sorted (*map* (*suffix T*) *xs*)
using *assms*
proof (*safe*)

```
  assume A: valid-list T and
         B: set xs ⊆ {0..<length T} and
         C: sorted-wrt list-less-ns (map (suffix T) xs)
  show sorted-wrt (<) (map (suffix T) xs)
  proof (intro sorted-wrt-mapI)
    fix i j
    assume i < j j < length xs
    with sorted-wrt-nth-less[OF C ‹i < j›]
    have R1: list-less-ns (suffix T (xs ! i)) (suffix T (xs ! j))
      by auto

    from B ‹i < j› ‹j < length xs›
    have xs ! i < length T
      by (meson atLeastLessThan-iff less-trans nth-mem subsetD)
    with valid-suffix[OF A]
    have R2: valid-list (suffix T (xs ! i))
      by simp

    from B ‹j < length xs›
    have xs ! j < length T
      by (meson atLeastLessThan-iff less-trans nth-mem subsetD)
    with valid-suffix[OF A]
    have R3: valid-list (suffix T (xs ! j))
      by simp

    from R1 valid-list-list-less-equiv-list-less-ns[OF R2 R3]
    show suffix T (xs ! i) < suffix T (xs ! j)
      by simp
  qed
next
  assume A: valid-list T and
         B: set xs ⊆ {0..<length T} and
         C: sorted-wrt (<) (map (suffix T) xs)
  show sorted-wrt list-less-ns (map (suffix T) xs)
  proof (intro sorted-wrt-mapI)
    fix i j
    assume i < j j < length xs
    with sorted-wrt-nth-less[OF C ‹i < j›]
    have R1: suffix T (xs ! i) < suffix T (xs ! j)
      by auto

    from B ‹i < j› ‹j < length xs›
    have xs ! i < length T
      by (meson atLeastLessThan-iff less-trans nth-mem subsetD)
    with valid-suffix[OF A]
    have R2: valid-list (suffix T (xs ! i))
      by simp

    from B ‹j < length xs›
```

```
    have xs ! j < length T
      by (meson atLeastLessThan-iff less-trans nth-mem subsetD)
    with valid-suffix[OF A]
    have R3: valid-list (suffix T (xs ! j))
      by simp

    from R1 valid-list-list-less-equiv-list-less-ns[OF R2 R3]
    show list-less-ns (suffix T (xs ! i)) (suffix T (xs ! j))
      by simp
  qed
qed

lemma Min-valid-suffix:
  assumes valid-list T
  and     length T = Suc n
shows ordlistns.Min {suffix T i |i. i < length T} = suffix T n
proof −
  from assms
  have suffix T n = [bot]
    by (metis add-diff-cancel-left' butlast-snoc length-butlast lessI list-slice-n-n
       nth-append-length plus-1-eq-Suc suffix-cons-Suc suffix-eq-list-slice valid-list-ex-def)

  have ∀ i < n. (suffix T i) ! 0 ≠ bot
   by (metis add-diff-cancel-left' assms last-suffix-index less-SucI list.sel(1) nat-neq-iff
         nth-Cons-0 plus-1-eq-Suc suffix-cons-Suc)
  hence A: ∀ i < n. bot < (suffix T i) ! 0
    using bot.not-eq-extremum by blast

  have B: ∀ i < length T. length (suffix T i) > 0
    by auto

  show ?thesis
  proof (intro ordlistns.Min-eqI conjI)
    show finite {suffix T i |i. i < length T}
      by simp
  next
    fix ys
    assume ys ∈ {suffix T i |i. i < length T}
    hence ∃ i < length T. ys = suffix T i
      by blast
    then obtain i where
      i < length T
      ys = suffix T i
      by blast

    with ‹ys = suffix T i›
    have R1: i = n ⟹ list-less-eq-ns (suffix T n) ys
      by simp
```

from ‹*i* < *length T*› *assms*(*2*)
have *R2-1*: *i* ≠ *n* ⟹ *i* < *n*
by *linarith*

from *A* ‹*suffix T n* = [*bot*]› ‹*i* < *length T*› ‹*ys* = *suffix T i*›
have *R2-2*: *i* < *n* ⟹ *list-less-eq-ns* (*suffix T n*) *ys*
by (*metis list-less-ns-cons-diff nth-Cons-0 ordlistns.less-imp-le suffix-cons-ex*)

from *R1 R2-2*[*OF R2-1*]
show *list-less-eq-ns* (*suffix T n*) *ys*
by *blast*
next
show *suffix T n* ∈ {*suffix T i* |*i*. *i* < *length T*}
using *assms*(*2*) **by** *auto*
qed
qed

end
theory *Prefix*
imports *Main*
begin

35 Prefix Definition

abbreviation *prefix* :: *'a list* ⇒ *nat* ⇒ *'a list*
where
prefix xs i ≡ *take i xs*

lemma *prefix-neq*:
assumes *i* < *length s*
and *j* < *length s*
and *i* ≠ *j*
shows *prefix s i* ≠ *prefix s j*
by (*metis assms length-take less-imp-le min-absorb2*)

lemma *not-prefix-app*:
(∀ *k*. *s1* ≠ *prefix s2 k*) ⟷ (∀ *xs*. *s2* ≠ *s1* @ *xs*)
by (*metis append-eq-conv-conj append-take-drop-id*)

lemma *not-prefix-imp-not-nil*:
∀ *k*. *s1* ≠ *prefix s2 k* ⟹ *s1* ≠ []
by (*metis take0*)

end
theory *Prefix-Util*
imports *Prefix ../order/Suffix-Util*
begin

lemma *prefix-suffix-not-suffix*:

assumes *valid-list s*
and $i < length\ s$
and $j < length\ s$
and $i \neq j$
shows $\neg(\exists k.\ prefix\ (suffix\ s\ i)\ k = suffix\ s\ j)$
using *suffix-has-no-prefix-suffix assms*
by (*metis append-take-drop-id*)

end
theory *Suffix-Array*
imports
../util/Sorting-Util
../order/List-Lexorder-Util
../order/Suffix
../order/Valid-List
../order/List-Permutation-Util
begin

36 Axiomatic Suffix Array Specification

locale *Suffix-Array-General* =
fixes *sa* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list*
assumes *sa-g-permutation*: *sa s* <~~> [*0*..<*length s*]
and *sa-g-sorted*: *strict-sorted* (*map* (*suffix s*) (*sa s*))

locale *Suffix-Array-Restricted* =
fixes *sa* :: *nat list* ⇒ *nat list*
assumes *sa-r-permutation*: *valid-list s* ⟹ *sa s* <~~> [*0*..<*length s*]
and *sa-r-sorted*: *valid-list s* ⟹ *strict-sorted* (*map* (*suffix s*) (*sa s*))

37 Wrapper for Natural Number String only Algorithm

definition *sa-nat-wrapper* ::
('*a* :: *linorder list* ⇒ '*a* ⇒ *nat*) ⇒ (*nat list* ⇒ *nat list*) ⇒ '*a* :: *linorder list* ⇒ *nat list*
where
sa-nat-wrapper α *sa xs* =
tl (*sa* ((*map* (λ*x*. *Suc* (α *xs x*)) *xs*) @ [*bot*]))

end
theory *Suffix-Array-Properties*
imports
../util/Fun-Util
../order/Suffix-Util
Suffix-Array

begin

38 General Suffix Array Properties

context *Suffix-Array-General* **begin**

lemma *sa-length*:
length (*sa s*) = *length s*
by (*metis Suffix-Array-General-axioms Suffix-Array-General-def length-upt minus-nat.diff-0*
perm-length)

lemma *sa-distinct*:
distinct (*sa s*)
using *Suffix-Array-General.sa-g-permutation Suffix-Array-General-axioms*
perm-distinct-set-of-upt-iff **by** *blast*

lemma *sa-set-upt*:
set (*sa s*) = {*0*..<*length s*}
using *Suffix-Array-General.sa-g-permutation Suffix-Array-General-axioms*
perm-distinct-set-of-upt-iff **by** *blast*

lemma *sa-nth-ex*:
i < *length s* ⟹ ∃ *k* < *length s*. *sa s* ! *i* = *k*
by (*metis atLeastLessThan-iff nth-mem sa-length sa-set-upt*)

lemma *ex-sa-nth*:
k < *length s* ⟹ ∃ *i* < *length s*. *sa s* ! *i* = *k*
by (*metis atLeast0LessThan in-set-conv-nth lessThan-iff sa-length sa-set-upt*)

end

lemma *Suffix-Array-General-determinism*:
assumes *Suffix-Array-General f*
and *Suffix-Array-General g*
shows *f* = *g*
proof
fix *s*
from *distinct-suffixes*[*OF Suffix-Array-General.sa-distinct*[*OF assms*(*1*)], *of s s*]
Suffix-Array-General.sa-set-upt[*OF assms*(*1*), *of s*]
have *distinct* (*map* (*suffix s*) (*f s*))
using *atLeastLessThan-iff* **by** *blast*
moreover
from *distinct-suffixes*[*OF Suffix-Array-General.sa-distinct*[*OF assms*(*2*)], *of s s*]
Suffix-Array-General.sa-set-upt[*OF assms*(*2*), *of s*]
have *distinct* (*map* (*suffix s*) (*g s*))
using *atLeastLessThan-iff* **by** *blast*
moreover
from *Suffix-Array-General.sa-set-upt*[*OF assms*(*1*), *of s*]
Suffix-Array-General.sa-set-upt[*OF assms*(*2*), *of s*]
have *set* (*map* (*suffix s*) (*f s*)) = *set* (*map* (*suffix s*) (*g s*))

by *simp*
ultimately have *map* (*suffix s*) (*f s*) = *map* (*suffix s*) (*g s*)
using *strict-sorted-distinct-set-unique*[
OF Suffix-Array-General.sa-g-sorted[*of f*, *OF assms*(*1*)] -
Suffix-Array-General.sa-g-sorted[*of g*, *OF assms*(*2*)],
of s s]
by *blast*
moreover
from *Suffix-Array-General.sa-set-upt*[*OF assms*(*1*), *of s*]
Suffix-Array-General.sa-set-upt[*OF assms*(*2*), *of s*]
have *inj-on* (*suffix s*) (*set* (*f s*) ∪ *set* (*g s*))
by (*simp add*: *inj-on-def suffix-eq-index*)
ultimately show *f s* = *g s*
using *map-inj-on*[*of suffix s f s g s*]
by *blast*
qed

39 Properties of Suffix Arrays on Valid Lists

lemma *valid-list-bot-min*:
assumes *valid-list* (*s* @ [*bot*])
and *sa* (*s* @ [*bot*]) <~~> [*0*..<*length* (*s* @ [*bot*])]
and *strict-sorted* (*map* (*suffix* (*s* @ [*bot*])) (*sa* (*s* @ [*bot*])))
shows ∃ *xs*. *sa* (*s* @ [*bot*]) = *length s* # *xs*
proof −
have *suffix* (*s* @ [*bot*]) (*length s*) = [*bot*]
by *simp*

have *P*: ∀ *i* < *length s*. *suffix* (*s* @ [*bot*]) (*length s*) < *suffix* (*s* @ [*bot*]) *i*
proof(*safe*)
fix *i*
assume *i* < *length s*
have ∃ *a as*. *suffix* (*s* @ [*bot*]) *i* = *a* # *as* ∧ *a* ≠ *bot*
by (*metis Cons-nth-drop-Suc* ‹*i* < *length s*› *assms*(*1*) *butlast-snoc length-append-singleton*
less-SucI nth-butlast valid-list-ex-def)
then obtain *a as* **where**
suffix (*s* @ [*bot*]) *i* = *a* # *as* ∧ *a* ≠ *bot*
by *blast*
moreover
from ‹*suffix* (*s* @ [*bot*]) (*length s*) = [*bot*]›
have *suffix* (*s* @ [*bot*]) (*length s*) = *bot* # []
by *simp*
ultimately show *suffix* (*s* @ [*bot*]) (*length s*) < *suffix* (*s* @ [*bot*]) *i*
by (*simp add*: *bot.not-eq-extremum*)
qed

have *Min* (*set* (*map* (*suffix* (*s* @ [*bot*])) (*sa* (*s* @ [*bot*]))))
= *suffix* (*s* @ [*bot*]) (*length s*)
proof (*intro Min-eqI*)

```
    show finite (set (map (suffix (s @ [bot])) (sa (s @ [bot]))))
      by blast
  next
    fix y
    assume y ∈ set (map (suffix (s @ [bot])) (sa (s @ [bot])))
    with set-perm-upt[OF assms(2)]
    have ∃ i < length (s @ [bot]). y = suffix (s @ [bot]) i
      by auto
    then obtain i where
      i < length (s @ [bot])
      y = suffix (s @ [bot]) i
      by blast
    hence i < length s ∨ i = length s
      by (simp add: less-Suc-eq)
    moreover
    have i < length s ⟹ suffix (s @ [bot]) (length s) < y
      using P ‹y = suffix (s @ [bot]) i› dual-order.strict-iff-order by blast
    moreover
    have i = length s ⟹ suffix (s @ [bot]) (length s) ≤ y
      by (simp add: ‹y = suffix (s @ [bot]) i›)
    ultimately show suffix (s @ [bot]) (length s) ≤ y
      using nless-le by blast
  next
    from assms
    have length s ∈ set (sa (s @ [bot]))
     by (metis ex-perm-nth length-append-singleton lessI nth-mem perm-upt-length)
    then show suffix (s @ [bot]) (length s) ∈ set (map (suffix (s @ [bot])) (sa (s @
[bot])))
      by force
  qed
  hence map (suffix (s @ [bot])) (sa (s @ [bot])) ! 0 = suffix (s @ [bot]) (length s)
    using assms(2,3) strict-sorted-Min by fastforce
  hence suffix (s @ [bot]) ((sa (s @ [bot])) ! 0) = suffix (s @ [bot]) (length s)
    by (metis assms(1,2) nth-map perm-upt-length valid-list-length)
  hence (sa (s @ [bot])) ! 0 = length s
     by (metis Suc-le-eq ‹suffix (s @ [bot]) (length s) = [bot]› assms(1) drop-all
last-suffix-index
            list.distinct(1) list.sel(1) not-less-eq-eq)
  then show ?thesis
     by (metis append-eq-Cons-conv assms(1,2) id-take-nth-drop perm-upt-length
take0
            valid-list-length)
qed

lemma valid-list-bot-perm:
  assumes valid-list (s @ [bot])
  and     sa (s @ [bot]) <~~> [0..<length (s @ [bot])]
  and     strict-sorted (map (suffix (s @ [bot])) (sa (s @ [bot])))
shows ∃ xs. sa (s @ [bot]) = length s # xs ∧ xs <~~> [0..<length s]
```

proof −
from *valid-list-bot-min*[*OF assms*(*1*), *of sa*, *OF assms*(*2*,*3*)]
obtain *xs* **where**
sa (*s* @ [*bot*]) = *length s* # *xs*
by *blast*
with *assms*(*2*)
have *length s* # *xs* <~~> [*0*..<*length* (*s* @ [*bot*])]
by *simp*
then show *?thesis*
by (*metis* ‹*sa* (*s* @ [*bot*]) = *length s* # *xs*› *assms*(*1*) *length-append-singleton less-Suc-eq-le*
perm-append2-eq perm-append-single upt-Suc valid-list-length)
qed

lemma *valid-list-bot-perm-sort*:
assumes *valid-list* (*s* @ [*bot*])
and *sa* (*s* @ [*bot*]) <~~> [*0*..<*length* (*s* @ [*bot*])]
and *strict-sorted* (*map* (*suffix* (*s* @ [*bot*])) (*sa* (*s* @ [*bot*])))
shows ∃*xs*. *sa* (*s* @ [*bot*]) = *length s* # *xs* ∧ *xs* <~~> [*0*..<*length s*] ∧
strict-sorted (*map* (*suffix s*) *xs*)
proof −
from *valid-list-bot-perm*[*OF assms*(*1*), *of sa*, *OF assms*(*2*,*3*)]
obtain *xs* **where**
sa (*s* @ [*bot*]) = *length s* # *xs*
xs <~~> [*0*..<*length s*]
by *blast*
with *assms*(*3*)
have *strict-sorted* (*map* (*suffix* (*s* @ [*bot*])) (*length s* # *xs*))
by *simp*
hence *strict-sorted* ((*suffix* (*s* @ [*bot*]) (*length s*)) # *map* (*suffix* (*s* @ [*bot*])) *xs*)
by *simp*
hence *P*: *strict-sorted* (*map* (*suffix* (*s* @ [*bot*])) *xs*)
using *strict-sorted-simps*(*2*) **by** *blast*

have *strict-sorted* (*map* (*suffix s*) *xs*)
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-nth-less*[*OF P*, *of i j*]
have *suffix* (*s* @ [*bot*]) (*xs* ! *i*) < *suffix* (*s* @ [*bot*]) (*xs* ! *j*)
by *auto*
moreover
have *xs* ! *i* < *length s*
using ‹*i* < *j*› ‹*j* < *length xs*› ‹*xs* <~~> [*0*..<*length s*]› *perm-distinct-set-of-upt-iff*
by *auto*
hence *suffix* (*s* @ [*bot*]) (*xs* ! *i*) = *suffix s* (*xs* ! *i*) @ [*bot*]
using *suffix-app* **by** *blast*
moreover
have *xs* ! *j* < *length s*

```
      using ‹j < length xs› ‹xs <~~> [0..<length s]› perm-distinct-set-of-upt-iff
by auto
    hence suffix (s @ [bot]) (xs ! j) = suffix s (xs ! j) @ [bot]
      using suffix-app by blast
    moreover
    have valid-list (suffix s (xs ! i) @ [bot])
      using ‹xs ! i < length s› assms valid-suffix by fastforce
    moreover
    have valid-list (suffix s (xs ! j) @ [bot])
      using ‹xs ! j < length s› assms valid-suffix by fastforce
    ultimately show suffix s (xs ! i) < suffix s (xs ! j)
      by (simp add: valid-list-list-less-imp)
  qed
  with ‹sa (s @ [bot]) = length s # xs› ‹xs <~~> [0..<length s]›
  show ?thesis
    by blast
qed

theorem Suffix-Array-Restricted-valid-list-bot-perm-sort:
  assumes valid-list (s @ [bot])
  and     Suffix-Array-Restricted sa
shows ∃xs. sa (s @ [bot]) = length s # xs ∧ xs <~~> [0..<length s] ∧
          strict-sorted (map (suffix s) xs)
proof (rule valid-list-bot-perm-sort[OF assms(1)])
  from assms
  show sa (s @ [bot]) <~~> [0..<length (s @ [bot])]
    using Suffix-Array-Restricted-def by blast
next
  from assms
  show strict-sorted (map (suffix (s @ [bot])) (sa (s @ [bot])))
    using Suffix-Array-Restricted-def by blast
qed

lemma Suffix-Array-Restricted-wrapper-permutation:
  assumes Linorder-to-Nat-List α s
  and Suffix-Array-Restricted sa
shows sa-nat-wrapper α sa s <~~> [0..<length s]
proof −
  let ?α = α s
  let ?f = λx. Suc (?α x)
  let ?s = map ?f s

  have valid-list (?s @ [bot])
    using valid-list-Suc-mapping by blast
 with Suffix-Array-Restricted-valid-list-bot-perm-sort[OF - ‹Suffix-Array-Restricted
-›]
  obtain xs where
    sa (?s @ [bot]) = length ?s # xs
    xs <~~> [0..<length ?s]
```

strict-sorted (*map* (*suffix ?s*) *xs*)
by *blast*
then show *?thesis*
by (*simp add*: *sa-nat-wrapper-def*)
qed

lemma *Suffix-Array-Restricted-wrapper-sorted*:
assumes *Linorder-to-Nat-List* α *s*
and *Suffix-Array-Restricted sa*
shows *strict-sorted* (*map* (*suffix s*) (*sa-nat-wrapper* α *sa s*))
proof −
let *?α* = α *s*
let *?f* = λ*x*. *Suc* (*?α x*)
let *?s* = *map ?f s*

have *valid-list* (*?s* @ [*bot*])
using *valid-list-Suc-mapping* **by** *blast*
with *Suffix-Array-Restricted-valid-list-bot-perm-sort*[*OF* - ‹*Suffix-Array-Restricted* -›]
obtain *xs* **where** *A*:
sa (*?s* @ [*bot*]) = *length ?s* # *xs*
xs <~~> [*0*..<*length ?s*]
strict-sorted (*map* (*suffix ?s*) *xs*)
by *blast*
hence *xs* <~~> [*0*..<*length s*]
by *simp*
with *ordlist-strict-mono-on-strict-sorted-1*[
OF Linorder-to-Nat-List.strict-mono-on-Suc-map-to-nat[*OF assms*(*1*)] - *A*(*3*)]
show *?thesis*
by (*simp add*: *A*(*1*) *sa-nat-wrapper-def*)
qed

40 Equivalence

lemma *Suffix-Array-General-imp-Restrict*:
Suffix-Array-General sa-nat ⟹ *Suffix-Array-Restricted sa-nat*
using *Suffix-Array-General-def Suffix-Array-Restricted.intro* **by** *blast*

interpretation *Linorder-to-Nat-List map-to-nat*
proof
show *strict-mono-on* (*set xs*) (*map-to-nat xs*)
by (*simp add*: *map-to-nat-strict-mono-on*)
qed

lemma *Suffix-Array-Restricted-imp-General*:
Suffix-Array-Restricted sa ⟹ *Suffix-Array-General* (*sa-nat-wrapper map-to-nat sa*)
using *Linorder-to-Nat-List-axioms Suffix-Array-General-def*

Suffix-Array-Restricted-wrapper-permutation Suffix-Array-Restricted-wrapper-sorted
by *blast*

lemma *Suffix-Array-General-Restrict-determinism*:
assumes *Suffix-Array-Restricted f*
and *Suffix-Array-General g*
shows *sa-nat-wrapper map-to-nat f = g*
by (*simp add*: *Suffix-Array-General-determinism Suffix-Array-Restricted-imp-General assms*)

end
theory *Simple-SACA*
imports
../order/Suffix
../order/List-Lexorder-Util
begin

fun *gen-suffixes* :: ('*a* :: {*linorder*,*order-bot*}) *list* ⇒ '*a list list*
where
gen-suffixes s = *map* (*suffix s*) [*0*..<(*length s*)]

fun *suffix-ids* :: ('*a* :: {*linorder*,*order-bot*}) *list* ⇒ '*a list list* ⇒ *nat list*
where
suffix-ids s ss = *map* (λ*x*. *length s* − *length x*) *ss*

fun *simple-saca* :: ('*a* :: {*linorder*,*order-bot*}) *list* ⇒ *nat list*
where
simple-saca s = *suffix-ids s* (*sort* (*gen-suffixes s*))

end
theory *Simple-SACA-Verification*
imports
Simple-SACA
../spec/Suffix-Array
begin

lemma *suf-length-app*:
i < *length xs* ⟹ *length* (*suffix* (*xs* @ *ys*) *i*) = *length* (*suffix xs i*) + *length ys*
apply (*induct xs*)
apply *simp*
apply *simp*
done

lemma *distinct-natlist-add*:
distinct (*xs* :: *nat list*) ⟹ *distinct* (*map* ((+) *n*) *xs*)
apply (*induct xs arbitrary*: *n*)
apply *simp*
apply *clarsimp*
done

lemma *nat-minus-cancel-right*:
 ⟦(*x*::*nat*) ≤ *n*; *y* ≤ *n*; *n* − *x* = *n* − *y*⟧ ⟹ *x* = *y*
 apply (*subst* (*asm*) *le-imp-diff-is-add*, *simp*)
 apply (*subst* (*asm*) *add.commute*)
 apply (*subst* (*asm*) *add-diff-assoc*, *simp*)
 apply (*subst* (*asm*) *add.commute*)
 apply (*drule sym*)
 apply (*subst* (*asm*) *Nat.le-imp-diff-is-add*, *simp*)
 apply *clarsimp*
 done

lemma *distinct-natlist-sub*:
 ⟦*distinct* (*xs* :: *nat list*); ∀ *x* ∈ *set xs*. *x* ≤ *n*⟧ ⟹ *distinct* (*map* ((−) *n*) *xs*)
 by (*meson distinct-map inj-onI nat-minus-cancel-right*)

lemma *map-suf-app*:
 n ≤ *length xs* ⟹
 map (*length* ∘ *suffix* (*xs* @ *ys*)) [*0*..<*n*] = *map* ((+) (*length ys*)) (*map* (*length* ∘ (*suffix xs*)) [*0*..<*n*])
 apply (*induct xs*)
 apply *simp*
 apply *clarsimp*
 apply (*subst add.commute*)
 apply *simp*
 done

lemma *distinct-map-length-gen-suffixes*:
 distinct (*map length* (*gen-suffixes s*))
 apply (*induct s rule*: *rev-induct*)
 apply *simp*
 apply (*simp only*: *gen-suffixes.simps map-map length-append*)
 apply (*subst upt-add-eq-append*; *simp only*: *map-append*)
 apply (*subst map-suf-app*; *simp only*: *distinct-append*)
 apply (*rule conjI*)
 apply (*rule distinct-natlist-add*; *simp*)
 apply (*rule conjI*; *clarsimp*)
 done

lemma *different-length-different-list*:
 length a ∉ *length* ‘ *set xs* ⟹ *a* ∉ *set xs*
 apply *blast*
 done

lemma *distinct-map-length-sort*:
 distinct (*map length xs*) ⟹ *distinct* (*map length* (*sort xs*))
 apply (*induct xs*)
 apply *simp*
 apply *clarsimp*

apply (*rule card-distinct*)
apply *simp*
apply (*drule distinct-card*)
apply *clarsimp*
apply (*frule different-length-different-list*)
apply (*subst insort-insert-insort*[*symmetric*]; *simp*)
apply (*subst set-insort-insert*)
apply *simp*
done

lemma *suffix-ids-def'*:
suffix-ids s xs = *map* (((−) (*length s*)) ∘ *length*) *xs*
apply *simp*
done

lemma *distinct-simple-saca*:
distinct (*simple-saca s*)
apply (*subst simple-saca.simps*)
apply (*subst suffix-ids-def'*)
apply (*subst map-map*[*symmetric*])
apply (*rule distinct-natlist-sub*)
apply (*rule distinct-map-length-sort*[*OF distinct-map-length-gen-suffixes*])
apply *clarsimp*
done

lemma *suf-suffix-id-suf*:
i < *length s* ⟹ *suffix s* (*length s* − *length* (*suffix s i*)) = *suffix s i*
apply (*induct s arbitrary*: *i*)
apply *simp*
apply *clarsimp*
done

lemma *in-set-ordlist-sort*:
(*x* ∈ *set xs*) = (*x* ∈ *set* (*sort xs*))
by *simp*

lemma *ordlist-sort-conv-nth*:
(∃ *i*<*length xs*. *xs* ! *i* = *x*) = (∃ *i*<*length xs*. (*sort xs*) ! *i* = *x*)
by (*metis in-set-conv-nth length-sort set-sort*)

lemma *ordlist-sort-nth-before*:
⟦*i* < *length xs*; (*sort xs*) ! *i* = *x*⟧ ⟹
∃ *j*<*length xs*. *xs* ! *j* = *x*
apply (*subst ordlist-sort-conv-nth*)
apply *blast*
done

lemma *suf-sort-suf-nth*:
i < *length s* ⟹

```
   suffix s (length s − length ((sort (gen-suffixes s)) ! i)) =
   sort (gen-suffixes s) ! i
proof −
  assume i < length s


  have ∃x. sort (gen-suffixes s) ! i = x
    by blast
  then obtain x where
    sort (gen-suffixes s) ! i = x
    by blast
  with ordlist-sort-nth-before[of i gen-suffixes s x]
  have ∃j<length (gen-suffixes s). gen-suffixes s ! j = x
    by (simp add: ‹i < length s›)
  then obtain j where
    j < length (gen-suffixes s)
    gen-suffixes s ! j = x
    by blast
  hence sort (gen-suffixes s) ! i = gen-suffixes s ! j
    using ‹sort (gen-suffixes s) ! i = x› by blast
  moreover
  have j < length s
    using ‹j < length (gen-suffixes s)› by auto
  hence gen-suffixes s ! j = suffix s j
    by simp
  ultimately show ?thesis
    by (metis ‹j < length s› suf-suffix-id-suf)
qed

lemma map-suf-simple-saca:
  map (suffix s) (simple-saca s) = sort (gen-suffixes s)
  apply (simp only: simple-saca.simps suffix-ids.simps)
  apply (subst list-eq-iff-nth-eq)
  apply (rule conjI)
   apply simp
  apply (clarsimp simp del:  gen-suffixes.simps)
  apply (rule suf-sort-suf-nth; simp)
  done

interpretation simple-saca: Suffix-Array-General simple-saca
proof
  fix s :: ('a :: {linorder, order-bot}) list

  show simple-saca s <~~> [0..<length s]
  proof −
    have set (simple-saca s) = {0..<length s}
      by force
   with perm-distinct-set-of-upt-iff[THEN iffD2, OF conjI, OF distinct-simple-saca]
    show ?thesis
```

```
    by blast
  qed

  show strict-sorted (map (suffix s) (simple-saca s))
  proof −
    from ‹simple-saca s <~~> [0..<length s]›
    have set (simple-saca s) = {0..<length s}
      using perm-distinct-set-of-upt-iff by blast
    hence ∀ x ∈ set (simple-saca s). x < length s
      using atLeastLessThan-iff by blast
    with distinct-simple-saca  distinct-suffixes
    have distinct (map (suffix s) (simple-saca s))
      by blast

    have sorted (map (suffix s) (simple-saca s))
      by (metis map-suf-simple-saca sorted-sort)
    with ‹distinct (map (suffix s) (simple-saca s))› show ?thesis
      using strict-sorted-iff by blast
  qed
qed

end
theory List-Type
  imports
    ../../util/Nat-Util
    ../../util/Set-Util
    ../../util/Fun-Util
    ../../util/List-Util
    ../../order/Suffix-Util
    ../../order/Valid-List-Util
    ../../spec/Suffix-Array-Properties
begin
```

This theory file contains the background theory for the SAIS algorithm (Nong et al., DCC 2009), which is essentially an optimisation of the KA algorithm (Ko et al, JDA 2005).

41 Small and Large List Types

datatype *SL-types* = *S-type* | *L-type*

This section contains a generalisation of the suffix types to sequences of any type and any element comparison function that satisfies certain properties given the theorem. Typical constraints involve either one or a combination of *totalp-on*, *irreflp-on*, *transp-on* and *asymp-on*.

definition
list-type :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ *SL-types*
where

list-type cmp xs =
(*if nslexordp cmp xs* (*suffix xs* (*Suc 0*))
then S-type
else L-type)

lemma *list-type-cons-same*:
⟦*irreflp-on A cmp*; *x* ∈ *A*⟧ ⟹ *list-type cmp* (*x* # *x* # *xs*) = *list-type cmp* (*x* # *xs*)
by (*clarsimp simp*: *list-type-def irreflp-onD*)

lemma *list-type-nil*:
list-type cmp [] = *L-type*
by (*clarsimp simp*: *list-type-def nslexordp-def*)

lemma *list-type-singleton*:
list-type cmp [*x*] = *S-type*
by (*simp add*: *nslexordp-def list-type-def*)

lemma *list-type-s-type-eq*:
list-type cmp xs = *S-type* ⟷ *nslexordp cmp xs* (*suffix xs* (*Suc 0*))
by (*simp add*: *list-type-def*)

lemma *list-type-l-type-eq*:
list-type cmp xs = *L-type* ⟷ ¬*nslexordp cmp xs* (*suffix xs* (*Suc 0*))
by (*simp add*: *list-type-def*)

lemma *list-type-cons-diff1*:
cmp x y ⟹ *list-type cmp* (*x* # *y* # *xs*) = *S-type*
by (*simp add*: *list-type-s-type-eq*)

lemma *list-type-cons-diff2*:
⟦¬*cmp x y*; *x* ≠ *y*⟧ ⟹ *list-type cmp* (*x* # *y* # *xs*) = *L-type*
by (*clarsimp simp add*: *list-type-l-type-eq*)

lemma *list-type-s-neq-nil*:
list-type cmp xs = *S-type* ⟹ *xs* ≠ []
by (*metis SL-types.simps*(*2*) *list-type-nil*)

lemma *list-type-s-hd-cmp*:
list-type cmp (*x* # *y* # *xs*) = *S-type* ⟹ *cmp x y* ∨ *x* = *y*
by (*metis SL-types.simps*(*2*) *list-type-cons-diff2*)

lemma *list-type-l-hd-cmp*:
list-type cmp (*x* # *y* # *xs*) = *L-type* ⟹ ¬*cmp x y* ∨ *x* = *y*
by (*metis SL-types.simps*(*2*) *list-type-cons-diff1*)

lemma *list-type-repl*:
⟦*irreflp-on A cmp*; *x* ∈ *A*; *set xs* = {*x*}⟧ ⟹ *list-type cmp* (*x* # *xs*) = *S-type*
apply (*induct xs*; *simp add*: *list-type-cons-same*)

using *list-type-singleton subset-singletonD* **by** *fastforce*

lemma *list-type-s-ex*:
assumes *list-type cmp* $(x \# xs) = S\text{-}type$
shows $(\forall\, a \in set\ xs.\ a = x) \vee (\exists\, b\ as\ bs.\ x \# xs = as\ @\ x \# b \# bs \wedge cmp\ x\ b \wedge (\forall\, k \in set\ as.\ k = x))$
proof −
from *list-type-s-type-eq*[*THEN iffD1*, *OF assms*(*1*)]
have *nslexordp cmp* $(x \# xs)$ *xs*
by *simp*
with *nslexordp-cons2-exE*[*of cmp x xs*]
show *?thesis*
by *blast*
qed

lemma *list-type-l-type-ex*:
assumes *list-type cmp* $(x \# xs) = L\text{-}type$
and *totalp-on A cmp*
and $x \in A$
and $set\ xs \subseteq A$
shows $\exists\, b\ as\ bs.\ x \# xs = as\ @\ x \# b \# bs \wedge cmp\ b\ x \wedge (\forall\, k \in set\ as.\ k = x)$
proof −
from *list-type-l-type-eq*[*THEN iffD1*, *OF assms*(*1*)]
have ¬ *nslexordp cmp* $(x \# xs)$ *xs*
by *simp*
moreover
have $x \# xs \neq xs$
by *fastforce*
ultimately have *nslexordp cmp xs* $(x \# xs)$
using *totalp-onD*[*OF nslexordp-totalp-on*[*OF assms*(*2*)]]
by (*metis assms*(*3*,*4*) *insert-subset list.simps*(*15*) *mem-Collect-eq*)
with *nslexordp-cons1-exE*[*of cmp xs x*]
show *?thesis*
by *blast*
qed

theorem *l-less-than-s-type-list-type*:
assumes *list-type cmp* $(a \# s1) = S\text{-}type$
and *list-type cmp* $(a \# s2) = L\text{-}type$
and *totalp-on A cmp*
and *transp-on A cmp*
and $a \in A$
and $set\ s1 \subseteq A$
and $set\ s2 \subseteq A$
shows *nslexordp cmp* $(a \# s2)$ $(a \# s1)$
proof −
from *list-type-l-type-ex*[*OF assms*(*2*,*3*,*5*,*7*)]
obtain *b as bs* **where**
$a \# s2 = as\ @\ a \# b \# bs$

```
    cmp b a
    ∀k∈set as. k = a
    by blast
  hence S2: a # s2 = replicate (length as) a @ a # b # bs
    by (simp add: replicate-length-same)

  let ?c1 = ∀x ∈ set s1. x = a
  and ?c2 = ∃d cs ds. a # s1 = cs @ a # d # ds ∧ cmp a d ∧ (∀k∈set cs. k =
a)

  from list-type-s-ex[OF assms(1)]
  have ?c1 ∨ ?c2
    by blast
  moreover
  have ?c1 ⟹ ?thesis
proof −
    assume ?c1

    have length s1 ≤ length as ⟹ ?thesis
    proof −
      assume length s1 ≤ length as
      have a # s1 = replicate (length s1) a @ [a]
        by (metis ‹?c1› replicate-append-same replicate-length-same)
      hence ∃es. replicate (length as) a @ [a] = a # s1 @ es
      by (metis ‹length s1 ≤ length as› le-add-diff-inverse list.simps(1) replicate-add
              replicate-append-same)
      then show ?thesis
        by (metis S2 SL-types.simps(2) append.assoc append.right-neutral assms(1)
assms(2)
                    list.sel(3) neq-Nil-conv nslexordpI2 nslexordp-cons-cons repli-
cate-app-Cons-same)
    qed
    moreover
    have length s1 > length as ⟹ ?thesis
    proof −
      assume length s1 > length as
      hence ∃es. s1 = replicate (length as) a @ a # es
        by (metis ‹?c1› add-Suc-right less-iff-Suc-add replicate-Suc replicate-add
              replicate-length-same)
      then obtain es where
        s1 = replicate (length as) a @ a # es
        by blast
      hence S1: a # s1 = replicate (length as) a @ a # a # es
        by (simp add: replicate-app-Cons-same)
      then show ?thesis
     by (metis S2 ‹cmp b a› nslexordpI1 nslexordp-cons-cons replicate-app-Cons-same)
    qed
    ultimately show ?thesis
      by linarith
```

qed
moreover
have *?c2* $\Longrightarrow$ *?thesis*
proof −
assume *?c2*
then obtain *d cs ds* **where**
$a \# s1 = cs \mathbin{@} a \# d \# ds$
cmp a d
$\forall k \in set\ cs.\ k = a$
by *blast*
hence *S1*: $a \# s1 = replicate\ (length\ cs)\ a \mathbin{@} a \# d \# ds$
by (*simp add*: *replicate-length-same*)

from *transp-onD*[*OF assms*(*4*) - *assms*(*5*) - ‹*cmp b a*› ‹*cmp a d*›]
have *cmp b d*
by (*metis S1 S2 add-diff-cancel-left′ assms*(*6*,*7*) *length-Cons length-append less-Suc-eq-le*
list.simps(*1*) *nth-append-length nth-mem replicate-app-Cons-same subsetD zero-le*
zero-less-diff)
hence *length cs* = *length as* $\Longrightarrow$ *?thesis*
by (*metis S1 S2 nslexordpI1 nslexordp-cons-cons replicate-app-Cons-same*)
moreover
have *length as* < *length cs* $\Longrightarrow$ *?thesis*
proof −
assume *length as* < *length cs*
hence $\exists es.\ replicate\ (length\ cs)\ a = replicate\ (length\ as)\ a \mathbin{@} a \# es \wedge (\forall k \in set\ es.\ k = a)$
by (*metis* (*no-types*, *lifting*) *Cons-nth-drop-Suc S1* ‹*a* # *s1* = *cs* @ *a* # *d* # *ds*› *add-Suc-right*
add-diff-cancel-left′ append-same-eq drop-replicate in-set-replicate less-iff-Suc-add nth-mem replicate-add)
then obtain *es* **where**
$replicate\ (length\ cs)\ a = replicate\ (length\ as)\ a \mathbin{@} a \# es$
$\forall k \in set\ es.\ k = a$
by *blast*
then show *?thesis*
by (*metis S1 S2* ‹*cmp b a*› *append.assoc append-Cons nslexordpI1 replicate-app-Cons-same*)
qed
moreover
have *length cs* < *length as* $\Longrightarrow$ *?thesis*
proof −
assume *length cs* < *length as*
hence $\exists es.\ replicate\ (length\ as)\ a = replicate\ (length\ cs)\ a \mathbin{@} a \# es \wedge (\forall k \in set\ es.\ k = a)$
by (*metis* (*no-types*, *lifting*) *Cons-nth-drop-Suc S2* ‹*a* # *s2* = *as* @ *a* # *b* # *bs*› *add-Suc-right*
add-diff-cancel-left′ append-same-eq drop-replicate

in-set-replicate less-iff-Suc-add nth-mem replicate-add)
then obtain *es* **where**
replicate (*length as*) *a* = *replicate* (*length cs*) *a* @ *a* # *es*
∀ *k* ∈ *set es*. *k* = *a*
by *blast*
then show *?thesis*
by (*metis S1 S2* ‹*cmp a d*› *append.assoc append-Cons nslexordpI1 replicate-app-Cons-same*)
qed
ultimately show *?thesis*
by *linarith*
qed
ultimately show *?thesis*
by *blast*
qed

lemma *list-type-cons-diff-type1*:
⟦*list-type cmp* (*a* # *b* # *xs*) = *S-type*; *list-type cmp* (*b* # *xs*) = *L-type*⟧ ⟹
cmp a b
by (*simp add*: *list-type-l-type-eq list-type-s-type-eq*)

lemma *list-type-cons-diff-type2*:
⟦*list-type cmp* (*a* # *b* # *xs*) = *L-type*; *list-type cmp* (*b* # *xs*) = *S-type*⟧ ⟹
¬*cmp a b* ∧ *a* ≠ *b*
by (*simp add*: *list-type-l-type-eq list-type-s-type-eq*)

42 Suffix Type

This section contains the suffix type definition.

definition *suffix-type* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ *SL-types*
where
suffix-type s i ≡
(*if list-less-ns* (*suffix s i*) (*suffix s* (*Suc i*)) *then S-type*
else L-type)

lemma *suffix-type-list-type-eq*:
suffix-type xs i = *list-type* (<) (*suffix xs i*)
by (*clarsimp simp*: *suffix-type-def list-type-def nslexordp-eq-list-less-ns*)

There are two types of suffixes (*SL-types*): *S-type* and *L-type*. An S-type suffix is a suffix that is strictly less than the suffix that occurs immediately after it, and an L-type suffix is a suffix that is strictly greater than the suffix that occurs immediately after it. The definition of less than used here is *list-less-ns*. Note that this definition of less than differs from lexicographical order(*list-less*, i.e. dictionary order, but it is equivalent when the both lists are valid (*valid-list*) as shown in ⟦*valid-list ?s1.0*; *valid-list ?s2.0*⟧ ⟹ (*?s1.0* < *?s2.0*) = *list-less-ns ?s1.0 ?s2.0*. There are three reasons for using the *list-less-ns* definition, and we explain in order of importance.

The first reason is that the original suffix types definition required a special case for the singleton suffix that only contains the sentinel symbol. While this special case makes sense in regards to the algorithms, i.e. it is necessary for the correctness of the algorithms, it does not naturally follow from the intuition of suffix types. In fact, it contradicts the intuitive definition that follows from the lexicographical order *list-less*. That is, a list that only consists of one element is always strictly greater than the empty list. With the alternate definition of less than *list-less-ns*, a proper prefix is always strictly greater, and so, a singleton list will always be strictly less than the empty list. Therefore, there is no need to have a special case for the singleton suffix that only contains the sentinel.

The second reason is that the SAIS algorithm uses a sublist order that depends on the suffix type definition (see Section S̈AIS Sublist Order). This definition is perfectly valid for the algorithm, since the ordering is only used for sublist of the same list. However, the ordering is not easily understandable when applied to arbitrary list, even though it is equivalent to *list-less-ns*, which we prove in a later section. As an ordering, *list-less-ns* is much easier to understand. It is also used within the definition of *suffix-type*. Therefore, it makes more sense to reuse *list-less-ns*, rather than having multiple definitions of the same thing.

The third reason is that the original suffix types definition does not handle the case where the suffix is not terminated by sentinel symbol. The reason for this is that it is assumed that all lists are terminated by the sentinel. This assumption is very important to the SAIS algorithm as it is central to its correctness argument. That being said, in terms of elegance and consistency, using *list-less-ns* requires the least amount of special cases.

42.1 General Suffix Type Simplifications

This section contains theorems that simplify the use of the definition *suffix-type*.

lemma *suffix-type-cons-suc*:
suffix-type (*a* # *s*) (*Suc i*) = *suffix-type s i*
by (*simp add*: *suffix-type-def*)

lemma *suffix-type-cons-same*:
suffix-type (*x* # *x* # *xs*) *0* = *suffix-type* (*x* # *xs*) *0*
by (*simp add*: *list-less-ns-cons-same suffix-type-def*)

lemma *suffix-type-suffix*:
suffix-type s i = *suffix-type* (*suffix s i*) *0*
by (*simp add*: *suffix-type-list-type-eq*)

lemma *suffix-type-suffix-gen*:
suffix-type (*suffix s n*) *i* = *suffix-type s* (*i* + *n*)

by (*simp add*: *suffix-type-list-type-eq*)

lemma *suffix-type-eq-Suc*:
suffix-type xs n = *suffix-type xs* (*Suc n*) ⟹
suffix-type xs n = *S-type* ∨ *suffix-type xs* (*Suc n*) = *L-type*
using *SL-types.exhaust* **by** *auto*

42.2 S-Type Simplifications

This subsection contains theorems about facts that can be derived S-type suffixes and vice versa.

lemma *suffix-is-bot*:
suffix s i = [*bot*] ⟹ *suffix-type s i* = *S-type*
by (*simp add*: *list-type-singleton suffix-type-list-type-eq*)

lemma *suffix-is-singleton*:
suffix s i = [*x*] ⟹ *suffix-type s i* = *S-type*
by (*simp add*: *list-type-singleton suffix-type-list-type-eq*)

lemma *suffix-type-last*:
length xs = *Suc n* ⟹ *suffix-type xs n* = *S-type*
by (*simp add*: *list-less-ns-nil suffix-type-def*)

lemma *s-type-list-less-ns*:
suffix-type s i = *S-type* ⟷ *list-less-ns* (*suffix s i*) (*suffix s* (*Suc i*))
by (*metis SL-types.simps*(*2*) *suffix-type-def*)

lemma *nth-less-imp-s-type*:
⟦*Suc i* < *length s*; *s* ! *i* < *s* ! *Suc i*⟧ ⟹ *suffix-type s i* = *S-type*
by (*metis Cons-nth-drop-Suc Suc-lessD less-imp-le list-less-ns-cons neq-iff s-type-list-less-ns*)

lemma *sl-type-hd-less*:
⟦*Suc i* < *length s*; *hd* (*suffix s i*) < *hd* (*suffix s* (*Suc i*))⟧ ⟹
suffix-type s i = *S-type*
by (*simp add*: *hd-drop-conv-nth nth-less-imp-s-type*)

lemma *suffix-type-cons-less*:
x < *y* ⟹ *suffix-type* (*x* # *y* # *xs*) *0* = *S-type*
by (*clarsimp simp*: *suffix-type-def list-less-ns-cons-diff*)

lemma *suffix-type-s-bound*:
suffix-type s i = *S-type* ⟹ *i* < *length s*
using *ordlistns.less-asym s-type-list-less-ns* **by** *fastforce*

lemma *s-type-letter-le-Suc*:
⟦*Suc i* < *length T*; *suffix-type T i* = *S-type*⟧ ⟹
T ! *i* ≤ *T* ! (*Suc i*)
by (*metis Cons-nth-drop-Suc Suc-lessD leI list-less-ns-cons-diff ordlistns.less-asym*
s-type-list-less-ns)

lemma *s-type-ex*:
assumes *suffix-type* (*x* # *xs*) *0* = *S-type*
shows (∀ *a* ∈ *set xs*. *a* = *x*) ∨ (∃ *b as bs*. *x* # *xs* = *as* @ *x* # *b* # *bs* ∧ *x* < *b* ∧ (∀ *k* ∈ *set as*. *k* = *x*))
by (*metis assms drop0 list-type-s-ex suffix-type-list-type-eq*)

42.3 L-Type Simplifications

This subsection contains theorems about facts that can be derived from L-type suffixes and vice versa.

lemma *suffix-is-nil*:
suffix s i = [] ⟹ *suffix-type s i* = *L-type*
by (*clarsimp simp*: *suffix-type-def split*: *if-splits*)

lemma *l-type-list-less-ns*:
suffix-type s i = *L-type* ⟷ *list-less-ns* (*suffix s* (*Suc i*)) (*suffix s i*) ∨ *suffix s i* = []
by (*metis Cons-nth-drop-Suc SL-types.distinct*(*1*) *drop-Nil drop-eq-Nil linorder-le-less-linear not-Cons-self2 ordlistns.less-imp-not-less ordlistns.neqE suffix-type-def suffix-type-suffix*)

lemma *nth-gr-imp-l-type*:
⟦*Suc i* < *length s*; *s* ! *i* > *s* ! *Suc i*⟧ ⟹ *suffix-type s i* = *L-type*
by (*metis Cons-nth-drop-Suc Suc-lessD list-less-ns-cons-diff ordlistns.less-asym suffix-type-def*)

lemma *sl-type-hd-greater*:
⟦*Suc i* < *length s*; *hd* (*suffix s i*) > *hd* (*suffix s* (*Suc i*))⟧ ⟹
suffix-type s i = *L-type*
by (*simp add*: *hd-drop-conv-nth nth-gr-imp-l-type*)

lemma *suffix-type-cons-greater*:
x > *y* ⟹ *suffix-type* (*x* # *y* # *xs*) *0* = *L-type*
by (*simp add*: *list-type-cons-diff2 suffix-type-list-type-eq*)

lemma *l-type-letter-gre-Suc*:
⟦*i* < *length T*; *suffix-type T i* = *L-type*⟧ ⟹
T ! (*Suc i*) ≤ *T* ! *i*
by (*metis SL-types.distinct*(*1*) *Suc-lessI not-less nth-less-imp-s-type suffix-type-last*)

lemma *l-type-ex*:
assumes *suffix-type* (*x* # *xs*) *0* = *L-type*
shows ∃ *b as bs*. *x* # *xs* = *as* @ *x* # *b* # *bs* ∧ *x* > *b* ∧ (∀ *k* ∈ *set as*. *k* = *x*)
by (*metis assms drop0 drop-Suc-Cons l-type-list-less-ns list.discI list-less-ns-cons1-exE*)

An overlooked property, but one that is crucial for completeness of the SAIS algorithm

lemma *suffix-max-hd-is-l-type*:

```
  assumes valid-list s
  and     i < length s
  and     length s > Suc 0
  and     hd (suffix s i) = Max (set s)
shows suffix-type s i = L-type
  using assms
proof (induct s arbitrary: i)
  case Nil
  then show ?case
    by simp
next
  case (Cons a s i)
  note IH = this
  show ?case
  proof (cases s)
    case Nil
    then show ?thesis
      using IH(4) by auto
  next
    case (Cons b xs)
    assume s = b # xs
    show ?thesis
    proof (cases xs)
      case Nil
      hence s = [b]
        by (simp add: local.Cons)
      moreover
      have b = bot
        by (metis IH(2) last.simps local.Cons local.Nil not-Cons-self
              valid-list-iff-butlast-app-last)
      moreover
      have a > b
      by (metis IH(2,4) One-nat-def antisym-conv3 bot.extremum-strict nth-Cons-0
valid-list-def
                zero-less-diff calculation(2))
      ultimately show ?thesis
     by (metis IH(2,3,5) Max-greD add-diff-cancel-left' last-suffix-index length-Cons
less-Suc0
            linorder-not-less list.size(3) not-less-eq not-less-iff-gr-or-eq nth-Cons-0
              plus-1-eq-Suc suffix-type-cons-greater)
    next
      case (Cons c ys)
      hence s = b # c # ys
        by (simp add: ‹s = b # xs›)
      show ?thesis
      proof (cases i)
        case 0
        hence a ≥ b
          by (metis IH(4,5) List.finite-set Max-ge ‹s = b # xs› drop0 list.sel(1)
```

```
nth-Cons-0
                nth-Cons-Suc nth-mem)
      hence a > b ∨ a = b
        using antisym-conv1 by blast
      then show ?thesis
      proof
        assume b < a
        then show ?thesis
          by (simp add: 0 ‹s = b # xs› suffix-type-cons-greater)
      next
        assume a = b
        hence Max (set s) = b
          using 0 IH(5) ‹s = b # xs› by auto
        with IH(1)[of 0]
        have suffix-type s 0 = L-type
            by (metis IH(2) Suc-less-eq2 ‹s = b # xs› drop0 drop-Suc-Cons
length-Cons list.sel(1)
                local.Cons valid-suffix zero-less-Suc)
        then show ?thesis
          by (simp add: 0 ‹a = b› ‹s = b # xs› suffix-type-cons-same)
      qed
    next
      case (Suc n)
      assume i = Suc n
      have valid-list s
        using IH(2) ‹s = b # xs› valid-list-consD by blast
      moreover
      have n < length s
        using IH(3) Suc by auto
      moreover
      have Suc 0 < length s
        by (simp add: ‹s = b # xs› local.Cons)
      moreover
      {
        have hd (suffix s n) = hd (suffix (a # s) i)
          using Suc by fastforce
        moreover
        have hd (suffix s n) ∈ set s
          by (simp add: ‹n < length s› hd-drop-conv-nth)
        with IH(5)
        have Max (set (a # s)) ∈ set s
          using calculation by argo
        hence Max (set (a # s)) = Max (set s)
          using IH(5) max.cobounded1[of a Max (set s)]
         by (metis List.finite-set Max-greD Max-insert Suc-lessD ‹Suc 0 < length
s›
                ‹n < length s› calculation hd-drop-conv-nth length-greater-0-conv
list.set(2)
                max-def set-empty)
```

```
        ultimately have hd (suffix s n) = Max (set s)
          using IH(5) by presburger
      }
      ultimately have suffix-type s n = L-type
        using Cons.hyps by blast
      then show ?thesis
        by (simp add: Suc suffix-type-cons-suc)
    qed
  qed
 qed
qed
```

42.4 General Suffix Type Theories

This subsection contains the background theory needed to prove that computing the suffix types of a list can be achieved in linear time by starting from the end of the list (lemma 1, Ko et al., JDA 2005).

The main intuition is that the suffix type of the (i+1)th suffix is known and the ith suffix starts with same symbol of the (i+1)th suffix, then the ith suffix will have the same type.

```
theorem sl-type-hd-equal:
  ⟦Suc i < length s; hd (suffix s i) = hd (suffix s (Suc i))⟧ ⟹
   suffix-type s i = suffix-type s (Suc i)
   by (metis Cons-nth-drop-Suc Suc-lessD hd-drop-conv-nth l-type-letter-gre-Suc
list-less-ns-cons
          suffix-type-def)

corollary sl-type-prefix-equal:
  ⟦i + n ≤ length s; ∀ j < n. hd (suffix s (i + j)) = hd (suffix s i)⟧ ⟹
   ∀ j < n. suffix-type s (i + j) = suffix-type s i
proof (induct n)
 case 0
 then show ?case
   by blast
next
 case (Suc n)
 note IH = this
 hence ∀ j<n. suffix-type s (i + j) = suffix-type s i
   by (metis add-Suc-right less-Suc-eq linorder-not-less)
 show ?case
 proof safe
   fix j
   assume j < Suc n
   then show suffix-type s (i + j) = suffix-type s i
   proof (cases j < n)
     assume j < n
     then show ?thesis
       by (simp add: ‹∀ j<n. suffix-type s (i + j) = suffix-type s i›)
```

```
    next
      assume ¬ j < n
      hence j = n
        using ‹j < Suc n› by auto
      show ?thesis
      proof (cases j)
        case 0
        then show ?thesis
          by simp
      next
        case (Suc m)
        then show ?thesis
          by (metis Suc.prems(1,2) Suc-lessD Suc-less-SucD ‹j < Suc n› ‹j = n›
add-Suc-right
                    ‹∀j<n. suffix-type s (i + j) = suffix-type s i› le-imp-less-Suc
                    sl-type-hd-equal)
      qed
    qed
  qed
qed

corollary sl-type-prefix-equal-nth:
  ⟦i + n ≤ length s; ∀j < n. (suffix s i) ! j = (suffix s i) ! 0⟧ ⟹
   ∀j < n. suffix-type s (i + j) = suffix-type s i
  by (rule sl-type-prefix-equal, assumption, clarsimp simp: hd-conv-nth)

corollary sl-type-prefix-replicate:
  ∀i < n. suffix-type (replicate n a @ as) i = suffix-type (replicate n a @ as) 0
  by (rule sl-type-prefix-equal-nth[where i = 0, simplified]; clarsimp simp: nth-append)

lemma suffix-type-neq:
  ⟦suffix-type T j ≠ suffix-type T (Suc j); Suc j < length T⟧ ⟹ T ! j ≠ T ! Suc j
  by (metis Cons-nth-drop-Suc Suc-lessD l-type-letter-gre-Suc list-less-ns-cons suf-
fix-type-def)
```

42.5 S/L-Type Ordering

This section contains the crucial theorem that L-type suffixes are always less than S-type suffixes if they start with the same symbol (lemma 2, Ko et al., JDA 2005).

```
theorem l-less-than-s-type-general:
  assumes suffix-type (a # s1) 0 = S-type
  and     suffix-type (a # s2) 0 = L-type
shows list-less-ns (a # s2) (a # s1)
proof −
  from suffix-type-list-type-eq[of a # s1 0]
  have suffix-type (a # s1) 0 = list-type (<) (a # s1)
    by simp
  hence list-type (<) (a # s1) = S-type
```

```
    using assms(1) by auto
  moreover
  from suffix-type-list-type-eq[of a # s2 0]
  have suffix-type (a # s2) 0 = list-type (<) (a # s2)
    by simp
  hence list-type (<) (a # s2) = L-type
    using assms(2) by auto
  ultimately show ?thesis
    using l-less-than-s-type-list-type[of (<) a s1 s2]
   by (meson UNIV-I nslexordp-eq-list-less-ns-app top-greatest totalp-on-less transp-on-less)
qed

corollary l-less-than-s-type-suffix:
  assumes i < length s
  and     j < length s
  and     s ! i = s ! j
  and     suffix-type s i = S-type
  and     suffix-type s j = L-type
shows list-less-ns (suffix s j) (suffix s i)
  by (metis Cons-nth-drop-Suc assms l-less-than-s-type-general suffix-type-suffix)

theorem l-less-than-s-type:
  assumes valid-list s
  and     i < length s
  and     j < length s
  and     hd (suffix s i) = hd (suffix s j)
  and     suffix-type s i = S-type
  and     suffix-type s j = L-type
shows list-less-ns (suffix s j) (suffix s i)
  by (metis hd-drop-conv-nth assms(2-) l-less-than-s-type-suffix)

corollary (in Suffix-Array-General) same-hd-s-after-l:
  assumes valid-list: valid-list s
  and     i-less-len-s: i < length s
  and     j-less-len-s: j < length s
  and     i-neq-j:      i ≠ j
  and     suf-i-type:   suffix-type s ((sa s)! i) = L-type
  and     suf-j-type:   suffix-type s ((sa s)! j) = S-type
  and     hd-eq:        hd (suffix s ((sa s) ! i)) = hd (suffix s ((sa s) ! j))
shows i < j
proof -
  have A: (sa s) ! i < length s
    using i-less-len-s sa-nth-ex by auto

  from sa-set-upt[where s = s] sa-length[where s = s] j-less-len-s
  have B: (sa s) ! j < length s
    by (metis atLeast0LessThan lessThan-iff nth-mem)

  from l-less-than-s-type[OF valid-list B A hd-eq[symmetric] suf-j-type suf-i-type]
```

```
  have suf-i-less-suf-j: list-less-ns (suffix s ((sa s)! i)) (suffix s ((sa s)! j))
    by simp

  from sorted-nth-less-mono[OF strict-sorted-imp-sorted[OF sa-g-sorted],
                                simplified length-map sa-length,
                                OF i-less-len-s j-less-len-s i-neq-j]
     nth-map[where f = suffix s and xs = sa s, simplified sa-length, OF i-less-len-s]
     nth-map[where f = suffix s and xs = sa s, simplified sa-length, OF j-less-len-s]
      valid-list-list-less-equiv-list-less-ns[OF valid-suffix,
                                              OF valid-list A valid-suffix,
                                              OF valid-list B,
                                              THEN iffD2,
                                              OF suf-i-less-suf-j]
  show ?thesis by simp
qed
```

42.6 Implementation of Suffix Type Computation

This subsection contain a shallow embedding of a function that would compute the suffix types for a list.

```
fun abs-get-suffix-types :: ('a :: {linorder, order-bot}) list ⇒ SL-types list
  where
abs-get-suffix-types [] = [] |
abs-get-suffix-types ([-]) = [S-type] |
abs-get-suffix-types (a # b # xs) =
  (let ys = abs-get-suffix-types (b # xs)
   in
     (if a < b then S-type # ys
      else if a > b then L-type # ys
      else hd (ys) # ys))

lemma length-abs-get-suffix-types:
  length (abs-get-suffix-types s) = length s
  by (induct s rule: abs-get-suffix-types.induct; clarsimp simp: Let-def)

lemma abs-get-suffix-types-correct-nth:
  i < length s ⟹ abs-get-suffix-types s ! i = suffix-type s i
proof (induct s arbitrary: i rule: abs-get-suffix-types.induct)
  case 1
  then show ?case
    by simp
next
  case (2 uu i)
  then show ?case
    by (simp add: suffix-is-singleton)
next
  case (3 a b xs i)
  note IH = this
```

have $i \neq 0 \Longrightarrow$ *?case*
proof −
  assume $i \neq 0$
  hence $\exists n.\ i = Suc\ n$
    using *not0-implies-Suc* **by** *blast*
  then obtain n **where**
    $i = Suc\ n$
    by *blast*
  hence *abs-get-suffix-types* $(a \# b \# xs)\ !\ i =$ *abs-get-suffix-types* $(b \# xs)\ !\ n$
    by (*clarsimp simp*: *Let-def*)
  moreover
  have *suffix-type* $(a \# b \# xs)\ i =$ *suffix-type* $(b \# xs)\ n$
    by (*simp add*: ‹$i = Suc\ n$› *suffix-type-cons-suc*)
  moreover
  from *IH*(*1*)[*of n*] *IH*(*2*) ‹$i = Suc\ n$›
  have *abs-get-suffix-types* $(b \# xs)\ !\ n =$ *suffix-type* $(b \# xs)\ n$
    by *simp*
  ultimately show *?thesis*
    by *simp*
qed
moreover
have $i = 0 \Longrightarrow$ *?case*
proof −
  assume $i = 0$

  have $a < b \vee b < a \vee a = b$
    by *fastforce*
  then show *?case*
  proof (*elim disjE*)
    assume $a < b$
    then show *?case*
      by (*simp add*: ‹$i = 0$› *suffix-type-cons-less*)
  next
    assume $b < a$
    then show *?case*
      using ‹$i = 0$› *order-less-imp-triv suffix-type-cons-greater* **by** *fastforce*
  next
    assume $a = b$

    have *abs-get-suffix-types* $(b \# xs) \neq []$
      by (*metis Zero-neq-Suc length-Cons length-abs-get-suffix-types list.size*(*3*))
    hence *abs-get-suffix-types* $(a \# b \# xs)\ !\ i =$ *abs-get-suffix-types* $(b \# xs)\ !\ 0$
      unfolding *abs-get-suffix-types.simps*(*3*)[*of a b xs*, *simplified Let-def*]
      by (*simp add*: ‹$a = b$› ‹$i = 0$› *hd-conv-nth*)
    moreover
    have *suffix-type* $(a \# b \# xs)\ i =$ *suffix-type* $(b \# xs)\ 0$
      by (*simp add*: ‹$a = b$› ‹$i = 0$› *suffix-type-cons-same*)
    ultimately show *?case*
      using *IH*(*1*)[*of 0*, *simplified*] **by** *presburger*

qed
qed
ultimately show *?case*
by *blast*
qed

lemma *get-suffix-types-correct*:
$\forall i < length\ s.\ (abs\text{-}get\text{-}suffix\text{-}types\ s)\ !\ i = suffix\text{-}type\ s\ i$
by (*simp add*: *abs-get-suffix-types-correct-nth*)

43 SAIS Sublist Order

This section contains the sublist ordering used in SAIS (definition 2.3, Nong et al., DCC 2009). Note that this generalised so that it is not a ternary relation but a binary relation.

fun *ss-order-less* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ '*a list* ⇒ *bool*
where
ss-order-less [] - = *False* |
ss-order-less - [] = *True* |
ss-order-less (*a* # *as*) (*b* # *bs*) =
(*if a* < *b then True*
else if a > *b then False*
else if suffix-type (*a* # *as*) *0* = *suffix-type* (*b* # *bs*) *0 then ss-order-less as bs*
else if suffix-type (*a* # *as*) *0* = *L-type then True*
else False)

As described in section "Suffix Type", the SAIS sublist ordering (*ss-order-less*) is equivalent to *list-less-ns*.

lemma *ss-order-less-equiv-list-less-ns*:
ss-order-less s1 s2 = *list-less-ns s1 s2*
proof (*induct rule*: *ss-order-less.induct*)
case (*1 uu*)
then show *?case*
by *simp*
next
case (*2 v va*)
then show *?case*
by (*simp add*: *list-less-ns-nil*)
next
case (*3 a as b bs*)
then show *?case*
by (*metis antisym-conv3 l-less-than-s-type-general list-less-ns-cons-diff list-less-ns-cons-same*
ordlistns.less-asym ss-order-less.simps(*3*) *suffix-type-def*)
qed

44 Sorting

lemma *sorted-letters-s-types*:

```
  assumes ∀ k≥i. k < j ⟶ suffix-type T k = S-type
  and     j ≤ length T
  shows sorted (list-slice T i j)
proof (intro sorted-iff-nth-mono[THEN iffD2] allI impI)
  fix x y
  assume x ≤ y y < length (list-slice T i j)

  have list-slice T i j ! x = T ! (i + x)
    by (meson ‹x ≤ y› ‹y < length (list-slice T i j)› dual-order.strict-trans2
nth-list-slice)
  moreover
  have list-slice T i j ! y = T ! (i + y)
    using ‹y < length (list-slice T i j)› nth-list-slice
    by blast
  moreover
  have i + y < j
    using ‹y < length (list-slice T i j)›
    by (simp add: assms(2))
  have i + x ≤ i + y
    by (simp add: ‹x ≤ y›)
  with ‹i + y < j›
  have T ! (i + x) ≤ T ! (i + y)
  proof (induct y - x arbitrary: x y)
    case 0
    then show ?case by simp
  next
    case (Suc z)
    note IH = this
    from IH(2)
    have ∃ y'. y = Suc y'
      by presburger
    then obtain y' where
      y = Suc y'
      by blast
    hence z = y' - x
      using IH(2) by linarith
    moreover
    have i + y' < j
      using Suc.prems(1) ‹y = Suc y'› by linarith
    moreover
    have i + x ≤ i + y'
      using Suc.hyps(2) ‹y = Suc y'› by linarith
    ultimately have T ! (i + x) ≤ T ! (i + y')
      using Suc.hyps(1) by blast
    moreover
    from assms(1)
    have suffix-type T (i + y') = S-type
      by (simp add: ‹i + y' < j›)
    hence T ! (i + y') ≤ T ! (i + y)
```

```
      using Suc.prems(1) ‹y = Suc y'› assms(2) less-le-trans s-type-letter-le-Suc
by fastforce
    ultimately  show ?case
      using order.trans by blast
  qed
  ultimately show list-slice T i j ! x ≤ list-slice T i j ! y
    by simp
qed

lemma sorted-letters-l-types:
  assumes ∀ k≥i. k < j ⟶ suffix-type T k = L-type
  and     j ≤ length T
shows sorted ((rev (list-slice T i j)))
proof (intro sorted-rev-iff-nth-mono[THEN iffD2] allI impI)
  fix x y
  assume x ≤ y y < length (list-slice T i j)

  have length (list-slice T i j) = j − i
    by (simp add: assms(2))

  have i + y < j
    using ‹length (list-slice T i j) = j − i› ‹y < length (list-slice T i j)› by linarith
  with ‹x ≤ y›
  have T ! (i + y) ≤ T ! (i + x)
  proof (induct y − x arbitrary: x y)
    case 0
    then show ?case by simp
  next
    case (Suc z x y)
    note IH = this
    have ∃ y'. y = Suc y' ∧ x ≤ y' ∧ z = y' − x
    by (metis Suc.hyps(2) Suc.prems(1) add-Suc-right add-diff-cancel-left' add-diff-cancel-right'
              diff-le-self ordered-cancel-comm-monoid-diff-class.add-diff-inverse)
    then obtain y' where
      y = Suc y' x ≤ y' z = y' − x
      by blast
    with IH(1) IH(4)
    have T ! (i + x) ≥ T ! (i + y')
      by simp
    moreover
    from assms(1)
    have suffix-type T (i + y') = L-type
      using Suc.prems(2) ‹y = Suc y'› by auto
    hence T ! (i + y') ≥ T ! (i + y)
    using Suc.prems(2) ‹y = Suc y'› assms(2) nth-less-imp-s-type order-less-le-trans
by fastforce
    ultimately show ?case
      by auto
  qed
```

moreover
have *list-slice T i j ! x = T ! (i + x)*
by (*metis ‹x ≤ y› ‹y < length (list-slice T i j)› nth-list-slice order-le-less-trans*)
moreover
have *list-slice T i j ! y = T ! (i + y)*
using *‹y < length (list-slice T i j)› nth-list-slice* **by** *blast*
ultimately show *list-slice T i j ! y ≤ list-slice T i j ! x*
by *presburger*
qed

45 LMS-Types

This section contains the definition of an LMS-type; standing for large, middle and small. It also contains lemmas pertaining to these types.

definition
abs-is-lms :: (*'a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ *bool*
where
abs-is-lms s i ≡
(*suffix-type s i = S-type*) ∧
(∃ *j*. *i = Suc j* ∧
suffix-type s j = L-type)

LMS-types are subtypes of *S-type*. This is because these are *S-type*, but they are also immediately succeed *L-type*.

45.1 LMS-Type Simplifications

This subsection contains theorems about facts that can be derived from the *abs-is-lms* definition and vice versa.

lemma *lms-type-list-less-ns*:
abs-is-lms s i = (∃ *j*. *i = Suc j* ∧ *list-less-ns* (*suffix s i*) (*suffix s j*) ∧
list-less-ns (*suffix s i*) (*suffix s* (*Suc i*)))
by (*metis SL-types.simps*(*2*) *abs-is-lms-def l-type-list-less-ns ordlistns.antisym-conv3*

s-type-list-less-ns)

lemma *abs-is-lms-0*:
¬*abs-is-lms s 0*
apply (*clarsimp simp*: *abs-is-lms-def*)
done

lemma *abs-is-lms-cons-suc*:
i > 0 ⟹ *abs-is-lms* (*a # s*) (*Suc i*) = *abs-is-lms s i*
apply (*drule gr0-implies-Suc*; *clarsimp*)
apply (*clarsimp simp*: *abs-is-lms-def suffix-type-cons-suc*)
done

lemma *i-s-type-imp-Suc-i-not-lms*:
suffix-type s i = S-type $\Longrightarrow$ $\neg$*abs-is-lms s* (*Suc i*)
by (*simp add*: *abs-is-lms-def*)

lemma *suffix-type-same-imp-not-lms*:
suffix-type s i = suffix-type s (*Suc i*) $\Longrightarrow$ $\neg$*abs-is-lms s* (*Suc i*)
by (*simp add*: *abs-is-lms-def*)

lemma *abs-is-lms-consec*:
abs-is-lms xs i $\Longrightarrow$ $\neg$*abs-is-lms xs* (*Suc i*)
abs-is-lms xs (*Suc i*) $\Longrightarrow$ $\neg$*abs-is-lms xs i*
by (*clarsimp simp*: *abs-is-lms-def*)+

lemma *abs-is-lms-gre-length*:
$n \geq$ *length xs* $\Longrightarrow$ $\neg$*abs-is-lms xs n*
by (*metis SL-types.distinct*(*1*) *drop-eq-Nil abs-is-lms-def l-type-list-less-ns*)

lemma *abs-is-lms-suffix*:
abs-is-lms (*suffix s n*) *i* $\Longrightarrow$ *abs-is-lms s* (*i* + *n*)
by (*clarsimp simp*: *abs-is-lms-def suffix-type-suffix-gen*)

lemma *abs-is-lms-i-gr-0*:
i > *0* $\Longrightarrow$ *abs-is-lms* (*suffix s n*) *i* = *abs-is-lms s* (*i* + *n*)
apply *safe*
apply (*erule abs-is-lms-suffix*)
apply (*clarsimp simp*: *abs-is-lms-def*)
apply (*rule conjI*)
apply (*subst suffix-type-suffix-gen*; *simp*)
by (*metis* (*no-types*, *opaque-lifting*) *add.commute add-Suc-right add-right-cancel less-Suc-eq-le*
less-iff-Suc-add ordered-cancel-comm-monoid-diff-class.diff-add suffix-type-suffix-gen)

lemma *set-abs-is-lms-suffix*:
{*i*. *abs-is-lms* (*suffix s n*) (*i* − *n*)} = {*i*. *abs-is-lms s i* ∧ *i* > *n*}
apply *safe*
apply (*metis abs-is-lms-0 abs-is-lms-suffix le-less-linear nat-diff-split*
ordered-cancel-comm-monoid-diff-class.diff-add)
apply (*metis bot-nat-0.not-eq-extremum abs-is-lms-0 zero-less-diff*)
apply (*cases n*)
apply *clarsimp*
by (*metis* (*no-types*, *lifting*) *Suc-diff-le abs-is-lms-def less-Suc-eq-le less-or-eq-imp-le*
ordered-cancel-comm-monoid-diff-class.diff-add suffix-type-suffix-gen)

lemma *abs-is-lms-set-less-length*:
$n \geq$ *length xs* $\Longrightarrow$ {*i*. *abs-is-lms xs i* ∧ *i* < *n*} = {*i*. *abs-is-lms xs i*}
by (*meson dual-order.trans abs-is-lms-gre-length le-less-linear*)

lemma *abs-is-lms-suffix-Suc*:
abs-is-lms (*suffix s n*) (*Suc i*) = *abs-is-lms s* (*Suc* (*i* + *n*))

apply *safe*
apply (*drule abs-is-lms-suffix*; *simp*)
apply (*clarsimp simp*: *abs-is-lms-def*)
apply (*subst suffix-type-suffix-gen*)+
apply *simp*
done

45.2 LMS-Type Sets and Subsets

This subsection contains lemmas about sets and subsets of LMS-types.

lemma *set-lms-gr-0*:
$\{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge 0 < i\} = \{i.\ abs\text{-}is\text{-}lms\ xs\ i\}$
using *bot-nat-0.not-eq-extremum abs-is-lms-0* **by** *blast*

lemma *set-lms-n-subset*:
$\{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > n\} \subseteq \{i.\ abs\text{-}is\text{-}lms\ xs\ i\}$
by *blast*

lemma *set-lms-Suc-subset*:
$\{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > Suc\ n\} \subseteq \{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > n\}$
by (*simp add*: *Collect-mono*)

lemma *set-lms-Suc-insert*:
$abs\text{-}is\text{-}lms\ xs\ (Suc\ n) \Longrightarrow \{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > n\} = insert\ (Suc\ n)\ \{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > Suc\ n\}$
using *Collect-cong* **by** *auto*

lemma *lms-finite*:
finite $\{i.\ abs\text{-}is\text{-}lms\ xs\ i\}$
by (*metis finite-nat-set-iff-bounded abs-is-lms-def mem-Collect-eq suffix-type-s-bound*)

lemma *lms-set-empty*:
$[\![length\ xs = Suc\ n;\ m \geq n]\!] \Longrightarrow \{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i > m\ \} = \{\}$
by (*metis* (*no-types*, *lifting*) *Collect-empty-eq Suc-leI diff-diff-cancel abs-is-lms-gre-length less-imp-diff-less*)

45.3 Implementation of LMS-Types Computation

This section contains a shallow embedding of a function that would compute all the LMS-types of an ordered list.

fun *get-lms* :: $('a :: \{linorder,\ order\text{-}bot\})\ list \Rightarrow nat \Rightarrow nat\ list$
where
get-lms xs 0 = [] |
get-lms xs (*Suc n*) = (*if abs-is-lms xs n then n # get-lms xs n else get-lms xs n*)

lemma *get-lms-correct*:
get-lms xs n = *rev* (*filter* (*abs-is-lms xs*) [*0*..<*n*])
apply (*induct n*)

apply *simp*
apply *clarsimp*
done

45.3.1 Properties

This subsection contains miscellaneous lemmas about facts that can be derived from the shallow embedding and vice versa.

lemma *get-lms-element-bound*:
$x \in set\ (get\text{-}lms\ xs\ n) \Longrightarrow x < n \wedge x > 0$
apply (*induct n*; *simp*)
apply (*clarsimp split*: *if-splits*)
apply (*erule disjE*; *clarsimp*)
apply (*cases x*; *clarsimp simp*: *abs-is-lms-0*)
done

lemma *distinct-get-lms*:
distinct (*get-lms xs n*)
apply (*induct n*; *clarsimp*)
apply (*drule get-lms-element-bound*)
by *blast*

lemma *get-lms-abs-is-lms*:
$x \in set\ (get\text{-}lms\ xs\ n) \longleftrightarrow abs\text{-}is\text{-}lms\ xs\ x \wedge x < n$
apply (*subst get-lms-correct*)
apply *clarsimp*
by *blast*

lemma *lms-le-length*:
$x \in set\ (get\text{-}lms\ xs\ n) \Longrightarrow x < length\ xs$
by (*simp add*: *get-lms-abs-is-lms abs-is-lms-def suffix-type-s-bound*)

lemma *get-lms-set*:
$set\ (get\text{-}lms\ xs\ n) = \{i.\ abs\text{-}is\text{-}lms\ xs\ i \wedge i < n\}$
apply (*induct n*)
apply *simp*
apply *safe*
using *get-lms-abs-is-lms* **by** *blast*+

lemma *get-lms-set-n-gre-length*:
$n \geq length\ xs \Longrightarrow set\ (get\text{-}lms\ xs\ n) = \{i.\ abs\text{-}is\text{-}lms\ xs\ i\}$
apply (*simp add*: *get-lms-set*)
by (*meson dual-order.trans abs-is-lms-gre-length not-less*)

45.4 Cardinality LMS-Types

This section contains lemmas about how many LMS-types exist (lemma 2.1, Nonge et al., DCC2009). These lemmas are particularly important when

proving that the SAIS is O(n) in space (bytes) and time complexity (lemma 3.1, Nong et al., DCC 2009).

```
lemma num-lms-bound-1:
  length (get-lms xs n) ≤ n div 2
proof −
  have card (set (get-lms xs n)) ≤ n div 2
  proof (intro ballI subset-upt-no-Suc[of set (get-lms xs n) n])
    show set (get-lms xs n) ⊆ {1..<n}
      by (simp add: Suc-leI get-lms-element-bound subset-code(1))
  next
    fix x
    assume x ∈ set (get-lms xs n)
    hence abs-is-lms xs x
      by (simp add: get-lms-abs-is-lms)
    then show Suc x ∉ set (get-lms xs n)
      using get-lms-abs-is-lms abs-is-lms-consec(2) by blast
  qed
  with distinct-card[OF distinct-get-lms[of xs n]]
  show ?thesis
    by presburger
qed

lemma num-lms-bound-2:
  length (get-lms xs n) ≤ length xs div 2
proof −
  have card (set (get-lms xs n)) ≤ length xs div 2
  proof (intro ballI subset-upt-no-Suc[of set (get-lms xs n) length xs])
    show set (get-lms xs n) ⊆ {1..<length xs}
        by (metis One-nat-def Suc-leI atLeastLessThan-iff get-lms-element-bound
lms-le-length subsetI)
  next
    fix x
    assume x ∈ set (get-lms xs n)
    hence abs-is-lms xs x
      by (simp add: get-lms-abs-is-lms)
    then show Suc x ∉ set (get-lms xs n)
      using get-lms-abs-is-lms abs-is-lms-consec(2) by blast
  qed
  with distinct-card[OF distinct-get-lms[of xs n]]
  show ?thesis
    by simp
qed

lemma card-abs-is-lms-bound:
  xs ≠ [] ⟹ card {i. abs-is-lms xs i} < length xs
   by (metis (no-types, opaque-lifting) One-nat-def distinct-card distinct-get-lms
div-less-dividend
      get-lms-set-n-gre-length le-less-linear le-less-trans length-greater-0-conv lessI
      num-lms-bound-2 numeral-2-eq-2)
```

lemma *card-abs-is-lms-bound-length-div-2*:
card {*i*. *abs-is-lms xs i*} ≤ *length xs div 2*
by (*metis distinct-card distinct-get-lms get-lms-set-n-gre-length linear num-lms-bound-2*)

lemma *length-filter-lms*:
T ≠ [] ⟹ *length* (*filter* (*abs-is-lms T*) [*0*..<*length T*]) < *length T*
by (*metis diff-zero abs-is-lms-0 length-filter-less length-greater-0-conv length-upt nth-Cons-0*
nth-mem upt-rec)

45.5 General Properties about LMS-types

lemma *abs-is-lms-imp-le-nth*:
⟦*abs-is-lms T i*; *Suc i* < *length T*⟧ ⟹ *T* ! *i* ≤ *T* ! *Suc i*
by (*metis SL-types.simps*(*2*) *abs-is-lms-def not-less nth-gr-imp-l-type*)

lemma *abs-is-lms-neq*:
abs-is-lms T (*Suc i*) ⟹ *T* ! *Suc i* < *T* ! *i*
unfolding *abs-is-lms-def*
proof(*safe*)
assume *suffix-type T* (*Suc i*) = *S-type suffix-type T i* = *L-type*

from ‹*suffix-type T* (*Suc i*) = *S-type*›
have *Suc i* < *length T*
by (*simp add*: *suffix-type-s-bound*)
with *suffix-type-neq* ‹*suffix-type T* (*Suc i*) = *S-type*› ‹*suffix-type T i* = *L-type*›
show *?thesis*
by (*metis SL-types.simps*(*2*) *not-less-iff-gr-or-eq nth-less-imp-s-type*)
qed

lemma *abs-is-lms-last*:
⟦*valid-list T*; *length T* = *Suc* (*Suc n*)⟧ ⟹ *abs-is-lms T* (*Suc n*)
proof (*induct n arbitrary*: *T*)
case *0*
note *IH* = *this*

have *T* ! *Suc 0* = *bot*
by (*metis IH Zero-neq-Suc diff-Suc-1 last-conv-nth list.size*(*3*) *valid-list-def*)
with *IH*(*1*)[*simplified valid-list-ex-def*]
have *T* ! *0* > *T* ! *Suc 0*
by (*metis IH*(*2*) *One-nat-def bot.not-eq-extremum butlast-snoc diff-Suc-1' le-less-Suc-eq*
length-butlast less-or-eq-imp-le nth-butlast)
with *suffix-type-last*[*OF IH*(*2*)]
show *?case*
using *abs-is-lms-def nth-gr-imp-l-type suffix-type-s-bound* **by** *blast*
next
case (*Suc n*)
note *IH* = *this*

show *?case*
proof (*cases T*)
case *Nil*
then show *?thesis*
by (*simp add*: *Suc.prems*(*1*) *valid-list-not-nil*)
next
case (*Cons a T'*)
with *IH valid-list-consD*[*of a T'*]
have *abs-is-lms T' (Suc n)*
by *fastforce*
then show *?thesis*
by (*simp add*: *abs-is-lms-def local.Cons suffix-type-cons-suc*)
qed
qed

lemma *abs-is-lms-imp-less-length*:
abs-is-lms T i $\Longrightarrow$ *i* < *length T*
using *abs-is-lms-gre-length le-less-linear* **by** *blast*

lemma *s-type-and-not-lms-Suc*:
⟦¬*abs-is-lms T* (*Suc i*); *suffix-type T* (*Suc i*) = *S-type*⟧ $\Longrightarrow$ *suffix-type T i* = *S-type*
by (*meson abs-is-lms-def suffix-type-def*)

lemma *no-lms-imp-all-s-type*:
assumes *j* < *length T*
and *i* ≤ *j*
and ∀ *k*>*i*. *k* ≤ *j* ⟶ ¬*abs-is-lms T k*
and *suffix-type T j* = *S-type*
and *i* ≤ *k*
and *k* ≤ *j*
shows *suffix-type T k* = *S-type*
using *assms*
proof (*induct j* − *k arbitrary*: *j*)
case *0*
then show *?case*
by *auto*
next
case (*Suc x*)
note *IH* = *this*

have ∃ *j'*. *j* = *Suc j'*
using *Suc.hyps*(*2*) **by** *presburger*
then obtain *j'* **where**
j = *Suc j'*
by *blast*
hence *x* = *j'* − *k*
using *Suc.hyps*(*2*) **by** *linarith*
moreover

```
  have j' < length T
    using Suc.prems(1) Suc-lessD ‹j = Suc j'› by blast
  moreover
  have i ≤ j'
    using Suc.hyps(2) ‹j = Suc j'› assms(5) by linarith
  moreover
  have ∀ k>i. k ≤ j' ⟶ ¬ abs-is-lms T k
    by (simp add: Suc.prems(3) ‹j = Suc j'›)
  moreover
  from IH(5)
  have ¬abs-is-lms T (Suc j')
    by (simp add: ‹j = Suc j'› calculation(3) le-imp-less-Suc)
  with IH(6) ‹j = Suc j'› s-type-and-not-lms-Suc[of T j']
  have suffix-type T j' = S-type
    by blast
  moreover
  have k ≤ j'
    using Suc.hyps(2) ‹j = Suc j'› by linarith
  ultimately show ?case
    using IH(1)[of j'] assms(5) by blast
qed

lemma first-l-type-after-s-type:
  assumes j < length T
  and     i ≤ j
  and     ∀ k>i. k ≤ j ⟶ ¬abs-is-lms T k
  and     suffix-type T j = L-type
  and     suffix-type T i = S-type
shows ∃ l≥i. l ≤ j ∧ (∀ k<l. i ≤ k ⟶ suffix-type T k = S-type) ∧ suffix-type T l
= L-type
  using assms
proof (induct j - i arbitrary: j)
  case 0
  then show ?case
    by auto
next
  case (Suc x)
  note IH = this

  have ∃ j'. j = Suc j'
   by (metis SL-types.distinct(1) Suc.prems(2) Suc.prems(4) assms(5) diff-is-0-eq
diff-zero
            less-imp-Suc-add neq0-conv)
  then obtain j' where
    j = Suc j'
    by blast
  hence x = j' - i
    using Suc.hyps(2) ‹j = Suc j'› by linarith
```

```
  have j' < length T
    using Suc.prems(1) Suc-lessD ‹j = Suc j'› by blast

  have i ≤ j'
    using Suc.hyps(2) ‹j = Suc j'› by linarith

  have P: ∀ k>i. k ≤ j' ⟶ ¬ abs-is-lms T k
    by (simp add: Suc.prems(3) ‹j = Suc j'›)

  have suffix-type T j' = S-type ∨ suffix-type T j' = L-type
    using SL-types.exhaust by blast
  moreover
  have suffix-type T j' = S-type ⟹ ?case
  proof −
    assume suffix-type T j' = S-type
    hence ∀ k<j. i ≤ k ⟶ suffix-type T k = S-type
      using P ‹j' < length T› no-lms-imp-all-s-type ‹j = Suc j'› less-Suc-eq-le by
auto
    then show ?thesis
      using Suc.prems(2) Suc.prems(4) by blast
  qed
  moreover
  have suffix-type T j' = L-type ⟹ ?case
  proof −
    assume suffix-type T j' = L-type
    with IH(1)[OF ‹x = -› ‹j' < -› ‹i ≤ j'› P - assms(5)]
    have ∃ l≥i. l ≤ j' ∧ (∀ k<l. i ≤ k ⟶ suffix-type T k = S-type) ∧ suffix-type
T l = L-type .
    then show ?thesis
      using ‹j = Suc j'› by auto
  qed
  ultimately show ?case
    by blast
qed

lemma no-lms-imp-and-s-imp-all-s-below:
  assumes ∀ k. i ≤ k ∧ k < j ⟶ ¬abs-is-lms T k
  and     suffix-type T k = S-type
  and     i ≤ k
  and     k < j
shows ⟦i ≤ k'; k' ≤ k⟧ ⟹ suffix-type T k' = S-type
proof (induct k − k' arbitrary: k')
  case 0
  with assms
  show ?case
    by auto
next
  case (Suc x)
  note IH = this
```

from *IH*(*2*)
have $x = k - Suc\ k'$
by *linarith*

from *IH*(*3*)
have $i \leq Suc\ k'$
by *simp*

from *IH*(*2*)
have $Suc\ k' \leq k$
by *linarith*

from *IH*(*1*)[*OF* ‹$x = k - Suc\ k'$› ‹$i \leq Suc\ k'$› ‹$Suc\ k' \leq k$›]
have *suffix-type T* (*Suc k'*) = *S-type*
by *assumption*
with *assms*(*1*) ‹$i \leq k'$› ‹$k' \leq k$› ‹$Suc\ k' \leq k$› ‹$i \leq Suc\ k'$› *assms*(*4*)
show *?case*
by (*meson SL-types.exhaust abs-is-lms-def order.strict-trans1*)
qed

lemma *no-lms-imp-and-l-imp-all-l-above*:
assumes $\forall k.\ i \leq k \wedge k < j \longrightarrow \neg$*abs-is-lms T k*
and *suffix-type T k* = *L-type*
and $i \leq k$
and $k < j$
shows $[\![k \leq k'; k' < j]\!] \Longrightarrow$ *suffix-type T k'* = *L-type*
proof (*induct k' − k arbitrary*: *k'*)
case *0*
with *assms*
show *?case*
by *auto*
next
case (*Suc x*)
note *IH* = *this*
from *IH*(*2*)
have $\exists n.\ k' = Suc\ n$
by (*metis less-Suc-eq less-Suc-eq-0-disj zero-diff*)
then obtain *n* **where**
$k' = Suc\ n$
by *blast*
with *IH*(*2*)
have $x = n - k$
by *linarith*

from *IH*(*2*) ‹$k' = Suc\ n$›
have $k \leq n$
by *linarith*

```
  from IH(4) ‹k' = Suc n›
  have n < j
    by linarith

  from IH(1)[OF ‹x = n − k› ‹k ≤ n› ‹n < j›]
  have suffix-type T n = L-type
    by assumption
  with assms(1) ‹k' = Suc n› ‹k' < j› ‹k ≤ k'› assms(3)
  show ?case
    by (meson SL-types.exhaust abs-is-lms-def le-trans)
qed

lemma lms-sublist-helper:
  assumes ∀ k. suffix-type T k = S-type ⟶ Suc k < n ⟶ i ≤ k ⟶ suffix-type
T (Suc k) ≠ L-type
  and     suffix-type T i = S-type
shows ⟦i ≤ k; k < n⟧ ⟹ suffix-type T k = S-type
proof (induct k − i arbitrary: k)
  case 0
  then show ?case
    using assms(2) by auto
next
  case (Suc x)
  note IH = this
  from IH(2)
  have ∃ k'. k = Suc k'
    by presburger
  then obtain k' where
    k = Suc k'
    by blast
  with IH(2)
  have x = k' − i
    by linarith

  from IH(2) ‹k = Suc k'›
  have i ≤ k'
    by linarith

  from IH(4) ‹k = Suc k'›
  have k' < n
    by linarith

  from IH(1)[OF ‹x = k' − i› ‹i ≤ k'› ‹k' < n›] assms(1) IH(4) ‹k = Suc k'› ‹i
≤ k'›
  show ?case
    using SL-types.exhaust by blast
qed

end
```

theory *Buckets*
imports
../../util/Continuous-Interval
List-Type
begin

46 Buckets

46.1 Entire Bucket

definition *bucket* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
where
bucket α *T b* ≡ {*k* |*k*. *k* < *length T* ∧ α (*T* ! *k*) = *b*}

definition *bucket-size* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
bucket-size α *T b* ≡ *card* (*bucket* α *T b*)

definition *bucket-upt* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
where
bucket-upt α *T b* = {*k* |*k*. *k* < *length T* ∧ α (*T* ! *k*) < *b*}

definition *bucket-start* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
bucket-start α *T b* ≡ *card* (*bucket-upt* α *T b*)

definition *bucket-end* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
bucket-end α *T b* ≡ *card* (*bucket-upt* α *T* (*Suc b*))

lemma *bucket-upt-subset*:
bucket-upt α *T b* ⊆ {*0*..<*length T*}
by (*rule subsetI*, *simp add*: *bucket-upt-def*)

lemma *bucket-upt-subset-Suc*:
bucket-upt α *T b* ⊆ *bucket-upt* α *T* (*Suc b*)
by (*rule subsetI*, *simp add*: *bucket-upt-def*)

lemma *bucket-upt-un-bucket*:
bucket-upt α *T b* ∪ *bucket* α *T b* = *bucket-upt* α *T* (*Suc b*)
apply (*clarsimp simp*: *bucket-upt-def bucket-def*)
apply (*intro equalityI subsetI*; *clarsimp*)
apply (*erule disjE*; *clarsimp*)
done

lemma *bucket-0*:
assumes *valid-list T* α *bot* = *0 strict-mono* α *length T* = *Suc k*

shows *bucket* α *T 0* = $\{k\}$
proof *safe*
fix *x*
assume *x* ∈ *bucket* α *T 0*
then show *x* = *k*
by (*metis* (*mono-tags*, *lifting*) *assms bucket-def diff-Suc-1 le-neq-trans le-simps*(*2*)
mem-Collect-eq strict-mono-eq valid-list-def)
next
show *k* ∈ *bucket* α *T 0*
by (*metis* (*mono-tags*, *lifting*) *assms*(*1*,*2*,*4*) *bucket-def diff-Suc-1 last-conv-nth lessI*
list.size(*3*) *mem-Collect-eq order-less-irrefl valid-list-def*)
qed

lemma *finite-bucket*:
finite (*bucket* α *T x*)
by (*clarsimp simp*: *bucket-def*)

lemma *finite-bucket-upt*:
finite (*bucket-upt* α *T b*)
by (*meson bucket-upt-subset subset-eq-atLeast0-lessThan-finite*)

lemma *bucket-start-Suc*:
bucket-start α *T* (*Suc b*) = *bucket-start* α *T b* + *bucket-size* α *T b*
apply (*clarsimp simp*: *bucket-start-def bucket-size-def*)
apply (*subst card-Un-disjoint*[*symmetric*])
apply (*meson bucket-upt-subset subset-eq-atLeast0-lessThan-finite*)
apply (*simp add*: *finite-bucket*)
apply (*rule disjointI'*)
apply (*metis* (*mono-tags*, *lifting*) *bucket-def bucket-upt-def less-irrefl-nat mem-Collect-eq*)
apply (*simp add*: *bucket-upt-un-bucket*)
done

lemma *bucket-start-le*:
b ≤ *b'* ⟹ *bucket-start* α *T b* ≤ *bucket-start* α *T b'*
apply (*clarsimp simp*: *bucket-start-def*)
by (*meson bucket-upt-subset bucket-upt-subset-Suc card-mono lift-Suc-mono-le*
subset-eq-atLeast0-lessThan-finite)

lemma *bucket-start-Suc-eq-bucket-end*:
bucket-start α *T* (*Suc b*) = *bucket-end* α *T b*
by (*simp add*: *bucket-end-def bucket-start-def*)

lemma *bucket-end-le-length*:
bucket-end α *T b* ≤ *length T*
apply (*clarsimp simp*: *bucket-end-def*)
apply (*insert card-mono*[*OF* - *bucket-upt-subset*[*of* α *T Suc b*]])
apply (*erule meta-impE*, *simp*)
apply (*erule order.trans*)

apply *simp*
done

lemma *bucket-start-le-end*:
bucket-start α T $b \leq$ *bucket-end* α T b
by (*metis Suc-n-not-le-n bucket-start-Suc-eq-bucket-end bucket-start-le nat-le-linear*)

lemma *le-bucket-start-le-end*:
$b \leq b' \Longrightarrow$ *bucket-start* α T $b \leq$ *bucket-end* α T b'
using *bucket-start-le bucket-start-le-end le-trans* **by** *blast*

lemma *bucket-end-le*:
$b \leq b' \Longrightarrow$ *bucket-end* α T $b \leq$ *bucket-end* α T b'
by (*metis bucket-start-Suc-eq-bucket-end bucket-start-le-end lift-Suc-mono-le*)

lemma *less-bucket-end-le-start*:
$b < b' \Longrightarrow$ *bucket-end* α T $b \leq$ *bucket-start* α T b'
by (*metis Suc-leI bucket-start-Suc-eq-bucket-end bucket-start-le*)

lemma *bucket-end-def'*:
bucket-end α T b = *bucket-start* α T b + *bucket-size* α T b
by (*metis bucket-start-Suc bucket-start-Suc-eq-bucket-end*)

lemma *valid-list-bucket-start-0*:
$[\![$*valid-list* T; *strict-mono* α; α *bot* = 0$]\!] \Longrightarrow$
bucket-start α T 0 = 0
by (*clarsimp simp*: *bucket-start-def bucket-upt-def*)

lemma *bucket-upt-0*:
bucket-upt α T 0 = {}
by (*clarsimp simp*: *bucket-upt-def*)

lemma *bucket-start-0*:
bucket-start α T 0 = 0
by (*clarsimp simp*: *bucket-start-def bucket-upt-def*)

lemma *valid-list-bucket-upt-Suc-0*:
$[\![$*valid-list* T; *strict-mono* α; α *bot* = 0; *length* T = *Suc* $n]\!] \Longrightarrow$
bucket-upt α T (*Suc* 0) = {n}
apply (*clarsimp simp*: *bucket-upt-def*)
apply (*intro equalityI subsetI*)
apply (*clarsimp simp*: *valid-list-def*)
apply (*metis less-antisym strict-mono-eq*)
apply (*clarsimp simp*: *valid-list-ex-def*)
done

lemma *valid-list-bucket-end-0*:
$[\![$*valid-list* T; *strict-mono* α; α *bot* = 0$]\!] \Longrightarrow$
bucket-end α T 0 = 1

apply (*clarsimp simp*: *bucket-end-def*)
apply (*frule valid-list-length-ex*)
apply *clarsimp*
apply (*frule* (*3*) *valid-list-bucket-upt-Suc-0*)
apply *simp*
done

lemma *nth-Max*:
$T \neq [] \Longrightarrow \exists\, i < \mathit{length}\ T.\ T\ !\ i = \mathit{Max}\ (\mathit{set}\ T)$
by (*metis List.finite-set Max-in in-set-conv-nth set-empty*)

lemma *bucket-upt-Suc-Max*:
strict-mono $\alpha \Longrightarrow \mathit{bucket\text{-}upt}\ \alpha\ T\ (\mathit{Suc}\ (\alpha\ (\mathit{Max}\ (\mathit{set}\ T)))) = \{0..<\mathit{length}\ T\}$
apply (*intro equalityI subsetI*)
apply (*erule bucket-upt-subset*[*THEN subsetD*])
by (*clarsimp simp*: *bucket-upt-def less-Suc-eq-le strict-mono-less-eq*)

lemma *bucket-end-Max*:
strict-mono $\alpha \Longrightarrow \mathit{bucket\text{-}end}\ \alpha\ T\ (\alpha\ (\mathit{Max}\ (\mathit{set}\ T))) = \mathit{length}\ T$
apply (*clarsimp simp*: *bucket-end-def*)
apply (*drule bucket-upt-Suc-Max*[**where** $T = T$])
apply *clarsimp*
done

lemma *bucket-end-eq-length*:
⟦*strict-mono* α; $b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$; $T \neq []$; $\mathit{bucket\text{-}end}\ \alpha\ T\ b = \mathit{length}\ T$⟧ ⟹
$b = \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
proof −
assume *strict-mono* α $b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$ $\mathit{bucket\text{-}end}\ \alpha\ T\ b = \mathit{length}\ T$ $T \neq []$
show $b = \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
proof (*rule ccontr*)
assume $b \neq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
with ‹$b \leq$ -›
have $b < \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
by *simp*
hence $\exists\, b'.\ b' = \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
by *blast*
then obtain b' **where**
$b' = \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
by *blast*
with ‹$b <$ -›
have $b < b'$
by *blast*
hence $\mathit{bucket\text{-}end}\ \alpha\ T\ b \leq \mathit{bucket\text{-}start}\ \alpha\ T\ b'$
by (*simp add*: *less-bucket-end-le-start*)
moreover
from *nth-Max*[*OF* ‹$T \neq []$›]
obtain i **where**
$i < \mathit{length}\ T$

$T ! i = Max\ (set\ T)$
by *blast*
with ‹$b' = \alpha\ (Max\ (set\ T))$› ‹*strict-mono* α›
have $i \in bucket\ \alpha\ T\ b'$
by (*simp add*: *bucket-def*)
hence *bucket-start* $\alpha\ T\ b' <$ *bucket-end* $\alpha\ T\ b'$
by (*metis add-diff-cancel-left' bucket-end-def' bucket-size-def bucket-start-le-end*
card-gt-0-iff diff-is-0-eq' empty-iff finite-bucket nat-less-le)
moreover
have *bucket-end* $\alpha\ T\ b' \leq$ *length* T
using *bucket-end-le-length* **by** *blast*
ultimately
show *False*
using ‹*bucket-end* $\alpha\ T\ b =$ *length* T›
by *linarith*
qed
qed

46.2 L-types

definition *l-bucket* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
where
l-bucket $\alpha\ T\ b = \{k\ |k.\ k \in bucket\ \alpha\ T\ b \wedge$ *suffix-type* $T\ k =$ *L-type*$\}$

definition *l-bucket-size* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
l-bucket-size $\alpha\ T\ b \equiv$ *card* (*l-bucket* $\alpha\ T\ b$)

definition *l-bucket-end* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
l-bucket-end $\alpha\ T\ b =$ *bucket-start* $\alpha\ T\ b +$ *l-bucket-size* $\alpha\ T\ b$

lemma *l-bucket-subset-bucket*:
l-bucket $\alpha\ T\ b \subseteq$ *bucket* $\alpha\ T\ b$
by (*rule subsetI*, *simp add*: *l-bucket-def*)

lemma *bucket-upt-int-l-bucket*:
strict-mono $\alpha \Longrightarrow$ *bucket-upt* $\alpha\ T\ b \cap$ *l-bucket* $\alpha\ T\ b = \{\}$
apply (*rule disjoint-subset2*[**where** $B =$ *bucket* $\alpha\ T\ b$])
apply (*simp add*: *l-bucket-def*)
apply (*simp add*: *bucket-def bucket-upt-def*)
apply (*rule disjointI'*)
apply *clarsimp*
done

lemma *subset-l-bucket*:
⟦∀ $k <$ *length ls*. *ls* ! $k <$ *length* $T \wedge$ *suffix-type* T (*ls* ! k) = *L-type* $\wedge\ \alpha\ (T\ !\ (ls$

! k)) = x;
$distinct\ ls$⟧ ⟹
$set\ ls \subseteq l\text{-}bucket\ \alpha\ T\ x$
apply (*rule subsetI*)
apply (*clarsimp simp*: *l-bucket-def bucket-def in-set-conv-nth*)
done

lemma *finite-l-bucket*:
finite (*l-bucket* α T x)
apply (*clarsimp simp*: *finite-bucket l-bucket-def*)
done

lemma *l-bucket-list-eq*:
⟦∀ k < *length ls*. *ls* ! k < *length* T ∧ *suffix-type* T (*ls* ! k) = *L-type* ∧ α (T ! (*ls*
! k)) = x;
$distinct\ ls$; *length ls* = *l-bucket-size* α T x⟧ ⟹
$set\ ls = l\text{-}bucket\ \alpha\ T\ x$
apply (*frule* (*1*) *subset-l-bucket*)
apply (*frule distinct-card*)
apply (*insert finite-l-bucket*[*of* α T x])
by (*simp add*: *card-subset-eq l-bucket-size-def*)

lemma *l-bucket-le-bucket-size*:
l-bucket-size α T $b \leq$ *bucket-size* α T b
apply (*clarsimp simp*: *l-bucket-size-def bucket-size-def*)
apply (*rule card-mono*[*OF finite-bucket l-bucket-subset-bucket*])
done

lemma *l-bucket-not-empty*:
⟦i < *length* T; *suffix-type* T i = *L-type*⟧ ⟹ *0* < *l-bucket-size* α T (α (T ! i))
apply (*clarsimp simp*: *l-bucket-size-def*)
apply (*subst card-gt-0-iff*)
apply (*intro conjI finite-l-bucket*)
apply (*clarsimp simp*: *l-bucket-def bucket-def*)
apply *blast*
done

lemma *l-bucket-end-le-bucket-end*:
l-bucket-end α T $b \leq$ *bucket-end* α T b
apply (*clarsimp simp*: *l-bucket-end-def*)
apply (*rule order.trans*[**where** b = *bucket-start* α T b + *bucket-size* α T b])
apply (*simp add*: *l-bucket-le-bucket-size*)
by (*metis bucket-start-Suc bucket-start-Suc-eq-bucket-end le-refl*)

lemma *l-bucket-Max*:
assumes *valid-list T*
and *Suc 0* < *length T*
and *strict-mono* α
shows *l-bucket* α T (α (*Max* (*set* T))) = *bucket* α T (α (*Max* (*set* T)))

proof (*intro subsetI equalityI*)
let *?b* = α (*Max* (*set T*))
fix *x*
assume *x* ∈ *l-bucket* α *T ?b*
then show *x* ∈ *bucket* α *T ?b*
using *l-bucket-subset-bucket* **by** *blast*
next
let *?b* = α (*Max* (*set T*))
fix *x*
assume *x* ∈ *bucket* α *T ?b*
hence *x* < *length T* α (*T* ! *x*) = *?b*
using *bucket-def* **by** *blast+*
with *suffix-max-hd-is-l-type*[*OF assms*(*1*) - *assms*(*2*)]
have *suffix-type T x* = *L-type*
by (*metis Cons-nth-drop-Suc assms*(*3*) *list.sel*(*1*) *strict-mono-eq*)
then show *x* ∈ *l-bucket* α *T ?b*
using ‹*x* ∈ *bucket* α *T* (α (*Max* (*set T*)))› *l-bucket-def* **by** *blast*
qed

46.3 LMS-types

definition *lms-bucket* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
where
lms-bucket α *T b* = {*k* |*k*. *k* ∈ *bucket* α *T b* ∧ *abs-is-lms T k*}

definition *lms-bucket-size* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
lms-bucket-size α *T b* ≡ *card* (*lms-bucket* α *T b*)

lemma *lms-bucket-subset-bucket*:
lms-bucket α *T b* ⊆ *bucket* α *T b*
by (*simp add*: *lms-bucket-def*)

lemma *finite-lms-bucket*:
finite (*lms-bucket* α *T b*)
by (*clarsimp simp*: *lms-bucket-def finite-bucket*)

lemma *disjoint-l-lms-bucket*:
l-bucket α *T b* ∩ *lms-bucket* α *T b* = {}
apply (*rule disjointI'*)
by (*clarsimp simp*: *l-bucket-def lms-bucket-def abs-is-lms-def*)

46.4 S-types

definition *s-bucket* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
where
s-bucket α *T b* = {*k* |*k*. *k* ∈ *bucket* α *T b* ∧ *suffix-type T k* = *S-type*}

definition *s-bucket-size* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
  where
s-bucket-size α *T b* ≡ *card* (*s-bucket* α *T b*)

definition *s-bucket-start* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
  where
s-bucket-start α *T b* ≡ *bucket-start* α *T b* + *l-bucket-size* α *T b*

lemma *finite-s-bucket*:
 finite (*s-bucket* α *T b*)
  by (*clarsimp simp*: *s-bucket-def finite-bucket*)

lemma *disjoint-l-s-bucket*:
 l-bucket α *T b* ∩ *s-bucket* α *T b* = {}
  apply (*rule disjointI'*)
  by (*clarsimp simp*: *l-bucket-def s-bucket-def*)

lemma *lms-subset-s-bucket*:
  lms-bucket α *T b* ⊆ *s-bucket* α *T b*
  by (*clarsimp simp*: *s-bucket-def lms-bucket-def abs-is-lms-def*)

lemma *l-un-s-bucket*:
  bucket α *T b* = *l-bucket* α *T b* ∪ *s-bucket* α *T b*
  apply (*rule equalityI*; *clarsimp simp*: *l-bucket-def s-bucket-def*)
  by (*meson suffix-type-def*)

lemma *s-bucket-Max*:
  assumes *valid-list T*
  and *length T* > *Suc 0*
  and *strict-mono* α
shows *s-bucket* α *T* (α (*Max* (*set T*))) = {}
proof −
  let *?b* = α (*Max* (*set T*))
  from *l-bucket-Max*[*OF assms*]
  have *l-bucket* α *T ?b* = *bucket* α *T ?b* **.**
  moreover
  from *l-un-s-bucket*
  have *bucket* α *T ?b* = *l-bucket* α *T ?b* ∪ *s-bucket* α *T ?b* **.**
  hence *s-bucket* α *T ?b* ⊆ *bucket* α *T ?b*
    by *blast*
  moreover
  from *disjoint-l-s-bucket*
  have *l-bucket* α *T ?b* ∩ *s-bucket* α *T ?b* = {} **.**
  ultimately
  show *?thesis*
    by *blast*

qed

lemma *s-bucket-0*:
assumes *valid-list T*
and *strict-mono α*
and *α bot = 0*
and *length T = Suc n*
shows *s-bucket α T 0 = {n}*
proof −
have *suffix-type T n = S-type*
using *assms(4) suffix-type-last* **by** *blast*
moreover
have *T ! n = bot*
by (*metis assms(1) assms(4) diff-Suc-1 last-conv-nth length-greater-0-conv valid-list-def*)
hence *α (T ! n) = 0*
by (*simp add*: *assms(3)*)
ultimately have *n ∈ s-bucket α T 0*
by (*simp add*: *assms(4) bucket-def s-bucket-def*)
hence *{n} ⊆ s-bucket α T 0*
by *blast*
moreover
have *s-bucket α T 0 ⊆ {n}*
unfolding *s-bucket-def*
proof *safe*
fix *k*
assume *k ∈ bucket α T 0 suffix-type T k = S-type*
hence *k ≤ n*
by (*simp add*: *assms(4) bucket-def*)
have *α (T ! k) = 0*
using ‹*k ∈ bucket α T 0*› *bucket-def* **by** *blast*
hence *T ! k = bot*
by (*metis assms(2) assms(3) strict-mono-eq*)
show *k = n*
proof (*rule ccontr*)
assume *k ≠ n*
hence *k < n*
by (*simp add*: ‹*k ≤ n*› *le-neq-implies-less*)
then show *False*
using ‹*k ∈ bucket α T 0*› ‹*k ≠ n*› *assms bucket-0* **by** *blast*
qed
qed
ultimately show *?thesis*
by *blast*
qed

lemma *s-bucket-successor*:
⟦*valid-list T*; *strict-mono α*; *α bot = 0*; *b ≠ 0*; *x ∈ s-bucket α T b*⟧ ⟹
Suc x ∈ s-bucket α T b ∨ (∃ *b′*. *b < b′* ∧ *Suc x ∈ bucket α T b′*)

```
proof −
  assume valid-list T strict-mono α α bot = 0 b ≠ 0 x ∈ s-bucket α T b
  hence suffix-type T x = S-type
    by (simp add: s-bucket-def)

  from valid-list-length-ex[OF ‹valid-list -›]
  obtain n where
    length T = Suc n
    by blast
  moreover
  from ‹x ∈ s-bucket α T b›
  have x < length T α (T ! x) = b
    by (simp add: s-bucket-def bucket-def)+
  ultimately have Suc x < length T
    by (metis Suc-lessI ‹α bot = 0› ‹b ≠ 0› ‹valid-list T› diff-Suc-1 last-conv-nth
list.size(3)
          valid-list-def)

  have T ! x ≤ T ! Suc x
   by (simp add: ‹Suc x < length T› ‹suffix-type T x = S-type› s-type-letter-le-Suc)
  hence T ! x < T ! Suc x ∨ T ! x = T ! Suc x
    using le-neq-trans by blast
  moreover
  have T ! x < T ! Suc x ⟹ ?thesis
  proof −
    assume T ! x < T ! Suc x
    hence α (T ! x) < α (T ! Suc x)
      by (simp add: ‹strict-mono α› strict-mono-less)
    hence b < α (T ! Suc x)
      by (simp add: ‹α (T ! x) = b›)
    with ‹Suc x < length T›
    have Suc x ∈ bucket α T (α (T ! Suc x))
      by (simp add: bucket-def)
    with ‹b < α (T ! Suc x)›
    show ?thesis
      by blast
  qed
  moreover
  have T ! x = T ! Suc x ⟹ ?thesis
  proof −
    assume T ! x = T ! Suc x
    hence α (T ! Suc x) = b
      using ‹α (T ! x) = b› by auto
    moreover
    from ‹Suc x < length T› ‹T ! x = T ! Suc x› ‹suffix-type T x = S-type›
    have suffix-type T (Suc x) = S-type
      using suffix-type-neq by force
    ultimately show ?thesis
      by (simp add: ‹Suc x < length T› bucket-def s-bucket-def)
```

```
  qed
  ultimately show ?thesis
    by blast
qed

lemma subset-s-bucket-successor:
  ⟦valid-list T; strict-mono α; α bot = 0; b ≠ 0; A ⊆ s-bucket α T b; A ≠ {}⟧ ⟹
    ∃ x ∈ A. Suc x ∈ s-bucket α T b − A ∨ (∃ b'. b < b' ∧ Suc x ∈ bucket α T b')
proof −
  assume valid-list T strict-mono α α bot = 0 b ≠ 0 A ⊆ s-bucket α T b A ≠ {}

  have finite A
    using ‹A ⊆ s-bucket α T b› finite-s-bucket finite-subset by blast

  let ?B = s-bucket α T b − A

  have ∃ x ∈ A. Suc x ∉ A
  proof (rule ccontr)
    assume ¬ (∃ x∈A. Suc x ∉ A)
    hence ∀ x∈A. Suc x ∈ A
      by clarsimp
    hence ¬ finite A
      using Max.coboundedI Max-in Suc-n-not-le-n ‹A ≠ {}› by blast
    with ‹finite A›
    show False
      by blast
  qed
  then obtain x where
    x ∈ A
    Suc x ∉ A
    by blast
  with s-bucket-successor[OF ‹valid-list -› ‹strict-mono -› ‹α - = -› ‹b ≠ -›, of x]
      ‹A ⊆ s-bucket α T b›
  have Suc x ∈ s-bucket α T b ∨ (∃ b'. b < b' ∧ Suc x ∈ bucket α T b')
    by blast
  moreover
  have Suc x ∈ s-bucket α T b ⟹ ?thesis
  proof −
    assume Suc x ∈ s-bucket α T b
    with ‹Suc x ∉ A›
    show ?thesis
      using ‹x ∈ A› by blast
  qed
  moreover
  have (∃ b'. b < b' ∧ Suc x ∈ bucket α T b') ⟹ ?thesis
    using ‹x ∈ A› by blast
  ultimately show ?thesis
    by blast
qed
```

lemma *valid-list-s-bucket-start-0*:
 ⟦*valid-list T*; *strict-mono α*; *α bot* = *0*⟧ ⟹
  s-bucket-start α T 0 = *0*
 apply (*clarsimp simp*: *s-bucket-start-def bucket-start-0*)
 apply (*frule valid-list-length-ex*)
 apply *clarsimp*
 apply (*frule* (*3*) *bucket-0*)
 apply (*frule suffix-type-last*)
 apply (*clarsimp simp*: *l-bucket-size-def l-bucket-def*)
 done

definition *pure-s-bucket* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat set*
 where
pure-s-bucket α T b = {*k* |*k*. *k* ∈ *s-bucket α T b* ∧ *k* ∉ *lms-bucket α T b*}

definition *pure-s-bucket-size* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
 where
pure-s-bucket-size α T b ≡ *card* (*pure-s-bucket α T b*)

lemma *finite-pure-s-bucket*:
 finite (*pure-s-bucket α T b*)
 by (*clarsimp simp*: *pure-s-bucket-def finite-s-bucket*)

lemma *pure-s-subset-s-bucket*:
 pure-s-bucket α T b ⊆ *s-bucket α T b*
 by (*clarsimp simp*: *s-bucket-def pure-s-bucket-def*)

lemma *disjoint-lms-pure-s-bucket*:
 lms-bucket α T b ∩ *pure-s-bucket α T b* = {}
 apply (*rule disjointI'*)
 by (*clarsimp simp*: *lms-bucket-def pure-s-bucket-def*)

lemma *disjoint-pure-s-lms-bucket*:
 pure-s-bucket α T b ∩ *lms-bucket α T b* = {}
 apply (*subst Int-commute*)
 by (*rule disjoint-lms-pure-s-bucket*)

lemma *s-eq-pure-s-un-lms-bucket*:
 s-bucket α T b = *pure-s-bucket α T b* ∪ *lms-bucket α T b*
 apply (*intro equalityI*; *clarsimp simp*: *pure-s-subset-s-bucket lms-subset-s-bucket*)
 apply (*clarsimp simp*: *s-bucket-def lms-bucket-def pure-s-bucket-def*)
 done

lemma *l-pl-pure-s-pl-lms-size*:
 bucket-size α T b = *l-bucket-size α T b* + *pure-s-bucket-size α T b* + *lms-bucket-size α T b*

apply (*clarsimp simp*: *bucket-size-def l-bucket-size-def pure-s-bucket-size-def*
lms-bucket-size-def)
apply (*subst add.assoc*)
apply (*subst card-Un-disjoint*[*symmetric*,
OF finite-pure-s-bucket finite-lms-bucket disjoint-pure-s-lms-bucket])
apply (*subst s-eq-pure-s-un-lms-bucket*[*symmetric*])
apply (*subst card-Un-disjoint*[*symmetric*,
OF finite-l-bucket finite-s-bucket disjoint-l-s-bucket])
apply (*clarsimp simp*: *l-un-s-bucket*)
done

lemma *s-bucket-start-eq-l-bucket-end*:
s-bucket-start α *T b* = *l-bucket-end* α *T b*
by (*simp add*: *l-bucket-end-def s-bucket-start-def*)

lemma *s-eq-pure-pl-lms-size*:
s-bucket-size α *T b* = *pure-s-bucket-size* α *T b* + *lms-bucket-size* α *T b*
by (*simp add*: *card-Un-disjoint disjoint-pure-s-lms-bucket finite-lms-bucket finite-pure-s-bucket*
lms-bucket-size-def pure-s-bucket-size-def s-bucket-size-def s-eq-pure-s-un-lms-bucket)

lemma *bucket-end-eq-s-start-pl-size*:
bucket-end α *T b* = *s-bucket-start* α *T b* + *s-bucket-size* α *T b*
by (*simp add*: *bucket-end-def' l-bucket-end-def l-pl-pure-s-pl-lms-size*
s-bucket-start-eq-l-bucket-end s-eq-pure-pl-lms-size)

lemma *bucket-start-le-s-bucket-start*:
bucket-start α *T b* $\leq$ *s-bucket-start* α *T b*
by (*simp add*: *s-bucket-start-def*)

lemma *bucket-0-size1*:
assumes *valid-list T*
and *strict-mono* α
and α *bot* = *0*
shows *bucket-size* α *T 0* = *Suc 0* $\wedge$ *l-bucket-size* α *T 0* = *0*
proof −
from *valid-list-length-ex*[*OF assms*(*1*)]
obtain *n* **where**
length T = *Suc n*
by *blast*
with *bucket-0*[*OF assms*(*1*,*3*,*2*)]
have *bucket* α *T 0* = {*n*}
by *blast*
hence *bucket-size* α *T 0* = *Suc 0*
by (*simp add*: *bucket-size-def*)
moreover
have *suffix-type T n* = *S-type*
by (*simp add*: ‹*length T* = *Suc n*› *suffix-type-last*)
hence *n* $\notin$ *l-bucket* α *T 0*

```
    by (simp add: l-bucket-def)
  hence l-bucket-size α T 0 = 0
  proof −
    have l-bucket α T 0 ⊆ {n}
      by (metis ‹bucket α T 0 = {n}› l-bucket-subset-bucket)
    hence ∀ n. n ∉ l-bucket α T 0
      using ‹n ∉ l-bucket α T 0› by blast
    then show ?thesis
      by (simp add: l-bucket-size-def)
  qed
  ultimately
  show ?thesis
    by blast
qed

lemma bucket-0-size2:
  assumes valid-list T
  and     strict-mono α
  and     α bot = 0
  and     length T = Suc (Suc n)
shows bucket-size α T 0 = Suc 0 ∧ l-bucket-size α T 0 = 0 ∧ lms-bucket-size α
T 0 = Suc 0 ∧
      pure-s-bucket-size α T 0 = 0
proof −
  from bucket-0[OF assms(1,3,2,4)]
  have bucket α T 0 = {Suc n} .

  have abs-is-lms T (Suc n)
    using assms(1,4) abs-is-lms-last by blast
  hence lms-bucket α T 0 = {Suc n}
    using lms-bucket-subset-bucket[of α T 0] ‹bucket α T 0 = {Suc n}›
    by (simp add: lms-bucket-def subset-antisym)
  hence lms-bucket-size α T 0 = Suc 0
    by (simp add: lms-bucket-size-def)
  moreover
  from ‹bucket α T 0 = {Suc n}› ‹lms-bucket α T 0 = {Suc n}›
  have pure-s-bucket α T 0 = {}
    by (metis disjoint-insert(1) disjoint-l-lms-bucket disjoint-lms-pure-s-bucket
        l-bucket-subset-bucket l-un-s-bucket pure-s-subset-s-bucket singletonI
        subset-singletonD sup-bot.left-neutral)
  hence pure-s-bucket-size α T 0 = 0
    by (simp add: pure-s-bucket-size-def)
  moreover
  from bucket-0-size1[OF assms(1−3)]
  have bucket-size α T 0 = Suc 0 ∧ l-bucket-size α T 0 = 0 .
  ultimately
  show ?thesis
    by blast
qed
```

definition *lms-bucket-start* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat* ⇒ *nat*
where
lms-bucket-start α T b = *bucket-start α T b* + *l-bucket-size α T b* + *pure-s-bucket-size α T b*

lemma *l-bucket-end-le-lms-bucket-start*:
l-bucket-end α T b ≤ *lms-bucket-start α T b*
by (*simp add*: *l-bucket-end-def lms-bucket-start-def*)

lemma *lms-bucket-start-le-bucket-end*:
lms-bucket-start α T b ≤ *bucket-end α T b*
by (*simp add*: *bucket-end-def' lms-bucket-start-def l-pl-pure-s-pl-lms-size*)

lemma *lms-bucket-pl-size-eq-end*:
lms-bucket-start α T b + *lms-bucket-size α T b* = *bucket-end α T b*
by (*simp add*: *bucket-end-def' l-pl-pure-s-pl-lms-size lms-bucket-start-def*)

47 Continuous Buckets

lemma *continuous-buckets*:
continuous-list (*map* (λ*b*. (*bucket-start α T b*, *bucket-end α T b*)) [*i*..<*j*])
by (*clarsimp simp*: *continuous-list-def bucket-start-Suc-eq-bucket-end*)

lemma *index-in-bucket-interval-gen*:
⟦*i* < *length T*; *strict-mono α*⟧ ⟹
∃ *b* ≤ *α* (*Max* (*set T*)). *bucket-start α T b* ≤ *i* ∧ *i* < *bucket-end α T b*
apply (*insert continuous-buckets*[*of α T 0 Suc* (*α* (*Max* (*set T*)))])
apply (*drule continuous-list-interval-2*[**where** *n* = *α* (*Max* (*set T*)) **and** *i* = *i*])
apply *clarsimp*
apply (*subst nth-map*)
apply *clarsimp*
apply (*clarsimp split*: *prod.split simp*: *upt-rec bucket-start-0*)
apply (*subst nth-map*)
apply *clarsimp*
apply (*clarsimp split*: *prod.split simp*: *nth-append bucket-end-Max*)
apply *clarsimp*
apply (*clarsimp simp*: *nth-append split*: *if-splits prod.splits*)
apply (*meson dual-order.order-iff-strict*)
by *blast*

lemma *index-in-bucket-interval*:
⟦*i* < *length T*; *valid-list T*; *α bot* = *0*; *strict-mono α*⟧ ⟹
∃ *b* ≤ *α* (*Max* (*set T*)). *bucket-start α T b* ≤ *i* ∧ *i* < *bucket-end α T b*
using *index-in-bucket-interval-gen* **by** *blast*

48 Bucket Initialisation

definition *lms-bucket-init* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
  where
lms-bucket-init α *T B* =
  (α (*Max* (*set T*)) < *length B* ∧
   (∀ *b* ≤ α (*Max* (*set T*)). *B* ! *b* = *bucket-end* α *T b*))

lemma *lms-bucket-init-length*:
  lms-bucket-init α *T B* ⟹ α (*Max* (*set T*)) < *length B*
  using *lms-bucket-init-def* **by** *blast*

lemma *lms-bucket-initD*:
  ⟦*lms-bucket-init* α *T B*; *b* ≤ α (*Max* (*set T*))⟧ ⟹ *B* ! *b* = *bucket-end* α *T b*
  using *lms-bucket-init-def* **by** *blast*

definition *l-bucket-init* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
  where
l-bucket-init α *T B* =
  (α (*Max* (*set T*)) < *length B* ∧
   (∀ *b* ≤ α (*Max* (*set T*)). *B* ! *b* = *bucket-start* α *T b*))

lemma *l-bucket-init-length*:
  l-bucket-init α *T B* ⟹ α (*Max* (*set T*)) < *length B*
  using *l-bucket-init-def* **by** *blast*

lemma *l-bucket-initD*:
  ⟦*l-bucket-init* α *T B*; *b* ≤ α (*Max* (*set T*))⟧ ⟹ *B* ! *b* = *bucket-start* α *T b*
  using *l-bucket-init-def* **by** *blast*

definition *s-bucket-init*
  where
s-bucket-init α *T B* =
  (α (*Max* (*set T*)) < *length B* ∧
   (∀ *b*≤α (*Max* (*set T*)).
    (*b* > *0* ⟶ *B* ! *b* = *bucket-end* α *T b*) ∧
    (*b* = *0* ⟶ *B* ! *b* = *0*)
)
)

lemma *s-bucket-init-length*:
  s-bucket-init α *T B* ⟹ α (*Max* (*set T*)) < *length B*
  using *s-bucket-init-def* **by** *blast*

lemma *s-bucket-initD*:
  ⟦*s-bucket-init* α *T B*; *b* ≤ α (*Max* (*set T*)); *b* > *0*⟧ ⟹ *B* ! *b* = *bucket-end* α *T b*

⟦*s-bucket-init* α *T B*; *b* ≤ α (*Max* (*set T*)); *b* = *0*⟧ ⟹ *B* ! *b* = *0*
using *s-bucket-init-def* **by** *auto*

49 Bucket Range

definition *in-s-current-bucket*
where
in-s-current-bucket α *T B b i* ≡ (*b* ≤ α (*Max* (*set T*)) ∧ *B* ! *b* ≤ *i* ∧ *i* < *bucket-end* α *T b*)

lemma *in-s-current-bucketD*:
in-s-current-bucket α *T B b i* ⟹ *b* ≤ α (*Max* (*set T*))
in-s-current-bucket α *T B b i* ⟹ *B* ! *b* ≤ *i*
in-s-current-bucket α *T B b i* ⟹ *i* < *bucket-end* α *T b*
by (*simp-all add*: *in-s-current-bucket-def*)

definition *in-s-current-buckets*
where
in-s-current-buckets α *T B i* ≡ (∃ *b*. *in-s-current-bucket* α *T B b i*)

lemma *in-s-current-bucket-list-slice*:
assumes *length SA* = *length T*
and *in-s-current-bucket* α *T B b i*
and *SA* ! *i* = *x*
shows *x* ∈ *set* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))
by (*metis assms bucket-end-le-length in-s-current-bucket-def list-slice-nth-mem*)

definition *in-l-bucket*
where
in-l-bucket α *T b i* ≡ (*b* ≤ α (*Max* (*set T*)) ∧ *bucket-start* α *T b* ≤ *i* ∧ *i* < *l-bucket-end* α *T b*)

end
theory *LMS-List-Slice-Util*
imports *List-Type*
begin

50 Helpers

lemma *filter-abs-is-lms-upt-0*:
filter (*abs-is-lms xs*) [*0*..<*n*] = *filter* (*abs-is-lms xs*) [*Suc 0*..<*n*]
by (*metis filter.simps*(*2*) *abs-is-lms-0 tl-upt upt-0 upt-rec*)

lemma *filter-abs-is-lms-upt-hd*:
⟦*abs-is-lms xs i*; *i* < *n*⟧ ⟹
filter (*abs-is-lms xs*) [*i*..<*n*] = *i* # *filter* (*abs-is-lms xs*) [*Suc i*..<*n*]
by (*metis filter.simps*(*2*) *upt-rec*)

51 LMS Slice

51.1 Find the next LMS position

fun
abs-find-index' :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *nat* ⇒ *nat*
where
abs-find-index' P xs i =
(*case xs of*
[] ⇒ *i*
| *x*#*xs'* ⇒
(*if P x*
then i
else abs-find-index' P xs' (*Suc i*)))

definition
abs-find-next-lms :: (*'a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ *nat*
where
abs-find-next-lms T i =
(*case find* (λ*j*. *abs-is-lms T j*) [*Suc i*..<*length T*] *of*
Some j ⇒ *j*
| - ⇒ *length T*)

lemma *abs-find-next-lms-le-length*:
abs-find-next-lms T i ≤ *length T*
unfolding *abs-find-next-lms-def*
apply (*clarsimp split*: *option.split*)
by (*metis find-Some-iff abs-is-lms-gre-length not-less order.order-iff-strict*)

lemma *abs-find-next-lms-abs-is-lms*:
abs-is-lms T (*Suc i*) ⟹ *abs-find-next-lms T i* = *Suc i*
unfolding *abs-find-next-lms-def*
apply (*frule abs-is-lms-imp-less-length*)
apply (*clarsimp split*: *option.split simp*: *upt-rec*)
done

lemma *Suc-not-lms-imp-abs-find-next-eq-Suc*:
¬ *abs-is-lms T* (*Suc i*) ⟹ *abs-find-next-lms T i* = *abs-find-next-lms T* (*Suc i*)
unfolding *abs-find-next-lms-def*
by (*simp add*: *upt-rec*)

lemma *abs-find-next-lms-lower-bound-1*:
i < *length T* ⟹ *i* < *abs-find-next-lms T i*
unfolding *abs-find-next-lms-def*
apply (*clarsimp split*: *option.split*)
using *findSomeD* **by** *fastforce*

lemma *abs-find-next-lms-lower-bound-2*:
length T ≤ *i* ⟹ *length T* ≤ *abs-find-next-lms T i*
unfolding *abs-find-next-lms-def*

by (*clarsimp split*: *option.split*)

lemma *abs-find-next-lms-le-Suc*:
abs-find-next-lms T i ≤ *abs-find-next-lms T* (*Suc i*)
apply (*cases Suc i* < *length T*)
apply (*metis find.simps*(*2*) *abs-find-next-lms-def abs-find-next-lms-abs-is-lms abs-find-next-lms-lower-bound-1*
le-less upt-rec)
by (*simp add*: *abs-find-next-lms-def*)

lemma *no-lms-between-i-and-next*:
⟦*i* < *k*; *k* < *abs-find-next-lms T i*⟧ ⟹ ¬*abs-is-lms T k*
unfolding *abs-find-next-lms-def*
apply (*clarsimp split*: *option.splits*)
apply (*drule findNoneD*)
apply (*erule ballE*[*of* - - *k*])
apply *blast*
apply *simp*
apply (*drule find-Some-iff*[*THEN iffD1*])
apply *clarsimp*
apply (*erule allE*[*of* - *k* − *Suc i*])
apply *clarsimp*
done

lemma *abs-find-next-lms-less-length-abs-is-lms*:
abs-find-next-lms T i < *length T* ⟹
abs-is-lms T (*abs-find-next-lms T i*)
unfolding *abs-find-next-lms-def*
apply (*clarsimp split*: *option.splits*)
apply (*drule find-Some-iff*[*THEN iffD1*])
apply *clarsimp*
done

lemma *abs-find-next-lms-strict-upper-imp-lower-bound*:
abs-find-next-lms T i < *length T* ⟹
i < *abs-find-next-lms T i*
unfolding *abs-find-next-lms-def*
apply (*clarsimp split*: *option.splits*)
using *findSomeD* **by** *fastforce*

lemma *abs-find-next-lms-suffix*:
assumes *i* ≤ *length T*
shows *abs-find-next-lms T i* =
i + *abs-find-next-lms* (*suffix T i*) *0*
proof −
from *abs-is-lms-i-gr-0*[*of* - *i T*] *no-lms-between-i-and-next*[*of i* - *T*]
have *P*: ⋀*k*. ⟦*0* < *k*; *k* < *abs-find-next-lms T i* − *i*⟧ ⟹ ¬*abs-is-lms* (*suffix T i*) *k*
by (*meson less-add-same-cancel2 less-diff-conv*)

```
  have abs-find-next-lms T i = length T ∨ abs-find-next-lms T i < length T
    using abs-find-next-lms-le-length le-neq-implies-less by blast
  moreover
  have abs-find-next-lms T i = length T ⟹ ?thesis
  proof −
    assume abs-find-next-lms T i = length T
    hence ⋀k. ⟦0 < k; k < length T − i⟧ ⟹ ¬abs-is-lms (suffix T i) k
      using P by presburger
    hence abs-find-next-lms (suffix T i) 0 = length T − i
      by (metis abs-find-next-lms-le-length abs-find-next-lms-less-length-abs-is-lms
                abs-find-next-lms-strict-upper-imp-lower-bound le-neq-implies-less
length-drop)
    then show ?thesis
      by (simp add: ‹abs-find-next-lms T i = length T› assms)
  qed
  moreover
  have abs-find-next-lms T i < length T ⟹ ?thesis
  proof −
    assume abs-find-next-lms T i < length T
    hence abs-is-lms T (abs-find-next-lms T i)
      using abs-find-next-lms-less-length-abs-is-lms by blast
    hence abs-is-lms (suffix T i) (abs-find-next-lms T i − i)
      by (simp add: abs-is-lms-i-gr-0 ‹abs-find-next-lms T i < length T›
                abs-find-next-lms-strict-upper-imp-lower-bound less-or-eq-imp-le)
    with P
    show ?thesis
    by (metis add.commute abs-find-next-lms-le-length abs-find-next-lms-less-length-abs-is-lms
          abs-find-next-lms-strict-upper-imp-lower-bound abs-is-lms-imp-less-length
            le-neq-implies-less length-drop less-or-eq-imp-le nat-neq-iff
          no-lms-between-i-and-next ordered-cancel-comm-monoid-diff-class.diff-add
            zero-less-diff)
  qed
  ultimately show ?thesis
    by blast
qed

lemma abs-find-next-lms-cons-Suc:
  assumes i ≤ length xs
  shows abs-find-next-lms (x # xs) (Suc i) =
        Suc (abs-find-next-lms xs i)
proof −
  have abs-find-next-lms xs i = length xs ∨ abs-find-next-lms xs i < length xs
    using abs-find-next-lms-le-length le-neq-implies-less by blast
  moreover
  have abs-find-next-lms xs i = length xs ⟹ ?thesis
    by (metis Suc-le-mono add.assoc assms drop-Suc-Cons
            abs-find-next-lms-suffix length-Cons plus-1-eq-Suc)
  moreover
```

have *abs-find-next-lms xs i < length xs ⟹ ?thesis*
by (*metis* (*no-types*, *lifting*) *Suc-le-mono add.assoc length-Cons*
assms drop-Suc-Cons abs-find-next-lms-suffix plus-1-eq-Suc)
ultimately show *?thesis*
by *blast*
qed

lemma *abs-find-next-lms-funpow-Suc*:
((*abs-find-next-lms T*)^^(*Suc k*)) *i* =
abs-find-next-lms T (((*abs-find-next-lms T*)^^*k*) *i*)
by *simp*

lemma *abs-find-next-lms-funpow-le*:
i < length T ⟹
((*abs-find-next-lms T*)^^*k*) *i* ≤
((*abs-find-next-lms T*)^^(*Suc k*)) *i*
apply (*induct k*; *clarsimp*)
apply (*simp add*: *abs-find-next-lms-lower-bound-1 less-or-eq-imp-le*)
by (*simp add*: *abs-find-next-lms-le-Suc lift-Suc-mono-le*)

lemma *no-lms-between-i-and-next-funpow*:
⟦((*abs-find-next-lms T*)^^*k*) *i* <
((*abs-find-next-lms T*)^^(*Suc k*)) *i*;
((*abs-find-next-lms T*)^^*k*) *i < j*;
j < ((*abs-find-next-lms T*)^^(*Suc k*)) *i*⟧ ⟹
¬ *abs-is-lms T j*
by (*simp add*: *no-lms-between-i-and-next*)

lemma *abs-find-next-lms-eq-Suc*:
xs ≠ [] ⟹ ∃ *k*. *abs-find-next-lms xs i* = *Suc k*
by (*metis abs-find-next-lms-less-length-abs-is-lms*
abs-is-lms-0 length-greater-0-conv not0-implies-Suc)

lemma *filter-no-lms1*:
⟦*abs-is-lms xs i*; *i < k*; *k* ≤ *abs-find-next-lms xs i*⟧ ⟹
filter (*abs-is-lms xs*) [*Suc i*..<*k*] = []
proof (*induct k*)
case *0*
then show *?case*
by *simp*
next
case (*Suc k*)
then show *?case*
by (*metis Suc-leD Suc-le-eq append-Nil filter.simps*(*1*,*2*)
upt-Suc filter-append no-lms-between-i-and-next)
qed

lemma *filter-no-lms2*:
⟦¬*abs-is-lms xs i*; *i < k*; *k* ≤ *abs-find-next-lms xs i*⟧ ⟹

filter (*abs-is-lms xs*) [*i*..<*k*] = []
proof (*induct k*)
case *0*
then show *?case*
by *simp*
next
case (*Suc k*)
then show *?case*
by (*metis Suc-le-eq append-Nil filter.simps*(*1*)
filter.simps(*2*) *filter-append*
not-less-eq-eq upt.simps(*2*) *nle-le*
no-lms-between-i-and-next upt-conv-Cons)
qed

51.2 LMS Prefix

fun
closest-lms ::
('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ *nat*
where
closest-lms T i =
(*if abs-is-lms T i*
then i
else abs-find-next-lms T i)

definition
lms-prefix ::
('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ '*a list*
where
lms-prefix T i =
list-slice T i (*Suc* (*closest-lms T i*))

lemma *lms-lms-prefix*:
abs-is-lms T i ⟹ *lms-prefix T i* = [*T* ! *i*]
unfolding *lms-prefix-def*
by (*simp add*: *abs-is-lms-imp-less-length list-slice-Suc*)

lemma *suffix-to-lms-prefix*:
i < *length T* ⟹
suffix T i =
lms-prefix T i @
(*list-slice T* (*Suc* (*closest-lms T i*)) (*length T*))
unfolding *lms-prefix-def*
apply *clarsimp*
apply (*intro impI conjI*)
apply (*rule suffix-to-list-slice-app*)
apply *linarith*
by (*meson abs-find-next-lms-lower-bound-1 less-SucI*

less-or-eq-imp-le suffix-to-list-slice-app)

lemma *abs-find-next-lms-funpow-all-lms*:
⟦*abs-is-lms xs* ((*abs-find-next-lms xs* ^^ *Suc k*) *x*);
i ≤ *k*⟧ ⟹
abs-is-lms xs ((*abs-find-next-lms xs* ^^ *Suc i*) *x*)
proof (*induct k arbitrary*: *i*)
case *0*
then show *?case*
by *blast*
next
case (*Suc k*)
note *IH* = *this*
hence (*abs-find-next-lms xs* ^^ *Suc* (*Suc k*)) *x* < *length xs*
using *abs-is-lms-imp-less-length* **by** *blast*
moreover
have *x* < *length xs*
by (*metis calculation abs-find-next-lms-funpow-Suc*
abs-find-next-lms-le-length abs-find-next-lms-lower-bound-2
abs-find-next-lms-strict-upper-imp-lower-bound funpow-swap1
linorder-not-less nat-neq-iff)
ultimately
have (*abs-find-next-lms xs* ^^ *Suc k*) *x* < *length xs*
using *abs-find-next-lms-funpow-le order.strict-trans1* **by** *blast*
hence *P*: *abs-is-lms xs* ((*abs-find-next-lms xs* ^^ *Suc k*) *x*)
by (*simp add*: *abs-find-next-lms-less-length-abs-is-lms*)

from *IH*(*3*)
have *i* ≤ *k* ∨ *i* = *Suc k*
by (*meson le-Suc-eq le-neq-implies-less*)
moreover
from *IH*(*1*)[*OF P*, *of i*]
have *i* ≤ *k* ⟹ *?case*
by *blast*
moreover
from *IH*(*2*)
have *i* = *Suc k* ⟹ *?case*
by *simp*
ultimately show *?case*
by *blast*
qed

51.3 LMS Slice

definition
lms-slice :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ '*a list*
where
lms-slice T i =
list-slice T i (*Suc* (*abs-find-next-lms T i*))

lemma *suffix-to-lms-slice*:
  $i < length\ T \Longrightarrow$
   $suffix\ T\ i =$
   $lms\text{-}slice\ T\ i\ @$
     $(list\text{-}slice\ T\ (Suc\ (abs\text{-}find\text{-}next\text{-}lms\ T\ i))\ (length\ T))$
  unfolding *lms-slice-def*
  apply (*rule suffix-to-list-slice-app*)
  by (*simp add*: *abs-find-next-lms-lower-bound-1*
                le-Suc-eq less-or-eq-imp-le)

lemma *suffix-to-lms-slice-app-suffix*:
  $i < length\ T \Longrightarrow$
   $suffix\ T\ i =$
   $lms\text{-}slice\ T\ i\ @$
     $(suffix\ T\ (Suc\ (abs\text{-}find\text{-}next\text{-}lms\ T\ i)))$
  by (*metis suffix-eq-list-slice suffix-to-lms-slice*)

lemma *lms-slice-cons*:
  $[\![i < length\ T;\ suffix\text{-}type\ T\ i = S\text{-}type]\!] \Longrightarrow$
   $lms\text{-}slice\ T\ i =$
   $T\ !\ i\ \#\ lms\text{-}slice\ T\ (Suc\ i)$
  using *abs-is-lms-def Suc-not-lms-imp-abs-find-next-eq-Suc*
        abs-find-next-lms-lower-bound-1 i-s-type-imp-Suc-i-not-lms
        list-slice-Suc Suc-not-lms-imp-abs-find-next-eq-Suc
  by (*clarsimp simp*: *lms-slice-def*) *fastforce*

lemma *lms-slice-hd*:
  $i < length\ T \Longrightarrow$
   $\exists xs.\ lms\text{-}slice\ T\ i = T\ !\ i\ \#\ xs$
 by (*simp add*: *abs-find-next-lms-lower-bound-1 less-SucI list-slice-Suc lms-slice-def*)

lemma *lms-slice-suffix*:
  assumes $i \leq length\ T$
  shows $lms\text{-}slice\ (suffix\ T\ i)\ 0 =$
        $lms\text{-}slice\ T\ i$
proof −
  from *list-slice-suffix*[*of T i Suc* (*abs-find-next-lms T i*)]
      lms-slice-def[*of T i*]
      abs-find-next-lms-suffix[*OF assms*]
      lms-slice-def[*of suffix T i 0*]
  show *?thesis*
    by (*metis add-Suc-right add-diff-cancel-left'*)
qed

lemma *lms-slice-suffix-gen*:
  assumes $i \leq length\ T$
  and $j \leq length\ T - i$
shows $lms\text{-}slice\ (suffix\ T\ i)\ j =$

lms-slice T (i + j)
proof −
have *lms-slice T (i + j) =*
lms-slice (suffix T (i + j)) 0
by (*metis add.commute assms lms-slice-suffix le-diff-conv2*)
hence *lms-slice T (i + j) =*
lms-slice (suffix (suffix T i) j) 0
by (*simp add*: *add.commute*)
moreover
have *lms-slice (suffix T i) j = lms-slice (suffix (suffix T i) j) 0*
by (*metis assms*(*2*) *length-drop lms-slice-suffix*)
ultimately show *?thesis*
by *simp*
qed

lemma *lms-slice-cons-Suc*:
i ≤ length xs ⟹ lms-slice (x # xs) (Suc i) = lms-slice xs i
by (*metis Suc-le-mono drop-Suc-Cons length-Cons lms-slice-suffix*)

51.4 LMS Substring butlast

definition *lms-slice-butlast* :: (*'a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat* ⇒ *'a list*
where
lms-slice-butlast T i = list-slice T i (*abs-find-next-lms T i*)

lemma *lms-slice-to-butlast-app*:
abs-find-next-lms T i < length T ⟹
lms-slice T i = lms-slice-butlast T i @ [*T ! abs-find-next-lms T i*]
unfolding *lms-slice-def lms-slice-butlast-def*
apply (*subst list-slice-append*[*of - abs-find-next-lms T i*])
apply (*simp add*: *abs-find-next-lms-strict-upper-imp-lower-bound order.strict-implies-order*)
apply *simp*
by (*simp add*: *list-slice-Suc*)

lemma *lms-slice-eq-butlast*:
length T ≤ abs-find-next-lms T i ⟹
lms-slice T i = lms-slice-butlast T i
by (*metis le-SucI list-slice-end-gre-length lms-slice-butlast-def lms-slice-def*)

lemma *lms-slice-eq-suffix*:
length T ≤ abs-find-next-lms T i ⟹
lms-slice T i = suffix T i
by (*simp add*: *list-slice.simps lms-slice-butlast-def lms-slice-eq-butlast*)

lemma *suffix-abs-find-next-lms*:
abs-find-next-lms T i < length T ⟹
suffix T i = lms-slice-butlast T i @ suffix T (*abs-find-next-lms T i*)
by (*simp add*: *abs-find-next-lms-strict-upper-imp-lower-bound less-or-eq-imp-le*
list-slice-append

lms-slice-butlast-def suffix-eq-list-slice)

51.5 Suffix Types

lemma *suffix-type-lms-slice-l-s*:
assumes *suffix-type T i = L-type*
and *suffix-type T (Suc i) = S-type*
shows *suffix-type (lms-slice T i) 0 = suffix-type T i*
proof −
have *Suc i < length T*
by (*simp add*: *assms*(*2*) *suffix-type-s-bound*)

have *abs-is-lms T (Suc i)*
by (*simp add*: *assms abs-is-lms-def*)
hence *abs-find-next-lms T i = Suc i*
by (*simp add*: *abs-find-next-lms-abs-is-lms*)
hence *lms-slice T i = [T ! i, T ! Suc i]*
by (*metis Suc-n-not-le-n* ‹*Suc i < length T*› *le-Suc-eq not-le*
list-slice-Suc list-slice-n-n lms-slice-def)
moreover
have *T ! i > T ! Suc i*
using ‹*abs-is-lms T (Suc i)*› *abs-is-lms-neq* **by** *blast*
ultimately show *?thesis*
by (*simp add*: ‹*suffix-type T i = L-type*› *suffix-type-cons-greater*)
qed

lemma *abs-find-next-lms-same-types*:
assumes ∀ *k*. *i ≤ k ∧ k < length T ⟶ suffix-type T k = suffix-type T i*
and *i ≤ j*
shows *abs-find-next-lms T j = length T*
proof (*cases find* (*abs-is-lms T*) [*Suc j..<length T*])
assume *find* (*abs-is-lms T*) [*Suc j..<length T*] = *None*
then show *abs-find-next-lms T j = length T*
by (*simp add*: *abs-find-next-lms-def*)
next
fix *x*
assume *find* (*abs-is-lms T*) [*Suc j..<length T*] = *Some x*
hence *x < length T Suc j ≤ x abs-is-lms T x*
using ‹*find* (*abs-is-lms T*) [*Suc j..<length T*] = *Some x*› *findSomeD* **by** *force+*
hence ∃ *y*. *x = Suc y*
using *abs-is-lms-def* **by** *blast*
then obtain *y* **where**
x = Suc y
by *blast*
hence *suffix-type T y = L-type suffix-type T (Suc y) = S-type*
using ‹*abs-is-lms T x*› *abs-is-lms-def* **by** *auto*

have *i ≤ y*
using ‹*Suc j ≤ x*› ‹*x = Suc y*› *assms*(*2*) *le-trans* **by** *blast*

```
  moreover
  have y < length T
    using Suc-lessD ‹x < length T› ‹x = Suc y› by blast
  ultimately have suffix-type T i = L-type
    by (metis ‹suffix-type T y = L-type› assms(1))

  have i ≤ Suc y
    by (simp add: ‹i ≤ y› le-SucI)
  moreover
  have Suc y < length T
    using ‹x < length T› ‹x = Suc y› by blast
  ultimately have suffix-type T i = S-type
    by (metis ‹suffix-type T (Suc y) = S-type› assms(1))
  with ‹suffix-type T i = L-type›
  have x = length T
    by simp
  then show ?thesis
    using ‹x < length T› by blast
qed

lemma lms-slice-same-types:
  assumes ∀ k. i ≤ k ∧ k < length T ⟶ suffix-type T k = suffix-type T i
  and     i ≤ j
shows lms-slice T j = suffix T j
proof −
  have abs-find-next-lms T j = length T
    using assms abs-find-next-lms-same-types by blast
  then show ?thesis
   by (metis le0 le-add-same-cancel2 list-slice-end-gre-length lms-slice-def plus-1-eq-Suc
            suffix-eq-list-slice)
qed

lemma all-l-types-up-to-next-lms:
  ⟦i ≤ k; k < abs-find-next-lms T i; suffix-type T i = L-type⟧ ⟹ suffix-type T k
= L-type
proof(induct k − i arbitrary: k)
  case 0
  then show ?case by simp
next
  case (Suc x)
  have ∃ k'. k = Suc k'
    using Suc.hyps(2) Suc-le-D diff-le-self by presburger
  then obtain k' where
    k = Suc k'
    by blast
  hence x = k' − i
    using Suc.hyps(2) by linarith
  moreover
  have i ≤ k'
```

```
    using Suc.hyps(2) ‹k = Suc k'› by linarith
  moreover
  have k' < abs-find-next-lms T i
    using Suc.prems(2) Suc-lessD ‹k = Suc k'› by blast
  ultimately have suffix-type T k' = L-type
    using Suc.hyps(1) Suc.prems(3) by blast

  have i < k
    by (simp add: ‹i ≤ k'› ‹k = Suc k'› le-imp-less-Suc)
  with  Suc.prems(2) no-lms-between-i-and-next[of i k T]
  have ¬ abs-is-lms T k
    by blast
  with ‹suffix-type T k' = L-type› ‹k = Suc k'›
  show ?case
    using SL-types.exhaust abs-is-lms-def by blast
qed

lemma abs-find-next-lms-eq-length:
  assumes abs-find-next-lms T i = length T
  and     i < length T
shows suffix-type T i = S-type
proof (rule ccontr)
  assume suffix-type T i ≠ S-type
  hence suffix-type T i = L-type
    using SL-types.exhaust by blast
  moreover
  have ∃ k. abs-find-next-lms T i = Suc k
    by (metis assms(1) assms(2) not-less0 old.nat.exhaust)
  then obtain k where
    abs-find-next-lms T i = Suc k
    by blast
  moreover
  have i ≤ k
    using assms(1,2) calculation by linarith
  ultimately have suffix-type T k = L-type
    by (metis all-l-types-up-to-next-lms lessI)
  moreover
  have length T = Suc k
    using ‹abs-find-next-lms T i = Suc k› assms(1) by auto
  ultimately show False
    using suffix-type-last[of T k]
    by simp
qed

lemma abs-find-next-lms-eq-length-all-s-types:
  assumes abs-find-next-lms T i = length T
  and     i ≤ j
  and     j < length T
shows suffix-type T j = S-type
```

by (*metis assms abs-find-next-lms-eq-length abs-find-next-lms-le-length abs-find-next-lms-less-length-abs-is-lms*
abs-find-next-lms-lower-bound-1 le-neq-implies-less no-lms-between-i-and-next
order.strict-trans1)

lemma *abs-find-next-lms-first-l-after-s-type*:
assumes *abs-find-next-lms T i* < *length T*
and *suffix-type T i* = *S-type*
shows ∃ *j*>*i*. *j* < *abs-find-next-lms T i* ∧ (∀ *k*<*j*. *i* ≤ *k* ⟶ *suffix-type T k* = *S-type*) ∧
suffix-type T j = *L-type*
proof −
have ∃ *j*. *abs-find-next-lms T i* = *Suc j*
by (*metis assms*(*2*) *abs-find-next-lms-lower-bound-1 not-less0 old.nat.exhaust suffix-type-s-bound*)
then obtain *j* **where**
abs-find-next-lms T i = *Suc j*
by *blast*
hence *abs-is-lms T* (*Suc j*)
using *assms*(*1*) *abs-find-next-lms-less-length-abs-is-lms* **by** *fastforce*
hence *suffix-type T j* = *L-type*
using *SL-types.exhaust i-s-type-imp-Suc-i-not-lms* **by** *auto*
moreover
have *j* < *length T*
using ‹*abs-find-next-lms T i* = *Suc j*› *assms*(*1*) **by** *linarith*
moreover
have *i* ≤ *j*
by (*metis* ‹*abs-find-next-lms T i* = *Suc j*› *assms*(*1*) *abs-find-next-lms-strict-upper-imp-lower-bound less-Suc-eq-le*)
moreover
have ∀ *k*>*i*. *k* ≤ *j* ⟶ ¬ *abs-is-lms T k*
by (*simp add*: ‹*abs-find-next-lms T i* = *Suc j*› *no-lms-between-i-and-next*)
ultimately show *?thesis*
using *first-l-type-after-s-type*[*OF - - - -assms*(*2*), *of j*]
by (*metis SL-types.simps*(*2*) ‹*abs-find-next-lms T i* = *Suc j*› *assms*(*2*) *dual-order.order-iff-strict le-imp-less-Suc*)
qed

lemma *lms-slice-type*:
assumes *i* < *length T*
shows *suffix-type* (*lms-slice T i*) *0* = *suffix-type T i*
proof −
have ∃ *k*. *abs-find-next-lms T i* = *Suc k*
by (*meson Nat.lessE assms abs-find-next-lms-lower-bound-1*)
then obtain *k* **where**
abs-find-next-lms T i = *Suc k*
by *blast*

have *suffix-type T i* = *L-type* ∨ *suffix-type T i* = *S-type*
using *SL-types.exhaust* **by** *blast*

```
  moreover
  have suffix-type T i = L-type ⟹ ?thesis
  proof −
    assume suffix-type T i = L-type

    have suffix-type T (Suc i) = L-type ∨ suffix-type T (Suc i) = S-type
      using SL-types.exhaust by blast
    moreover
    have suffix-type T (Suc i) = S-type ⟹ ?thesis
      by (simp add: ‹suffix-type T i = L-type› suffix-type-lms-slice-l-s)
    moreover
    have suffix-type T (Suc i) = L-type ⟹ ?thesis
    proof −
      assume suffix-type T (Suc i) = L-type

      from ‹abs-find-next-lms T i = Suc k›
      have P: ∀ k'≥i. k' < Suc k ⟶ suffix-type T k' = L-type
        by (simp add: ‹suffix-type T i = L-type› all-l-types-up-to-next-lms)

      have lms-slice T i = list-slice T i (Suc (Suc k))
        by (simp add: lms-slice-def ‹abs-find-next-lms T i = Suc k›)
      moreover
      {
        have i < k
          by (metis SL-types.simps(2) Suc-lessI ‹abs-find-next-lms T i = Suc k›
              ‹suffix-type T (Suc i) = L-type› ‹suffix-type T i = L-type› assms
            abs-find-next-lms-less-length-abs-is-lms abs-find-next-lms-lower-bound-1
less-antisym
              suffix-type-last suffix-type-same-imp-not-lms)
       hence list-slice T i (Suc (Suc k)) = list-slice T i k @ list-slice T k (Suc (Suc
k))
          by (meson dual-order.order-iff-strict le-Suc-eq list-slice-append)
        moreover
        have list-slice T k (Suc (Suc k)) = [T ! k, T ! (Suc k)]
          by (metis SL-types.simps(2) ‹abs-find-next-lms T i = Suc k› ‹suffix-type
T i = L-type›
              all-l-types-up-to-next-lms assms dual-order.order-iff-strict
              abs-find-next-lms-le-length less-Suc-eq-le list-slice-Suc list-slice-n-n
              not-less-eq suffix-type-last)
        moreover
        have list-slice T i k = T ! i # list-slice T (Suc i) k
          using ‹i < k› assms list-slice-Suc by blast
        ultimately have
         list-slice T i (Suc (Suc k)) = T ! i # (list-slice T (Suc i) k) @ [T ! k, T !
Suc k]
          by simp
      }
      ultimately have
        lms-slice T i = T ! i # (list-slice T (Suc i) k) @ [T ! k, T ! Suc k]
```

by *simp*

let *?bs = list-slice T (Suc i) k*

have *abs-is-lms T (Suc k)*
by (*metis SL-types.simps(2)* ‹*abs-find-next-lms T i = Suc k*› ‹*suffix-type T i = L-type*›
all-l-types-up-to-next-lms assms dual-order.order-iff-strict suffix-type-last
abs-find-next-lms-le-length abs-find-next-lms-less-length-abs-is-lms less-Suc-eq-le)
hence *T ! k > T ! Suc k*
using *abs-is-lms-neq* **by** *blast*
moreover
from *sorted-letters-l-types*[*OF P*[*simplified* ‹*abs-find-next-lms T i = Suc k*›]]
have *sorted (rev (list-slice T i (Suc k)))*
using ‹*abs-is-lms T (Suc k)*› *abs-is-lms-gre-length linear* **by** *blast*
moreover
have *list-slice T i (Suc k) = T ! i # (list-slice T (Suc i) k) @ [T ! k]*
by (*metis SL-types.simps(2)* ‹*abs-find-next-lms T i = Suc k*› ‹*abs-is-lms T (Suc k)*›
‹*suffix-type T (Suc i) = L-type*› *assms dual-order.order-iff-strict not-less-eq*
abs-find-next-lms-le-length abs-find-next-lms-lower-bound-1 abs-is-lms-def list-slice-Suc not-less
list-slice-append list-slice-n-n)
ultimately have
list-less-ns (?bs @ [T ! k, T ! Suc k]) (T ! i # ?bs @ [T ! k, T ! Suc k])
using *rev-sorted-list-less-ns*[*of T ! i ?bs T ! k T ! Suc k* [] []]
by *simp*
moreover
have *suffix (lms-slice T i) 0 = T ! i # ?bs @ [T ! k, T ! Suc k]*
by (*simp add*: ‹*lms-slice T i = T ! i # ?bs @ [T ! k, T ! Suc k]*›)
moreover
have *suffix (lms-slice T i) (Suc 0) = ?bs @ [T ! k, T ! Suc k]*
by (*simp add*: ‹*lms-slice T i = T ! i # list-slice T (Suc i) k @ [T ! k, T ! Suc k]*›)
ultimately show *?thesis*
by (*metis* ‹*suffix-type T i = L-type*› *ordlistns.less-asym suffix-type-def*)
qed
ultimately show *?thesis*
by *blast*
qed
moreover
have *suffix-type T i = S-type ⟹ ?thesis*
proof −
assume *suffix-type T i = S-type*
hence *lms-slice T i = T ! i # lms-slice T (Suc i)*
using *assms lms-slice-cons* **by** *blast*

have *abs-find-next-lms T i < length T ∨ abs-find-next-lms T i = length T*
by (*simp add*: *abs-find-next-lms-le-length nat-less-le*)
moreover
have *abs-find-next-lms T i < length T ⟹ ?thesis*
proof −
assume *abs-find-next-lms T i < length T*
with *abs-find-next-lms-first-l-after-s-type*[*OF* - ‹*suffix-type T i = S-type*›]
obtain *j* **where**
i < j
j < abs-find-next-lms T i
∀ *k*<*j*. *i ≤ k ⟶ suffix-type T k = S-type*
suffix-type T j = L-type
by *blast*
hence *sorted* (*list-slice T i j*)
by (*meson* ‹*abs-find-next-lms T i < length T*› *dual-order.order-iff-strict order.strict-trans*
sorted-letters-s-types)
have ∃ *l*. *j = Suc l*
using ‹*i < j*› *less-imp-Suc-add* **by** *blast*
then obtain *l* **where**
j = Suc l
by *blast*

let *?xs = list-slice T* (*Suc i*) (*Suc l*)
and *?ys = list-slice T* (*Suc* ((*Suc l*))) (*Suc* (*Suc k*))

have *lms-slice T i = list-slice T i j* @ *list-slice T j* (*Suc* (*Suc k*))
by (*metis* ‹*abs-find-next-lms T i = Suc k*› ‹*i < j*› ‹*j < abs-find-next-lms T i*› *less-SucI*
less-imp-le-nat list-slice-append lms-slice-def)
moreover
have *list-slice T i j = T* ! *i* # *?xs*
using ‹*i < j*› ‹*j = Suc l*› *assms list-slice-Suc* **by** *blast*
moreover
have *list-slice T j* (*Suc* (*Suc k*)) = *T* ! *Suc l* # *?ys*
by (*metis* ‹*abs-find-next-lms T i < length T*› ‹*abs-find-next-lms T i = Suc k*› ‹*j < abs-find-next-lms T i*›
‹*j = Suc l*› *less-SucI list-slice-Suc order.strict-trans*)
ultimately have *lms-slice T i = T* ! *i* # *?xs* @ [*T* ! *Suc l*] @ *?ys*
by *auto*

have *i = l ∨ i < l*
using ‹*i < j*› *less-antisym* ‹*j = Suc l*› **by** *blast*
moreover
have *i = l ⟹ ?thesis*
proof −
assume *i = l*
hence *?xs* = []

```
      using list-slice-n-n by blast
    hence lms-slice T i = T ! i # [T ! Suc l] @ ?ys
      using ‹lms-slice T i = T ! i # ?xs @ [T ! Suc l] @ ?ys›
      by simp
    moreover
    have T ! i < T ! Suc l
      by (metis SL-types.simps(2) ‹abs-find-next-lms T i < length T› ‹i = l› ‹j
< abs-find-next-lms T i›
          ‹j = Suc l› ‹suffix-type T i = S-type› ‹suffix-type T j = L-type›
suffix-type-neq
          le-imp-less-or-eq order.strict-trans s-type-letter-le-Suc)
    ultimately show ?thesis
      by (simp add: ‹suffix-type T i = S-type› suffix-type-cons-less)
  qed
  moreover
  have i < l ⟹ ?thesis
  proof −
    let ?zs = list-slice T (Suc i) l
    assume i < l
    hence ?xs = ?zs @ [T ! l]
      by (metis Suc-leI Suc-n-not-le-n ‹abs-find-next-lms T i < length T› ‹j <
abs-find-next-lms T i›
          ‹j = Suc l› lessI less-le-trans linear list-slice-Suc list-slice-append
          list-slice-n-n)
    hence lms-slice T i = T ! i # ?zs @ [T ! l, T ! Suc l] @ ?ys
      using ‹lms-slice T i = T ! i # ?xs @ [T ! Suc l] @ ?ys›
      by simp
    moreover
    have suffix-type T l = S-type
     by (simp add: ‹∀ k<j. i ≤ k ⟶ suffix-type T k = S-type› ‹i < l› ‹j = Suc
l› less-or-eq-imp-le)
    hence T ! l < T ! Suc l
       by (metis SL-types.simps(2) ‹abs-find-next-lms T i < length T› ‹j <
abs-find-next-lms T i›
           ‹j = Suc l› ‹suffix-type T j = L-type› order.strict-iff-order or-
der.strict-trans
          s-type-letter-le-Suc suffix-type-neq)
    ultimately show ?thesis
      using ‹sorted (list-slice T i j)›
          sorted-list-less-ns[of T ! i ?zs T ! l T ! Suc l ?ys ?ys]
       by (metis ‹?xs = ?zs @ [T ! l]› ‹list-slice T i j = T ! i # ?xs› suffix-0
length-Cons
          ‹suffix-type T i = S-type› list.inject suffix-0 suffix-cons-Suc suffix-type-def
           zero-less-Suc)
  qed
  ultimately show ?thesis
    by blast
qed
moreover
```

```
    have abs-find-next-lms T i = length T ⟹ ?thesis
    proof −
      assume abs-find-next-lms T i = length T
      hence P: ∀ k'≥i. k' < length T ⟶ suffix-type T k' = S-type
        using abs-find-next-lms-eq-length-all-s-types by blast

      have lms-slice T i = T ! i # list-slice T (Suc i) (length T)
           by (metis ‹abs-find-next-lms T i = length T› assms leI less-not-refl
list-slice-Suc
             lms-slice-butlast-def lms-slice-eq-butlast)
      with sorted-letters-s-types[OF P]
          sorted-cons-list-less-ns[of T ! i list-slice T (Suc i) (length T)]
      show ?thesis
        by (metis assms list-slice-Suc suffix-eq-list-slice suffix-type-suffix)
    qed

    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
    by blast
qed

lemma lms-slice-l-less-than-s-type-gen:
  assumes suffix-type (a # as) 0 = L-type
  and     suffix-type (a # bs) 0 = S-type
shows list-less-ns (lms-slice (a # as) 0) (lms-slice (a # bs) 0)
proof −
  from lms-slice-type[of 0 a # as] assms(1)
  have suffix-type (lms-slice (a # as) 0) 0 = L-type
    by simp
  moreover
  have ∃ xs. lms-slice (a # as) 0 = a # xs
    by (simp add: list-slice-Suc lms-slice-def)
  then obtain xs where
    lms-slice (a # as) 0 = a # xs
    by blast
  moreover
  from lms-slice-type[of 0 a # bs] assms(2)
  have suffix-type (lms-slice (a # bs) 0) 0 = S-type
    by simp
  moreover
  have ∃ xs. lms-slice (a # bs) 0 = a # xs
    by (simp add: assms(2) lms-slice-cons)
  then obtain ys where
    lms-slice (a # bs) 0 = a # ys
    by blast
  ultimately show ?thesis
    by (simp add: l-less-than-s-type-general)
```

qed

lemma *lms-slice-l-less-than-s-type*:
assumes $i < length\ T$
and $j < length\ T$
and $T\ !\ i = T\ !\ j$
and *suffix-type* $T\ i = L\text{-}type$
and *suffix-type* $T\ j = S\text{-}type$
shows *list-less-ns* (*lms-slice* $T\ i$) (*lms-slice* $T\ j$)
by (*metis assms abs-find-next-lms-lower-bound-1 l-less-than-s-type-general less-SucI list-slice-Suc*
lms-slice-def lms-slice-type)

lemma *lms-prefix-type*:
assumes $i < length\ T$
shows *suffix-type* (*lms-prefix* $T\ i$) 0 = *suffix-type* $T\ i$
proof −
have *abs-is-lms* $T\ i \vee \neg$*abs-is-lms* $T\ i$
by *blast*
moreover
have $\neg$*abs-is-lms* $T\ i \Longrightarrow$ *?thesis*
by (*metis assms closest-lms.simps lms-prefix-def lms-slice-def*
lms-slice-type)
moreover
have *abs-is-lms* $T\ i \Longrightarrow$ *?thesis*
by (*simp add*: *abs-is-lms-def lms-lms-prefix suffix-type-last*)
ultimately show *?thesis*
by *blast*
qed

lemma *lms-prefix-l-less-than-s-type-gen*:
assumes *suffix-type* $(a\ \#\ as)\ 0 = L\text{-}type$
and *suffix-type* $(a\ \#\ bs)\ 0 = S\text{-}type$
shows *list-less-ns* (*lms-prefix* $(a\ \#\ as)\ 0$) (*lms-prefix* $(a\ \#\ bs)\ 0$)
by (*metis assms closest-lms.simps lms-prefix-def abs-is-lms-def lessI lms-slice-def*
lms-slice-l-less-than-s-type-gen not-gr-zero not-less-iff-gr-or-eq)

lemma *lms-prefix-l-less-than-s-type*:
assumes $i < length\ T$
and $j < length\ T$
and $T\ !\ i = T\ !\ j$
and *suffix-type* $T\ i = L\text{-}type$
and *suffix-type* $T\ j = S\text{-}type$
shows *list-less-ns* (*lms-prefix* $T\ i$) (*lms-prefix* $T\ j$)
proof −
let *?a* = $T\ !\ i$

have $\exists\, as.$ *lms-prefix* $T\ i$ = *?a* # *as*
by (*simp add*: *assms*(*1*) *lms-prefix-def abs-find-next-lms-lower-bound-1 less-SucI*

```
        list-slice-Suc)
  then obtain as where
    lms-prefix T i = ?a # as
    by blast
  hence suffix-type (?a # as) 0 = L-type
    using assms(1,4) lms-prefix-type by fastforce

  have ∃ bs. lms-prefix T j = ?a # bs
   by (simp add: assms(2,3) lms-prefix-def abs-find-next-lms-lower-bound-1 less-SucI
        list-slice-Suc)
  then obtain bs where
    lms-prefix T j = ?a # bs
    by blast
  hence suffix-type (?a # bs) 0 = S-type
    using assms(2) assms(5) lms-prefix-type by fastforce
  with l-less-than-s-type-general[OF - ‹suffix-type (?a # as) 0 = -›, of bs]
  have list-less-ns (?a # as) (?a # bs) .
  then show ?thesis
    by (simp add: ‹lms-prefix T i = T ! i # as› ‹lms-prefix T j = T ! i # bs›)
qed

lemma l-type-lms-prefix-cons:
  assumes suffix-type T i = L-type
  and     i < length T
shows lms-prefix T i = T ! i # lms-prefix T (Suc i)
proof −
  have suffix-type T (Suc i) = L-type ∨ suffix-type T (Suc i) = S-type
    using SL-types.exhaust by blast
  moreover
  have suffix-type T (Suc i) = L-type ⟹ ?thesis
  proof −
    assume suffix-type T (Suc i) = L-type
    hence ¬abs-is-lms T (Suc i)
      by (simp add: ‹suffix-type T i = L-type› suffix-type-same-imp-not-lms)
    with Suc-not-lms-imp-abs-find-next-eq-Suc
    have abs-find-next-lms T i = abs-find-next-lms T (Suc i) .
    hence closest-lms T i = abs-find-next-lms T (Suc i)
      by (simp add: ‹suffix-type T i = L-type› abs-is-lms-def)
    hence lms-prefix T i = list-slice T i (Suc (abs-find-next-lms T (Suc i)))
      by (simp add: lms-prefix-def)
    moreover
    have Suc i < Suc (abs-find-next-lms T (Suc i))
    using ‹abs-find-next-lms T i = abs-find-next-lms T (Suc i)› assms(2) abs-find-next-lms-lower-bound-1
      by force
    ultimately have
      lms-prefix T i = T ! i # list-slice T (Suc i) (Suc (abs-find-next-lms T (Suc
i)))
      by (simp add:  assms(2) list-slice-Suc)
    then show ?thesis
```

```
    by (simp add: ‹¬ abs-is-lms T (Suc i)› lms-prefix-def)
  qed
  moreover
  have suffix-type T (Suc i) = S-type ⟹ ?thesis
  proof −
    assume suffix-type T (Suc i) = S-type
    hence abs-is-lms T (Suc i)
      by (simp add: assms(1) abs-is-lms-def)
    hence abs-find-next-lms T i = Suc i
      by (simp add: abs-find-next-lms-abs-is-lms)
    hence lms-prefix T i = [T ! i, T ! Suc i]
    by (metis Suc-lessD ‹abs-is-lms T (Suc i)› assms(2) closest-lms.simps lms-prefix-def
          abs-is-lms-consec(1) lessI list-slice-Suc lms-lms-prefix)
    moreover
    have lms-prefix T (Suc i) = [T ! Suc i]
      by (simp add: ‹abs-is-lms T (Suc i)› lms-lms-prefix)
    ultimately show ?thesis
      by simp
  qed
  ultimately show ?thesis
    by blast
qed
```

52 Ordering LMS-substrings

This section contains theorems about how LMS-substrings and suffixes are ordered.

```
lemma lms-slice-eq-suffix-less:
  assumes lms-slice T i = lms-slice T j
  shows list-less-ns (suffix T i) (suffix T j) ⟷
        list-less-ns (suffix T (abs-find-next-lms T i)) (suffix T (abs-find-next-lms T
j))
proof −
  have ⟦abs-find-next-lms T i < length T; abs-find-next-lms T j < length T⟧ ⟹
?thesis
  proof −
    assume A: abs-find-next-lms T i < length T abs-find-next-lms T j < length T
    have suffix T i = lms-slice-butlast T i @ suffix T (abs-find-next-lms T i)
      using A(1) suffix-abs-find-next-lms by blast
    moreover
    have suffix T j = lms-slice-butlast T j @ suffix T (abs-find-next-lms T j)
      using A(2) suffix-abs-find-next-lms by blast
    moreover
    have lms-slice-butlast T i = lms-slice-butlast T j
      by (metis A(1) A(2) assms butlast-snoc lms-slice-to-butlast-app)
    ultimately show ?thesis
      by (simp add: list-less-ns-app-same)
  qed
```

moreover
have ⟦*abs-find-next-lms T i = length T*; *abs-find-next-lms T j = length T*⟧ ⟹ *?thesis*
by (*metis assms dual-order.refl lms-slice-eq-suffix ordlistns.less-irrefl*)
moreover
have ⟦*abs-find-next-lms T i = length T*; *abs-find-next-lms T j < length T*⟧ ⟹ *?thesis*
proof −
assume *A*: *abs-find-next-lms T i = length T abs-find-next-lms T j < length T*
have *suffix T i = lms-slice T i*
by (*simp add*: *A*(*1*) *lms-slice-eq-suffix*)
moreover
have *suffix T j = lms-slice T j* @ *suffix T* (*Suc* (*abs-find-next-lms T j*))
by (*meson A*(*2*) *abs-find-next-lms-lower-bound-2 linorder-not-less suffix-to-lms-slice-app-suffix*)
ultimately show *?thesis*
by (*metis A*(*1*) *append.right-neutral assms cancel-comm-monoid-add-class.diff-cancel length-0-conv length-drop list-less-ns-app ordlistns.not-less-iff-gr-or-eq ordlistns.top.extremum-strict*)
qed
moreover
have ⟦*abs-find-next-lms T i < length T*; *abs-find-next-lms T j = length T*⟧ ⟹ *?thesis*
proof −
assume *A*: *abs-find-next-lms T i < length T abs-find-next-lms T j = length T*
have *suffix T i = lms-slice T i* @ *suffix T* (*Suc* (*abs-find-next-lms T i*))
by (*meson A*(*1*) *abs-find-next-lms-lower-bound-2 linorder-not-less suffix-to-lms-slice-app-suffix*)
moreover
have *suffix T j = lms-slice T j*
by (*simp add*: *A*(*2*) *lms-slice-eq-suffix*)
ultimately show *?thesis*
by (*metis A append-Nil2 assms drop-eq-Nil2 abs-find-next-lms-lower-bound-2 linorder-le-less-linear list-less-ns-app nless-le ordlistns.top.not-eq-extremum suffix-neq-nil suffixes-neq*)
qed
ultimately show *?thesis*
by (*meson abs-find-next-lms-le-length le-neq-implies-less*)
qed

lemma *lms-slice-eq-suffix-less-funpow′*:
assumes ∀ *k* < *n*. *lms-slice T* (((*abs-find-next-lms T*) ^^ *k*) *i*) = *lms-slice T* (((*abs-find-next-lms T*) ^^ *k*) *j*)
and *k* < *n*
shows *list-less-ns* (*suffix T i*) (*suffix T j*) ⟷ *list-less-ns* (*suffix T* (((*abs-find-next-lms T*) ^^ *k*) *i*)) (*suffix T* (((*abs-find-next-lms T*) ^^ *k*) *j*))
using *assms*

```
proof (induct n arbitrary: k)
  case 0
  then show ?case
    by blast
next
  case (Suc n)
  note IH = this
  have k < n ⟹ ?case
    by (simp add: Suc.hyps Suc.prems(1))
  moreover
  have k = n ⟹ ?case
  proof −
    assume k = n

    have k = 0 ⟹ ?thesis
      by auto
    moreover
    have ∃ m. k = Suc m ⟹ ?thesis
    proof −
      assume ∃ m. k = Suc m
      then obtain m where k = Suc m
        by blast
      hence m < n
        using ‹k = n› by blast
      with IH(1)[of m]
      have list-less-ns (suffix T i) (suffix T j) =
           list-less-ns (suffix T ((abs-find-next-lms T ^^ m) i))
                        (suffix T ((abs-find-next-lms T ^^ m) j))
        using Suc.prems(1) less-Suc-eq-le less-or-eq-imp-le by presburger
      moreover
      have list-less-ns (suffix T ((abs-find-next-lms T ^^ m) i))
                        (suffix T ((abs-find-next-lms T ^^ m) j)) =
           list-less-ns (suffix T (abs-find-next-lms T ((abs-find-next-lms T ^^ m)
i)))
                        (suffix T (abs-find-next-lms T ((abs-find-next-lms T ^^ m) j)))
       using Suc.prems(1,2) Suc-lessD ‹k = Suc m› lms-slice-eq-suffix-less by blast
      ultimately show ?thesis
        by (simp add: ‹k = Suc m›)
    qed
    ultimately show ?thesis
      using not0-implies-Suc by blast
  qed
  ultimately show ?case
    using Suc.prems(2) less-Suc-eq by blast
qed

lemma lms-slice-eq-suffix-less-funpow:
  assumes ∀ k < n. lms-slice T (((abs-find-next-lms T)^^k) i) =
                   lms-slice T (((abs-find-next-lms T)^^k) j)
```

```
  shows list-less-ns (suffix T i) (suffix T j) ⟷
      list-less-ns (suffix T (((abs-find-next-lms T) ^^ n) i)) (suffix T (((abs-find-next-lms
T) ^^ n) j))
proof (cases n)
  case 0
  then show ?thesis
    by auto
next
  case (Suc m)
  then show ?thesis
    by (metis assms abs-find-next-lms-funpow-Suc lessI lms-slice-eq-suffix-less
          lms-slice-eq-suffix-less-funpow')
qed

lemma list-slice-single:
  i < length xs ⟹ list-slice xs i (Suc i) = [xs ! i]
  by (simp add: list-slice-Suc)

lemma less-lms-slice-imp-suffix:
  assumes i < length T
  and     j < length T
  and     list-less-ns (lms-slice T i) (lms-slice T j)
  shows list-less-ns (suffix T i) (suffix T j)
proof −

  let ?c1 = ∃ b c as bs cs. lms-slice T i = as @ b # bs ∧
                        lms-slice T j = as @ c # cs ∧ b < c
  let ?c2 = ∃ c cs. lms-slice T i = lms-slice T j @ c # cs

  from list-less-ns-exE[OF assms(3)[simplified list-less-ns-alt-def]]
  have ?c1 ∨ ?c2 .
  moreover
  have ?c1 ⟹ ?thesis
   by (metis append.assoc append-Cons assms(1,2) list-less-ns-app-same list-less-ns-cons-diff
          suffix-to-lms-slice-app-suffix)
  moreover
  have ?c2 ⟹ ?thesis
  proof −
    assume ?c2
    then obtain c cs where
      lms-slice T i = lms-slice T j @ c # cs
      by blast
    moreover
    from lms-slice-hd[OF assms(2)]
    obtain xs where
      Sj: lms-slice T j = T ! j # xs
      by blast
    ultimately have Si: lms-slice T i = T ! j # xs @ c # cs
      by simp
```

let *?i* = *abs-find-next-lms T i*
let *?j* = *abs-find-next-lms T j*
have ∃ *k*. *?j* = *Suc k*
by (*meson Nat.lessE assms*(*1*) *dual-order.strict-trans1*
abs-find-next-lms-strict-upper-imp-lower-bound linorder-not-less)
then obtain *k* **where**
?j = *Suc k*
by *blast*
hence *j* ≤ *k*
using *assms*(*2*) *abs-find-next-lms-lower-bound-1* **by** *force*

have *?j* = *length T* ⟹ *?thesis*
by (*metis append-Nil2 assms*(*1*,*3*) *abs-find-next-lms-le-length list-less-ns-app*
lms-slice-eq-suffix ordlistns.dual-order.strict-trans
suffix-to-lms-slice-app-suffix)
moreover
have *?j* < *length T* ⟹ *?thesis*
proof −
assume *?j* < *length T*
with *lms-slice-to-butlast-app*
have *lms-slice T j* = *lms-slice-butlast T j* @ [*T* ! *Suc k*]
using ‹*?j* = *Suc k*› **by** *fastforce*
moreover
have ∃ *ys*. *lms-slice-butlast T j* = *ys* @ [*T* ! *k*]
proof (*cases j* = *k*)
case *True*
hence *lms-slice-butlast T j* = *list-slice T k* (*Suc k*)
by (*metis* ‹*?j* = *Suc k*› *lms-slice-butlast-def*)
moreover
have *list-slice T k* (*Suc k*) = [*T* ! *k*]
using *True assms*(*2*) *list-slice-single* **by** *auto*
ultimately show *?thesis*
by *simp*
next
case *False*
hence *j* < *k*
by (*simp add*: ‹*j* ≤ *k*› *le-neq-implies-less*)
hence *list-slice T j* (*Suc k*) = *list-slice T j k* @ [*T* ! *k*]
by (*metis* ‹*?j* < *length T*› ‹*?j* = *Suc k*› ‹*j* ≤ *k*› *le-SucI le-add2*
list-slice-append
list-slice-single not-less plus-1-eq-Suc)
then show *?thesis*
by (*simp add*: ‹*?j* = *Suc k*› *lms-slice-butlast-def*)
qed
then obtain *ys* **where**
lms-slice-butlast T j = *ys* @ [*T* ! *k*]
by *blast*
ultimately have *Sj'*: *lms-slice T j* = *ys* @ [*T* ! *k*, *T* ! *Suc k*]

by *simp*

have *Si'*: *lms-slice T i = ys @ T ! k # T ! Suc k # c # cs*
using *Si Sj Sj'* **by** *auto*

have *suffix-type T k = L-type*
by (*metis ‹?j < length T› ‹?j = Suc k› abs-find-next-lms-less-length-abs-is-lms abs-is-lms-neq nth-gr-imp-l-type*)

have *abs-is-lms T (Suc k)*
by (*metis ‹?j < length T› ‹?j = Suc k› abs-find-next-lms-less-length-abs-is-lms*)
hence *T ! k > T ! Suc k*
using *abs-is-lms-neq* **by** *blast*

have *Si*: *suffix T i = ys @ T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)*
by (*simp add*: *Si' assms(1) suffix-to-lms-slice-app-suffix*)
moreover
have *suffix T j = ys @ T ! k # T ! Suc k # suffix T (Suc ?j)*
by (*simp add*: *‹lms-slice T j = ys @ [T ! k, T ! Suc k]› assms(2) suffix-to-lms-slice-app-suffix*)
ultimately have *list-less-ns (suffix T i) (suffix T j) ⟷ list-less-ns (T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)) (T ! k # T ! Suc k # suffix T (Suc ?j))*
by (*simp add*: *list-less-ns-app-same*)
moreover
have *suffix-type (T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)) (Suc 0) = L-type ⟹ ?thesis*
by (*metis Cons-nth-drop-Suc ‹?j < length T› ‹?j = Suc k› ‹abs-is-lms T (Suc k)› calculation abs-is-lms-def l-less-than-s-type-general list-less-ns-cons-same suffix-type-cons-suc suffix-type-suffix*)
moreover
have *suffix-type (T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)) (Suc 0) = S-type ⟹ ?thesis*
proof −
assume *suffix-type (T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)) (Suc 0) = S-type*
with *Si*
have *suffix-type T (Suc (i + length ys)) = S-type*
by (*metis One-nat-def plus-1-eq-Suc suffix-cons-app suffix-type-suffix-gen*)
moreover
have *suffix-type (T ! k # T ! Suc k # c # cs @ suffix T (Suc ?i)) 0 = L-type*
using *‹T ! Suc k < T ! k›* *suffix-type-cons-greater* **by** *blast*
hence *suffix-type T (i + length ys) = L-type*
by (*simp add*: *Si suffix-cons-app suffix-type-list-type-eq*)
ultimately have *abs-is-lms T (Suc (i + length ys))*
using *abs-is-lms-def* **by** *blast*

```
        moreover
        {
          have i < ?i
            by (simp add: assms(1) abs-find-next-lms-lower-bound-1)

          from ‹lms-slice T i = ys @ T ! k # T ! Suc k # c # cs›
          have Suc (Suc (length ys)) < length (lms-slice T i)
            by simp

          have ?i = length T ⟹ Suc (i + length ys) < ?i
         by (simp add: ‹abs-is-lms T (Suc (i + length ys))› abs-is-lms-imp-less-length)
          moreover
          have ?i < length T ⟹ Suc (i + length ys) < ?i
          proof −
            assume ?i < length T
            hence length (lms-slice T i) = Suc ?i − i
              by (simp add: lms-slice-def Suc-leI)
            hence Suc (Suc (length ys)) < Suc ?i − i
              using ‹Suc (Suc (length ys)) < length (lms-slice T i)› by presburger
            then show ?thesis
              by linarith
          qed
          ultimately have Suc (i + length ys) < ?i
            using abs-find-next-lms-le-length le-neq-implies-less by blast
        }
        ultimately show ?thesis
          using no-lms-between-i-and-next[of i Suc (i + length ys)] less-add-Suc1
by blast
      qed
      ultimately show ?thesis
        using SL-types.exhaust by blast
    qed
    ultimately show ?thesis
      using abs-find-next-lms-le-length le-neq-implies-less by blast
  qed
  ultimately show ?thesis
    by blast
qed

lemma lms-slice-list-less-ns-suffix:
  assumes abs-is-lms T i
  and     abs-is-lms T j
  and     list-less-ns (lms-slice T i) (lms-slice T j)
shows list-less-ns (suffix T i) (suffix T j)
  by (simp add: assms abs-is-lms-imp-less-length less-lms-slice-imp-suffix)

lemma less-suffix-imp-lms-slice:
  assumes i < length T
  and     j < length T
```

and *lms-slice T i* ≠ *lms-slice T j*
and *list-less-ns* (*suffix T i*) (*suffix T j*)
shows *list-less-ns* (*lms-slice T i*) (*lms-slice T j*)
by (*meson assms less-lms-slice-imp-suffix ordlistns.less-asym ordlistns.neqE*)

lemma *not-lms-imp-next-eq-lms-prefix*:
¬*abs-is-lms T i* ⟹ *lms-slice T i* = *lms-prefix T i*
by (*simp add*: *lms-prefix-def lms-slice-def*)

lemma *lms-slice-last*:
assumes *valid-list T*
and *length T* = *Suc n*
shows *lms-slice T n* = [*bot*]
by (*metis add-diff-cancel-left′ assms butlast-snoc abs-find-next-lms-lower-bound-1 le-Suc-eq*
length-butlast less-Suc-eq list-slice-Suc list-slice-start-gre-length lms-slice-def nth-append-length plus-1-eq-Suc valid-list-ex-def)

lemma *Min-valid-lms-slice*:
assumes *valid-list T*
and *length T* = *Suc n*
shows *ordlistns.Min* {*lms-slice T i* |*i*. *i* < *length T*} = *lms-slice T n*
proof −
from *lms-slice-last*[*OF assms*]
have *lms-slice T n* = [*bot*]
by *assumption*

have ∀ *i* < *n*. (*lms-slice T i*) ! *0* ≠ *bot*
by (*metis add-diff-cancel-left′ assms abs-find-next-lms-lower-bound-1 less-SucI list-slice-Suc*
lms-slice-def nth-Cons-0 plus-1-eq-Suc valid-list-def)
hence *A*: ∀ *i* < *n*. *bot* < (*lms-slice T i*) ! *0*
using *bot.not-eq-extremum* **by** *blast*

have *B*: ∀ *i* < *length T*. *length* (*lms-slice T i*) > *0*
by (*simp add*: *abs-find-next-lms-lower-bound-1 less-SucI list-slice-Suc lms-slice-def*)

show *?thesis*
proof (*intro ordlistns.Min-eqI conjI*)
show *finite* {*lms-slice T i* |*i*. *i* < *length T*}
using *finite-image-set* **by** *blast*
next
fix *ys*
assume *ys* ∈ {*lms-slice T i* |*i*. *i* < *length T*}
hence ∃ *i* < *length T*. *ys* = *lms-slice T i*
by *blast*
then obtain *i* **where**
i < *length T*
ys = *lms-slice T i*

```
    by blast

  with ‹ys = lms-slice T i›
  have R1: i = n ⟹ list-less-eq-ns (lms-slice T n) ys
    by simp

  from ‹i < length T› assms(2)
  have R2-1: i ≠ n ⟹ i < n
    by linarith

  from A ‹lms-slice T n = [bot]› ‹i < length T› ‹ys = lms-slice T i›
  have R2-2: i < n ⟹ list-less-eq-ns (lms-slice T n) ys
    using list-less-eq-ns-def by fastforce

  from R1 R2-2[OF R2-1]
  show list-less-eq-ns (lms-slice T n) ys
    by blast
 next
  show lms-slice T n ∈ {lms-slice T i |i. i < length T}
    using assms(2) by auto
 qed
qed

lemma unique-valid-lms-slice:
 assumes valid-list T
 and     length T = Suc n
shows ∀ i < n. lms-slice T i ≠ lms-slice T n
proof (intro allI impI)
 fix i
 assume i < n
 from lms-slice-last[OF assms]
 have lms-slice T n = [bot]
   by assumption

 have ∀ i < n. (lms-slice T i) ! 0 ≠ bot
   by (metis add-diff-cancel-left' assms abs-find-next-lms-lower-bound-1 less-SucI
list-slice-Suc
       lms-slice-def nth-Cons-0 plus-1-eq-Suc valid-list-def)
 hence ∀ i < n. bot < (lms-slice T i) ! 0
   using bot.not-eq-extremum by blast
 with ‹lms-slice T n = [bot]› ‹i < n›
 show lms-slice T i ≠ lms-slice T n
   by auto
qed

lemma strict-Min-valid-lms-slice:
 assumes valid-list T
 and     length T = Suc n
shows ∀ i < n. list-less-ns (lms-slice T n) (lms-slice T i)
```

by (*metis add-diff-cancel-left′ assms bot.not-eq-extremum abs-find-next-lms-lower-bound-1 less-Suc-eq*
list-less-ns-cons-diff list-slice-Suc lms-slice-def lms-slice-last plus-1-eq-Suc valid-list-def)

lemma *ordlistns-lms-slice-imp-suffix-strict-sorted*:
assumes *set xs* ⊆ {*i*. *abs-is-lms T i*} *ordlistns.strict-sorted* (*map* (*lms-slice T*) *xs*)
shows *ordlistns.strict-sorted* (*map* (*suffix T*) *xs*)
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length xs*
with *sorted-wrt-mapD*[*OF assms*(*2*), *of i j*]
have *list-less-ns* (*lms-slice T* (*xs* ! *i*)) (*lms-slice T* (*xs* ! *j*))
by *blast*
moreover
have *abs-is-lms T* (*xs* ! *i*)
using ‹*i* < *j*› ‹*j* < *length xs*› *assms*(*1*) *subsetD* **by** *fastforce*
hence *xs* ! *i* < *length T*
by (*simp add*: *abs-is-lms-imp-less-length*)
moreover
have *abs-is-lms T* (*xs* ! *j*)
using ‹*j* < *length xs*› *assms*(*1*) *nth-mem* **by** *auto*
hence *xs* ! *j* < *length T*
by (*simp add*: *abs-is-lms-imp-less-length*)
ultimately show *list-less-ns* (*suffix T* (*xs* ! *i*)) (*suffix T* (*xs* ! *j*))
using *less-lms-slice-imp-suffix* **by** *blast*
qed

53 Mapping from suffix to lists of LMS-Substrings

This section contains the mapping from LMS-type suffixes to suffixes of the reduced sequence. The mapping is constructed in 3 major steps. 1) From suffix ID to a sequence of LMS-type suffix IDs 2) From a sequence of LMS-type suffix IDs to a sequence of LMS-substrings 3) From a LMS-type suffix to a reduced suffix using the mappings 1, 2 and *ordlistns.elem-rank* The mapping is also shown to be monotonic.

abbreviation *lms-substrs xs* ≡ *lms-slice xs* ‘ {*i*. *abs-is-lms xs i*}
abbreviation *lms-suffixes xs* ≡ *suffix xs* ‘ {*i*. *abs-is-lms xs i*}

abbreviation *nth-lms xs i* ≡ (*abs-find-next-lms xs* ^^ *Suc i*) *0*

abbreviation *lms0 xs* ≡ *abs-find-next-lms xs 0*
abbreviation *lms0-suffix xs* ≡ *suffix xs* (*lms0 xs*)
abbreviation *lms0-substr xs* ≡ *lms-slice xs* (*lms0 xs*)

53.1 LMS Sequence

definition *lms-seq* :: *'a* :: {*linorder*,*order-bot*} *list* ⇒ *nat* ⇒ *nat list*
where
lms-seq xs i = *filter* (*abs-is-lms xs*) [*i*..<*length xs*]

lemma *lms-seq-distinct*:
distinct (*lms-seq xs i*)
by (*simp add*: *lms-seq-def*)

lemma *lms-seq-sorted*:
sorted (*lms-seq xs i*)
by (*simp add*: *filter-sorted lms-seq-def*)

lemma *lms-seq-strict-sorted*:
strict-sorted (*lms-seq xs i*)
by (*simp add*: *lms-seq-distinct lms-seq-sorted sorted-and-distinct-imp-strict-sorted*)

lemma *lms-seq-abs-is-lms-hd*:
abs-is-lms xs i ⟹ ∃ *ys*. *lms-seq xs i* = *i* # *ys*
by (*simp add*: *filter-abs-is-lms-upt-hd abs-is-lms-imp-less-length lms-seq-def*)

lemma *length-lms-seq*:
assumes *abs-is-lms xs i*
shows *length* (*lms-seq xs i*) = *card* {*j*. *abs-is-lms xs j* ∧ *i* ≤ *j*}
proof −
from *distinct-length-filter*[*of* [*i*..<*length xs*] *abs-is-lms xs*]
have *length* (*lms-seq xs i*) = *card* ({*x*. *abs-is-lms xs x*} ∩ {*i*..<*length xs*})
by (*simp add*: *lms-seq-def*)
moreover
have {*x*. *abs-is-lms xs x*} ∩ {*i*..<*length xs*} = {*x*. *abs-is-lms xs x* ∧ *i* ≤ *x*}
by (*safe*; *clarsimp simp*: *abs-is-lms-imp-less-length*)
ultimately show *?thesis*
by *simp*
qed

lemma *length-lms-seq-less*:
assumes *abs-is-lms xs i*
and *abs-is-lms xs j*
and *i* < *j*
shows *length* (*lms-seq xs j*) < *length* (*lms-seq xs i*)
proof −
have {*k*. *abs-is-lms xs k* ∧ *j* ≤ *k*} ⊆ {*j*. *abs-is-lms xs j* ∧ *i* ≤ *j*}
using *assms*(*3*) **by** *force*
moreover
have *i* ∈ {*j*. *abs-is-lms xs j* ∧ *i* ≤ *j*}
using *assms*(*1*) *less-or-eq-imp-le* **by** *blast*
moreover
have *i* ∉ {*k*. *abs-is-lms xs k* ∧ *j* ≤ *k*}
using *assms*(*3*) *linorder-not-less* **by** *blast*

ultimately have $\{k.\ abs\text{-}is\text{-}lms\ xs\ k \wedge j \leq k\} \subset \{j.\ abs\text{-}is\text{-}lms\ xs\ j \wedge i \leq j\}$
by *blast*
hence $card\ \{k.\ abs\text{-}is\text{-}lms\ xs\ k \wedge j \leq k\} < card\ \{j.\ abs\text{-}is\text{-}lms\ xs\ j \wedge i \leq j\}$
by (*simp add*: *lms-finite psubset-card-mono*)
then show *?thesis*
by (*simp add*: *assms*(*1*) *assms*(*2*) *length-lms-seq*)
qed

lemma *lms-seq-nth-0*:
$lms\text{-}seq\ xs\ (Suc\ k) \neq [] \Longrightarrow lms\text{-}seq\ xs\ (Suc\ k)\ !\ 0 = abs\text{-}find\text{-}next\text{-}lms\ xs\ k$
unfolding *lms-seq-def*
apply (*simp add*: *abs-find-next-lms-funpow-Suc abs-find-next-lms-def*)
apply (*drule filter-find*)
by (*clarsimp split*: *option.splits*)

lemma *lms-seq-eq-cons-lms*:
assumes $abs\text{-}is\text{-}lms\ xs\ i\ i < k\ k \leq abs\text{-}find\text{-}next\text{-}lms\ xs\ i$
shows $lms\text{-}seq\ xs\ i = i \#\ lms\text{-}seq\ xs\ k$
proof −
have $filter\ (abs\text{-}is\text{-}lms\ xs)\ [Suc\ i..<k] = []$
using *assms*(*1*) *assms*(*2*) *assms*(*3*) *filter-no-lms1* **by** *blast*
moreover
have $[i..<length\ xs] = i \#\ [Suc\ i..<k] @ [k..<length\ xs]$
by (*metis Suc-leI assms dual-order.trans abs-find-next-lms-le-length abs-is-lms-imp-less-length le-add-diff-inverse upt-add-eq-append upt-conv-Cons*)
hence $lms\text{-}seq\ xs\ i = i \#\ (filter\ (abs\text{-}is\text{-}lms\ xs)\ [Suc\ i..<k]) @ (filter\ (abs\text{-}is\text{-}lms\ xs)\ [k..<length\ xs])$
by (*simp add*: *assms*(*1*) *lms-seq-def*)
ultimately show *?thesis*
by (*metis append-Nil lms-seq-def*)
qed

lemma *lms-seq-not-lms*:
assumes $\neg abs\text{-}is\text{-}lms\ xs\ i\ i < k\ k \leq abs\text{-}find\text{-}next\text{-}lms\ xs\ i$
shows $lms\text{-}seq\ xs\ i = lms\text{-}seq\ xs\ k$
proof −
have $filter\ (abs\text{-}is\text{-}lms\ xs)\ [i..<k] = []$
using *assms filter-no-lms2* **by** *blast*
moreover
have $[i..<length\ xs] = [i..<k] @ [k..<length\ xs]$
by (*metis assms*(*2*,*3*) *dual-order.trans abs-find-next-lms-le-length le-add-diff-inverse less-or-eq-imp-le upt-add-eq-append*)
ultimately show *?thesis*
by (*simp add*: *lms-seq-def*)
qed

lemma *lms-seq-eq-cons*:
assumes $lms\text{-}seq\ xs\ (Suc\ i) \neq []$
shows $lms\text{-}seq\ xs\ (Suc\ i) = abs\text{-}find\text{-}next\text{-}lms\ xs\ i \#\ lms\text{-}seq\ xs\ (Suc\ (abs\text{-}find\text{-}next\text{-}lms$

xs i))
proof −
from *lms-seq-nth-0*[*OF assms*]
have *lms-seq xs* (*Suc i*) ! *0* = *abs-find-next-lms xs i* .
moreover
have *i* < *abs-find-next-lms xs i*
by (*metis Suc-lessD assms filter.simps*(*1*) *abs-find-next-lms-lower-bound-1 lms-seq-def upt-rec*)
hence *Suc i* ≤ *abs-find-next-lms xs i*
by *simp*
hence *lms-seq xs* (*Suc i*) = *lms-seq xs* (*abs-find-next-lms xs i*)
by (*metis abs-find-next-lms-abs-is-lms abs-find-next-lms-le-Suc lms-seq-not-lms order-le-imp-less-or-eq*)
ultimately show *?thesis*
by (*metis Suc-leI assms filter.simps*(*1*) *abs-find-next-lms-less-length-abs-is-lms abs-find-next-lms-lower-bound-1 lessI lms-seq-def lms-seq-eq-cons-lms upt-rec*)
qed

lemma *lms-seq-nth-abs-is-lms*:
i < *length* (*lms-seq xs k*) ⟹ *abs-is-lms xs* ((*lms-seq xs k*) ! *i*)
unfolding *lms-seq-def*
using *nth-mem* **by** *fastforce*

lemma *lms-seq-0*:
lms-seq xs 0 = *lms-seq xs* (*Suc 0*)
by (*metis filter-abs-is-lms-upt-0 lms-seq-def*)

lemma *lms-seq-nth*:
i < *length* (*lms-seq xs* (*Suc k*)) ⟹ *lms-seq xs* (*Suc k*) ! *i* = ((*abs-find-next-lms xs*)^^(*Suc i*)) *k*
proof (*induct i arbitrary*: *k*)
case *0*
then show *?case*
by (*simp add*: *lms-seq-nth-0*)
next
case (*Suc i*)
note *IH* = *this*
let *?j* = *abs-find-next-lms xs k*
have *lms-seq xs* (*Suc k*) = *?j* # *lms-seq xs* (*Suc ?j*)
using *Suc.prems lms-seq-eq-cons* **by** *force*
with *IH*(*1*)[*of ?j*]
have *lms-seq xs* (*Suc ?j*) ! *i* = (*abs-find-next-lms xs* ^^ *Suc i*) *?j*
using *Suc.prems* **by** *fastforce*
then show *?case*
by (*simp add*: ‹*lms-seq xs* (*Suc k*) = *?j* # *lms-seq xs* (*Suc ?j*)› *funpow-swap1*)
qed

lemma *inj-on-lms-seq*:

inj-on (*lms-seq xs*) {*i*. *abs-is-lms xs i*}
by (*metis* (*mono-tags*, *lifting*) *inj-onI list.inject lms-seq-abs-is-lms-hd mem-Collect-eq*)

lemma *list-app-imp-suffix*:
xs = *ys* @ *zs* ⟹ *suffix xs* (*length ys*) = *zs*
by *auto*

abbreviation *nth-lms-seq xs i* ≡ *lms-seq xs* (*nth-lms xs i*)

abbreviation *lms0-seq xs* ≡ *lms-seq xs* (*lms0 xs*)

lemma *lms-seq-0-zeroth-lms*:
lms-seq xs 0 = *lms0-seq xs*
by (*metis gr-zeroI abs-is-lms-0 le-refl lms-seq-not-lms*)

lemma *lms-seq-set*:
set (*lms-seq xs i*) = {*k*. *abs-is-lms xs k* ∧ *i* ≤ *k*}
by (*intro equalityI subsetI*; *clarsimp simp add*: *abs-is-lms-def suffix-type-s-bound lms-seq-def*)

lemma *lms-seq-last-eq-length*:
length (*lms-seq xs i*) = *Suc n* ⟹
abs-find-next-lms xs ((*lms-seq xs i*) ! *n*) = *length xs*
proof −
let *?k* = (*lms-seq xs i*) ! *n*
assume *length* (*lms-seq xs i*) = *Suc n*
hence *i* ≤ *?k*
by (*metis* (*no-types*, *lifting*) *lessI lms-seq-set mem-Collect-eq nth-mem*)
have ∀ *j* < *length xs*. *?k* < *j* ⟶ ¬*abs-is-lms xs j*
proof *safe*
fix *j*
assume *j* < *length xs ?k* < *j abs-is-lms xs j*
hence *j* ∈ *set* (*lms-seq xs i*)
using ‹*i* ≤ *lms-seq xs i* ! *n*› *lms-seq-set* **by** *fastforce*
hence ∃ *j'*< *length* (*lms-seq xs i*). (*lms-seq xs i*) ! *j'* = *j*
by (*simp add*: *in-set-conv-nth*)
then obtain *j'* **where**
j' < *length* (*lms-seq xs i*)
(*lms-seq xs i*) ! *j'* = *j*
by *blast*
with *strict-sorted-nth-less-mono*[*OF lms-seq-strict-sorted*[*of xs i*], *of n j'*]
have *n* < *j'*
using ‹*length* (*lms-seq xs i*) = *Suc n*› ‹*lms-seq xs i* ! *n* < *j*› **by** *fastforce*
then show *False*
using ‹*j'* < *length* (*lms-seq xs i*)› ‹*length* (*lms-seq xs i*) = *Suc n*› **by** *linarith*
qed
then show *?thesis*
by (*meson abs-find-next-lms-le-length abs-find-next-lms-less-length-abs-is-lms*
abs-find-next-lms-strict-upper-imp-lower-bound order.strict-iff-order)

qed

lemma *lms0-seq-has-all-lms*:
set (*lms0-seq xs*) = {*i*. *abs-is-lms xs i*}
by (*metis* (*mono-tags*, *lifting*) *Collect-cong linorder-le-less-linear lms-seq-set mem-Collect-eq no-lms-between-i-and-next set-lms-gr-0*)

lemma *lms0-seq-length*:
length (*lms0-seq xs*) = *card* {*i*. *abs-is-lms xs i*}
by (*metis distinct-card lms0-seq-has-all-lms lms-seq-distinct*)

lemma *lms0-seq-nth*:
i < *card* {*i*. *abs-is-lms xs i*} ⟹ *lms0-seq xs* ! *i* = *nth-lms xs i*
by (*metis lms0-seq-length lms-seq-0 lms-seq-0-zeroth-lms lms-seq-nth*)

lemma *lms-seq-Suc1*:
assumes *abs-is-lms xs i*
shows *lms-seq xs i* = *i* # *lms-seq xs* (*Suc i*)
by (*simp add*: *assms filter-abs-is-lms-upt-hd abs-is-lms-imp-less-length lms-seq-def*)

lemma *lms-seq-Suc2*:
assumes ¬*abs-is-lms xs i*
shows *lms-seq xs i* = *lms-seq xs* (*Suc i*)
by (*metis* (*no-types*, *lifting*) *assms dual-order.strict-trans2 filter.simps*(*2*) *lessI less-or-eq-imp-le lms-seq-def upt-rec*)

lemma *lms-seq-suf*:
i ≤ *j* ⟹ ∃ *ys*. *lms-seq xs i* = *ys* @ *lms-seq xs j*
proof (*induct j* − *i arbitrary*: *i j*)
case *0*
then show *?case*
by *force*
next
case (*Suc x*)
note *IH* = *this*
hence ∃ *j'*. *j* = *Suc j'*
by (*metis Suc-le-D diff-le-self*)
then obtain *j'* **where**
A: *j* = *Suc j'*
by *blast*
with *IH*(*1*)[*of j' i*] *IH*(*2*,*3*)
have *B*: ∃ *ys*. *lms-seq xs i* = *ys* @ *lms-seq xs j'*
by (*metis Suc-diff-le Suc-eq-plus1 add.commute add-left-imp-eq antisym-conv2 le-Suc-eq zero-less-Suc zero-less-diff*)
show *?case*
proof (*cases abs-is-lms xs j'*)
assume *abs-is-lms xs j'*
with *A B lms-seq-Suc1*[*of xs j'*]

```
    show ?thesis
      by auto
  next
    assume ¬abs-is-lms xs j'
    with A B lms-seq-Suc2[of xs j']
    show ?thesis
      by simp
  qed
qed

lemma lms-lms-seq-is-suffix:
  assumes abs-is-lms xs i
  shows ∃ k < length (lms0-seq xs).
          suffix (lms0-seq xs) k = lms-seq xs i
proof −
  have lms0 xs ≤ i
   by (metis assms bot-nat-0.not-eq-extremum abs-is-lms-0 linorder-not-less no-lms-between-i-and-next)
  with lms-seq-suf[of lms0 xs i xs]
  show ?thesis
   by (metis assms length-Cons length-append length-greater-0-conv less-add-same-cancel1
                 less-numeral-extra(1) list.size(3) list-app-imp-suffix lms-seq-Suc1
plus-1-eq-Suc
              zero-eq-add-iff-both-eq-0)
qed

lemma nth-lms:
  i < card {i. abs-is-lms xs i} ⟹
   abs-is-lms xs (nth-lms xs i)
proof −
  assume i < card {i. abs-is-lms xs i}
  hence i < length (lms0-seq xs)
    by (metis distinct-card lms0-seq-has-all-lms lms-seq-distinct)
  moreover
  have ∃ k. lms0 xs = Suc k
    by (metis calculation filter.simps(1) abs-find-next-lms-eq-Suc less-nat-zero-code
list.size(3)
              lms-seq-def upt.simps(1))
  then obtain k where lms0 xs = Suc k by blast
  ultimately show ?thesis
    by (metis lms-seq-0 lms-seq-0-zeroth-lms lms-seq-nth lms-seq-nth-abs-is-lms)
qed

lemma card-abs-find-next-lms-funpow:
  i < card {k. abs-is-lms xs k} ⟹
   card {k. abs-is-lms xs k ∧ k < nth-lms xs i} = i
proof (induct i)
  case 0
  then show ?case
    by (metis (mono-tags, lifting) Collect-empty-eq card-eq-0-iff comp-apply fun-
```

pow.simps(2)
funpow-0 gr-zeroI abs-is-lms-0 no-lms-between-i-and-next)
next
case (*Suc i*)
note *IH* = *this*
let *?i* = *nth-lms xs i*
let *?j* = *nth-lms xs* (*Suc i*)
let *?A* = {*k*. *abs-is-lms xs k* ∧ *k* < *?i*}
let *?B* = {*k*. *abs-is-lms xs k* ∧ *k* < *?j*}

from *IH*
have *A*: *card ?A* = *i*
using *Suc-lessD* **by** *blast*
moreover
have *P*: *?i* < *?j*
by (*metis Suc.prems Suc-lessD abs-find-next-lms-funpow-Suc abs-find-next-lms-lower-bound-1*
abs-is-lms-imp-less-length nth-lms)
with *no-lms-between-i-and-next-funpow*
have *B*: ∀ *j*. *?i* < *j* ∧ *j* < *?j* ⟶ *j* ∉ *?B*
by *blast*

have *C*: *?i* ∈ *?B*
using *P Suc.prems Suc-lessD nth-lms* **by** *fastforce*

have *D*: *?A* ⊆ *?B*
using *P* **by** *force*

have *?B* = *insert ?i ?A*
using *B nat-neq-iff C D*
by (*intro equalityI subsetI insert-iff*[*THEN iffD2*]; *auto*)
moreover
have *?i* ∉ *?A*
by *blast*
hence *card* (*insert ?i ?A*) = *Suc* (*card ?A*)
by *simp*
ultimately show *?case*
by *simp*
qed

lemma *lms-seq-nth-suffix*:
i < *card* {*i*. *abs-is-lms xs i*} ⟹
suffix (*lms0-seq xs*) *i* = *nth-lms-seq xs i*
proof −
let *?i* = *lms0 xs*
let *?j* = *nth-lms xs i*
assume *A*: *i* < *card* {*i*. *abs-is-lms xs i*}
from *nth-lms*[*OF A*]
have *abs-is-lms xs ?j* .
moreover

have *?i* ≤ *?j*
by (*metis bot-nat-0.not-eq-extremum calculation abs-is-lms-0 linorder-not-less no-lms-between-i-and-next*)
hence [*?i..<length xs*] = [*?i..<?j*] @ [*?j..<length xs*]
by (*metis abs-find-next-lms-funpow-Suc abs-find-next-lms-le-length le-add-diff-inverse upt-add-eq-append*)
moreover
have *length* (*filter* (*abs-is-lms xs*) [*?i..<?j*]) = *card* {*k*. *abs-is-lms xs k* ∧ *k* < *?j*}
proof −
from *filter-no-lms2*[*of xs 0 ?i*]
have *filter* (*abs-is-lms xs*) [*0..<?i*] = []
using *abs-is-lms-0* **by** *fastforce*
moreover
have [*0..<?j*] = [*0..<?i*] @ [*?i..<?j*]
by (*metis* ‹*?i* ≤ *?j*› *bot-nat-0.not-eq-extremum le-add-diff-inverse less-or-eq-imp-le upt-add-eq-append*)
ultimately have *filter* (*abs-is-lms xs*) [*0..<?j*] = *filter* (*abs-is-lms xs*) [*?i..<?j*]
by *fastforce*
moreover
have *length* (*filter* (*abs-is-lms xs*) [*0..<?j*]) = *card* {*k*. *abs-is-lms xs k* ∧ *k* < *?j*}
proof −
from *length-filter-conv-card*[*of abs-is-lms xs* [*0..<?j*], *simplified length-upt*]
have *length* (*filter* (*abs-is-lms xs*) [*0..<?j*]) = *card* {*k*. *k* < *?j* ∧ *abs-is-lms xs* ([*0..<?j*] ! *k*)}
by *simp*
moreover
have {*k*. *k* < *?j* ∧ *abs-is-lms xs* ([*0..<?j*] ! *k*)} = {*k*. *abs-is-lms xs k* ∧ *k* < *?j*}
by *force*
ultimately show *?thesis*
by *simp*
qed
ultimately show *?thesis*
by (*metis* (*no-types*, *lifting*) *A Collect-cong card-abs-find-next-lms-funpow filter-append list-app-imp-suffix lms-seq-def*)
qed

53.2 LMS-Substring Sequence

definition *lms-substr-seq* :: *'a* :: {*linorder*,*order-bot*} *list* ⇒ *nat* ⇒ *'a list list*
where
lms-substr-seq xs i = *map* (*lms-slice xs*) (*lms-seq xs i*)

lemma *lms-substr-seq-length*:

length (*lms-substr-seq xs i*) = *length* (*lms-seq xs i*)
by (*simp add*: *lms-substr-seq-def*)

lemma *inj-on-map-lms-slice-lms-seq*:
inj-on (*map* (*lms-slice xs*)) (*lms-seq xs* ‘ {*i*. *abs-is-lms xs i*})
proof (*intro inj-onI*)
fix *x y*
assume *x* ∈ *lms-seq xs* ‘ {*i*. *abs-is-lms xs i*}
y ∈ *lms-seq xs* ‘ {*i*. *abs-is-lms xs i*}
map (*lms-slice xs*) *x* = *map* (*lms-slice xs*) *y*
then obtain *i j* **where**
x = *lms-seq xs i y* = *lms-seq xs j*
map (*lms-slice xs*) (*lms-seq xs i*) = *map* (*lms-slice xs*) (*lms-seq xs j*)
abs-is-lms xs i abs-is-lms xs j
by *clarsimp*+
then have *lms-seq xs i* = *lms-seq xs j*
by (*metis length-lms-seq-less length-map nat-neq-iff*)
then show *x* = *y*
by (*simp add*: ‹*x* = *lms-seq xs i*› ‹*y* = *lms-seq xs j*›)
qed

lemma *inj-on-lms-substr-seq*:
inj-on (*lms-substr-seq xs*) {*i*. *abs-is-lms xs i*}
unfolding *lms-substr-seq-def*
using *comp-inj-on*[*OF inj-on-lms-seq inj-on-map-lms-slice-lms-seq*, *simplified o-def*]
by *blast*

lemma *lms-substr-seq-nth*:
i < *length* (*lms-substr-seq xs* (*Suc k*)) ⟹
lms-substr-seq xs (*Suc k*) ! *i* = *lms-slice xs* ((*abs-find-next-lms xs* ^^ *Suc i*) *k*)
by (*simp add*: *lms-seq-nth lms-substr-seq-def*)

lemma *lms-substr-seq-nth-abs-is-lms*:
i < *length* (*lms-substr-seq xs k*) ⟹
(*lms-substr-seq xs k*) ! *i* ∈ *lms-substrs xs*
by (*simp add*: *lms-seq-nth-abs-is-lms lms-substr-seq-def*)

definition *suffix-to-id*
where
suffix-to-id xs ys = *length xs* − *length ys*

lemma *suffix-lengths-neq*:
⟦*i* < *j*; *j* < *length xs*⟧ ⟹ *length* (*suffix xs i*) > *length* (*suffix xs j*)
by *simp*

lemma *inj-on-suffix-to-id*:
inj-on (*suffix-to-id xs*) (*suffix xs* ‘ {*i*. *abs-is-lms xs i*})
by (*intro inj-onI*; *clarsimp simp*: *suffix-to-id-def abs-is-lms-imp-less-length less-or-eq-imp-le*)

lemma *suffix-id-suffix*:
i < *length xs* ⟹ *suffix-to-id xs* (*suffix xs i*) = *i*
by (*simp add*: *suffix-to-id-def*)

lemma *suffix-to-id-image*:
suffix-to-id xs ‘ *suffix xs* ‘ {*i*. *abs-is-lms xs i*} = {*i*. *abs-is-lms xs i*}
proof *safe*
fix *i*
assume *abs-is-lms xs i*
then show *abs-is-lms xs* (*suffix-to-id xs* (*suffix xs i*))
by (*simp add*: *abs-is-lms-imp-less-length suffix-id-suffix*)
next
fix *i*
assume *abs-is-lms xs i*
then show *i* ∈ *suffix-to-id xs* ‘ *lms-suffixes xs*
by (*simp add*: *image-iff abs-is-lms-imp-less-length suffix-id-suffix*)
qed

abbreviation *lms-substr-seq-id xs* ≡ (*lms-substr-seq xs*) ∘ (*suffix-to-id xs*)

lemma *lms-subtrs-seq-id-suffix*:
lms-substr-seq-id xs (*suffix xs i*) = *lms-substr-seq xs i*
apply *simp*
apply (*cases i* < *length xs*)
apply (*simp add*: *suffix-id-suffix*)
by (*simp add*: *lms-seq-def lms-substr-seq-def suffix-to-id-def*)

lemma *lms-substr-seq-id-nth-abs-is-lms*:
i < *length* (*lms-substr-seq-id xs* (*suffix xs k*)) ⟹
(*lms-substr-seq-id xs* (*suffix xs k*)) ! *i* ∈ *lms-substrs xs*
by (*simp add*: *lms-seq-nth-abs-is-lms lms-substr-seq-def*)

lemma *inj-on-lms-substr-seq-o-suffix-to-id*:
inj-on (*lms-substr-seq-id xs*) (*lms-suffixes xs*)
proof −
have *lms-substr-seq xs* = *map* (*lms-slice xs*) ∘ *lms-seq xs*
using *lms-substr-seq-def* **by** *fastforce*
with *comp-inj-on*[*OF inj-on-lms-seq inj-on-map-lms-slice-lms-seq*, *of xs*]
have *inj-on* (*lms-substr-seq xs*) {*i*. *abs-is-lms xs i*}
by *simp*
with *comp-inj-on*[*OF inj-on-suffix-to-id*[*of xs*], *simplified suffix-to-id-image*[*of xs*]]
show *?thesis*
by *blast*
qed

lemma *list-less-ns-lms-substr-seq-suffix*:
assumes *abs-is-lms xs i*
and *abs-is-lms xs j*
and *nslexordp list-less-ns* (*lms-substr-seq xs i*) (*lms-substr-seq xs j*)

shows *list-less-ns* (*suffix xs i*) (*suffix xs j*)
proof −
have ∃ *i'*. *i* = *Suc i'*
using *assms*(*1*) *abs-is-lms-def* **by** *blast*
then obtain *i'* **where**
i = *Suc i'*
by *blast*
hence *abs-find-next-lms xs i'* = *i*
using *assms*(*1*) *abs-find-next-lms-abs-is-lms* **by** *blast*

have *A*: ⋀*k*. *k* < *length* (*lms-substr-seq xs i*) ⟹
lms-substr-seq xs i ! *k* = *lms-slice xs* ((*abs-find-next-lms xs* ^^ *k*) *i*)
by (*simp add*: ‹*abs-find-next-lms xs i'* = *i*› ‹*i* = *Suc i'*› *funpow-swap1 lms-substr-seq-nth*)

have ∃ *j'*. *j* = *Suc j'*
using *assms*(*2*) *abs-is-lms-def* **by** *blast*
then obtain *j'* **where**
j = *Suc j'*
by *blast*
hence *abs-find-next-lms xs j'* = *j*
using *assms*(*2*) *abs-find-next-lms-abs-is-lms* **by** *blast*

have *B*: ⋀*k*. *k* < *length* (*lms-substr-seq xs j*) ⟹
lms-substr-seq xs j ! *k* = *lms-slice xs* ((*abs-find-next-lms xs* ^^ *k*) *j*)
by (*simp add*: ‹*abs-find-next-lms xs j'* = *j*› ‹*j* = *Suc j'*› *funpow-swap1 lms-substr-seq-nth*)

let *?c1* = ∃ *b c as bs cs*.
lms-substr-seq xs i = *as* @ *b* # *bs* ∧ *lms-substr-seq xs j* = *as* @ *c* # *cs*
∧
list-less-ns b c
let *?c2* = ∃ *c cs*. *lms-substr-seq xs i* = *lms-substr-seq xs j* @ *c* # *cs*
from *nslexordpE*[*OF assms*(*3*)]
have *?c1* ∨ *?c2* **.**
moreover
have *?c1* ⟹ *?thesis*
proof −
assume *?c1*
then obtain *b c as bs cs* **where**
lms-substr-seq xs i = *as* @ *b* # *bs*
lms-substr-seq xs j = *as* @ *c* # *cs*
list-less-ns b c
by *blast*

let *?b* = *lms-slice xs* ((*abs-find-next-lms xs* ^^ (*length as*)) *i*)
from *lms-substr-seq-nth*[*of length as xs i'*] ‹*i* = *Suc i'*›
‹*lms-substr-seq xs i* = *as* @ *b* # *bs*›
have *b* = *lms-slice xs* ((*abs-find-next-lms xs* ^^ *Suc* (*length as*)) *i'*)
by *simp*
with ‹*abs-find-next-lms xs i'* = *i*›

```
have b = ?b
  by (simp add: funpow-swap1)

let ?c = lms-slice xs ((abs-find-next-lms xs ^^ (length as)) j)
from lms-substr-seq-nth[of length as xs j'] ‹j = Suc j'›
      ‹lms-substr-seq xs j = as @ c # cs›
have c = lms-slice xs ((abs-find-next-lms xs ^^ Suc (length as)) j')
  by simp
with ‹abs-find-next-lms xs j' = j›
have c = lms-slice xs ((abs-find-next-lms xs ^^ (length as)) j)
  by (simp add: funpow-swap1)

have P: ∀ k<length as. lms-slice xs ((abs-find-next-lms xs ^^ k) i) =
                       lms-slice xs ((abs-find-next-lms xs ^^ k) j)
proof (safe)
  fix k
  assume k < length as
  with ‹lms-substr-seq xs i = as @ b # bs› A[of k]
  have as ! k = lms-slice xs ((abs-find-next-lms xs ^^ k) i)
    by (simp add: nth-append)
  moreover
  from ‹lms-substr-seq xs j = as @ c # cs› ‹k < length as› B[of k]
  have as ! k = lms-slice xs ((abs-find-next-lms xs ^^ k) j)
    by (simp add: nth-append)
  ultimately show
    lms-slice xs ((abs-find-next-lms xs ^^ k) i) =
     lms-slice xs ((abs-find-next-lms xs ^^ k) j)
    by simp
qed

have Q: list-less-ns (suffix xs ((abs-find-next-lms xs ^^ (length as)) i))
                     (suffix xs ((abs-find-next-lms xs ^^ (length as)) j))
by (metis ‹b = ?b› ‹c = ?c› ‹list-less-ns b c› drop-eq-Nil abs-find-next-lms-lower-bound-2
          less-lms-slice-imp-suffix list-less-ns-nil lms-slice-eq-suffix
          not-le-imp-less ordlistns.less-asym)

show ?thesis
proof (cases length as)
  case 0
  then show ?thesis
    using Q by force
next
  case (Suc n)
  assume length as = Suc n
  moreover
  from lms-slice-eq-suffix-less-funpow[OF P]
  have list-less-ns (suffix xs i) (suffix xs j) =
       list-less-ns (suffix xs ((abs-find-next-lms xs ^^ (length as)) i))
                    (suffix xs ((abs-find-next-lms xs ^^ (length as)) j))
```

```
      using lessI by presburger
    ultimately show ?thesis
      using Q by blast
  qed
qed
moreover
have ?c2 ⟹ ?thesis
proof −
  assume ?c2
  then obtain c cs where
    lms-substr-seq xs i = lms-substr-seq xs j @ c # cs
    by blast

  have lms-substr-seq xs j ≠ []
     by (metis assms(2) list.distinct(1) list.map-disc-iff lms-seq-abs-is-lms-hd
lms-substr-seq-def)
  hence ∃ n. length (lms-substr-seq xs j) = Suc n
    using not0-implies-Suc by auto
  then obtain n where
    length (lms-substr-seq xs j) = Suc n
    by blast

  have P: ∀ k < Suc n. lms-slice xs ((abs-find-next-lms xs ^^ k) i) =
                  lms-slice xs ((abs-find-next-lms xs ^^ k) j)
  proof safe
    fix k
    assume k < Suc n
    hence lms-substr-seq xs j ! k  = lms-slice xs ((abs-find-next-lms xs ^^ k) j)
      by (simp add: B ‹length (lms-substr-seq xs j) = Suc n›)
    moreover
    from ‹lms-substr-seq xs i = lms-substr-seq xs j @ c # cs› ‹k < Suc n›
    have lms-substr-seq xs i ! k = lms-slice xs ((abs-find-next-lms xs ^^ k) j)
      by (simp add: B ‹length (lms-substr-seq xs j) = Suc n› nth-append)
    moreover
    have lms-substr-seq xs i ! k = lms-slice xs ((abs-find-next-lms xs ^^ k) i)
      using A ‹k < Suc n› ‹length (lms-substr-seq xs j) = Suc n›
          ‹lms-substr-seq xs i = lms-substr-seq xs j @ c # cs› by auto
    ultimately show
      lms-slice xs ((abs-find-next-lms xs ^^ k) i) =
       lms-slice xs ((abs-find-next-lms xs ^^ k) j)
      by simp
  qed

  have list-less-ns (suffix xs i) (suffix xs j) =
        list-less-ns (suffix xs ((abs-find-next-lms xs ^^ Suc n) i))
                     (suffix xs ((abs-find-next-lms xs ^^ Suc n) j))
    using lms-slice-eq-suffix-less-funpow[OF P]
    by blast
  moreover
```

from *lms-seq-last-eq-length*[*of xs j n*]
have (*abs-find-next-lms xs* ^^ *Suc n*) *j* = *length xs*
by (*metis* ‹*abs-find-next-lms xs j'* = *j*› ‹*j* = *Suc j'*› ‹*length* (*lms-substr-seq xs j*) = *Suc n*›
funpow-swap1 length-map lessI lms-seq-nth lms-substr-seq-def)
hence *suffix xs* ((*abs-find-next-lms xs* ^^ *Suc n*) *j*) = []
by *force*
ultimately show *?thesis*
by (*metis P* ‹*lms-substr-seq xs i* = *lms-substr-seq xs j* @ *c* # *cs*› *append-self-conv assms*(*1*,*2*)
abs-is-lms-imp-less-length list.distinct(*1*) *list-less-ns-nil suffix-id-suffix*
lms-slice-eq-suffix-less-funpow ordlistns.not-less-iff-gr-or-eq)
qed
ultimately show *?thesis*
by *blast*
qed

lemma *monotone-on-lms-substr-seq-id*:
monotone-on (*lms-suffixes xs*) *list-less-ns* (*nslexordp list-less-ns*) (*lms-substr-seq-id xs*)
(**is** *monotone-on ?A ?orda ?ordb ?f*)
proof −
let *?B* = *?f* ‘ *?A*

from *inj-on-imp-bij-betw*[*OF inj-on-lms-substr-seq-o-suffix-to-id*]
have *A*: *bij-betw ?f ?A ?B* **.**
with *bij-betw-inv-alt*
have ∃ *g*. *bij-betw g ?B ?A* ∧ *inverses-on ?f g ?A ?B*
by *blast*
then obtain *g* **where** *B*:
bij-betw g ?B ?A
inverses-on ?f g ?A ?B
by *blast*

have *C*: *monotone-on ?B ?ordb ?orda g*
proof (*intro monotone-onI*)
fix *x y*
assume *x* ∈ *?B y* ∈ *?B nslexordp list-less-ns x y*
moreover
have ∃ *i*. *abs-is-lms xs i* ∧ *g x* = *suffix xs i*
using ‹*x* ∈ *?B*› *bij-betw-apply B*(*1*) **by** *fastforce*
then obtain *i* **where**
abs-is-lms xs i g x = *suffix xs i*
by *blast*
moreover
have ∃ *j*. *abs-is-lms xs j* ∧ *g y* = *suffix xs j*
using ‹*y* ∈ *?B*› *bij-betw-apply B*(*1*) **by** *fastforce*
then obtain *j* **where**
abs-is-lms xs j g y = *suffix xs j*

by *blast*
ultimately show *list-less-ns* (*g x*) (*g y*)
using *list-less-ns-lms-substr-seq-suffix*[*of xs i j*]
by (*metis* (*mono-tags*, *lifting*) *B*(*2*) *comp-def inverses-onD2 abs-is-lms-imp-less-length suffix-id-suffix*)
qed

from *nslexordp-asymp*[*of list-less-ns*]
have *asymp-on ?B ?ordb*
using *asympD* **by** *fastforce*

from *totalp-on-subset*[*OF nslexordp-totalp*[*of list-less-ns*]]
have *totalp-on ?B ?ordb*
using *ordlistns.totalp-on-less* **by** *blast*

note *D* = ‹*asymp-on ?B ?ordb*› ‹*totalp-on ?B ?ordb*›

from *monotone-on-bij-betw-inv*[*OF C D - - B*(*1*) *A inverses-on-sym*[*THEN iffD1*, *OF B*(*2*)], *simplified*]
show *?thesis* **.**
qed

lemma *list-less-ns-suffix-lms-substr-seq*:
assumes *abs-is-lms xs i*
and *abs-is-lms xs j*
and *list-less-ns* (*suffix xs i*) (*suffix xs j*)
shows *nslexordp list-less-ns* (*lms-substr-seq xs i*) (*lms-substr-seq xs j*)
using *monotone-onD*[*OF monotone-on-lms-substr-seq-id - - assms*(*3*)] *assms*(*1*,*2*) *abs-is-lms-imp-less-length suffix-id-suffix* **by** *fastforce*

lemma *lms-substr-seq-suf*:
i ≤ *j* ⟹ ∃ *ys*. *lms-substr-seq xs i* = *ys* @ *lms-substr-seq xs j*
unfolding *lms-substr-seq-def*
by (*frule lms-seq-suf*[*of - - xs*]; *clarsimp*)

lemma *lms-lms-substr-seq-is-suffix*:
assumes *abs-is-lms xs i*
shows ∃ *k* < *length* (*lms-substr-seq xs* (*abs-find-next-lms xs 0*)).
suffix (*lms-substr-seq xs* (*abs-find-next-lms xs 0*)) *k* = *lms-substr-seq xs i*
unfolding *lms-substr-seq-def*
by (*metis assms length-map lms-lms-seq-is-suffix*[*of xs i*] *suffix-map*)

lemma *lms-substr-seq-nth-suffix*:
i < *card* {*i*. *abs-is-lms xs i*} ⟹
suffix (*lms-substr-seq xs* (*abs-find-next-lms xs 0*)) *i* =
lms-substr-seq xs ((*abs-find-next-lms xs* ^^ *Suc i*) *0*)
by (*simp add*: *lms-seq-nth-suffix lms-substr-seq-def suffix-map*)

53.3 LMS Map

lemma *finite-lms-substrs*:
finite (*lms-substrs xs*)
by (*simp add*: *lms-finite*)

definition *lms-map* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ '*a list* ⇒ *nat list*
where
lms-map xs ≡ (*map* (*ordlistns.elem-rank* (*lms-substrs xs*))) ∘ (*lms-substr-seq-id xs*)

lemma *lms-substr-seq-o-suffix-to-id-range*:
(*lms-substr-seq xs* ∘ *suffix-to-id xs*) ‘ *lms-suffixes xs* ⊆ {*ys*. *set ys* ⊆ *lms-substrs xs*}
unfolding *lms-substr-seq-def suffix-to-id-def*
by (*safe*; *clarsimp simp*: *lms-seq-set*)

lemma *lms-map-o-def*:
lms-map xs ys = *map* (*ordlistns.elem-rank* (*lms-substrs xs*)) (*lms-substr-seq-id xs ys*)
by (*simp add*: *lms-map-def*)

lemma *inj-on-lms-map*:
inj-on (*lms-map xs*) (*lms-suffixes xs*)
proof −
note *A* = *comp-inj-on*[*OF inj-on-lms-substr-seq-o-suffix-to-id*]
note *B* = *inj-on-subset*[*OF bij-betw-imp-inj-on*[*OF bij-betw-map*[*OF ordlistns.bij-betw-elem-rank-upt*]]]
from *A*[*OF B*, *OF finite-lms-substrs lms-substr-seq-o-suffix-to-id-range*, *of xs*]
show *?thesis*
by (*simp add*: *lms-map-def*)
qed

lemma *lms-map-length*:
length (*lms-map xs ys*) = *length* (*lms-substr-seq xs* (*suffix-to-id xs ys*))
by (*simp add*: *lms-map-def*)

lemma *lms-map-nth-suffix*:
i < *card* {*i*. *abs-is-lms xs i*} ⟹
suffix (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs* 0))) *i* =
lms-map xs (*suffix xs* ((*abs-find-next-lms xs* ^^ *Suc i*) 0))
by (*simp add*: *abs-find-next-lms-le-length lms-map-def lms-seq-nth-suffix lms-substr-seq-def suffix-map*
suffix-to-id-def)

lemma *lms-lms-map-is-suffix*:
assumes *abs-is-lms xs i*
shows ∃ *k* < *length* (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs* 0))).
suffix (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs* 0))) *k* = *lms-map xs* (*suffix xs i*)

proof −

have *suffix-to-id xs* (*suffix xs i*) = *i*
by (*simp add*: *assms abs-is-lms-imp-less-length suffix-id-suffix*)
moreover
have *suffix-to-id xs* (*suffix xs* (*abs-find-next-lms xs 0*)) = *abs-find-next-lms xs 0*
by (*simp add*: *abs-find-next-lms-le-length suffix-to-id-def*)
moreover
from *lms-lms-substr-seq-is-suffix*[*OF assms*]
obtain *k* **where**
k < *length* (*lms-substr-seq xs* (*abs-find-next-lms xs 0*))
suffix (*lms-substr-seq xs* (*abs-find-next-lms xs 0*)) *k* = *lms-substr-seq xs i*
by *blast*
moreover
have *k* < *length* (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs 0*)))
by (*simp add*: *calculation*(*2*) *calculation*(*3*) *lms-map-length*)
moreover
have *suffix* (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs 0*))) *k* = *lms-map xs* (*suffix xs i*)
by (*simp add*: *lms-map-def calculation suffix-map*)
ultimately show *?thesis*
by *blast*
qed

lemma *length-reduced-seq*:
length (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs 0*))) = *card* (*lms-suffixes xs*)
apply (*simp add*: *lms-map-length lms-substr-seq-length*)
apply (*cases abs-find-next-lms xs 0* < *length xs*)
apply (*simp add*: *suffix-id-suffix*)
apply (*subst distinct-card*[*OF lms-seq-distinct*[*of xs abs-find-next-lms xs 0*], *symmetric*])
apply (*simp add*: *lms0-seq-has-all-lms*)
apply (*metis card-image inj-on-suffix-to-id suffix-to-id-image*)
by (*metis card-eq-0-iff diff-diff-cancel empty-set filter.simps*(*1*) *abs-find-next-lms-le-length*
lms0-seq-has-all-lms image-is-empty length-drop linorder-le-less-linear list.size(*3*)
lms-seq-def suffix-to-id-def upt-eq-Nil-conv)

corollary *lms-lms-map-in-suffixes*:
abs-is-lms xs i ⟹
lms-map xs (*suffix xs i*) ∈
suffix (*lms-map xs* (*suffix xs* (*abs-find-next-lms xs 0*))) ‘ {*0*..<*card* (*lms-suffixes xs*)}
by (*metis atLeastLessThan-iff imageI length-reduced-seq lms-lms-map-is-suffix zero-le*)

lemma *card-lms-suffixes*:
card (*lms-suffixes xs*) = *card* {*i*. *abs-is-lms xs i*}
by (*metis card-image inj-on-suffix-to-id suffix-to-id-image*)

lemma *lms-map-image*:

```
  lms-map xs ‘ lms-suffixes xs =
   suffix (lms-map xs (suffix xs (abs-find-next-lms xs 0))) ‘ {0..<card (lms-suffixes
xs)}
proof (safe)
  fix i
  assume abs-is-lms xs i
  then show lms-map xs (suffix xs i) ∈
        suffix (lms-map xs (lms0-suffix xs)) ‘ {0..<card (lms-suffixes xs)}
    using lms-lms-map-in-suffixes by blast
next
  fix i
  assume i ∈ {0..<card (lms-suffixes xs)}
  with card-lms-suffixes
  have i < card {i. abs-is-lms xs i}
    by (metis atLeastLessThan-iff)
  with lms-map-nth-suffix[of i xs]
  have suffix (lms-map xs (lms0-suffix xs)) i = lms-map xs (suffix xs (nth-lms xs
i))
    by blast
  moreover
  have suffix xs (nth-lms xs i) ∈ lms-suffixes xs
    using ‹i < card {i. abs-is-lms xs i}› nth-lms by fastforce
 ultimately show suffix (lms-map xs (lms0-suffix xs)) i ∈ lms-map xs ‘ lms-suffixes
xs
    by blast
qed

lemma monotone-on-lms-map:
  monotone-on (lms-suffixes xs) list-less-ns list-less-ns (lms-map xs)
proof (intro monotone-onI)
  fix x y
  assume x ∈ lms-suffixes xs y ∈ lms-suffixes xs list-less-ns x y
  with monotone-onD[OF monotone-on-lms-substr-seq-id, of x xs y]
  have nslexordp list-less-ns (lms-substr-seq-id xs x) (lms-substr-seq-id xs y)
    by blast
  moreover
 have ⋀x. lms-map xs x = map (ordlistns.elem-rank (lms-substrs xs)) (lms-substr-seq-id
xs x)
    by (simp add: lms-map-def)
  moreover
  {
    have set (lms-substr-seq-id xs x) ⊆ lms-substrs xs
      by (simp add: Collect-mono-iff image-mono lms-seq-set lms-substr-seq-def)
    moreover
    have set (lms-substr-seq-id xs y) ⊆ lms-substrs xs
      by (simp add: Collect-mono-iff image-mono lms-seq-set lms-substr-seq-def)
    ultimately have
      nslexordp list-less-ns (lms-substr-seq-id xs x) (lms-substr-seq-id xs y) =
       nslexordp (<) (map (ordlistns.elem-rank (lms-substrs xs)) (lms-substr-seq-id
```

xs x))
(*map* (*ordlistns.elem-rank* (*lms-substrs xs*)) (*lms-substr-seq-id xs y*))
using *monotone-on-iff-nslexordp*[*OF ordlistns.strict-mono-on-elem-rank*, *simplified*,
OF finite-lms-substrs[*of xs*] *ordlistns.bij-betw-elem-rank-upt*,
OF finite-lms-substrs[*of xs*]]
by *blast*
}
ultimately show *list-less-ns* (*lms-map xs x*) (*lms-map xs y*)
by (*simp add*: *nslexordp-eq-list-less-ns*)
qed

lemma *list-less-ns-lms-map-suffix*:
assumes *abs-is-lms xs i*
and *abs-is-lms xs j*
and *list-less-ns* (*lms-map xs* (*suffix xs i*)) (*lms-map xs* (*suffix xs j*))
shows *list-less-ns* (*suffix xs i*) (*suffix xs j*)
using *monotone-on-iff*[*OF monotone-on-lms-map*, *simplified*] *assms* **by** *blast*

abbreviation
lms0-map xs ≡
lms-map xs (*lms0-suffix xs*)

lemma *sorted-reduced-seq-imp-lms*:
assumes *ordlistns.strict-sorted* (*map* (*suffix* (*lms0-map xs*)) *ys*)
and ∀ *y* ∈ *set ys*. *y* < *card* {*i*. *abs-is-lms xs i*}
shows *ordlistns.strict-sorted* (*map* (*suffix xs*) (*map* ((!) (*lms0-seq xs*)) *ys*))
proof (*intro sorted-wrt-mapI*)
fix *i j*
assume *i* < *j j* < *length* (*map* ((!) (*lms0-seq xs*)) *ys*)
hence *A*: *i* < *j j* < *length ys*
by *simp-all*
with *sorted-wrt-mapD*[*OF assms*(*1*)]
have *list-less-ns* (*suffix* (*lms0-map xs*) (*ys* ! *i*)) (*suffix* (*lms0-map xs*) (*ys* ! *j*)) **.**
moreover
from *lms-map-nth-suffix*[*of ys* ! *i xs*]
have *suffix* (*lms0-map xs*) (*ys* ! *i*) = *lms-map xs* (*suffix xs* (*nth-lms xs* (*ys* ! *i*)))
using *A*(*1*) *A*(*2*) *assms*(*2*) **by** *fastforce*
moreover
from *lms-map-nth-suffix*[*of ys* ! *j xs*]
have *suffix* (*lms0-map xs*) (*ys* ! *j*) = *lms-map xs* (*suffix xs* (*nth-lms xs* (*ys* ! *j*)))
using *A*(*2*) *assms*(*2*) **by** *fastforce*
moreover
have *abs-is-lms xs* (*nth-lms xs* (*ys* ! *i*))
by (*meson A*(*1*) *A*(*2*) *assms*(*2*) *nth-lms nth-mem order.strict-trans*)
hence *suffix xs* (*nth-lms xs* (*ys* ! *i*)) ∈ *lms-suffixes xs*
by *blast*
moreover

```
  have abs-is-lms xs (nth-lms xs (ys ! j))
    by (meson A(2) assms(2) nth-lms nth-mem order.strict-trans)
  hence suffix xs (nth-lms xs (ys ! j)) ∈ lms-suffixes xs
    by blast
  ultimately have
    list-less-ns (suffix xs (nth-lms xs (ys ! i))) (suffix xs (nth-lms xs (ys ! j)))
    using monotone-on-iff[OF monotone-on-lms-map, simplified] by auto
  moreover
  from lms0-seq-nth[of ys ! i xs]
  have lms0-seq xs ! (ys ! i) = nth-lms xs (ys ! i)
    using A(1) A(2) assms(2) by force
  moreover
  from lms0-seq-nth[of ys ! j xs]
  have lms0-seq xs ! (ys ! j) = nth-lms xs (ys ! j)
    using A(2) assms(2) by force
  ultimately have
    list-less-ns (suffix xs (lms0-seq xs ! (ys ! i))) (suffix xs (lms0-seq xs ! (ys ! j)))
    by presburger
  then show list-less-ns (suffix xs (map ((!) (lms0-seq xs)) ys ! i))
                         (suffix xs (map ((!) (lms0-seq xs)) ys ! j))
    using A(1) A(2) by fastforce
qed

lemma sorted-distinct-lms-substr:
  assumes ordlistns.sorted (map (lms-slice xs) ys)
  and     distinct (map (lms-slice xs) ys)
  and     ∀ y ∈ set ys. y < length xs
shows ordlistns.sorted (map (suffix xs) ys)
proof (intro sorted-wrt-mapI)
  fix i j
  assume i < j j < length ys
  with sorted-wrt-mapD[OF assms(1)]
  have list-less-eq-ns (lms-slice xs (ys ! i)) (lms-slice xs (ys ! j)) .
  moreover
  have lms-slice xs (ys ! i) ≠ lms-slice xs (ys ! j)
    using ‹i < j› ‹j < length ys› assms(2) nth-eq-iff-index-eq by fastforce
  ultimately have  list-less-ns (lms-slice xs (ys ! i)) (lms-slice xs (ys ! j))
    using ordlistns.nless-le by blast
  then show list-less-eq-ns (suffix xs (ys ! i)) (suffix xs (ys ! j))
   by (metis ‹i < j› ‹j < length ys› less-lms-slice-imp-suffix assms(3) dual-order.strict-trans
             nth-mem ordlistns.dual-order.strict-implies-order)
qed

lemma distinct-lms0-map:
  assumes distinct (lms0-map xs)
  shows distinct (map (lms-slice xs) (lms0-seq xs))
proof (intro distinct-conv-nth[THEN iffD2] allI impI)
  fix i j
  assume i < length (map (lms-slice xs) (lms0-seq xs))
```

$j < length\ (map\ (lms\text{-}slice\ xs)\ (lms0\text{-}seq\ xs))$
$i \neq j$
hence *A*: $i < length\ (lms0\text{-}seq\ xs)\ j < length\ (lms0\text{-}seq\ xs)\ i \neq j$
by *simp-all*
with *distinct-conv-nth*[*THEN iffD1*, *OF assms*]
have *B*: $lms0\text{-}map\ xs\ !\ i \neq lms0\text{-}map\ xs\ !\ j$
by (*metis card-lms-suffixes length-reduced-seq lms0-seq-length*)
moreover
have $lms\text{-}substr\text{-}seq\text{-}id\ xs\ (lms0\text{-}suffix\ xs) = map\ (lms\text{-}slice\ xs)\ (lms0\text{-}seq\ xs)$
by (*metis lms-substr-seq-def lms-subtrs-seq-id-suffix*)
hence $lms0\text{-}map\ xs = map\ (ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ xs))\ (map\ (lms\text{-}slice\ xs)\ (lms0\text{-}seq\ xs))$
by (*simp add*: *lms-map-o-def*)
with *A*
have $lms0\text{-}map\ xs\ !\ i = ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ xs)\ (lms\text{-}slice\ xs\ (lms0\text{-}seq\ xs\ !\ i))$
$lms0\text{-}map\ xs\ !\ j = ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ xs)\ (lms\text{-}slice\ xs\ (lms0\text{-}seq\ xs\ !\ j))$
by *auto*
ultimately have $lms\text{-}slice\ xs\ (lms0\text{-}seq\ xs\ !\ i) \neq lms\text{-}slice\ xs\ (lms0\text{-}seq\ xs\ !\ j)$
by *fastforce*
then show $map\ (lms\text{-}slice\ xs)\ (lms0\text{-}seq\ xs)\ !\ i \neq map\ (lms\text{-}slice\ xs)\ (lms0\text{-}seq\ xs)\ !\ j$
by (*simp add*: *A*(*1*) *A*(*2*))
qed

lemma *sorted-distinct-lms-substr-perm*:
assumes $ordlistns.sorted\ (map\ (lms\text{-}slice\ xs)\ ys)$
and $distinct\ (lms0\text{-}map\ xs)$
and $ys <^\sim\sim> lms0\text{-}seq\ xs$
shows $ordlistns.sorted\ (map\ (suffix\ xs)\ ys)$
by (*metis sorted-distinct-lms-substr*[*OF assms*(*1*)] *distinct-lms0-map*[*OF assms*(*2*)] *assms*(*3*)
distinct-map abs-is-lms-imp-less-length lms0-seq-has-all-lms mem-Collect-eq perm-distinct-iff perm-set-eq)

lemma *list-less-ns-suffix-lms-map*:
assumes *abs-is-lms xs i*
and *abs-is-lms xs j*
and $list\text{-}less\text{-}ns\ (suffix\ xs\ i)\ (suffix\ xs\ j)$
shows $list\text{-}less\text{-}ns\ (lms\text{-}map\ xs\ (suffix\ xs\ i))\ (lms\text{-}map\ xs\ (suffix\ xs\ j))$
using *monotone-on-iff*[*OF monotone-on-lms-map*, *simplified*] *assms* **by** *blast*

lemma *valid-list-lms-map*:
assumes $valid\text{-}list\ (a\ \#\ b\ \#\ xs)$
and $abs\text{-}is\text{-}lms\ (a\ \#\ b\ \#\ xs)\ i$
shows $valid\text{-}list\ (lms\text{-}map\ (a\ \#\ b\ \#\ xs)\ (suffix\ (a\ \#\ b\ \#\ xs)\ i))$
proof −
let $?xs = a\ \#\ b\ \#\ xs$

have $\exists n.\ length\ ?xs = Suc\ n$
by *simp*
then obtain n **where**
$length\ ?xs = Suc\ n$
by *blast*
hence *abs-is-lms ?xs n*
using *assms(1) abs-is-lms-last* **by** *fastforce*

have *lms-slice ?xs n* = [*bot*]
using ‹*length ?xs* = *Suc n*› *assms(1) lms-slice-last* **by** *blast*

have $\exists m.\ i = Suc\ m$
using *assms(2) lms-type-list-less-ns* **by** *auto*
then obtain m **where**
$i = Suc\ m$
by *blast*

have *P*: $\forall x \in \{k.\ abs\text{-}is\text{-}lms\ ?xs\ k\}.\ x \neq n \longrightarrow$
list-less-ns (*lms-slice ?xs n*) (*lms-slice ?xs x*)
proof *safe*
fix x
assume *abs-is-lms ?xs x x* ≠ *n*
hence $\exists ys.\ lms\text{-}slice\ ?xs\ x = ?xs\ !\ x\ \#\ ys$
using *abs-is-lms-imp-less-length lms-slice-hd* **by** *blast*
then obtain *ys* **where**
lms-slice ?xs x = *?xs* ! *x* # *ys*
by *blast*
moreover
have *bot* < *?xs* ! *x*
by (*metis* ‹*abs-is-lms ?xs x*› ‹*length ?xs* = *Suc n*› ‹*x* ≠ *n*› *assms(1) bot.not-eq-extremum*
diff-Suc-1 hd-drop-conv-nth abs-is-lms-imp-less-length last-suffix-index)
ultimately show *list-less-ns* (*lms-slice ?xs n*) (*lms-slice ?xs x*)
by (*simp add*: ‹*lms-slice ?xs n* = [*bot*]› *list-less-ns-cons-diff*)
qed

have *Q*: *ordlistns.elem-rank* (*lms-substrs ?xs*) (*lms-slice ?xs n*) = *0*
unfolding *ordlistns.elem-rank-def elm-rank-def*
proof −
have {*y* ∈ *lms-substrs ?xs. list-less-ns y* (*lms-slice ?xs n*)} = {}
using *P* **by** *auto*
then show *card* {*y* ∈ *lms-substrs ?xs. list-less-ns y* (*lms-slice ?xs n*)} = *0*
by (*metis card.empty*)
qed

have *R*: $\forall x \in lms\text{-}substrs\ ?xs.\ list\text{-}less\text{-}ns\ (lms\text{-}slice\ ?xs\ n)\ x \longrightarrow$
ordlistns.elem-rank (*lms-substrs ?xs*) *x* > *0*
using ‹*abs-is-lms ?xs n*› *finite-lms-substrs ordlistns.strict-mono-onD*
ordlistns.strict-mono-on-elem-rank **by** *fastforce*

have $[i..<length\ ?xs] = [i..<n] \ @ \ [n..<Suc\ n]$
using ‹$length\ ?xs = Suc\ n$› $assms(2)$ $abs\text{-}is\text{-}lms\text{-}imp\text{-}less\text{-}length$ **by** $fastforce$
hence $last\ (lms\text{-}seq\ ?xs\ i) = n$
unfolding $lms\text{-}seq\text{-}def$
by ($simp\ add$: ‹$abs\text{-}is\text{-}lms\ ?xs\ n$›)
hence $last\ (lms\text{-}substr\text{-}seq\ ?xs\ i) = lms\text{-}slice\ ?xs\ n$
by ($metis\ assms(2)\ last\text{-}map\ list.discI\ lms\text{-}seq\text{-}Suc1\ lms\text{-}substr\text{-}seq\text{-}def$)
hence $last\ (lms\text{-}substr\text{-}seq\text{-}id\ ?xs\ (suffix\ ?xs\ i)) = lms\text{-}slice\ ?xs\ n$
by ($metis\ lms\text{-}subtrs\text{-}seq\text{-}id\text{-}suffix$)
hence $last\ (lms\text{-}map\ ?xs\ (suffix\ ?xs\ i)) = 0$
unfolding $lms\text{-}map\text{-}o\text{-}def$
by ($metis\ assms(2)\ Q\ last\text{-}map\ length\text{-}0\text{-}conv\ list.discI\ lms\text{-}seq\text{-}Suc1\ lms\text{-}substr\text{-}seq\text{-}length$
$lms\text{-}subtrs\text{-}seq\text{-}id\text{-}suffix$)
moreover
have $butlast\ (lms\text{-}seq\ ?xs\ i) = filter\ (abs\text{-}is\text{-}lms\ ?xs)\ [i..<n]$
unfolding $lms\text{-}seq\text{-}def$
using ‹$[i..<length\ ?xs] = [i..<n] \ @ \ [n..<Suc\ n]$› ‹$abs\text{-}is\text{-}lms\ ?xs\ n$› **by** $auto$
hence $\forall x \in set\ (butlast\ (lms\text{-}seq\ ?xs\ i)).\ x \neq n \wedge abs\text{-}is\text{-}lms\ ?xs\ x$
by $clarsimp$
hence $\forall x \in set\ (butlast\ (lms\text{-}substr\text{-}seq\ ?xs\ i)).$
$list\text{-}less\text{-}ns\ (lms\text{-}slice\ ?xs\ n)\ x$
by ($clarsimp\ simp$: $P\ map\text{-}butlast[symmetric]\ lms\text{-}substr\text{-}seq\text{-}def$)
hence S: $\forall x \in set\ (butlast\ (lms\text{-}substr\text{-}seq\text{-}id\ ?xs\ (suffix\ ?xs\ i))).$
$list\text{-}less\text{-}ns\ (lms\text{-}slice\ ?xs\ n)\ x$
by ($metis\ lms\text{-}subtrs\text{-}seq\text{-}id\text{-}suffix$)
have $\forall x \in set\ (butlast\ (lms\text{-}map\ ?xs\ (suffix\ ?xs\ i))).\ x > 0$
proof $safe$
fix x
assume $x \in set\ (butlast\ (lms\text{-}map\ ?xs\ (suffix\ ?xs\ i)))$
moreover
have $butlast\ (lms\text{-}map\ ?xs\ (suffix\ ?xs\ i)) =$
$map\ (ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ ?xs))\ (butlast\ (lms\text{-}substr\text{-}seq\text{-}id\ ?xs$
$(suffix\ ?xs\ i)))$
unfolding $lms\text{-}map\text{-}o\text{-}def$
by ($simp\ add$: $map\text{-}butlast$)
ultimately have
$\exists y \in set\ (butlast\ (lms\text{-}substr\text{-}seq\text{-}id\ ?xs\ (suffix\ ?xs\ i))).$
$x = ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ ?xs)\ y$
by ($simp\ add$: $in\text{-}set\text{-}mapD$)
then obtain y **where** A:
$y \in set\ (butlast\ (lms\text{-}substr\text{-}seq\text{-}id\ ?xs\ (suffix\ ?xs\ i)))$
$x = ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ ?xs)\ y$
by $blast$
moreover
have $y \in lms\text{-}substrs\ (a\ \#\ b\ \#\ xs)$
by ($metis\ calculation(1)\ in\text{-}set\text{-}butlastD\ in\text{-}set\text{-}conv\text{-}nth\ lms\text{-}substr\text{-}seq\text{-}id\text{-}nth\text{-}abs\text{-}is\text{-}lms$)
ultimately show $0 < x$
using $S\ R$ **by** $blast$
qed

moreover
have *lms-map ?xs* (*suffix ?xs i*) ≠ []
unfolding *lms-map-o-def*
by (*metis assms*(*2*) *list.discI list.map-disc-iff lms-seq-Suc1 lms-substr-seq-def*
lms-subtrs-seq-id-suffix)
ultimately show *?thesis*
unfolding *valid-list-iff-butlast-app-last*
by (*metis bot-nat-def less-numeral-extra*(*3*))
qed

end
theory *Abs-SAIS*
imports *../prop/Buckets*
../prop/LMS-List-Slice-Util
../../util/Repeat
begin

54 Induce Sorting

54.1 Bucket Insert

fun *abs-bucket-insert* ::
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list ⇒
nat list ⇒
nat list
where
abs-bucket-insert α T - SA [] = *SA* |
abs-bucket-insert α T B SA (*x* # *xs*) =
(*let b* = *α* (*T* ! *x*);
k = *B* ! *b* − *Suc 0*;
SA′ = *SA*[*k* := *x*];
B′ = *B*[*b* := *k*]
in abs-bucket-insert α T B′ *SA*′ *xs*)

54.2 Induce L-types

fun *abs-induce-l-step* ::
nat list × *nat list* × *nat* ⇒
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) × '*a list* ⇒
nat list × *nat list* × *nat*
where
abs-induce-l-step (*B*, *SA*, *i*) (*α*, *T*) =
(*if i* < *length SA* ∧ *SA* ! *i* < *length T*
then
(*case SA* ! *i of*
Suc j ⇒

```
        (case suffix-type T j of
           L-type ⇒
             (let k = α (T ! j);
                  l = B ! k
              in (B[k := Suc l], SA[l := j], Suc i))
           | - ⇒ (B, SA, Suc i))
       | - ⇒ (B, SA, Suc i))
  else (B, SA, Suc i))
```

definition *abs-induce-l-base* ::
  (('a :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
   'a *list* ⇒
   nat list ⇒
   nat list ⇒
   nat list × *nat list* × *nat*
  where
abs-induce-l-base α *T B SA* = *repeat* (*length T*) *abs-induce-l-step* (*B*, *SA*, *0*) (α, *T*)

definition *abs-induce-l* ::
  (('a :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
   'a *list* ⇒
   nat list ⇒
   nat list ⇒
   nat list
  where
abs-induce-l α *T B SA* =
  (*let* (*B'*, *SA'*, *i*) = *abs-induce-l-base* α *T B SA*
   in SA')

54.3 Induce S-types

fun *abs-induce-s-step* ::
  nat list × *nat list* × *nat* ⇒
   (('a :: {*linorder*, *order-bot*}) ⇒ *nat*) × 'a *list* ⇒
   nat list × *nat list* × *nat*
  where
abs-induce-s-step (*B*, *SA*, *i*) (α, *T*) =
  (*case i of*
    Suc n ⇒
      (*if Suc n* < *length SA* ∧ *SA* ! *Suc n* < *length T then*
        (*case SA* ! *Suc n of*
          Suc j ⇒
            (*case suffix-type T j of*
              S-type ⇒
                (*let b* = α (*T* ! *j*);
                     k = *B* ! *b* − *Suc 0*
                 in (*B*[*b* := *k*], *SA*[*k* := *j*], *n*)
)

```
            | - ⇒ (B, SA, n)
          )
        | - ⇒ (B, SA, n)
      )
      else
        (B, SA, n)
    )
  | -   ⇒ (B, SA, 0)
)
```

definition *abs-induce-s-base* ::
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list ⇒
nat list × *nat list* × *nat*
where
abs-induce-s-base α *T B SA* = *repeat* (*length T*) *abs-induce-s-step* (*B*, *SA*, *length T*) (α, *T*)

definition *abs-induce-s* ::
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list ⇒
nat list
where
abs-induce-s α *T B SA* =
(*let* (*B'*, *SA'*, *i*) = *abs-induce-s-base* α *T B SA*
in SA')

54.4 Induce Sorting

definition *abs-sa-induce* ::
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list
where
abs-sa-induce α *T LMS* =
(*let*
B0 = *map* (*bucket-end* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))];
B1 = *map* (*bucket-start* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))];

— Initialise SA
SA = *replicate* (*length T*) (*length T*);

— Insert the LMS types into the suffix array
SA = *abs-bucket-insert* α *T B0 SA* (*rev LMS*);

— Insert the L types into the suffix array
SA = abs-induce-l α T B1 SA

— Insert the S types into the suffix array
in abs-induce-s α T (B0[0 := 0]) SA)

55 Rename Mapping

fun *abs-rename-mapping′* ::
('a :: {linorder, order-bot}) list ⇒
nat list ⇒
nat list ⇒
nat ⇒
nat list
where
abs-rename-mapping′ - [] names - = names |
abs-rename-mapping′ - [x] names i = names[x := i] |
abs-rename-mapping′ T (a # b # xs) names i =
(if lms-slice T a = lms-slice T b
then abs-rename-mapping′ T (b # xs) (names[a := i]) i
else abs-rename-mapping′ T (b # xs) (names[a := i]) (Suc i))

definition *abs-rename-mapping :: ('a :: {linorder, order-bot}) list ⇒ nat list ⇒ nat list*
where
abs-rename-mapping T LMS = abs-rename-mapping′ T LMS (replicate (length T) (length T)) 0

56 Rename String

fun *rename-string :: nat list ⇒ nat list ⇒ nat list*
where
rename-string [] - = [] |
rename-string (x#xs) names = (names ! x) # rename-string xs names

57 Order LMS

fun *order-lms :: nat list ⇒ nat list ⇒ nat list*
where
order-lms LMS [] = [] |
order-lms LMS (x # xs) = LMS ! x # order-lms LMS xs

58 Extract LMS

abbreviation *abs-extract-lms :: ('a :: {linorder, order-bot}) list ⇒ nat list ⇒ nat list*

where
abs-extract-lms ≡ *filter* ∘ *abs-is-lms*

59 SAIS Definition

function *abs-sais* ::
nat list ⇒
nat list
where
abs-sais [] = [] |
abs-sais [*x*] = [*0*] |
abs-sais (*a* # *b* # *xs*) =
(*let*
T = *a* # *b* # *xs*;

— Extract the LMS types
LMS0 = *abs-extract-lms T* [*0*..<*length T*];

— Induce the prefix ordering based on LMS
SA = *abs-sa-induce id T LMS0*;

— Extract the LMS types
LMS = *abs-extract-lms T SA*;

— Create a new alphabet
names = *abs-rename-mapping T LMS*;

— Make a reduced string
T′ = *rename-string LMS0 names*;

— Obtain the correct ordering of LMS-types
LMS = (*if distinct T′ then LMS else order-lms LMS0* (*abs-sais T′*))

— Induce the suffix ordering based of LMS
in abs-sa-induce id T LMS)
by *pat-completeness blast*+

end
theory *Abs-Bucket-Insert-Verification*
imports
../abs−def/Abs-SAIS
../../util/List-Util
../../util/List-Slice

begin

60 Bucket Insert with Ghost State

fun *bucket-insert-abs'* ::
  (('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒
   '*a list* ⇒
   nat list ⇒
   nat list ⇒
   nat list ⇒
   nat list ⇒
   nat list × *nat list* × *nat list*
where
  bucket-insert-abs' α *T B SA gs* [] = (*SA*, *B*, *gs*) |
  bucket-insert-abs' α *T B SA gs* (*x* # *xs*) =
    (*let b* = α (*T* ! *x*);
        k = *B* ! *b* − *Suc 0*;
        SA' = *SA*[*k* := *x*];
        B' = *B*[*b* := *k*];
        gs' = *gs* @ [*x*]
     in bucket-insert-abs' α *T B' SA' gs' xs*)

61 Simple Properties

lemma *abs-bucket-insert-length*:
  length (*abs-bucket-insert* α *T B SA xs*) = *length SA*
  by (*induct xs arbitrary*: *B SA*; *simp add*: *Let-def*)

lemma *abs-bucket-insert-equiv*:
  abs-bucket-insert α *T B SA xs* = *fst* (*bucket-insert-abs'* α *T B SA gs xs*)
  by (*induct xs arbitrary*: *B SA gs*; *simp add*: *Let-def*)

62 Invariants

62.1 Defintions and Simple Helper Lemmas

62.1.1 Distinctness

definition *lms-distinct-inv* ::
  ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
  lms-distinct-inv T SA LMS =
     distinct ((*filter* (λ*x*. *x* < *length T*) *SA*) @ *LMS*)

lemma *lms-inv-distinct-inv-helper*:
  assumes *lms-distinct-inv T SA LMS*
  shows *distinct* (*filter* (λ*x*. *x* < *length T*) *SA*) ∧
          distinct LMS ∧
          set (*filter* (λ*x*. *x* < *length T*) *SA*) ∩ *set LMS* = {}
  using *assms distinct-append lms-distinct-inv-def* **by** *blast*

62.1.2 LMS Bucket Ptr

definition *cur-lms-types* ::
  ('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat set*
where
  cur-lms-types α *T SA b* =
    {*i*|*i*. *i* ∈ *set SA* ∧
    i ∈ *lms-bucket* α *T b* }

lemma *cur-lms-subset-SA*:
  cur-lms-types α *T SA b* ⊆ *set SA*
  using *cur-lms-types-def* **by** *blast*

lemma *cur-lms-subset-lms-bucket*:
  cur-lms-types α *T SA b* ⊆ *lms-bucket* α *T b*
  using *cur-lms-types-def* **by** *blast*

definition *num-lms-types* ::
  ('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat*
where
  num-lms-types α *T SA b* =
    card (*cur-lms-types* α *T SA b*)

lemma *num-lms-types-upper-bound*:
  num-lms-types α *T SA b* ≤ *lms-bucket-size* α *T b*
  by (*metis not-le cur-lms-subset-lms-bucket num-lms-types-def*
        finite-lms-bucket lms-bucket-size-def card-mono)

definition *lms-bucket-ptr-inv* ::
  ('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒
   '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
  lms-bucket-ptr-inv α *T B SA* ≡
    (∀ *b* ≤ α (*Max* (*set T*)).
      B ! *b* + *num-lms-types* α *T SA b* = *bucket-end* α *T b*)

lemma *lms-bucket-ptr-invD*:
  assumes *lms-bucket-ptr-inv* α *T B SA*
  and *b* ≤ α (*Max* (*set T*))
  shows *B* ! *b* + *num-lms-types* α *T SA b* = *bucket-end* α *T b*
  using *assms lms-bucket-ptr-inv-def* **by** *blast*

lemma *lms-bucket-ptr-lower-bound*:
  assumes *lms-bucket-ptr-inv* α *T B SA*
  and *b* ≤ α (*Max* (*set T*))
  shows *lms-bucket-start* α *T b* ≤ *B* ! *b*
proof −
  from *lms-bucket-ptr-invD*[*OF assms*]
  have *B* ! *b* + *num-lms-types* α *T SA b* = *bucket-end* α *T b* .
  then show *?thesis*

by (*metis add.commute add-le-cancel-left lms-bucket-pl-size-eq-end num-lms-types-upper-bound*)
qed

lemma *lms-bucket-ptr-upper-bound*:
assumes *lms-bucket-ptr-inv* α T B SA
and $b \leq \alpha$ (*Max* (*set* T))
shows B ! $b \leq$ *bucket-end* α T b
by (*metis assms le-iff-add lms-bucket-ptr-inv-def*)

62.1.3 Unknowns

definition *lms-unknowns-inv* ::
($'a$ :: {*linorder*, *order-bot*} $\Rightarrow$ *nat*) $\Rightarrow$
$'a$ *list* $\Rightarrow$ *nat list* $\Rightarrow$ *nat list* $\Rightarrow$ *bool*
where
lms-unknowns-inv α T B SA $\equiv$
($\forall\, b \leq \alpha$ (*Max* (*set* T))).
($\forall\, i$. *lms-bucket-start* α T $b \leq i \wedge$
$i < B$! $b \longrightarrow$ SA ! i = *length* T))

lemma *lms-unknowns-invD*:
assumes *lms-unknowns-inv* α T B SA
and $b \leq \alpha$ (*Max* (*set* T))
and *lms-bucket-start* α T $b \leq i$
and $i < B$! b
shows SA ! i = *length* T
using *assms lms-unknowns-inv-def* **by** *blast*

62.1.4 Locations

definition *lms-locations-inv* ::
($'a$ :: {*linorder*, *order-bot*} $\Rightarrow$ *nat*) $\Rightarrow$
$'a$ *list* $\Rightarrow$ *nat list* $\Rightarrow$ *nat list* $\Rightarrow$ *bool*
where
lms-locations-inv α T B SA $\equiv$
($\forall\, b \leq \alpha$ (*Max* (*set* T))).
($\forall\, i$. B ! $b \leq i \wedge$
$i <$ *bucket-end* α T $b \longrightarrow$ SA ! $i \in$ *lms-bucket* α T b))

lemma *lms-locations-invD*:
assumes *lms-locations-inv* α T B SA
and $b \leq \alpha$ (*Max* (*set* T))
and B ! $b \leq i$
and $i <$ *bucket-end* α T b
shows SA ! $i \in$ *lms-bucket* α T b
using *assms lms-locations-inv-def* **by** *blast*

62.1.5 Unchanged

definition *lms-unchanged-inv* ::

$('a :: \{linorder,\ order\text{-}bot\} \Rightarrow nat) \Rightarrow$
$'a\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow bool$
where
lms-unchanged-inv $\alpha\ T\ B\ SA\ SA' \equiv$
$(\forall\, b \leq \alpha\ (Max\ (set\ T)).$
$(\forall\, i.\ bucket\text{-}start\ \alpha\ T\ b \leq i \wedge$
$i < B\ !\ b \longrightarrow\ SA'\ !\ i = SA\ !\ i))$

lemma *lms-unchanged-invD*:
assumes *lms-unchanged-inv* $\alpha\ T\ B\ SA\ SA'$
and $b \leq \alpha\ (Max\ (set\ T))$
and $bucket\text{-}start\ \alpha\ T\ b \leq i$
and $i < B\ !\ b$
shows $SA'\ !\ i = SA\ !\ i$
using *assms lms-unchanged-inv-def* **by** *blast*

62.1.6 Inserted

definition *lms-inserted-inv* ::
$nat\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow bool$
where
lms-inserted-inv LMS SA LMSa LMSb $\equiv$
$LMS = LMSa\ @\ LMSb \wedge$
$set\ LMSa \subseteq set\ SA$

lemma *lms-inserted-invD*:
$\bigwedge LMS\ SA\ LMSa\ LMSb.\ lms\text{-}inserted\text{-}inv\ LMS\ SA\ LMSa\ LMSb \Longrightarrow LMS = LMSa\ @\ LMSb$
$\bigwedge LMS\ SA\ LMSa\ LMSb.\ lms\text{-}inserted\text{-}inv\ LMS\ SA\ LMSa\ LMSb \Longrightarrow set\ LMSa \subseteq set\ SA$
unfolding *lms-inserted-inv-def* **by** *blast+*

62.1.7 Sorted

definition *lms-sorted-inv* :: $('a :: \{linorder,\ order\text{-}bot\})\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow bool$
where
lms-sorted-inv T LMS SA $\equiv$
$(\forall\, j < length\ SA.$
$\forall\, i < j.$
$SA\ !\ i \in set\ LMS \wedge SA\ !\ j \in set\ LMS \longrightarrow$
$(T\ !\ (SA\ !\ i) \neq T\ !\ (SA\ !\ j) \longrightarrow T\ !\ (SA\ !\ i) < T\ !\ (SA\ !\ j)) \wedge$
$(T\ !\ (SA\ !\ i) = T\ !\ (SA\ !\ j) \longrightarrow$
$(\exists\, j' < length\ LMS.\ \exists\, i' < j'.\ LMS\ !\ i' = SA\ !\ j \wedge LMS\ !\ j' = SA\ !\ i))$
$)$

lemma *lms-sorted-invD*:
$[\![lms\text{-}sorted\text{-}inv\ T\ LMS\ SA;\ j < length\ SA;\ i < j;\ SA\ !\ i \in set\ LMS;\ SA\ !\ j \in set\ LMS]\!] \Longrightarrow$
$(T\ !\ (SA\ !\ i) \neq T\ !\ (SA\ !\ j) \longrightarrow T\ !\ (SA\ !\ i) < T\ !\ (SA\ !\ j)) \wedge$

$(T\ !\ (SA\ !\ i) = T\ !\ (SA\ !\ j) \longrightarrow$
$(\exists j' < length\ LMS.\ \exists i' < j'.\ LMS\ !\ i' = SA\ !\ j \wedge LMS\ !\ j' = SA\ !\ i))$
using *lms-sorted-inv-def* **by** *blast*

lemma *lms-sorted-invD1*:
$[\![lms\text{-}sorted\text{-}inv\ T\ LMS\ SA;\ j < length\ SA;\ i < j;$
$SA\ !\ i \in set\ LMS;\ SA\ !\ j \in set\ LMS;$
$T\ !\ (SA\ !\ i) \neq T\ !\ (SA\ !\ j)]\!] \Longrightarrow$
$T\ !\ (SA\ !\ i) < T\ !\ (SA\ !\ j)$
using *lms-sorted-inv-def* **by** *blast*

lemma *lms-sorted-invD2*:
$[\![lms\text{-}sorted\text{-}inv\ T\ LMS\ SA;\ j < length\ SA;\ i < j;\ SA\ !\ i \in set\ LMS;\ SA\ !\ j \in set\ LMS;$
$T\ !\ (SA\ !\ i) = T\ !\ (SA\ !\ j)]\!] \Longrightarrow$
$\exists j' < length\ LMS.\ \exists i' < j'.\ LMS\ !\ i' = SA\ !\ j \wedge LMS\ !\ j' = SA\ !\ i$
using *lms-sorted-inv-def* **by** *blast*

62.2 Combined Invariant

definition *lms-inv* ::
$('a :: \{linorder,\ order\text{-}bot\} \Rightarrow nat) \Rightarrow$
$'a\ list \Rightarrow$
$nat\ list \Rightarrow$
$nat\ list \Rightarrow$
$nat\ list \Rightarrow$
$nat\ list \Rightarrow$
$nat\ list \Rightarrow$
$nat\ list \Rightarrow$
$bool$
where
$lms\text{-}inv\ \alpha\ T\ B\ LMS\ LMSa\ LMSb\ SA0\ SA \equiv$
$lms\text{-}distinct\text{-}inv\ T\ SA\ LMSb \wedge$
$lms\text{-}bucket\text{-}ptr\text{-}inv\ \alpha\ T\ B\ SA \wedge$
$lms\text{-}unknowns\text{-}inv\ \alpha\ T\ B\ SA \wedge$
$lms\text{-}locations\text{-}inv\ \alpha\ T\ B\ SA \wedge$
$lms\text{-}unchanged\text{-}inv\ \alpha\ T\ B\ SA0\ SA \wedge$
$lms\text{-}inserted\text{-}inv\ LMS\ SA\ LMSa\ LMSb \wedge$
$lms\text{-}sorted\text{-}inv\ T\ LMS\ SA \wedge$
$strict\text{-}mono\ \alpha \wedge$
$\alpha\ (Max\ (set\ T)) < length\ B \wedge$
$set\ LMS \subseteq \{i.\ abs\text{-}is\text{-}lms\ T\ i\} \wedge$
$length\ SA0 = length\ T \wedge$
$length\ SA = length\ T \wedge$
$(\forall i < length\ T.\ SA0\ !\ i = length\ T)$

lemma *lms-invD*:
$lms\text{-}inv\ \alpha\ T\ B\ LMS\ LMSa\ LMSb\ SA0\ SA \Longrightarrow lms\text{-}distinct\text{-}inv\ T\ SA\ LMSb$
$lms\text{-}inv\ \alpha\ T\ B\ LMS\ LMSa\ LMSb\ SA0\ SA \Longrightarrow lms\text{-}bucket\text{-}ptr\text{-}inv\ \alpha\ T\ B\ SA$

lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ lms-unknowns-inv α T B SA
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ lms-locations-inv α T B SA
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ lms-unchanged-inv α T B SA0 SA
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ lms-inserted-inv LMS SA LMSa LMSb
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ lms-sorted-inv T LMS SA
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ strict-mono α
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ α (Max (set T)) < length B
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ set LMS ⊆ {i. abs-is-lms T i}
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ length SA0 = length T
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ length SA = length T
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ ∀ i < length T. SA0 ! i = length T
unfolding *lms-inv-def* **by** *blast+*

lemma *lms-inv-lms-helper*:
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ ∀ x ∈ set LMS. abs-is-lms T x
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ ∀ x ∈ set LMSa. abs-is-lms T x
lms-inv α T B LMS LMSa LMSb SA0 SA ⟹ ∀ x ∈ set LMSb. abs-is-lms T x
using *lms-invD(10) lms-inserted-invD(1)[OF lms-invD(6)]* **by** *fastforce+*

62.3 Helpers

lemma *lms-distinct-bucket-ptr-lower-bound*:
assumes *b = α (T ! x)*
and *lms-distinct-inv T SA (x # LMS)*
and *lms-bucket-ptr-inv α T B SA*
and *strict-mono α*
and *∀ i ∈ set (x # LMS). abs-is-lms T i*
shows *lms-bucket-start α T b < B ! b*
proof (*rule ccontr*)
assume *¬ lms-bucket-start α T b < B ! b*
hence *B ! b ≤ lms-bucket-start α T b*
by *simp*
moreover
from *lms-bucket-ptr-invD[OF assms(3), of b] assms(1,4,5)*
have *B ! b + num-lms-types α T SA b = bucket-end α T b*
by (*simp add*: *abs-is-lms-imp-less-length strict-mono-leD*)
ultimately
have *lms-bucket-start α T b + num-lms-types α T SA b ≥ bucket-end α T b*
by *linarith*
with *lms-bucket-pl-size-eq-end*
have *num-lms-types α T SA b ≥ lms-bucket-size α T b*
by (*metis add-le-cancel-left*)
with *card-seteq[OF finite-lms-bucket cur-lms-subset-lms-bucket]*
have *cur-lms-types α T SA b = lms-bucket α T b*
by (*simp add*: *lms-bucket-size-def num-lms-types-def*)
with *cur-lms-subset-SA*
have *lms-bucket α T b ⊆ set SA*

```
    by blast
  moreover
  from assms(1,5)
  have x ∈ lms-bucket α T b
    by (simp add: bucket-def abs-is-lms-imp-less-length lms-bucket-def)
  ultimately
  have x ∈ set SA
    by blast
  moreover
  from assms(2,5)
  have x ∉ set SA
    by (simp add: abs-is-lms-imp-less-length lms-distinct-inv-def)
  ultimately
  show False
    by blast
qed

lemma lms-next-insert-at-unknown:
  assumes b = α (T ! x)
  and     k = (B ! b) − Suc 0
  and     lms-distinct-inv T SA (x # LMS)
  and     lms-bucket-ptr-inv α T B SA
  and     lms-unknowns-inv α T B SA
  and     strict-mono α
  and     length SA = length T
  and     ∀ i ∈ set (x # LMS). abs-is-lms T i
shows k < length SA ∧ SA ! k = length T
proof −

  from lms-distinct-bucket-ptr-lower-bound[OF assms(1,3−4,6,8)]
  have lms-bucket-start α T b < B ! b
    by assumption
   with assms(2)
  have lms-bucket-start α T b ≤ k k < B ! b
    by linarith+
  with lms-unknowns-invD[OF assms(5), of b k] assms(1,6,8)
  have SA ! k = length T
    by (simp add: abs-is-lms-imp-less-length strict-mono-less-eq)
  moreover
  from ‹k < B ! b› lms-bucket-ptr-invD[OF assms(4), of b] assms(1,6,8)
  have k < bucket-end α T b
    by (simp add: assms(7) abs-is-lms-imp-less-length strict-mono-less-eq)
  with assms(7)
  have  k < length SA
    by (metis bucket-end-le-length dual-order.strict-trans1)
  ultimately
  show ?thesis
    by blast
qed
```

lemma *lms-distinct-slice*:
assumes *lms-distinct-inv T SA LMS*
and *lms-bucket-ptr-inv α T B SA*
and *lms-locations-inv α T B SA*
and *length SA = length T*
and $b \leq \alpha$ (*Max* (*set T*))
shows *distinct* (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*))
proof −
from *assms*(*4*)
have *bucket-end α T b* ≤ *length SA*
by (*simp add*: *bucket-end-le-length*)

from *lms-bucket-ptr-upper-bound*[*OF assms*(*2*,*5*)]
have *B* ! *b* ≤ *bucket-end α T b* **.**

from *lms-locations-invD*[*OF assms*(*3*,*5*)]
have ∀ *i*. *B* ! *b* ≤ *i* ∧ *i* < *bucket-end α T b* ⟶ *SA* ! *i* ∈ *lms-bucket α T b*
by *blast*
hence ∀ *i*. *B* ! *b* ≤ *i* ∧ *i* < *bucket-end α T b* ⟶ *SA* ! *i* < *length T*
by (*simp add*: *abs-is-lms-imp-less-length lms-bucket-def*)

have ∀ *x* ∈ *set* (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)). *x* < *length T*
proof
fix *x*
assume *A*: *x* ∈ *set* (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*))
from *in-set-conv-nth*[*THEN iffD1*, *OF A*]
obtain *i* **where**
i < *length* (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*))
(*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)) ! *i* = *x*
by *blast*
with *nth-list-slice*
have (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)) ! *i* = *SA* ! (*B* ! *b* + *i*)
by *blast*
moreover
from ‹*i* < *length* (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*))›
have *B* ! *b* + *i* < *bucket-end α T b*
by (*simp add*: ‹*bucket-end α T b* ≤ *length SA*›
length-list-slice)
with ‹∀ *i*. *B* ! *b* ≤ *i* ∧ *i* < *bucket-end α T b* ⟶ *SA* ! *i* < *length T*›
have *SA* ! (*B* ! *b* + *i*) < *length T*
by *simp*
ultimately
show *x* < *length T*
using ‹(*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)) ! *i* = *x*› **by** *simp*
qed

from *lms-inv-distinct-inv-helper*[*OF assms*(*1*)]
have *distinct* (*filter* (λ*x*. *x* < *length T*) *SA*) *distinct LMS*

```
    set (filter (λx. x < length T) SA) ∩ set LMS = {}
   by blast+

  have SA = list-slice SA 0 (length SA)
   by (simp add: list-slice-0-length)
  hence SA = list-slice SA 0 (B ! b) @ list-slice SA (B ! b) (length SA)
   using append-take-drop-id
   by (simp add: list-slice.simps)
  moreover
  from list-slice-append[OF ‹B ! b ≤ bucket-end α T b› ‹bucket-end α T b ≤ length
SA›, of SA]
  have list-slice SA (B ! b) (length SA)
       = list-slice SA (B ! b) (bucket-end α T b) @ list-slice SA (bucket-end α T
b) (length SA)
   by assumption
  ultimately
  have SA = list-slice SA 0 (B ! b) @ list-slice SA (B ! b) (bucket-end α T b) @
          list-slice SA (bucket-end α T b) (length SA)
   by metis
  with ‹distinct (filter (λx. x < length T) SA)›
  have distinct (filter (λx. x < length T) (list-slice SA (B ! b) (bucket-end α T
b)))
   by (metis distinct-append filter-append)
  with ‹∀ x ∈ set (list-slice SA (B ! b) (bucket-end α T b)). x < length T›
  show distinct (list-slice SA (B ! b) (bucket-end α T b))
   by simp
qed

lemma lms-slice-subset-lms-bucket:
  assumes lms-locations-inv α T B SA
  and     length SA = length T
  and     b ≤ α (Max (set T ))
shows set (list-slice SA (B ! b) (bucket-end α T b)) ⊆ lms-bucket α T b
proof
  fix x
  assume A: x ∈ set (list-slice SA (B ! b) (bucket-end α T b))
  from in-set-conv-nth[THEN iffD1, OF A]
  obtain i where
   i < length (list-slice SA (B ! b) (bucket-end α T b))
   (list-slice SA (B ! b) (bucket-end α T b)) ! i = x
   by blast
  with nth-list-slice
  have (list-slice SA (B ! b) (bucket-end α T b)) ! i = SA ! (B ! b + i)
   by blast
  moreover
  from ‹i < length (list-slice SA (B ! b) (bucket-end α T b))›
  have B ! b + i < bucket-end α T b
   by (simp add: assms(2) bucket-end-le-length length-list-slice)
  moreover
```

```
  have B ! b ≤ B ! b + i
    by simp
  ultimately
  show x ∈ lms-bucket α T b
   using ‹list-slice SA (B ! b) (bucket-end α T b) ! i = x› assms(1,3) lms-locations-invD
    by fastforce
qed

lemma lms-val-location:
  assumes lms-locations-inv α T B SA
  and     lms-unchanged-inv α T B SA0 SA
  and     strict-mono α
  and     length SA = length T
  and     ∀ i < length T. SA0 ! i = length T
  and     i < length SA
  and     SA ! i < length T
shows ∃ b ≤ α (Max (set T)). B ! b ≤ i ∧ i < bucket-end α T b
proof −
  from assms
  have i < length T
    by simp
  with index-in-bucket-interval-gen[OF - assms(3)]
  obtain b where
    b ≤ α (Max (set T))
    bucket-start α T b ≤ i
    i < bucket-end α T b
    by blast
  moreover
  have B ! b ≤ i
  proof (rule ccontr)
    assume ¬B ! b ≤ i
    hence i < B ! b
      by simp
    with lms-unchanged-invD[OF assms(2) ‹b ≤ α (Max (set T))› ‹bucket-start α
T b ≤ i›]
    have SA ! i = SA0 ! i
      by blast
    with assms(5) ‹i < length T›
    have SA ! i = length T
      by auto
    with assms(7)
    show False
      by auto
  qed
  ultimately
  show ?thesis
    by auto
qed
```

lemma *lms-val-imp-abs-is-lms*:
  assumes *lms-locations-inv* α *T B SA*
  and *lms-unchanged-inv* α *T B SA0 SA*
  and *strict-mono* α
  and *length SA* = *length T*
  and $\forall i <$ *length T*. *SA0* ! i = *length T*
  and $i <$ *length SA*
  and *SA* ! $i <$ *length T*
shows *abs-is-lms T* (*SA* ! i)
proof −
  from *lms-val-location*[*OF assms*(1−7)]
  obtain b **where**
    $b \leq \alpha$ (*Max* (*set T*))
    B ! $b \leq i$
    $i <$ *bucket-end* α *T b*
    by *blast*
  with *lms-locations-invD*[*OF assms*(1)]
  have *SA* ! $i \in$ *lms-bucket* α *T b*
    by *blast*
  then show *abs-is-lms T* (*SA* ! i)
    using *lms-bucket-def* **by** *blast*
qed

lemma *lms-lms-prefix-sorted*:
  assumes *lms-bucket-ptr-inv* α *T B SA*
  and *lms-locations-inv* α *T B SA*
  and *lms-unchanged-inv* α *T B SA0 SA*
  and *strict-mono* α
  and *length SA* = *length T*
  and $\forall i <$ *length T*. *SA0* ! i = *length T*
  and *set LMS* = {i. *abs-is-lms T i*}
shows *ordlistns.sorted* (*map* (*lms-prefix T*) (*filter* (λx. $x <$ *length T*) *SA*))
proof (*intro sorted-wrt-mapI*)
  fix i j
  let *?fs* = *filter* (λx. $x <$ *length T*) *SA*
  let *?goal* = *list-less-eq-ns* (*lms-prefix T* (*?fs* ! i)) (*lms-prefix T* (*?fs* ! j))
  assume $i < j$ $j <$ *length ?fs*

  from *filter-nth-relative-2*[*OF* ‹$j <$ *length ?fs*› ‹$i < j$›]
  obtain i' j' **where**
    $j' <$ *length SA*
    $i' < j'$
    ?fs ! j = *SA* ! j'
    ?fs ! i = *SA* ! i'
    by *blast*
  then have ‹$i' <$ *length SA*›
    by *linarith*
  from ‹$i < j$› ‹$j <$ *length ?fs*› **have** ‹$i <$ *length ?fs*›
    by (*rule less-trans*)

with ‹*?fs* ! *i* = *SA* ! *i'*› [*symmetric*] **have** ‹*SA* ! *i'* ∈ *set ?fs*›
  by (*simp only*:) (*erule nth-mem*)
then have ‹*SA* ! *i'* < *length T*›
  by *simp*
with *lms-val-location*[*OF assms*(*2*−*6*) ‹*i'* < *length SA*›]
obtain *b* **where**
  b ≤ *α* (*Max* (*set T*))
  B ! *b* ≤ *i'*
  i' < *bucket-end α T b*
  by *blast*
with *lms-locations-invD*[*OF assms*(*2*)]
have *SA* ! *i'* ∈ *lms-bucket α T b*
  by *blast*
hence *α* (*T* ! (*SA* ! *i'*)) = *b abs-is-lms T* (*SA* ! *i'*)
  by (*simp add*: *bucket-def lms-bucket-def*)+

from *lms-lms-prefix*[*OF* ‹*abs-is-lms T* (*SA* ! *i'*)›] ‹*?fs* ! *i* = *SA* ! *i'*›
have *S1*: *lms-prefix T* (*?fs* ! *i*) = [*T* ! (*SA* ! *i'*)]
  by *simp*

have *SA* ! *j'* < *length T*
  using ‹*?fs* ! *j* = *SA* ! *j'*› ‹*j* < *length ?fs*› *nth-mem* **by** *fastforce*
with *lms-val-location*[*OF assms*(*2*−*6*) ‹*j'* < *length SA*›]
obtain *b'* **where**
  b' ≤ *α* (*Max* (*set T*))
  B ! *b'* ≤ *j'*
  j' < *bucket-end α T b'*
  by *blast*
with *lms-locations-invD*[*OF assms*(*2*)]
have *SA* ! *j'* ∈ *lms-bucket α T b'*
  by *blast*
hence *α* (*T* ! (*SA* ! *j'*)) = *b' abs-is-lms T* (*SA* ! *j'*)
  by (*simp add*: *bucket-def lms-bucket-def*)+

from *lms-lms-prefix*[*OF* ‹*abs-is-lms T* (*SA* ! *j'*)›] ‹*?fs* ! *j* = *SA* ! *j'*›
have *S2*: *lms-prefix T* (*?fs* ! *j*) = [*T* ! (*SA* ! *j'*)]
  by *simp*

have *b* ≤ *b'*
proof (*rule ccontr*)
  assume ¬*b* ≤ *b'*
  hence *b'* < *b*
    by *simp*
  hence *bucket-end α T b'* ≤ *bucket-start α T b*
    by (*simp add*: *less-bucket-end-le-start*)
  hence *bucket-end α T b'* ≤ *lms-bucket-start α T b*
    by (*metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans*)
  with *lms-bucket-ptr-lower-bound*[*OF assms*(*1*) ‹*b* ≤ *α* (*Max* (*set T*))›]
  have *bucket-end α T b'* ≤ *B* ! *b*

```
    by linarith
  with ‹j' < bucket-end α T b'› ‹B ! b ≤ i'› ‹i' < j'›
  show False
    by linarith
qed
moreover
have b < b' ⟹ ?goal
proof −
  assume b < b'
  with ‹α (T ! (SA ! i')) = b› ‹α (T ! (SA ! j')) = b'› assms(4)
  have T ! (SA ! i') < T ! (SA ! j')
    using strict-mono-less by blast
  with S1 S2
  show ?goal
    by (simp add: list-less-eq-ns-cons)
qed
moreover
have b = b' ⟹ ?goal
proof −
  assume b = b'
  with ‹α (T ! (SA ! i')) = b› ‹α (T ! (SA ! j')) = b'› assms(4)
  have T ! (SA ! i') = T ! (SA ! j')
    by (meson strict-mono-eq)
  with S1 S2
  show ?goal
    by simp
qed
ultimately
show ?goal
  using less-le by blast
qed

lemma lms-suffix-sorted:
  assumes lms-bucket-ptr-inv α T B SA
  and     lms-locations-inv α T B SA
  and     lms-unchanged-inv α T B SA0 SA
  and     lms-sorted-inv T LMS SA
  and     strict-mono α
  and     length SA = length T
  and     ∀ i < length T. SA0 ! i = length T
  and     set LMS = {i. abs-is-lms T i}
  and     ordlistns.sorted (map (suffix T) (rev LMS))
shows ordlistns.sorted (map (suffix T) (filter (λx. x < length T) SA))
proof (intro sorted-wrt-mapI)
  fix i j
  let ?fs = filter (λx. x < length T) SA
  let ?goal = list-less-eq-ns (suffix T (?fs ! i)) (suffix T (?fs ! j))
  assume i < j j < length ?fs
```

from *filter-nth-relative-2*[*OF* ‹$j < length\ ?fs$› ‹$i < j$›]
obtain $i'\ j'$ **where**
  $j' < length\ SA$
  $i' < j'$
  $?fs\ !\ j = SA\ !\ j'$
  $?fs\ !\ i = SA\ !\ i'$
  by *blast*
then have ‹$i' < length\ SA$›
  by *linarith*
from ‹$i < j$› ‹$j < length\ ?fs$› **have** ‹$i < length\ ?fs$›
  by (*rule less-trans*)
with ‹$?fs\ !\ i = SA\ !\ i'$› [*symmetric*] **have** ‹$SA\ !\ i' \in set\ ?fs$›
  by (*simp only*:) (*erule nth-mem*)
then have ‹$SA\ !\ i' < length\ T$›
  by *simp*
with *lms-val-location*[*OF assms*(*2*,*3*,*5*−*7*) ‹$i' < length\ SA$›]
obtain b **where**
  $b \leq \alpha\ (Max\ (set\ T))$
  $B\ !\ b \leq i'$
  $i' < bucket\text{-}end\ \alpha\ T\ b$
  by *blast*
with *lms-locations-invD*[*OF assms*(*2*)]
have $SA\ !\ i' \in lms\text{-}bucket\ \alpha\ T\ b$
  by *blast*
hence $\alpha\ (T\ !\ (SA\ !\ i')) = b\ abs\text{-}is\text{-}lms\ T\ (SA\ !\ i')$
  by (*simp add*: *bucket-def lms-bucket-def*)+
hence $SA\ !\ i' \in set\ LMS$
  using *assms*(*8*)
  by *blast*

have $SA\ !\ j' < length\ T$
  using ‹$?fs\ !\ j = SA\ !\ j'$› ‹$j < length\ ?fs$› *nth-mem* **by** *fastforce*
with *lms-val-location*[*OF assms*(*2*,*3*,*5*−*7*) ‹$j' < length\ SA$›]
obtain b' **where**
  $b' \leq \alpha\ (Max\ (set\ T))$
  $B\ !\ b' \leq j'$
  $j' < bucket\text{-}end\ \alpha\ T\ b'$
  by *blast*
with *lms-locations-invD*[*OF assms*(*2*)]
have $SA\ !\ j' \in lms\text{-}bucket\ \alpha\ T\ b'$
  by *blast*
hence $\alpha\ (T\ !\ (SA\ !\ j')) = b'\ abs\text{-}is\text{-}lms\ T\ (SA\ !\ j')$
  by (*simp add*: *bucket-def lms-bucket-def*)+
hence $SA\ !\ j' \in set\ LMS$
  using *assms*(*8*)
  by *blast*

have $b \leq b'$
proof (*rule ccontr*)

assume $\neg b \leq b'$
hence $b' < b$
by *simp*
hence *bucket-end* α T $b' \leq$ *bucket-start* α T b
by (*simp add*: *less-bucket-end-le-start*)
hence *bucket-end* α T $b' \leq$ *lms-bucket-start* α T b
by (*metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans*)
with *lms-bucket-ptr-lower-bound*[*OF assms*(1) ‹$b \leq \alpha$ (*Max* (*set* T))›]
have *bucket-end* α T $b' \leq B$! b
by *linarith*
with ‹$j' <$ *bucket-end* α T b'› ‹B ! $b \leq i'$› ‹$i' < j'$›
show *False*
by *linarith*
qed
moreover
have $b < b' \Longrightarrow$ *?goal*
proof −
assume $b < b'$
with ‹α (T ! (*SA* ! i')) = b› ‹α (T ! (*SA* ! j')) = b'› *assms*(5)
have T ! (*SA* ! i') < T ! (*SA* ! j')
using *strict-mono-less* **by** *blast*
with ‹*?fs* ! i = *SA* ! i'› ‹*?fs* ! j = *SA* ! j'› ‹*SA* ! $i' <$ *length* T› ‹*SA* ! $j' <$ *length* T›
show *?goal*
by (*metis list-less-ns-cons-diff ordlistns.less-imp-le suffix-cons-ex*)
qed
moreover
have $b = b' \Longrightarrow$ *?goal*
proof −
assume $b = b'$
with ‹α (T ! (*SA* ! i')) = b› ‹α (T ! (*SA* ! j')) = b'› *assms*(5)
have T ! (*SA* ! i') = T ! (*SA* ! j')
by (*meson strict-mono-eq*)
with *lms-sorted-invD2*[*OF assms*(4) ‹$j' <$ *length SA*› ‹$i' < j'$› ‹*SA* ! $i' \in$ *set LMS*›
‹*SA* ! $j' \in$ *set LMS*›]
obtain m n **where**
$n <$ *length LMS*
$m < n$
LMS ! m = *SA* ! j'
LMS ! n = *SA* ! i'
by *blast*
hence *rev LMS* ! (*length LMS* − *Suc* m) = *SA* ! j' *rev LMS* ! (*length LMS* − *Suc* n) = *SA* ! i'
by (*metis length-rev order.strict-trans rev-nth rev-rev-ident*)+
moreover
from ‹$m < n$› ‹$n <$ *length LMS*›
have *length LMS* − *Suc* $n \leq$ *length LMS* − *Suc* m
by *linarith*

```
    moreover
    have length LMS − Suc m < length (rev LMS)
      using ‹n < length LMS› by auto
    ultimately
    have list-less-eq-ns (suffix T (SA ! i')) (suffix T (SA ! j'))
      using ordlistns.sorted-nth-mono[OF assms(9)]
      by fastforce
    with ‹?fs ! i = SA ! i'› ‹?fs ! j = SA ! j'›
    show ?goal
      by simp
  qed
  ultimately
  show ?goal
    using less-le by blast
qed

lemma next-index-outside:
  assumes b = α (T ! x)
  and     k = B ! b − Suc 0
  and     lms-distinct-inv T SA (x # LMS)
  and     lms-bucket-ptr-inv α T B SA
  and     strict-mono α
  and     ∀ a ∈ set (x # LMS). abs-is-lms T a
  and     b' ≤ α (Max (set T))
  and     b ≠ b'
shows k < bucket-start α T b' ∨ bucket-end α T b' ≤ k
proof −

  from lms-distinct-bucket-ptr-lower-bound[OF assms(1,3−6)]
  have lms-bucket-start α T b < B ! b .
  hence k < B ! b
    using assms(2) by linarith

  from ‹lms-bucket-start α T b < B ! b›
  have lms-bucket-start α T b ≤ k
    using assms(2) by linarith

  have x < length T
    by (simp add: assms(6) abs-is-lms-imp-less-length)
  hence b ≤ α (Max (set T))
    by (simp add: assms(1,5) strict-mono-leD)

  from assms(8)
  have b < b' ∨ b' < b
    by linarith
  moreover
  have b < b' ⟹ k < bucket-start α T b'
  proof −
    assume b < b'
```

hence *bucket-end* α T $b \leq$ *bucket-start* α T b'
by (*simp add*: *less-bucket-end-le-start*)
with *lms-bucket-ptr-upper-bound*[*OF assms*(*4*) ‹$b \leq \alpha$ (*Max* (*set* T))›]
have B ! $b \leq$ *bucket-start* α T b'
by *linarith*
with ‹$k < B$! b›
show *?thesis*
by *linarith*
qed
moreover
have $b' < b \Longrightarrow$ *bucket-end* α T $b' \leq k$
proof −
assume $b' < b$
hence *bucket-end* α T $b' \leq$ *bucket-start* α T b
by (*simp add*: *less-bucket-end-le-start*)
hence *bucket-end* α T $b' \leq$ *lms-bucket-start* α T b
by (*metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans*)
with ‹*lms-bucket-start* α T $b \leq k$›
show *?thesis*
using *le-trans* **by** *blast*
qed
ultimately show *?thesis*
by *blast*
qed

62.4 Establishment and Maintenance Steps

62.4.1 Distinctness

lemma *lms-distinct-inv-established*:
assumes *distinct LMS*
and $\forall i <$ *length SA*. *SA* ! $i =$ *length* T
shows *lms-distinct-inv* T *SA LMS*
proof −
from *assms*(*2*)
have *filter* (λx. $x <$ *length* T) *SA* = []
by (*metis filter-False in-set-conv-nth nat-neq-iff*)
then show *?thesis*
unfolding *lms-distinct-inv-def*
using *distinct-append assms*(*1*)
by *simp*
qed

lemma *lms-distinct-inv-maintained-step*:
assumes *lms-distinct-inv* T *SA* (x # *LMS*)
shows *lms-distinct-inv* T (*SA*[k := x]) *LMS*
unfolding *lms-distinct-inv-def*
proof(*intro distinct-conv-nth*[*THEN iffD2*] *allI impI*)
let *?xs* = *filter* (λx. $x <$ *length* T) *SA* **and**
?ys = *filter* (λx. $x <$ *length* T) (*SA*[k := x])

fix $i\ j$
assume $i \neq j\ i < length\ (filter\ (\lambda x.\ x < length\ T)\ (SA[k := x])\ @\ LMS)$
$j < length\ (filter\ (\lambda x.\ x < length\ T)\ (SA[k := x])\ @\ LMS)$
hence $i < length\ ?ys + length\ LMS\ \ j < length\ ?ys + length\ LMS$
by *simp-all*

let $?goal = (?ys\ @\ LMS)\ !\ i \neq (?ys\ @\ LMS)\ !\ j$

from *assms(1) distinct-append*
have *distinct LMS distinct ?xs x ∉ set ?xs x ∉ set LMS set ?xs ∩ set LMS = {}*
by (*simp add*: *lms-distinct-inv-def*)+

from ‹*distinct LMS*› ‹$i < length\ ?ys + length\ LMS$› ‹$j < length\ ?ys + length\ LMS$› ‹$i \neq j$›
have *R1*: $[\![length\ ?ys \leq i;\ length\ ?ys \leq j]\!] \Longrightarrow ?goal$
by (*metis le-Suc-ex nat-add-left-cancel-less nth-append-length-plus nth-eq-iff-index-eq*)

have $set\ ?ys \subseteq \{x\} \cup set\ ?xs$
by (*meson filter-nth-update-subset*)

have *R2*:
$\bigwedge i\ j.\ [\![i < length\ ?ys;\ length\ ?ys \leq j;\ j < length\ ?ys + length\ LMS]\!] \Longrightarrow$
$(?ys\ @\ LMS)\ !\ i \neq (?ys\ @\ LMS)\ !\ j$
proof −
fix $i\ j$
assume $i < length\ ?ys\ length\ ?ys \leq j\ j < length\ ?ys + length\ LMS$
hence $?ys\ !\ i \in \{x\} \cup set\ ?xs$
using ‹$set\ ?ys \subseteq \{x\} \cup set\ ?xs$› *nth-mem* **by** *blast*
hence $(?ys\ @\ LMS)\ !\ i \in \{x\} \cup set\ ?xs$
by (*simp add*: ‹$i < length\ ?ys$› *nth-append*)
moreover
from ‹$length\ ?ys \leq j$› ‹$j < length\ ?ys + length\ LMS$›
have $(?ys\ @\ LMS)\ !\ j \in set\ LMS$
by (*metis add.commute in-set-conv-nth leD less-diff-conv2 nth-append*)
moreover
from ‹$set\ ?xs \cap set\ LMS = \{\}$› ‹$x \notin set\ LMS$›
have $(\{x\} \cup set\ ?xs) \cap set\ LMS = \{\}$
by *blast*
ultimately
show $(?ys\ @\ LMS)\ !\ i \neq (?ys\ @\ LMS)\ !\ j$
by (*metis disjoint-iff-not-equal*)
qed

have *R3*: $[\![i < length\ ?ys;\ j < length\ ?ys]\!] \Longrightarrow ?goal$
proof −
assume $i < length\ ?ys\ j < length\ ?ys$
with *filter-nth-relative-neq-2*[*OF* - - ‹$i \neq j$›]
obtain $i'\ j'$ **where**
$i' < length\ (SA[k := x])$

```
  j' < length (SA[k := x])
  (SA[k := x]) ! i' = ?ys ! i
  (SA[k := x]) ! j' = ?ys ! j
  i' ≠ j'
  by blast

have ?ys ! i < length T
  using ‹i < length ?ys› nth-mem by fastforce
hence (SA[k := x]) ! i' < length T
  using ‹(SA[k := x]) ! i' = ?ys ! i› by simp

have ?ys ! j < length T
  using ‹j < length ?ys› nth-mem by fastforce
hence (SA[k := x]) ! j' < length T
  using ‹(SA[k := x]) ! j' = ?ys ! j› by simp

have R4:
  ⋀i j. ⟦i ≠ k; j = k; i < length (SA[k := x]); j < length (SA[k := x]);
         (SA[k := x]) ! i < length T⟧ ⟹
         (SA[k := x]) ! i ≠ (SA[k := x]) ! j
proof −
  fix i j
  assume i ≠ k j = k i < length (SA[k := x]) j < length (SA[k := x])
        (SA[k := x]) ! i < length T

  from ‹j = k› ‹j < length (SA[k := x])›
  have (SA[k := x]) ! j = x
    by simp
  moreover
  from ‹i ≠ k› ‹i < length (SA[k := x])›
  have (SA[k := x]) ! i = SA ! i
    by simp
  with ‹(SA[k := x]) ! i < length T›
  have SA ! i < length T
    by simp
  with filter-nth-1[of i SA λx. x < length T] ‹i < length (SA[k := x])›
  obtain i' where
    i' < length ?xs
    ?xs ! i' = SA ! i
    by auto
  with ‹(SA[k := x]) ! i = SA ! i›
  have (SA[k := x]) ! i ∈ set ?xs
    using nth-mem by fastforce
  ultimately
  show (SA[k := x]) ! i ≠ (SA[k := x]) ! j
    using ‹x ∉ set ?xs› by auto
qed

have ⟦i' ≠ k; j' ≠ k⟧ ⟹ (SA[k := x]) ! i' ≠ (SA[k := x]) ! j'
```

```
    proof −
      assume i' ≠ k j' ≠ k
      with ‹(SA[k := x]) ! i' = ?ys ! i› ‹(SA[k := x]) ! j' = ?ys ! j›
      have ?ys ! i = SA ! i' ?ys ! j = SA ! j'
        by auto
      with ‹?ys ! i < length T› ‹?ys ! j < length T› ‹i' < length (SA[k := x])›
          ‹j' < length (SA[k := x])›
          filter-nth-relative-neq-1[of i' SA λx. x < length T j', OF - - - - ‹i' ≠ j'›]
      obtain i'' j'' where
        i'' < length ?xs
        j'' < length ?xs
        ?xs ! i'' = SA ! i'
        ?xs ! j'' = SA ! j'
        i'' ≠ j''
        by auto
      with ‹distinct ?xs›
      have SA ! i' ≠ SA ! j'
        by (metis distinct-Ex1 in-set-conv-nth)
      then show ?thesis
        using ‹SA[k := x] ! i' = ?ys ! i› ‹?ys ! i = SA ! i'› ‹j' ≠ k› by auto
    qed
    moreover
    from ‹i' ≠ j'›
    have ⟦i' = k; j' = k⟧ ⟹ False
      by blast
    ultimately
    have (SA[k := x]) ! i' ≠ (SA[k := x]) ! j'
      using R4[of i' j', OF - - ‹i' < length (SA[k := x])› ‹j' < length (SA[k := x])›
            ‹(SA[k := x]) ! i' < length T›]
          R4[of j' i', OF - - ‹j' < length (SA[k := x])› ‹i' < length (SA[k := x])›
            ‹(SA[k := x]) ! j' < length T›]
      by metis
    with ‹(SA[k := x]) ! i' = ?ys ! i› ‹i < length ?ys›
        ‹(SA[k := x]) ! j' = ?ys ! j› ‹j < length ?ys›
    show ?goal
      by (simp add: nth-append)
  qed

  from R1 R2[of i j, OF - - ‹j < length ?ys + length LMS›]
      R2[of j i, OF - - ‹i < length ?ys + length LMS›] R3
  show ?goal
    by presburger
qed

lemma lms-distinct-inv-maintained:
  assumes lms-distinct-inv T SA LMS
  shows lms-distinct-inv T (abs-bucket-insert α T B SA LMS) []
  using assms
proof (induct rule: abs-bucket-insert.induct[of - α T B SA LMS])
```

case (*1* α *T uu SA*)
then show *?case*
by *simp*
next
case (*2* α *T B SA x xs*)
note *IH* = *this*
let *?b* = α (*T* ! *x*)
let *?k* = *B* ! *?b* − *Suc 0*
from *IH*(*1*)[*OF* - - - - *lms-distinct-inv-maintained-step*[*OF IH*(*2*), *of ?k*], *of ?b ?k B*[*?b* := *?k*]]
show *?case*
by (*metis* (*full-types*) *One-nat-def abs-bucket-insert.simps*(*2*))
qed

lemma *abs-bucket-insert-lms-distinct-inv*:
assumes *distinct LMS*
and ∀ *i* < *length SA*. *SA* ! *i* = *length T*
shows *lms-distinct-inv T* (*abs-bucket-insert* α *T B SA LMS*) []
using *assms lms-distinct-inv-maintained lms-distinct-inv-established*
by *blast*

62.4.2 Bucket Ptr

lemma *lms-bucket-ptr-inv-established*:
assumes *lms-bucket-init* α *T B*
and ∀ *i* < *length SA*. *SA* ! *i* = *length T*
shows *lms-bucket-ptr-inv* α *T B SA*
unfolding *lms-bucket-ptr-inv-def*
proof (*intro allI impI*)
fix *b*
assume *b* ≤ α (*Max* (*set T*))
with *lms-bucket-initD*[*OF assms*(*1*)]
have *B* ! *b* = *bucket-end* α *T b*
by *simp*
moreover
from *assms*(*2*)
have ∀ *i* ∈ *set SA*. ¬*abs-is-lms T i*
by (*metis in-set-conv-nth abs-is-lms-imp-less-length less-not-refl2*)
hence *cur-lms-types* α *T SA b* = {}
by (*simp add*: *cur-lms-types-def lms-bucket-def*)
hence *num-lms-types* α *T SA b* = *0*
by (*simp add*: *num-lms-types-def*)
ultimately
show *B* ! *b* + *num-lms-types* α *T SA b* = *bucket-end* α *T b*
by *simp*
qed

lemma *lms-bucket-ptr-inv-maintained-step*:
assumes *b* = α (*T* ! *x*)

```
  and     k = B ! b − Suc 0
  and     lms-distinct-inv T SA (x # LMS)
  and     lms-bucket-ptr-inv α T B SA
  and     lms-unknowns-inv α T B SA
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA = length T
  and     ∀ a ∈ set (x # LMS). abs-is-lms T a
shows lms-bucket-ptr-inv α T (B[b := k]) (SA[k := x])
  unfolding lms-bucket-ptr-inv-def
proof (intro allI impI)
  fix b'
  assume b' ≤ α (Max (set T))

  let ?goal = B[b := k] ! b' + num-lms-types α T (SA[k := x]) b' = bucket-end α
T b'

  from assms(1,9)
  have x ∈ lms-bucket α T b
    by (simp add: bucket-def abs-is-lms-imp-less-length lms-bucket-def)

  from lms-next-insert-at-unknown[OF assms(1−6,8,9)]
  have k < length SA SA ! k = length T
    by blast+

  have b' ≠ b ⟹ ?goal
  proof −
    assume b' ≠ b
    with ‹x ∈ lms-bucket α T b›
    have x ∉ lms-bucket α T b'
      by (simp add: bucket-def lms-bucket-def)
    with cur-lms-subset-lms-bucket
    have x ∉ cur-lms-types α T SA b'
      by blast

    have cur-lms-types α T (SA[k := x]) b' = cur-lms-types α T SA b'
      unfolding cur-lms-types-def
    proof (intro equalityI subsetI)
      fix y
      assume y ∈ {i |i. i ∈ set (SA[k := x]) ∧ i ∈ lms-bucket α T b'}
      hence y ∈ set (SA[k := x]) y ∈ lms-bucket α T b'
        by simp-all
      with ‹x ∉ lms-bucket α T b'›
      have y ∈ set SA
      by (metis in-set-conv-nth length-list-update nth-list-update-eq nth-list-update-neq)
      then show y ∈ {i |i. i ∈ set SA ∧ i ∈ lms-bucket α T b'}
        by (simp add: ‹y ∈ lms-bucket α T b'›)
    next
      fix y
```

```
    assume y ∈ {i |i. i ∈ set SA ∧ i ∈ lms-bucket α T b'}
    hence y ∈ set SA y ∈ lms-bucket α T b'
      by simp-all
    with ‹x ∉ lms-bucket α T b'› ‹SA ! k = length T›
    have y ∈ set (SA[k := x])
     using in-set-list-update abs-is-lms-imp-less-length lms-bucket-def by fastforce
    then show y ∈ {i |i. i ∈ set (SA[k := x]) ∧ i ∈ lms-bucket α T b'}
      using ‹y ∈ lms-bucket α T b'› by blast
  qed
  hence num-lms-types α T (SA[k := x]) b' = num-lms-types α T SA b'
    by (simp add: num-lms-types-def)
  with lms-bucket-ptr-invD[OF assms(4) ‹b' ≤ α (Max (set T))›] ‹b' ≠ b›
  show ?thesis
    by simp
qed
moreover
have b' = b ⟹ ?goal
proof −
  assume b' = b
  with ‹b' ≤ α (Max (set T))›
  have b ≤ α (Max (set T))
    by simp

  from assms(3,9)
  have x ∉ set SA
    by (simp add: abs-is-lms-imp-less-length lms-distinct-inv-def)

  from ‹k < length SA›
  have x ∈ set (SA[k := x])
    by (simp add: set-update-memI)

  have b < length B
    using ‹b ≤ α (Max (set T))› assms(7) dual-order.strict-trans2 by blast
  hence B[b := k] ! b = k
    by simp

  have finite (cur-lms-types α T SA b)
    by (meson List.finite-set cur-lms-subset-SA finite-subset)
  moreover
  from ‹x ∉ set SA›
  have cur-lms-types α T SA b − {x} = cur-lms-types α T SA b
    using cur-lms-subset-SA by fastforce
  moreover
  have insert x (cur-lms-types α T SA b) = cur-lms-types α T (SA[k := x]) b
    unfolding cur-lms-types-def
  proof (intro equalityI subsetI)
    fix y
    assume y ∈ insert x {i |i. i ∈ set SA ∧ i ∈ lms-bucket α T b}
    hence y = x ∨ y ∈ set SA ∧ y ∈ lms-bucket α T b
```

by *blast*
with ‹*SA* ! *k* = *length T*› ‹*x* ∈ *lms-bucket α T b*› ‹*x* ∈ *set* (*SA*[*k* := *x*])›
have *y* ∈ *set* (*SA*[*k* := *x*]) ∧ *y* ∈ *lms-bucket α T b*
by (*metis* (*no-types*, *lifting*) *in-set-list-update abs-is-lms-imp-less-length less-irrefl-nat*
lms-bucket-def mem-Collect-eq)
then show *y* ∈ {*i* |*i*. *i* ∈ *set* (*SA*[*k* := *x*]) ∧ *i* ∈ *lms-bucket α T b*}
by *blast*
next
fix *y*
assume *y* ∈ {*i* |*i*. *i* ∈ *set* (*SA*[*k* := *x*]) ∧ *i* ∈ *lms-bucket α T b*}
hence *y* ∈ *set* (*SA*[*k* := *x*]) *y* ∈ *lms-bucket α T b*
by *simp-all*
moreover
have *y* ∈ *set SA* ⟹ *y* ∈ *insert x* {*i* |*i*. *i* ∈ *set SA* ∧ *i* ∈ *lms-bucket α T b*}
using *calculation*(*2*) **by** *blast*
moreover
from ‹*k* < *length SA* ∧ *SA* ! *k* = *length T*›
have *y* ∉ *set SA* ⟹ *y* ∈ *insert x* {*i* |*i*. *i* ∈ *set SA* ∧ *i* ∈ *lms-bucket α T b*}
by (*metis* (*no-types*, *lifting*) *calculation*(*1*) *in-set-conv-nth insert-iff length-list-update nth-list-update*)
ultimately show *y* ∈ *insert x* {*i* |*i*. *i* ∈ *set SA* ∧ *i* ∈ *lms-bucket α T b*}
by *blast*
qed
ultimately
have *num-lms-types α T* (*SA*[*k* := *x*]) *b* = *Suc* (*num-lms-types α T SA b*)
by (*metis num-lms-types-def card.insert-remove*)
with ‹*b*′ = *b*› ‹*B*[*b* := *k*] ! *b* = *k*› *assms*(*2*)
have *B*[*b* := *k*] ! *b*′ + *num-lms-types α T* (*SA*[*k* := *x*]) *b*′
= *B* ! *b* − *Suc 0* + *Suc* (*num-lms-types α T SA b*)
by *simp*
hence *B*[*b* := *k*] ! *b*′ + *num-lms-types α T* (*SA*[*k* := *x*]) *b*′ = *B* ! *b* + *num-lms-types α T SA b*
using *add-Suc-shift assms less-nat-zero-code lms-distinct-bucket-ptr-lower-bound neq0-conv*
by *fastforce*
with ‹*b*′ = *b*›
have *B*[*b* := *k*] ! *b*′ + *num-lms-types α T* (*SA*[*k* := *x*]) *b*′ = *B* ! *b*′ + *num-lms-types α T SA b*′
by *simp*
with *lms-bucket-ptr-invD*[*OF assms*(*4*) ‹*b*′ ≤ *α* (*Max* (*set T*))›]
show *?thesis*
by *simp*
qed
ultimately
show *?goal*
by *blast*
qed

62.4.3 Unknowns

lemma *lms-unknowns-inv-established*:
assumes *lms-bucket-init* α T B
and $\forall i < length\ SA.\ SA\ !\ i = length\ T$
and $length\ SA = length\ T$
shows *lms-unknowns-inv* α T B SA
unfolding *lms-unknowns-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix b i
assume $b \leq \alpha\ (Max\ (set\ T))$ *lms-bucket-start* α T $b \leq i$ $i < B\ !\ b$
with *lms-bucket-initD*[*OF assms*(*1*)]
have $B\ !\ b = bucket\text{-}end\ \alpha\ T\ b$
by *simp*
with ‹$i < B\ !\ b$›
have $i < length\ T$
by (*simp add*: *bucket-end-le-length less-le-trans*)
with *assms*(*3*)
have $i < length\ SA$
by *simp*
with *assms*(*2*)
show $SA\ !\ i = length\ T$
by *simp*
qed

lemma *lms-unknowns-inv-maintained-step*:
assumes $b = \alpha\ (T\ !\ x)$
and $k = B\ !\ b - Suc\ 0$
and *lms-distinct-inv* T SA $(x\ \#\ LMS)$
and *lms-bucket-ptr-inv* α T B SA
and *lms-unknowns-inv* α T B SA
and *strict-mono* α
and $\alpha\ (Max\ (set\ T)) < length\ B$
and $\forall a \in set\ (x\ \#\ LMS).\ abs\text{-}is\text{-}lms\ T\ a$
shows *lms-unknowns-inv* α T $(B[b := k])$ $(SA[k := x])$
unfolding *lms-unknowns-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix b' i
assume $b' \leq \alpha\ (Max\ (set\ T))$ *lms-bucket-start* α T $b' \leq i$ $i < B[b := k]\ !\ b'$

let *?goal* $= SA[k := x]\ !\ i = length\ T$

have $b' \neq b \Longrightarrow$ *?goal*
proof −
assume $b' \neq b$
with ‹$i < B[b := k]\ !\ b'$›
have $i < B\ !\ b'$
by *simp*
with *lms-unknowns-invD*[*OF assms*(*5*) ‹$b' \leq \alpha\ (Max\ (set\ T))$› ‹*lms-bucket-start* α T $b' \leq i$›]

```
    have SA ! i = length T
      by simp

    from ‹b' ≠ b›
    have b' < b ∨ b < b'
      by auto
    then show ?thesis
    proof
      assume b' < b
      from lms-distinct-bucket-ptr-lower-bound[OF assms(1,3,4,6,8)]
      have lms-bucket-start α T b < B ! b .
      hence bucket-start α T b < B ! b
      by (metis dual-order.strict-trans2 l-bucket-end-def l-bucket-end-le-lms-bucket-start
le-add1)
      moreover
      from lms-bucket-ptr-upper-bound[OF assms(4) ‹b' ≤ α (Max (set T))›]
      have B ! b' ≤ bucket-end α T b' .
      ultimately
      have B ! b' < B ! b
        by (meson ‹b' < b› dual-order.strict-trans2 less-bucket-end-le-start)
      hence i ≠ k
        using assms(2,3) ‹i < B ! b'›
        by linarith
      with ‹SA ! i = length T›
      show ?thesis
        by auto
    next
      assume b < b'
      from ‹lms-bucket-start α T b' ≤ i›
      have bucket-start α T b' ≤ i
        by (metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans)
      moreover
      from lms-bucket-ptr-upper-bound[OF assms(4), of b] assms(7)
      have B ! b ≤ bucket-end α T b
        using ‹b < b'› ‹b' ≤ α (Max (set T))› by linarith
      hence k < bucket-end α T b
        using assms lms-distinct-bucket-ptr-lower-bound by fastforce
      ultimately
      have i ≠ k
        by (meson ‹b < b'› leD le-trans less-bucket-end-le-start)
      with ‹SA ! i = length T›
      show ?thesis
        by auto
    qed
  qed
  moreover
  have b' = b ⟹ ?goal
  proof −
    assume b' = b
```

with ‹$b' \leq \alpha\ (Max\ (set\ T))$› ‹*lms-bucket-start* $\alpha\ T\ b' \leq i$› ‹$i < B[b := k]\ !\ b'$›
have $b \leq \alpha\ (Max\ (set\ T))$ *lms-bucket-start* $\alpha\ T\ b \leq i\ i < B[b := k]\ !\ b$
by *simp-all*
hence $i < B\ !\ b - Suc\ 0$
using *assms* **by** *auto*
with *lms-unknowns-invD*[*OF assms*(*5*) ‹$b \leq \alpha\ (Max\ (set\ T))$› ‹*lms-bucket-start* $\alpha\ T\ b \leq i$›]
have $SA\ !\ i = length\ T$
by *linarith*
with ‹$i < B\ !\ b - Suc\ 0$› *assms*(*2*)
show *?thesis*
by *simp*
qed
ultimately
show *?goal*
by *blast*
qed

62.4.4 Locations

lemma *lms-locations-inv-established*:
assumes *lms-bucket-init* $\alpha\ T\ B$
shows *lms-locations-inv* $\alpha\ T\ B\ SA$
unfolding *lms-locations-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix $b\ i$
assume $b \leq \alpha\ (Max\ (set\ T))\ B\ !\ b \leq i\ i < bucket\text{-}end\ \alpha\ T\ b$
with *lms-bucket-initD*[*OF assms*(*1*), *of b*]
have *False*
by *linarith*
then show $SA\ !\ i \in lms\text{-}bucket\ \alpha\ T\ b$
by *blast*
qed

lemma *lms-locations-inv-maintained-step*:
assumes $b = \alpha\ (T\ !\ x)$
and $k = (B\ !\ b) - Suc\ 0$
and *lms-distinct-inv* $T\ SA\ (x\ \#\ LMS)$
and *lms-bucket-ptr-inv* $\alpha\ T\ B\ SA$
and *lms-locations-inv* $\alpha\ T\ B\ SA$
and *strict-mono* α
and $\alpha\ (Max\ (set\ T)) < length\ B$
and $length\ SA = length\ T$
and $\forall a \in set\ (x\ \#\ LMS).\ abs\text{-}is\text{-}lms\ T\ a$
shows *lms-locations-inv* $\alpha\ T\ (B[b := k])\ (SA[k := x])$
unfolding *lms-locations-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix $b'\ i$
assume $b' \leq \alpha\ (Max\ (set\ T))\ B[b := k]\ !\ b' \leq i\ i < bucket\text{-}end\ \alpha\ T\ b'$

```
let ?goal = SA[k := x] ! i ∈ lms-bucket α T b'

have lms-bucket-start α T b < B ! b
  using assms lms-distinct-bucket-ptr-lower-bound by blast

have b' ≠ b ⟹ ?goal
proof −
  assume b' ≠ b
  with ‹B[b := k] ! b' ≤ i›
  have B ! b' ≤ i
    by simp

  from ‹b' ≠ b›
  have b' < b ∨ b < b'
    by auto
  then show ?thesis
  proof
    assume b' < b
    from ‹lms-bucket-start α T b < B ! b›
    have bucket-start α T b < B ! b
    by (metis dual-order.strict-trans2 l-bucket-end-def l-bucket-end-le-lms-bucket-start
              le-add1)
    with ‹i < bucket-end α T b'› ‹b' < b›
    have i ≠ k
      using assms(2,3)
      by (metis Suc-lessI Suc-pred dual-order.strict-trans1 less-bucket-end-le-start
              less-nat-zero-code not-less-iff-gr-or-eq)
    hence SA[k := x] ! i = SA ! i
      by auto
    with lms-locations-invD[OF assms(5) ‹b' ≤ α (Max (set T))› ‹B ! b' ≤ i›
                               ‹i < bucket-end α T b'›]
    show ?thesis
      by simp
  next
    assume b < b'
    from lms-bucket-ptr-lower-bound[OF assms(4) ‹b' ≤ α (Max (set T))›]
    have lms-bucket-start α T b' ≤ B ! b' .
    hence bucket-start α T b' ≤ B ! b'
      by (metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans)
    hence bucket-start α T b' ≤ i
      using ‹B ! b' ≤ i› le-trans by blast
    moreover
    from lms-bucket-ptr-upper-bound[OF assms(4), of b] assms(7)
    have B ! b ≤ bucket-end α T b
      using ‹b < b'› ‹b' ≤ α (Max (set T))› by linarith
    with ‹lms-bucket-start α T b < B ! b›
    have k < bucket-end α T b
      using assms(2) by linarith
```

```
      ultimately
      have i ≠ k
        by (meson ‹b < b'› leD le-less-trans less-bucket-end-le-start)
      hence SA[k := x] ! i = SA ! i
        by simp
      with lms-locations-invD[OF assms(5)‹b' ≤ α (Max (set T))› ‹B ! b' ≤ i›
                              ‹i < bucket-end α T b'›]
      show ?thesis
        by simp
    qed
  qed
  moreover
  have b' = b ⟹ ?goal
  proof −
    assume b' = b
    with ‹b' ≤ α (Max (set T))› ‹B[b := k] ! b' ≤ i› ‹i < bucket-end α T b'›
    have b ≤ α (Max (set T)) B[b := k] ! b ≤ i i < bucket-end α T b
      by simp-all
    hence k ≤ i
      by (simp add: assms(7) order.strict-trans1)
    hence k = i ∨ B ! b ≤ i
      using assms(2) by linarith
    then show ?thesis
    proof
      assume k = i
      hence SA[k := x] ! i = x
     using ‹i < bucket-end α T b› assms(8) bucket-end-le-length order.strict-trans2
by fastforce
      moreover
      from assms(1,9)
      have x ∈ lms-bucket α T b
        by (simp add: bucket-def abs-is-lms-imp-less-length lms-bucket-def)
      ultimately
      show ?thesis
        using ‹b' = b› by simp
    next
      assume B ! b ≤ i
     with lms-locations-invD[OF assms(5) ‹b ≤ α (Max (set T))› - ‹i < bucket-end
α T b›]
      have SA ! i ∈ lms-bucket α T b
        by blast
      moreover
      from ‹B ! b ≤ i› assms(2) ‹lms-bucket-start α T b < B ! b›
      have SA[k := x] ! i = SA ! i
        by auto
      ultimately
      show ?thesis
        using ‹b' = b› by simp
    qed
```

qed
ultimately
show *?goal*
by *blast*
qed

62.4.5 Unchanged

lemma *lms-unchanged-inv-established*:
lms-unchanged-inv α T B SA SA
unfolding *lms-unchanged-inv-def*
by *blast*

lemma *lms-unchanged-inv-maintained-step*:
assumes $b = \alpha\ (T\ !\ x)$
and $k = (B\ !\ b) - Suc\ 0$
and *lms-distinct-inv T SA* $(x\ \#\ LMS)$
and *lms-bucket-ptr-inv α T B SA*
and *lms-unchanged-inv α T B SA0 SA*
and *strict-mono α*
and $\alpha\ (Max\ (set\ T)) < length\ B$
and $length\ SA = length\ T$
and $\forall\, a \in set\ (x\ \#\ LMS).\ abs\text{-}is\text{-}lms\ T\ a$
shows *lms-unchanged-inv α T* $(B[b := k])$ *SA0* $(SA[k := x])$
unfolding *lms-unchanged-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix $b'\ i$
assume $b' \leq \alpha\ (Max\ (set\ T))$ *bucket-start α T* $b' \leq i\ i < B[b := k]\ !\ b'$

let *?goal* $= SA[k := x]\ !\ i = SA0\ !\ i$

have *lms-bucket-start α T* $b < B\ !\ b$
using *assms lms-distinct-bucket-ptr-lower-bound* **by** *blast*

have $b' \neq b \Longrightarrow$ *?goal*
proof −
assume $b' \neq b$
with ‹$i < B[b := k]\ !\ b'$›
have $i < B\ !\ b'$
by *simp*

from ‹$b' \neq b$›
have $b' < b \vee b < b'$
by *linarith*
then show *?thesis*
proof
assume $b' < b$
from ‹*lms-bucket-start α T* $b < B\ !\ b$›
have *bucket-start α T* $b < B\ !\ b$

```
      by (simp add: lms-bucket-start-def)
    with assms(2)
    have bucket-start α T b ≤ k
      by linarith
    moreover
    from lms-bucket-ptr-upper-bound[OF assms(4) ‹b' ≤ α (Max (set T))›]
    have B ! b' ≤ bucket-end α T b' .
    with ‹i < B ! b'›
    have i < bucket-end α T b'
      using less-le-trans by blast
    ultimately
    have i ≠ k
      using ‹b' < b›
    by (meson dual-order.strict-trans2 less-bucket-end-le-start order.strict-implies-not-eq)
    hence SA[k := x] ! i = SA ! i
      by simp
    with lms-unchanged-invD[OF assms(5) ‹b' ≤ α (Max (set T))› ‹bucket-start
α T b' ≤ i›
                            ‹i < B ! b'›]
    show ?thesis
      by simp
  next
    assume b < b'
    from ‹lms-bucket-start α T b < B ! b› assms(2)
    have k < B ! b
      by linarith
    with lms-bucket-ptr-upper-bound[OF assms(4), of b] ‹b' ≤ α (Max (set T))›
‹b < b'› assms(7)
    have k < bucket-end α T b
      by linarith
    with ‹bucket-start α T b' ≤ i› ‹b < b'›
    have i ≠ k
      by (meson dual-order.strict-trans1 leD less-bucket-end-le-start)
    hence SA[k := x] ! i = SA ! i
      by simp
    with lms-unchanged-invD[OF assms(5) ‹b' ≤ α (Max (set T))› ‹bucket-start
α T b' ≤ i›
                            ‹i < B ! b'›]
    show ?thesis
      by simp
  qed
qed
moreover
have b' = b ⟹ ?goal
proof −
  assume b' = b
  with ‹b' ≤ α (Max (set T))› ‹bucket-start α T b' ≤ i› ‹i < B[b := k] ! b'›
  have b ≤ α (Max (set T)) bucket-start α T b ≤ i i < B[b := k] ! b
    by simp-all
```

hence $i < k$
using *assms*(7) **by** *auto*
hence $i < B\ !\ b$
using *assms*(2) **by** *linarith*
with *lms-unchanged-invD*[*OF assms*(5) ‹$b \leq \alpha\ (Max\ (set\ T))$› ‹*bucket-start* α $T\ b \leq i$›]
have $SA\ !\ i = SA0\ !\ i$
by *simp*
with ‹$i < k$›
show *?thesis*
by *auto*
qed
ultimately
show *?goal*
by *blast*
qed

62.4.6 Inserted

lemma *lms-inserted-inv-established*:
shows *lms-inserted-inv LMS SA* [] *LMS*
unfolding *lms-inserted-inv-def*
by *simp*

lemma *lms-inserted-inv-maintained-step*:
assumes $b = \alpha\ (T\ !\ x)$
and $k = (B\ !\ b) - Suc\ 0$
and *lms-distinct-inv T SA* $(x\ \#\ LMSb)$
and *lms-bucket-ptr-inv* α *T B SA*
and *lms-unknowns-inv* α *T B SA*
and *lms-inserted-inv LMS SA LMSa* $(x\ \#\ LMSb)$
and *strict-mono* α
and *length SA* = *length T*
and $\forall\, a \in set\ LMS.\ abs\text{-}is\text{-}lms\ T\ a$
shows *lms-inserted-inv LMS* $(SA[k := x])$ $(LMSa\ @\ [x])$ *LMSb*
proof −

from *lms-inserted-invD*(1)[*OF assms*(6)] *assms*(9)
have $\forall\, a \in set\ (x\ \#\ LMSb).\ abs\text{-}is\text{-}lms\ T\ a$
by *auto*
with *lms-next-insert-at-unknown*[*OF assms*(1−5,7,8)]
have $k < length\ SA\ SA\ !\ k = length\ T$
by *blast*+

from *lms-inserted-invD*(1)[*OF assms*(6)] *assms*(9)
have $\forall\, a \in set\ LMSa.\ abs\text{-}is\text{-}lms\ T\ a$
by *auto*
hence $\forall\, a \in set\ LMSa.\ a < length\ T$
using *abs-is-lms-imp-less-length* **by** *blast*

```
  have set LMSa ⊆ set (SA[k := x])
  proof (intro subsetI)
    fix y
    assume y ∈ set LMSa
    with ‹∀ a ∈ set LMSa. a < length T›
    have y < length T
      by blast
    with ‹SA ! k = length T›
    have SA ! k ≠ y
      by simp
    moreover
    from lms-inserted-invD(2)[OF assms(6)] ‹y ∈ set LMSa›
    have y ∈ set SA
      by blast
    ultimately
    show y ∈ set (SA[k := x])
      using in-set-list-update by fast
  qed
  moreover
  from ‹k < length SA›
  have x ∈ set (SA[k := x])
    by (simp add: set-update-memI)
  ultimately
  have set (LMSa @ [x]) ⊆ set (SA[k := x])
    by simp
  with lms-inserted-invD(1)[OF assms(6)]
  show ?thesis
    by (simp add: lms-inserted-inv-def)
qed
```

62.4.7 Sorted

```
lemma lms-sorted-inv-established:
  assumes ∀ i < length SA. SA ! i = length T
  and     ∀ a ∈ set LMS. abs-is-lms T a
shows lms-sorted-inv T LMS SA
  unfolding lms-sorted-inv-def
proof (intro allI impI; elim conjE)
  fix i j
  assume A: j < length SA i < j  SA ! i ∈ set LMS SA ! j ∈ set LMS
  from A(1) assms(1)
  have SA ! j = length T
    by blast
  moreover
  from A(4) assms(2)
  have SA ! j < length T
    using abs-is-lms-imp-less-length by blast
  ultimately
```

```
  have False
    by auto
  then show
    (T ! (SA ! i) ≠ T ! (SA ! j) ⟶ T ! (SA ! i) < T ! (SA ! j)) ∧
     (T ! (SA ! i) = T ! (SA ! j) ⟶
      (∃j'<length LMS. ∃i'<j'. LMS ! i' = SA ! j ∧ LMS ! j' = SA ! i))
    by blast
qed

lemma lms-sorted-inv-maintained-step:
  assumes b = α (T ! x)
  and     k = (B ! b) - Suc 0
  and     lms-distinct-inv T SA (x # LMSb)
  and     lms-bucket-ptr-inv α T B SA
  and     lms-unknowns-inv α T B SA
  and     lms-locations-inv α T B SA
  and     lms-unchanged-inv α T B SA0 SA
  and     lms-inserted-inv LMS SA LMSa (x # LMSb)
  and     lms-sorted-inv T LMS SA
  and     strict-mono α
  and     length SA = length T
  and     ∀i < length T. SA0 ! i = length T
  and     ∀a ∈ set LMS. abs-is-lms T a
shows lms-sorted-inv T LMS (SA[k := x])
  unfolding lms-sorted-inv-def
proof (intro allI impI; elim conjE)
  fix i j
  assume A: j < length (SA[k := x]) i < j SA[k := x] ! i ∈ set LMS
          SA[k := x] ! j ∈ set LMS
  let ?goal1 = T ! (SA[k := x] ! i) ≠ T ! (SA[k := x] ! j) ⟶
               T ! (SA[k := x] ! i) < T ! (SA[k := x] ! j)
  and ?goal2 = T ! (SA[k := x] ! i) = T ! (SA[k := x] ! j) ⟶
               (∃j'<length LMS. ∃i'<j'.
                 LMS ! i' = SA[k := x] ! j ∧ LMS! j' = SA[k := x] ! i)

  let ?goal = ?goal1 ∧ ?goal2

  from assms(13) lms-inserted-invD[OF assms(8)]
  have ∀a∈set (x # LMSb). abs-is-lms T a
    by auto
  with lms-next-insert-at-unknown[OF assms(1-5,10,11)]
  have k < length SA SA ! k = length T
    by blast+

 from lms-distinct-bucket-ptr-lower-bound[OF assms(1,3,4,10) ‹∀a∈set (x # LMSb).
abs-is-lms T a›]
  have lms-bucket-start α T b < B ! b .

  from assms(1,10) ‹∀a∈set (x # LMSb). abs-is-lms T a›
```

```
  have b ≤ α (Max (set T))
    by (simp add: abs-is-lms-imp-less-length strict-mono-less-eq)

  have ⟦k ≠ i; k ≠ j⟧ ⟹ ?goal
  proof −
    assume k ≠ i k ≠ j
    with A
    have j < length SA SA ! i ∈ set LMS SA ! j ∈ set LMS
      by simp-all
    with lms-sorted-invD[OF assms(9) - ‹i < j›] ‹k ≠ i› ‹k ≠ j›
    show ?goal
      by simp
  qed
  moreover
  have ⟦k = i; k ≠ j⟧ ⟹ ?goal
  proof −
    assume k = i k ≠ j
    with A(1,4)
    have j < length SA SA ! j ∈ set LMS SA[k := x] ! j = SA ! j
      by simp-all

    from ‹k = i› assms(2)
    have i = B ! b − Suc 0
      by simp

    from ‹SA ! j ∈ set LMS› assms(13)
    have SA ! j < length T
      using abs-is-lms-imp-less-length by blast

     from index-in-bucket-interval-gen[of j T, OF - assms(10)] assms(11) ‹j <
length SA›
    obtain b′ where
      b′ ≤ α (Max (set T))
      bucket-start α T b′ ≤ j
      j < bucket-end α T b′
      by auto

    have B ! b′ ≤ j
    proof (rule ccontr)
      assume ¬ B ! b′ ≤ j
      hence j < B ! b′
        by simp
      with lms-unchanged-invD[OF assms(7) ‹b′ ≤ α (Max (set T))› ‹bucket-start
α T b′ ≤ j›]
          assms(11,12) ‹j < length SA›
      have SA ! j = length T
        by auto
      with ‹SA ! j < length T›
      show False
```

by *auto*
qed
with *lms-locations-invD*[*OF assms*(*6*) ‹$b' \leq \alpha\ (Max\ (set\ T))$› - ‹$j < bucket\text{-}end\ \alpha\ T\ b'$›]
have $SA\ !\ j \in lms\text{-}bucket\ \alpha\ T\ b'$
by *blast*
hence $\alpha\ (T\ !\ (SA\ !\ j)) = b'$
by (*simp add*: *bucket-def lms-bucket-def*)

from ‹*lms-bucket-start* $\alpha\ T\ b < B\ !\ b$› ‹$i = B\ !\ b - Suc\ 0$›
have *lms-bucket-start* $\alpha\ T\ b \leq i$
by *linarith*
hence *bucket-start* $\alpha\ T\ b \leq i$
by (*metis l-bucket-end-def l-bucket-end-le-lms-bucket-start le-add1 le-trans*)

have $b \leq b'$
proof (*rule ccontr*)
assume $\neg\ b \leq b'$
hence $b' < b$
by *simp*
hence *bucket-end* $\alpha\ T\ b' \leq$ *bucket-start* $\alpha\ T\ b$
by (*simp add*: *less-bucket-end-le-start*)
with ‹$j <$ *bucket-end* $\alpha\ T\ b'$›
have $j <$ *bucket-start* $\alpha\ T\ b$
by *linarith*
with ‹*bucket-start* $\alpha\ T\ b \leq i$›
have $j < i$
by *linarith*
with ‹$i < j$›
show *False*
by *linarith*
qed

from *assms*(*3*)[*simplified lms-distinct-inv-def distinct-append*] ‹$SA\ !\ j <$ *length* T›
‹$j <$ *length* SA›
have $SA\ !\ j \notin set\ (x\ \#\ LMSb)$
by (*simp add*: *disjoint-iff*)
with *lms-inserted-invD*[*OF assms*(*8*)] ‹$SA\ !\ j \in set\ LMS$›
have $SA\ !\ j \in set\ LMSa$
by *auto*
hence $\exists j' <$ *length LMSa*. $LMSa\ !\ j' = SA\ !\ j$
by (*simp add*: *in-set-conv-nth*)
then obtain j' **where**
$j' <$ *length LMSa*
$LMSa\ !\ j' = SA\ !\ j$
by *blast*
with *lms-inserted-invD*(*1*)[*OF assms*(*8*)] ‹$SA[k := x]\ !\ j = SA\ !\ j$›
have $LMS\ !\ j' = SA[k := x]\ !\ j$

by (*simp add*: *nth-append*)

from ‹α (T ! (SA ! j)) = b'› ‹$SA[k := x]$! j = SA ! j›
have α (T ! ($SA[k := x]$! j)) = b'
by *simp*

from *lms-inserted-invD*(*1*)[*OF assms*(*8*)]
have *length LMSa* < *length LMS*
by *simp*

from *assms*(*3*)[*simplified lms-distinct-inv-def distinct-append*] *lms-inserted-invD*[*OF assms*(*8*)]
have $x \notin$ *set LMSa*
using ‹$\forall$ *a*∈*set* (*x* # *LMSb*). *abs-is-lms T a*› *abs-is-lms-imp-less-length* **by** *fastforce*
with *lms-inserted-invD*(*1*)[*OF assms*(*8*)]
have *LMS* ! *length LMSa* = *x*
by *simp*
hence *LMS* ! *length LMSa* = $SA[k := x]$! i
using ‹k < *length SA*› ‹k = i› **by** *auto*

from ‹k = i› *assms*(*1*) ‹k < *length SA*›
have α (T ! ($SA[k := x]$! i)) = b
by *auto*

have b = b' ⟹ *?goal2*
proof
from ‹*LMS* ! j' = $SA[k := x]$! j› ‹*LMS* ! *length LMSa* = $SA[k := x]$! i› ‹j' < *length LMSa*›
‹*length LMSa* < *length LMS*›
show ∃j'<*length LMS*. ∃i'<j'. *LMS* ! i' = $SA[k := x]$! j ∧ *LMS* ! j' = $SA[k := x]$! i
by *blast*
qed
moreover
from ‹α (T ! ($SA[k := x]$! j)) = b'› ‹α (T ! ($SA[k := x]$! i)) = b› *assms*(*10*)
have b = b' ⟹ T ! ($SA[k := x]$! i) = T ! ($SA[k := x]$! j)
using *strict-mono-eq* **by** *auto*
moreover
have b < b' ⟹ *?goal1*
proof
assume b < b'
with ‹α (T ! ($SA[k := x]$! i)) = b› ‹α (T ! ($SA[k := x]$! j)) = b'› *assms*(*10*)
show T ! ($SA[k := x]$! i) < T ! ($SA[k := x]$! j)
using *strict-mono-less* **by** *blast*
qed
moreover
from ‹α (T ! ($SA[k := x]$! j)) = b'› ‹α (T ! ($SA[k := x]$! i)) = b› *assms*(*10*)
have b < b' ⟹ T ! ($SA[k := x]$! i) ≠ T ! ($SA[k := x]$! j)

```
    by auto
  moreover
  from ‹b ≤ b'›
  have b = b' ∨ b < b'
    by linarith
  ultimately
  show ?goal
    by blast
qed
moreover
have ⟦k ≠ i; k = j⟧ ⟹ ?goal
proof −
  assume k ≠ i k = j
  with ‹SA[k := x] ! i ∈ set LMS›
  have SA[k := x] ! i = SA ! i SA ! i ∈ set LMS
    by simp-all

  from ‹k = j› assms(2)
  have j = B ! b − Suc 0
    by simp

  from ‹j < length (SA[k := x])› ‹i < j› assms(11)
  have i < length T
    by simp
  with index-in-bucket-interval-gen[of i T, OF - assms(10)]
  obtain b' where
    b' ≤ α (Max (set T))
    bucket-start α T b' ≤ i
    i < bucket-end α T b'
    by auto

  from ‹SA ! i ∈ set LMS› assms(13)
  have SA ! i < length T
    using abs-is-lms-imp-less-length by blast

  have B ! b' ≤ i
  proof (rule ccontr)
    assume ¬B ! b' ≤ i
    hence i < B ! b'
      by (simp add: not-le)
    with lms-unchanged-invD[OF assms(7) ‹b' ≤ α (Max (set T))› ‹bucket-start
α T b' ≤ i›]
        assms(12) ‹i < length T›
    have SA ! i = length T
      by simp
    with ‹SA ! i < length T›
    show False
      by linarith
  qed
```

with *lms-locations-invD*[*OF assms*(6) ‹$b' \leq \alpha\ (Max\ (set\ T))$› - ‹$i < bucket\text{-}end\ \alpha\ T\ b'$›]
have $SA\ !\ i \in lms\text{-}bucket\ \alpha\ T\ b'$
by *blast*
hence $\alpha\ (T\ !\ (SA\ !\ i)) = b'$
by (*simp add*: *bucket-def lms-bucket-def*)
with ‹$SA[k := x]\ !\ i = SA\ !\ i$›
have $\alpha\ (T\ !\ (SA[k := x]\ !\ i)) = b'$
by *simp*

from ‹$i < j$› ‹$j = B\ !\ b - Suc\ 0$› ‹$lms\text{-}bucket\text{-}start\ \alpha\ T\ b < B\ !\ b$›
have $i < B\ !\ b$
using *diff-le-self dual-order.strict-trans1* **by** *blast*

have $i < bucket\text{-}start\ \alpha\ T\ b$
proof (*rule ccontr*)
assume $\neg i < bucket\text{-}start\ \alpha\ T\ b$
hence $bucket\text{-}start\ \alpha\ T\ b \leq i$
by (*simp add*: *not-le*)
with *lms-unchanged-invD*[*OF assms*(7) ‹$b \leq \alpha\ (Max\ (set\ T))$› - ‹$i < B\ !\ b$›]
assms(12) ‹$i < length\ T$›
have $SA\ !\ i = length\ T$
by *auto*
with ‹$SA\ !\ i < length\ T$›
show *False*
by *linarith*
qed
with ‹$bucket\text{-}start\ \alpha\ T\ b' \leq i$›
have $bucket\text{-}start\ \alpha\ T\ b' < bucket\text{-}start\ \alpha\ T\ b$
by *linarith*
hence $b' < b$
by (*meson bucket-start-le leD leI*)

from *assms*(1) ‹$k = j$› ‹$k < length\ SA$›
have $\alpha\ (T\ !\ (SA[k := x]\ !\ j)) = b$
by *simp*
with ‹$\alpha\ (T\ !\ (SA[k := x]\ !\ i)) = b'$› ‹$b' < b$› *assms*(10)
have $T\ !\ (SA[k := x]\ !\ i) < T\ !\ (SA[k := x]\ !\ j)$
using *strict-mono-less* **by** *blast*
then show *?goal*
by *auto*
qed
moreover
from ‹$i < j$›
have $[\![k = i;\ k = j]\!] \Longrightarrow$ *?goal*
by *blast*
ultimately
show *?goal*
by *blast*

qed

62.5 Combined Establishment and Maintenance

lemma *lms-inv-established*:
assumes $\forall\, i < \mathit{length}\ \mathit{SA}.\ \mathit{SA}\ !\ i = \mathit{length}\ T$
and $\forall\, x \in \mathit{set}\ \mathit{LMS}.\ \mathit{abs\text{-}is\text{-}lms}\ T\ x$
and *distinct LMS*
and *lms-bucket-init* α T B
and $\mathit{length}\ \mathit{SA} = \mathit{length}\ T$
and *strict-mono* α
shows *lms-inv* α T B *LMS* [] *LMS SA SA*
unfolding *lms-inv-def*
using *lms-distinct-inv-established*[*OF assms*(3,1)]
lms-bucket-ptr-inv-established[*OF assms*(4,1)]
lms-unknowns-inv-established[*OF assms*(4,1,5)]
lms-locations-inv-established[*OF assms*(4)]
lms-unchanged-inv-established
lms-inserted-inv-established
lms-sorted-inv-established[*OF assms*(1,2)]
lms-bucket-init-length[*OF assms*(4)]
assms
by *auto*

lemma *lms-inv-maintained-step*:
assumes *lms-inv* α T B *LMS LMSa* $(x \#\ \mathit{LMSb})$ *SA0 SA*
and $b = \alpha\ (T\ !\ x)$
and $k = (B\ !\ b) - \mathit{Suc}\ 0$
shows *lms-inv* α T $(B[b := k])$ *LMS* $(\mathit{LMSa}\ @\ [x])$ *LMSb SA0* $(\mathit{SA}[k := x])$
unfolding *lms-inv-def*
using *lms-distinct-inv-maintained-step*[*OF lms-invD*(1)[*OF assms*(1)]]
lms-bucket-ptr-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−3,8,9,12)[*OF assms*(1)]
lms-inv-lms-helper(3)[*OF assms*(1)]]
lms-unknowns-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−3,8,9)[*OF assms*(1)]
lms-inv-lms-helper(3)[*OF assms*(1)]]
lms-locations-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−2,4,8,9,12)[*OF assms*(1)]
lms-inv-lms-helper(3)[*OF assms*(1)]]
lms-unchanged-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−2,5,8,9,12)[*OF assms*(1)]
lms-inv-lms-helper(3)[*OF assms*(1)]]
lms-inserted-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−3,6,8,12)[*OF assms*(1)]
lms-inv-lms-helper(1)[*OF assms*(1)]]
lms-sorted-inv-maintained-step[*OF assms*(2−3)
lms-invD(1−8,12,13)[*OF assms*(1)]
lms-inv-lms-helper(1)[*OF assms*(1)]]

by (*metis assms(1) length-list-update lms-inv-def*)

lemma *lms-inv-maintained*:
assumes *bucket-insert-abs'* α *T B SA gs xs* = (*SA'*, *B'*, *gs'*)
and *lms-inv* α *T B LMS gs xs SA0 SA*
shows *lms-inv* α *T B' LMS gs'* [] *SA0 SA'*
using *assms*
proof (*induct arbitrary*: *SA' B' gs' LMS SA0*
rule: *bucket-insert-abs'.induct*[*of* - α *T B SA gs xs*])
case (*1* α *T B SA gs*)
note *BC* = *this*
from *BC*(*1*)[*simplified*]
have *B'* = *B SA'* = *SA gs'* = *gs*
by *auto*
with *BC*(*2*)
show *?case*
by *simp*
next
case (*2* α *T B SA gs x xs*)
note *IH* = *this*
let *?b* = α (*T* ! *x*)
let *?k* = *B* ! *?b* − *Suc 0*
from *lms-inv-maintained-step*[*OF IH*(*3*), *of ?b ?k*]
have *R1*: *lms-inv* α *T* (*B*[*?b* := *?k*]) *LMS* (*gs* @ [*x*]) *xs SA0* (*SA*[*?k* := *x*])
by *simp*

from *IH*(*2*)[*simplified*, *simplified Let-def*]
have *R2*: *bucket-insert-abs'* α *T* (*B*[*?b* := *?k*]) (*SA*[*?k* := *x*]) (*gs* @ [*x*]) *xs* = (*SA'*, *B'*, *gs'*) **.**

from *IH*(*1*)[*of ?b ?k SA*[*?k* := *x*] *B*[*?b* := *?k*] *gs* @ [*x*] *SA' B' gs' LMS SA0*,
simplified,
OF R2 R1]
show *?case* **.**
qed

lemma *lms-inv-holds*:
assumes ∀ *i* < *length SA*. *SA* ! *i* = *length T*
and ∀ *x* ∈ *set LMS*. *abs-is-lms T x*
and *distinct LMS*
and *lms-bucket-init* α *T B*
and *length SA* = *length T*
and *strict-mono* α
and *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs'*)
shows *lms-inv* α *T B' LMS gs'* [] *SA SA'*
using *lms-inv-maintained*[*OF assms*(*7*) *lms-inv-established*[*OF assms*(*1*−*6*)]] **.**

63 Exhaustiveness

definition *lms-type-exhaustive* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list* ⇒ *bool*
where
lms-type-exhaustive T SA = (∀ *i* < *length T*. *abs-is-lms T i* ⟶ *i* ∈ *set SA*)

lemma *lms-type-exhaustiveD*:
⟦*lms-type-exhaustive T SA*; *i* < *length T*; *abs-is-lms T i*⟧ ⟹ *i* ∈ *set SA*
using *lms-type-exhaustive-def* **by** *blast*

lemma *lms-all-inserted-imp-exhaustive*:
assumes *lms-inserted-inv LMS SA LMS* []
and *set LMS* = {*i*. *abs-is-lms T i*}
shows *lms-type-exhaustive T SA*
unfolding *lms-type-exhaustive-def*
proof (*intro allI impI*)
fix *i*
assume *i* < *length T abs-is-lms T i*
with *assms*(*2*)
have *i* ∈ *set LMS*
by *blast*
with *lms-inserted-invD*(*2*)[*OF assms*(*1*)]
show *i* ∈ *set SA*
by *blast*
qed

lemma *lms-type-exhaustive-imp-lms-bucket-subset*:
assumes *lms-type-exhaustive T SA*
and *b* ≤ *α* (*Max* (*set T*))
shows *lms-bucket α T b* ⊆ *set SA*
proof (*intro subsetI*)
fix *x*
assume *x* ∈ *lms-bucket α T b*
hence *x* < *length T*
by (*simp add*: *abs-is-lms-imp-less-length lms-bucket-def*)

from ‹*x* ∈ *lms-bucket α T b*›
have *abs-is-lms T x*
by (*simp add*: *lms-bucket-def*)

from *lms-type-exhaustiveD*[*OF assms*(*1*) ‹*x* < *length T*› ‹*abs-is-lms T x*›]
show *x* ∈ *set SA* **.**
qed

lemma *lms-B-val*:
assumes ∀ *i* < *length SA*. *SA* ! *i* = *length T*
and *distinct LMS*
and *lms-bucket-init α T B*

and *length SA = length T*
and *strict-mono α*
and *set LMS = {i. abs-is-lms T i}*
and *bucket-insert-abs' α T B SA [] LMS = (SA', B', gs')*
and *b ≤ α (Max (set T))*
shows *B' ! b = lms-bucket-start α T b*
proof −
from *assms(6)*
have *∀ x ∈ set LMS. abs-is-lms T x*
by *blast*
with *lms-inv-holds[OF assms(1) - assms(2−5,7)]*
have *lms-inv α T B' LMS gs' [] SA SA'* **.**
hence *lms-inserted-inv LMS SA' gs' []*
using *lms-inv-def* **by** *blast*
hence *gs' = LMS*
by (*simp add*: *lms-inserted-inv-def*)
with ‹*lms-inserted-inv LMS SA' gs' []*› *lms-all-inserted-imp-exhaustive[OF - assms(6), of SA']*
have *lms-type-exhaustive T SA'*
by *simp*
with *lms-type-exhaustive-imp-lms-bucket-subset assms(8)*
have *lms-bucket α T b ⊆ set SA'*
by *blast*
with *cur-lms-subset-lms-bucket*
have *cur-lms-types α T SA' b = lms-bucket α T b*
by (*simp add*: *cur-lms-types-def equalityI subset-eq*)
hence *num-lms-types α T SA' b = card (lms-bucket α T b)*
by (*simp add*: *num-lms-types-def*)
moreover
from ‹*lms-inv α T B' LMS gs' [] SA SA'*›
have *lms-bucket-ptr-inv α T B' SA'*
using *lms-inv-def* **by** *blast*
with *lms-bucket-ptr-invD assms(8)*
have *B' ! b + num-lms-types α T SA' b = bucket-end α T b*
by *blast*
ultimately
have *B' ! b + card (lms-bucket α T b) = bucket-end α T b*
by *simp*
then show *B' ! b = lms-bucket-start α T b*
by (*metis add-implies-diff lms-bucket-pl-size-eq-end lms-bucket-size-def*)
qed

64 Postconditions

definition *lms-vals-post* :: (*'a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *bool*
where
lms-vals-post α T SA =
(*∀ b ≤ α (Max (set T))*.

lms-bucket α *T b* = *set* (*list-slice SA* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
)

lemma *lms-vals-postD*:
⟦*lms-vals-post* α *T SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
lms-bucket α *T b* = *set* (*list-slice SA* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
using *lms-vals-post-def* **by** *blast*

definition
lms-pre :: ('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
lms-pre α *T B SA LMS* ≡
(∀ *i* < *length SA*. *SA* ! *i* = *length T*) ∧
length SA = *length T* ∧
lms-bucket-init α *T B* ∧
strict-mono α ∧
distinct LMS ∧
set LMS = {*i*. *abs-is-lms T i*}

lemma *lms-pre-elims*:
lms-pre α *T B SA LMS* ⟹ ∀ *i* < *length SA*. *SA* ! *i* = *length T*
lms-pre α *T B SA LMS* ⟹ *length SA* = *length T*
lms-pre α *T B SA LMS* ⟹ *lms-bucket-init* α *T B*
lms-pre α *T B SA LMS* ⟹ *strict-mono* α
lms-pre α *T B SA LMS* ⟹ *distinct LMS*
lms-pre α *T B SA LMS* ⟹ *set LMS* = {*i*. *abs-is-lms T i*}
using *lms-pre-def* **by** *blast*+

lemma *lms-vals-post-holds*:
assumes ∀ *i* < *length SA*. *SA* ! *i* = *length T*
and *distinct LMS*
and *lms-bucket-init* α *T B*
and *length SA* = *length T*
and *strict-mono* α
and *set LMS* = {*i*. *abs-is-lms T i*}
and *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs'*)
shows *lms-vals-post* α *T SA'*
unfolding *lms-vals-post-def*
proof(*intro allI impI*)
fix *b*
assume *b* ≤ α (*Max* (*set T*))

from *assms*(*6*)
have ∀ *x* ∈ *set LMS*. *abs-is-lms T x*
by *blast*
with *lms-inv-holds*[*OF assms*(*1*) - *assms*(*2*−*5*,*7*)]

have *R0*:*lms-inv* α *T B' LMS gs'* [] *SA SA'* **.**

from *lms-B-val*[*OF assms*(*1*−*7*) ‹$b \leq \alpha$ (*Max* (*set T*))›]
have *R1*: *B'* ! *b* = *lms-bucket-start* α *T b* **.**

have *bucket-end* α *T b* ≤ *length SA'*
by (*metis R0 bucket-end-le-length lms-inv-def*)

have *finite* (*lms-bucket* α *T b*)
by (*simp add*: *finite-lms-bucket*)
moreover
from *lms-slice-subset-lms-bucket*[*OF lms-invD*(*4*,*12*)[*OF R0*] ‹$b \leq \alpha$ (*Max* (*set T*))›]
have *set* (*list-slice SA'* (*B'* ! *b*) (*bucket-end* α *T b*)) ⊆ *lms-bucket* α *T b* **.**
with *R1*
have *set* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*)) ⊆ *lms-bucket* α *T b*
by *simp*
moreover
from *lms-distinct-slice*[*OF lms-invD*(*1*,*2*,*4*,*12*)[*OF R0*] ‹$b \leq \alpha$ (*Max* (*set T*))›]
have *distinct* (*list-slice SA'* (*B'* ! *b*) (*bucket-end* α *T b*)) **.**
with *R1*
have *distinct* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
by *simp*
with *distinct-card*
have *card* (*set* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*)))
= *length* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
by *blast*
with ‹*bucket-end* α *T b* ≤ *length SA'*›
have *card* (*set* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*)))
= *lms-bucket-size* α *T b*
by (*metis add-diff-cancel-left' length-list-slice lms-bucket-pl-size-eq-end min.absorb-iff1*)
hence *card* (*set* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*)))
= *card* (*lms-bucket* α *T b*)
by (*simp add*: *lms-bucket-size-def*)
ultimately
show *lms-bucket* α *T b* = *set* (*list-slice SA'* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
using *card-subset-eq* **by** *blast*
qed

corollary *abs-bucket-insert-vals*:
assumes *lms-pre* α *T B SA LMS*
shows *lms-vals-post* α *T* (*abs-bucket-insert* α *T B SA LMS*)
proof −
have ∃ *SA' B' gs*. *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)
by (*meson prod-cases3*)
then obtain *SA' B' gs* **where**
A: *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)

by *blast*
hence *abs-bucket-insert α T B SA LMS = SA'*
by (*metis abs-bucket-insert-equiv fst-conv*)
with *lms-vals-post-holds*[*OF lms-pre-elims*(*1*,*5*,*3*,*2*,*4*,*6*)[*OF assms*] *A*]
show *?thesis*
by *simp*
qed

definition *lms-unknowns-post*
where
lms-unknowns-post α T SA =
(∀ *b* ≤ *α* (*Max* (*set T*)).
(∀ *i*. *bucket-start α T b* ≤ *i* ∧ *i* < *lms-bucket-start α T b* ⟶ *SA* ! *i* = *length T*)
)

lemma *lms-unknowns-postD*:
⟦*lms-unknowns-post α T SA*; *b* ≤ *α* (*Max* (*set T*)); *bucket-start α T b* ≤ *i*;
i < *lms-bucket-start α T b*⟧ ⟹
SA ! *i* = *length T*
using *lms-unknowns-post-def* **by** *blast*

lemma *lms-unknowns-post-holds*:
assumes ∀ *i* < *length SA*. *SA* ! *i* = *length T*
and *distinct LMS*
and *lms-bucket-init α T B*
and *length SA* = *length T*
and *strict-mono α*
and *set LMS* = {*i*. *abs-is-lms T i*}
and *bucket-insert-abs' α T B SA* [] *LMS* = (*SA'*, *B'*, *gs'*)
shows *lms-unknowns-post α T SA'*
unfolding *lms-unknowns-post-def*
proof(*intro allI impI*; *elim conjE*)
fix *b i*
assume *b* ≤ *α* (*Max* (*set T*)) *bucket-start α T b* ≤ *i i* < *lms-bucket-start α T b*

from *assms*(*6*)
have ∀ *x* ∈ *set LMS*. *abs-is-lms T x*
by *blast*
with *lms-inv-holds*[*OF assms*(*1*) - *assms*(*2*−*5*,*7*)]
have *R0*:*lms-inv α T B' LMS gs'* [] *SA SA'* **.**

from ‹*i* < *lms-bucket-start α T b*›
have *i* < *length SA*
by (*metis assms*(*4*) *bucket-end-le-length dual-order.strict-trans1 lms-bucket-start-le-bucket-end*)
moreover
from *lms-B-val*[*OF assms*(*1*−*7*) ‹*b* ≤ *α* (*Max* (*set T*))›]
have *B'* ! *b* = *lms-bucket-start α T b* **.**
with ‹*i* < *lms-bucket-start α T b*›

```
  have i < B' ! b
    by simp
  with lms-unchanged-invD[OF lms-invD(5)[OF R0] ‹b ≤ α (Max (set T))› ‹bucket-start
α T b ≤ i›]
  have SA' ! i = SA ! i
    by blast
  ultimately
  show SA' ! i = length T
    using assms(1) by auto
qed

corollary abs-bucket-insert-unknowns:
  assumes lms-pre α T B SA LMS
  shows lms-unknowns-post α T (abs-bucket-insert α T B SA LMS)
proof −
  have ∃ SA' B' gs. bucket-insert-abs' α T B SA [] LMS = (SA', B', gs)
    by (meson prod-cases3)
  then obtain SA' B' gs where
    A: bucket-insert-abs' α T B SA [] LMS = (SA', B', gs)
    by blast
  hence abs-bucket-insert α T B SA LMS = SA'
    by (metis abs-bucket-insert-equiv fst-conv)
  with lms-unknowns-post-holds[OF lms-pre-elims(1,5,3,2,4,6)[OF assms] A]
  show ?thesis
    by simp
qed

corollary abs-bucket-insert-values:
  assumes lms-pre α T B SA LMS
  shows ∀ b ≤ α (Max (set T)).
      (∀ i. bucket-start α T b ≤ i ∧ i < lms-bucket-start α T b ⟶ (abs-bucket-insert
α T B SA LMS) ! i = length T) ∧
          lms-bucket α T b = set (list-slice (abs-bucket-insert α T B SA LMS)
(lms-bucket-start α T b) (bucket-end α T b))
 by (meson assms abs-bucket-insert-unknowns abs-bucket-insert-vals lms-unknowns-postD
lms-vals-postD)

lemma lms-lms-prefix-sorted-holds:
  assumes ∀ i < length SA. SA ! i = length T
  and     distinct LMS
  and     lms-bucket-init α T B
  and     length SA = length T
  and     strict-mono α
  and     set LMS = {i. abs-is-lms T i}
  and     bucket-insert-abs' α T B SA [] LMS = (SA', B', gs')
shows ordlistns.sorted (map (lms-prefix T) (filter (λx. x < length T) SA'))
proof −
  from assms(6)
  have ∀ x ∈ set LMS. abs-is-lms T x
```

```
    by blast
  with lms-inv-holds[OF assms(1) - assms(2−5,7)]
  have R0:lms-inv α T B′ LMS gs′ [] SA SA′ .

  from lms-lms-prefix-sorted[OF lms-invD(2,4,5,8,12,13)[OF R0] assms(6)]
  show ?thesis .
qed

lemma lms-suffix-sorted-holds:
  assumes ∀ i < length SA. SA ! i = length T
  and     distinct LMS
  and     lms-bucket-init α T B
  and     length SA = length T
  and     strict-mono α
  and     set LMS = {i. abs-is-lms T i}
  and     bucket-insert-abs′ α T B SA [] LMS = (SA′, B′, gs′)
  and     ordlistns.sorted (map (suffix T) (rev LMS))
shows ordlistns.sorted (map (suffix T) (filter (λx. x < length T) SA′))
proof −
  from assms(6)
  have ∀ x ∈ set LMS. abs-is-lms T x
    by blast
  with lms-inv-holds[OF assms(1) - assms(2−5,7)]
  have R0:lms-inv α T B′ LMS gs′ [] SA SA′ .

 from lms-suffix-sorted[OF lms-invD(2,4,5,7,8,12,13)[OF R0] assms(6) assms(8)]
  show ?thesis .
qed

lemma lms-bot-is-first:
  assumes ∀ i < length SA. SA ! i = length T
  and     distinct LMS
  and     lms-bucket-init α T B
  and     length SA = length T
  and     strict-mono α
  and     set LMS = {i. abs-is-lms T i}
  and     bucket-insert-abs′ α T B SA [] LMS = (SA′, B′, gs′)
  and     valid-list T
  and     length T = Suc (Suc n)
  and     α bot = 0
shows SA′ ! 0 = Suc n
proof −
  have abs-is-lms T (Suc n)
    by (simp add: assms(8,9) abs-is-lms-last)
  moreover
  have α (T ! (Suc n)) = 0
   by (metis assms(8−10) diff-Suc-1 last-conv-nth length-greater-0-conv valid-list-def)
  ultimately
  have Suc n ∈ lms-bucket α T 0
```

```
    by (simp add: assms(5,8−10) bucket-0 lms-bucket-def)

  have lms-bucket-size α T 0 = Suc 0 l-bucket-size α T 0 = 0 pure-s-bucket-size
α T 0 = 0
    using assms(5,8−10) bucket-0-size2 by blast+
  hence lms-bucket α T 0 = {Suc n}
   by (metis One-nat-def add.commute assms(5,8−10) atLeastLessThan-singleton
bucket-0
        lms-bucket-size-def lms-bucket-subset-bucket plus-1-eq-Suc subset-card-intvl-is-intvl)

  from assms(6)
  have ∀ x ∈ set LMS. abs-is-lms T x
    by blast
  with lms-inv-holds[OF assms(1) - assms(2−5,7)]
  have R0:lms-inv α T B′ LMS gs′ [] SA SA′ .


  from ‹l-bucket-size α T 0 = 0› ‹pure-s-bucket-size α T 0 = 0›
  have lms-bucket-start α T 0 = 0
    by (simp add: bucket-start-0 lms-bucket-start-def)
  moreover
  from lms-B-val[OF assms(1−7), of 0]
  have B′ ! 0 = lms-bucket-start α T 0
    by simp
  moreover
  have 0 < bucket-end α T 0
    by (simp add: assms(5,8,10) valid-list-bucket-end-0)
  ultimately
  show ?thesis
    using lms-locations-invD[OF lms-invD(4)[OF R0], of 0 0] ‹lms-bucket α T 0
= {Suc n}›
    by auto
qed

corollary abs-bucket-insert-bot-first:
  assumes lms-pre α T B SA LMS
  and     valid-list T
  and     length T = Suc (Suc n)
  and     α bot = 0
shows (abs-bucket-insert α T B SA LMS) ! 0 = Suc n
proof −
  have ∃ SA′ B′ gs. bucket-insert-abs′ α T B SA [] LMS = (SA′, B′, gs)
    by (meson prod-cases3)
  then obtain SA′ B′ gs where
    A: bucket-insert-abs′ α T B SA [] LMS = (SA′, B′, gs)
    by blast
  hence abs-bucket-insert α T B SA LMS = SA′
    by (metis abs-bucket-insert-equiv fst-conv)
  with lms-bot-is-first[OF lms-pre-elims(1,5,3,2,4,6)[OF assms(1)] A assms(2−)]
```

show *?thesis*
by *simp*
qed

— Used in SAIS algorithm as part of inducing the prefix ordering based on LMS
theorem *lms-prefix-sorted-bucket*:
assumes *lms-pre* α *T B SA LMS*
and $b \leq \alpha$ (*Max* (*set T*))
shows *ordlistns.sorted* (*map* (*lms-prefix T*)
(*list-slice* (*abs-bucket-insert* α *T B SA LMS*) (*lms-bucket-start* α *T b*)
(*bucket-end* α *T b*)))
(**is** *ordlistns.sorted* (*map ?f ?SA*))
proof −
have $\exists$ *SA' B' gs. bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)
by (*meson prod-cases3*)
then obtain *SA' B' gs* **where**
A: *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)
by *blast*
hence *abs-bucket-insert* α *T B SA LMS* = *SA'*
by (*metis abs-bucket-insert-equiv fst-conv*)

from *lms-vals-postD*[*OF abs-bucket-insert-vals*[*OF assms*(*1*)] *assms*(*2*)]
have *P*: $\forall$ *x* $\in$ *set ?SA* . *x* < *length T*
using *abs-is-lms-imp-less-length lms-bucket-def* **by** *blast*

from *lms-lms-prefix-sorted-holds*[*OF lms-pre-elims*(*1*,*5*,*3*,*2*,*4*,*6*)[*OF assms*(*1*)]
A]
have *ordlistns.sorted* (*map* (*lms-prefix T*) (*filter* (λ*x*. *x* < *length T*) *SA'*)) **.**
hence *ordlistns.sorted* (*map* (*lms-prefix T*) (*filter* (λ*x*. *x* < *length T*) *?SA*))
using ‹*abs-bucket-insert* α *T B SA LMS* = *SA'*› *ordlistns.sorted-map-filter-list-slice*

by *blast*
then show *?thesis*
by (*simp add*: *P*)
qed

— Used in SAIS algorithm as part of inducing the suffix ordering based on LMS
theorem *lms-suffix-sorted-bucket*:
assumes *lms-pre* α *T B SA LMS*
and *ordlistns.sorted* (*map* (*suffix T*) (*rev LMS*))
and $b \leq \alpha$ (*Max* (*set T*))
shows *ordlistns.sorted* (*map* (*suffix T*)
(*list-slice* (*abs-bucket-insert* α *T B SA LMS*) (*lms-bucket-start* α *T b*)
(*bucket-end* α *T b*)))
(**is** *ordlistns.sorted* (*map ?f ?SA*))
proof −
have $\exists$ *SA' B' gs. bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)
by (*meson prod-cases3*)
then obtain *SA' B' gs* **where**

A: *bucket-insert-abs'* α *T B SA* [] *LMS* = (*SA'*, *B'*, *gs*)
by *blast*
hence *abs-bucket-insert* α *T B SA LMS* = *SA'*
by (*metis abs-bucket-insert-equiv fst-conv*)

from *lms-vals-postD*[*OF abs-bucket-insert-vals*[*OF assms*(*1*)] *assms*(*3*)]
have *P*: $\forall x \in$ *set ?SA* . *x* < *length T*
using *abs-is-lms-imp-less-length lms-bucket-def* **by** *blast*

from *lms-suffix-sorted-holds*[*OF lms-pre-elims*(*1*,*5*,*3*,*2*,*4*,*6*)[*OF assms*(*1*)] *A assms*(*2*)]
have *ordlistns.sorted* (*map* (*suffix T*) (*filter* (λ*x*. *x* < *length T*) *SA'*)) **.**
hence *ordlistns.sorted* (*map* (*suffix T*) (*filter* (λ*x*. *x* < *length T*) *?SA*))
using ‹*abs-bucket-insert* α *T B SA LMS* = *SA'*› *ordlistns.sorted-map-filter-list-slice*
by *blast*
then show *?thesis*
by (*simp add*: *P*)
qed

end
theory *Abs-Induce-L-Verification*
imports *../abs−def/Abs-SAIS*
begin

65 Abstract Induce L-types Simple Properties

lemma *abs-induce-l-step-ex*:
∃ *B' SA' i'*. *abs-induce-l-step a b* = (*B'*, *SA'*, *i'*)
by (*cases a*; *cases b*; *clarsimp split*: *prod.splits nat.splits SL-types.splits simp*: *Let-def*)

lemma *abs-induce-l-step-B-length*:
abs-induce-l-step (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*) ⟹ *length B'* = *length B*
by (*clarsimp split*: *prod.splits nat.splits SL-types.splits if-splits simp*: *Let-def*)

lemma *abs-induce-l-step-SA-length*:
abs-induce-l-step (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*) ⟹ *length SA'* = *length SA*
by (*clarsimp split*: *prod.splits nat.splits SL-types.splits if-splits simp*: *Let-def*)

lemma *abs-induce-l-step-Suc*:
∃ *B' SA'*. *abs-induce-l-step* (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *Suc i*)
by (*clarsimp simp*: *Let-def split*: *prod.splits nat.splits SL-types.splits*)

lemma *abs-induce-l-step-B-val-1*:
⟦*length SA* ≤ *i*; *abs-induce-l-step* (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*)⟧ ⟹
B' = *B*
⟦*i* < *length SA*; *length T* ≤ *SA* ! *i*; *abs-induce-l-step* (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*)⟧ ⟹
B' = *B*
⟦*i* < *length SA*; *SA* ! *i* < *length T*; *SA* ! *i* = *0*;

abs-induce-l-step $(B,\ SA,\ i)\ (\alpha,\ T) = (B',\ SA',\ i')\rrbracket \Longrightarrow$
$B' = B$
$\llbracket i < \textit{length } SA;\ SA\ !\ i < \textit{length } T;\ SA\ !\ i = \textit{Suc } j;\ \textit{suffix-type } T\ j = \textit{S-type};$
abs-induce-l-step $(B,\ SA,\ i)\ (\alpha,\ T) = (B',\ SA',\ i')\rrbracket \Longrightarrow$
$B' = B$
by (*clarsimp simp*: *Let-def split*: *prod.splits nat.splits SL-types.splits if-splits*)+

lemma *abs-induce-l-step-B-val-2*:
$\llbracket$*strict-mono* α;
$\alpha\ (\textit{Max } (\textit{set } T)) < \textit{length } B$;
$i < \textit{length } SA$;
$SA\ !\ i < \textit{length } T$;
$SA\ !\ i = \textit{Suc } j$;
suffix-type $T\ j$ = *L-type*;
abs-induce-l-step $(B,\ SA,\ i)\ (\alpha,\ T) = (B',\ SA',\ i')\rrbracket \Longrightarrow$
$B' = B[\alpha\ (T\ !\ j) := \textit{Suc } (B\ !\ \alpha\ (T\ !\ j))]$
by (*clarsimp simp*: *Let-def split*: *prod.splits nat.splits SL-types.splits if-splits*)

lemma *repeat-abs-induce-l-step-index*:
$\exists B'\ SA'.$ *repeat* n *abs-induce-l-step* $(B,\ SA,\ m)\ (\alpha,\ T) = (B',\ SA',\ n + m)$
proof (*induct* n)
case *0*
then show *?case*
by (*simp add*: *repeat-0*)
next
case (*Suc* n)
from *this*
obtain $B'\ SA'$ **where**
A: *repeat* n *abs-induce-l-step* $(B,\ SA,\ m)\ (\alpha,\ T) = (B',\ SA',\ n + m)$
by *blast*

from *repeat-step*[*of* n *abs-induce-l-step* $(B,\ SA,\ m)\ (\alpha,\ T)$]
have B: *repeat* (*Suc* n) *abs-induce-l-step* $(B,\ SA,\ m)\ (\alpha,\ T) =$
abs-induce-l-step (*repeat* n *abs-induce-l-step* $(B,\ SA,\ m)\ (\alpha,\ T))\ (\alpha,\ T)$
by *assumption*

from *abs-induce-l-step-Suc*[*of* $B'\ SA'\ n + m\ \alpha\ T$]
obtain $B''\ SA''$ **where**
abs-induce-l-step $(B',\ SA',\ n + m)\ (\alpha,\ T) = (B'',\ SA'',\ \textit{Suc } (n + m))$
by *blast*
with $A\ B$
show *?case*
by *simp*
qed

lemma *abs-induce-l-step-lengths*:
abs-induce-l-step $(B,\ SA,\ i)\ (\alpha,\ T) = (B',\ SA',\ i') \Longrightarrow$
$\textit{length } B' = \textit{length } B \wedge \textit{length } SA' = \textit{length } SA$
by (*clarsimp split*: *if-splits nat.splits SL-types.splits simp*: *Let-def*)

lemma *repeat-abs-induce-l-step-lengths*:
repeat n abs-induce-l-step (B, SA, i) $(\alpha, T) = (B', SA', i') \Longrightarrow$
length $B' =$ *length* $B \wedge$ *length* $SA' =$ *length* SA
proof −
let $?P = \lambda(a, b, c).$ *length* $a =$ *length* $B \wedge$ *length* $b =$ *length* SA

from *abs-induce-l-step-lengths*
have A: $\bigwedge a.$ $?P\ a \Longrightarrow ?P$ $(abs\text{-}induce\text{-}l\text{-}step\ a\ (\alpha, T))$
by (*clarsimp simp*: *Let-def split*: *prod.splits if-splits nat.splits SL-types.splits*)

assume *repeat n abs-induce-l-step* (B, SA, i) $(\alpha, T) = (B', SA', i')$

with *repeat-maintain-inv*[*of* $?P$ *abs-induce-l-step* (α, T) (B, SA, i) *n*,
OF A]
show *?thesis*
by *auto*
qed

lemma *abs-induce-l-index*:
$\exists B'\ SA'.$ *abs-induce-l-base* α T B $SA = (B', SA',$ *length* $T)$
by (*metis add.right-neutral abs-induce-l-base-def repeat-abs-induce-l-step-index*)

lemma *abs-induce-l-length*:
length (*abs-induce-l* α T B SA) = *length* SA
unfolding *abs-induce-l-def abs-induce-l-base-def*
by (*rule repeat-maintain-inv*)
(*fastforce*
simp del: *abs-induce-l-step.simps*
split: *prod.splits*
dest: *abs-induce-l-step-SA-length*)+

66 Precondition Definitions

definition *lms-init* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
lms-init α T $SA =$
$(\forall\, b \leq \alpha\ (Max\ (set\ T)).$
lms-bucket α T $b =$
set (*list-slice SA* (*lms-bucket-start* α T b) (*bucket-end* α T b))
)

lemma *lms-init-D*:
$[\![$*lms-init* α T *SA*; $b \leq \alpha\ (Max\ (set\ T))]\!] \Longrightarrow$
lms-bucket α T $b =$ *set* (*list-slice SA* (*lms-bucket-start* α T b) (*bucket-end* α T b))
using *lms-init-def* **by** *blast*

lemma *lms-init-nth*:
  ⟦*lms-init* α *T SA*;
    $b \leq \alpha$ (*Max* (*set T*));
    lms-bucket-start α *T b* $\leq i$;
    $i <$ *bucket-end* α *T b*;
    length SA = *length T*⟧ $\Longrightarrow$
    abs-is-lms T (*SA* ! *i*) $\wedge$ α (*T* ! (*SA* ! *i*)) = *b*
  by (*fastforce*
        dest: *lms-init-D list-slice-nth-mem*[**where** *xs* = *SA*]
        simp: *bucket-end-le-length lms-bucket-def bucket-def*)

lemma *lms-init-imp-distinct-bucket*:
  ⟦*lms-init* α *T SA*;
    $b \leq \alpha$ (*Max* (*set T*));
    length SA = *length T*⟧ $\Longrightarrow$
    distinct (*list-slice SA* (*lms-bucket-start* α *T b*) (*bucket-end* α *T b*))
  by (*metis bucket-end-def' min.absorb1 diff-diff-add bucket-end-le-length*
        l-pl-pure-s-pl-lms-size lms-bucket-start-def length-list-slice
        diff-add-inverse lms-init-D lms-bucket-size-def card-distinct)

lemma *lms-init-imp-all-lms-in-SA*:
  assumes *lms-init* α *T SA*
  and *strict-mono* α
  shows {*k* |*k*. *abs-is-lms T k*} $\subseteq$ *set SA*
proof
  fix *x*
  assume *x* $\in$ {*k* |*k*. *abs-is-lms T k*}
  hence *x* $<$ *length T*
    using *abs-is-lms-gre-length le-less-linear* **by** *blast*

  from ‹*x* $\in$ {*k* |*k*. *abs-is-lms T k*}›
  have *abs-is-lms T x*
    by *blast*
  with ‹*x* $<$ *length T*›
  have *x* $\in$ *lms-bucket* α *T* (α (*T* ! *x*))
    unfolding *lms-bucket-def bucket-def*
    by *blast*

  from ‹*strict-mono* α› ‹*x* $<$ *length T*›
  have α (*T* ! *x*) $\leq$ α (*Max* (*set T*))
    by (*simp add*: *strict-mono-leD*)

  from *lms-init-D*[*OF* ‹*lms-init* α *T SA*› ‹α (*T* ! *x*) $\leq$ α (*Max* (*set T*))›]
      ‹*x* $\in$ *lms-bucket* α *T* (α (*T* ! *x*))›
  show *x* $\in$ *set SA*
    using *list-slice-subset* **by** *force*
qed

definition *s-init* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
s-init α *T SA* =
(∀ *b* ≤ α (*Max* (*set T*)).
∀ *i* < *length SA*. *l-bucket-end* α *T b* ≤ *i* ∧ *i* < *lms-bucket-start* α *T b* ⟶ *SA* ! *i* = *length T*
)

lemma *s-init-D*:
⟦*s-init* α *T SA*;
b ≤ α (*Max* (*set T*));
i < *length SA*;
l-bucket-end α *T b* ≤ *i*;
i < *lms-bucket-start* α *T b*⟧ ⟹
SA ! *i* = *length T*
using *s-init-def* **by** *blast*

definition *l-init* :: ('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
l-init α *T SA* =
(∀ *b* ≤ α (*Max* (*set T*)).
∀ *i* < *length SA*. *bucket-start* α *T b* ≤ *i* ∧ *i* < *l-bucket-end* α *T b* ⟶ *SA* ! *i* = *length T*
)

lemma *l-init-D*:
⟦*l-init* α *T SA*;
b ≤ α (*Max* (*set T*));
i < *length SA*;
bucket-start α *T b* ≤ *i*;
i < *l-bucket-end* α *T b*⟧ ⟹
SA ! *i* = *length T*
using *l-init-def* **by** *blast*

lemma *init-imp-lms-range*:
assumes *lms-init* α *T SA*
and *l-init* α *T SA*
and *s-init* α *T SA*
and *length SA* = *length T*
and *strict-mono* α
and *i* < *length SA*
and *SA* ! *i* = *j*
and *j* < *length T*
shows *lms-bucket-start* α *T* (α (*T* ! *j*)) ≤ *i* ∧ *i* < *bucket-end* α *T* (α (*T* ! *j*))
proof −
from ‹*i* < *length SA*› ‹*length SA* = *length T*›
have *i* < *length T*
by *simp*

```
  from index-in-bucket-interval-gen[OF ‹i < length T› ‹strict-mono α›]
  obtain b where
    b ≤ α (Max (set T))
    bucket-start α T b ≤ i
    i < bucket-end α T b
    by blast

  have i < l-bucket-end α T b ⟶ False
  proof
    assume i < l-bucket-end α T b
    with l-init-D[OF ‹l-init α T SA› ‹b ≤ α (Max (set T))› ‹i < length SA›]
        ‹bucket-start α T b ≤ i›
        ‹SA ! i = j›
        ‹j < length T›
    show False
      by simp
  qed

  have l-bucket-end α T b ≤ i ∧ i < lms-bucket-start α T b ⟶ False
  proof(intro impI; elim conjE)
    assume l-bucket-end α T b ≤ i i < lms-bucket-start α T b
    with s-init-D[OF ‹s-init α T SA› ‹b ≤ α (Max (set T))› ‹i < length SA›]
        ‹SA ! i = j›
        ‹j < length T›
    show False
      by simp
  qed
  with ‹i < l-bucket-end α T b ⟶ False›
      ‹b ≤ α (Max (set T))›
      ‹i < bucket-end α T b›
      ‹lms-init α T SA›
      ‹length SA = length T›
      ‹SA ! i = j›
  show ?thesis
    by (metis lms-init-nth not-less)
qed

lemma init-imp-only-lms-types:
  assumes lms-init α T SA
  and     l-init α T SA
  and     s-init α T SA
  and     length SA = length T
  and     strict-mono α
  shows ∀ i < length SA. SA ! i < length T ⟶ abs-is-lms T (SA ! i)
proof (intro allI impI)
  fix i
  assume i < length SA SA ! i < length T
  with init-imp-lms-range[OF ‹lms-init α T SA›
                             ‹l-init α T SA›
```

‹s-init α T SA›
‹length SA = length T›
‹strict-mono α›
‹i < length SA›]
have *lms-bucket-start α T (α (T ! (SA ! i))) ≤ i ∧ i < bucket-end α T (α (T ! (SA ! i)))*
by *simp*
with ‹*SA ! i < length T*› ‹*lms-init α T SA*› ‹*length SA = length T*› ‹*strict-mono α*›
show *abs-is-lms T (SA ! i)*
by (*meson Max-greD lms-init-nth strict-mono-less-eq*)
qed

lemma *init-imp-only-s-types*:
assumes *lms-init α T SA*
and *l-init α T SA*
and *s-init α T SA*
and *length SA = length T*
and *strict-mono α*
shows ∀ *i < length SA. SA ! i < length T ⟶ suffix-type T (SA ! i) = S-type*
using *assms init-imp-only-lms-types abs-is-lms-def* **by** *blast*

definition *lms-sorted-init* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒
('*a list* ⇒ *nat* ⇒ '*a list*) ⇒
'*a list* ⇒
nat list ⇒
bool
where
lms-sorted-init α f T SA =
(∀ *b* ≤ *α* (*Max* (*set T*)).
ordlistns.sorted (*map* (*f T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
)

lemma *lms-sorted-init-D*:
⟦*lms-sorted-init α f T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*f T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
using *lms-sorted-init-def* **by** *blast*

definition *l-suffix-sorted-pre* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
l-suffix-sorted-pre α T SA =
(∀ *b* ≤ *α* (*Max* (*set T*)).
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
)

lemma *l-suffix-sorted-preD*:
⟦*l-suffix-sorted-pre α T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
using *l-suffix-sorted-pre-def* **by** *blast*

definition *l-prefix-sorted-pre* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
l-prefix-sorted-pre α T SA =
(∀ *b* ≤ *α* (*Max* (*set T*)).
ordlistns.sorted (*map* (*lms-prefix T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
)

lemma *l-prefix-sorted-preD*:
⟦*l-prefix-sorted-pre α T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*lms-prefix T*) (*list-slice SA* (*lms-bucket-start α T b*) (*bucket-end α T b*)))
using *l-prefix-sorted-pre-def* **by** *blast*

definition *l-perm-pre* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list ⇒
bool
where
l-perm-pre α T B SA =
(*lms-init α T SA* ∧
l-init α T SA ∧
s-init α T SA ∧
l-bucket-init α T B ∧
T ≠ [] ∧
strict-mono α ∧
length SA = *length T* ∧
α (*Max* (*set T*)) < *length B*)

lemma *l-perm-pre-elims*:
l-perm-pre α T B SA ⟹ *lms-init α T SA*
l-perm-pre α T B SA ⟹ *l-init α T SA*
l-perm-pre α T B SA ⟹ *s-init α T SA*
l-perm-pre α T B SA ⟹ *l-bucket-init α T B*
l-perm-pre α T B SA ⟹ *T* ≠ []
l-perm-pre α T B SA ⟹ *strict-mono α*
l-perm-pre α T B SA ⟹ *length SA* = *length T*
l-perm-pre α T B SA ⟹ *α* (*Max* (*set T*)) < *length B*
unfolding *l-perm-pre-def* **by** *blast+*

67 Invariant Definitions

This section contains all the various invariants that we need for the *abs-induce-l* subroutine.

67.1 Distinctness

definition *l-distinct-inv* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list* ⇒ *bool*
  where
l-distinct-inv T SA = *distinct* (*filter* (λ*x*. *x* < *length T*) *SA*)

lemma *l-distinct-inv-D*:
  assumes *l-distinct-inv T SA*
  and *i* < *length SA*
  and *j* < *length SA*
  and *i* ≠ *j*
  and *SA* ! *i* < *length T*
  and *SA* ! *j* < *length T*
  shows *SA* ! *i* ≠ *SA* ! *j*
proof −
  from *filter-nth-relative-neq-1*[**where** *P* = λ*x*. *x* < *length T*,
                                OF ‹*i* < *length SA*›
                                  ‹*SA* ! *i* < *length T*›
                                  ‹*j* < *length SA*›
                                  ‹*SA* ! *j* < *length T*›
                                  ‹*i* ≠ *j*›]
  obtain *i*′ *j*′ **where**
    i′ < *length* (*filter* (λ*x*. *x* < *length T*) *SA*)
    j′ < *length* (*filter* (λ*x*. *x* < *length T*) *SA*)
    filter (λ*x*. *x* < *length T*) *SA* ! *i*′ = *SA* ! *i*
    filter (λ*x*. *x* < *length T*) *SA* ! *j*′ = *SA* ! *j*
    i′ ≠ *j*′
    by *blast*

  from *distinct-conv-nth*[*THEN iffD1*,
                        OF l-distinct-inv-def[*THEN iffD1*],
                        OF ‹*l-distinct-inv T SA*›]
      ‹*filter* (λ*x*. *x* < *length T*) *SA* ! *i*′ = *SA* ! *i*›
      ‹*filter* (λ*x*. *x* < *length T*) *SA* ! *j*′ = *SA* ! *j*›
      ‹*i*′ < *length* (*filter* (λ*x*. *x* < *length T*) *SA*)›
      ‹*i*′ ≠ *j*′›
      ‹*j*′ < *length* (*filter* (λ*x*. *x* < *length T*) *SA*)›
  show *?thesis*
    by *fastforce*
qed

67.2 Predecessor

definition *l-pred-inv* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list* ⇒ *nat* ⇒ *bool*

where
l-pred-inv T SA k =
(∀ *i* < *length SA*. *SA* ! *i* < *length T* ∧ *suffix-type T* (*SA* ! *i*) = *L-type* ⟶
(∃ *j* < *length SA*. *SA* ! *j* = *Suc* (*SA* ! *i*) ∧ *j* < *i* ∧ *j* < *k*))

lemma *l-pred-inv-D*:
⟦*l-pred-inv T SA k*; *i* < *length SA*; *SA* ! *i* < *length T*; *suffix-type T* (*SA* ! *i*) = *L-type*⟧ ⟹
∃ *j* < *length SA*. *SA* ! *j* = *Suc* (*SA* ! *i*) ∧ *SA* ! *j* < *length T* ∧ *j* < *i* ∧ *j* < *k*
by (*metis SL-types.simps*(*2*) *Suc-lessI l-pred-inv-def suffix-type-last*)

67.3 L Bucket Ptr

We prove that the pointer for each bucket is related to the number of L-types currently in SA. That is, if we subtract the original pointer with the current, we should have the number of L-types currently in SA for each symbol.

definition *cur-l-types* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat set*
where
cur-l-types α *T SA b* = {*i*|*i*. *i* ∈ *set SA* ∧ *i* ∈ *l-bucket* α *T b* }

definition *num-l-types* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat*
where
num-l-types α *T SA b* = *card* (*cur-l-types* α *T SA b*)

definition *l-bucket-ptr-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-bucket-ptr-inv α *T B SA* ≡
(∀ *b* ≤ α (*Max* (*set T*)). *B* ! *b* = *bucket-start* α *T b* + *num-l-types* α *T SA b*)

lemma *l-bucket-ptr-inv-D*:
⟦*l-bucket-ptr-inv* α *T B SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
B ! *b* = *bucket-start* α *T b* + *num-l-types* α *T SA b*
using *l-bucket-ptr-inv-def* **by** *blast*

67.4 Unknowns

definition *l-unknowns-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-unknowns-inv α *T B SA* ≡
(∀ *a* ≤ α (*Max* (*set T*)). ∀ *k*. *B* ! *a* ≤ *k* ∧ *k* < *l-bucket-end* α *T a* ⟶ *SA* ! *k* = *length T*)

lemma *l-unknowns-inv-D*:
⟦*l-unknowns-inv* α *T B SA*; *b* ≤ α (*Max* (*set T*)); *B* ! *b* ≤ *k*; *k* < *l-bucket-end* α *T b*⟧ ⟹

SA ! *k* = *length T*
using *l-unknowns-inv-def* **by** *blast*

67.5 Indexes

definition *l-index-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-index-inv α T B SA ≡
(∀ *i* < *length SA*.
(∀ *j*. *SA* ! *i* = *Suc j* ∧ *Suc j* < *length T* ∧ *suffix-type T j* = *L-type* ⟶
i < *B* ! (*α* (*T* ! *j*))
)
)

lemma *l-index-inv-D*:
⟦*l-index-inv α T B SA*; *i* < *length SA*; *SA* ! *i* = *Suc j*; *Suc j* < *length T*; *suffix-type T j* = *L-type*⟧ ⟹
i < *B* ! (*α* (*T* ! *j*))
using *l-index-inv-def* **by** *blast*

67.6 Unchanged

definition *l-unchanged-inv* ::
('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-unchanged-inv α T SA SA'=
((*length SA'* = *length SA*) ∧
(∀ *b* ≤ *α* (*Max* (*set T*)).
(∀ *i* < *length SA*. *l-bucket-end α T b* ≤ *i* ∧ *i* < *bucket-end α T b* ⟶ *SA* ! *i* = *SA'* ! *i*)
))

lemma *l-unchanged-inv-trans*:
⟦*l-unchanged-inv α T SA0 SA1*; *l-unchanged-inv α T SA1 SA2*⟧ ⟹
l-unchanged-inv α T SA0 SA2
by (*simp add*: *l-unchanged-inv-def*)

lemma *l-unchanged-inv-D*:
⟦*l-unchanged-inv α T SA SA'*; *length SA'* = *length SA*; *b* ≤ *α* (*Max* (*set T*));
i < *length SA*; *l-bucket-end α T b* ≤ *i*; *i* < *bucket-end α T b*⟧ ⟹
SA ! *i* = *SA'* ! *i*
using *l-unchanged-inv-def* **by** *blast*

67.7 L Locations

definition *l-locations-inv* ::
('*a* :: {*linorder*,*order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-locations-inv α T B SA =

```
  (∀ b ≤ α (Max (set T)).
    (∀ i < length SA. bucket-start α T b ≤ i ∧ i < B ! b ⟶
      SA ! i < length T ∧ suffix-type T (SA ! i) = L-type ∧ α (T ! (SA ! i)) = b
    )
  )

lemma l-locations-inv-D:
  ⟦l-locations-inv α T B SA;
    b ≤ α (Max (set T));
    i < length SA;
    bucket-start α T b ≤ i;
    i < B ! b⟧ ⟹
    SA ! i < length T ∧ suffix-type T (SA ! i) = L-type ∧ α (T ! (SA ! i)) = b
  using l-locations-inv-def by blast

lemma l-locations-list-slice:
  assumes l-locations-inv α T B SA
  and     b ≤ α (Max (set T))
shows set (list-slice SA (bucket-start α T b) (B ! b)) ⊆ l-bucket α T b
      (is set ?xs ⊆ l-bucket α T b)
proof
  fix x
  assume x ∈ set ?xs

  from nth-mem-list-slice[OF ‹x ∈ set ?xs›]
  obtain i where
    i < length SA
    bucket-start α T b ≤ i
    i < B ! b
    SA ! i = x
    by blast
  with l-locations-inv-D[OF assms, of i]
  have x < length T suffix-type T x = L-type α (T ! x) = b
    by blast+
  then show x ∈ l-bucket α T b
    by (simp add: bucket-def l-bucket-def)
qed
```

67.8 Seen

In this section, we prove that the seen invariant is maintained. In English, this invariant states for all L-type suffixes, excluding the one that starts at position 0, in the suffix array (SA) and that are less than the current index, their left neighbour is also in SA.

```
definition l-seen-inv :: ('a :: {linorder, order-bot}) list ⇒ nat list ⇒ nat ⇒ bool
  where
l-seen-inv T SA n ≡ ∀ i < n. i < length SA ∧ SA ! i < length T ⟶
                      (∀ j. SA ! i = Suc j ∧ suffix-type T j = L-type ⟶
                        (∃ k < length SA. SA ! k = j))
```

lemma *l-seen-inv-nth-ex*:
⟦*l-seen-inv T SA n*; *i* < *n*; *i* < *length SA*; *SA* ! *i* < *length T*; *SA* ! *i* = *Suc j*;
suffix-type T j = *L-type*⟧ ⟹
∃ *k* < *length SA*. *SA* ! *k* = *j*
using *l-seen-inv-def* **by** *blast*

67.9 Sortedness

definition *abs-induce-l-sorted* ::
(('*a* :: {*linorder*,*order-bot*}) *list* ⇒ *nat* ⇒ '*a list*) ⇒ '*a list* ⇒ *nat list* ⇒ *bool*
where
abs-induce-l-sorted f T SA = *ordlistns.sorted* (*map* (*f T*) (*filter* (λ*x*. *x* < *length T*) *SA*))

lemma *abs-induce-l-sorted-nth*:
assumes *abs-induce-l-sorted f T SA*
and *i* < *j*
and *j* < *length SA*
and *SA* ! *i* < *length T*
and *SA* ! *j* < *length T*
shows *list-less-eq-ns* (*f T* (*SA* ! *i*)) (*f T* (*SA* ! *j*))
proof −
let *?SA* = *filter* (λ*x*. *x* < *length T*) *SA* **and**
?le = (λ*x y*. *list-less-eq-ns* (*f T x*) (*f T y*))
from *filter-nth-relative-1*[**where** *P* = (λ*x*. *x* < *length T*),
OF ‹*j* < *length SA*›
‹*SA* ! *j* < *length T*›
‹*i* < *j*›
‹*SA* ! *i* < *length T*›]
obtain *i′ j′* **where**
j′ < *length ?SA*
i′ < *j′*
?SA ! *j′* = *SA* ! *j*
?SA ! *i′* = *SA* ! *i*
by *blast*

from *ordlistns.sorted-map*
‹*abs-induce-l-sorted f T SA*›
have *sorted-wrt ?le ?SA*
unfolding *abs-induce-l-sorted-def*
by *blast*

from *sorted-wrt-nth-less*[*OF* ‹*sorted-wrt ?le ?SA*›
‹*i′* < *j′*›
‹*j′* < *length ?SA*›]
have *list-less-eq-ns* (*f T* (*?SA* ! *i′*)) (*f T* (*?SA* ! *j′*))
by *assumption*
with ‹*?SA* ! *i′* = *SA* ! *i*› ‹*?SA* ! *j′* = *SA* ! *j*›

show *?thesis*
by *simp*
qed

definition *l-suffix-sorted-inv* ::
(('*a* :: {*linorder*,*order-bot*}) ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-suffix-sorted-inv α *T B SA* =
(∀ *b* ≤ α (*Max* (*set T*)).
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*bucket-start* α *T b*) (*B* ! *b*))))

lemma *l-suffix-sorted-invD*:
⟦*l-suffix-sorted-inv* α *T B SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*bucket-start* α *T b*) (*B* ! *b*)))
using *l-suffix-sorted-inv-def* **by** *blast*

definition *l-prefix-sorted-inv* ::
(('*a* :: {*linorder*,*order-bot*}) ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
l-prefix-sorted-inv α *T B SA* =
(∀ *b* ≤ α (*Max* (*set T*)).
ordlistns.sorted (*map* (*lms-prefix T*) (*list-slice SA* (*bucket-start* α *T b*) (*B* ! *b*))))

lemma *l-prefix-sorted-invD*:
⟦*l-prefix-sorted-inv* α *T B SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*lms-prefix T*) (*list-slice SA* (*bucket-start* α *T b*) (*B* ! *b*)))
using *l-prefix-sorted-inv-def* **by** *blast*

67.10 Permutation

definition *l-perm-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒
'*a list* ⇒
nat list ⇒
nat list ⇒
nat list ⇒
nat ⇒
bool
where
l-perm-inv α *T B SA SA' i* ≡
α (*Max* (*set T*)) < *length B* ∧
length SA = *length T* ∧
length SA' = *length SA* ∧
l-distinct-inv T SA' ∧
l-unknowns-inv α *T B SA'* ∧
l-bucket-ptr-inv α *T B SA'* ∧
l-index-inv α *T B SA'* ∧
l-unchanged-inv α *T SA SA'* ∧
l-locations-inv α *T B SA'* ∧

l-pred-inv T SA′ i ∧
l-seen-inv T SA′ i ∧
strict-mono α ∧
T ≠ [] ∧
lms-init α T SA ∧
s-init α T SA

lemma *l-perm-inv-elims*:
l-perm-inv α T B SA SA′ i ⟹ *α* (*Max* (*set T*)) < *length B*
l-perm-inv α T B SA SA′ i ⟹ *length SA* = *length T*
l-perm-inv α T B SA SA′ i ⟹ *length SA′* = *length SA*
l-perm-inv α T B SA SA′ i ⟹ *l-distinct-inv T SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-unknowns-inv α T B SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-bucket-ptr-inv α T B SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-index-inv α T B SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-unchanged-inv α T SA SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-locations-inv α T B SA′*
l-perm-inv α T B SA SA′ i ⟹ *l-pred-inv T SA′ i*
l-perm-inv α T B SA SA′ i ⟹ *l-seen-inv T SA′ i*
l-perm-inv α T B SA SA′ i ⟹ *strict-mono α*
l-perm-inv α T B SA SA′ i ⟹ *T* ≠ []
l-perm-inv α T B SA SA′ i ⟹ *lms-init α T SA*
l-perm-inv α T B SA SA′ i ⟹ *s-init α T SA*
by (*simp add*: *l-perm-inv-def*)+

68 Invariant Helpers

68.1 Distinctness of New Insert

We prove that the next item to be inserted cannot already be in the suffix array.

lemma *l-distinct-pred-inv-helper*:
assumes *i* < *length SA*
and *SA* ! *i* = *Suc j*
and *Suc j* < *length T*
and *suffix-type T j* = *L-type*
and *l-distinct-inv T SA*
and *l-pred-inv T SA i*
shows *j* ∉ *set SA*
proof
assume *j* ∈ *set SA*
then obtain *l* **where**
l < *length SA*
SA ! *l* = *j*
by (*meson in-set-conv-nth*)

from *l-pred-inv-D*[*OF* ‹*l-pred-inv T SA i*› ‹*l* < *length SA*›]
‹*SA* ! *l* = *j*›

```
      ‹SA ! i = Suc j›
      ‹Suc j < length T›
      ‹suffix-type T j = L-type›
  obtain i' where
    i' < length SA
    SA ! i' = Suc j
    i' < l
    i' < i
    by auto

  from ‹SA ! i = Suc j› ‹SA ! i' = Suc j›
  have SA ! i = SA ! i'
    by simp

  from ‹SA ! i' = Suc j› ‹Suc j < length T›
  have SA ! i' < length T
    by simp

  from ‹i' < i›
  have i ≠ i'
    by simp

  from ‹SA ! i = Suc j› ‹Suc j < length T›
  have SA ! i < length T
    by simp

  from l-distinct-inv-D[OF ‹l-distinct-inv T SA›
                           ‹i < length SA›
                           ‹i' < length SA›
                           ‹i ≠ i'›
                           ‹SA ! i < length T›
                           ‹SA ! i' < length T›]
      ‹SA ! i = SA ! i'›
  show False
    by simp
qed

lemma l-distinct-slice:
  assumes l-distinct-inv T SA
  and     l-locations-inv α T B SA
  and     length SA = length T
  and     b ≤ α (Max (set T))
shows distinct (list-slice SA (bucket-start α T b) (B ! b))
      (is distinct ?xs)
proof (intro distinct-conv-nth[THEN iffD2] allI impI)
  fix i j
  assume i < length ?xs j < length ?xs i ≠ j

  let ?i = bucket-start α T b + i
```

and *?j = bucket-start α T b + j*

have *SA ! ?i = ?xs ! i*
using ‹*i < length ?xs*› *nth-list-slice* **by** *force*

have *SA ! ?j = ?xs ! j*
using ‹*j < length ?xs*› *nth-list-slice* **by** *force*

from ‹*i ≠ j*›
have *?i ≠ ?j*
by *auto*
moreover
from ‹*i < length ?xs*›
have *?i < length SA ?i < B ! b*
by *auto*
with *l-locations-inv-D*[*OF assms*(*2*,*4*), *of ?i*]
have *SA ! ?i < length T*
using *le-add1* **by** *blast*
moreover
from ‹*j < length ?xs*›
have *?j < length SA ?j < B ! b*
by *auto*
with *l-locations-inv-D*[*OF assms*(*2*,*4*), *of ?j*]
have *SA ! ?j < length T*
using *le-add1* **by** *blast*
ultimately have *SA ! ?i ≠ SA ! ?j*
using *l-distinct-inv-D*[*OF assms*(*1*) ‹*?i < length SA*› ‹*?j < length SA*›]
by *blast*
with ‹*SA ! ?i = ?xs ! i*› ‹*SA ! ?j = ?xs ! j*›
show *?xs ! i ≠ ?xs ! j*
by *simp*
qed

68.2 Bucket Ranges

lemma *num-l-types-le-l-bucket-size*:
num-l-types α T SA b ≤ l-bucket-size α T b
by (*metis* (*no-types*, *lifting*) *card-mono cur-l-types-def finite-l-bucket l-bucket-size-def mem-Collect-eq num-l-types-def subsetI*)

lemma *num-l-types-less-l-bucket-size*:
⟦*j ∉ set SA*; *suffix-type T j = L-type*; *α* (*T ! j*) = *b*; *j < length T*⟧ ⟹
num-l-types α T SA b < l-bucket-size α T b
apply (*clarsimp simp*: *num-l-types-def cur-l-types-def l-bucket-size-def*)
apply (*rule psubset-card-mono*)
apply (*simp add*: *finite-l-bucket*)
apply (*rule psubsetI*)
apply (*rule subsetI*)
apply *blast*

apply *clarsimp*
apply (*drule equalityD2*)
apply (*drule subsetD*[**where** $c = j$])
apply (*simp add*: *bucket-def l-bucket-def*)
apply *blast*
done

lemma *l-bucket-ptr-inv-imp-le-l-bucket-end*:
⟦*l-bucket-ptr-inv* α *T B SA*; $b \leq \alpha$ (*Max* (*set T*))⟧ ⟹
B ! $b \leq$ *l-bucket-end* α *T b*
apply (*drule* (*1*) *l-bucket-ptr-inv-D*)
by (*simp add*: *l-bucket-end-def num-l-types-le-l-bucket-size*)

lemma *l-bucket-ptr-inv-imp-less-l-bucket-end*:
⟦*l-bucket-ptr-inv* α *T B SA*; $j <$ *length T*; *suffix-type T j* = *L-type*; $j \notin$ *set SA*; *strict-mono* α⟧ ⟹
B ! (α (T ! j)) < *l-bucket-end* α *T* (α (T ! j))
apply (*frule num-l-types-less-l-bucket-size*[**where** $\alpha = \alpha$ **and** $b = \alpha$ (T ! j)]; *assumption?*)
apply *simp*
apply (*drule l-bucket-ptr-inv-D*[**where** $b = \alpha$ (T ! j)])
apply (*simp add*: *strict-mono-leD*)
by (*simp add*: *l-bucket-end-def*)

lemma *bucket-size-imp-less-length*:
⟦*l-bucket-ptr-inv* α *T B SA*; $j <$ *length T*; *suffix-type T j* = *L-type*; $j \notin$ *set SA*; *strict-mono* α⟧ ⟹
B ! (α (T ! j)) < *length T*
apply (*drule* (*4*) *l-bucket-ptr-inv-imp-less-l-bucket-end*)
apply (*erule order.strict-trans2*)
apply (*rule order.trans*[*OF l-bucket-end-le-bucket-end bucket-end-le-length*])
done

lemma *l-bucket-ptr-inv-imp-ge-bucket-start*:
⟦*l-bucket-ptr-inv* α *T B SA*; $b \leq \alpha$ (*Max* (*set T*))⟧ ⟹
bucket-start α *T b* ≤ B ! b
by (*simp add*: *l-bucket-ptr-inv-D*)

lemma *l-bucket-ptr-inv-le-bucket-pointers*:
⟦*l-bucket-ptr-inv* α *T B SA*; $a < b$; $b \leq \alpha$ (*Max* (*set T*))⟧ ⟹
B ! $a \leq B$! b
apply (*frule l-bucket-ptr-inv-imp-le-l-bucket-end*[**where** $b = a$])
apply *arith*
apply (*frule l-bucket-ptr-inv-D*[**where** $b = b$])
apply *assumption*
apply *simp*
apply (*erule order.trans*)
apply (*rule order.trans*[**where** b = *bucket-start* α *T b*])
apply (*erule order.trans*[*OF l-bucket-end-le-bucket-end less-bucket-end-le-start*])

apply *simp*
done

68.3 No Overwrite

We prove that the next location is set as unknown.

lemma *l-unknowns-l-bucket-ptr-inv-helper*:
⟦*l-unknowns-inv* α *T B SA*;
l-bucket-ptr-inv α *T B SA*;
$j <$ *length T*;
suffix-type T j = *L-type*;
$j \notin$ *set SA*;
strict-mono α;
$k = \alpha$ (*T* ! *j*);
$l = B$! k⟧ $\Longrightarrow$
SA ! *l* = *length T*
apply (*drule* (*4*) *l-bucket-ptr-inv-imp-less-l-bucket-end*)
apply (*drule l-unknowns-inv-D*[**where** $b = k$ **and** $k = l$]; *simp add*: *strict-mono-leD*)
done

lemma *unchanged-slice*:
assumes *l-unchanged-inv* α *T SA0 SA*
and *length SA* = *length SA0*
and *length SA* = *length T*
and $b \leq \alpha$ (*Max* (*set T*))
and *l-bucket-end* α *T b* $\leq i$
and $j \leq$ *bucket-end* α *T b*
shows *list-slice SA0 i j* = *list-slice SA i j*
proof (*intro list-eq-iff-nth-eq*[*THEN iffD2*] *allI impI conjI*)
from ‹*length SA* = *length SA0*›
length-list-slice
show *length* (*list-slice SA0 i j*) = *length* (*list-slice SA i j*)
by *simp*
next
fix *k*
assume $k <$ *length* (*list-slice SA0 i j*)
with ‹*l-bucket-end* α *T b* $\leq i$›
have *l-bucket-end* α *T b* $\leq i + k$
by *simp*

from ‹*length SA* = *length T*›
have *bucket-end* α *T b* $\leq$ *length SA*
by (*simp add*: *bucket-end-le-length*)
with ‹$j \leq$ *bucket-end* α *T b*›
have $j \leq$ *length SA*
by *simp*
with ‹*length SA* = *length SA0*›
have *length* (*list-slice SA0 i j*) = $j - i$

```
    by simp
  with ‹k < length (list-slice SA0 i j)›
  have i < j
    by linarith

  from ‹j ≤ length SA›
       ‹k < length (list-slice SA0 i j)›
       ‹length (list-slice SA0 i j) = j − i›
       ‹length SA = length SA0›
  have i + k < length SA0
    by linarith

  from ‹j ≤ bucket-end α T b›
       ‹k < length (list-slice SA0 i j)›
       ‹length (list-slice SA0 i j) = j − i›
  have i + k < bucket-end α T b
    by linarith

  from l-unchanged-inv-D[OF ‹l-unchanged-inv α T SA0 SA›
                            ‹length SA = length SA0›
                            ‹b ≤ α (Max (set T))›
                            ‹i + k < length SA0›
                            ‹l-bucket-end α T b ≤ i + k›
                            ‹i + k < bucket-end α T b›]
       ‹k < length (list-slice SA0 i j)›
       ‹length SA = length SA0›
  show list-slice SA0 i j ! k = list-slice SA i j ! k
    by (metis length-list-slice nth-list-slice)
qed

lemma lms-init-unchanged:
  assumes l-unchanged-inv α T SA0 SA
  and     length SA = length SA0
  and     length SA = length T
  and     lms-init α T SA0
  shows lms-init α T SA
  unfolding lms-init-def
proof (intro allI impI)
  fix b
  assume b ≤ α (Max (set T))

  have l-bucket-end α T b ≤ lms-bucket-start α T b
    by (simp add: l-bucket-end-le-lms-bucket-start)

  from unchanged-slice[OF ‹l-unchanged-inv α T SA0 SA›
                          ‹length SA = length SA0›
                          ‹length SA = length T›
                          ‹b ≤ α (Max (set T))›
                          ‹l-bucket-end α T b ≤ lms-bucket-start α T b›
```

```
                    order.refl]
      lms-init-D[OF ‹lms-init α T SA0› ‹b ≤ α (Max (set T))›]
  show lms-bucket α T b = set (list-slice SA (lms-bucket-start α T b) (bucket-end
α T b))
    by simp
qed

lemma s-init-unchanged:
  assumes l-unchanged-inv α T SA0 SA
  and     length SA = length SA0
  and     length SA = length T
  and     s-init α T SA0
  shows s-init α T SA
  unfolding s-init-def
proof (intro allI impI; elim conjE)
  fix b i
  assume b ≤ α (Max (set T))
         i < length SA
         l-bucket-end α T b ≤ i
         i < lms-bucket-start α T b

  have lms-bucket-start α T b ≤ bucket-end α T b
    by (simp add: bucket-end-def' l-pl-pure-s-pl-lms-size lms-bucket-start-def)
  with ‹i < lms-bucket-start α T b›
  have i < bucket-end α T b
    by simp

  from ‹i < length SA› ‹length SA = length SA0›
  have i < length SA0
    by simp

  from l-unchanged-inv-D[OF ‹l-unchanged-inv α T SA0 SA›
                            ‹length SA = length SA0›
                            ‹b ≤ α (Max (set T))›
                            ‹i < length SA0›
                            ‹l-bucket-end α T b ≤ i›
                            ‹i < bucket-end α T b›]
  have SA0 ! i = SA ! i
    by assumption

  from s-init-D[OF ‹s-init α T SA0›
                   ‹b ≤ α (Max (set T))›
                   ‹i < length SA0›
                   ‹l-bucket-end α T b ≤ i›
                   ‹i < lms-bucket-start α T b›]
  have SA0 ! i = length T
    by simp
  with ‹SA0 ! i = SA ! i›
  show SA ! i = length T
```

```
    by simp
qed

lemma l-suffix-sorted-pre-maintained:
  assumes l-unchanged-inv α T SA0 SA
  and     length SA = length SA0
  and     length SA = length T
  and     l-suffix-sorted-pre α T SA0
shows l-suffix-sorted-pre α T SA
  unfolding l-suffix-sorted-pre-def
proof (safe)
  fix b
  assume b ≤ α (Max (set T))
  let ?xs = list-slice SA0 (lms-bucket-start α T b) (bucket-end α T b) and
      ?ys = list-slice SA (lms-bucket-start α T b) (bucket-end α T b)

  have l-bucket-end α T b ≤ lms-bucket-start α T b
    using l-bucket-end-le-lms-bucket-start by auto
  with unchanged-slice[OF assms(1-3) ‹b ≤ -›]
  have ?xs = ?ys
    by blast
  then show ordlistns.sorted (map (suffix T) ?ys)
    by (metis ‹b ≤ α (Max (set T))› assms(4) l-suffix-sorted-pre-def)
qed

lemma l-prefix-sorted-pre-maintained:
  assumes l-unchanged-inv α T SA0 SA
  and     length SA = length SA0
  and     length SA = length T
  and     l-prefix-sorted-pre α T SA0
shows l-prefix-sorted-pre α T SA
  unfolding l-prefix-sorted-pre-def
proof (safe)
  fix b
  assume b ≤ α (Max (set T))
  let ?xs = list-slice SA0 (lms-bucket-start α T b) (bucket-end α T b) and
      ?ys = list-slice SA (lms-bucket-start α T b) (bucket-end α T b)

  have l-bucket-end α T b ≤ lms-bucket-start α T b
    using l-bucket-end-le-lms-bucket-start by auto
  with unchanged-slice[OF assms(1-3) ‹b ≤ -›]
  have ?xs = ?ys
    by blast
  then show ordlistns.sorted (map (lms-prefix T) ?ys)
    by (metis ‹b ≤ α (Max (set T))› assms(4) l-prefix-sorted-pre-def)
qed

lemma unknown-range-values:
  assumes l-unchanged-inv α T SA0 SA
```

and *l-unknowns-inv α T B SA*
and *length SA = length SA0*
and *length SA = length T*
and *lms-init α T SA0*
and *s-init α T SA0*
and $b \leq \alpha$ (*Max* (*set T*))
and *B* ! *b* ≤ *i*
and *i* < *lms-bucket-start α T b*
shows *SA* ! *i* = *length T*
proof −
have *i* < *length T*
by (*meson assms*(*9*) *bucket-end-le-length leD le-less-linear le-less-trans lms-bucket-start-le-bucket-end*)
hence *i* < *length SA*
by (*simp add*: *assms*(*4*))

have *i* < *l-bucket-end α T b* ∨ *l-bucket-end α T b* ≤ *i*
using *not-less* **by** *blast*
moreover
have *i* < *l-bucket-end α T b* ⟹ *?thesis*
proof −
assume *i* < *l-bucket-end α T b*
with *l-unknowns-inv-D*[*OF assms*(*2*,*7*,*8*)]
show *?thesis* **.**
qed
moreover
have *l-bucket-end α T b* ≤ *i* ⟹ *?thesis*
proof −
assume *l-bucket-end α T b* ≤ *i*
with *s-init-D*[*OF s-init-unchanged*[*OF assms*(*1*,*3*−*4*,*6*)] *assms*(*7*) ‹*i* < *length SA*› - *assms*(*9*)]
show *?thesis* **.**
qed
ultimately show *?thesis*
by *blast*
qed

68.4 Bucket Values

lemma *same-bucket-same-hd*:
assumes *l-unchanged-inv α T SA0 SA*
and *l-locations-inv α T B SA*
and *l-bucket-ptr-inv α T B SA*
and *l-unknowns-inv α T B SA*
and *length SA = length T*
and *length SA = length SA0*
and *lms-init α T SA0*
and *s-init α T SA0*
and $b \leq \alpha$ (*Max* (*set T*))
and *i* < *length SA*

```
  and     SA ! i < length T
  and     bucket-start α T b ≤ i
  and     i < bucket-end α T b
  shows α (T ! (SA ! i)) = b
proof −
  have i < B ! b ∨ B ! b ≤ i
    by linarith
  then show ?thesis
  proof
    assume i < B ! b
    from l-locations-inv-D[OF ‹l-locations-inv α T B SA›
                         ‹b ≤ α (Max (set T))›
                         ‹i < length SA›
                         ‹bucket-start α T b ≤ i›
                         ‹i < B ! b›]
    show ?thesis
      by blast
  next
    assume B ! b ≤ i

    from ‹i < length SA› ‹length SA = length SA0›
    have i < length SA0
      by simp

    have lms-bucket-start α T b ≤ i
    proof −
      have i < l-bucket-end α T b ⟶ SA ! i = length T
      proof
        assume i < l-bucket-end α T b
        from l-unknowns-inv-D[OF ‹l-unknowns-inv α T B SA›
                               ‹b ≤ α (Max (set T))›
                               ‹B ! b ≤ i›
                               ‹i < l-bucket-end α T b›]
        show SA ! i = length T
          by assumption
      qed

     have l-bucket-end α T b ≤ i ∧ i < lms-bucket-start α T b ⟶ SA ! i = length
T
      proof (intro impI; elim conjE)
        assume i < lms-bucket-start α T b
               l-bucket-end α T b ≤ i

        from ‹s-init α T SA0›
             ‹l-bucket-end α T b ≤ i›
             ‹i < lms-bucket-start α T b›
             ‹b ≤ α (Max (set T))›
             ‹i < length SA0›
        have SA0 ! i = length T
```

```
      by (metis s-init-def)
    with l-unchanged-inv-D[OF ‹l-unchanged-inv α T SA0 SA›
                              ‹length SA = length SA0›
                              ‹b ≤ α (Max (set T))›
                              ‹i < length SA0›
                              ‹l-bucket-end α T b ≤ i›
                              ‹i < bucket-end α T b›]
    show SA ! i = length T
      by simp
  qed

  from ‹i < l-bucket-end α T b ⟶ SA ! i = length T›
      ‹l-bucket-end α T b ≤ i ∧ i < lms-bucket-start α T b ⟶ SA ! i = length
T›
       ‹SA ! i < length T›
  show lms-bucket-start α T b ≤ i
    by auto
 qed
 hence l-bucket-end α T b ≤ i
  using l-bucket-end-le-lms-bucket-start le-trans by blast

 from ‹length SA = length T›
 have bucket-end α T b ≤ length SA
  by (simp add: bucket-end-le-length)
 with ‹lms-init α T SA0›
      ‹lms-bucket-start α T b ≤ i›
      ‹i < bucket-end α T b›
      ‹b ≤ α (Max (set T))›
      ‹length SA = length SA0›
 have SA0 ! i ∈ lms-bucket α T b
  by (metis list-slice-nth-mem lms-init-def)
 with l-unchanged-inv-D[OF ‹l-unchanged-inv α T SA0 SA›
                           ‹length SA = length SA0›
                           ‹b ≤ α (Max (set T))›
                           ‹i < length SA0›
                           ‹l-bucket-end α T b ≤ i›
                           ‹i < bucket-end α T b›]
 have SA ! i ∈ lms-bucket α T b
  by simp
 then show ?thesis
  by (simp add: bucket-def lms-bucket-def)
 qed
qed

lemma same-hd-same-bucket:
 assumes l-unchanged-inv α T SA0 SA
 and     l-locations-inv α T B SA
 and     l-bucket-ptr-inv α T B SA
 and     l-unknowns-inv α T B SA
```

```
  and     strict-mono α
  and     length SA = length T
  and     length SA = length SA0
  and     lms-init α T SA0
  and     s-init α T SA0
  and     i < length SA
  and     SA ! i < length T
  and     b = α (T ! (SA ! i))
  shows bucket-start α T b ≤ i ∧ i < bucket-end α T b
proof −
  from ‹length SA = length T› ‹i < length SA›
  have i < length T
    by simp
  from index-in-bucket-interval-gen[OF ‹i < length T› ‹strict-mono α›]
  obtain b' where
    b' ≤ α (Max (set T))
    bucket-start α T b' ≤ i
    i < bucket-end α T b'
    by blast

  from same-bucket-same-hd[OF ‹l-unchanged-inv α T SA0 SA›
                              ‹l-locations-inv α T B SA›
                              ‹l-bucket-ptr-inv α T B SA›
                              ‹l-unknowns-inv α T B SA›
                              ‹length SA = length T›
                              ‹length SA = length SA0›
                              ‹lms-init α T SA0›
                              ‹s-init α T SA0›
                              ‹b' ≤ α (Max (set T))›
                              ‹i < length SA›
                              ‹SA ! i < length T›
                              ‹bucket-start α T b' ≤ i›
                              ‹i < bucket-end α T b'›]
      ‹b = α (T ! (SA ! i))›
      ‹bucket-start α T b' ≤ i›
      ‹i < bucket-end α T b'›
  show ?thesis
    by blast
qed


lemma less-bucket-less-hd:
  assumes l-unchanged-inv α T SA0 SA
  and     l-locations-inv α T B SA
  and     l-bucket-ptr-inv α T B SA
  and     l-unknowns-inv α T B SA
  and     strict-mono α
  and     length SA = length T
  and     length SA = length SA0
```

and *lms-init α T SA0*
and *s-init α T SA0*
and *i < length SA*
and *SA ! i < length T*
and *i < bucket-start α T b*
shows *α (T ! (SA ! i)) < b*
proof −
from *same-hd-same-bucket*[*OF* ‹*l-unchanged-inv α T SA0 SA*›
‹*l-locations-inv α T B SA*›
‹*l-bucket-ptr-inv α T B SA*›
‹*l-unknowns-inv α T B SA*›
‹*strict-mono α*›
‹*length SA = length T*›
‹*length SA = length SA0*›
‹*lms-init α T SA0*›
‹*s-init α T SA0*›
‹*i < length SA*›
‹*SA ! i < length T*›,
of α (T ! (SA ! i))]
have *bucket-start α T (α (T ! (SA ! i))) ≤ i ∧ i < bucket-end α T (α (T ! (SA ! i)))*
by *simp*
then show *?thesis*
by (*meson* ‹*i < bucket-start α T b*› *bucket-start-le leD le-less-linear le-trans*)
qed

lemma *gr-bucket-gr-hd*:
assumes *l-unchanged-inv α T SA0 SA*
and *l-locations-inv α T B SA*
and *l-bucket-ptr-inv α T B SA*
and *l-unknowns-inv α T B SA*
and *strict-mono α*
and *length SA = length T*
and *length SA = length SA0*
and *lms-init α T SA0*
and *s-init α T SA0*
and *i < length SA*
and *SA ! i < length T*
and *bucket-end α T b ≤ i*
shows *b < α (T ! (SA ! i))*
proof −
from *same-hd-same-bucket*[*OF* ‹*l-unchanged-inv α T SA0 SA*›
‹*l-locations-inv α T B SA*›
‹*l-bucket-ptr-inv α T B SA*›
‹*l-unknowns-inv α T B SA*›
‹*strict-mono α*›
‹*length SA = length T*›
‹*length SA = length SA0*›
‹*lms-init α T SA0*›

```
                    ‹s-init α T SA0›
                    ‹i < length SA›
                    ‹SA ! i < length T›,
                   of α (T ! (SA ! i))]
  have bucket-start α T (α (T ! (SA ! i))) ≤ i ∧ i < bucket-end α T (α (T ! (SA
! i)))
    by simp
  then show ?thesis
    by (meson ‹bucket-end α T b ≤ i› bucket-end-le leD le-less-linear less-le-trans)
qed
```

68.5 Seen

We have two helper lemmas in the case of updating the suffix array SA, and in the case when the current index is incremented. The two lemmas are used in conjunction in the case that the SA is updated and the current index is incremented.

```
lemma l-seen-inv-upd:
  assumes l-seen-inv T SA n n ≤ k SA ! k = length T
  shows l-seen-inv T (SA[k := x]) n
  unfolding l-seen-inv-def
proof safe
  fix i j
  assume A: i < n i < length (SA[k := x]) SA[k := x] ! i < length T SA[k := x]
! i = Suc j
          suffix-type T j = L-type
  hence i ≠ k
    using assms(2) leD by blast
  hence B: i < length SA SA ! i < length T SA ! i = Suc j
    using A by auto
  with l-seen-inv-nth-ex[OF assms(1) A(1) B A(5)]
  have ∃k'<length SA. SA ! k' = j
    by blast
  then obtain k' where
    k' < length SA SA ! k' = j
    by blast
  then show ∃k'<length (SA[k := x]). SA[k := x] ! k' = j
   by (metis B(2,3) Suc-lessD assms(3) length-list-update less-irrefl-nat nth-list-update-neq)
qed

lemma l-seen-inv-Suc:
  assumes l-seen-inv T SA n SA ! n = Suc j k < length SA SA ! k = j
  shows l-seen-inv T SA (Suc n)
  unfolding l-seen-inv-def
proof safe
  fix i j'
  assume A: i < Suc n i < length SA SA ! i < length T SA ! i = Suc j'
          suffix-type T j' = L-type
  have i < n ∨ ¬ i < n
```

by *blast*
then show $\exists k{<}$*length SA*. *SA* ! $k = j'$
proof
assume $i < n$
with *l-seen-inv-nth-ex*[*OF assms*(*1*) - *A*(*2*−)]
show $\exists k{<}$*length SA*. *SA* ! $k = j'$
by *blast*
next
assume $\neg\ i < n$
then show $\exists k{<}$*length SA*. *SA* ! $k = j'$
using *A*(*1*) *A*(*4*) *assms*(*2*−) *not-less-less-Suc-eq* **by** *force*
qed
qed

69 Distinctness

lemma *distinct-app3*:
distinct (*xs* @ *ys* @ *zs*) ⟷
distinct xs ∧ *distinct ys* ∧ *distinct zs* ∧
set xs ∩ *set ys* = {} ∧ *set xs* ∩ *set zs* = {} ∧ *set ys* ∩ *set zs* = {}
by *auto*

69.1 Establishment

lemma *abs-is-lms-imp-in-lms-bucket*:
abs-is-lms T i ⟹ *i* ∈ *lms-bucket* α *T* (α (*T* ! *i*))
apply (*clarsimp simp*: *lms-bucket-def bucket-def*)
by (*simp add*: *abs-is-lms-def suffix-type-s-bound*)

lemma *l-distinct-inv-established*:
assumes *lms-init* α *T SA*
and *l-init* α *T SA*
and *s-init* α *T SA*
and *length SA* = *length T*
and *strict-mono* α
and *l-bucket-init* α *T B*
shows *l-distinct-inv T SA*
unfolding *l-distinct-inv-def*
proof (*intro distinct-conv-nth*[*THEN iffD2*] *allI impI*)
let *?P* = (λ*x*. *x* < *length T*)
let *?SA* = *filter* (λ*x*. *x* < *length T*) *SA*

fix *i j*
assume *i* < *length ?SA j* < *length ?SA i* ≠ *j*

from ‹*i* < *length ?SA*›
have *?SA* ! *i* < *length T*
using *mem-Collect-eq nth-mem* **by** *fastforce*

from ‹$j < \mathit{length}\ \mathit{?SA}$›
have $\mathit{?SA}\ !\ j < \mathit{length}\ T$
using *mem-Collect-eq nth-mem* **by** *fastforce*

from *filter-nth-relative-neq-2*[*OF* ‹$i < \mathit{length}\ \mathit{?SA}$› ‹$j < \mathit{length}\ \mathit{?SA}$› ‹$i \neq j$›]
obtain $i'\ j'$ **where**
$i' < \mathit{length}\ SA$
$j' < \mathit{length}\ SA$
$SA\ !\ i' = \mathit{?SA}\ !\ i$
$SA\ !\ j' = \mathit{?SA}\ !\ j$
$i' \neq j'$
by *blast*

from ‹$\mathit{?SA}\ !\ i < \mathit{length}\ T$› ‹$SA\ !\ i' = \mathit{?SA}\ !\ i$›
have $SA\ !\ i' < \mathit{length}\ T$
by *simp*

from ‹$\mathit{?SA}\ !\ j < \mathit{length}\ T$› ‹$SA\ !\ j' = \mathit{?SA}\ !\ j$›
have $SA\ !\ j' < \mathit{length}\ T$
by *simp*

from *init-imp-lms-range assms* ‹$i' < \mathit{length}\ SA$› ‹$SA\ !\ i' < \mathit{length}\ T$›
have *lms-bucket-start* $\alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ i'))) \leq i'\ i' < \mathit{bucket\text{-}end}\ \alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ i')))$
by *blast+*

from *init-imp-lms-range assms* ‹$j' < \mathit{length}\ SA$› ‹$SA\ !\ j' < \mathit{length}\ T$›
have *lms-bucket-start* $\alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ j'))) \leq j'\ j' < \mathit{bucket\text{-}end}\ \alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ j')))$
by *blast+*

have $\alpha\ (T\ !\ (SA\ !\ i')) = \alpha\ (T\ !\ (SA\ !\ j')) \vee \alpha\ (T\ !\ (SA\ !\ i')) \neq \alpha\ (T\ !\ (SA\ !\ j'))$
by *simp*
then show $\mathit{?SA}\ !\ i \neq \mathit{?SA}\ !\ j$
proof
assume $\alpha\ (T\ !\ (SA\ !\ i')) = \alpha\ (T\ !\ (SA\ !\ j'))$
with ‹*lms-bucket-start* $\alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ i'))) \leq i'$›
have *lms-bucket-start* $\alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ j'))) \leq i'$
by *simp*

from ‹$\alpha\ (T\ !\ (SA\ !\ i')) = \alpha\ (T\ !\ (SA\ !\ j'))$›
‹$i' < \mathit{bucket\text{-}end}\ \alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ i')))$›
have $i' < \mathit{bucket\text{-}end}\ \alpha\ T\ (\alpha\ (T\ !\ (SA\ !\ j')))$
by *simp*

with *list-slice-nth-eq-iff-index-eq*[*OF lms-init-imp-distinct-bucket*,
OF ‹*lms-init* $\alpha\ T\ SA$›
-

```
                                    ‹length SA = length T›
                                    -
                                    ‹lms-bucket-start α T (α (T ! (SA ! j'))) ≤ i'›
                                    -
                                    ‹lms-bucket-start α T (α (T ! (SA ! j'))) ≤ j'›
                                    ‹j' < bucket-end α T (α (T ! (SA ! j')))›]
        ‹i' ≠ j'›
        ‹length SA = length T›
        ‹SA ! j' < length T›
        ‹strict-mono α›
    have SA ! i' ≠ SA ! j'
      by (simp add: bucket-end-le-length strict-mono-less-eq)
    with ‹SA ! i' = ?SA ! i› ‹SA ! j' = ?SA ! j›
    show ?thesis
      by simp
  next
    assume α (T ! (SA ! i')) ≠ α (T ! (SA ! j'))
    with ‹SA ! i' = ?SA ! i› ‹SA ! j' = ?SA ! j›
    show ?thesis
      by auto
  qed
qed

corollary l-distinct-inv-perm-established:
  assumes l-perm-pre α T B SA
  shows l-distinct-inv T SA
  using assms l-distinct-inv-established l-perm-pre-def by blast
```

69.2 Maintenance

```
lemma l-distinct-inv-maintained:
  assumes i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     l-distinct-inv T SA
  and     l-pred-inv T SA i
  shows l-distinct-inv T (SA[l := j])
proof −
  let ?P = (λx. x < length T)

  from ‹Suc j < length T›
  have j < length T
    by simp

  — We case on whether the update occurs on an index within bounds
  have l < length SA ∨ l ≥ length SA
    by arith
  then show ?thesis
```

proof
assume $l < length\ SA$

from *l-distinct-pred-inv-helper*[*OF* ‹$i < length\ SA$›
‹$SA\ !\ i = Suc\ j$›
‹$Suc\ j < length\ T$›
‹*suffix-type* $T\ j = L\text{-}type$›
‹*l-distinct-inv* $T\ SA$›
‹*l-pred-inv* $T\ SA\ i$›]
have $j \notin set\ SA$
by *assumption*

let *?xs* = *filter ?P* (*take l SA*) **and**
?ys = *filter ?P* (*drop* (*Suc l*) *SA*)
— We have that $j \notin set\ SA$ and show that if we now add j to SA, it will maintain distinctness. This is straightforward but does require some massaging, i.e. casing on whether $SA\ !\ l < length\ T$ or not, due to the use of *filter* in the *l-distinct-inv ?T ?SA* = *distinct* (*filter* ($\lambda x.\ x < length\ ?T$) *?SA*) definition.

from ‹$j < length\ T$› ‹$l < length\ SA$›
have *f-SA′*: *filter ?P* ($SA[l := j]$) = *?xs* @ [*j*] @ *?ys*
by (*simp add*: *filter-update-nth-success*)

from ‹$j \notin set\ SA$›
have *set ?xs* ∩ *set* [*j*] = {}
using *in-set-takeD* **by** *fastforce*

from ‹$j \notin set\ SA$›
have *set* [*j*] ∩ *set ?ys* = {}
using *in-set-dropD* **by** *force*

have $SA\ !\ l < length\ T \vee SA\ !\ l \geq length\ T$
by *arith*
then show *?thesis*
proof
assume $SA\ !\ l < length\ T$
with ‹$l < length\ SA$›
have *f-SA*: *filter ?P SA* = *?xs* @ [$SA\ !\ l$] @ *?ys*
by (*meson filter-take-nth-drop-success*)

from *f-SA* ‹*l-distinct-inv T SA*› *distinct-app3*[*of ?xs* [$SA\ !\ l$] *?ys*]
have *distinct ?xs*
distinct ?ys
set ?xs ∩ *set ?ys* = {}
unfolding *l-distinct-inv-def*
by *auto*

with ‹*set ?xs* ∩ *set* [*j*] = {}›
‹*set* [*j*] ∩ *set ?ys* = {}›

```
        f-SA'
    show ?thesis
      unfolding l-distinct-inv-def
      by auto
  next
    assume SA ! l ≥ length T
    with ‹l < length SA›
    have f-SA: filter ?P SA = ?xs @ ?ys
      by (simp add: filter-take-nth-drop-fail)

    from f-SA ‹l-distinct-inv T SA› distinct-append[of ?xs ?ys]
    have distinct ?xs
        distinct ?ys
        set ?xs ∩ set ?ys = {}
      unfolding l-distinct-inv-def
      by auto
    with ‹set ?xs ∩ set [j] = {}›
        ‹set [j] ∩ set ?ys = {}›
        f-SA'
    show ?thesis
      unfolding l-distinct-inv-def
      by auto
  qed
 next
```

— We now handle the case *length SA* ≤ *l*, which is straightforward. In the actual *abs-induce-l* subroutine, *l* will always be less than *length SA*, but for this proof, we make no such assumption, nor do we need to prove it.

```
  assume l ≥ length SA
  hence SA[l := j] = SA
    by (simp add: list-update-beyond)
  with ‹l-distinct-inv T SA›
  show ?thesis
    by simp
 qed
qed

corollary l-distinct-inv-perm-maintained:
 assumes l-perm-inv α T B SA0 SA i
 and     i < length SA
 and     SA ! i = Suc j
 and     Suc j < length T
 and     suffix-type T j = L-type
shows l-distinct-inv T (SA[l := j])
 by (meson assms l-distinct-inv-maintained l-perm-inv-elims(4,10))
```

70 Unknowns

70.1 Establishment

lemma *l-unknowns-inv-established*:
  assumes *l-init* α *T SA*
        l-bucket-init α *T B*
        length SA = *length T*
  shows *l-unknowns-inv* α *T B SA*
  unfolding *l-unknowns-inv-def*
proof (*intro allI impI*; *elim conjE*)
  fix *a k*
  assume $a \leq \alpha$ (*Max* (*set T*))
      B ! *a* ≤ *k*
      k < *l-bucket-end* α *T a*

  from ‹*B* ! *a* ≤ *k*› *l-bucket-initD*[*OF* ‹*l-bucket-init* α *T B*› ‹$a \leq \alpha$ (*Max* (*set T*))›]
  have *bucket-start* α *T a* ≤ *k*
    by *simp*

  from ‹*length SA* = *length T*› ‹*k* < *l-bucket-end* α *T a*›
  have *k* < *length SA*
    by (*metis bucket-end-le-length l-bucket-end-le-bucket-end less-le-trans*)

  from *l-init-D*[*OF* ‹*l-init* α *T SA*›
            ‹$a \leq \alpha$ (*Max* (*set T*))›
            ‹*k* < *length SA*›
            ‹*bucket-start* α *T a* ≤ *k*›
            ‹*k* < *l-bucket-end* α *T a*›]
  show *SA* ! *k* = *length T*
    by *assumption*
qed

corollary *l-unknowns-inv-perm-established*:
  assumes *l-perm-pre* α *T B SA*
  shows *l-unknowns-inv* α *T B SA*
  using *assms l-perm-pre-elims*(*2*,*4*,*7*) *l-unknowns-inv-established* **by** *blast*

70.2 Maintenance

lemma *l-unknowns-inv-maintained*:
  assumes *l-unknowns-inv* α *T B SA*
  and *length B* > α (*Max* (*set T*))
  and *i* < *length SA*
  and *SA* ! *i* = *Suc j*
  and *Suc j* < *length T*
  and *suffix-type T j* = *L-type*
  and *k* = α (*T* ! *j*)
  and *l* = *B* ! *k*
  and *strict-mono* α

```
  and     l-distinct-inv T SA
  and     l-pred-inv T SA i
  and     l-bucket-ptr-inv α T B SA
  shows l-unknowns-inv α T (B[k := Suc (B ! k)]) (SA[l := j])
  unfolding l-unknowns-inv-def
proof (intro allI impI; elim conjE)
  fix a k'
  assume a ≤ α (Max (set T))
         B[k := Suc (B ! k)] ! a ≤ k'
         k' < l-bucket-end α T a

  from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                     ‹SA ! i = Suc j›
                                     ‹Suc j < length T›
                                     ‹suffix-type T j = L-type›
                                     ‹l-distinct-inv T SA›
                                     ‹l-pred-inv T SA i›]
  have j ∉ set SA
    by assumption

  from ‹Suc j < length T›
  have j < length T
    by simp

  from l-unknowns-l-bucket-ptr-inv-helper[OF ‹l-unknowns-inv α T B SA›
                                             ‹l-bucket-ptr-inv α T B SA›
                                             ‹j < length T›
                                             ‹suffix-type T j = L-type›
                                             ‹j ∉ set SA›
                                             ‹strict-mono α›
                                             ‹k = α (T ! j)›
                                             ‹l = B ! k›]
  have SA ! l = length T
    by assumption

  from ‹strict-mono α› ‹k = α (T ! j)› ‹j < length T›
  have k ≤ α (Max (set T))
    by (simp add: strict-mono-leD)

  from l-unknowns-inv-D[OF ‹l-unknowns-inv α T B SA›
                           ‹a ≤ α (Max (set T))›
                           -
                           ‹k' < l-bucket-end α T a›]
       ‹B[k := Suc (B ! k)] ! a ≤ k'›
       ‹a ≤ α (Max (set T))›
       ‹α (Max (set T)) < length B›
  have SA ! k' = length T
    by (metis Suc-le-mono le-SucI le-less-trans nth-list-update nth-list-update-neq)
```

```
  from ‹j < length T› ‹strict-mono α› ‹l-bucket-ptr-inv α T B SA› ‹k = α (T !
j)› ‹l = B ! k›
  have bucket-start α T k ≤ l
   using Max-greD l-bucket-ptr-inv-imp-ge-bucket-start strict-mono-less-eq by blast

  from ‹j < length T›
       ‹j ∉ set SA›
       ‹strict-mono α›
       ‹l-bucket-ptr-inv α T B SA›
       ‹suffix-type T j = L-type›
       ‹k = α (T ! j)›
       ‹l = B ! k›
  have l < l-bucket-end α T k
    using  l-bucket-ptr-inv-imp-less-l-bucket-end by blast

  have a = k ∨ a ≠ k
    by simp
  then show SA[l := j] ! k' = length T
  proof
    assume a = k
    hence k' ≠ l
      using ‹B[k := Suc (B ! k)] ! a ≤ k'› ‹a ≤ α (Max (set T))› ‹α (Max (set
T)) < length B›
            ‹l = B ! k› by auto
    then show ?thesis
      using ‹SA ! k' = length T› by auto
  next
    assume a ≠ k
    hence a < k ∨ a > k
      by arith
    then show ?thesis
    proof
      assume a < k

      from less-bucket-end-le-start[OF ‹a < k›]
      have bucket-end α T a ≤ bucket-start α T k
        by blast
      with ‹bucket-start α T k ≤ l›
      have bucket-end α T a ≤ l
        by simp
      with l-bucket-end-le-bucket-end
      have l-bucket-end α T a ≤ l
        using le-trans by blast
      with ‹k' < l-bucket-end α T a›
      have k' < l
        using less-le-trans by blast
      then show ?thesis
        using ‹SA ! k' = length T› by auto
    next
```

assume $a > k$

from ‹$B[k := Suc\ (B\ !\ k)]\ !\ a \leq k'$›
‹$a \leq \alpha\ (Max\ (set\ T))$›
‹$a \neq k$›
‹*l-bucket-ptr-inv* $\alpha\ T\ B\ SA$›
l-bucket-ptr-inv-imp-ge-bucket-start
have *bucket-start* $\alpha\ T\ a \leq k'$
by *force*

from *less-bucket-end-le-start*[*OF* ‹$k < a$›]
have *bucket-end* $\alpha\ T\ k \leq$ *bucket-start* $\alpha\ T\ a$
by *blast*
with ‹*bucket-start* $\alpha\ T\ a \leq k'$›
have *bucket-end* $\alpha\ T\ k \leq k'$
by *simp*
with *l-bucket-end-le-bucket-end*
have *l-bucket-end* $\alpha\ T\ k \leq k'$
using *le-trans* **by** *blast*
with ‹$l <$ *l-bucket-end* $\alpha\ T\ k$›
have $l < k'$
using *less-le-trans* **by** *blast*
then show *?thesis*
using ‹$SA\ !\ k' =$ *length* T› **by** *auto*
qed
qed
qed

corollary *l-unknowns-inv-perm-maintained*:
assumes *l-perm-inv* $\alpha\ T\ B\ SA0\ SA\ i$
and $i <$ *length* SA
and $SA\ !\ i = Suc\ j$
and $Suc\ j <$ *length* T
and *suffix-type* $T\ j =$ *L-type*
and $k = \alpha\ (T\ !\ j)$
and $l = B\ !\ k$
shows *l-unknowns-inv* $\alpha\ T\ (B[k := Suc\ (B\ !\ k)])\ (SA[l := j])$
by (*metis assms l-perm-inv-elims*(*1*,*4*−*6*,*10*,*12*) *l-unknowns-inv-maintained*)

71 Number of L-types

71.1 Establishment

We first prove that this invariant is established from the precondition, i.e., that initially, there are only LMS-types, which are just a special type of S-types, and that the initial pointer is the start of the bucket.

lemma *l-bucket-ptr-inv-established*:
assumes *lms-init* $\alpha\ T\ SA$

```
  and     l-init α T SA
  and     s-init α T SA
  and     length SA = length T
  and     strict-mono α
  and     l-bucket-init α T B
  shows l-bucket-ptr-inv α T B SA
proof −
  have ∀ b ≤ α (Max (set T)). cur-l-types α T SA b = {}
    unfolding cur-l-types-def
  proof (intro allI impI equalityI subsetI)
    fix b x
    assume x ∈ {i |i. i ∈ set SA ∧ i ∈ l-bucket α T b}
    hence x ∈ set SA x ∈ l-bucket α T b
      by blast+

    have x < length T ∨ x ≥ length T
      using not-less by blast
    then show x ∈ {}
    proof
      assume x < length T
      with ‹x ∈ set SA› init-imp-only-s-types assms
      have suffix-type T x = S-type
        by (metis in-set-conv-nth)
      hence x ∉ l-bucket α T b
        by (simp add: l-bucket-def)
      with ‹x ∈ l-bucket α T b›
      show ?thesis
        by blast
    next
      assume  length T ≤ x
      hence x ∉ l-bucket α T b
        by (simp add: bucket-def l-bucket-def)
      with ‹x ∈ l-bucket α T b›
      show ?thesis
        by blast
    qed
  next
    fix b :: nat
    fix x :: nat
    assume x ∈ {}
    then show x ∈ {i |i. i ∈ set SA ∧ i ∈ l-bucket α T b}
      by blast
  qed
  hence ∀ b ≤ α (Max (set T)). num-l-types α T SA b = 0
    by (simp add: num-l-types-def)
  with ‹l-bucket-init α T B›
  show ?thesis
    by (simp add: l-bucket-ptr-inv-def l-bucket-init-def)
qed
```

corollary *l-bucket-ptr-inv-perm-established*:
assumes *l-perm-pre* α *T B SA*
shows *l-bucket-ptr-inv* α *T B SA*
using *assms l-bucket-ptr-inv-established l-perm-pre-def* **by** *blast*

71.2 Maintenance

We now prove that the invariant is maintained.

lemma *set-update-mem-neqI*:
$[\![x \in set\ xs;\ xs\ !\ i \neq x]\!] \Longrightarrow x \in set\ (xs[i := y])$
by (*metis in-set-conv-nth length-list-update nth-list-update-neq*)

lemma *cur-l-types-update-1*:
$[\![SA\ !\ l = length\ T;\ l < length\ SA;\ j \notin set\ SA;\ suffix\text{-}type\ T\ j = L\text{-}type;\ j < length\ T;$
$\alpha\ (T\ !\ j) = b]\!] \Longrightarrow$
cur-l-types α *T* $(SA[l := j])$ *b* = *insert j* (*cur-l-types* α *T SA b*)
apply (*intro equalityI subsetI*)
apply (*metis* (*no-types*, *lifting*) *cur-l-types-def in-set-conv-nth insertCI length-list-update mem-Collect-eq nth-list-update nth-list-update-neq*)
by (*metis* (*mono-tags*, *lifting*) *bucket-def cur-l-types-def insert-iff l-bucket-def less-irrefl-nat mem-Collect-eq set-update-memI set-update-mem-neqI*)

lemma *cur-l-types-update-2*:
assumes *SA* ! *l* = *length T* α (*T* ! *j*) $\neq$ *b*
shows *cur-l-types* α *T* $(SA[l := j])$ *b* = *cur-l-types* α *T SA b*
proof (*cases l* < *length SA*)
assume *l* < *length SA*
show *?thesis*
proof *safe*
fix *x*
assume *x* ∈ *cur-l-types* α *T* $(SA[l := j])$ *b*
show *x* ∈ *cur-l-types* α *T SA b*
proof (*cases x* = *j*)
assume *x* = *j*
then show *?thesis*
using ‹*x* ∈ *cur-l-types* α *T* $(SA[l := j])$ *b*› *assms*(*2*) *bucket-def cur-l-types-def l-bucket-def*
by *fastforce*
next
assume *x* ≠ *j*
then show *?thesis*
by (*metis* (*no-types*, *lifting*) ‹*x* ∈ *cur-l-types* α *T* $(SA[l := j])$ *b*› *cur-l-types-def in-set-conv-nth length-list-update mem-Collect-eq nth-list-update nth-list-update-neq*)
qed
next

```
    fix x
    assume x ∈ cur-l-types α T SA b
    then show x ∈ cur-l-types α T (SA[l := j]) b
    by (simp add: assms(1) bucket-def cur-l-types-def l-bucket-def set-update-mem-neqI)
  qed
qed (simp add: list-update-beyond)

lemma num-l-types-update-1:
  ⟦SA ! l = length T; l < length SA; j ∉ set SA; suffix-type T j = L-type; j < length
T;
    α (T ! j) = b⟧ ⟹
    num-l-types α T (SA[l := j]) b = Suc (num-l-types α T SA b)
  apply (clarsimp simp: num-l-types-def)
  apply (subst cur-l-types-update-1; simp?)
  apply (subst card-insert-disjoint)
   apply (metis (no-types, lifting) List.finite-set cur-l-types-def finite-nat-set-iff-bounded
                                    mem-Collect-eq)
   apply (simp add: cur-l-types-def)
  apply simp
  done

lemma num-l-types-update-2:
  ⟦SA ! l = length T; α (T ! j) ≠ b⟧ ⟹
    num-l-types α T (SA[l := j]) b = num-l-types α T SA b
  by (simp add: cur-l-types-update-2 num-l-types-def)

lemma l-bucket-ptr-inv-maintained:
  assumes l-bucket-ptr-inv α T B SA
  and     length SA = length T
  and     length B > α (Max (set T))
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
  and     strict-mono α
  and     l-distinct-inv T SA
  and     l-pred-inv T SA i
  and     l-unknowns-inv α T B SA
  shows l-bucket-ptr-inv α T (B[k := Suc (B ! k)]) (SA[l := j])
  unfolding l-bucket-ptr-inv-def
proof safe
  fix b
  assume b ≤ α (Max (set T))

  from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                     ‹SA ! i = Suc j›
                                     ‹Suc j < length T›
```

```
                                ‹suffix-type T j = L-type›
                                ‹l-distinct-inv T SA›
                                ‹l-pred-inv T SA i›]
  have j ∉ set SA
    by assumption

  from ‹Suc j < length T›
  have j < length T
    by arith

  from l-unknowns-l-bucket-ptr-inv-helper[OF ‹l-unknowns-inv α T B SA›
                                          ‹l-bucket-ptr-inv α T B SA›
                                          ‹j < length T›
                                          ‹suffix-type T j = L-type›
                                          ‹j ∉ set SA›
                                          ‹strict-mono α›
                                          ‹k = α (T ! j)›
                                          ‹l = B ! k›]
  have SA ! l = length T
    by assumption

  let ?G = B[k := Suc (B ! k)] ! b = bucket-start α T b + num-l-types α T (SA[l
:= j]) b
  have b = k ∨ b ≠ k
    by blast
  then show ?G
  proof
    assume b = k
    hence B[k := Suc (B ! k)] ! b = Suc (B ! k)
      using ‹b ≤ α (Max (set T))› ‹length B > α (Max (set T))› by auto

    from num-l-types-less-l-bucket-size[OF ‹j ∉ set SA› ‹suffix-type T j = L-type›]
        ‹Suc j < length T›
        ‹b = k›
        ‹k = α (T ! j)›
    have num-l-types α T SA b < l-bucket-size α T b
      by simp

    from ‹l-bucket-ptr-inv α T B SA› ‹b ≤ α (Max (set T))›
    have B ! b = bucket-start α T b + num-l-types α T SA b
      by (metis l-bucket-ptr-inv-def)
    with ‹num-l-types α T SA b < l-bucket-size α T b›
         bucket-end-le-length l-bucket-le-bucket-size bucket-end-def'
    have B ! b < length T
      by (metis add-less-cancel-left less-le-trans)
    with ‹k = α (T ! j)› ‹b = k› ‹l = B ! k› ‹length SA = length T›
    have l < length SA
      by simp
```

```
    from ‹SA ! l = length T›
        ‹b = k›
        ‹b ≤ α (Max (set T))›
        ‹j ∉ set SA›
        ‹l < length SA›
        ‹k = α (T ! j)›
        num-l-types-update-1
        ‹suffix-type T j = L-type›
        ‹Suc j < length T›
        Suc-lessD
    have num-l-types α T (SA[l := j]) b = Suc (num-l-types α T SA b)
      by blast
    from ‹B[k := Suc (B ! k)] ! b = Suc (B ! k)›
        ‹b = k›
        ‹b ≤ α (Max (set T))›
        ‹num-l-types α T (SA[l := j]) b = Suc (num-l-types α T SA b)›
        ‹l-bucket-ptr-inv α T B SA›
        l-bucket-ptr-inv-def
    show ?thesis
      by fastforce
  next
    assume b ≠ k
    hence B[k := Suc (B ! k)] ! b = B ! b
      by auto

    from num-l-types-update-2[OF ‹SA ! l = length T›]
        ‹b ≤ α (Max (set T))›
        ‹b ≠ k›
        ‹k = α (T ! j)›
    have num-l-types α T (SA[l := j]) b = num-l-types α T SA b
      by simp

    from ‹B[k := Suc (B ! k)] ! b = B ! b›
        ‹b ≤ α (Max (set T))›
        ‹num-l-types α T (SA[l := j]) b = num-l-types α T SA b›
        ‹l-bucket-ptr-inv α T B SA›
        l-bucket-ptr-inv-def
    show ?thesis
      by fastforce
  qed
qed

corollary l-bucket-ptr-inv-perm-maintained:
  assumes l-perm-inv α T B SA0 SA i
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
```

and $l = B\ !\ k$
shows *l-bucket-ptr-inv* $\alpha\ T\ (B[k := Suc\ (B\ !\ k)])\ (SA[l := j])$
by (*metis assms l-bucket-ptr-inv-maintained l-perm-inv-elims*(*1*,*2*,*3*−*6*,*10*,*12*))

72 L Locations

72.1 Establishment

lemma *l-locations-inv-established*:
assumes *l-bucket-init* $\alpha\ T\ B$
shows *l-locations-inv* $\alpha\ T\ B\ SA$
using *assms l-bucket-initD l-locations-inv-def* **by** *fastforce*

corollary *l-locations-inv-perm-established*:
assumes *l-perm-pre* $\alpha\ T\ B\ SA$
shows *l-locations-inv* $\alpha\ T\ B\ SA$
using *assms l-locations-inv-established l-perm-pre-elims*(*4*) **by** *blast*

72.2 Maintenance

lemma *l-locations-inv-maintained*:
assumes *l-locations-inv* $\alpha\ T\ B\ SA$
and $length\ B > \alpha\ (Max\ (set\ T))$
and $i < length\ SA$
and $SA\ !\ i = Suc\ j$
and $Suc\ j < length\ T$
and $suffix\text{-}type\ T\ j = L\text{-}type$
and $k = \alpha\ (T\ !\ j)$
and $l = B\ !\ k$
and *strict-mono* α
and *l-distinct-inv* $T\ SA$
and *l-pred-inv* $T\ SA\ i$
and *l-bucket-ptr-inv* $\alpha\ T\ B\ SA$
shows *l-locations-inv* $\alpha\ T\ (B[k := Suc\ (B\ !\ k)])\ (SA[l := j])$
unfolding *l-locations-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix $b\ i'$
assume $b \leq \alpha\ (Max\ (set\ T))$
$i' < length\ (SA[l := j])$
$bucket\text{-}start\ \alpha\ T\ b \leq i'$
$i' < B[k := Suc\ (B\ !\ k)]\ !\ b$
hence $i' < length\ SA$
by *simp*

have $b = k \vee b \neq k$
by *blast*
then
show $SA[l := j]\ !\ i' < length\ T \wedge suffix\text{-}type\ T\ (SA[l := j]\ !\ i') = L\text{-}type \wedge$
$\alpha\ (T\ !\ (SA[l := j]\ !\ i')) = b$

```
 proof
   assume b = k
   with ‹bucket-start α T b ≤ i'›
   have bucket-start α T k ≤ i'
     by simp

   from ‹b = k› ‹i' < B[k := Suc (B ! k)] ! b›
   have i' < Suc (B ! k)
     using ‹b ≤ α (Max (set T))› ‹length B > α (Max (set T))› by auto
   hence i' < B ! k ∨ i' = B ! k
     by (simp add: less-Suc-eq)
   then show ?thesis
   proof
     assume i' < B ! k
     with ‹l = B ! k› ‹b = k› ‹b ≤ α (Max (set T))› ‹bucket-start α T k ≤ i'› ‹i'
< length SA›
     show ?thesis
       using ‹l-locations-inv α T B SA› l-locations-inv-def by fastforce
   next
     assume i' = B ! k
     with ‹l = B ! k›
     have i' = l
       by simp

     from ‹i' < length SA› ‹i' = l›
     have SA[l := j] ! i' = j
       by simp
     with ‹suffix-type T j = L-type› ‹Suc j < length T› ‹i' < length SA› ‹b = k›
‹k = α (T ! j)›
     show ?thesis
       by auto
   qed
 next
   assume b ≠ k
   hence B[k := Suc (B ! k)] ! b = B ! b
     by simp

   from l-bucket-ptr-inv-imp-le-l-bucket-end[OF ‹l-bucket-ptr-inv α T B SA›
                                               ‹b ≤ α (Max (set T))›]
   have B ! b ≤ l-bucket-end α T b
     by simp

   from ‹Suc j < length T›
   have j < length T
     by simp

   from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                      ‹SA ! i = Suc j›
                                      ‹Suc j < length T›
```

```
                                ‹suffix-type T j = L-type›
                                ‹l-distinct-inv T SA›
                                ‹l-pred-inv T SA i›]
have j ∉ set SA
  by assumption

from l-bucket-ptr-inv-imp-less-l-bucket-end[OF ‹l-bucket-ptr-inv α T B SA›
                                               ‹j < length T›
                                               ‹suffix-type T j = L-type›
                                               ‹j ∉ set SA›
                                               ‹strict-mono α›]
    ‹k = α (T ! j)›
    ‹l = B ! k›
have l < l-bucket-end α T k
  by simp

from ‹k = α (T ! j)› ‹j < length T› ‹strict-mono α›
have k ≤ α (Max (set T))
  by (simp add: strict-mono-less-eq)

from l-bucket-ptr-inv-imp-ge-bucket-start[OF ‹l-bucket-ptr-inv α T B SA›
                                             ‹k ≤ α (Max (set T))›]
    ‹l = B ! k›
have bucket-start α T k ≤ l
  by simp

from ‹B[k := Suc (B ! k)] ! b = B ! b›
    ‹i' < B[k := Suc (B ! k)] ! b›
    l-bucket-ptr-inv-imp-le-l-bucket-end[OF ‹l-bucket-ptr-inv α T B SA›
                                            ‹b ≤ α (Max (set T))›]
    l-bucket-end-le-bucket-end
have i' < bucket-end α T b
  by (metis less-le-trans)

from ‹b ≠ k›
have b < k ∨ k < b
  by linarith
hence i' ≠ l
proof
  assume b < k
  hence bucket-end α T b ≤ bucket-start α T k
    by (simp add: less-bucket-end-le-start)
    with ‹i' < bucket-end α T b› ‹bucket-start α T k ≤ l›
    have i' < l
      by linarith
    then show ?thesis
      by simp
  next
    assume k < b
```

```
      hence bucket-end α T k ≤ bucket-start α T b
        by (simp add: less-bucket-end-le-start)
      hence l-bucket-end α T k ≤ bucket-start α T b
        using l-bucket-end-le-bucket-end le-trans by blast
      with ‹bucket-start α T b ≤ i'› ‹l < l-bucket-end α T k›
      have l < i'
        by simp
      then show ?thesis
        by simp
    qed
    with ‹B[k := Suc (B ! k)] ! b = B ! b›
         ‹b ≤ α (Max (set T))›
         ‹bucket-start α T b ≤ i'›
         ‹i' < B[k := Suc (B ! k)] ! b›
         ‹i' < length SA›
         ‹l-locations-inv α T B SA›
    show ?thesis
    using l-locations-inv-D by fastforce
  qed
qed

corollary l-locations-inv-perm-maintained:
  assumes l-perm-inv α T B SA0 SA i
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
shows l-locations-inv α T (B[k := Suc (B ! k)]) (SA[l := j])
  by (metis assms l-locations-inv-maintained l-perm-inv-elims(1,4,6,9,10,12))
```

73 Unchanged

73.1 Establishment

```
lemma l-unchanged-inv-established:
  l-unchanged-inv α T SA SA
  using l-unchanged-inv-def by blast
```

73.2 Maintenance

```
lemma l-unchanged-inv-maintained:
  assumes l-unchanged-inv α T SA0 SA
  and     length B > α (Max (set T))
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
```

```
  and     k = α (T ! j)
  and     l = B ! k
  and     strict-mono α
  and     l-distinct-inv T SA
  and     l-pred-inv T SA i
  and     l-bucket-ptr-inv α T B SA
  shows l-unchanged-inv α T SA0 (SA[l := j])
proof −
  have l-unchanged-inv α T SA (SA[l := j])
    unfolding l-unchanged-inv-def
  proof safe
    show length (SA[l := j]) = length SA
      by simp
  next
    fix b i'
    assume b ≤ α (Max (set T))
           i' < length SA
           l-bucket-end α T b ≤ i'
           i' < bucket-end α T b

    from ‹strict-mono α›
         ‹l-bucket-ptr-inv α T B SA›
         ‹Suc j < length T›
         ‹k = α (T ! j)›
         ‹l = B ! k›
    have bucket-start α T k ≤ l
    by (metis Max-greD Suc-lessD l-bucket-ptr-inv-imp-ge-bucket-start strict-mono-leD)

    from ‹l-distinct-inv T SA›
         ‹l-pred-inv T SA i›
         ‹i < length SA›
         ‹SA ! i = Suc j›
         ‹Suc j < length T›
         ‹suffix-type T j = L-type›
    have j ∉ set SA
      using l-distinct-pred-inv-helper by blast

    from ‹j ∉ set SA›
         ‹strict-mono α›
         ‹l-bucket-ptr-inv α T B SA›
         ‹Suc j < length T›
         ‹suffix-type T j = L-type›
         ‹k = α (T ! j)›
         ‹l = B ! k›
    have l < l-bucket-end α T k
      using Suc-lessD l-bucket-ptr-inv-imp-less-l-bucket-end by blast

    have b = k ∨ b ≠ k
      by blast
```

```
    then show SA ! i' = SA[l := j] ! i'
    proof
      assume b = k
      then show ?thesis
        using ‹l < l-bucket-end α T k› ‹l-bucket-end α T b ≤ i'› by auto
    next
      assume b ≠ k
      hence b < k ∨ k < b
        using less-linear by blast
      then show ?thesis
      proof
        assume b < k
        hence bucket-end α T b ≤ bucket-start α T k
          by (simp add: less-bucket-end-le-start)
        with ‹i' < bucket-end α T b› ‹bucket-start α T k ≤ l›
        have i' < l
          by linarith
        then show ?thesis
          by simp
      next
        assume k < b
        hence bucket-end α T k ≤ bucket-start α T b
          by (simp add: less-bucket-end-le-start)
        hence l-bucket-end α T k ≤ bucket-start α T b
          using l-bucket-end-le-bucket-end le-trans by blast
        hence l-bucket-end α T k ≤ l-bucket-end α T b
          by (simp add: l-bucket-end-def)
        with ‹l < l-bucket-end α T k› ‹l-bucket-end α T b ≤ i'›
        have l < i'
          by linarith
        then show ?thesis
          by simp
      qed
    qed
  qed
  with ‹l-unchanged-inv α T SA0 SA›
  show ?thesis
    using l-unchanged-inv-trans by blast
qed

corollary l-unchanged-inv-perm-maintained:
  assumes l-perm-inv α T B SA0 SA i
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
shows l-unchanged-inv α T SA0 (SA[l := j])
```

by (*metis assms l-perm-inv-elims*(*1*,*4*,*6*,*8*,*10*,*12*) *l-unchanged-inv-maintained*)

74 Invariant about the Current Index

74.1 Establishment

The first invariant is that current index is always less than the index where the update will occur.

lemma *l-index-inv-established*:
assumes *lms-init* α *T SA*
and *l-init* α *T SA*
and *s-init* α *T SA*
and *length SA* = *length T*
and *strict-mono* α
and *l-bucket-init* α *T B*
shows *l-index-inv* α *T B SA*
unfolding *l-index-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix *i j*
assume *i* < *length SA SA* ! *i* = *Suc j Suc j* < *length T suffix-type T j* = *L-type*
with *init-imp-only-s-types*[*OF* ‹*lms-init* α *T SA*›
‹*l-init* α *T SA*›
‹*s-init* α *T SA*›
‹*length SA* = *length T*›
‹*strict-mono* α›,
THEN spec, *of i*]
have *suffix-type T* (*Suc j*) = *S-type*
by *simp*
with ‹*suffix-type T j* = *L-type*› ‹*Suc j* < *length T*›
have *T* ! *j* ≠ *T* ! *Suc j*
by (*simp add*: *suffix-type-neq*)
with ‹*suffix-type T j* = *L-type*› ‹*Suc j* < *length T*›
have *T* ! *Suc j* < *T* ! *j*
using *nth-less-imp-s-type* **by** *fastforce*
from *same-hd-same-bucket*[*OF l-unchanged-inv-established*
l-locations-inv-established[*OF* ‹*l-bucket-init* α *T B*›]
l-bucket-ptr-inv-established[*OF* ‹*lms-init* α *T SA*›
‹*l-init* α *T SA*›
‹*s-init* α *T SA*›
‹*length SA* = *length T*›
‹*strict-mono* α›]
l-unknowns-inv-established
- - - - -
‹*i* < *length SA*›]
assms
‹*SA* ! *i* = *Suc j*›
‹*Suc j* < *length T*›
have *bucket-start* α *T* (α (*T* ! *Suc j*)) ≤ *i* *i* < *bucket-end* α *T* (α (*T* ! *Suc j*))

by *simp+*
with ‹T ! *Suc* j < T ! j› ‹*strict-mono* α›
have i < *bucket-start* α T (α (T ! j))
by (*meson less-bucket-end-le-start less-le-trans strict-mono-less*)

from ‹*l-bucket-init* α T B› ‹*Suc* j < *length* T› ‹*strict-mono* α›
have B ! α (T ! j) = *bucket-start* α T (α (T ! j))
by (*simp add*: *Suc-lessD l-bucket-initD strict-mono-leD*)
with ‹i < *bucket-start* α T (α (T ! j))›
show i < B ! α (T ! j)
by *simp*
qed

corollary *l-index-inv-perm-established*:
assumes *l-perm-pre* α T B *SA*
shows *l-index-inv* α T B *SA*
using *assms l-index-inv-established l-perm-pre-def* **by** *blast*

74.2 Maintenance

lemma *l-index-inv-maintained*:
assumes *l-index-inv* α T B *SA*
and *length* B > α (*Max* (*set* T))
and i < *length SA*
and *SA* ! i = *Suc* j
and *Suc* j < *length* T
and *suffix-type* T j = *L-type*
and k = α (T ! j)
and l = B ! k
and *strict-mono* α
and *l-distinct-inv* T *SA*
and *l-pred-inv* T *SA* i
and *l-bucket-ptr-inv* α T B *SA*
and *l-unknowns-inv* α T B *SA*
shows *l-index-inv* α T (B[k := *Suc* (B ! k)]) (*SA*[l := j])
unfolding *l-index-inv-def*
proof(*intro impI allI*; *elim conjE*)
fix l' j'
assume l' < *length* (*SA*[l := j])
SA[l := j] ! l' = *Suc* j'
Suc j' < *length* T
suffix-type T j' = *L-type*

from ‹l' < *length* (*SA*[l := j])›
have l' < *length SA*
by *simp*

have l' = l ∨ l' ≠ l
by *simp*

```
then show l' < B[k := Suc (B ! k)] ! α (T ! j')
proof
  assume l' = l
  with ‹SA[l := j] ! l' = Suc j'› ‹l' < length SA›
  have j = Suc j'
    by simp
  with ‹suffix-type T j' = L-type› ‹Suc j' < length T›
  have T ! j ≤ T ! j'
    by (simp add: Suc-lessD l-type-letter-gre-Suc)
  with ‹strict-mono α›
  have α (T ! j) ≤ α (T ! j')
    using strict-mono-less-eq by blast
  hence α (T ! j) = α (T ! j') ∨ α (T ! j) < α (T ! j')
    using le-imp-less-or-eq by blast
  then show ?thesis
  proof
    assume α (T ! j) = α (T ! j')
    with ‹k = α (T ! j)›
         ‹Suc j' < length T›
         ‹j = Suc j'›
         ‹strict-mono α›
         ‹length B > α (Max (set T))›
    have B[k := Suc (B ! k)] ! α (T ! j') = Suc (B ! k)
     by (metis Max-greD less-le-trans not-less nth-list-update-eq strict-mono-less)
    with ‹l = B ! k› ‹l' = l›
    show ?thesis
      by simp
  next
    assume α (T ! j) < α (T ! j')

    from ‹α (T ! j) < α (T ! j')› ‹k = α (T ! j)›
    have B[k := Suc (B ! k)] ! α (T ! j') = B ! α (T ! j')
      by simp

    from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                       ‹SA ! i = Suc j›
                                       ‹Suc j < length T›
                                       ‹suffix-type T j = L-type›
                                       ‹l-distinct-inv T SA›
                                       ‹l-pred-inv T SA i›]
    have j ∉ set SA
      by assumption

    from Suc-lessD[OF ‹Suc j < length T›]
    have j < length T
      by assumption

    from l-bucket-ptr-inv-imp-less-l-bucket-end[OF ‹l-bucket-ptr-inv α T B SA›
                                                   ‹j < length T›
```

```
                                        ‹suffix-type T j = L-type›
                                        ‹j ∉ set SA›
                                        ‹strict-mono α›]
      have B ! α (T ! j) < l-bucket-end α T (α (T ! j))
        by assumption
      with ‹l' = l› ‹k = α (T ! j)› ‹l = B ! k›
      have l' < l-bucket-end α T (α (T ! j))
        by simp
      hence l' < bucket-end α T (α (T ! j))
        using l-bucket-end-le-bucket-end less-le-trans by blast
      with less-bucket-end-le-start[OF ‹α (T ! j) < α (T ! j')›]
      have l' < bucket-start α T (α (T ! j'))
        using less-le-trans by blast
      with l-bucket-ptr-inv-imp-ge-bucket-start[OF ‹l-bucket-ptr-inv α T B SA›]
          ‹Suc j' < length T›
          ‹strict-mono α›
      have l' < B ! α (T ! j')
        by (meson Max-greD Suc-lessD less-le-trans strict-mono-less-eq)
      with ‹B[k := Suc (B ! k)] ! α (T ! j') = B ! α (T ! j')›
      show ?thesis
        by simp
    qed
  next
    assume l' ≠ l
    with ‹SA[l := j] ! l' = Suc j'›
    have SA ! l' = Suc j'
      by auto

    from l-index-inv-D[OF ‹l-index-inv α T B SA›
                          ‹l' < length SA›
                          ‹SA ! l' = Suc j'›
                          ‹Suc j' < length T›
                          ‹suffix-type T j' = L-type›]

    show ?thesis
    by (metis Suc-leI Suc-le-lessD Suc-lessD list-update-beyond not-less nth-list-update-eq
              nth-list-update-neq)
  qed
qed

corollary l-index-inv-perm-maintained:
  assumes l-perm-inv α T B SA0 SA i
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
shows l-index-inv α T (B[k := Suc (B ! k)]) (SA[l := j])
```

using *assms l-index-inv-maintained l-perm-inv-def* **by** *blast*

75 Predecessor Invariant

75.1 Establishment

The proof for the establishment is simple because initially, SA contains no L-types.

lemma *l-pred-inv-established*:
assumes *lms-init* α *T SA*
and *l-init* α *T SA*
and *s-init* α *T SA*
and *length SA* = *length T*
and *strict-mono* α
shows *l-pred-inv T SA 0*
using *assms init-imp-only-s-types l-pred-inv-def* **by** *fastforce*

corollary *l-pred-inv-perm-established*:
assumes *l-perm-pre* α *T B SA*
shows *l-pred-inv T SA 0*
using *assms l-perm-pre-def l-pred-inv-established* **by** *blast*

75.2 Maintenance

In this section, we prove that the predecessor invariant *l-pred-inv ?T ?SA ?k* = ($\forall$ *i*<*length ?SA*. *?SA* ! *i* < *length ?T* $\wedge$ *suffix-type ?T* (*?SA* ! *i*) = *L-type* $\longrightarrow$ ($\exists$ *j*<*length ?SA*. *?SA* ! *j* = *Suc* (*?SA* ! *i*) $\wedge$ *j* < *i* $\wedge$ *j* < *?k*))is maintained. In English, this invariant states that for all L-type suffixes in the suffix array (SA), their right neighbour is in SA and occurs before them.

We now prove that the invariant is maintained for each branch of the *abs-induce-l-step*

lemma *l-pred-inv-maintained-no-update*:
assumes *l-pred-inv T SA i*
shows *l-pred-inv T SA* (*Suc i*)
using *assms*
unfolding *l-pred-inv-def*
using *less-Suc-eq* **by** *auto*

lemma *l-pred-inv-maintained*:
assumes *l-pred-inv T SA i*
and *i* < *length SA*
and *SA* ! *i* = *Suc j*
and *Suc j* < *length T*
and *suffix-type T j* = *L-type*
and *k* = α (*T* ! *j*)
and *l* = *B* ! *k*
and *strict-mono* α

```
  and     l-distinct-inv T SA
  and     l-bucket-ptr-inv α T B SA
  and     l-unknowns-inv α T B SA
  and     l-index-inv α T B SA
  shows l-pred-inv T (SA[l := j]) (Suc i)
proof −

  from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                     ‹SA ! i = Suc j›
                                     ‹Suc j < length T›
                                     ‹suffix-type T j = L-type›
                                     ‹l-distinct-inv T SA›
                                     ‹l-pred-inv T SA i›]
  have j ∉ set SA
    by assumption

  from ‹Suc j < length T›
  have j < length T
    by simp

  from l-unknowns-l-bucket-ptr-inv-helper[OF ‹l-unknowns-inv α T B SA›
                                             ‹l-bucket-ptr-inv α T B SA›
                                             ‹j < length T›
                                             ‹suffix-type T j = L-type›
                                             ‹j ∉ set SA›
                                             ‹strict-mono α›
                                             ‹k = α (T ! j)›
                                             ‹l = B !  k›]
  have SA ! l = length T
    by assumption

  from l-index-inv-D[OF ‹l-index-inv α T B SA›
                        ‹i < length SA›
                        ‹SA ! i = Suc j›
                        ‹Suc j < length T›
                        ‹suffix-type T j = L-type›]
       ‹k = α (T ! j)›
       ‹l = B ! k›
  have l > i
    by simp

  show ?thesis
    unfolding l-pred-inv-def
  proof (intro impI allI; elim conjE)
    fix i′
    assume i′ < length (SA[l := j])
           SA[l := j] ! i′ < length T
           suffix-type T (SA[l := j] ! i′) = L-type
    have i′ = l ∨ i′ ≠ l
```

```
      by blast
    then show
     ∃ja<length (SA[l := j]). SA[l := j] ! ja = Suc (SA[l := j] ! i') ∧ ja < i' ∧ ja
< Suc i
    proof
      assume i' = l
      with ‹i' < length (SA[l := j])›
      have SA[l := j] ! i' = j
        by simp

      from ‹l > i› ‹SA ! i = Suc j›
      have SA[l := j] ! i = Suc j
        by simp
      with ‹l > i› ‹i' = l› ‹SA[l := j] ! i' = j› ‹i < length SA›
      show ?thesis
        by auto
    next
      assume i' ≠ l
      with ‹i' < length (SA[l := j])›
      have  i' < length SA
        by simp

      from ‹i' ≠ l› ‹SA[l := j] ! i' < length T›
      have SA ! i' < length T
        by simp

      from ‹i' ≠ l› ‹suffix-type T (SA[l := j] ! i') = L-type›
      have suffix-type T (SA ! i') = L-type
        by simp

      from ‹i' < length SA› ‹SA ! i' < length T› ‹suffix-type T (SA ! i') = L-type›
          ‹l-pred-inv T SA i›[simplified l-pred-inv-def, THEN spec, of i']
      obtain j' where
        j' < length SA
        SA ! j' = Suc (SA ! i')
        j' < i'
        j' < i
        by blast

      with ‹SA ! l = length T› ‹i' ≠ l› ‹suffix-type T (SA ! i') = L-type› ‹i < l›
      show ?thesis
        by auto
    qed
  qed
qed

corollary l-pred-inv-perm-maintained:
  assumes l-perm-inv α T B SA0 SA i
  and     i < length SA
```

and $SA\ !\ i = Suc\ j$
and $Suc\ j < length\ T$
and $suffix\text{-}type\ T\ j = L\text{-}type$
and $k = \alpha\ (T\ !\ j)$
and $l = B\ !\ k$
shows $l\text{-}pred\text{-}inv\ T\ (SA[l := j])\ (Suc\ i)$
by (*metis assms l-perm-inv-elims*(4−7,10,12) *l-pred-inv-maintained*)

76 Seen Invariant

76.1 Establishment

We first show that the invariant is initially true, i.e. *l-seen-inv T SA 0*.

lemma *l-seen-inv-established*:
$l\text{-}seen\text{-}inv\ T\ SA\ 0$
by (*simp add*: *l-seen-inv-def*)

76.2 Maintenance

We now show that the invariant is maintained after each call of *abs-induce-l-step*.

lemma *l-seen-inv-maintained-no-update*:
$[\![l\text{-}seen\text{-}inv\ T\ SA\ i;\ length\ T \leq SA\ !\ i]\!] \Longrightarrow l\text{-}seen\text{-}inv\ T\ SA\ (Suc\ i)$
$[\![l\text{-}seen\text{-}inv\ T\ SA\ i;\ length\ SA \leq i]\!] \Longrightarrow l\text{-}seen\text{-}inv\ T\ SA\ (Suc\ i)$
$[\![l\text{-}seen\text{-}inv\ T\ SA\ i;\ SA\ !\ i < length\ T;\ SA\ !\ i = 0]\!] \Longrightarrow l\text{-}seen\text{-}inv\ T\ SA\ (Suc\ i)$
$[\![l\text{-}seen\text{-}inv\ T\ SA\ i;\ SA\ !\ i < length\ T;\ SA\ !\ i = Suc\ j;\ suffix\text{-}type\ T\ j = S\text{-}type]\!]$
$\Longrightarrow$
$l\text{-}seen\text{-}inv\ T\ SA\ (Suc\ i)$
unfolding *l-seen-inv-def*
using *less-Suc-eq* **by** *auto*

lemma *l-seen-inv-maintained*:
assumes $l\text{-}seen\text{-}inv\ T\ SA\ i$
and $i < length\ SA$
and $SA\ !\ i = Suc\ j$
and $Suc\ j < length\ T$
and $suffix\text{-}type\ T\ j = L\text{-}type$
and $k = \alpha\ (T\ !\ j)$
and $l = B\ !\ k$
and $length\ SA = length\ T$
and $strict\text{-}mono\ \alpha$
and $l\text{-}distinct\text{-}inv\ T\ SA$
and $l\text{-}pred\text{-}inv\ T\ SA\ i$
and $l\text{-}unknowns\text{-}inv\ \alpha\ T\ B\ SA$
and $l\text{-}bucket\text{-}ptr\text{-}inv\ \alpha\ T\ B\ SA$
and $l\text{-}index\text{-}inv\ \alpha\ T\ B\ SA$
shows $l\text{-}seen\text{-}inv\ T\ (SA[l := j])\ (Suc\ i)$
proof −
from *l-distinct-pred-inv-helper*[*OF* ‹$i < length\ SA$›

```
                                ‹SA ! i = Suc j›
                                ‹Suc j < length T›
                                ‹suffix-type T j = L-type›
                                ‹l-distinct-inv T SA›
                                ‹l-pred-inv T SA i›]
have j ∉ set SA
  by assumption

from ‹Suc j < length T›
have j < length T
  by simp

from bucket-size-imp-less-length[OF ‹l-bucket-ptr-inv α T B SA›
                                ‹j < length T›
                                ‹suffix-type T j = L-type›
                                ‹j ∉ set SA›
                                ‹strict-mono α›]
    ‹k = α (T ! j)›
    ‹l = B ! k›
have l < length T
  by simp

from l-index-inv-D[OF ‹l-index-inv α T B SA›
                ‹i < length SA›
                ‹SA ! i = Suc j›
                ‹Suc j < length T›
                ‹suffix-type T j = L-type›]
    ‹k = α (T ! j)›
    ‹l = B ! k›
have l > i
  by simp

from l-unknowns-l-bucket-ptr-inv-helper[OF ‹l-unknowns-inv α T B SA›
                                    ‹l-bucket-ptr-inv α T B SA›
                                    ‹j < length T›
                                    ‹suffix-type T j = L-type›
                                    ‹j ∉ set SA›
                                    ‹strict-mono α›
                                    ‹k = α (T ! j)›
                                    ‹l = B !  k›]
have SA ! l = length T
  by assumption

with ‹SA ! i = Suc j› ‹Suc j < length T› ‹i < l›
have (SA[l := j]) ! i < length T
  by auto

with ‹SA ! i = Suc j› ‹i < l›
have (SA[l := j]) ! i = Suc j
```

by *auto*

from *l-seen-inv-upd*[*OF* ‹*l-seen-inv T SA i*›]
‹*l* > *i*›
‹*SA* ! *l* = *length T*›
‹*l* < *length T*›
‹*length SA* = *length T*›
have *l-seen-inv T* (*SA*[*l* := *j*]) *i*
by *auto*
with *l-seen-inv-Suc*[*OF* - ‹(*SA*[*l* := *j*]) ! *i* = *Suc j*›]
‹*l* < *length T*›
‹*length SA* = *length T*›
show *?thesis*
by (*metis length-list-update nth-list-update-eq*)
qed

corollary *l-seen-inv-perm-maintained*:
assumes *l-perm-inv* α *T B SA0 SA i*
and *i* < *length SA*
and *SA* ! *i* = *Suc j*
and *Suc j* < *length T*
and *suffix-type T j* = *L-type*
and *k* = α (*T* ! *j*)
and *l* = *B* ! *k*
shows *l-seen-inv T* (*SA*[*l* := *j*]) (*Suc i*)
by (*metis assms l-perm-inv-elims*(*2*,*3*−*7*,*10*−*12*) *l-seen-inv-maintained*)

77 Permutation

77.1 Establishment

lemma *l-perm-inv-established*:
assumes *l-perm-pre* α *T B SA*
shows *l-perm-inv* α *T B SA SA 0*
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-pre-elims*[*OF assms*] *l-distinct-inv-perm-established*[*OF assms*]
l-unknowns-inv-perm-established[*OF assms*] *l-bucket-ptr-inv-perm-established*[*OF assms*]
l-index-inv-perm-established[*OF assms*] *l-unchanged-inv-established*
l-locations-inv-perm-established[*OF assms*] *l-pred-inv-perm-established*[*OF assms*]
l-seen-inv-established)

77.2 Maintenance

lemma *l-perm-inv-maintained*:
assumes *l-perm-inv* α *T B SA0 SA i*
and *i* < *length SA*
and *SA* ! *i* = *Suc j*

and *Suc j < length T*
and *suffix-type T j = L-type*
and *k = α (T ! j)*
and *l = B ! k*
shows *l-perm-inv α T (B[k := Suc (B ! k)]) SA0 (SA[l := j]) (Suc i)*
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-inv-elims*[*OF assms*(*1*)] *l-distinct-inv-perm-maintained*[*OF assms*(*1−5*)]
l-unknowns-inv-perm-maintained[*OF assms*] *l-bucket-ptr-inv-perm-maintained*[*OF assms*]
l-index-inv-perm-maintained[*OF assms*] *l-unchanged-inv-perm-maintained*[*OF assms*]
l-locations-inv-perm-maintained[*OF assms*] *l-pred-inv-perm-maintained*[*OF assms*]
l-seen-inv-perm-maintained[*OF assms*])

lemma *l-perm-inv-maintained-no-upd-1*:
assumes *l-perm-inv α T B SA0 SA i*
and *length SA ≤ i*
shows *l-perm-inv α T B SA0 SA* (*Suc i*)
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-inv-elims*[*OF assms*(*1*)] *l-pred-inv-maintained-no-update*
l-seen-inv-maintained-no-update(*2*)[*OF l-perm-inv-elims*(*11*)[*OF assms*(*1*)] *assms*(*2*)])

lemma *l-perm-inv-maintained-no-upd-2*:
assumes *l-perm-inv α T B SA0 SA i*
and *length T ≤ SA ! i*
shows *l-perm-inv α T B SA0 SA* (*Suc i*)
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-inv-elims*[*OF assms*(*1*)] *l-pred-inv-maintained-no-update*
l-seen-inv-maintained-no-update(*1*)[*OF l-perm-inv-elims*(*11*)[*OF assms*(*1*)] *assms*(*2*)])

lemma *l-perm-inv-maintained-no-upd-3*:
assumes *l-perm-inv α T B SA0 SA i*
and *SA ! i < length T*
and *SA ! i = 0*
shows *l-perm-inv α T B SA0 SA* (*Suc i*)
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-inv-elims*[*OF assms*(*1*)] *l-pred-inv-maintained-no-update*
l-seen-inv-maintained-no-update(*3*)[*OF l-perm-inv-elims*(*11*)[*OF assms*(*1*)] *assms*(*2−*)])

lemma *l-perm-inv-maintained-no-upd-4*:
assumes *l-perm-inv α T B SA0 SA i*
and *SA ! i < length T*
and *SA ! i = Suc j*
and *suffix-type T j = S-type*

shows *l-perm-inv α T B SA0 SA* (*Suc i*)
unfolding *l-perm-inv-def*
by (*simp add*: *l-perm-inv-elims*[*OF assms*(*1*)] *l-pred-inv-maintained-no-update*
l-seen-inv-maintained-no-update(*4*)[*OF l-perm-inv-elims*(*11*)[*OF assms*(*1*)] *assms*(*2*−)])

lemmas *l-perm-inv-maintained-no-update* =
l-perm-inv-maintained-no-upd-1 l-perm-inv-maintained-no-upd-2 l-perm-inv-maintained-no-upd-3 l-perm-inv-maintained-no-upd-4

lemma *abs-induce-l-perm-step*:
assumes *l-perm-inv α T B SA0 SA i*
and *abs-induce-l-step* (*B*, *SA*, *i*) (*α*, *T*) = (*B′*, *SA′*, *i′*)
shows *l-perm-inv α T B′ SA0 SA′ i′*
proof (*cases i* < *length SA* ∧ *SA* ! *i* < *length T*)
assume *A*: *i* < *length SA* ∧ *SA* ! *i* < *length T*
show *?thesis*
proof (*cases SA* ! *i*)
case *0*
then show *?thesis*
using *A l-perm-inv-maintained-no-update*(*3*)[*OF assms*(*1*)] *assms*(*2*)
by *force*
next
case (*Suc j*)
assume *B*: *SA* ! *i* = *Suc j*
show *?thesis*
proof (*cases suffix-type T j*)
case *S-type*
then show *?thesis*
using *A B l-perm-inv-maintained-no-update*(*4*)[*OF assms*(*1*)] *assms*(*2*)
by *force*
next
case *L-type*
then show *?thesis*
using *A B l-perm-inv-maintained*[*OF assms*(*1*)] *assms*(*2*)
by (*clarsimp simp*: *Let-def*)
qed
qed
next
assume *A*: ¬(*i* < *length SA* ∧ *SA* ! *i* < *length T*)
show *?thesis*
using *l-perm-inv-maintained-no-update*(*1*,*2*)[*OF assms*(*1*)] *A assms*(*2*)
by *force*
qed

lemma *abs-induce-l-base-perm-inv-maintained*:
assumes *l-perm-inv α T B SA0 SA 0*
and *abs-induce-l-base α T B SA* = (*B′*, *SA′*, *i*)

shows *l-perm-inv α T B′ SA0 SA′ i*
proof −
let *?P* = λ(*B*, *SA*, *i*). *l-perm-inv α T B SA0 SA i*

from *assms*(*2*)
have *repeat* (*length T*) *abs-induce-l-step* (*B*, *SA*, *0*) (*α*, *T*) = (*B′*, *SA′*, *i*)
by (*simp add*: *abs-induce-l-base-def*)
moreover
have ⋀*a*. *?P a* ⟹ *?P* (*abs-induce-l-step a* (*α*, *T*))
using *abs-induce-l-perm-step* **by** *blast*
ultimately show *?thesis*
using *repeat-maintain-inv*[*of ?P abs-induce-l-step* (*α*, *T*) (*B*, *SA*, *0*) *length T*]
using *assms*(*1*) **by** *auto*
qed

78 Sorted

lemma *l-suffix-sorted-inv-established*:
assumes *l-bucket-init α T B*
shows *l-suffix-sorted-inv α T B SA*
unfolding *l-suffix-sorted-inv-def*
proof(*intro allI impI*)
fix *b*
assume *b* ≤ *α* (*Max* (*set T*))
with *l-bucket-initD*[*OF assms*, *of b*]
have *B* ! *b* = *bucket-start α T b* **.**
then
show *ordlistns.sorted*
(*map* (*suffix T*)
(*list-slice SA* (*bucket-start α T b*) (*B* ! *b*)))
by *simp*
qed

lemma *l-prefix-sorted-inv-established*:
assumes *l-bucket-init α T B*
shows *l-prefix-sorted-inv α T B SA*
unfolding *l-prefix-sorted-inv-def*
proof(*intro allI impI*)
fix *b*
assume *b* ≤ *α* (*Max* (*set T*))
with *l-bucket-initD*[*OF assms*, *of b*]
have *B* ! *b* = *bucket-start α T b* **.**
then show *ordlistns.sorted* (*map* (*lms-prefix T*) (*list-slice SA* (*bucket-start α T b*) (*B* ! *b*)))
by *simp*
qed

lemma *l-sorted-inv-maintained-step*:
assumes *l-perm-inv α T B SA0 SA i*

```
  and     i < length SA
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
  and     b ≤ α (Max (set T))
  and     b ≠ k
  and     ordlistns.sorted (map f (list-slice SA (bucket-start α T b) (B ! b)))
shows ordlistns.sorted (map f (list-slice (SA[l := j]) (bucket-start α T b) (B[k :=
Suc l] ! b)))
proof −
  let ?xs = list-slice (SA[l := j]) (bucket-start α T b) (B[k := Suc l] ! b) and
      ?ys = list-slice SA (bucket-start α T b) (B ! b)

  have i < length T
    by (metis assms(1,2) l-perm-inv-def)
  hence k ≤ α (Max (set T))
    using assms(1,4,6) l-perm-inv-def strict-mono-less-eq by fastforce

  from ‹Suc j < length T›
  have j < length T
    by simp

  from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                     ‹SA ! i = Suc j›
                                     ‹Suc j < length T›
                                     ‹suffix-type T j = L-type›
                                     l-perm-inv-elims(4,10)[OF assms(1)]]
  have j ∉ set SA
    by assumption

 from l-bucket-ptr-inv-imp-less-l-bucket-end[OF l-perm-inv-elims(6)[OF assms(1)]
                                               ‹j < length T›
                                               ‹suffix-type T j = L-type›
                                               ‹j ∉ set SA›
                                               l-perm-inv-elims(12)[OF assms(1)]]
      ‹k = α (T ! j)›
      ‹l = B ! k›
  have l < l-bucket-end α T k
    by simp
  hence l < length SA
   by (metis assms(1) bucket-end-le-length dual-order.strict-trans1 l-bucket-end-le-bucket-end
l-perm-inv-def)

  have B[k := Suc l] ! b = B ! b
    using assms(9) by auto

  have l < bucket-start α T b ∨ B ! b ≤ l
```

```
  proof −
    have b < k ∨ k < b
      using ‹b ≠ k› less-linear by blast
    moreover
    have b < k ⟹ ?thesis
    proof −
      assume b < k
      hence bucket-end α T b ≤ bucket-start α T k
        by (simp add: less-bucket-end-le-start)
      hence l-bucket-end α T b ≤ bucket-start α T k
        using l-bucket-end-le-bucket-end le-trans by blast
     with l-bucket-ptr-inv-imp-le-l-bucket-end[OF l-perm-inv-elims(6)[OF assms(1)]
‹b ≤ -›]
      have B ! b ≤ bucket-start α T k
        by linarith
     with l-bucket-ptr-inv-imp-ge-bucket-start[OF l-perm-inv-elims(6)[OF assms(1)]
‹k ≤ -›]
      show ?thesis
        using assms(7) le-trans by blast
    qed
    moreover
    have k < b ⟹ ?thesis
    proof −
      assume k < b
      hence bucket-end α T k ≤ bucket-start α T b
        by (simp add: less-bucket-end-le-start)
      hence l-bucket-end α T k ≤ bucket-start α T b
        using l-bucket-end-le-bucket-end le-trans by blast
      with ‹l < l-bucket-end α T k›
      show ?thesis
        using less-le-trans by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  with ‹B[k := Suc l] ! b = B ! b›
       list-slice-update-unchanged-1
       list-slice-update-unchanged-2
  have ?xs = ?ys
    by auto
  then show ?thesis
    using assms(10) by auto
qed

lemma l-suffix-sorted-inv-maintained-step:
  assumes l-perm-inv α T B SA0 SA i
  and     l-suffix-sorted-pre α T SA0
  and     l-suffix-sorted-inv α T B SA
  and     i < length SA
```

```
  and     SA ! i = Suc j
  and     Suc j < length T
  and     suffix-type T j = L-type
  and     k = α (T ! j)
  and     l = B ! k
shows l-suffix-sorted-inv α T (B[k := Suc l]) (SA[l := j])
  unfolding l-suffix-sorted-inv-def
proof (safe)
  fix b
  assume b ≤ α (Max (set T))
  let ?xs = list-slice (SA[l := j]) (bucket-start α T b) (B[k := Suc l] ! b) and
      ?ys = list-slice SA (bucket-start α T b) (B ! b)

  have i < length T
    by (metis assms(1,4) l-perm-inv-def)
  hence k ≤ α (Max (set T))
    using assms(1,6,8) l-perm-inv-def strict-mono-less-eq by fastforce

   from ‹Suc j < length T›
    have j < length T
      by simp

    from l-distinct-pred-inv-helper[OF ‹i < length SA›
                                       ‹SA ! i = Suc j›
                                       ‹Suc j < length T›
                                       ‹suffix-type T j = L-type›
                                       l-perm-inv-elims(4,10)[OF assms(1)]]
    have j ∉ set SA
      by assumption

   from l-bucket-ptr-inv-imp-less-l-bucket-end[OF l-perm-inv-elims(6)[OF assms(1)]
                                                  ‹j < length T›
                                                  ‹suffix-type T j = L-type›
                                                  ‹j ∉ set SA›
                                                  l-perm-inv-elims(12)[OF assms(1)]]
        ‹k = α (T ! j)›
        ‹l = B ! k›
    have l < l-bucket-end α T k
      by simp
    hence l < length SA
     by (metis assms(1) bucket-end-le-length dual-order.strict-trans1 l-bucket-end-le-bucket-end
l-perm-inv-def)

  have b = k ∨ b ≠ k
    by simp
  moreover
  have b ≠ k ⟹ ordlistns.sorted (map (suffix T) ?xs)
   using ‹b ≤ α (Max (set T))› assms l-sorted-inv-maintained-step l-suffix-sorted-inv-def
by blast
```

```
  moreover
  have b = k ⟹ ordlistns.sorted (map (suffix T) ?xs)
  proof −
    assume b = k
    hence B[k := Suc l] ! b = Suc l
      using ‹b ≤ α (Max (set T))› assms(1) l-perm-inv-elims(1) by fastforce

    have SA[l := j] ! l = j
      by (simp add: ‹l < length SA›)

    from list-slice-update-unchanged-2[of B ! b j - bucket-start α T b]
    have list-slice (SA[l := j]) (bucket-start α T b) (B ! b) = ?ys
      using ‹b = k› assms(9)
      by (simp add: list-slice-update-unchanged-2)
    hence ?xs = ?ys @ list-slice (SA[l := j]) (B ! b) (B[k := Suc l] ! k)
      by (metis Suc-n-not-le-n ‹B[k := Suc l] ! b = Suc l› ‹b = k› ‹k ≤ α (Max
(set T))› assms(1,9)
                l-bucket-ptr-inv-imp-ge-bucket-start l-perm-inv-elims(6) linear
list-slice-append)
    moreover
    have list-slice (SA[l := j]) (B ! b) (B[k := Suc l] ! k) = [j]
      by (metis ‹B[k := Suc l] ! b = Suc l› ‹SA[l := j] ! l = j›
            ‹b = k› ‹l < length SA› assms(9)
            length-list-update lessI list-slice-Suc list-slice-n-n)
    ultimately have ?xs = ?ys @ [j]
      by simp
    hence map (suffix T) ?xs = (map (suffix T) ?ys) @ [suffix T j]
      by simp
    moreover
    have ordlistns.sorted ((map (suffix T) ?ys) @ [suffix T j])
    proof −
      from l-suffix-sorted-invD[OF assms(3) ‹b ≤ -›]
      have ordlistns.sorted (map (suffix T) ?ys) .
      moreover
      have ordlistns.sorted [suffix T j]
        by simp
      moreover
      have ∀ x ∈ set (map (suffix T) ?ys).
           ∀ y ∈ set [suffix T j].
                list-less-eq-ns x y
      proof(safe)
        fix x y
        assume
          x ∈ set (map (suffix T) ?ys)
          y ∈ set [suffix T j]
        hence y = suffix T j
          by simp

        have A: length ?ys = B ! b − bucket-start α T b
```

using ‹$b = k$› ‹$l < \mathit{length}\ \mathit{SA}$› *assms*(9) *min-def* **by** *auto*

from *in-set-conv-nth*[*THEN iffD1*, *OF* ‹$x \in \mathit{set}\ (\mathit{map}\ (\mathit{suffix}\ T)\ ?ys)$›]
have $\exists i'.\ x = \mathit{suffix}\ T\ (\mathit{SA}\ !\ i') \wedge$
$\mathit{bucket\text{-}start}\ \alpha\ T\ b \leq i' \wedge i' < B\ !\ b$
by (*metis A add.commute le-add1 length-map*
less-diff-conv nth-list-slice nth-map)
then obtain j' **where** j'-*assms*:
$x = \mathit{suffix}\ T\ (\mathit{SA}\ !\ j')$
$\mathit{bucket\text{-}start}\ \alpha\ T\ b \leq j'$
$j' < B\ !\ b$
by *blast*
hence j'-*less*: $j' < \mathit{length}\ \mathit{SA}$
using ‹$b = k$› ‹$l < \mathit{length}\ \mathit{SA}$› *assms*(9) *dual-order.strict-trans*
by *blast*
with *l-locations-inv-D*
[*OF l-perm-inv-elims*(9)[*OF assms*(1)] ‹$b \leq$ -› - j'-*assms*(2,3)]
have $\mathit{SA}\ !\ j' < \mathit{length}\ T$
$\mathit{suffix\text{-}type}\ T\ (\mathit{SA}\ !\ j') = \mathit{L\text{-}type}$
$\alpha\ (T\ !\ (\mathit{SA}\ !\ j')) = b$
by *blast*+
with *l-pred-inv-D*[*OF l-perm-inv-elims*(10)[*OF assms*(1)] j'-*less*]
have $\exists j{<}\mathit{length}\ \mathit{SA}.$
$\mathit{SA}\ !\ j = \mathit{Suc}\ (\mathit{SA}\ !\ j') \wedge$
$\mathit{SA}\ !\ j < \mathit{length}\ T \wedge$
$j < j' \wedge$
$j < i$
by *blast*
then obtain i' **where** i'-*assms*:
$i' < \mathit{length}\ \mathit{SA}$
$\mathit{SA}\ !\ i' = \mathit{Suc}\ (\mathit{SA}\ !\ j')$
$\mathit{SA}\ !\ i' < \mathit{length}\ T$
$i' < j'$
$i' < i$
by *blast*

have $\alpha\ (T\ !\ j) = b$
using ‹$b = k$› *assms*(8) **by** *blast*
with ‹$\alpha\ (T\ !\ (\mathit{SA}\ !\ j')) = b$›
have $T\ !\ (\mathit{SA}\ !\ j') = T\ !\ j$
by (*metis* (*no-types*, *lifting*) *assms*(1) *l-perm-inv-elims*(12)
less-le not-le strict-mono-less-eq)
moreover
have $x = T\ !\ (\mathit{SA}\ !\ j')\ \#\ \mathit{suffix}\ T\ (\mathit{SA}\ !\ i')$
using ‹$\mathit{SA}\ !\ i' = \mathit{Suc}\ (\mathit{SA}\ !\ j')$›
‹$\mathit{SA}\ !\ j' < \mathit{length}\ T$›
‹$x = \mathit{suffix}\ T\ (\mathit{SA}\ !\ j')$›
suffix-cons-Suc
by *auto*

moreover
have $y = T\ !\ j\ \#\ suffix\ T\ (SA\ !\ i)$
using ‹$j < length\ T$› ‹$y = suffix\ T\ j$›
suffix-cons-Suc
‹$SA\ !\ i = Suc\ j$›
by *auto*
ultimately
have *list-less-eq-ns x y* =
list-less-eq-ns (*suffix T* (*SA* ! *i'*)) (*suffix T* (*SA* ! *i*))
using *list-less-eq-ns-cons*
[*of T* ! (*SA* ! *j'*)
suffix T (*SA* ! *i'*)
T ! *j*
suffix T (*SA* ! *i*)]
by *simp*
moreover
have *list-less-eq-ns* (*suffix T* (*SA* ! *i'*)) (*suffix T* (*SA* ! *i*))
proof −
have *i'* < *length T*
using ‹*i* < *length T*› ‹*i'* < *i*› *order.strict-trans* **by** *blast*
with *index-in-bucket-interval-gen*
[*of i' T α*,
OF - l-perm-inv-elims(*12*)[*OF assms*(*1*)]]
obtain *b0* **where** *b0-assms*:
b0 ≤ *α* (*Max* (*set T*))
bucket-start α T b0 ≤ *i'*
i' < *bucket-end α T b0*
by *blast*
with *same-bucket-same-hd*
[*OF l-perm-inv-elims*(*8*,*9*,*6*,*5*)[*OF assms*(*1*)],
of b0 i']
l-perm-inv-elims(*2*,*3*,*14*,*15*)[*OF assms*(*1*)]
have *α* (*T* ! (*SA* ! *i'*)) = *b0*
using ‹*SA* ! *i'* < *length T*› ‹*i'* < *length SA*› **by** *auto*

from *index-in-bucket-interval-gen*[*OF* ‹*i* < *length T*› *l-perm-inv-elims*(*12*)[*OF assms*(*1*)]]
obtain *b1* **where** *b1-assms*:
b1 ≤ *α* (*Max* (*set T*))
bucket-start α T b1 ≤ *i*
i < *bucket-end α T b1*
by *blast*
with *same-bucket-same-hd*
[*OF l-perm-inv-elims*(*8*,*9*,*6*,*5*)[*OF assms*(*1*)],
of b1 i]
l-perm-inv-elims(*2*,*3*,*14*,*15*)[*OF assms*(*1*)]
have *α* (*T* ! (*SA* ! *i*)) = *b1*
using *assms*(*4*−*6*) **by** *auto*

have *b0* $\leq$ *b1*
proof (*rule ccontr*)
  assume $\neg$ *b0* $\leq$ *b1*
  hence *b1* $<$ *b0*
    by *auto*
  hence *bucket-end* α *T b1* $\leq$ *bucket-start* α *T b0*
    by (*simp add*: *less-bucket-end-le-start*)
  with ‹*i* $<$ *bucket-end* α *T b1*› ‹*bucket-start* α *T b0* $\leq$ *i'*› ‹*i'* $<$ *i*›
  show *False*
    by *linarith*
qed
hence *b0* $<$ *b1* $\vee$ *b0* $=$ *b1*
  by *linarith*
moreover
have *b0* $<$ *b1* $\Longrightarrow$ *?thesis*
proof $-$
  assume *b0* $<$ *b1*
  with ‹α (*T* ! (*SA* ! *i'*)) $=$ *b0*›
      ‹α (*T* ! (*SA* ! *i*)) $=$ *b1*›
  have *T* ! (*SA* ! *i'*) $<$ *T* ! (*SA* ! *i*)
    using *assms*(*1*) *l-perm-inv-elims*(*12*) *strict-mono-less* **by** *blast*
  moreover
  have $\exists$ *as*. *suffix T* (*SA* ! *i'*) $=$ *T* ! (*SA* ! *i'*) # *as*
    using ‹*SA* ! *i'* $<$ *length T*› *suffix-cons-Suc* **by** *blast*
  then obtain *as* **where** *as-assm*:
    suffix T (*SA* ! *i'*) $=$ *T* ! (*SA* ! *i'*) # *as*
    by *blast*
  moreover
  have $\exists$ *bs*. *suffix T* (*SA* ! *i*) $=$ *T* ! (*SA* ! *i*) # *bs*
    by (*metis Cons-nth-drop-Suc assms*(*5*,*6*))
  then obtain *bs* **where** *bs-assm*:
    suffix T (*SA* ! *i*) $=$ *T* ! (*SA* ! *i*) # *bs*
    by *blast*
  ultimately show *?thesis*
    by (*simp add*: *less-le list-less-eq-ns-cons*)
qed
moreover
have *b0* $=$ *b1* $\Longrightarrow$ *?thesis*
proof $-$
  assume *b0* $=$ *b1*
  hence α (*T* ! (*SA* ! *i'*)) $=$ α (*T* ! (*SA* ! *i*))
    by (*simp add*: ‹α (*T* ! (*SA* ! *i'*)) $=$ *b0*› ‹α (*T* ! (*SA* ! *i*)) $=$ *b1*›)
  hence *T* ! (*SA* ! *i'*) $=$ *T* ! (*SA* ! *i*)
    by (*metis* (*no-types*, *opaque-lifting*) *assms*(*1*) *strict-mono-less*
        l-perm-inv-elims(*12*) *not-less-iff-gr-or-eq*)

  have *i* $<$ *bucket-end* α *T b0*
    by (*simp add*: ‹*b0* $=$ *b1*› ‹*i* $<$ *bucket-end* α *T b1*›)

```
from unknown-range-values[OF l-perm-inv-elims(8,5)[OF assms(1)] - -
                             l-perm-inv-elims(14,15)[OF assms(1)]
                             ‹b0 ≤ α -›]
    l-perm-inv-elims(2,3)[OF assms(1)]
have i < B ! b0 ∨ lms-bucket-start α T b0 ≤ i
  using assms(5) assms(6) not-le by force

from unknown-range-values[OF l-perm-inv-elims(8,5)[OF assms(1)] - -
                             l-perm-inv-elims(14,15)[OF assms(1)]
                             ‹b0 ≤ α -›]
    l-perm-inv-elims(2,3)[OF assms(1)]
    ‹SA ! i' < length T›
have i' < B ! b0 ∨
      lms-bucket-start α T b0 ≤ i'
  using not-le by force
moreover
have i' < B ! b0 ⟹ ?thesis
proof −
  assume i' < B ! b0
  have i < B ! b0 ⟹ ?thesis
  proof −
    assume i-b0-assm: i < B ! b0
    let ?xs = list-slice SA (bucket-start α T b0) (B ! b0)

    have i' ≤ i
      by (simp add: ‹i' < i› less-imp-le-nat)
    from l-suffix-sorted-invD[OF assms(3) ‹b0 ≤ α -›]
    have ordlistns.sorted (map (suffix T) ?xs) .
    with ordlistns.list-slice-sorted-nth-mono
           [OF - b0-assms(2) ‹i' ≤ i› i-b0-assm,
            of map (suffix T) SA]
    show ?thesis
      by (metis i'-assms(1) assms(4) length-map
            map-list-slice nth-map)
  qed
  moreover
  have lms-bucket-start α T b0 ≤ i ⟹ ?thesis
  proof −
    assume lms-bucket-start α T b0 ≤ i
    with lms-init-D
           [OF lms-init-unchanged
                 [OF l-perm-inv-elims(8)[OF assms(1)]],
            OF - - l-perm-inv-elims(14)[OF assms(1)]
              ‹b0 ≤ α -›]
        l-perm-inv-elims(2,3)[OF assms(1)]
        ‹i < bucket-end α T b0›
    have SA ! i ∈ lms-bucket α T b0
      by (metis bucket-end-le-length list-slice-nth-mem)
    hence suffix-type T (SA ! i) = S-type
```

```
    by (metis ‹b0 ≤ α (Max (set T))›
              ‹i < bucket-end α T b0›
              ‹length SA = length SA0›
              ‹length SA0 = length T›
              ‹lms-bucket-start α T b0 ≤ i›
              ‹lms-init α T SA0›
              assms(1) abs-is-lms-def l-perm-inv-elims(8)
              lms-init-nth lms-init-unchanged)
  moreover
  from l-locations-inv-D
       [OF l-perm-inv-elims(9)[OF assms(1)]
           ‹b0 ≤ α -›
           i'-assms(1)
           b0-assms(2)
           ‹i' < B ! b0›]
  have suffix-type T (SA ! i') = L-type SA ! i' < length T
    by blast+
  ultimately show ?thesis
    using l-less-than-s-type-suffix[of SA ! i T SA ! i']
    by (simp add: ordlistns.less-imp-le ‹α (T ! (SA ! i')) = b0›
                  ‹T ! (SA ! i') = T ! (SA ! i)› assms(5,6))
qed
ultimately
show ?thesis
  using ‹i < B ! b0 ∨ lms-bucket-start α T b0 ≤ i›
  by blast
qed
moreover
have lms-bucket-start α T b0 ≤ i' ⟹ ?thesis
proof −
  assume lms-bucket-start α T b0 ≤ i'

  let ?xs =
    list-slice SA (lms-bucket-start α T b0) (bucket-end α T b0)

  from l-suffix-sorted-pre-maintained
       [OF l-perm-inv-elims(8)[OF assms(1)]]
       l-perm-inv-elims(2,3)[OF assms(1)]
       l-suffix-sorted-preD[of α T SA b0, OF - ‹b0 ≤ α -›]
  have ordlistns.sorted (map (suffix T) ?xs)
    by (simp add: assms(2))
  with ordlistns.list-slice-sorted-nth-mono
       [of map (suffix T) SA
           lms-bucket-start α T b0
           bucket-end α T b0 i' i]
       ‹i < bucket-end α T b0›
       ‹i' < i›
       i'-assms(1)
       ‹lms-bucket-start α T b0 ≤ i'›
```

```
        show ?thesis
          by (metis assms(4) length-map map-list-slice
                not-less not-less-iff-gr-or-eq nth-map)
      qed
      ultimately show ?thesis
        by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show list-less-eq-ns x y
    by simp
 qed
 ultimately show ?thesis
  using ordlistns.sorted-append[of map (suffix T) ?ys [suffix T j]]
  by blast
qed
ultimately show ?thesis
 by simp
qed
ultimately show ordlistns.sorted (map (suffix T) ?xs)
 by blast
qed

lemma l-prefix-sorted-inv-maintained-step:
 assumes l-perm-inv α T B SA0 SA i
 and     l-prefix-sorted-pre α T SA0
 and     l-prefix-sorted-inv α T B SA
 and     i < length SA
 and     SA ! i = Suc j
 and     Suc j < length T
 and     suffix-type T j = L-type
 and     k = α (T ! j)
 and     l = B ! k
shows l-prefix-sorted-inv α T (B[k := Suc l]) (SA[l := j])
 unfolding l-prefix-sorted-inv-def
proof (safe)
 fix b
 assume b ≤ α (Max (set T))
 let ?xs = list-slice (SA[l := j]) (bucket-start α T b) (B[k := Suc l] ! b)
 and ?ys = list-slice SA (bucket-start α T b) (B ! b)

 have i < length T
  by (metis assms(1,4) l-perm-inv-def)
 hence k ≤ α (Max (set T))
  using assms(1,6,8) l-perm-inv-def strict-mono-less-eq by fastforce

 from ‹Suc j < length T›
  have j < length T
```

```
  by simp

  from l-distinct-pred-inv-helper
        [OF ‹i < length SA›
            ‹SA ! i = Suc j›
            ‹Suc j < length T›
            ‹suffix-type T j = L-type›
            l-perm-inv-elims(4,10)[OF assms(1)]]
  have j ∉ set SA
    by assumption

  from l-bucket-ptr-inv-imp-less-l-bucket-end
        [OF l-perm-inv-elims(6)[OF assms(1)]
            ‹j < length T›
            ‹suffix-type T j = L-type›
            ‹j ∉ set SA›
            l-perm-inv-elims(12)[OF assms(1)]]
       ‹k = α (T ! j)›
       ‹l = B ! k›
  have l < l-bucket-end α T k
    by simp
  hence l < length SA
    by (metis bucket-end-le-length dual-order.strict-trans1
          assms(1) l-bucket-end-le-bucket-end l-perm-inv-def)

have b = k ∨ b ≠ k
  by simp
moreover
have b ≠ k ⟹
      ordlistns.sorted (map (lms-prefix T) ?xs)
  using ‹b ≤ α (Max (set T))›
        assms l-sorted-inv-maintained-step l-prefix-sorted-inv-def
  by blast
moreover
have b = k ⟹
      ordlistns.sorted (map (lms-prefix T) ?xs)
proof −
  assume b = k
  hence B[k := Suc l] ! b = Suc l
    using ‹b ≤ α (Max (set T))› assms(1) l-perm-inv-elims(1)
    by fastforce

  have SA[l := j] ! l = j
    by (simp add: ‹l < length SA›)

  from list-slice-update-unchanged-2
  have list-slice (SA[l := j]) (bucket-start α T b) (B ! b) = ?ys
    using ‹b = k› assms(9)
    by fast
```

hence *?xs = ?ys @ list-slice (SA[l := j]) (B ! b) (B[k := Suc l] ! k)*
by (*metis ‹B[k := Suc l] ! b = Suc l› ‹b = k›*
‹k ≤ α (Max (set T))›
assms(1,9) Suc-n-not-le-n list-slice-append linear
l-bucket-ptr-inv-imp-ge-bucket-start l-perm-inv-elims(6))
moreover
have *list-slice (SA[l := j]) (B ! b) (B[k := Suc l] ! k) = [j]*
by (*metis ‹B[k := Suc l] ! b = Suc l›*
‹SA[l := j] ! l = j›
‹b = k›
‹l < length SA›
assms(9) length-list-update lessI list-slice-Suc list-slice-n-n)
ultimately
have *?xs = ?ys @ [j]*
by *simp*
hence *map (lms-prefix T) ?xs = (map (lms-prefix T) ?ys) @ [lms-prefix T j]*
by *simp*
moreover
have *ordlistns.sorted ((map (lms-prefix T) ?ys) @ [lms-prefix T j])*
proof −
from *l-prefix-sorted-invD[OF assms(3) ‹b ≤ -›]*
have *ordlistns.sorted (map (lms-prefix T) ?ys)* **.**
moreover
have *ordlistns.sorted [lms-prefix T j]*
by *simp*
moreover
have *∀ x ∈ set (map (lms-prefix T) ?ys). ∀ y ∈ set [lms-prefix T j].*
list-less-eq-ns x y
proof(*safe*)
fix *x y*
assume *x ∈ set (map (lms-prefix T) ?ys) y ∈ set [lms-prefix T j]*
hence *y = lms-prefix T j*
by *simp*

have *A*: *length ?ys = B ! b − bucket-start α T b*
using *‹b = k› ‹l < length SA› assms(9) min-def* **by** *auto*

from *in-set-conv-nth[THEN iffD1, OF ‹x ∈ set (map (lms-prefix T) ?ys)›]*
have *∃ i′. x = lms-prefix T (SA ! i′) ∧*
bucket-start α T b ≤ i′ ∧
i′ < B ! b
by (*metis A add.commute le-add1 length-map less-diff-conv*
nth-list-slice nth-map)
then obtain *j′* **where** *j′-assms*:
x = lms-prefix T (SA ! j′)
bucket-start α T b ≤ j′
j′ < B ! b
by *blast*
hence *j′ < length SA*

```
      using ‹b = k› ‹l < length SA›
            assms(9) dual-order.strict-trans by blast
    with l-locations-inv-D
            [OF l-perm-inv-elims(9)[OF assms(1)]
                ‹b ≤ -› -
                j'-assms(2,3)]
    have SA ! j' < length T
        suffix-type T (SA ! j') = L-type
        α (T ! (SA ! j')) = b
      by blast+
    with l-pred-inv-D[OF l-perm-inv-elims(10)[OF assms(1)] ‹j' < length SA›]
    have ∃j<length SA. SA ! j = Suc (SA ! j') ∧ SA ! j < length T ∧ j < j' ∧
j < i
      by blast
    then obtain i' where
      i' < length SA
      SA ! i' = Suc (SA ! j')
      SA ! i' < length T
      i' < j'
      i' < i
      by blast

    have α (T ! j) = b
      using ‹b = k› assms(8) by blast
    with ‹α (T ! (SA ! j')) = b›
    have T ! (SA ! j') = T ! j
      by (metis (no-types, lifting) assms(1) l-perm-inv-elims(12) less-le not-le
strict-mono-less-eq)
    moreover
    have x = T ! (SA ! j') # lms-prefix T (SA ! i')
      by (simp add: ‹SA ! i' = Suc (SA ! j')› ‹SA ! j' < length T›
                ‹suffix-type T (SA ! j') = L-type› ‹x = lms-prefix T (SA ! j')›
                l-type-lms-prefix-cons)
    moreover
    have y = T ! j # lms-prefix T (SA ! i)
      by (simp add: ‹j < length T› ‹y = lms-prefix T j› assms(5,7)
                l-type-lms-prefix-cons)
    ultimately have
      list-less-eq-ns x y =
        list-less-eq-ns (lms-prefix T (SA ! i')) (lms-prefix T (SA ! i))
      using list-less-eq-ns-cons[of T ! (SA ! j') lms-prefix T (SA ! i') T ! j
                                    lms-prefix T (SA ! i)]
      by simp
    moreover
    have list-less-eq-ns (lms-prefix T (SA ! i')) (lms-prefix T (SA ! i))
    proof −
      have i' < length T
        using ‹i < length T› ‹i' < i› order.strict-trans by blast
     with index-in-bucket-interval-gen[of i' T α, OF - l-perm-inv-elims(12)[OF
```

assms(*1*)]]
obtain *b0* **where**
b0 ≤ α (*Max* (*set T*))
bucket-start α *T b0* ≤ *i'*
i' < *bucket-end* α *T b0*
by *blast*
with *same-bucket-same-hd*[*OF l-perm-inv-elims*(*8*,*9*,*6*,*5*)[*OF assms*(*1*)],
of b0 i']
l-perm-inv-elims(*2*,*3*,*14*,*15*)[*OF assms*(*1*)]
have α (*T* ! (*SA* ! *i'*)) = *b0*
using ‹*SA* ! *i'* < *length T*› ‹*i'* < *length SA*› **by** *auto*

from *index-in-bucket-interval-gen*[*OF* ‹*i* < *length T*› *l-perm-inv-elims*(*12*)[*OF*
assms(*1*)]]
obtain *b1* **where**
b1 ≤ α (*Max* (*set T*))
bucket-start α *T b1* ≤ *i*
i < *bucket-end* α *T b1*
by *blast*
with *same-bucket-same-hd*[*OF l-perm-inv-elims*(*8*,*9*,*6*,*5*)[*OF assms*(*1*)],
of b1 i]
l-perm-inv-elims(*2*,*3*,*14*,*15*)[*OF assms*(*1*)]
have α (*T* ! (*SA* ! *i*)) = *b1*
using *assms*(*4*−*6*) **by** *auto*

have *b0* ≤ *b1*
proof (*rule ccontr*)
assume ¬*b0* ≤ *b1*
hence *b1* < *b0*
by *auto*
hence *bucket-end* α *T b1* ≤ *bucket-start* α *T b0*
by (*simp add*: *less-bucket-end-le-start*)
with ‹*i* < *bucket-end* α *T b1*› ‹*bucket-start* α *T b0* ≤ *i'*› ‹*i'* < *i*›
show *False*
by *linarith*
qed
hence *b0* < *b1* ∨ *b0* = *b1*
by *linarith*
moreover
have *b0* < *b1* ⟹ *?thesis*
proof −
assume *b0* < *b1*
with ‹α (*T* ! (*SA* ! *i'*)) = *b0*› ‹α (*T* ! (*SA* ! *i*)) = *b1*›
have *T* ! (*SA* ! *i'*) < *T* ! (*SA* ! *i*)
using *assms*(*1*) *l-perm-inv-elims*(*12*) *strict-mono-less* **by** *blast*
moreover
have ∃ *as*. *lms-prefix T* (*SA* ! *i'*) = *T* ! (*SA* ! *i'*) # *as*
by (*metis* ‹*SA* ! *i'* < *length T*› *lms-slice-hd*
lms-lms-prefix not-lms-imp-next-eq-lms-prefix)

```
  then obtain as where
    lms-prefix T (SA ! i') = T ! (SA ! i') # as
    by blast
  moreover
  have ∃ bs. lms-prefix T (SA ! i) = T ! (SA ! i) # bs
    by (metis (full-types) SL-types.exhaust assms(5−7) abs-is-lms-def
                           l-type-lms-prefix-cons lms-lms-prefix)
  then obtain bs where
    lms-prefix T (SA ! i) = T ! (SA ! i) # bs
    by blast
  ultimately show ?thesis
    by (simp add: less-le list-less-eq-ns-cons)
qed
moreover
have b0 = b1 ⟹ ?thesis
proof −
  assume b0 = b1
  hence α (T ! (SA ! i')) = α (T ! (SA ! i))
    by (simp add: ‹α (T ! (SA ! i')) = b0› ‹α (T ! (SA ! i)) = b1›)
  hence T ! (SA ! i') = T ! (SA ! i)
    by (metis (no-types, opaque-lifting) assms(1) strict-mono-less
          l-perm-inv-elims(12) not-less-iff-gr-or-eq)

  have i < bucket-end α T b0
    by (simp add: ‹b0 = b1› ‹i < bucket-end α T b1›)

  from unknown-range-values
        [OF l-perm-inv-elims(8,5)[OF assms(1)] - -
            l-perm-inv-elims(14,15)[OF assms(1)] ‹b0 ≤ α ->]
      l-perm-inv-elims(2,3)[OF assms(1)]
  have i < B ! b0 ∨ lms-bucket-start α T b0 ≤ i
    using assms(5) assms(6) not-le by force

  from unknown-range-values
        [OF l-perm-inv-elims(8,5)[OF assms(1)] - -
            l-perm-inv-elims(14,15)[OF assms(1)] ‹b0 ≤ α ->]
      l-perm-inv-elims(2,3)[OF assms(1)]
      ‹SA ! i' < length T›
  have i' < B ! b0 ∨ lms-bucket-start α T b0 ≤ i'
    using not-le by force
  moreover
  have i' < B ! b0 ⟹ ?thesis
  proof −
    assume i' < B ! b0
    have i < B ! b0 ⟹ ?thesis
    proof −
      assume i < B ! b0
      let ?xs = list-slice SA (bucket-start α T b0) (B ! b0)
```

```
      have i' ≤ i
        by (simp add: ‹i' < i› less-imp-le-nat)
      from l-prefix-sorted-invD[OF assms(3) ‹b0 ≤ α -›]
      have ordlistns.sorted (map (lms-prefix T) ?xs) .
      with ordlistns.list-slice-sorted-nth-mono
                [OF - ‹bucket-start α T b0 ≤ i'› ‹i' ≤ i›
                    ‹i < B ! b0›,
                 of map (lms-prefix T) SA]
      show ?thesis
          by (metis ‹i' < length SA› assms(4) length-map map-list-slice
nth-map)
    qed
    moreover
    have lms-bucket-start α T b0 ≤ i ⟹ ?thesis
    proof −
      assume lms-bucket-start α T b0 ≤ i
      with lms-init-D[OF lms-init-unchanged
                         [OF l-perm-inv-elims(8)[OF assms(1)]],
                      OF - - l-perm-inv-elims(14)[OF assms(1)] ‹b0 ≤ α -›]
          l-perm-inv-elims(2,3)[OF assms(1)]
          ‹i < bucket-end α T b0›
      have SA ! i ∈ lms-bucket α T b0
        by (metis bucket-end-le-length list-slice-nth-mem)
      hence suffix-type T (SA ! i) = S-type
        by (metis ‹b0 ≤ α (Max (set T))›
                  ‹i < bucket-end α T b0›
                  ‹length SA = length SA0›
                  ‹length SA0 = length T›
                  ‹lms-bucket-start α T b0 ≤ i›
                  ‹lms-init α T SA0› assms(1)
                  abs-is-lms-def l-perm-inv-elims(8) lms-init-nth
                  lms-init-unchanged)
      moreover
       from l-locations-inv-D[OF l-perm-inv-elims(9)[OF assms(1)] ‹b0 ≤
α -›
                                 ‹i' < length SA› ‹bucket-start α T b0 ≤ i'›
                                 ‹i' < B ! b0›]
      have suffix-type T (SA ! i') = L-type SA ! i' < length T
        by blast+
      ultimately show ?thesis
        using lms-prefix-l-less-than-s-type[of SA ! i T SA ! i']
        by (simp add: ‹T ! (SA ! i') = T ! (SA ! i)› assms(5−6)
                  lms-prefix-l-less-than-s-type
                  ordlistns.dual-order.strict-implies-order)
    qed
    ultimately show ?thesis
      using ‹i < B ! b0 ∨ lms-bucket-start α T b0 ≤ i› by blast
  qed
  moreover
```

```
      have lms-bucket-start α T b0 ≤ i' ⟹ ?thesis
      proof −
        assume lms-bucket-start α T b0 ≤ i'

        let ?xs = list-slice SA (lms-bucket-start α T b0) (bucket-end α T b0)

      from l-prefix-sorted-pre-maintained[OF l-perm-inv-elims(8)[OF assms(1)]]
           l-perm-inv-elims(2,3)[OF assms(1)]
           l-prefix-sorted-preD[of α T SA b0, OF - ‹b0 ≤ α ›]
        have ordlistns.sorted (map (lms-prefix T) ?xs)
          by (simp add: assms(2))
        with ordlistns.list-slice-sorted-nth-mono
                    [of map (lms-prefix T) SA
                        lms-bucket-start α T b0
                        bucket-end α T b0 i' i]
             ‹i < bucket-end α T b0› ‹i' < i›
             ‹i' < length SA›
             ‹lms-bucket-start α T b0 ≤ i'›
        show ?thesis
          by (metis assms(4) length-map map-list-slice
                    not-less not-less-iff-gr-or-eq nth-map)
      qed
      ultimately show ?thesis
        by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show list-less-eq-ns x y
    by simp
qed
ultimately show ?thesis
  using ordlistns.sorted-append
              [of map (lms-prefix T) ?ys
                  [lms-prefix T j]]
  by blast
qed
ultimately show ?thesis
  by simp
qed
ultimately show ordlistns.sorted (map (lms-prefix T) ?xs)
  by blast
qed

lemma abs-induce-l-suffix-sorted-step:
  assumes l-perm-inv α T B SA0 SA i
  and     l-suffix-sorted-pre α T SA0
  and     l-suffix-sorted-inv α T B SA
  and     abs-induce-l-step (B, SA, i) (α, T) = (B', SA', i')
```

```
  shows l-suffix-sorted-inv α T B′ SA′
proof (cases i < length SA ∧ SA ! i < length T)
  assume ¬ (i < length SA ∧ SA ! i < length T)
  then show ?thesis
    using assms(3,4) by force
next
  assume A: i < length SA ∧ SA ! i < length T
  show ?thesis
  proof (cases SA ! i)
    case 0
    then show ?thesis
      using assms(3,4) A by force
  next
    case (Suc j)
    assume B: SA ! i = Suc j
    show ?thesis
    proof (cases suffix-type T j)
      case S-type
      then show ?thesis
        using assms(3,4) A B by force
    next
      case L-type
      then show ?thesis
        using assms(3,4) A B
              l-suffix-sorted-inv-maintained-step
                [OF assms(1−3), of j α (T ! j) B ! α (T ! j)]
        by (clarsimp simp: Let-def)
    qed
  qed
qed

lemma abs-induce-l-prefix-sorted-step:
  assumes l-perm-inv α T B SA0 SA i
  and     l-prefix-sorted-pre α T SA0
  and     l-prefix-sorted-inv α T B SA
  and     abs-induce-l-step (B, SA, i) (α, T) = (B′, SA′, i′)
  shows l-prefix-sorted-inv α T B′ SA′
proof (cases i < length SA ∧ SA ! i < length T)
  assume ¬ (i < length SA ∧ SA ! i < length T)
  then show ?thesis
    using assms(3,4) by force
next
  assume A: i < length SA ∧ SA ! i < length T
  show ?thesis
  proof (cases SA ! i)
    case 0
    then show ?thesis
      using assms(3,4) A by force
  next
```

```
    case (Suc j)
    assume B: SA ! i = Suc j
    show ?thesis
    proof (cases suffix-type T j)
      case S-type
      then show ?thesis
        using assms(3,4) A B by force
    next
      case L-type
      then show ?thesis
        using assms(3,4) A B
          l-prefix-sorted-inv-maintained-step[OF assms(1−3), of j α (T ! j) B !
α (T ! j)]
        by (clarsimp simp: Let-def)
    qed
  qed
qed

lemma abs-induce-l-base-suffix-sorted-inv-maintained:
  assumes l-perm-inv α T B SA0 SA 0
  and     l-suffix-sorted-pre α T SA0
  and     l-suffix-sorted-inv α T B SA
  and     abs-induce-l-base α T B SA = (B', SA', i)
  shows l-suffix-sorted-inv α T B' SA'
proof −
  let ?P = λ(B, SA, i). l-perm-inv α T B SA0 SA i ∧ l-suffix-sorted-inv α T B
SA

  from assms(4)
  have repeat (length T) abs-induce-l-step (B, SA, 0) (α, T) = (B', SA', i)
    by (simp add: abs-induce-l-base-def)
  moreover
  have ⋀a. ?P a ⟹ ?P (abs-induce-l-step a (α, T))
    using abs-induce-l-perm-step abs-induce-l-suffix-sorted-step assms(2) by blast
  ultimately show ?thesis
    using repeat-maintain-inv[of ?P abs-induce-l-step (α, T) (B, SA, 0) length T]
    using assms(1,2,3) by auto
qed

lemma abs-induce-l-base-prefix-sorted-inv-maintained:
  assumes l-perm-inv α T B SA0 SA 0
  and     l-prefix-sorted-pre α T SA0
  and     l-prefix-sorted-inv α T B SA
  and     abs-induce-l-base α T B SA = (B', SA', i)
  shows l-prefix-sorted-inv α T B' SA'
proof −
  let ?P = λ(B, SA, i). l-perm-inv α T B SA0 SA i ∧ l-prefix-sorted-inv α T B
SA
```

from *assms*(*4*)
have *repeat* (*length T*) *abs-induce-l-step* (*B*, *SA*, *0*) (*α*, *T*) = (*B'*, *SA'*, *i*)
by (*simp add*: *abs-induce-l-base-def*)
moreover
have ⋀*a*. *?P a* ⟹ *?P* (*abs-induce-l-step a* (*α*, *T*))
using *abs-induce-l-perm-step abs-induce-l-prefix-sorted-step assms*(*2*) **by** *blast*
ultimately show *?thesis*
using *repeat-maintain-inv*[*of ?P abs-induce-l-step* (*α*, *T*) (*B*, *SA*, *0*) *length T*]
using *assms*(*1*,*2*,*3*) **by** *auto*
qed

79 L-type Exhaustiveness

The *abs-induce-l* function is exhaustive if it has inserted all the L-types

definition *l-type-exhaustive* :: ('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *nat list* ⇒ *bool*
where
l-type-exhaustive T SA = (∀ *i* < *length T*. *suffix-type T i* = *L-type* ⟶ *i* ∈ *set SA*)

There two cases when the *abs-induce-l* function is not exhaustive: when there is an L-type that is not in SA but its successor (right neighbour) is in SA, and the other is when there is an L-type that is not in SA and its successor is also not in SA. We will show that both cases will be False.

lemma *not-l-type-exhaustive-imp-ex*:
¬*l-type-exhaustive T SA* ⟹
(∃ *i* < *length T*. *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA* ∧ *Suc i* ∈ *set SA*) ∨
((∃ *i* < *length T*. *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA*) ∧
¬(∃ *i*. *i* < *length T* ∧ *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA* ∧ *Suc i* ∈ *set SA*))
using *l-type-exhaustive-def*
by *blast*

lemma *l-type-exhaustive-imp-l-bucket*:
⟦*strict-mono α*; *l-type-exhaustive T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
{*i*. *i* ∈ *set SA* ∧ *i* ∈ *l-bucket α T b*} = *l-bucket α T b*
by (*intro equalityI subsetI*; *clarsimp simp add*: *bucket-def l-bucket-def l-type-exhaustive-def*)

lemma *l-type-exhaustive-imp-all-l-types*:
l-type-exhaustive T SA ⟹
{*i*. *i* ∈ *set SA* ∧ *i* ∈ *l-bucket α T* (*α* (*T* ! *i*))} = {*i*. *i* < *length T* ∧ *suffix-type T i* = *L-type*}
apply (*intro equalityI subsetI*; *clarsimp*)
apply (*simp add*: *bucket-def l-bucket-def*)
by (*simp add*: *l-type-exhaustive-def bucket-def l-bucket-def*)

79.1 Case 1

In the case 1, we have that ∃ *k*<*length T*. *suffix-type T k* = *L-type* ∧ *k* ∉ *set SA* ∧ *Suc k* ∈ *set SA*. From this, we know that ∃ *j*<*length SA*. *SA* ! *j* = *Suc k*

lemma
Suc k ∈ set SA⟹ ∃ j < length SA. SA ! j = Suc k
by (*simp add*: *in-set-conv-nth*)

After executing the *abs-induce-l* function, we know that we have seen

79.2 Case 2

In the case 2, we have that $\exists\, k{<}length\ T.\ suffix\text{-}type\ T\ k = L\text{-}type \wedge k \notin set\ SA \wedge Suc\ k \notin set\ SA$.

lemma *finite-and-Suc-imp-False*:
assumes *finite-A*: *finite A*
and *not-empty*: *A ≠ {}*
and *Suc-A*: *∀ a ∈ A. Suc a ∈ A*
shows *False*
proof −
from *Max-in*[*OF finite-A not-empty*]
have *Max A ∈ A* **by** *assumption*

with *Suc-A bspec*
have *Suc* (*Max A*) *∈ A* **by** *blast*

with ‹*Max A ∈ A*› *finite-A*
show *?thesis*
using *Max-ge Suc-n-not-le-n* **by** *blast*
qed

lemma *not-exhaustive-neighbour-is-l-type*:
assumes *A*: *A = {k |k. suffix-type T k = L-type ∧ k ∉ B ∧ Suc k ∉ B ∧ k < length T}*
and *subset-B*: *{k |k. abs-is-lms T k} ⊆ B*
and *k ∈ A*
shows *suffix-type T* (*Suc k*) = *L-type*
proof −
from *A* ‹*k ∈ A*›
have *Suc k ∉ B*
by *blast*
with *subset-B*
have *¬abs-is-lms T* (*Suc k*)
by *blast*

from *A* ‹*k ∈ A*›
have *suffix-type T k = L-type*
by *blast*

with ‹*~abs-is-lms T* (*Suc k*)›
show *?thesis*
by (*meson abs-is-lms-def suffix-type-def*)
qed

lemma *no-exhausted-neighbour*:
assumes *A*: $A = \{k \mid k.\ \textit{suffix-type}\ T\ k = \textit{L-type} \wedge k \notin B \wedge \textit{Suc}\ k \notin B \wedge k < \textit{length}\ T\}$
and *B*: $\{k \mid k.\ \textit{abs-is-lms}\ T\ k\} \subseteq B$
and *C*: $\neg(\exists\, k.\ k < \textit{length}\ T \wedge \textit{suffix-type}\ T\ k = \textit{L-type} \wedge k \notin B \wedge \textit{Suc}\ k \in B)$
and *D*: $\textit{suffix-type}\ T\ i = \textit{L-type}$
and *E*: $i \notin B$
and *F*: $i < \textit{length}\ T$
shows $i \in A$
proof −
from *C*[*simplified*] *D E F*
have $\textit{Suc}\ i \notin B$
by *blast*

with *A D E F*
show *?thesis*
by *blast*
qed

lemma *l-type-less-length-imp-neightbour-less-length*:
$[\![\textit{suffix-type}\ T\ i = \textit{L-type};\ i < \textit{length}\ T]\!] \Longrightarrow \textit{Suc}\ i < \textit{length}\ T$
by (*metis SL-types.simps*(*2*) *Suc-lessI suffix-type-last*)

lemma *no-exhausted-neighbour-imp-False*:
assumes *A*: $A = \{k \mid k.\ \textit{suffix-type}\ T\ k = \textit{L-type} \wedge k \notin B \wedge \textit{Suc}\ k \notin B \wedge k < \textit{length}\ T\}$
and *B*: $\{k \mid k.\ \textit{abs-is-lms}\ T\ k\} \subseteq B$
and *C*: $\neg(\exists\, k.\ k < \textit{length}\ T \wedge \textit{suffix-type}\ T\ k = \textit{L-type} \wedge k \notin B \wedge \textit{Suc}\ k \in B)$
and *nempty*: $A \neq \{\}$
shows *False*
proof −

from *A*
have *finite A*
by *auto*

have $\forall\, a \in A.\ \textit{Suc}\ a \in A$
proof
fix *a*
assume $a \in A$
with *not-exhaustive-neighbour-is-l-type*[*OF A B*]
have $\textit{suffix-type}\ T\ (\textit{Suc}\ a) = \textit{L-type}$
by *blast*

from ‹$a \in A$› *A*
have $\textit{Suc}\ a < \textit{length}\ T$
by (*simp add*: *l-type-less-length-imp-neightbour-less-length*)

from ‹$a \in A$› A
have $Suc\ a \notin B$
by *blast*

from *no-exhausted-neighbour*[*OF A B C* ‹*suffix-type T* (*Suc a*) = *L-type*› ‹*Suc a* ∉ *B*› ‹*Suc a* < *length T*›]
show $Suc\ a \in A$
by *blast*
qed

with ‹*finite A*› *nempty*
show *?thesis*
using *finite-and-Suc-imp-False* **by** *blast*
qed

79.3 Exhaustiveness Proof

lemma *abs-induce-l-exhaustive*:
assumes *l-seen-inv T SA* (*length SA*)
and *lms-init* α *T SA0*
and *length SA* = *length SA0*
and *length SA* = *length T*
and *strict-mono* α
and *l-unchanged-inv* α *T SA0 SA*
shows *l-type-exhaustive T SA*
proof(*rule ccontr*)

let *?P* = ∃ *i*<*length T*. *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA* ∧ *Suc i* ∈ *set SA*
and
?Q1 = ∃ *i*<*length T*. *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA* **and**
?Q2 = ¬(∃ *i*. *i* < *length T* ∧ *suffix-type T i* = *L-type* ∧ *i* ∉ *set SA* ∧ *Suc i* ∈ *set SA*)

assume ¬*l-type-exhaustive T SA*
with *not-l-type-exhaustive-imp-ex*
have *?P* ∨ (*?Q1* ∧ *?Q2*)
by *blast*
then show *False*
proof
assume *?P*
then obtain *i* **where**
i < *length T*
suffix-type T i = *L-type*
i ∉ *set SA*
Suc i ∈ *set SA*
by *blast*

from ‹*Suc i* ∈ *set SA*›
have ∃ *k* < *length SA*. *SA* ! *k* = *Suc i*

by (*simp add*: *in-set-conv-nth*)
then obtain *k* **where**
k < *length SA*
SA ! *k* = *Suc i*
by *blast*

from *l-type-less-length-imp-neightbour-less-length*[*OF* ‹*suffix-type T i* = *L-type*›
‹*i* < *length T*›]
have *Suc i* < *length T*
by *assumption*

from ‹*l-seen-inv T SA* (*length SA*)›
have ∀ *i* < *length SA*. *SA* ! *i* < *length T* ⟶
(∀ *j*. *SA* ! *i* = *Suc j* ∧ *suffix-type T j* = *L-type* ⟶
(∃ *k* < *length SA*. *SA* ! *k* = *j*))
using *l-seen-inv-def* **by** *blast*
with ‹*k* < *length SA*› ‹*SA* ! *k* = *Suc i*› ‹*Suc i* < *length T*›
have ∀ *j*. *SA* ! *k* = *Suc j* ∧ *suffix-type T j* = *L-type* ⟶ (∃ *k*<*length SA*. *SA* !
k = *j*)
by *auto*
with ‹*SA* ! *k* = *Suc i*› ‹*suffix-type T i* = *L-type*›
have ∃ *k*<*length SA*. *SA* ! *k* = *i*
by *blast*
with ‹*i* ∉ *set SA*›
show *?thesis*
using *nth-mem* **by** *blast*
next
assume *?Q1* ∧ *?Q2*
then have *?Q1 ?Q2*
by *blast*+
then have ∃ *A*. *A* = {*k* |*k*. *suffix-type T k* = *L-type* ∧ *k* ∉ *set SA* ∧ *Suc k* ∉
set SA ∧ *k* < *length T*}
by *blast*
then obtain *A* **where**
A-eq: *A* = {*k* |*k*. *suffix-type T k* = *L-type* ∧ *k* ∉ *set SA* ∧ *Suc k* ∉ *set SA* ∧
k < *length T*}
by *blast*

from ‹*?Q1*› ‹*?Q2*› *A-eq*
have *A* ≠ {}
by *blast*

from *lms-init-unchanged*[*OF* ‹*l-unchanged-inv* α *T SA0 SA*›
‹*length SA* = *length SA0*›
‹*length SA* = *length T*›
‹*lms-init* α *T SA0*›]
lms-init-imp-all-lms-in-SA[*OF* - ‹*strict-mono* α›]
have *lms-subset-SA* : {*k* |*k*. *abs-is-lms T k*} ⊆ *set SA*
by *blast*

from *no-exhausted-neighbour-imp-False*[*OF A-eq lms-subset-SA* ‹*?Q2*› ‹$A \neq \{\}$›]
show *?thesis*
by *assumption*
qed
qed

80 Correctness and Exhaustiveness

lemma *abs-induce-l-perm-inv-imp-exhaustiveness*:
assumes *abs-induce-l-base* α T B $SA = (B', SA', i)$
and *l-perm-inv* α T B' SA SA' i
shows *l-type-exhaustive* T SA'
proof −
from *abs-induce-l-index*[*of* α T B SA] *assms*(*1*)
have i = *length* T
by *simp*
hence i = *length* SA'
using *assms*(*2*) *l-perm-inv-elims*(*2*,*3*) **by** *fastforce*
hence *P*: *l-perm-inv* α T B' SA SA' (*length* SA')
using *assms*(*2*) **by** *blast*

have *length* SA' = *length* T
using ‹i = *length* SA'› ‹i = *length* T› **by** *blast*
with *abs-induce-l-exhaustive*[*OF l-perm-inv-elims*(*11*,*14*,*3*)[*OF P*] - *l-perm-inv-elims*(*12*,*8*)[*OF P*]]
show *?thesis* **.**
qed

lemma *abs-induce-l-perm-inv-B-val*:
assumes *abs-induce-l-base* α T B $SA = (B', SA', i)$
and *l-perm-inv* α T B' SA SA' i
and $b \leq \alpha$ (*Max* (*set* T))
shows B' ! b = *l-bucket-end* α T b
proof −
from *abs-induce-l-perm-inv-imp-exhaustiveness*[*OF assms*(*1*−*2*)]
have *l-type-exhaustive* T SA'
by *assumption*

have *strict-mono* α
using *assms*(*2*) *l-perm-inv-elims*(*12*) **by** *blast*

from *l-bucket-ptr-inv-D*[*OF l-perm-inv-elims*(*6*)[*OF assms*(*2*)] *assms*(*3*)]
have B' ! b = *bucket-start* α T b + *num-l-types* α T SA' b
by *blast*
moreover
from *l-type-exhaustive-imp-l-bucket*[*OF* ‹*strict-mono* α› ‹*l-type-exhaustive* T SA'› *assms*(*3*)]

have *cur-l-types* α T SA' b = *l-bucket* α T b
unfolding *cur-l-types-def*
by *blast*
hence *num-l-types* α T SA' b = *l-bucket-size* α T b
by (*simp add*: *l-bucket-size-def num-l-types-def*)
ultimately
show *?thesis*
by (*simp add*: *l-bucket-end-def*)
qed

theorem *abs-induce-l-distinct-l-bucket*:
assumes *l-perm-pre* α T B SA
and $b \leq \alpha$ (*Max* (*set* T))
shows *distinct* (*list-slice* (*abs-induce-l* α T B SA) (*bucket-start* α T b) (*l-bucket-end* α T b))
proof −
from *abs-induce-l-index*[*of* α T B SA]
obtain B' SA' **where**
A: *abs-induce-l-base* α T B SA = (B', SA', *length* T)
by *blast*
with *abs-induce-l-base-perm-inv-maintained*[*OF l-perm-inv-established*[*OF assms*(*1*)], *of* B' SA' *length* T]
have B: *l-perm-inv* α T B' SA SA' (*length* T) **.**
with *abs-induce-l-perm-inv-B-val*[*OF A - assms*(*2*)]
have B' ! b = *l-bucket-end* α T b **.**
with *l-distinct-slice*[*OF l-perm-inv-elims*(*4*,*9*)[*OF B*] - *assms*(*2*)] *l-perm-inv-elims*(*2*,*3*)[*OF B*]
have *distinct* (*list-slice* SA' (*bucket-start* α T b) (*l-bucket-end* α T b))
by *simp*
then show *?thesis*
by (*simp add*: A *abs-induce-l-def*)
qed

theorem *abs-induce-l-list-slice-l-bucket*:
assumes *l-perm-pre* α T B SA
and $b \leq \alpha$ (*Max* (*set* T))
shows *set* (*list-slice* (*abs-induce-l* α T B SA) (*bucket-start* α T b) (*l-bucket-end* α T b)) = *l-bucket* α T b
(**is** *set ?xs* = *l-bucket* α T b)
proof −
from *abs-induce-l-index*[*of* α T B SA]
obtain B' SA' **where**
A: *abs-induce-l-base* α T B SA = (B', SA', *length* T)
by *blast*
with *abs-induce-l-base-perm-inv-maintained*[*OF l-perm-inv-established*[*OF assms*(*1*)], *of* B' SA' *length* T]
have B: *l-perm-inv* α T B' SA SA' (*length* T) **.**
with *abs-induce-l-perm-inv-B-val*[*OF A - assms*(*2*)]
have B' ! b = *l-bucket-end* α T b **.**

with *l-locations-list-slice*[*OF l-perm-inv-elims*(*9*)[*OF B*] *assms*(*2*)]
have *set* (*list-slice SA′* (*bucket-start α T b*) (*l-bucket-end α T b*)) ⊆ *l-bucket α T b*
by *simp*
hence *set ?xs* ⊆ *l-bucket α T b*
by (*simp add*: *A abs-induce-l-def*)
moreover
from *distinct-card*[*OF abs-induce-l-distinct-l-bucket*[*OF assms*]]
have *card* (*set ?xs*) = *length ?xs* **.**
hence *card* (*set ?xs*) = *l-bucket-end α T b* − *bucket-start α T b*
by (*metis B bucket-end-le-length abs-induce-l-length l-bucket-end-le-bucket-end l-perm-inv-elims*(*2*) *le-trans length-list-slice min-def*)
hence *card* (*set ?xs*) = *card* (*l-bucket α T b*)
by (*simp add*: *l-bucket-end-def l-bucket-size-def*)
ultimately show *?thesis*
using *card-subset-eq*[*OF finite-l-bucket*]
by *blast*
qed

lemma *abs-induce-l-unchanged*:
assumes *l-perm-pre α T B SA*
and *b* ≤ *α* (*Max* (*set T*))
and *s-bucket-start α T b* ≤ *i*
and *i* < *bucket-end α T b*
shows (*abs-induce-l α T B SA*) ! *i* = *SA* ! *i*
proof −
from *abs-induce-l-index*[*of α T B SA*]
obtain *B′ SA′* **where**
A: *abs-induce-l-base α T B SA* = (*B′*, *SA′*, *length T*)
by *blast*
with *abs-induce-l-base-perm-inv-maintained*[*OF l-perm-inv-established*[*OF assms*(*1*)], *of B′ SA′ length T*]
have *B*: *l-perm-inv α T B′ SA SA′* (*length T*) **.**

have *i* < *length SA*
by (*metis B assms*(*4*) *bucket-end-le-length l-perm-inv-elims*(*2*) *le-less-trans not-less-eq*)
moreover
have *l-bucket-end α T b* ≤ *i*
by (*metis assms*(*3*) *s-bucket-start-eq-l-bucket-end*)
ultimately have *SA′* ! *i* = *SA* ! *i*
using *l-unchanged-inv-D*[*OF l-perm-inv-elims*(*8*,*3*)[*OF B*] *assms*(*2*) - - *assms*(*4*)]
by *auto*
then show *?thesis*
by (*simp add*: *A abs-induce-l-def*)
qed

— Used in SAIS algorithm as part of inducing the suffix ordering based on LMS

theorem *abs-induce-l-suffix-sorted-l-bucket*:
assumes *l-perm-pre* α *T B SA*
and *l-suffix-sorted-pre* α *T SA*
and $b \leq \alpha$ (*Max* (*set T*))
shows *ordlistns.sorted* (*map* (*suffix T*)
(*list-slice* (*abs-induce-l* α *T B SA*) (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
proof −

from *l-perm-inv-established*[*OF assms*(*1*)]
have *A*: *l-perm-inv* α *T B SA SA 0* **.**

from *l-suffix-sorted-inv-established*[*OF l-perm-pre-elims*(*4*)[*OF assms*(*1*)], *of SA*]
have *D*: *l-suffix-sorted-inv* α *T B SA* **.**

from *abs-induce-l-index*[*of* α *T B SA*]
obtain *B′ SA′* **where**
B: *abs-induce-l-base* α *T B SA* = (*B′*, *SA′*, *length T*)
by *blast*
with *abs-induce-l-base-perm-inv-maintained*[*OF A*, *of B′ SA′ length T*]
have *C*: *l-perm-inv* α *T B′ SA SA′* (*length T*) **.**
with *abs-induce-l-perm-inv-B-val*[*OF B - assms*(*3*)]
have *B′* ! *b* = *l-bucket-end* α *T b* **.**
moreover
from *abs-induce-l-base-suffix-sorted-inv-maintained*[*OF A assms*(*2*) *D B*]
have *l-suffix-sorted-inv* α *T B′ SA′* **.**
ultimately show *?thesis*
using *l-suffix-sorted-invD*[*of* α *T B′ SA′*, *OF - assms*(*3*)]
by (*simp add*: *B abs-induce-l-def*)
qed

— Used in SAIS algorithm as part of inducing the prefix ordering based on LMS
theorem *abs-induce-l-prefix-sorted-l-bucket*:
assumes *l-perm-pre* α *T B SA*
and *l-prefix-sorted-pre* α *T SA*
and $b \leq \alpha$ (*Max* (*set T*))
shows *ordlistns.sorted* (*map* (*lms-prefix T*)
(*list-slice* (*abs-induce-l* α *T B SA*) (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
proof −

from *l-perm-inv-established*[*OF assms*(*1*)]
have *A*: *l-perm-inv* α *T B SA SA 0* **.**

from *l-prefix-sorted-inv-established*[*OF l-perm-pre-elims*(*4*)[*OF assms*(*1*)], *of SA*]
have *D*: *l-prefix-sorted-inv* α *T B SA* **.**

from *abs-induce-l-index*[*of* α *T B SA*]
obtain *B′ SA′* **where**

B: *abs-induce-l-base* α *T B SA* = (*B'*, *SA'*, *length T*)
by *blast*
with *abs-induce-l-base-perm-inv-maintained*[*OF A*, *of B' SA' length T*]
have *C*: *l-perm-inv* α *T B' SA SA'* (*length T*) **.**
with *abs-induce-l-perm-inv-B-val*[*OF B - assms*(*3*)]
have *B'* ! *b* = *l-bucket-end* α *T b* **.**
moreover
from *abs-induce-l-base-prefix-sorted-inv-maintained*[*OF A assms*(*2*) *D B*]
have *l-prefix-sorted-inv* α *T B' SA'* **.**
ultimately show *?thesis*
using *l-prefix-sorted-invD*[*of* α *T B' SA'*, *OF - assms*(*3*)]
by (*simp add*: *B abs-induce-l-def*)
qed

end
theory *Abs-Induce-S-Verification*
imports *../abs−def/Abs-SAIS*
begin

81 Abstract Induce S Simple Properties

lemma *abs-induce-s-step-ex*:
∃ *B' SA' i'*. *abs-induce-s-step a b* = (*B'*, *SA'*, *i'*)
by (*meson prod-cases3*)

lemma *abs-induce-s-step-B-length*:
abs-induce-s-step (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*) ⟹ *length B'* = *length B*
by (*clarsimp split*: *prod.splits if-splits nat.splits SL-types.splits simp*: *Let-def*)

lemma *abs-induce-s-step-SA-length*:
abs-induce-s-step (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*) ⟹ *length SA'* = *length SA*
by (*clarsimp split*: *prod.splits if-splits nat.splits SL-types.splits simp*: *Let-def*)

lemma *abs-induce-s-step-Suc*:
abs-induce-s-step (*B*, *SA*, *Suc i*) (α, *T*) = (*B'*, *SA'*, *i'*) ⟹ *i'* = *i*
by (*clarsimp split*: *prod.splits if-splits nat.splits SL-types.splits simp*: *Let-def*)

lemma *abs-induce-s-step-0*:
abs-induce-s-step (*B*, *SA*, *0*) (α, *T*) = (*B*, *SA*, *0*)
by (*clarsimp split*: *prod.splits if-splits nat.splits SL-types.splits simp*: *Let-def*)

corollary *abs-induce-s-step-0-alt*:
assumes *abs-induce-s-step* (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*)
and *i* = *0*
shows *B* = *B'* ∧ *SA* = *SA'* ∧ *i'* = *0*
using *assms*(*1*) *assms*(*2*) **by** *auto*

lemma *repeat-abs-induce-s-step-index*:
∃ *B' SA'*. *repeat n abs-induce-s-step* (*B*, *SA*, *m*) (α, *T*) = (*B'*, *SA'*, *m* − *n*) ∧

```
        length SA' = length SA ∧ length B' = length B
proof(induct n arbitrary: m)
  case 0
  then show ?case by (clarsimp simp: repeat-0)
next
  case (Suc n)
  note IH = this

  from IH[of m]
  obtain B' SA' where
    repeat n abs-induce-s-step (B, SA, m) (α, T) = (B', SA', m − n)
    length SA' = length SA
    length B' = length B
    by blast
  with repeat-step[of n abs-induce-s-step (B, SA, m) (α, T)]
   have repeat (Suc n) abs-induce-s-step (B, SA, m) (α, T) = abs-induce-s-step
(B', SA', m − n) (α, T)
    by simp
  moreover
  have ∃ B'' SA''. abs-induce-s-step (B', SA', m − n) (α, T) = (B'', SA'', m −
Suc n) ∧
                  length SA'' = length SA' ∧ length B'' = length B'
  proof (cases n < m)
    assume n < m
    hence ∃ k. m − n = Suc k
      by presburger
    then obtain k where
      m − n = Suc k
      by blast

    from abs-induce-s-step-ex[of (B',SA', m − n) (α, T)]
    obtain B'' SA'' i' where
      P: abs-induce-s-step (B', SA', m − n) (α, T) = (B'', SA'', i')
      by blast
    with ‹m − n = Suc k› abs-induce-s-step-Suc[of B' SA' k α T B'' SA'' i']
    have i' = m − Suc n
      by auto
    with ‹abs-induce-s-step (B', SA', m − n) (α, T) = (B'', SA'', i')›
         abs-induce-s-step-SA-length[OF P]
         abs-induce-s-step-B-length[OF P]
    show ?thesis
      by simp
  next
    assume ¬n < m
    hence m ≤ n
      by simp
    hence m − n = 0
      by simp
    with abs-induce-s-step-0[of B' SA' α T]
```

show *?thesis*
by *simp*
qed
ultimately show *?case*
by (*simp add*: ‹*length SA′ = length SA*› ‹*length B′ = length B*›)
qed

lemma *abs-induce-s-base-index*:
∃ *B′ SA′*. *abs-induce-s-base α T B SA* = (*B′*, *SA′*, *0*)
unfolding *abs-induce-s-base-def*
by (*metis cancel-comm-monoid-add-class.diff-cancel repeat-abs-induce-s-step-index*)

lemma *abs-induce-s-length*:
length (*abs-induce-s α T B SA*) = *length SA*
unfolding *abs-induce-s-def abs-induce-s-base-def*
apply (*rule repeat-maintain-inv*; *simp add*: *Let-def*)
apply (*clarsimp split*: *prod.splits simp del*: *abs-induce-s-step.simps*)
apply (*drule abs-induce-s-step-SA-length*; *simp*)
done

82 Preconditions

definition *l-types-init*
where
l-types-init α T SA ≡
(∀ *b* ≤ *α* (*Max* (*set T*)).
set (*list-slice SA* (*bucket-start α T b*) (*l-bucket-end α T b*)) = *l-bucket α T b* ∧
distinct (*list-slice SA* (*bucket-start α T b*) (*l-bucket-end α T b*))
)

lemma *l-types-initD*:
⟦*l-types-init α T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
set (*list-slice SA* (*bucket-start α T b*) (*l-bucket-end α T b*)) = *l-bucket α T b*
⟦*l-types-init α T SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
distinct (*list-slice SA* (*bucket-start α T b*) (*l-bucket-end α T b*))
using *l-types-init-def* **by** *blast+*

lemma *l-types-init-nth*:
assumes *length SA* = *length T*
and *l-types-init α T SA*
and *b* ≤ *α* (*Max* (*set T*))
and *bucket-start α T b* ≤ *i*
and *i* < *l-bucket-end α T b*
shows *SA* ! *i* ∈ *l-bucket α T b*
proof −
have *l-bucket-end α T b* ≤ *length SA*
by (*metis assms*(*1*) *bucket-end-le-length dual-order.order-iff-strict l-bucket-end-le-bucket-end*
less-le-trans)
with *l-types-initD*(*1*)[*OF assms*(*2*,*3*)] *list-slice-nth-mem*[*OF assms*(*4*,*5*)]

show *?thesis*
by *blast*
qed

definition *s-type-init*
where
s-type-init T SA ≡ (∃ *n*. *length T* = *Suc n* ∧ *SA* ! *0* = *n*)

definition *s-perm-pre*
where
s-perm-pre α *T B SA n* ≡
s-bucket-init α *T B* ∧
s-type-init T SA ∧
strict-mono α ∧
α (*Max* (*set T*)) < *length B* ∧
length SA = *length T* ∧
l-types-init α *T SA* ∧
valid-list T ∧
α *bot* = *0* ∧
Suc 0 < *length T* ∧
length T ≤ *n*

definition *s-sorted-pre*
where
s-sorted-pre α *T SA* ≡
(∀ *b* ≤ α (*Max* (*set T*)).
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
)

lemma *s-sorted-preD*:
⟦*s-sorted-pre* α *T SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
using *s-sorted-pre-def* **by** *blast*

definition *s-prefix-sorted-pre*
where
s-prefix-sorted-pre α *T SA* ≡
(∀ *b* ≤ α (*Max* (*set T*)).
ordlistns.sorted (*map* (*lms-slice T*) (*list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
)

lemma *s-prefix-sorted-preD*:
⟦*s-prefix-sorted-pre* α *T SA*; *b* ≤ α (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*lms-slice T*) (*list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)))
using *s-prefix-sorted-pre-def* **by** *blast*

83 Invariants

83.1 Definitions

83.1.1 Distinctness

definition *s-distinct-inv*
  where
s-distinct-inv α *T B SA* $\equiv$
  $(\forall\, b \leq \alpha\ (Max\ (set\ T)).\ distinct\ (list\text{-}slice\ SA\ (B\ !\ b)\ (bucket\text{-}end\ \alpha\ T\ b)))$

lemma *s-distinct-invD*:
  $[\![s\text{-}distinct\text{-}inv\ \alpha\ T\ B\ SA;\ b \leq \alpha\ (Max\ (set\ T))]\!] \Longrightarrow$
   $distinct\ (list\text{-}slice\ SA\ (B\ !\ b)\ (bucket\text{-}end\ \alpha\ T\ b))$
  using *s-distinct-inv-def* **by** *blast*

83.1.2 S Bucket Ptr

definition *s-bucket-ptr-inv* ::
  $('a :: \{linorder,\ order\text{-}bot\} \Rightarrow nat) \Rightarrow 'a\ list \Rightarrow nat\ list \Rightarrow bool$
  where
s-bucket-ptr-inv α *T B* $\equiv$
  $(\forall\, b \leq \alpha\ (Max\ (set\ T)).$
    $s\text{-}bucket\text{-}start\ \alpha\ T\ b \leq B\ !\ b\ \wedge$
    $B\ !\ b \leq bucket\text{-}end\ \alpha\ T\ b\ \wedge$
    $(b = 0 \longrightarrow B\ !\ b = 0))$

lemma *s-bucket-ptr-lower-bound*:
  assumes *s-bucket-ptr-inv* α *T B*
  and $b \leq \alpha\ (Max\ (set\ T))$
shows *s-bucket-start* α *T b* $\leq B\ !\ b$
  using *assms*(*1*) *assms*(*2*) *s-bucket-ptr-inv-def* **by** *blast*

lemma *s-bucket-ptr-upper-bound*:
  assumes *s-bucket-ptr-inv* α *T B*
  and $b \leq \alpha\ (Max\ (set\ T))$
shows $B\ !\ b \leq bucket\text{-}end\ \alpha\ T\ b$
  by (*metis assms s-bucket-ptr-inv-def*)

lemma *s-bucket-ptr-0*:
  assumes *s-bucket-ptr-inv* α *T B*
  and $b = 0$
shows $B\ !\ b = 0$
  using *assms s-bucket-ptr-inv-def* **by** *auto*

83.1.3 Locations

definition *s-locations-inv* ::
  $('a :: \{linorder,\ order\text{-}bot\} \Rightarrow nat) \Rightarrow 'a\ list \Rightarrow nat\ list \Rightarrow nat\ list \Rightarrow bool$
  where

s-locations-inv α *T B SA* $\equiv$
(∀ $b \leq \alpha$ (*Max* (*set T*)).
(∀ *i*. *B* ! $b \leq i \wedge i <$ *bucket-end* α *T b* $\longrightarrow$ *SA* ! $i \in$ *s-bucket* α *T b*))

lemma *s-locations-invD*:
⟦*s-locations-inv* α *T B SA*; $b \leq \alpha$ (*Max* (*set T*)); *B* ! $b \leq i$; $i <$ *bucket-end* α *T b*⟧ $\Longrightarrow$
SA ! $i \in$ *s-bucket* α *T b*
using *s-locations-inv-def* **by** *blast*

lemma *s-locations-inv-in-list-slice*:
assumes *s-locations-inv* α *T B SA*
and $b \leq \alpha$ (*Max* (*set T*))
and $x \in$ *set* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))
shows $x \in$ *s-bucket* α *T b*
proof −
from *in-set-conv-nth*[*THEN iffD1*, *OF assms*(*3*)]
obtain *i* **where**
$i <$ *length* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))
list-slice SA (*B* ! *b*) (*bucket-end* α *T b*) ! $i = x$
by *blast*
with *nth-list-slice*
have *SA* ! (*B* ! *b* + *i*) = *x*
by *fastforce*
moreover
have *B* ! *b* + *i* < *bucket-end* α *T b*
using ‹$i <$ *length* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))› **by** *auto*
ultimately
show *?thesis*
using *assms*(*1*,*2*) *le-add1 s-locations-invD* **by** *blast*
qed

lemma *s-locations-inv-subset-s-bucket*:
assumes *s-locations-inv* α *T B SA*
and $b \leq \alpha$ (*Max* (*set T*))
shows *set* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*)) $\subseteq$ *s-bucket* α *T b*
using *assms s-locations-inv-in-list-slice* **by** *blast*

83.1.4 Unchanged

definition *s-unchanged-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat list* ⇒ *bool*
where
s-unchanged-inv α *T B SA SA'* $\equiv$
(∀ $b \leq \alpha$ (*Max* (*set T*)). (∀ *i*. *bucket-start* α *T b* $\leq i \wedge i <$ *B* ! *b* $\longrightarrow$ *SA'* ! *i* = *SA* ! *i*))

lemma *s-unchanged-invD*:

⟦*s-unchanged-inv α T B SA SA'*; *b* ≤ *α* (*Max* (*set T*)); *bucket-start α T b* ≤ *i*; *i* < *B* ! *b*⟧ ⟹
SA' ! *i* = *SA* ! *i*
using *s-unchanged-inv-def* **by** *blast*

83.1.5 Seen

definition *s-seen-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat* ⇒ *bool*
where
s-seen-inv α T B SA n ≡
∀ *i* < *length SA*. *n* ≤ *i* ⟶
(*suffix-type T* (*SA* ! *i*) = *S-type* ⟶ *in-s-current-bucket α T B* (*α* (*T* ! (*SA* ! *i*))) *i*) ∧
(*suffix-type T* (*SA* ! *i*) = *L-type* ⟶ *in-l-bucket α T* (*α* (*T* ! (*SA* ! *i*))) *i*) ∧
SA ! *i* < *length T*

lemma *s-seen-invD*:
⟦*s-seen-inv α T B SA n*; *i* < *length SA*; *n* ≤ *i*⟧ ⟹ *SA* ! *i* < *length T*
⟦*s-seen-inv α T B SA n*; *i* < *length SA*; *n* ≤ *i*; *suffix-type T* (*SA* ! *i*) = *L-type*⟧ ⟹
in-l-bucket α T (*α* (*T* ! (*SA* ! *i*))) *i*
⟦*s-seen-inv α T B SA n*; *i* < *length SA*; *n* ≤ *i*; *suffix-type T* (*SA* ! *i*) = *S-type*⟧ ⟹
in-s-current-bucket α T B (*α* (*T* ! (*SA* ! *i*))) *i*
unfolding *s-seen-inv-def* **by** *blast+*

83.1.6 Predecessor

definition *s-pred-inv* ::
(('*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat* ⇒ *bool*
where
s-pred-inv α T B SA n =
(∀ *b i*. *in-s-current-bucket α T B b i* ∧ *b* ≠ *0* ⟶
(∃ *j* < *length SA*. *SA* ! *j* = *Suc* (*SA* ! *i*) ∧ *i* < *j* ∧ *n* < *j*)
)

lemma *s-pred-invD*:
⟦*s-pred-inv α T B SA k*; *in-s-current-bucket α T B b i*; *b* ≠ *0*⟧ ⟹
∃ *j* < *length SA*. *SA* ! *j* = *Suc* (*SA* ! *i*) ∧ *i* < *j* ∧ *k* < *j*
using *s-pred-inv-def* **by** *blast*

83.1.7 Successor

definition *s-suc-inv* ::
('*a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ '*a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat* ⇒ *bool*
where
s-suc-inv α T B SA n ≡
∀ *i* < *length SA*. *n* < *i* ⟶

$(\forall j.\ SA\ !\ i = Suc\ j \wedge \textit{suffix-type}\ T\ j = \textit{S-type} \longrightarrow$
$(\exists k.\ \textit{in-s-current-bucket}\ \alpha\ T\ B\ (\alpha\ (T\ !\ j))\ k \wedge SA\ !\ k = j \wedge k < i))$

lemma *s-suc-invD*:
$[\![\textit{s-suc-inv}\ \alpha\ T\ B\ SA\ n;\ i < \textit{length}\ SA;\ n < i;\ SA\ !\ i = Suc\ j;\ \textit{suffix-type}\ T\ j = \textit{S-type}]\!] \Longrightarrow$
$\exists k.\ \textit{in-s-current-bucket}\ \alpha\ T\ B\ (\alpha\ (T\ !\ j))\ k \wedge SA\ !\ k = j \wedge k < i$
using *s-suc-inv-def* **by** *blast*

83.1.8 Combined Permutation Invariant

definition *s-perm-inv* ::
$('a :: \{\textit{linorder}, \textit{order-bot}\} \Rightarrow \textit{nat}) \Rightarrow 'a\ \textit{list} \Rightarrow \textit{nat list} \Rightarrow \textit{nat list} \Rightarrow \textit{nat list} \Rightarrow \textit{nat} \Rightarrow \textit{bool}$
where
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \equiv$
$\textit{s-distinct-inv}\ \alpha\ T\ B\ SA' \wedge$
$\textit{s-bucket-ptr-inv}\ \alpha\ T\ B \wedge$
$\textit{s-locations-inv}\ \alpha\ T\ B\ SA' \wedge$
$\textit{s-unchanged-inv}\ \alpha\ T\ B\ SA\ SA' \wedge$
$\textit{s-seen-inv}\ \alpha\ T\ B\ SA'\ n \wedge$
$\textit{s-pred-inv}\ \alpha\ T\ B\ SA'\ n \wedge$
$\textit{s-suc-inv}\ \alpha\ T\ B\ SA'\ n \wedge$
$\textit{strict-mono}\ \alpha \wedge$
$\alpha\ (\textit{Max}\ (\textit{set}\ T)) < \textit{length}\ B \wedge$
$\textit{length}\ SA = \textit{length}\ T \wedge$
$\textit{length}\ SA' = \textit{length}\ T \wedge$
$\textit{l-types-init}\ \alpha\ T\ SA \wedge$
$\textit{valid-list}\ T \wedge$
$\alpha\ \textit{bot} = 0 \wedge$
$\textit{Suc}\ 0 < \textit{length}\ T$

lemma *s-perm-inv-elims*:
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-distinct-inv}\ \alpha\ T\ B\ SA'$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-bucket-ptr-inv}\ \alpha\ T\ B$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-locations-inv}\ \alpha\ T\ B\ SA'$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-unchanged-inv}\ \alpha\ T\ B\ SA\ SA'$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-seen-inv}\ \alpha\ T\ B\ SA'\ n$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-pred-inv}\ \alpha\ T\ B\ SA'\ n$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{s-suc-inv}\ \alpha\ T\ B\ SA'\ n$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{strict-mono}\ \alpha$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \alpha\ (\textit{Max}\ (\textit{set}\ T)) < \textit{length}\ B$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{length}\ SA = \textit{length}\ T$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{length}\ SA' = \textit{length}\ T$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{l-types-init}\ \alpha\ T\ SA$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{valid-list}\ T$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \alpha\ \textit{bot} = 0$
$\textit{s-perm-inv}\ \alpha\ T\ B\ SA\ SA'\ n \Longrightarrow \textit{Suc}\ 0 < \textit{length}\ T$
by (*simp-all add*: *s-perm-inv-def*)

fun *s-perm-inv-alt* ::
(*'a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *nat list* × *nat list* × *nat* ⇒ *bool*
where
s-perm-inv-alt α T SA (*B*, *SA'*, *n*) = *s-perm-inv α T B SA SA' n*

83.1.9 Sorted

definition *s-sorted-inv*
where
s-sorted-inv α T B SA ≡
(∀ *b* ≤ *α* (*Max* (*set T*)).
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)))
)

lemma *s-sorted-invD*:
⟦*s-sorted-inv α T B SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*suffix T*) (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)))
using *s-sorted-inv-def* **by** *blast*

fun *s-sorted-inv-alt* ::
(*'a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *nat list* × *nat list* × *nat* ⇒ *bool*
where
s-sorted-inv-alt α T SA (*B*, *SA'*, *n*) =
(*s-perm-inv α T B SA SA' n* ∧ *s-sorted-pre α T SA* ∧ *s-sorted-inv α T B SA'*)

definition *s-prefix-sorted-inv*
where
s-prefix-sorted-inv α T B SA ≡
(∀ *b* ≤ *α* (*Max* (*set T*)).
ordlistns.sorted (*map* (*lms-slice T*) (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)))
)

lemma *s-prefix-sorted-invD*:
⟦*s-prefix-sorted-inv α T B SA*; *b* ≤ *α* (*Max* (*set T*))⟧ ⟹
ordlistns.sorted (*map* (*lms-slice T*) (*list-slice SA* (*B* ! *b*) (*bucket-end α T b*)))
using *s-prefix-sorted-inv-def* **by** *blast*

fun *s-prefix-sorted-inv-alt* ::
(*'a* :: {*linorder*, *order-bot*} ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *nat list* × *nat list* × *nat* ⇒ *bool*
where
s-prefix-sorted-inv-alt α T SA (*B*, *SA'*, *n*) =
(*s-perm-inv α T B SA SA' n* ∧ *s-prefix-sorted-pre α T SA* ∧ *s-prefix-sorted-inv α T B SA'*)

83.2 Helpers

lemma *s-current-bucket-pairwise-distinct*:
assumes *s-distinct-inv* α *T B SA*
and *s-locations-inv* α *T B SA*
and $b \leq \alpha$ (*Max* (*set T*))
and $b' \leq \alpha$ (*Max* (*set T*))
and $b \neq b'$
shows *distinct* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*) @ *list-slice SA* (*B* ! *b'*) (*bucket-end* α *T b'*))
proof (*intro distinct-append*[*THEN iffD2*] *conjI disjointI'*)
from *s-distinct-invD*[*OF assms*(*1*,*3*)]
show *distinct* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*)) **.**
next
from *s-distinct-invD*[*OF assms*(*1*,*4*)]
show *distinct* (*list-slice SA* (*B* ! *b'*) (*bucket-end* α *T b'*)) **.**
next
fix *x y*
assume *A*: $x \in$ *set* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))
$y \in$ *set* (*list-slice SA* (*B* ! *b'*) (*bucket-end* α *T b'*))

from *s-locations-inv-in-list-slice*[*OF assms*(*2*,*3*) *A*(*1*)]
have $x \in$ *s-bucket* α *T b* **.**
moreover
from *s-locations-inv-in-list-slice*[*OF assms*(*2*,*4*) *A*(*2*)]
have $y \in$ *s-bucket* α *T b'* **.**
ultimately
show $x \neq y$
using *assms*(*5*)
by (*metis* (*mono-tags*, *lifting*) *bucket-def s-bucket-def mem-Collect-eq*)
qed

lemma *s-unchanged-list-slice*:
assumes *s-unchanged-inv* α *T B SA0 SA*
and *length SA0* = *length T*
and *length SA* = *length T*
and $b \leq \alpha$ (*Max* (*set T*))
and *bucket-start* α *T b* $\leq i$
and $j \leq B$! b
shows *list-slice SA i j* = *list-slice SA0 i j*
proof (*intro list-eq-iff-nth-eq*[*THEN iffD2*] *conjI allI impI*)
show *length* (*list-slice SA i j*) = *length* (*list-slice SA0 i j*)
by (*simp add*: *assms*)
next
fix *k*
assume *k* < *length* (*list-slice SA i j*)
hence *k* < *length* (*list-slice SA0 i j*)
by (*simp add*: *assms*)

from *nth-list-slice*[*OF* ‹*k* < *length* (*list-slice SA i j*)›]

```
  have list-slice SA i j ! k = SA ! (i + k) .
  moreover
  from nth-list-slice[OF ‹k < length (list-slice SA0 i j)›]
  have list-slice SA0 i j ! k = SA0 ! (i + k) .
  moreover
  {
    have bucket-start α T b ≤ i + k
      by (simp add: assms(5) trans-le-add1)
    moreover
    have i + k < j
      using ‹k < length (list-slice SA i j)› by auto
    hence i + k < B ! b
      using assms(6) order.strict-trans2 by blast
    ultimately
    have SA ! (i + k) = SA0 ! (i + k)
      using s-unchanged-invD[OF assms(1,4)]
      by blast
  }
  ultimately show list-slice SA i j ! k = list-slice SA0 i j ! k
    by simp
qed

lemma l-types-init-maintained:
  assumes s-bucket-ptr-inv α T B
  and     s-unchanged-inv α T B SA0 SA
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
shows l-types-init α T SA
  unfolding l-types-init-def
proof(intro allI impI)
  fix b
  let ?xs = list-slice SA (bucket-start α T b) (l-bucket-end α T b)
  let ?ys = list-slice SA0 (bucket-start α T b) (l-bucket-end α T b)
  assume b ≤ α (Max (set T))
  with s-bucket-ptr-lower-bound[OF assms(1), of b]
  have l-bucket-end α T b ≤ B ! b
    by (simp add: s-bucket-start-eq-l-bucket-end)
  with s-unchanged-list-slice[OF assms(2−4) ‹b ≤ α (Max (set T))›,
        of bucket-start α T b l-bucket-end α T b]
  have ?xs = ?ys
    by blast
  with assms(5)[simplified l-types-init-def, THEN spec, THEN mp, OF ‹b ≤ α
(Max (set T))›]
  show set ?xs = l-bucket α T b ∧ distinct ?xs
    by simp
qed

lemma s-sorted-pre-maintained:
```

assumes *s-bucket-ptr-inv α T B*
and *s-unchanged-inv α T B SA0 SA*
and *length SA0 = length T*
and *length SA = length T*
and *s-sorted-pre α T SA0*
shows *s-sorted-pre α T SA*
unfolding *s-sorted-pre-def*
proof(*intro allI impI*)
fix *b*
let *?xs = list-slice SA (bucket-start α T b) (l-bucket-end α T b)*
let *?ys = list-slice SA0 (bucket-start α T b) (l-bucket-end α T b)*
assume *b ≤ α (Max (set T))*
with *s-bucket-ptr-lower-bound*[*OF assms*(*1*), *of b*]
have *l-bucket-end α T b ≤ B ! b*
by (*simp add*: *s-bucket-start-eq-l-bucket-end*)
with *s-unchanged-list-slice*[*OF assms*(*2−4*) ‹*b ≤ α (Max (set T))*›,
of bucket-start α T b l-bucket-end α T b]
have *?xs = ?ys*
by *blast*
then show *ordlistns.sorted (map (suffix T) ?xs)*
using ‹*b ≤ α (Max (set T))*› *assms*(*5*) *s-sorted-pre-def* **by** *auto*
qed

lemma *s-prefix-sorted-pre-maintained*:
assumes *s-bucket-ptr-inv α T B*
and *s-unchanged-inv α T B SA0 SA*
and *length SA0 = length T*
and *length SA = length T*
and *s-prefix-sorted-pre α T SA0*
shows *s-prefix-sorted-pre α T SA*
unfolding *s-prefix-sorted-pre-def*
proof(*intro allI impI*)
fix *b*
let *?xs = list-slice SA (bucket-start α T b) (l-bucket-end α T b)*
let *?ys = list-slice SA0 (bucket-start α T b) (l-bucket-end α T b)*
assume *b ≤ α (Max (set T))*
with *s-bucket-ptr-lower-bound*[*OF assms*(*1*), *of b*]
have *l-bucket-end α T b ≤ B ! b*
by (*simp add*: *s-bucket-start-eq-l-bucket-end*)
with *s-unchanged-list-slice*[*OF assms*(*2−4*) ‹*b ≤ α (Max (set T))*›,
of bucket-start α T b l-bucket-end α T b]
have *?xs = ?ys*
by *blast*
then show *ordlistns.sorted (map (lms-slice T) ?xs)*
using ‹*b ≤ α (Max (set T))*› *assms*(*5*) *s-prefix-sorted-pre-def* **by** *auto*
qed

lemma *s-next-item-not-seen*:
assumes *s-distinct-inv α T B SA*

```
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     strict-mono α
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
shows j ∉ set (list-slice SA (B ! b) (bucket-end α T b))
proof
  let ?xs = list-slice SA (B ! b) (bucket-end α T b)
  let ?b = α (T ! (Suc j))
  assume j ∈ set ?xs

  from s-seen-invD(1)[OF assms(5,14)] assms(13,15)
  have Suc j < length T
    by simp
  hence ?b ≤ α (Max (set T))
    using assms(7)
    by (simp add: strict-mono-leD)

  have bucket-end α T ?b ≤ length SA
    by (simp add: assms(9) bucket-end-le-length)
  hence l-bucket-end α T ?b ≤ length SA
    by (metis dual-order.trans l-bucket-end-le-bucket-end)

  have j < length T
    by (simp add: assms(16) suffix-type-s-bound)

  from valid-list-length-ex[OF assms(11)]
  obtain n' where
    length T = Suc n'
    by blast
  with ‹Suc j < length T›
  have j < n'
    by linarith
  hence T ! j ≠ bot
    by (metis ‹length T = Suc n'› assms(11) diff-Suc-1 valid-list-def)
  hence b ≠ 0
    using assms(7,12,17) strict-mono-eq by fastforce
```

with *in-set-conv-nth*[*THEN iffD1*, *OF* ‹$j \in set\ ?xs$›]
obtain a **where**
$a < length\ ?xs$
$?xs\ !\ a = j$
by *blast*
hence $SA\ !\ (B\ !\ b + a) = j$
using *nth-list-slice* **by** *fastforce*

from *assms*(9)
have *bucket-end* $\alpha\ T\ b \leq length\ SA$
by (*simp add*: *bucket-end-le-length*)
with ‹$a < length\ ?xs$›
have $B\ !\ b + a < bucket\text{-}end\ \alpha\ T\ b$
by *auto*
with *assms*(7,17) ‹$j < length\ T$›
have *in-s-current-bucket* $\alpha\ T\ B\ b\ (B\ !\ b + a)$
unfolding *in-s-current-bucket-def*
by (*simp add*: *strict-mono-less-eq*)
with *s-pred-invD*[*OF assms*(6) - ‹$b \neq 0$›, *of* $B\ !\ b + a$] ‹$SA\ !\ (B\ !\ b + a) = j$›
obtain m **where**
$m < length\ SA$
$SA\ !\ m = Suc\ j$
$B\ !\ b + a < m$
$i < m$
by *blast*
hence $i \neq m$
by *blast*

have *suffix-type* $T\ (Suc\ j) = L\text{-}type \Longrightarrow False$
proof −
assume *suffix-type* $T\ (Suc\ j) = L\text{-}type$
with *s-seen-invD*(2)[*OF assms*(5) ‹$m < length\ SA$›] ‹$i < m$› ‹$SA\ !\ m = Suc\ j$›
have *in-l-bucket* $\alpha\ T\ ?b\ m$
by *simp*
hence *bucket-start* $\alpha\ T\ ?b \leq m\ m < l\text{-}bucket\text{-}end\ \alpha\ T\ ?b$
using *in-l-bucket-def* **by** *blast*+
moreover
from ‹*suffix-type* $T\ (Suc\ j) = L\text{-}type$› *assms*(13,15)
s-seen-invD(2)[*OF assms*(5) ‹$Suc\ n < length\ SA$›]
have *in-l-bucket* $\alpha\ T\ ?b\ i$
by *simp*
hence *bucket-start* $\alpha\ T\ ?b \leq i\ i < l\text{-}bucket\text{-}end\ \alpha\ T\ ?b$
using *in-l-bucket-def* **by** *blast*+
ultimately
show *False*
using *list-slice-nth-eq-iff-index-eq*[
OF l-types-initD(2)[*OF l-types-init-maintained*[*OF assms*(2,4,8−10)]
‹$?b \leq \alpha$ -›]
‹*l-bucket-end* $\alpha\ T\ ?b \leq length\ SA$›,

```
          of i m]
        ‹i ≠ m› assms(13,15) ‹SA ! m = Suc j›
    by simp
 qed
 moreover
 have suffix-type T (Suc j) = S-type ⟹ False
 proof −
  assume suffix-type T (Suc j) = S-type
  with s-seen-invD(3)[OF assms(5) ‹m < length SA›] ‹i < m› ‹SA ! m = Suc j›
  have in-s-current-bucket α T B ?b m
    by simp
  hence B ! ?b ≤ m m < bucket-end α T ?b
    using in-s-current-bucket-def by blast+
  moreover
  from ‹suffix-type T (Suc j) = S-type› assms(13,15)
      s-seen-invD(3)[OF assms(5) ‹Suc n < length SA›]
  have in-s-current-bucket α T B ?b i
    by simp
  hence B ! ?b ≤ i i < bucket-end α T ?b
    using in-s-current-bucket-def by blast+
  ultimately
  show False
    using list-slice-nth-eq-iff-index-eq[
            OF s-distinct-invD[OF assms(1) ‹?b ≤ α -›] ‹bucket-end α T ?b ≤
length SA›,
          of i m]
        ‹i ≠ m› assms(13,15) ‹SA ! m = Suc j›
    by simp
 qed
 ultimately
 show False
  using SL-types.exhaust by blast
qed

lemma s-bucket-ptr-strict-lower-bound:
 assumes s-distinct-inv α T B SA
 and     s-bucket-ptr-inv α T B
 and     s-locations-inv α T B SA
 and     s-unchanged-inv α T B SA0 SA
 and     s-seen-inv α T B SA i
 and     s-pred-inv α T B SA i
 and     strict-mono α
 and     length SA0 = length T
 and     length SA = length T
 and     l-types-init α T SA0
 and     valid-list T
 and     α bot = 0
 and     i = Suc n
 and     Suc n < length SA
```

```
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
shows s-bucket-start α T b < B ! b
proof −
  have j < length T
    by (simp add: assms(16) suffix-type-s-bound)
  hence b ≤ α (Max (set T))
    by (simp add: assms(7,17) strict-mono-leD)

  let ?xs = list-slice SA (B ! b) (bucket-end α T b)

  have bucket-end α T b ≤ length SA
    by (simp add: assms(9) bucket-end-le-length)
  hence length ?xs = bucket-end α T b − B ! b
    by auto

  from s-next-item-not-seen[OF assms(1−17)]
  have j ∉ set ?xs .
  moreover
  have j ∈ s-bucket α T b
    by (simp add: assms(16,17) bucket-def s-bucket-def suffix-type-s-bound)
  ultimately
  have set ?xs ⊂ s-bucket α T b
    using s-locations-inv-subset-s-bucket[OF assms(3) ‹b ≤ -›]
    by blast
  hence card (set ?xs) < s-bucket-size α T b
   using psubset-card-mono[OF finite-s-bucket, simplified s-bucket-size-def[symmetric]]
    by blast
  moreover
  from s-distinct-invD[OF assms(1) ‹b ≤ -›]
  have distinct ?xs .
  ultimately
  have length ?xs < s-bucket-size α T b
    by (simp add: distinct-card)
  with ‹length ?xs = -›
  have bucket-end α T b − B ! b < s-bucket-size α T b
    by simp
  hence bucket-end α T b < B ! b + s-bucket-size α T b
    by linarith
  hence
     s-bucket-start α T b + bucket-end α T b < B ! b + s-bucket-start α T b +
s-bucket-size α T b
    by simp
  then show s-bucket-start α T b < B ! b
    by (simp add: bucket-end-eq-s-start-pl-size)
qed

lemma outside-another-bucket:
```

assumes $b \neq b'$
and *bucket-start* α T $b \leq i$
and $i <$ *bucket-end* α T b
shows $\neg$(*bucket-start* α T $b' \leq i \wedge i <$ *bucket-end* α T b')
by (*meson assms dual-order.antisym less-bucket-end-le-start not-le order.strict-trans1*)

lemma *s-B-val*:
assumes *s-distinct-inv* α T B SA
and *s-bucket-ptr-inv* α T B
and *s-locations-inv* α T B SA
and *s-unchanged-inv* α T B $SA0$ SA
and *s-seen-inv* α T B SA i
and *s-pred-inv* α T B SA i
and *strict-mono* α
and *length* $SA0$ = *length* T
and *length* SA = *length* T
and *l-types-init* α T $SA0$
and *valid-list* T
and *length* $T >$ *Suc 0*
and $b \leq \alpha$ (*Max* (*set* T))
and $i < B$! b
shows B ! b = *s-bucket-start* α T b
proof (*rule ccontr*; *drule neq-iff*[*THEN iffD1*]; *elim disjE*)
assume B ! $b <$ *s-bucket-start* α T b
with *s-bucket-ptr-lower-bound*[*OF assms*(*2*,*13*)]
show *False*
by *linarith*
next
let *?k* = B ! b − *Suc 0*
assume *s-bucket-start* α T $b < B$! b
hence *s-bucket-start* α T $b \leq$ *?k*
by *linarith*
hence *bucket-start* α T $b \leq$ *?k*
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

have *?k* $< B$! b
using ‹*s-bucket-start* α T $b < B$! b› **by** *auto*

from *assms*(*14*)
have $i \leq$ *?k*
by *linarith*

from *s-bucket-ptr-upper-bound*[*OF assms*(*2*,*13*)]
have B ! $b \leq$ *bucket-end* α T b **.**
hence *?k* $<$ *bucket-end* α T b
using ‹*s-bucket-start* α T $b < B$! b› **by** *linarith*

have *bucket-end* α T $b \leq$ *length* SA
by (*simp add*: *assms*(*9*) *bucket-end-le-length*)

moreover
have *bucket-end* α *T b* = *length SA* $\Longrightarrow$ *False*
proof −
assume *bucket-end* α *T b* = *length SA*
with ‹*s-bucket-start* α *T b* < *B* ! *b*› ‹*B* ! *b* ≤ *bucket-end* α *T b*› *assms*(*7*,*9*,*13*)
have *b* = α (*Max* (*set T*))
using *bucket-end-eq-length* **by** *fastforce*
hence *s-bucket* α *T b* = {}
by (*simp add*: *assms*(*7*,*11*,*12*) *s-bucket-Max*)
hence *s-bucket-size* α *T b* = *0*
by (*simp add*: *s-bucket-size-def*)
hence *s-bucket-start* α *T b* = *bucket-end* α *T b*
by (*simp add*: *bucket-end-eq-s-start-pl-size*)
with ‹*B* ! *b* ≤ *bucket-end* α *T b*› ‹*s-bucket-start* α *T b* < *B* ! *b*›
show *False*
by *simp*
qed
moreover
have *bucket-end* α *T b* < *length SA* $\Longrightarrow$ *False*
proof −
assume *bucket-end* α *T b* < *length SA*
hence *B* ! *b* < *length SA*
using ‹*B* ! *b* ≤ *bucket-end* α *T b*›
by *linarith*
hence *?k* < *length SA*
using *less-imp-diff-less* **by** *blast*
with *s-seen-invD*[*OF assms*(*5*) - ‹*i* ≤ *?k*›]
have *SA* ! *?k* < *length T*
by *blast*

let *?b* = α (*T* ! (*SA* ! *?k*))

from ‹*SA* ! *?k* < *length T*› *assms*(*7*)
have *?b* ≤ α (*Max* (*set T*))
using *Max-greD strict-mono-leD* **by** *blast*
with *s-bucket-ptr-lower-bound*[*OF assms*(*2*)]
have *s-bucket-start* α *T ?b* ≤ *B* ! *?b*
by *blast*
hence *bucket-start* α *T ?b* ≤ *B* ! *?b*
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

have *suffix-type T* (*SA* ! *?k*) = *L-type* $\Longrightarrow$ *False*
proof −
assume *suffix-type T* (*SA* ! *?k*) = *L-type*
with *s-seen-invD*(*2*)[*OF assms*(*5*) ‹*?k* < *length SA*› ‹*i* ≤ *?k*›]
have *in-l-bucket* α *T ?b ?k*
by *blast*
hence *bucket-start* α *T ?b* ≤ *?k ?k* < *l-bucket-end* α *T ?b*
using *in-l-bucket-def* **by** *blast+*

```
    hence ?k < bucket-end α T ?b
      using l-bucket-end-le-bucket-end less-le-trans by blast

    from ‹?k < l-bucket-end - - -› ‹s-bucket-start - - - ≤ ?k›
    have b = ?b ⟹ False
      by (simp add: s-bucket-start-eq-l-bucket-end)
    moreover
    from outside-another-bucket[OF - ‹bucket-start - - b ≤ ?k› ‹?k < bucket-end
- - b›]
        ‹bucket-start - - ?b ≤ ?k› ‹?k < bucket-end - - ?b›
    have b ≠ ?b ⟹ False
      by blast
    ultimately show False
      by blast
  qed
  moreover
  have suffix-type T (SA ! ?k) = S-type ⟹ False
  proof −
    assume suffix-type T (SA ! ?k) = S-type
    with s-seen-invD(3)[OF assms(5) ‹?k < length SA› ‹i ≤ ?k›]
    have in-s-current-bucket α T B ?b ?k
      by blast
    hence B ! ?b ≤ ?k ?k < bucket-end α T ?b
      using in-s-current-bucket-def by blast+
    hence bucket-start α T ?b ≤ ?k
      using ‹bucket-start α T ?b ≤ B ! ?b› by linarith

    from ‹B ! ?b ≤ ?k› ‹?k < B ! b›
    have b = ?b ⟹ False
      by simp
    moreover
    from outside-another-bucket[OF - ‹bucket-start - - b ≤ ?k› ‹?k < bucket-end
- - b›]
        ‹bucket-start α T ?b ≤ ?k› ‹?k < bucket-end α T ?b›
    have b ≠ ?b ⟹ False
      by blast
    ultimately show False
      by blast
  qed
  ultimately show False
    using SL-types.exhaust by blast
qed
ultimately show False
  by linarith
qed

lemma s-bucket-eq-list-slice:
  assumes s-distinct-inv α T B SA
  and     s-locations-inv α T B SA
```

and *length SA = length T*
and $b \leq \alpha$ (*Max* (*set T*))
and *B ! b = s-bucket-start* α *T b*
shows *set* (*list-slice SA* (*s-bucket-start* α *T b*) (*bucket-end* α *T b*)) = *s-bucket* α *T b*
(**is** *set ?xs = s-bucket* α *T b*)
using *card-subset-eq*[
OF finite-s-bucket s-locations-inv-subset-s-bucket[*OF assms*(*2*,*4*), *simplified assms*(*5*)]]
distinct-card[*OF s-distinct-invD*[*OF assms*(*1*,*4*), *simplified assms*(*5*)]]
bucket-end-eq-s-start-pl-size[*of* α *T b*]
s-bucket-size-def[*of* α *T b*]
by (*metis assms*(*3*) *bucket-end-le-length diff-add-inverse length-list-slice min-def*)

lemma *bucket-eq-list-slice*:
assumes *s-distinct-inv* α *T B SA*
and *s-bucket-ptr-inv* α *T B*
and *s-locations-inv* α *T B SA*
and *s-unchanged-inv* α *T B SA0 SA*
and *length SA0 = length T*
and *length SA = length T*
and *l-types-init* α *T SA0*
and $b \leq \alpha$ (*Max* (*set T*))
and *B ! b = s-bucket-start* α *T b*
shows *set* (*list-slice SA* (*bucket-start* α *T b*) (*bucket-end* α *T b*)) = *bucket* α *T b*
(**is** *set ?xs = bucket* α *T b*)
proof −
let *?ys = list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)
and *?zs = list-slice SA* (*s-bucket-start* α *T b*) (*bucket-end* α *T b*)

have *?xs = ?ys @ ?zs*
by (*metis list-slice-append bucket-start-le-s-bucket-start l-bucket-end-le-bucket-end*
s-bucket-start-eq-l-bucket-end)
hence *set ?xs = set ?ys* ∪ *set ?zs*
by *simp*
with *l-types-initD*(*1*)[*OF l-types-init-maintained*[*OF assms*(*2*,*4*−*7*)] *assms*(*8*)]
s-bucket-eq-list-slice[*OF assms*(*1*,*3*,*6*,*8*,*9*)]
have *set ?xs = l-bucket* α *T b* ∪ *s-bucket* α *T b*
by *simp*
then show *?thesis*
using *l-un-s-bucket* **by** *blast*
qed

lemma *s-index-lower-bound*:
assumes *s-bucket-ptr-inv* α *T B*
and *s-seen-inv* α *T B SA n*
and *i < length SA*
and *n* ≤ *i*
shows *bucket-start* α *T* (α (*T !* (*SA ! i*))) ≤ *i*

```
      (is bucket-start α T ?b ≤ i)
proof −

  have ?b ≤ α (Max (set T))
    by (meson SL-types.exhaust assms(2−) in-l-bucket-def in-s-current-bucketD(1)
s-seen-invD(2,3))

  have suffix-type T (SA ! i) = S-type ∨ suffix-type T (SA ! i) = L-type
    using SL-types.exhaust by blast
  moreover
  have suffix-type T (SA ! i) = S-type ⟹ ?thesis
  proof −
    assume suffix-type T (SA ! i) = S-type
    with s-seen-invD(3)[OF assms(2−)] s-bucket-ptr-lower-bound[OF assms(1)]
    show ?thesis
    by (meson ‹?b ≤ α (Max (set T))› bucket-start-le-s-bucket-start dual-order.trans
          in-s-current-bucketD(2))
  qed
  moreover
  have suffix-type T (SA ! i) = L-type ⟹ ?thesis
  proof −
    assume suffix-type T (SA ! i) = L-type
    with s-seen-invD(2)[OF assms(2−)]
    show ?thesis
      using in-l-bucket-def by blast
  qed
  ultimately show ?thesis
    by blast
qed

lemma s-index-upper-bound:
  assumes s-bucket-ptr-inv α T B
  and     s-seen-inv α T B SA n
  and     i < length SA
  and     n ≤ i
shows i < bucket-end α T (α (T ! (SA ! i)))
      (is i < bucket-end α T ?b)
proof −

  have ?b ≤ α (Max (set T))
    by (meson SL-types.exhaust assms(2−) in-l-bucket-def in-s-current-bucketD(1)
s-seen-invD(2,3))

  have suffix-type T (SA ! i) = S-type ∨ suffix-type T (SA ! i) = L-type
    using SL-types.exhaust by blast
  moreover
  have suffix-type T (SA ! i) = S-type ⟹ ?thesis
  proof −
    assume suffix-type T (SA ! i) = S-type
```

```
    with s-seen-invD(3)[OF assms(2−)] s-bucket-ptr-upper-bound[OF assms(1)]
    show ?thesis
      by (simp add: in-s-current-bucket-def)
  qed
  moreover
  have suffix-type T (SA ! i) = L-type ⟹ ?thesis
  proof −
    assume suffix-type T (SA ! i) = L-type
    with s-seen-invD(2)[OF assms(2−)]
    show ?thesis
      using in-l-bucket-def l-bucket-end-le-bucket-end less-le-trans by blast
  qed
  ultimately show ?thesis
    by blast
qed
```

83.3 Establishment and Maintenance Steps

83.3.1 Distinctness

```
lemma s-distinct-inv-established:
  assumes s-bucket-init α T B
  and     valid-list T
  and     strict-mono α
  and     α bot = 0
  shows s-distinct-inv α T B SA
  unfolding s-distinct-inv-def
proof (intro allI impI)
  fix b
  let ?goal = distinct (list-slice SA (B ! b) (bucket-end α T b))
  assume b ≤ α (Max (set T))

  have b > 0 ⟹ ?goal
  proof −
    assume b > 0
    with s-bucket-initD(1)[OF assms(1) ‹b ≤ -›]
    have B ! b = bucket-end α T b
      by blast
    then show ?goal
      using list-slice-n-n
      by (metis distinct.simps(1))
  qed
  moreover
  have b = 0 ⟹ ?goal
  proof −
    assume b = 0
    with s-bucket-initD(2)[OF assms(1) ‹b ≤ -›]
    have B ! b = 0 .
    moreover
    from ‹b = 0› assms(2−4)
```

```
    have bucket-end α T b = Suc 0
      by (simp add: valid-list-bucket-end-0)
    ultimately show ?thesis
      by (simp add: distinct-conv-nth)
  qed
  ultimately show ?goal
    by blast
qed

lemma s-distinct-inv-maintained-step:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
shows s-distinct-inv α T (B[b := k]) (SA[k := j])
  unfolding s-distinct-inv-def
proof (intro allI impI)
  fix b'

  let ?goal = distinct (list-slice (SA[k := j]) (B[b := k] ! b') (bucket-end α T b'))

  assume b' ≤ α (Max (set T))

  from s-next-item-not-seen[OF assms(1−7,9−18)]
  have j ∉ set (list-slice SA (B ! b) (bucket-end α T b)).

  from s-bucket-ptr-strict-lower-bound[OF assms(1−7,9−18)]
  have s-bucket-start α T b < B ! b .
  hence s-bucket-start α T b ≤ k k < B ! b
    using assms(19) by linarith+
  hence bucket-start α T b ≤ k
    using bucket-start-le-s-bucket-start le-trans by blast

  from assms(17)
```

```
  have j < length T
    by (simp add: suffix-type-s-bound)
  hence b ≤ α (Max (set T))
    by (simp add: assms(7,18) strict-mono-less-eq)
  with s-bucket-ptr-upper-bound[OF assms(2)] ‹k < B ! b›
  have k < bucket-end α T b
    using less-le-trans by auto
  hence k < length SA
    using assms(10) bucket-end-le-length less-le-trans by fastforce

  have b = b' ⟹ ?goal
  proof −
    assume b = b'
    hence B[b := k] ! b' = k
      using ‹b ≤ α (Max (set T))› assms(8) by auto
    from s-distinct-invD[OF assms(1) ‹b ≤ -›]
    have distinct (list-slice SA (B ! b) (bucket-end α T b)) .
    moreover
    from ‹B[b := k] ! b' = k› ‹b = b'› ‹k < B ! b› ‹k < bucket-end α T b› ‹k <
length SA› assms(19)
    have list-slice (SA[k := j]) (B[b := k] ! b') (bucket-end α T b')
          = j # list-slice SA (B ! b) (bucket-end α T b)
     by (metis Suc-pred diff-is-0-eq' dual-order.order-iff-strict gr0I length-list-update
           list-slice-Suc list-slice-update-unchanged-1 nth-list-update-eq)
    ultimately show ?thesis
      using ‹j ∉ set (list-slice SA (B ! b) (bucket-end α T b))›
      by simp
  qed
  moreover
  have b ≠ b' ⟹ ?goal
  proof −
    assume b ≠ b'
    hence B[b := k] ! b' = B ! b'
      by simp

   from outside-another-bucket[OF ‹b ≠ b'› ‹bucket-start - - - ≤ k› ‹k < bucket-end
- - -›]
    have k < bucket-start α T b' ∨ bucket-end α T b' ≤ k
      using leI by auto
    moreover
    have k < bucket-start α T b' ⟹
         list-slice (SA[k := j]) (B[b := k] ! b') (bucket-end α T b')
           = list-slice SA (B ! b') (bucket-end α T b')
    proof −
      assume k < bucket-start α T b'
      hence k < B ! b'
        by (meson ‹b' ≤ α (Max (set T))› assms(2) bucket-start-le-s-bucket-start
less-le-trans
              s-bucket-ptr-inv-def)
```

with *list-slice-update-unchanged-1* ‹$B[b := k] \,!\, b' = B \,!\, b'$›
show *?thesis*
by *simp*
qed
moreover
from ‹$B[b := k] \,!\, b' = B \,!\, b'$›
have *bucket-end* α T $b' \leq k \Longrightarrow$
list-slice ($SA[k := j]$) ($B[b := k] \,!\, b'$) (*bucket-end* α T b')
$=$ *list-slice* SA ($B \,!\, b'$) (*bucket-end* α T b')
by (*simp add*: *list-slice-update-unchanged-2*)
moreover
from *s-distinct-invD*[*OF assms*(*1*) ‹$b' \leq$ -›]
have *distinct* (*list-slice SA* ($B \,!\, b'$) (*bucket-end* α T b')) **.**
ultimately show *?goal*
by *auto*
qed
ultimately
show *?goal*
by *blast*
qed

corollary *s-distinct-inv-maintained-perm-step*:
assumes *s-perm-inv* α T B *SA0 SA* i
and $i = Suc\ n$
and $Suc\ n < length\ SA$
and $SA \,!\, Suc\ n = Suc\ j$
and *suffix-type* $T\ j =$ *S-type*
and $b = \alpha\ (T \,!\, j)$
and $k = B \,!\, b - Suc\ 0$
shows *s-distinct-inv* α T ($B[b := k]$) ($SA[k := j]$)
using *s-distinct-inv-maintained-step*[*OF s-perm-inv-elims*(*1*−*6*,*8*−*14*)[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

83.3.2 Bucket Pointer

lemma *s-bucket-ptr-inv-established*:
assumes *s-bucket-init* α T B
and *valid-list* T
and *strict-mono* α
and α *bot* $= 0$
shows *s-bucket-ptr-inv* α T B
unfolding *s-bucket-ptr-inv-def*
proof(*intro allI impI*)
fix b
let *?goal* $=$ *s-bucket-start* α T $b \leq B \,!\, b \wedge B \,!\, b \leq$ *bucket-end* α T $b \wedge (b = 0 \longrightarrow B \,!\, b = 0)$
assume $b \leq \alpha$ (*Max* (*set* T))

```
  have b > 0 ⟹ ?goal
  proof −
    assume b > 0
    with s-bucket-initD(1)[OF assms(1) ‹b ≤ -›]
    have B ! b = bucket-end α T b .
    then show ?thesis
      by (metis ‹0 < b› l-bucket-end-le-bucket-end less-numeral-extra(3) order-refl
          s-bucket-start-eq-l-bucket-end)
  qed
  moreover
  have b = 0 ⟹ ?goal
  proof −
    assume b = 0
    with s-bucket-initD(2)[OF assms(1) ‹b ≤ -›]
    have B ! b = 0 .
    with ‹b = 0›
        valid-list-bucket-end-0[OF assms(2−)]
        valid-list-s-bucket-start-0[OF assms(2−)]
    show ?thesis
      by auto
  qed
  ultimately show ?goal
    by auto
qed

lemma s-bucket-ptr-inv-maintained-step:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
shows s-bucket-ptr-inv α T (B[b := k])
  unfolding s-bucket-ptr-inv-def
proof(intro allI impI)
  fix b'
```

assume $b' \leq \alpha\ (Max\ (set\ T))$

let *?goal* = *s-bucket-start* $\alpha\ T\ b' \leq B[b := k]\ !\ b' \wedge\ B[b := k]\ !\ b' \leq$ *bucket-end* $\alpha\ T\ b' \wedge$
$(b' = 0 \longrightarrow B[b := k]\ !\ b' = 0)$

from *valid-list-length-ex*[*OF assms*(12)]
obtain m **where**
length $T = Suc\ m$
by *blast*
moreover
have $j <$ *length* T
by (*simp add*: *assms*(17) *suffix-type-s-bound*)
moreover
from *s-seen-invD*(1)[*OF assms*(5,15)] *assms*(14,16)
have *Suc* $j <$ *length* T
by *simp*
ultimately have $j < m$
by *linarith*
hence $b \neq 0$
by (*metis* ‹*length* T = *Suc* m› *assms*(7,12,13,18) *diff-Suc-1 strict-mono-eq valid-list-def*)

have $b \neq b' \Longrightarrow$ *?goal*
proof −
assume $b \neq b'$
hence $B[b := k]\ !\ b' = B\ !\ b'$
by *simp*
with *s-bucket-ptr-lower-bound*[*OF assms*(2) ‹$b' \leq \alpha\ (Max\ (set\ T))$›]
s-bucket-ptr-upper-bound[*OF assms*(2) ‹$b' \leq \alpha\ (Max\ (set\ T))$›]
s-bucket-ptr-0[*OF assms*(2)]
show *?thesis*
by *auto*
qed
moreover
have $b = b' \Longrightarrow$ *?goal*
proof −
assume $b = b'$
from *s-bucket-ptr-strict-lower-bound*[*OF assms*(1−7,9−13,14−18)]
have *s-bucket-start* $\alpha\ T\ b < B\ !\ b$.
hence *s-bucket-start* $\alpha\ T\ b \leq k$
using *assms*(19) **by** *linarith*
moreover
from ‹$b = b'$› ‹$b' \leq \alpha\ (Max\ (set\ T))$› *assms*(2,19)
have $k \leq$ *bucket-end* $\alpha\ T\ b$
using *le-diff-conv s-bucket-ptr-inv-def trans-le-add1* **by** *blast*
moreover
have $B[b := k]\ !\ b' = k$
using ‹$b = b'$› ‹$b' \leq \alpha\ (Max\ (set\ T))$› *assms*(8) **by** *auto*

ultimately show *?thesis*
using ‹$b = b'$› ‹$b \neq 0$› **by** *auto*
qed
ultimately show *?goal*
by *linarith*
qed

corollary *s-bucket-ptr-inv-maintained-perm-step*:
assumes *s-perm-inv* α T B *SA0 SA* i
and $i = Suc\ n$
and $Suc\ n < length\ SA$
and $SA\ !\ Suc\ n = Suc\ j$
and *suffix-type* $T\ j = S\text{-}type$
and $b = \alpha\ (T\ !\ j)$
and $k = B\ !\ b - Suc\ 0$
shows *s-bucket-ptr-inv* α T $(B[b := k])$
using *s-bucket-ptr-inv-maintained-step*[*OF s-perm-inv-elims*(*1*−*6*,*8*−*14*)[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

83.3.3 Locations

lemma *s-locations-inv-established*:
assumes *s-bucket-init* α T B
and *s-type-init* T *SA*
and *valid-list* T
and *strict-mono* α
and α *bot* $= 0$
shows *s-locations-inv* α T B *SA*
unfolding *s-locations-inv-def*
proof(*safe*)
fix b i
assume $b \leq \alpha\ (Max\ (set\ T))\ B\ !\ b \leq i\ i < bucket\text{-}end\ \alpha\ T\ b$
hence $b > 0 \Longrightarrow SA\ !\ i \in s\text{-}bucket\ \alpha\ T\ b$
by (*metis assms*(*1*) *not-le s-bucket-init-def*)
moreover
have $b = 0 \Longrightarrow SA\ !\ i \in s\text{-}bucket\ \alpha\ T\ b$
proof −
assume $b = 0$
have $0 \leq i$
by *blast*
moreover
have *bucket-end* α T $0 = 1$
using *assms*(*3*−*5*) *valid-list-bucket-end-0* **by** *blast*
with ‹$i < bucket\text{-}end\ \alpha\ T\ b$› ‹$b = 0$›
have $i < 1$
by *simp*
ultimately have $i = 0$
by *blast*

```
    moreover
    from s-type-init-def[of T SA] assms(2)
    obtain n where
      length T = Suc n
      SA ! 0 = n
      by blast
    with suffix-type-last[of T n]
    have suffix-type T n = S-type
      by blast
    moreover
    have T ! n = bot
    by (metis ‹length T = Suc n› assms(3) diff-Suc-1 last-conv-nth less-numeral-extra(3)
          list.size(3) valid-list-def)
    hence α (T ! n) = 0
      by (simp add: assms(5))
    ultimately show ?thesis
     by (simp add: ‹SA ! 0 = n›‹b = 0› ‹length T = Suc n› bucket-def s-bucket-def)
  qed
  ultimately show SA ! i ∈ s-bucket α T b
    by linarith
qed

lemma s-locations-inv-maintained-step:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
shows s-locations-inv α T (B[b := k]) (SA[k := j])
  unfolding s-locations-inv-def
proof(safe)
  fix b′ i′
  assume b′ ≤ α (Max (set T)) B[b := k] ! b′ ≤ i′ i′ < bucket-end α T b′

  from s-bucket-ptr-strict-lower-bound[OF assms(1−7,9−18)]
```

have *s-bucket-start* α T b $<$ B ! b**.**
hence *s-bucket-start* α T $b \leq k$
using *assms*(*19*) **by** *linarith*

from ‹*s-bucket-start* α T b $<$ B ! b›
have $k < B$! b
using *assms*(*19*) **by** *linarith*

have $j <$ *length* T
by (*simp add*: *assms*(*17*) *suffix-type-s-bound*)
hence $b \leq \alpha$ (*Max* (*set* T))
by (*simp add*: *assms*(*18*) *assms*(*7*) *strict-mono-less-eq*)
with *s-bucket-ptr-upper-bound*[*OF assms*(*2*)]
have B ! $b \leq$ *bucket-end* α T b
by *blast*

have $b \neq b' \Longrightarrow SA[k := j]$! $i' \in$ *s-bucket* α T b'
proof −
assume $b \neq b'$
hence $B[b := k]$! $b' = B$! b'
by *simp*
with ‹$B[b := k]$! $b' \leq i'$›
have B ! $b' \leq i'$
by *simp*
with *s-locations-invD*[*OF assms*(*3*) ‹$b' \leq \alpha$ (*Max* (*set* T))› - ‹$i' <$ *bucket-end* α T b'›]
have *SA* ! $i' \in$ *s-bucket* α T b'
by *linarith*
moreover
from *s-bucket-ptr-lower-bound*[*OF assms*(*2*) ‹$b' \leq \alpha$ (*Max* (*set* T))›] ‹B ! $b' \leq i'$›
have *bucket-start* α T $b' \leq i'$
by (*meson bucket-start-le-s-bucket-start dual-order.trans*)
with *outside-another-bucket*[*OF* ‹$b \neq b'$›[*symmetric*] - ‹$i' <$ -›]
‹B ! $b \leq$ *bucket-end* α T b› ‹$k < B$! b› ‹*s-bucket-start* α T $b \leq k$›
have $k \neq i'$
using *bucket-start-le-s-bucket-start le-trans less-le-trans* **by** *blast*
hence $SA[k := j]$! $i' = SA$! i'
by *simp*
ultimately show *?thesis*
by *simp*
qed
moreover
have $b = b' \Longrightarrow SA[k := j]$! $i' \in$ *s-bucket* α T b'
proof −
assume $b = b'$
hence $k \leq i'$
using ‹$B[b := k]$! $b' \leq i'$› ‹$b' \leq \alpha$ (*Max* (*set* T))› *assms*(*8*) **by** *auto*
hence $k = i' \vee k < i'$

```
    by linarith
  moreover
  have k = i' ⟹ ?thesis
  proof −
    assume k = i'
    hence SA[k := j] ! i' = j
    by (metis ‹i' < bucket-end α T b'› assms(10) bucket-end-le-length dual-order.strict-trans1
          nth-list-update)
    with assms(17,18) ‹b = b'› ‹j < length T›
    show ?thesis
      by (simp add: bucket-def s-bucket-def)
  qed
  moreover
  have k < i' ⟹ ?thesis
  proof −
    assume k < i'
    hence B ! b ≤ i'
      using assms(19) by linarith
    with s-locations-invD[OF assms(3) ‹b' ≤ -› - ‹i' < bucket-end - - -›] ‹b = b'›
    have SA ! i' ∈ s-bucket α T b'
      by blast
    moreover
    have SA[k := j] ! i' = SA ! i'
      using ‹k < i'› by auto
    ultimately show ?thesis
      by simp
  qed
  ultimately show ?thesis
    by linarith
 qed
 ultimately show SA[k := j] ! i' ∈ s-bucket α T b'
  by blast
qed

corollary s-locations-inv-maintained-perm-step:
 assumes s-perm-inv α T B SA0 SA i
 and     i = Suc n
 and     Suc n < length SA
 and     SA ! Suc n = Suc j
 and     suffix-type T j = S-type
 and     b = α (T ! j)
 and     k = B ! b − Suc 0
shows s-locations-inv α T (B[b := k]) (SA[k := j])
 using s-locations-inv-maintained-step[OF s-perm-inv-elims(1−6,8−14)[OF assms(1)]
assms(2−)]
 by blast
```

83.3.4 Unchanged

lemma *s-unchanged-inv-established*:
shows *s-unchanged-inv α T B SA SA*
by (*simp add*: *s-unchanged-inv-def*)

lemma *s-unchanged-inv-maintained-step*:
assumes *s-distinct-inv α T B SA*
and *s-bucket-ptr-inv α T B*
and *s-locations-inv α T B SA*
and *s-unchanged-inv α T B SA0 SA*
and *s-seen-inv α T B SA i*
and *s-pred-inv α T B SA i*
and *strict-mono α*
and α (*Max* (*set T*)) < *length B*
and *length SA0* = *length T*
and *length SA* = *length T*
and *l-types-init α T SA0*
and *valid-list T*
and α *bot* = *0*
and *i* = *Suc n*
and *Suc n* < *length SA*
and *SA* ! *Suc n* = *Suc j*
and *suffix-type T j* = *S-type*
and *b* = α (*T* ! *j*)
and *k* = *B* ! *b* − *Suc 0*
shows *s-unchanged-inv α T* (*B*[*b* := *k*]) *SA0* (*SA*[*k* := *j*])
unfolding *s-unchanged-inv-def*
proof(*safe*)
fix *b′ i′*
assume *b′* ≤ α (*Max* (*set T*)) *bucket-start α T b′* ≤ *i′ i′* < *B*[*b* := *k*] ! *b′*

from *s-bucket-ptr-strict-lower-bound*[*OF assms*(*1*−*7*,*9*−*18*)]
have *s-bucket-start α T b* < *B* ! *b*.
hence *s-bucket-start α T b* ≤ *k*
using *assms*(*19*) **by** *linarith*
hence *bucket-start α T b* ≤ *k*
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

from ‹*s-bucket-start α T b* < *B* ! *b*›
have *k* < *B* ! *b*
using *assms*(*19*) **by** *linarith*

have *j* < *length T*
by (*simp add*: *assms*(*17*) *suffix-type-s-bound*)
hence *b* ≤ α (*Max* (*set T*))
by (*simp add*: *assms*(*7*,*18*) *strict-mono-less-eq*)
with *s-bucket-ptr-upper-bound*[*OF assms*(*2*)]
have *B* ! *b* ≤ *bucket-end α T b*
by *blast*

```
  with ‹k < B ! b›
  have k < bucket-end α T b
    by linarith

  have b = b' ⟹ SA[k := j] ! i' = SA0 ! i'
  proof −
    assume b = b'
    hence B[b := k] ! b' = k
      using ‹b' ≤ α (Max (set T))› assms(8) by auto
    with ‹i' < B[b := k] ! b'›
    have i' < k
      by linarith
    hence SA[k := j] ! i' = SA ! i'
      by simp
    moreover
    from ‹i' < k› ‹k < B ! b› ‹b = b'›
    have i' < B ! b'
      by simp
    with s-unchanged-invD[OF assms(4) ‹b' ≤ -› ‹bucket-start - - - ≤ i'›]
    have SA ! i' = SA0 ! i'
      by simp
    ultimately show ?thesis
      by simp
  qed
  moreover
  have b ≠ b' ⟹ SA[k := j] ! i' = SA0 ! i'
  proof −
    assume b ≠ b'
    with ‹i' < B[b := k] ! b'›
    have i' < B ! b'
      by simp
    with s-unchanged-invD[OF assms(4) ‹b' ≤ -› ‹bucket-start - - b' ≤ -›]
    have SA ! i' = SA0 ! i'
      by blast
    moreover
    from s-bucket-ptr-upper-bound[OF assms(2) ‹b' ≤ -›] ‹i' < B ! b'›
    have i' < bucket-end α T b'
      by linarith
   with outside-another-bucket[OF ‹b ≠ b'› ‹bucket-start - - - ≤ k› ‹k < bucket-end
- - -›]
        ‹bucket-start - - b' ≤ i'›
    have k ≠ i'
      by blast
    hence SA[k := j] ! i' = SA ! i'
      by simp
    ultimately show ?thesis
      by simp
  qed
  ultimately show SA[k := j] ! i' = SA0 ! i'
```

by *blast*
qed

corollary *s-unchanged-inv-maintained-perm-step*:
assumes *s-perm-inv* α *T B SA0 SA i*
and $i = Suc\ n$
and $Suc\ n < length\ SA$
and $SA\ !\ Suc\ n = Suc\ j$
and *suffix-type T j* = *S-type*
and $b = \alpha\ (T\ !\ j)$
and $k = B\ !\ b - Suc\ 0$
shows *s-unchanged-inv* α *T* $(B[b := k])$ *SA0* $(SA[k := j])$
using *s-unchanged-inv-maintained-step*[*OF s-perm-inv-elims*(*1*−*6*,*8*−*14*)[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

83.3.5 Seen

lemma *s-seen-inv-established*:
assumes *length SA* = *length T*
and *length T* ≤ *n*
shows *s-seen-inv* α *T B SA n*
unfolding *s-seen-inv-def*
using *assms* **by** *auto*

lemma *s-seen-inv-maintained-step-c1*:
assumes *s-bucket-ptr-inv* α *T B*
and *s-unchanged-inv* α *T B SA0 SA*
and *s-seen-inv* α *T B SA i*
and *strict-mono* α
and *length SA0* = *length T*
and *length SA* = *length T*
and *l-types-init* α *T SA0*
and *valid-list T*
and $Suc\ 0 < length\ T$
and $i = Suc\ n$
and $length\ SA \leq Suc\ n$
shows *s-seen-inv* α *T B SA n*
unfolding *s-seen-inv-def*
proof (*intro allI impI*)
fix *j*
assume $j < length\ SA\ n \leq j$
hence $n < length\ SA$
by *simp*
with *assms*(*10*,*11*)
have *length SA* = *Suc n*
by *linarith*

let *?b* = $\alpha\ (T\ !\ (SA\ !\ j))$

```
  let ?g1 = (suffix-type T (SA ! j) = S-type ⟶ in-s-current-bucket α T B ?b j)
  and ?g2 = (suffix-type T (SA ! j) = L-type ⟶ in-l-bucket α T ?b j)
  and ?g3 = SA ! j < length T

  from ‹n ≤ j›
  have n = j ∨ Suc n ≤ j
    using dual-order.antisym not-less-eq-eq by auto
  moreover
  have n = j ⟹ ?g1 ∧ ?g2 ∧ ?g3
  proof −
    let ?b-max = α (Max (set T))
    assume n = j
    hence j < bucket-end α T ?b-max
      using ‹j < length SA› assms(4,6) bucket-end-Max by fastforce
    hence j < l-bucket-end α T ?b-max
      using l-bucket-Max[OF assms(8,9,4)]
     by (simp add: bucket-end-def' bucket-size-def l-bucket-end-def l-bucket-size-def)
    moreover
    from ‹n = j› ‹j < length SA› ‹length SA = Suc n› assms(4,6,10)
    have bucket-start α T ?b-max ≤ j
     by (metis Suc-leI antisym bucket-end-eq-length bucket-end-le-length gr-implies-not0
              index-in-bucket-interval-gen length-0-conv)
    moreover
    have ?b-max ≤ α (Max (set T))
      by simp
    ultimately have SA ! j ∈ l-bucket α T ?b-max
    using l-types-init-nth[OF assms(6) l-types-init-maintained[OF assms(1,2,5−7)]
]
      by blast
    hence ?b-max = α (T ! (SA ! j))
      by (simp add: bucket-def l-bucket-def)
    moreover
    from ‹SA ! j ∈ l-bucket α T ?b-max›
    have ?g3
      by (simp add: bucket-def l-bucket-def)
    moreover
    from ‹SA ! j ∈ l-bucket α T ?b-max›
    have suffix-type T (SA ! j) = L-type
      by (simp add: bucket-def l-bucket-def)
    moreover
    have in-l-bucket α T (α (T ! (SA ! j))) j
        using ‹bucket-start - - -  ≤ j› ‹j < l-bucket-end - - -› calculation(1)
in-l-bucket-def
      by fastforce
    hence ?g2
      using calculation(3) by blast
    moreover
    from calculation(3)
    have ?g1
```

```
    by simp
  ultimately show ?thesis
    by simp
 qed
 moreover
 have Suc n ≤ j ⟹ ?g1 ∧ ?g2 ∧ ?g3
 proof −
  assume Suc n ≤ j
  with s-seen-invD[OF assms(3) ‹j < length SA›] assms(10)
  show ?thesis
   by blast
 qed
 ultimately show ?g1 ∧ ?g2 ∧ ?g3
  by blast
qed

corollary s-seen-inv-maintained-perm-step-c1:
 assumes s-perm-inv α T B SA0 SA i
 and     i = Suc n
 and     length SA ≤ Suc n
 shows s-seen-inv α T B SA n
 using s-seen-inv-maintained-step-c1[OF s-perm-inv-elims(2,4,5,8,10−13,15)[OF
assms(1)] assms(2−)]
 by blast

lemma s-seen-inv-maintained-step-c1-alt:
 assumes s-bucket-ptr-inv α T B
 and     s-unchanged-inv α T B SA0 SA
 and     s-seen-inv α T B SA i
 and     strict-mono α
 and     length SA0 = length T
 and     length SA = length T
 and     l-types-init α T SA0
 and     valid-list T
 and     Suc 0 < length T
 and     i = Suc n
 and     length T ≤ SA ! Suc n
 shows s-seen-inv α T B SA n
proof (cases length SA ≤ Suc n)
 assume length SA ≤ Suc n
 then show ?thesis
  using assms(1−10) s-seen-inv-maintained-step-c1 by blast
next
 assume ¬ length SA ≤ Suc n
 hence Suc n < length SA
  by simp
 show ?thesis
  unfolding s-seen-inv-def
 proof (intro allI impI)
```

```
    fix j
    assume j < length SA n ≤ j
    hence n < length SA
      by simp

    let ?b = α (T ! (SA ! j))
    let ?g1 = (suffix-type T (SA ! j) = S-type ⟶ in-s-current-bucket α T B ?b j)
    and ?g2 = (suffix-type T (SA ! j) = L-type ⟶ in-l-bucket α T ?b j)
    and ?g3 = SA ! j < length T

   from ‹n ≤ j›
    have n = j ∨ Suc n ≤ j
      using dual-order.antisym not-less-eq-eq by auto
    moreover
    have n = j ⟹ ?g1 ∧ ?g2 ∧ ?g3
    by (metis Suc-le-mono ‹Suc n < length SA› ‹n ≤ j› assms(3,10,11) linorder-not-le
            s-seen-invD(1))
    moreover
    have Suc n ≤ j ⟹ ?g1 ∧ ?g2 ∧ ?g3
    proof −
      assume Suc n ≤ j
      with s-seen-invD[OF assms(3) ‹j < length SA›] assms(10)
      show ?thesis
        by blast
    qed
    ultimately show ?g1 ∧ ?g2 ∧ ?g3
      by blast
  qed
qed

corollary s-seen-inv-maintained-perm-step-c1-alt:
  assumes s-perm-inv α T B SA0 SA i
  and     i = Suc n
  and     length T ≤ SA ! Suc n
  shows s-seen-inv α T B SA n
  using s-seen-inv-maintained-step-c1-alt[OF s-perm-inv-elims(2,4,5,8,10−13,15)[OF
assms(1)] assms(2−)]
  by blast

lemma s-seen-inv-maintained-step-c2:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     s-suc-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
```

```
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     Suc 0 < length T
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = 0
  shows s-seen-inv α T B SA n
 unfolding s-seen-inv-def
proof (intro allI impI)
  fix j
  assume j < length SA n ≤ j
  hence n < length SA
    by simp
  hence n < length T
    by (simp add: assms(11))


  let ?b = α (T ! (SA ! j))
  let ?g1 = (suffix-type T (SA ! j) = S-type ⟶ in-s-current-bucket α T B ?b j)
  and ?g2 = (suffix-type T (SA ! j) = L-type ⟶ in-l-bucket α T ?b j)
  and ?g3 = SA ! j < length T

 from ‹n ≤ j›
  have n = j ∨ Suc n ≤ j
    by linarith
  moreover
  have n = j ⟹ ?g1 ∧ ?g2 ∧ ?g3
  proof −
    assume n = j
    with index-in-bucket-interval-gen[OF ‹n < length T› assms(8)]
    obtain b where
      b ≤ α (Max (set T))
      bucket-start α T b ≤ j
      j < bucket-end α T b
      by blast

    have j < l-bucket-end α T b ∨ s-bucket-start α T b ≤ j
      by (metis not-le s-bucket-start-eq-l-bucket-end)
    moreover
    have j < l-bucket-end α T b ⟹ ?thesis
    proof −
      assume j < l-bucket-end α T b
    with l-types-init-nth[OF assms(11) l-types-init-maintained[OF assms(2,4,10−12)]
‹b ≤ -›
          ‹bucket-start - - - ≤ j›]
      have SA ! j ∈ l-bucket α T b .
```

```
  hence suffix-type T (SA ! j) = L-type SA ! j < length T
    by (simp add: l-bucket-def bucket-def)+
  moreover
  have ?g1
    by(simp add: calculation(1) SL-types.exhaust)
  moreover
  from ‹SA ! j ∈ l-bucket α T b›
  have b = α (T ! (SA ! j))
    by (metis (mono-tags, lifting)  bucket-def l-bucket-def mem-Collect-eq)
  with ‹bucket-start α T b ≤ j› ‹j < l-bucket-end α T b› ‹b ≤ -›
  have in-l-bucket α T (α (T ! (SA ! j))) j
    using in-l-bucket-def by blast
  ultimately show ?thesis
    by blast
qed
moreover
have s-bucket-start α T b ≤ j ⟹ ?thesis
proof −
  assume s-bucket-start α T b ≤ j
  hence s-bucket-start α T b < i
    by (simp add: ‹n = j› assms(16))

  have B ! b ≤ i
  proof(rule ccontr)
    assume ¬B ! b ≤ i
    hence i < B ! b
      by simp
    with s-B-val[OF assms(1−6,8,10−13,15) ‹b ≤ α -›]
    have B ! b = s-bucket-start α T b .
    with ‹s-bucket-start α T b < i› ‹i < B ! b›
    show False
      by linarith
  qed
  hence B ! b < i ∨ B ! b = i
    using dual-order.order-iff-strict by blast
  moreover
  have B ! b < i ⟹ ?thesis
  proof −
    assume B ! b < i
    hence B ! b ≤ j
      by (simp add: ‹n = j› assms(16))
    with s-locations-invD[OF assms(3) ‹b ≤ -› - ‹j < bucket-end - - -›]
    have SA ! j ∈ s-bucket α T b .
    hence SA ! j < length T suffix-type T (SA ! j) = S-type
      by (simp add: s-bucket-def bucket-def)+
    moreover
    from calculation(2)
    have ?g2
      by simp
```

```
    moreover
    from ‹SA ! j ∈ s-bucket α T b›
    have α (T ! (SA ! j)) = b
      by (simp add: s-bucket-def bucket-def)
    with ‹B ! b ≤ j› ‹j < bucket-end α T b› ‹b ≤ -›
    have in-s-current-bucket α T B (α (T ! (SA ! j))) j
      by (simp add: in-s-current-bucket-def)
    ultimately show ?thesis
      by blast
  qed
  moreover
  have B ! b = i ⟹ ?thesis
  proof −
    assume B ! b = i
    hence s-bucket-start α T b < B ! b
      using ‹s-bucket-start α T b < i› by blast

    have b ≠ 0
     using ‹B ! b = i› assms(2,16) less-Suc-eq-0-disj s-bucket-ptr-0 by fastforce

    let ?xs = list-slice SA (B ! b) (bucket-end α T b)
    let ?B = set ?xs
    let ?A = s-bucket α T b − ?B

   from s-locations-inv-subset-s-bucket[OF assms(3) ‹b ≤ -›]
    have ?B ⊆ s-bucket α T b .
    hence ?A ⊆ s-bucket α T b
      by blast

    have card (s-bucket α T b) = bucket-end α T b − s-bucket-start α T b
      by (simp add: bucket-end-eq-s-start-pl-size s-bucket-size-def)

    from s-distinct-invD[OF assms(1) ‹b ≤ -›]
    have card ?B = bucket-end α T b − B ! b
        by (metis assms(11) bucket-end-le-length distinct-card length-list-slice
min.absorb-iff1)
    hence card ?B < card (s-bucket α T b)
      using ‹card (s-bucket α T b) = bucket-end α T b − s-bucket-start α T b›
          ‹j < bucket-end α T b› ‹s-bucket-start α T b < B ! b› ‹s-bucket-start
α T b ≤ j›
      by linarith
    with card-psubset[OF finite-s-bucket ‹?B ⊆ s-bucket α T b›]
    have ?B ⊂ s-bucket α T b .
    hence ?A ≠ {}
      by blast
    with subset-s-bucket-successor[OF assms(13,8,14) ‹b ≠ -› ‹?A ⊆ -›]
    obtain x where
      x ∈ ?A
      Suc x ∈ ?B ∨ (∃ b'. b < b' ∧ Suc x ∈ bucket α T b')
```

```
      by blast
    hence suffix-type T x = S-type α (T ! x) = b x < length T
      by (simp add: s-bucket-def bucket-def)+

    from ‹x ∈ ?A›
    have x ∉ ?B
      by blast

    have Suc x ∈ ?B ⟹ ?thesis
    proof −
      assume Suc x ∈ ?B
      from nth-mem-list-slice[OF ‹Suc x ∈ ?B›]
      obtain i' where
        i' < length SA
        B ! b ≤ i'
        i' < bucket-end α T b
        SA ! i' = Suc x
        by blast

      have i ≠ i'
      proof (rule ccontr)
        assume ¬ i ≠ i'
        hence i = i'
          by auto
        with assms(16,18) ‹SA ! i' = Suc x›
        show False
          by simp
      qed
      with ‹B ! b = i› ‹B ! b ≤ i'›
      have i < i'
        by simp
          with s-suc-invD[OF assms(7) ‹i' < length SA› - ‹SA ! i' = Suc x›
‹suffix-type T x = -›,
            simplified ‹α (T ! x) = b›]
      obtain k where
        in-s-current-bucket α T B b k
        SA ! k = x
        k < i'
        by blast
      hence x ∈ ?B
        by (meson assms(11) in-s-current-bucket-list-slice)
      with ‹x ∉ ?B›
      have False
        by blast
      then show ?thesis
        by blast
    qed
    moreover
    have ∃ b'. b < b' ∧ Suc x ∈ bucket α T b' ⟹ ?thesis
```

```
proof −
  assume ∃ b'. b < b' ∧ Suc x ∈ bucket α T b'
  then obtain b' where
    b < b'
    Suc x ∈ bucket α T b'
    by blast
  hence bucket-end α T b ≤ bucket-start α T b'
    by (simp add: less-bucket-end-le-start)
  with s-bucket-ptr-upper-bound[OF assms(2) ‹b ≤ - ›] ‹B ! b = i›
  have i ≤ bucket-start α T b'
    by linarith

  from ‹Suc x ∈ bucket α T b'›
  have b' ≤ α (Max (set T))
 by (metis (mono-tags, lifting) Max-greD assms(8) bucket-def mem-Collect-eq
        strict-mono-less-eq)
  with ‹i ≤ bucket-start α T b'› s-bucket-ptr-lower-bound[OF assms(2), of
b']
  have i ≤ B ! b'
    by (metis nat-le-iff-add s-bucket-start-def trans-le-add1)
  hence i = B ! b' ∨ i < B ! b'
    using antisym-conv1 by blast
  hence B ! b' = s-bucket-start α T b'
  proof
    assume i = B ! b'
   with ‹i ≤ bucket-start α T b'› s-bucket-ptr-lower-bound[OF assms(2) ‹b'
≤ -›]
    show ?thesis
      by (simp add: s-bucket-start-def)
  next
    assume i < B ! b'
    with s-B-val[OF assms(1−6,8,10−13,15) ‹b' ≤ -›]
    show ?thesis .
  qed

  from ‹Suc x ∈ bucket α T b'›
  have Suc x ∈ l-bucket α T b' ∨ Suc x ∈ s-bucket α T b'
    by (simp add: l-un-s-bucket)
  moreover
  have Suc x ∈ l-bucket α T b' ⟹ ?thesis
  proof −
    assume Suc x ∈ l-bucket α T b'
   with l-types-initD(1)[OF l-types-init-maintained[OF assms(2,4,10−12)]
‹b' ≤ -›]
     have Suc x ∈ set (list-slice SA (bucket-start α T b') (l-bucket-end α T
b'))
      by simp
     with nth-mem-list-slice[of Suc x SA bucket-start α T b' l-bucket-end α
T b']
```

```
      obtain i' where
        i' < length SA
        bucket-start α T b' ≤ i'
        i' < l-bucket-end α T b'
        SA ! i' = Suc x
        by blast

      have i ≠ i'
      proof (rule ccontr)
        assume ¬ i ≠ i'
        hence i = i'
          by auto
        with ‹SA ! i' = Suc x› assms(16,18)
        show False
          by simp
      qed
      hence i < i'
        using ‹bucket-start α T b' ≤ i'› ‹i ≤ bucket-start α T b'› by auto
      with s-suc-invD[OF assms(7) ‹i' < length -› - ‹SA ! i' = -› ‹suffix-type
T x = -›,
           simplified ‹α (T ! x) = b›]
      obtain k where
        in-s-current-bucket α T B b k
        SA ! k = x
        k < i'
        by blast
      hence x ∈ ?B
        by (meson assms(11) in-s-current-bucket-list-slice)
      with ‹x ∉ ?B›
      have False
        by blast
      then show ?thesis
        by blast
    qed
    moreover
    have Suc x ∈ s-bucket α T b' ⟹ ?thesis
    proof −
      assume Suc x ∈ s-bucket α T b'

    let ?ys = list-slice SA (s-bucket-start α T b') (bucket-end α T b')

      from distinct-card[OF s-distinct-invD[OF assms(1) ‹b' ≤ -›],
          simplified ‹B ! b' = s-bucket-start - - -›]
      have card (set ?ys) = card (s-bucket α T b')
        by (metis add-diff-cancel-left' assms(11) bucket-end-eq-s-start-pl-size
            bucket-end-le-length length-list-slice min-def s-bucket-size-def)
      with card-subset-eq[
          OF finite-s-bucket s-locations-inv-subset-s-bucket[OF assms(3) ‹b' ≤
-›],
```

```
            simplified ‹B ! b' = s-bucket-start α T b'›]
          have set ?ys = s-bucket α T b'
            by blast
          with ‹Suc x ∈ s-bucket α T b'›
          have Suc x ∈ set ?ys
            by simp
          with nth-mem-list-slice[of Suc x]
          obtain i' where
            i' < length SA
            s-bucket-start α T b' ≤ i'
            i' < bucket-end α T b'
            SA ! i' = Suc x
            by blast

          from ‹SA ! i' = Suc x› assms(16,18)
          have i ≠ i'
            using nat.discI by blast
          hence i < i'
            using ‹B ! b' = s-bucket-start α T b'› ‹i ≤ B ! b'› ‹s-bucket-start α T
b' ≤ i'›
            by linarith
          with s-suc-invD[OF assms(7) ‹i' < length -› - ‹SA ! i' = -› ‹suffix-type
T x = -›,
            simplified ‹α (T ! x) = b›]
          obtain k where
            in-s-current-bucket α T B b k
            SA ! k = x
            k < i'
            by blast
          hence x ∈ ?B
            by (meson assms(11) in-s-current-bucket-list-slice)
          with ‹x ∉ ?B›
          have False
            by blast
          then show ?thesis
            by blast
        qed
        ultimately show ?thesis
          by blast
      qed
      ultimately show ?thesis
        using ‹Suc x ∈ ?B ∨ (∃b'. b < b' ∧ Suc x ∈ bucket α T b')› by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
    by blast
qed
```

```
  moreover
  have Suc n ≤ j ⟹ ?g1 ∧ ?g2 ∧ ?g3
  proof −
    assume Suc n ≤ j
    with s-seen-invD[OF assms(5) ‹j < length SA›] assms(16)
    show ?thesis
      by blast
  qed
  ultimately show ?g1 ∧ ?g2 ∧ ?g3
    by blast
qed

corollary s-seen-inv-maintained-perm-step-c2:
  assumes s-perm-inv α T B SA0 SA i
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = 0
shows s-seen-inv α T B SA n
 using s-seen-inv-maintained-step-c2[OF s-perm-inv-elims[OF assms(1)] assms(2−)]
  by blast

lemma s-seen-inv-maintained-step-c3:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     s-suc-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     Suc 0 < length T
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = L-type
shows s-seen-inv α T B SA n
 unfolding s-seen-inv-def
proof (intro allI impI)
  fix k
  assume k < length SA n ≤ k
  hence n < length SA
    by simp
  hence n < length T
```

```
  by (simp add: assms(11))

 let ?b = α (T ! (SA ! k))
 let ?g1 = (suffix-type T (SA ! k) = S-type ⟶ in-s-current-bucket α T B ?b k)
 and ?g2 = (suffix-type T (SA ! k) = L-type ⟶ in-l-bucket α T ?b k)
 and ?g3 = SA ! k < length T

from ‹n ≤ k›
 have n = k ∨ Suc n ≤ k
   by linarith
 moreover
 have n = k ⟹ ?g1 ∧ ?g2 ∧ ?g3
 proof −
   assume n = k
   with index-in-bucket-interval-gen[OF ‹n < length T› assms(8)]
   obtain b where
     b ≤ α (Max (set T))
     bucket-start α T b ≤ k
     k < bucket-end α T b
     by blast

   have k < l-bucket-end α T b ∨ s-bucket-start α T b ≤ k
     by (metis not-le s-bucket-start-eq-l-bucket-end)
   moreover
   have k < l-bucket-end α T b ⟹ ?thesis
   proof −
     assume k < l-bucket-end α T b
    with l-types-init-nth[OF assms(11) l-types-init-maintained[OF assms(2,4,10−12)]
          ‹b ≤ -› ‹bucket-start - - - ≤ -›]
     have SA ! k ∈ l-bucket α T b .
     hence SA ! k < length T suffix-type T (SA ! k) = L-type
       by (simp add: l-bucket-def bucket-def)+
     moreover
     from calculation(2)
     have ?g1
       by simp
     moreover
     from ‹SA ! k ∈ l-bucket α T b›
     have b = (α (T ! (SA ! k)))
       by (simp add: l-bucket-def bucket-def)
     with ‹b ≤ -› ‹bucket-start - - - ≤ -› ‹k < l-bucket-end - - -›
     have in-l-bucket α T (α (T ! (SA ! k))) k
       using in-l-bucket-def by blast
     ultimately show ?thesis
       by blast
   qed
   moreover
   have s-bucket-start α T b ≤ k ⟹ ?thesis
   proof −
```

assume *s-bucket-start* α T $b \leq k$
hence *s-bucket-start* α T $b < i$
  by (*simp add*: ‹$n = k$› *assms*(*16*))

have B ! $b \leq i$
proof(*rule ccontr*)
  assume $\neg B$! $b \leq i$
  hence $i < B$! b
    by *simp*
  with *s-B-val*[*OF assms*(*1−6*,*8*,*10−13*,*15*) ‹$b \leq \alpha$ -›]
  have B ! b = *s-bucket-start* α T b **.**
  with ‹*s-bucket-start* α T $b < i$› ‹$i < B$! b›
  show *False*
    by *linarith*
qed
hence $i = B$! $b \vee B$! $b < i$
  by *linarith*
moreover
have B ! $b < i \Longrightarrow$ *?thesis*
proof −
  assume B ! $b < i$
  hence B ! $b \leq k$
    by (*simp add*: ‹$n = k$› *assms*(*16*))
  with *s-locations-invD*[*OF assms*(*3*) ‹$b \leq$ -› - ‹$k <$ *bucket-end* - - -›]
  have *SA* ! $k \in$ *s-bucket* α T b **.**
  hence *SA* ! $k <$ *length* T *suffix-type* T (*SA* ! k) = *S-type*
    by (*simp add*: *s-bucket-def bucket-def*)+
  moreover
  from *calculation*(*2*)
  have *?g2*
    by *simp*
  moreover
  from ‹*SA* ! $k \in$ *s-bucket* α T b›
  have α (T ! (*SA* ! k)) = b
    by (*simp add*: *s-bucket-def bucket-def*)
  with ‹B ! $b \leq k$› ‹$k <$ *bucket-end* α T b› ‹$b \leq$ -›
  have *in-s-current-bucket* α T B (α (T ! (*SA* ! k))) k
    by (*simp add*: *in-s-current-bucket-def*)
  ultimately show *?thesis*
    by *blast*
qed
moreover
have $i = B$! $b \Longrightarrow$ *?thesis*
proof −
  assume $i = B$! b
  hence $k < B$! b
    using ‹$n = k$› *assms*(*16*) **by** *linarith*

  have *s-bucket-start* α T $b < B$! b

using ‹$i = B\ !\ b$› ‹*s-bucket-start* $\alpha\ T\ b < i$› **by** *blast*

have $b \neq 0$
by (*metis* ‹$k < B\ !\ b$› *assms*(*2*) *not-less-zero s-bucket-ptr-0*)

let *?xs* = *list-slice SA* ($B\ !\ b$) (*bucket-end* $\alpha\ T\ b$)
let *?B* = *set ?xs*
let *?A* = *s-bucket* $\alpha\ T\ b - ?B$

from *s-locations-inv-subset-s-bucket*[*OF assms*(*3*) ‹$b \leq$ -›]
have *?B* $\subseteq$ *s-bucket* $\alpha\ T\ b$ **.**
hence *?A* $\subseteq$ *s-bucket* $\alpha\ T\ b$
by *blast*

have *card* (*s-bucket* $\alpha\ T\ b$) = *bucket-end* $\alpha\ T\ b$ − *s-bucket-start* $\alpha\ T\ b$
by (*simp add*: *bucket-end-eq-s-start-pl-size s-bucket-size-def*)

from *s-distinct-invD*[*OF assms*(*1*) ‹$b \leq$ -›]
have *card ?B* = *bucket-end* $\alpha\ T\ b - B\ !\ b$
by (*metis assms*(*11*) *bucket-end-le-length distinct-card length-list-slice min.absorb-iff1*)
hence *card ?B* < *card* (*s-bucket* $\alpha\ T\ b$)
using ‹*card* (*s-bucket* $\alpha\ T\ b$) = *bucket-end* $\alpha\ T\ b$ − *s-bucket-start* $\alpha\ T\ b$›
‹$k <$ *bucket-end* $\alpha\ T\ b$› ‹*s-bucket-start* $\alpha\ T\ b < B\ !\ b$› ‹*s-bucket-start* $\alpha\ T\ b \leq k$›
by *linarith*
with *card-psubset*[*OF finite-s-bucket* ‹*?B* $\subseteq$ *s-bucket* $\alpha\ T\ b$›]
have *?B* $\subset$ *s-bucket* $\alpha\ T\ b$ **.**
hence *?A* $\neq$ {}
by *blast*
with *subset-s-bucket-successor*[*OF assms*(*13*,*8*,*14*) ‹$b \neq 0$› ‹*?A* $\subseteq$ *s-bucket* $\alpha\ T\ b$›]
obtain x **where**
$x \in ?A$
Suc $x \in$ *s-bucket* $\alpha\ T\ b - ?A \vee (\exists\, b'.\ b < b' \wedge Suc\ x \in bucket\ \alpha\ T\ b')$
by *blast*
hence *Suc* $x \in ?B \vee (\exists\, b'.\ b < b' \wedge Suc\ x \in bucket\ \alpha\ T\ b')$
by *blast*

from ‹$x \in ?A$› ‹*?A* $\subseteq$ *s-bucket* $\alpha\ T\ b$›
have *suffix-type* $T\ x$ = *S-type* α ($T\ !\ x$) = b
by (*simp add*: *s-bucket-def bucket-def*)+

have $x \notin ?B$
using ‹$x \in ?A$› **by** *blast*

from ‹*Suc* $x \in ?B \vee (\exists\, b'.\ b < b' \wedge Suc\ x \in bucket\ \alpha\ T\ b')$›
have *False*
proof

assume *Suc x ∈ ?B*
from *nth-mem-list-slice*[*OF* ‹*Suc x ∈ ?B*›]
obtain *i'* **where**
i' < length SA
B ! b ≤ i'
i' < bucket-end α T b
SA ! i' = Suc x
by *blast*

have *i ≠ i'*
proof (*rule ccontr*)
assume ¬ *i ≠ i'*
hence *i = i'*
by *auto*
hence *j = x*
using ‹*SA ! i' = Suc x*› *assms*(*16*,*18*) **by** *auto*
with *assms*(*19*) ‹*suffix-type T x = -*›
show *False*
by *simp*
qed
with ‹*B ! b ≤ i'*› ‹*i = B ! b*›
have *i < i'*
using *nat-less-le* **by** *blast*
with *s-suc-invD*[*OF assms*(*7*) ‹*i' < length SA* › - ‹*SA ! i' = -*› ‹*suffix-type T x = -*›]
‹*α* (*T ! x*) = *b*›
obtain *k* **where**
in-s-current-bucket α T B b k
SA ! k = x
k < i'
by *blast*
hence *x ∈ ?B*
by (*meson assms*(*11*) *in-s-current-bucket-list-slice*)
with ‹*x ∉ ?B*›
show *False*
by *blast*
next
assume ∃ *b'*. *b < b' ∧ Suc x ∈ bucket α T b'*
then obtain *b'* **where**
b < b'
Suc x ∈ bucket α T b'
by *blast*
hence *b' ≤ α* (*Max* (*set T*))
by (*metis* (*mono-tags*, *lifting*) *Max-greD assms*(*8*) *bucket-def mem-Collect-eq*
strict-mono-less-eq)

have *suffix-type T* (*Suc x*) = *S-type ∨ suffix-type T* (*Suc x*) = *L-type*
by (*simp add*: *suffix-type-def*)
hence *Suc x ∈ l-bucket α T b' ∨ Suc x ∈ s-bucket α T b'*

```
    using ‹Suc x ∈ bucket α T b'› l-bucket-def s-bucket-def by fastforce
  moreover
  have Suc x ∈ l-bucket α T b' ⟹ False
  proof −
    assume Suc x ∈ l-bucket α T b'
   with l-types-initD(1)[OF l-types-init-maintained[OF assms(2,4,10−12)]
‹b' ≤ -›]
    have Suc x ∈ set (list-slice SA (bucket-start α T b') (l-bucket-end α T
b'))
      by blast
    with nth-mem-list-slice[of Suc x]
    obtain i' where
      i' < length SA
      bucket-start α T b' ≤ i'
      i' < l-bucket-end α T b'
      SA ! i' = Suc x
      by blast

    have i ≠ i'
    proof (rule ccontr)
      assume ¬ i ≠ i'
      hence i = i'
        by auto
      hence j = x
        using ‹SA ! i' = Suc x› assms(16,18) by auto
      with assms(19) ‹suffix-type T x = -›
      show False
        by simp
    qed
    moreover
    from ‹b < b'›
    have bucket-end α T b ≤ bucket-start α T b'
      by (simp add: less-bucket-end-le-start)
    hence B ! b ≤ i'
      using s-bucket-ptr-upper-bound[OF assms(2) ‹b ≤ α (Max (set T))›]
            ‹bucket-start α T b' ≤ i'›
      by linarith
    ultimately have i < i'
      using ‹i = B ! b› nat-less-le by blast
  with s-suc-invD[OF assms(7) ‹i' < length SA › - ‹SA ! i' = -› ‹suffix-type
T x = -›]
        ‹α (T ! x) = b›
    obtain k where
      in-s-current-bucket α T B b k
      SA ! k = x
      k < i'
      by blast
    hence x ∈ ?B
      by (meson assms(11) in-s-current-bucket-list-slice)
```

```
  with ‹x ∉ ?B›
  show False
    by blast
qed
moreover
have Suc x ∈ s-bucket α T b' ⟹ False
proof −
  assume Suc x ∈ s-bucket α T b'

  have i ≤ bucket-end α T b
    by (simp add: Suc-le-eq ‹k < bucket-end α T b› ‹n = k› assms(16))
  hence i ≤ bucket-start α T b'
    using ‹b < b'› less-bucket-end-le-start order.trans by blast
  hence i ≤ B ! b'
    using s-bucket-ptr-lower-bound[OF assms(2) ‹b' ≤ -›]
by (metis l-bucket-end-def le-trans nat-le-iff-add s-bucket-start-eq-l-bucket-end)
  hence i < B ! b' ∨ i = B ! b'
    using nat-less-le by blast
  hence B ! b' = s-bucket-start α T b'
  proof
    assume i < B ! b'
    with s-B-val[OF assms(1−6,8,10−13,15) ‹b' ≤ -›]
    show B ! b' = s-bucket-start α T b'
      by blast
  next
    assume i = B ! b'
    with s-bucket-ptr-lower-bound[OF assms(2) ‹b' ≤ -›]
        ‹i ≤ bucket-start α T b'›
    show B ! b' = s-bucket-start α T b'
      by (simp add: s-bucket-start-def)
  qed

  let ?ys = list-slice SA (s-bucket-start α T b') (bucket-end α T b')

  from distinct-card[OF s-distinct-invD[OF assms(1) ‹b' ≤ -›],
       simplified ‹B ! b' = s-bucket-start - - -›]
  have card (set ?ys) = card (s-bucket α T b')
    by (metis add-diff-cancel-left' assms(11) bucket-end-eq-s-start-pl-size
        bucket-end-le-length length-list-slice min-def s-bucket-size-def)
  with card-subset-eq[
      OF finite-s-bucket s-locations-inv-subset-s-bucket[OF assms(3) ‹b' ≤
-›],
       simplified ‹B ! b' = s-bucket-start α T b'›]
  have set ?ys = s-bucket α T b'
    by blast
  with ‹Suc x ∈ s-bucket α T b'›
  have Suc x ∈ set ?ys
    by simp
  with nth-mem-list-slice[of Suc x]
```

```
          obtain i' where
            i' < length SA
            s-bucket-start α T b' ≤ i'
            i' < bucket-end α T b'
            SA ! i' = Suc x
            by blast

          have i ≠ i'
          proof (rule ccontr)
            assume ¬ i ≠ i'
            hence i = i'
              by auto
            hence j = x
              using ‹SA ! i' = Suc x› assms(16,18) by auto
            with assms(19) ‹suffix-type T x = -›
            show False
              by simp
          qed
          moreover
          have i ≤ i'
            using ‹B ! b' = s-bucket-start α T b'› ‹i ≤ B ! b'› ‹s-bucket-start α T
b' ≤ i'›
            by linarith
          ultimately have i < i'
            using dual-order.order-iff-strict by blast
        with s-suc-invD[OF assms(7) ‹i' < length SA › - ‹SA ! i' = -› ‹suffix-type
T x = -›]
              ‹α (T ! x) = b›
          obtain k where
            in-s-current-bucket α T B b k
            SA ! k = x
            k < i'
            by blast
          hence x ∈ ?B
            by (meson assms(11) in-s-current-bucket-list-slice)
          with ‹x ∉ ?B›
          show False
            by blast
        qed
        ultimately show False
          by blast
      qed
      then show ?thesis
        by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
```

```
    by blast
  qed
  moreover
  have Suc n ≤ k ⟹ ?g1 ∧ ?g2 ∧ ?g3
  proof −
    assume Suc n ≤ k
    with s-seen-invD[OF assms(5) ‹k < length SA›] assms(16)
    show ?thesis
      by blast
  qed
  ultimately show ?g1 ∧ ?g2 ∧ ?g3
    by blast
qed

corollary s-seen-inv-maintained-perm-step-c3:
  assumes s-perm-inv α T B SA0 SA i
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = L-type
shows s-seen-inv α T B SA n
  using s-seen-inv-maintained-step-c3[OF s-perm-inv-elims[OF assms(1)] assms(2−)]
  by blast

lemma s-seen-inv-maintained-step-c4:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     s-suc-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     Suc 0 < length T
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
shows s-seen-inv α T (B[b := k]) (SA[k := j]) n
  unfolding s-seen-inv-def
proof(intro allI impI)
```

fix i'
assume $i' < \mathit{length}\ (SA[k := j])\ n \leq i'$

let *?g1* $= (\mathit{suffix\text{-}type}\ T\ (SA[k := j]\ !\ i') = \mathit{S\text{-}type} \longrightarrow$
$\mathit{in\text{-}s\text{-}current\text{-}bucket}\ \alpha\ T\ (B[b := k])\ (\alpha\ (T\ !\ (SA[k := j]\ !\ i'))\ i')$ **and**
?g2 $= (\mathit{suffix\text{-}type}\ T\ (SA[k := j]\ !\ i') = \mathit{L\text{-}type} \longrightarrow$
$\mathit{in\text{-}l\text{-}bucket}\ \alpha\ T\ (\alpha\ (T\ !\ (SA[k := j]\ !\ i'))\ i')$ **and**
?g3 $= SA[k := j]\ !\ i' < \mathit{length}\ T$

from *s-bucket-ptr-strict-lower-bound*[*OF assms*(*1−6*,*8*,*10−14*,*16−20*)]
have $\mathit{s\text{-}bucket\text{-}start}\ \alpha\ T\ b < B\ !\ b$**.**
hence $\mathit{s\text{-}bucket\text{-}start}\ \alpha\ T\ b \leq k$
using *assms*(*21*) **by** *linarith*
hence $\mathit{bucket\text{-}start}\ \alpha\ T\ b \leq k$
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

from ‹$\mathit{s\text{-}bucket\text{-}start}\ \alpha\ T\ b < B\ !\ b$›
have $k < B\ !\ b$
using *assms*(*21*) **by** *linarith*

have $j < \mathit{length}\ T$
by (*simp add*: *assms*(*19*) *suffix-type-s-bound*)
hence $b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
by (*simp add*: *assms*(*8*,*20*) *strict-mono-less-eq*)
with *s-bucket-ptr-upper-bound*[*OF assms*(*2*)]
have $B\ !\ b \leq \mathit{bucket\text{-}end}\ \alpha\ T\ b$
by *blast*
with ‹$k < B\ !\ b$›
have $k < \mathit{bucket\text{-}end}\ \alpha\ T\ b$
by *linarith*

have $B\ !\ b \leq i$
proof(*rule ccontr*)
assume $\neg B\ !\ b \leq i$
hence $i < B\ !\ b$
by *simp*
with *s-B-val*[*OF assms*(*1−6*,*8*,*10−13*,*15*) ‹$b \leq \alpha$ -›]
have $B\ !\ b = \mathit{s\text{-}bucket\text{-}start}\ \alpha\ T\ b$ **.**
with ‹$\mathit{s\text{-}bucket\text{-}start}\ \alpha\ T\ b < B\ !\ b$›
show *False*
by *linarith*
qed
hence $k < i$
using ‹$k < B\ !\ b$› *less-le-trans* **by** *blast*

have $k = i' \Longrightarrow n = i'$
using ‹$k < i$› ‹$n \leq i'$› *assms*(*16*) *le-less-Suc-eq* **by** *blast*

have $k \leq i'$

```
  using ‹k < i› ‹n ≤ i'› assms(16) by linarith

have i' < length T
  using ‹i' < length (SA[k := j])› assms(11) by auto
with index-in-bucket-interval-gen[OF - assms(8), of i' T]
obtain b' where
  b' ≤ α (Max (set T))
  bucket-start α T b' ≤ i'
  i' < bucket-end α T b'
  by blast
hence n < bucket-end α T b'
  using ‹n ≤ i'› dual-order.strict-trans2 by blast
hence i ≤ bucket-end α T b'
  using assms(16) by linarith

have b ≤ b'
proof (rule ccontr)
  assume ¬b ≤ b'
  hence b' < b
    by linarith
  hence bucket-end α T b' ≤ bucket-start α T b
    by (simp add: less-bucket-end-le-start)
  with ‹i ≤ bucket-end α T b'› ‹bucket-start α T b ≤ k› ‹k < B ! b›
  have i < B ! b
    by linarith
  with ‹B ! b ≤ i›
  show False
    by linarith
qed

have in-s-current-bucket α T (B[b := k]) b k
  unfolding in-s-current-bucket-def
  using ‹b ≤ α (Max (set T))› ‹k < bucket-end α T b› assms(9) by auto

have b < b' ⟹ ?g1 ∧ ?g2 ∧ ?g3
proof −
  assume b < b'
  hence bucket-end α T b ≤ bucket-start α T b'
    by (simp add: less-bucket-end-le-start)
  with ‹k < bucket-end - - b› ‹bucket-start - - b' ≤ i'›
  have k < i'
    by linarith
  hence SA[k := j] ! i' = SA ! i'
    by simp

  from ‹b < b'›
  have B[b := k] ! b' = B ! b'
    by simp
```

```
    have i' < l-bucket-end α T b' ∨ B ! b' ≤ i' ∨ (s-bucket-start α T b' ≤ i' ∧ i'
< B ! b')
      by (metis not-le s-bucket-start-eq-l-bucket-end)
    moreover
    have B ! b' ≤ i' ⟹ ?thesis
    proof −
      assume B ! b' ≤ i'
      with s-locations-invD[OF assms(3) ‹b' ≤ -› - ‹i' < bucket-end - - -›]
      have SA ! i' ∈ s-bucket α T b' .
     hence suffix-type T (SA ! i') = S-type α (T ! (SA ! i')) = b' SA ! i' < length
T
        by (simp add: s-bucket-def bucket-def)+
      moreover
      from ‹B[b := k] ! b' = B ! b'› ‹b' ≤ α -› ‹B ! b' ≤ i'› ‹i' < bucket-end α T
b'›
      have in-s-current-bucket α T (B[b := k]) b' i'
        by (simp add: in-s-current-bucket-def)
      ultimately show ?thesis
        by (simp add: ‹SA[k := j] ! i' = SA ! i'›)
    qed
    moreover
    have i' < l-bucket-end α T b' ⟹ ?thesis
    proof −
      assume i' < l-bucket-end α T b'
      hence in-l-bucket α T b' i'
        by (simp add: ‹bucket-start α T b' ≤ i'› ‹b' ≤ α -› in-l-bucket-def)
      moreover
     from l-types-init-nth[OF assms(11) l-types-init-maintained[OF assms(2,4,10−12)]
          ‹b' ≤ α -› ‹bucket-start - - - ≤ i'› ‹i' < l-bucket-end - - -›]
      have SA ! i' ∈ l-bucket α T b' .
       hence SA ! i' < length T α (T ! (SA ! i')) = b' suffix-type T (SA ! i') =
L-type
        by (simp add: l-bucket-def bucket-def)+
      ultimately show ?thesis
        using ‹SA[k := j] ! i' = SA ! i'›
        by simp
    qed
    moreover
    have ⟦s-bucket-start α T b' ≤ i'; i' < B ! b'⟧ ⟹ ?thesis
    proof −
      assume s-bucket-start α T b' ≤ i' i' < B ! b'
      have B ! b' = i
      proof (rule ccontr)
        assume B ! b' ≠ i
        hence i < B ! b' ∨ B ! b' < i
          by linarith
        moreover
        have B ! b' < i ⟹ False
          using ‹i' < B ! b'› ‹n ≤ i'› assms(16) by linarith
```

```
    moreover
    have i < B ! b' ⟹ False
    proof −
      assume i < B ! b'
      with s-B-val[OF assms(1−6,8,10−13,15) ‹b' ≤ α -›]
      have B ! b' = s-bucket-start α T b' .
      with ‹s-bucket-start α T b' ≤ i'› ‹i' < B ! b'›
      show False
        by linarith
    qed
    ultimately show False
      by linarith
  qed

  have s-bucket-start α T b' < B ! b'
    using ‹i' < B ! b'› ‹s-bucket-start α T b' ≤ i'› by linarith
  hence s-bucket-start α T b' < bucket-end α T b'
    using ‹B ! b' = i› ‹i ≤ bucket-end α T b'› order.strict-trans2 by blast
  hence s-bucket α T b' ≠ {}
  by (metis add.commute bucket-end-eq-s-start-pl-size distinct-card distinct-conv-nth
        empty-set less-irrefl-nat less-nat-zero-code list.size(3) plus-nat.add-0
        s-bucket-size-def)

  have bucket-end α T b' ≤ length SA
    by (simp add: assms(11) bucket-end-le-length)

  let ?xs = list-slice SA (B ! b') (bucket-end α T b')

  have set ?xs ⊂ s-bucket α T b'
  proof
    from s-locations-inv-subset-s-bucket[OF assms(3) ‹b' ≤ -›]
    show set ?xs ⊆ s-bucket α T b' .
  next
    from ‹s-bucket-start α T b' < B ! b'› ‹s-bucket-start α T b' < bucket-end
α T b'›
    have bucket-end α T b' − B ! b' < bucket-end α T b' − s-bucket-start α T
b'
      using diff-less-mono2 by blast
    hence length ?xs < s-bucket-size α T b'
      by (metis ‹bucket-end α T b' ≤ length SA› add-diff-cancel-left'
          bucket-end-eq-s-start-pl-size length-list-slice min-def)
    hence card (set ?xs) ≠ card (s-bucket α T b')
      by (metis card-length not-le s-bucket-size-def)
    then show set ?xs ≠ s-bucket α T b'
      by auto
  qed

  have P0: ∀ i0 < length T. α (T ! i0) = b' ⟶ T ! i0 ≠ bot
    using ‹b < b'› assms(8,20) strict-mono-less by fastforce
```

hence *P1*: $\forall\, i0 < \mathit{length}\ T.\ \alpha\ (T\ !\ i0) = b' \longrightarrow \mathit{Suc}\ i0 < \mathit{length}\ T$
by (*metis Suc-leI assms*(13) *diff-Suc-1 last-conv-nth le-imp-less-or-eq length-greater-0-conv valid-list-def*)

let *?S* = *s-bucket* α *T* b' − *set ?xs*

from ‹*set ?xs* ⊂ *s-bucket* α *T* b'›
have *?S* ≠ {}
by *blast*
have *?S* ⊆ *s-bucket* α *T* b'
by *blast*
hence *P2*: $\forall\, x \in ?S.\ \alpha\ (T\ !\ x) = b' \wedge \mathit{suffix\text{-}type}\ T\ x = \mathit{S\text{-}type} \wedge x < \mathit{length}\ T$
by (*simp add*: *bucket-def s-bucket-def*)

have *P3*: $\forall\, x \in ?S.\ \mathit{Suc}\ x < \mathit{length}\ T \wedge \alpha\ (T\ !\ \mathit{Suc}\ x) \geq b'$
proof
fix *x*
assume *x* ∈ *?S*
with *P2*
have $\alpha\ (T\ !\ x) = b'$ *suffix-type T x* = *S-type x* < *length T*
by *blast*+
with *P1*
have *Suc x* < *length T*
by *blast*
moreover
from ‹*suffix-type T x* = *S-type*› ‹*x* < *length T*›
have *T* ! *x* ≤ *T* ! *Suc x*
using *calculation nth-gr-imp-l-type* **by** *fastforce*
hence $\alpha\ (T\ !\ \mathit{Suc}\ x) \geq b'$
using ‹$\alpha\ (T\ !\ x) = b'$› *assms*(8) *strict-mono-leD* **by** *blast*
ultimately show *Suc x* < *length T* ∧ $\alpha\ (T\ !\ \mathit{Suc}\ x) \geq b'$
by *blast*
qed

have *finite ?S*
by (*simp add*: *finite-s-bucket*)

have $\exists\, x \in ?S.\ \alpha\ (T\ !\ \mathit{Suc}\ x) > b' \vee \mathit{Suc}\ x \in \mathit{set}\ ?xs$
proof (*rule ccontr*)
assume ¬ ($\exists\, x \in ?S.\ b' < \alpha\ (T\ !\ \mathit{Suc}\ x) \vee \mathit{Suc}\ x \in \mathit{set}\ ?xs$)
hence $\forall\, x \in ?S.\ \alpha\ (T\ !\ \mathit{Suc}\ x) \leq b' \wedge \mathit{Suc}\ x \notin \mathit{set}\ ?xs$
using *not-le-imp-less* **by** *blast*
with *P3*
have *P4*: $\forall\, x \in ?S.\ \alpha\ (T\ !\ \mathit{Suc}\ x) = b' \wedge \mathit{Suc}\ x \notin \mathit{set}\ ?xs$
using *dual-order.antisym* **by** *blast*
hence *P5*: $\forall\, x \in ?S.$ *suffix-type T* (*Suc x*) = *S-type*
by (*metis P2 P3 assms*(8) *strict-mono-eq suffix-type-neq*)

```
      hence P6: ∀ x ∈ ?S. Suc x ∈ ?S
        by (metis (mono-tags, lifting) Diff-iff P3 P4 bucket-def mem-Collect-eq
s-bucket-def)
      with ‹?S ≠ {}› ‹finite ?S›
      show False
        using Suc-le-lessD infinite-growing by blast
    qed
    then obtain x where
      x ∈ ?S
      α (T ! Suc x) > b' ∨ Suc x ∈ set ?xs
      by blast
    with P3
    have Suc x < length T
      by blast

    from ‹x ∈ ?S›
    have suffix-type T x = S-type α (T ! x) = b' x < length T
      using P2 by blast+

    have P4: ∀ b0 ≤ α (Max (set T)). b' < b0 ⟶ B ! b0 = s-bucket-start α T
b0
    proof(safe)
      fix b0
      assume b0 ≤ α (Max (set T)) b' < b0
      hence bucket-end α T b' ≤ bucket-start α T b0
        by (simp add: less-bucket-end-le-start)
      with s-bucket-ptr-upper-bound[OF assms(2) ‹b' ≤ - ›]
          s-bucket-ptr-lower-bound[OF assms(2) ‹b0 ≤ - ›]
      have B ! b' ≤ B ! b0
        by (meson bucket-start-le-s-bucket-start le-trans)
      hence B ! b' = B ! b0 ∨ B ! b' < B ! b0
        by linarith
      moreover
      have B ! b' = B ! b0 ⟹ B ! b0 = s-bucket-start α T b0
        by (metis ‹B ! b' = i› ‹bucket-end α T b' ≤ bucket-start α T b0› le-trans
        ‹i ≤ bucket-end α T b'› ‹s-bucket-start α T b0 ≤ B ! b0› dual-order.antisym
            bucket-start-le-s-bucket-start)
      moreover
      have B ! b' < B ! b0 ⟹ i < B ! b0
        by (simp add: ‹B ! b' = i›)
      with s-B-val[OF assms(1−6,8,10−13,15) ‹b0 ≤ -›]
      have B ! b' < B ! b0 ⟹ B ! b0 = s-bucket-start α T b0
        by blast
      ultimately show B ! b0 = s-bucket-start α T b0
        by blast
    qed

    from ‹α (T ! Suc x) > b' ∨ Suc x ∈ set ?xs›
    show ?thesis
```

```
    proof
      let ?b = α (T ! Suc x)
      let ?ys = list-slice SA (bucket-start α T ?b) (l-bucket-end α T ?b)
      and ?zs = list-slice SA (s-bucket-start α T ?b) (bucket-end α T ?b)

      assume b' < ?b
      with P4 ‹Suc x < length T›
      have B ! ?b = s-bucket-start α T ?b
        by (simp add: assms(8) strict-mono-less-eq)

      from ‹Suc x < length T›
      have ?b ≤ α (Max (set T))
        by (simp add: assms(8) strict-mono-leD)

      have bucket-end α T b' ≤ bucket-start α T ?b
        using ‹b' < α (T ! Suc x)› less-bucket-end-le-start by blast
      hence i ≤ bucket-start α T ?b
        using ‹i ≤ bucket-end α T b'› order.trans by blast

      have set ?zs = s-bucket α T ?b
      proof (rule card-subset-eq[OF finite-s-bucket])
        show set ?zs ⊆ s-bucket α T ?b
          by (metis Max-greD ‹B ! ?b = s-bucket-start α T ?b› ‹Suc x < length
T› assms(3,8)
              s-locations-inv-subset-s-bucket strict-mono-leD)
      next
        from distinct-card[OF s-distinct-invD[OF assms(1) ‹?b ≤ -›]]
            ‹B ! ?b = s-bucket-start α T ?b›
        have card (set ?zs) = length ?zs
          by simp
        moreover
        have length ?zs = bucket-end α T ?b − s-bucket-start α T ?b
          by (metis assms(11) bucket-end-le-length length-list-slice min-def)
        moreover
        have s-bucket-size α T ?b = bucket-end α T ?b − s-bucket-start α T ?b
          by (simp add: bucket-end-eq-s-start-pl-size)
        hence card (s-bucket α T ?b) = bucket-end α T ?b − s-bucket-start α T
?b
          by (simp add: s-bucket-size-def)
        ultimately show card (set ?zs) = card (s-bucket α T ?b)
          by simp
      qed

      have suffix-type T (Suc x) = L-type ⟹ ?thesis
      proof −
        assume suffix-type T (Suc x) = L-type
         with l-types-initD(1)[OF l-types-init-maintained[OF assms(2,4,10−12)]
‹?b ≤ -›]
        have Suc x ∈ set ?ys
```

```
      by (simp add: ‹Suc x < length T› bucket-def l-bucket-def)

    from nth-mem-list-slice[OF ‹Suc x ∈ set ?ys›]
    obtain i0 where
      i0 < length SA
      bucket-start α T ?b ≤ i0
      i0 < l-bucket-end α T ?b
      SA ! i0 = Suc x
      by blast
    hence i ≤ i0
      using ‹i ≤ bucket-start α T ?b› dual-order.trans by blast
    hence i = i0 ∨ i < i0
      by linarith
    then show ?thesis
    proof
      assume i = i0
      hence x = j
        using ‹SA ! i0 = Suc x› assms(16,18) by auto
      then show ?thesis
        using ‹α (T ! x) = b'› ‹b < b'› assms(20) by blast
    next
      assume i < i0
     with s-suc-invD[OF assms(7) ‹i0 < length -› - ‹SA ! i0 = -› ‹suffix-type
T x = S-type›]
          ‹α (T ! x) = b'›
      obtain i1 where
        in-s-current-bucket α T B b' i1
        SA ! i1 = x
        i1 < i0
        by auto
      with in-s-current-bucket-list-slice[OF assms(11)]
      have x ∈ set ?xs
        by blast
      then show ?thesis
        using ‹x ∈ ?S› by blast
    qed
  qed
  moreover
  have suffix-type T (Suc x) = S-type ⟹ ?thesis
  proof −
    assume suffix-type T (Suc x) = S-type
    with ‹set ?zs = s-bucket α T ?b› ‹Suc x < length T›
    have Suc x ∈ set ?zs
      by (simp add: s-bucket-def bucket-def)

    from nth-mem-list-slice[OF ‹Suc x ∈ set ?zs›]
    obtain i0 where
      i0 < length SA
      s-bucket-start α T ?b ≤ i0
```

```
        i0 < bucket-end α T ?b
        SA ! i0 = Suc x
        by blast
      hence i ≤ i0
          by (meson ‹i ≤ bucket-start α T ?b› bucket-start-le-s-bucket-start
dual-order.trans)
      hence i = i0 ∨ i < i0
        by linarith
      then show ?thesis
      proof
        assume i = i0
        hence x = j
          using ‹SA ! i0 = Suc x› assms(16,18) by auto
        then show ?thesis
          using ‹α (T ! x) = b'› ‹b < b'› assms(20) by blast
      next
        assume i < i0
       with s-suc-invD[OF assms(7) ‹i0 < length -› - ‹SA ! i0 = -› ‹suffix-type
T x = S-type›]
            ‹α (T ! x) = b'›
        obtain i1 where
          in-s-current-bucket α T B b' i1
          SA ! i1 = x
          i1 < i0
          by auto
        with in-s-current-bucket-list-slice[OF assms(11)]
        have x ∈ set ?xs
          by blast
        then show ?thesis
          using ‹x ∈ ?S› by blast
      qed
    qed
    ultimately show ?thesis
      using SL-types.exhaust by blast
  next
    assume Suc x ∈ set ?xs

    from nth-mem-list-slice[OF ‹Suc x ∈ set ?xs›]
    obtain i0 where
      i0 < length SA
      B ! b' ≤ i0
      i0 < bucket-end α T b'
      SA ! i0 = Suc x
      by blast
    with ‹B ! b' = i›
    have i ≤ i0
      by blast
    hence i = i0 ∨ i < i0
      by linarith
```

```
      then show ?thesis
      proof
        assume i = i0
        hence x = j
          using ‹SA ! i0 = Suc x› assms(16,18) by auto
        then show ?thesis
          using ‹α (T ! x) = b'› ‹b < b'›  assms(20) by blast
      next
        assume i < i0
        with s-suc-invD[OF assms(7) ‹i0 < length -› - ‹SA ! i0 = -› ‹suffix-type
T x = -›]
            ‹α (T ! x) = b'›
        obtain i1 where
          in-s-current-bucket α T B b' i1
          SA ! i1 = x
          i1 < i0
          by blast
        with in-s-current-bucket-list-slice[OF assms(11)]
        have x ∈ set ?xs
          by blast
        then show ?thesis
          using ‹x ∈ ?S› by blast
      qed
    qed
  qed
  ultimately show ?thesis
    by linarith
qed
moreover
have b = b' ⟹ ?g1 ∧ ?g2 ∧ ?g3
proof −
  assume b = b'
  have k = i' ⟹ ?thesis
  proof −
    assume k = i'
    hence SA[k := j] ! i' = j
      using ‹i' < length (SA[k := j])› by auto
    with ‹suffix-type T j = S-type› ‹j < length T› ‹in-s-current-bucket α T (B[b
:= k]) b k›
         assms(20) ‹k = i'›
    show ?thesis
      by simp
  qed
  moreover
  have k < i' ⟹ ?thesis
  proof −
    assume k < i'
    hence B ! b ≤ i'
      using assms(21) by linarith
```

with *s-locations-invD*[*OF assms*(*3*) ‹$b' \leq \alpha$ -› - ‹$i' <$ *bucket-end* - - -›] ‹$b = b'$›
have $SA\ !\ i' \in$ *s-bucket* $\alpha\ T\ b'$
by *blast*
hence *suffix-type* $T\ (SA\ !\ i') =$ *S-type* $\alpha\ (T\ !\ (SA\ !\ i')) = b'$
by (*simp add*: *s-bucket-def bucket-def*)+
moreover
have $SA[k := j]\ !\ i' = SA\ !\ i'$
using ‹$k < i'$› **by** *simp*
moreover
have *in-s-current-bucket* $\alpha\ T\ (B[b := k])\ b'\ i'$
by (*metis* (*no-types*, *lifting*) ‹$b = b'$› ‹$b' \leq \alpha\ (Max\ (set\ T))$› ‹$i' <$ *bucket-end* $\alpha\ T\ b'$›
‹$k \leq i'$› *assms*(*9*) *dual-order.strict-trans2 in-s-current-bucket-def nth-list-update-eq*)
ultimately show *?thesis*
by (*simp add*: *suffix-type-s-bound*)
qed
ultimately show *?thesis*
using ‹$k \leq i'$› *dual-order.order-iff-strict* **by** *blast*
qed
ultimately show *?g1* $\wedge$ *?g2* $\wedge$ *?g3*
using ‹$b \leq b'$› *dual-order.order-iff-strict* **by** *blast*
qed

corollary *s-seen-inv-maintained-perm-step-c4*:
assumes *s-perm-inv* $\alpha\ T\ B\ SA0\ SA\ i$
and $i = Suc\ n$
and $Suc\ n <$ *length* SA
and $SA\ !\ Suc\ n = Suc\ j$
and *suffix-type* $T\ j =$ *S-type*
and $b = \alpha\ (T\ !\ j)$
and $k = B\ !\ b - Suc\ 0$
shows *s-seen-inv* $\alpha\ T\ (B[b := k])\ (SA[k := j])\ n$
using *s-seen-inv-maintained-step-c4* [*OF s-perm-inv-elims*[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

lemmas *s-seen-inv-maintained-perm-step* =
s-seen-inv-maintained-perm-step-c1
s-seen-inv-maintained-perm-step-c2
s-seen-inv-maintained-perm-step-c3
s-seen-inv-maintained-perm-step-c4

83.3.6 Predecessor

lemma *s-pred-inv-established*:
assumes *s-bucket-init* $\alpha\ T\ B$
shows *s-pred-inv* $\alpha\ T\ B\ SA\ n$
unfolding *s-pred-inv-def*

proof (*safe*)
  fix *b i*
  assume *A*: *in-s-current-bucket* α *T B b i 0* $<$ *b*

  let *?goal* = $\exists j<$*length SA*. *SA* ! *j* = *Suc* (*SA* ! *i*) $\wedge$ *i* $<$ *j* $\wedge$ *n* $<$ *j*

  have *b* = *0* $\vee$ *0* $<$ *b*
    by *blast*
  moreover
  from *A*(*2*)
  have *b* = *0* $\Longrightarrow$ *?goal*
    by *blast*
  moreover
  have *0* $<$ *b* $\Longrightarrow$ *?goal*
  proof −
    assume *0* $<$ *b*
    with *s-bucket-initD*(*1*)[*OF assms*(*1*) *in-s-current-bucketD*(*1*)[*OF A*(*1*)]]
    have *B* ! *b* = *bucket-end* α *T b* **.**
    with *in-s-current-bucketD*(*2*,*3*)[*OF A*(*1*)]
    show *?goal*
      by *linarith*
  qed
  ultimately show *?goal*
    by *blast*
qed

lemma *s-pred-inv-maintained-step-alt*:
  assumes *s-pred-inv* α *T B SA i*
  and *i* = *Suc n*
shows *s-pred-inv* α *T B SA n*
  unfolding *s-pred-inv-def*
proof (*intro allI impI*; *elim conjE*)
  fix *b i′*
  assume *in-s-current-bucket* α *T B b i′ b* $\neq$ *0*
  with *s-pred-invD*[*OF assms*(*1*), *of b i′*] *assms*(*2*)
  show $\exists j<$*length SA*. *SA* ! *j* = *Suc* (*SA* ! *i′*) $\wedge$ *i′* $<$ *j* $\wedge$ *n* $<$ *j*
    using *Suc-lessD* **by** *blast*
qed

corollary *s-pred-inv-maintained-perm-step-alt*:
  assumes *s-perm-inv* α *T B SA0 SA i*
  and *i* = *Suc n*
shows *s-pred-inv* α *T B SA n*
  using *s-pred-inv-maintained-step-alt*[*OF s-perm-inv-elims*(*6*), *OF assms*]
  by *blast*

lemma *s-pred-inv-maintained-step*:
  assumes *s-distinct-inv* α *T B SA*
  and *s-bucket-ptr-inv* α *T B*

and *s-locations-inv α T B SA*
and *s-unchanged-inv α T B SA0 SA*
and *s-seen-inv α T B SA i*
and *s-pred-inv α T B SA i*
and *s-suc-inv α T B SA i*
and *strict-mono α*
and *α (Max (set T)) < length B*
and *length SA0 = length T*
and *length SA = length T*
and *l-types-init α T SA0*
and *valid-list T*
and *α bot = 0*
and *Suc 0 < length T*
and *i = Suc n*
and *Suc n < length SA*
and *SA ! Suc n = Suc j*
and *suffix-type T j = S-type*
and *b = α (T ! j)*
and *k = B ! b − Suc 0*
shows *s-pred-inv α T (B[b := k]) (SA[k := j]) n*
unfolding *s-pred-inv-def*
proof(*safe*)
fix *b′ i′*
assume *in-s-current-bucket α T (B[b := k]) b′ i′ 0 < b′*
hence *b′ ≠ 0*
by *linarith*

let *?goal = ∃j′<length (SA[k := j]). SA[k := j] ! j′ = Suc (SA[k := j] ! i′) ∧ i′ < j′ ∧ n < j′*

from *s-bucket-ptr-strict-lower-bound*[*OF assms*(*1−6*,*8*,*10−14*,*16−20*)]
have *s-bucket-start α T b < B ! b*.
hence *s-bucket-start α T b ≤ k*
using *assms*(*21*) **by** *linarith*
hence *bucket-start α T b ≤ k*
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

from ‹*s-bucket-start α T b < B ! b*›
have *k < B ! b*
using *assms*(*21*) **by** *linarith*

have *j < length T*
by (*simp add*: *assms*(*19*) *suffix-type-s-bound*)
hence *b ≤ α (Max (set T))*
by (*simp add*: *assms*(*8*,*20*) *strict-mono-less-eq*)
with *s-bucket-ptr-upper-bound*[*OF assms*(*2*)]
have *B ! b ≤ bucket-end α T b*
by *blast*
with ‹*k < B ! b*›

```
  have k < bucket-end α T b
    by linarith

  have B ! b ≤ i
  proof(rule ccontr)
    assume ¬B ! b ≤ i
    hence i < B ! b
      by simp
   with s-B-val[OF assms(1−6,8,10−13,15) ‹b ≤ α (Max (set T))›] ‹s-bucket-start
α T b < B ! b›
    show False
      by simp
  qed
  with ‹k < B ! b›
  have k < i
    by linarith

  have b ≠ b' ⟹ ?goal
  proof −
    assume b ≠ b'
    hence B[b := k] ! b' = B ! b'
      by simp
    with ‹in-s-current-bucket α T (B[b := k]) b' i'›
    have in-s-current-bucket α T B b' i'
      by (simp add: in-s-current-bucket-def)
    with s-pred-invD[OF assms(6) - ‹b' ≠ 0›]
    obtain j' where
      j' < length SA
      SA ! j' = Suc (SA ! i')
      i' < j'
      i < j'
      by blast
    moreover
    from ‹in-s-current-bucket α T B b' i'›
    have B ! b' ≤ i' i' < bucket-end α T b'
      by (simp-all add: in-s-current-bucket-def)
    with s-bucket-ptr-lower-bound[OF assms(2)]
        in-s-current-bucketD(1)[OF ‹in-s-current-bucket - - B - -›]
    have bucket-start α T b' ≤ i'
      by (meson bucket-start-le-s-bucket-start le-trans)
   with outside-another-bucket[OF ‹b ≠ b'› ‹bucket-start - - - ≤ k› ‹k < bucket-end
- - -›]
        ‹i' < bucket-end α T b'›
    have k ≠ i'
      by blast
    hence SA[k := j] ! i' = SA ! i'
      by simp
    moreover
    from ‹i < j'› assms(16)
```

```
    have n < j'
      using Suc-lessD by blast
    moreover
    have SA[k := j] ! j' = SA ! j'
      using ‹k < i› calculation(4) by auto
    ultimately show ?thesis
      by auto
  qed
  moreover
  have b = b' ⟹ ?goal
  proof −
    assume b = b'
    hence B[b := k] ! b' = k
      using ‹b ≤ α (Max (set T))› assms(9) by auto

    have k = i' ⟹ ?goal
    proof −
      assume k = i'
      hence SA[k := j] ! i' = j
        using ‹k < i› assms(16,17) by auto
      moreover
      have SA[k := j] ! i = SA ! i
        using ‹k < i› by auto
      ultimately show ?goal
        using assms(16−18) ‹k = i'› ‹k < i›
        by auto
    qed
    moreover
    have k ≠ i' ⟹ ?goal
    proof −
      assume k ≠ i'
      with ‹in-s-current-bucket α T (B[b := k]) b' i'› ‹B[b := k] ! b' = k›
      have k < i'
        by (simp add: in-s-current-bucket-def)
      hence B ! b' ≤ i'
        using assms(21) ‹b = b'› ‹k < B ! b› by simp
      hence in-s-current-bucket α T B b' i'
       using ‹in-s-current-bucket α T (B[b := k]) b' i'› in-s-current-bucket-def by
blast
      with s-pred-invD[OF assms(6) - ‹b' ≠ 0›]
      obtain j' where
        j' < length SA
        SA ! j' = Suc (SA ! i')
        i' < j'
        i < j'
        by blast
      moreover
      have SA[k := j] ! i' = SA ! i'
        using ‹k ≠ i'› by simp
```

moreover
have $SA[k := j] \,!\, j' = SA \,!\, j'$
using ‹$k < i'$› ‹$i' < j'$›
by *auto*
ultimately show *?goal*
using *assms*(*16*) **by** *auto*
qed
ultimately show *?goal*
by *blast*
qed
ultimately show *?goal*
by *blast*
qed

corollary *s-pred-inv-maintained-perm-step*:
assumes *s-perm-inv* α T B *SA0* SA i
and $i = Suc\ n$
and $Suc\ n < length\ SA$
and $SA \,!\, Suc\ n = Suc\ j$
and *suffix-type* $T\ j =$ *S-type*
and $b = \alpha\ (T \,!\, j)$
and $k = B \,!\, b - Suc\ 0$
shows *s-pred-inv* α T $(B[b := k])$ $(SA[k := j])$ n
using *s-pred-inv-maintained-step*[*OF s-perm-inv-elims*[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

83.3.7 Successor

lemma *s-suc-inv-established*:
assumes *length* $SA = length\ T$
and $length\ T \leq n$
shows *s-suc-inv* α T B SA n
unfolding *s-suc-inv-def*
using *assms*(*1*) *assms*(*2*) **by** *linarith*

lemma *s-suc-inv-maintained-step-c1*:
assumes $length\ SA \leq Suc\ n$
shows *s-suc-inv* α T B SA n
unfolding *s-suc-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix i' j
assume $i' < length\ SA$ $n < i'$ $SA \,!\, i' = Suc\ j$ *suffix-type* $T\ j =$ *S-type*
with *assms*
have *False*
using *less-trans-Suc not-less* **by** *blast*
then show $\exists k.$ *in-s-current-bucket* α T B $(\alpha\ (T \,!\, j))$ $k \wedge SA \,!\, k = j \wedge k < i'$
by *blast*
qed

corollary *s-suc-inv-maintained-perm-step-c1*:
assumes *s-perm-inv* α *T B SA0 SA i*
and $i = Suc\ n$
and $length\ SA \leq Suc\ n$
shows *s-suc-inv* α *T B SA n*
by (*simp add*: *assms*(*3*) *s-suc-inv-maintained-step-c1*)

lemma *s-suc-inv-maintained-step-c1-alt*:
assumes *s-suc-inv* α *T B SA i*
and *s-bucket-ptr-inv* α *T B*
and *s-locations-inv* α *T B SA*
and *strict-mono* α
and $\alpha\ (Max\ (set\ T)) < length\ B$
and *valid-list T*
and $\alpha\ bot = 0$
and $i = Suc\ n$
and $length\ T \leq SA\ !\ Suc\ n$
shows *s-suc-inv* α *T B SA n*
proof (*cases* $length\ SA \leq Suc\ n$)
case *True*
then show *?thesis*
by (*simp add*: *s-suc-inv-maintained-step-c1*)
next
case *False*
hence $Suc\ n < length\ SA$
by *simp*
show *?thesis*
unfolding *s-suc-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix $i'\ j$
let *?goal* = $\exists k.$ *in-s-current-bucket* α *T B* $(\alpha\ (T\ !\ j))\ k \wedge SA\ !\ k = j \wedge k < i'$
assume $i' < length\ SA\ n < i'\ SA\ !\ i' = Suc\ j$ *suffix-type T j = S-type*
hence $i' = Suc\ n \vee Suc\ n < i'$
using *Suc-lessI* **by** *blast*
moreover
from *s-suc-invD*[*OF assms*(*1*) ‹$i' < length\ SA$› - ‹$SA\ !\ i' = Suc\ j$› ‹*suffix-type T j = S-type*›]
have $Suc\ n < i' \Longrightarrow$ *?goal*
using ‹$i = Suc\ n$› **by** *blast*
moreover
have $i' = Suc\ n \Longrightarrow$ *?goal*
proof −
assume $i' = Suc\ n$
have $j < length\ T \vee length\ T \leq j$
using *linorder-not-le* **by** *blast*
moreover
have $length\ T \leq j \Longrightarrow$ *?goal*
by (*meson* ‹*suffix-type T j = S-type*› *linorder-not-le suffix-type-s-bound*)
moreover

```
      have j < length T ⟹ ?goal
      proof −
        assume j < length T
        hence length T = Suc j
          using ‹SA ! i′ = Suc j› ‹i′ = Suc n› ‹length T ≤ SA ! Suc n› by force
        hence T ! j = bot
            by (metis ‹valid-list T› diff-Suc-1 last-conv-nth length-greater-0-conv
valid-list-def)
        hence α (T ! j) = 0
          using ‹α bot = 0› by presburger
        hence in-s-current-bucket α T B (α (T ! j)) 0
          unfolding in-s-current-bucket-def
       using One-nat-def assms(2,4,6,7) lessI s-bucket-ptr-0 valid-list-bucket-end-0
          by fastforce
        moreover
        {
          have 0 < bucket-end α T 0
            using ‹α (T ! j) = 0› calculation in-s-current-bucket-def by fastforce
          with s-bucket-ptr-0[OF assms(2), of 0, simplified]
               s-locations-invD[OF assms(3), of 0 0, simplified]
          have SA ! 0 ∈ s-bucket α T 0
            by simp
          moreover
          have s-bucket α T 0 = {j}
          by (simp add: ‹length T = Suc j› assms(4) assms(6) assms(7) s-bucket-0)
          ultimately have SA ! 0 = j
            by blast
        }
        ultimately show ?goal
          using ‹i′ = Suc n› by blast
      qed
      ultimately show ?goal
        by blast
    qed
    ultimately show ?goal
      by blast
  qed
qed

corollary s-suc-inv-maintained-perm-step-c1-alt:
  assumes s-perm-inv α T B SA0 SA i
  and     i = Suc n
  and     length T ≤ SA ! Suc n
shows s-suc-inv α T B SA n
  using assms s-perm-inv-def s-suc-inv-maintained-step-c1-alt by blast

lemma s-suc-inv-maintained-step-c2:
  assumes s-suc-inv α T B SA i
  and     i = Suc n
```

and *Suc n < length SA*
and *SA ! Suc n = 0*
shows *s-suc-inv α T B SA n*
unfolding *s-suc-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix *i' j*
assume *i' < length SA n < i' SA ! i' = Suc j suffix-type T j = S-type*

let *?goal* = ∃*k*. *in-s-current-bucket α T B* (*α* (*T ! j*)) *k* ∧ *SA ! k* = *j* ∧ *k < i'*

from ‹*n < i'*› ‹*i = Suc n*›
have *i = i'* ∨ *i < i'*
by *linarith*
moreover
from *assms*(*2*,*4*) ‹*SA ! i' = Suc j*›
have *i = i'* ⟹ *?goal*
by *simp*
moreover
from *s-suc-invD*[*OF assms*(*1*) ‹*i' < -*› - ‹*SA ! i' = -*› ‹*suffix-type T j = -*›]
assms(*2*)
have *i < i'* ⟹ *?goal*
by *blast*
ultimately show *?goal*
by *blast*
qed

corollary *s-suc-inv-maintained-perm-step-c2*:
assumes *s-perm-inv α T B SA0 SA i*
and *i = Suc n*
and *Suc n < length SA*
and *SA ! Suc n = 0*
shows *s-suc-inv α T B SA n*
using *assms s-perm-inv-elims*(*7*) *s-suc-inv-maintained-step-c2* **by** *blast*

lemma *s-suc-inv-maintained-step-c3*:
assumes *s-suc-inv α T B SA i*
and *i = Suc n*
and *Suc n < length SA*
and *SA ! Suc n = Suc j*
and *suffix-type T j = L-type*
shows *s-suc-inv α T B SA n*
unfolding *s-suc-inv-def*
proof (*intro allI impI*; *elim conjE*)
fix *i' j'*
assume *i' < length SA n < i' SA ! i' = Suc j' suffix-type T j' = S-type*

let *?goal* = ∃*k*. *in-s-current-bucket α T B* (*α* (*T ! j'*)) *k* ∧ *SA ! k* = *j'* ∧ *k < i'*

from ‹*n < i'*› *assms*(*2*)

```
  have i = i' ∨ i < i'
    using Suc-lessI by blast
  moreover
  from assms(2,4,5) ‹SA ! i' = -› ‹suffix-type T j' = -›
  have i = i' ⟹ ?goal
    by simp
  moreover
  from s-suc-invD[OF assms(1) ‹i' < -› - ‹SA ! i' = -› ‹suffix-type T j' = -›]
  have i < i' ⟹ ?goal
    by blast
  ultimately show ?goal
    by blast
qed

corollary s-suc-inv-maintained-perm-step-c3:
  assumes s-perm-inv α T B SA0 SA i
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = L-type
shows s-suc-inv α T B SA n
  using assms s-perm-inv-elims(7) s-suc-inv-maintained-step-c3 by blast

lemma s-suc-inv-maintained-step-c4:
  assumes s-distinct-inv α T B SA
  and     s-bucket-ptr-inv α T B
  and     s-locations-inv α T B SA
  and     s-unchanged-inv α T B SA0 SA
  and     s-seen-inv α T B SA i
  and     s-pred-inv α T B SA i
  and     s-suc-inv α T B SA i
  and     strict-mono α
  and     α (Max (set T)) < length B
  and     length SA0 = length T
  and     length SA = length T
  and     l-types-init α T SA0
  and     valid-list T
  and     α bot = 0
  and     Suc 0 < length T
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b - Suc 0
shows s-suc-inv α T (B[b := k]) (SA[k := j]) n
  unfolding s-suc-inv-def
proof(safe)
  fix i' j'
```

assume $i' < \mathit{length}\ (SA[k := j])\ n < i'\ SA[k := j]\ !\ i' = \mathit{Suc}\ j'\ \mathit{suffix\text{-}type}\ T\ j' = \mathit{S\text{-}type}$
hence $i' < \mathit{length}\ SA$
by *simp*

let $?b = \alpha\ (T\ !\ j')$
and $?B = B[b := k]$
and $?SA = SA[k := j]$

let $?goal = \exists k'.\ \mathit{in\text{-}s\text{-}current\text{-}bucket}\ \alpha\ T\ ?B\ ?b\ k' \wedge\ ?SA\ !\ k' = j' \wedge k' < i'$

from ‹*suffix-type* $T\ j' =$ -›
have $j' < \mathit{length}\ T$
by (*simp add*: *suffix-type-s-bound*)
hence $?b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
using ‹*strict-mono* -›
by (*simp add*: *strict-mono-less-eq*)

from *s-bucket-ptr-strict-lower-bound*[*OF assms*(*1*−*6*,*8*,*10*−*14*,*16*−*20*)]
have *s-bucket-start* $\alpha\ T\ b < B\ !\ b$**.**
hence *s-bucket-start* $\alpha\ T\ b \leq k$
using *assms*(*21*) **by** *linarith*
hence *bucket-start* $\alpha\ T\ b \leq k$
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

from ‹*s-bucket-start* $\alpha\ T\ b < B\ !\ b$›
have $k < B\ !\ b$
using *assms*(*21*) **by** *linarith*

have $j < \mathit{length}\ T$
by (*simp add*: *assms*(*19*) *suffix-type-s-bound*)
hence $b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$
by (*simp add*: *assms*(*8*,*20*) *strict-mono-less-eq*)
with *s-bucket-ptr-upper-bound*[*OF assms*(*2*)]
have $B\ !\ b \leq$ *bucket-end* $\alpha\ T\ b$
by *blast*
with ‹$k < B\ !\ b$›
have $k <$ *bucket-end* $\alpha\ T\ b$
by *linarith*

have $B\ !\ b \leq i$
proof(*rule ccontr*)
assume $\neg B\ !\ b \leq i$
hence $i < B\ !\ b$
by *simp*
with *s-B-val*[*OF assms*(*1*−*6*,*8*,*10*−*13*,*15*) ‹$b \leq \alpha\ (\mathit{Max}\ (\mathit{set}\ T))$›] ‹*s-bucket-start* $\alpha\ T\ b < B\ !\ b$›
show *False*
by *simp*

qed
with ‹$k < B$! b›
have $k < i$
by *linarith*
hence $k \leq n$
by (*simp add*: *assms*(*16*))
with ‹$n < i'$›
have $k < i'$
using *dual-order.strict-trans2* **by** *blast*
hence $SA[k := j]$! $i' = SA$! i'
by *simp*
with ‹$SA[k := j]$! $i' = Suc\ j'$›
have SA ! $i' = Suc\ j'$
by *simp*

have $i \leq i'$
by (*simp add*: *Suc-leI* ‹$n < i'$› *assms*(*16*))
hence $i = i' \vee i < i'$
by (*simp add*: *nat-less-le*)
moreover
have $i = i' \Longrightarrow$ *?goal*
proof −
assume $i = i'$
hence $j = j'$
using ‹SA ! $i' = Suc\ j'$› *assms*(*16*,*18*) **by** *auto*
hence $SA[k := j]$! $k = j'$
using ‹$k \leq n$› *assms*(*17*) **by** *auto*
moreover
have *?b* = b
using ‹$j = j'$› *assms*(*20*) **by** *blast*
hence *in-s-current-bucket* α T *?B* *?b* k = *in-s-current-bucket* α T *?B* b k
by *simp*
moreover
from ‹α (T ! j') $\leq \alpha$ (*Max* (*set* T))›
‹*?b* = b›[*symmetric*]
have *in-s-current-bucket* α T *?B* b k
unfolding *in-s-current-bucket-def*
using ‹$k <$ *bucket-end* α T b› *assms*(*9*) **by** *auto*
ultimately show *?goal*
using ‹$k < i'$› **by** *blast*
qed
moreover
have $i < i' \Longrightarrow$ *?goal*
proof −
assume $i < i'$
with *s-suc-invD*[*OF assms*(*7*) ‹$i' <$ *length* SA› - ‹SA ! $i' = Suc\ j'$› ‹*suffix-type* $T\ j' =$ -›]
obtain k' **where**
in-s-current-bucket α T B *?b* k'

```
      SA ! k' = j'
      k' < i'
      by blast
    moreover
    from in-s-current-bucketD[OF ‹in-s-current-bucket α T B ?b k'›]
    have in-s-current-bucket α T ?B ?b k'
      unfolding in-s-current-bucket-def
    proof (safe)
      show ?B ! ?b ≤ k'
        by (metis ‹B ! ?b ≤ k'› ‹k < B ! b› dual-order.trans list-update-beyond
nat-le-linear
            not-less nth-list-update-eq nth-list-update-neq)
    qed
    moreover
    from in-s-current-bucketD(2)[OF ‹in-s-current-bucket α T B ?b k'›]
    have B ! ?b ≤ k' .
    hence s-bucket-start α T ?b ≤ k'
    by (meson ‹?b ≤ α (Max (set T))› assms(2) le-less-trans not-le s-bucket-ptr-inv-def)
    hence bucket-start α T ?b ≤ k'
      using bucket-start-le-s-bucket-start dual-order.trans by blast

    have b = ?b ∨ b ≠ ?b
      by blast
    hence k ≠ k'
    proof
      assume b = ?b
      with ‹B ! ?b ≤ k'› ‹k < B ! b›
      show ?thesis
        by simp
    next
      assume b ≠ ?b
      with outside-another-bucket[OF - ‹bucket-start - - - ≤ k› ‹k < bucket-end -
- -›]
        ‹bucket-start α T ?b ≤ k'› in-s-current-bucketD(3)[OF ‹in-s-current-bucket
α T B ?b k'›]
      show ?thesis
        by blast
    qed
    hence SA[k := j] ! k' = SA ! k'
      by simp
    ultimately show ?goal
      by blast
  qed
  ultimately show ?goal
    by blast
qed

corollary s-suc-inv-maintained-perm-step-c4:
  assumes s-perm-inv α T B SA0 SA i
```

and $i = Suc\ n$
and $Suc\ n < length\ SA$
and $SA\ !\ Suc\ n = Suc\ j$
and $suffix\text{-}type\ T\ j = S\text{-}type$
and $b = \alpha\ (T\ !\ j)$
and $k = B\ !\ b - Suc\ 0$
shows *s-suc-inv* $\alpha\ T\ (B[b := k])\ (SA[k := j])\ n$
using *s-suc-inv-maintained-step-c4*[*OF s-perm-inv-elims*[*OF assms*(*1*)] *assms*(*2*−)]
by *blast*

lemmas *s-suc-inv-maintained-perm-step* =
s-suc-inv-maintained-step-c1
s-suc-inv-maintained-perm-step-c2
s-suc-inv-maintained-perm-step-c3
s-suc-inv-maintained-perm-step-c4

83.3.8 Combined Permutation Invariant

lemma *s-perm-inv-established*:
assumes *s-bucket-init* $\alpha\ T\ B$
and *s-type-init T SA*
and *strict-mono* α
and $\alpha\ (Max\ (set\ T)) < length\ B$
and $length\ SA = length\ T$
and *l-types-init* $\alpha\ T\ SA$
and *valid-list T*
and $\alpha\ bot = 0$
and $Suc\ 0 < length\ T$
and $length\ T \leq n$
shows *s-perm-inv* $\alpha\ T\ B\ SA\ SA\ n$
unfolding *s-perm-inv-def*
by (*simp add*: *assms s-distinct-inv-established s-bucket-ptr-inv-established*
s-locations-inv-established s-unchanged-inv-established s-seen-inv-established
s-pred-inv-established s-suc-inv-established)

lemma *s-perm-inv-maintained-step-c1*:
assumes *s-perm-inv* $\alpha\ T\ B\ SA0\ SA\ i$
and $i = Suc\ n$
and $length\ SA \leq Suc\ n$
shows *s-perm-inv* $\alpha\ T\ B\ SA0\ SA\ n$
unfolding *s-perm-inv-def*
by (*clarsimp simp*: *s-perm-inv-elims*[*OF assms*(*1*)]
s-seen-inv-maintained-perm-step-c1[*OF assms*]
s-pred-inv-maintained-perm-step-alt[*OF assms*(*1*,*2*)]
s-suc-inv-maintained-step-c1[*OF assms*(*3*)])

lemma *s-perm-inv-maintained-step-c1-alt*:
assumes *s-perm-inv* $\alpha\ T\ B\ SA0\ SA\ i$
and $i = Suc\ n$

and *length T* ≤ *SA* ! *Suc n*
shows *s-perm-inv α T B SA0 SA n*
proof (*cases length T* ≤ *Suc n*)
case *True*
then show *?thesis*
by (*metis assms*(*1*) *assms*(*2*) *s-perm-inv-elims*(*11*) *s-perm-inv-maintained-step-c1*)
next
case *False*
hence *Suc n* < *length T*
by *simp*
then show *?thesis*
unfolding *s-perm-inv-def*
by (*metis assms s-perm-inv-def s-pred-inv-maintained-step-alt*
s-seen-inv-maintained-perm-step-c1-alt s-suc-inv-maintained-perm-step-c1-alt)
qed

lemma *s-perm-inv-maintained-step-c2*:
assumes *s-perm-inv α T B SA0 SA i*
and *i* = *Suc n*
and *Suc n* < *length SA*
and *SA* ! *Suc n* = *0*
shows *s-perm-inv α T B SA0 SA n*
unfolding *s-perm-inv-def*
by (*clarsimp simp*: *s-perm-inv-elims*[*OF assms*(*1*)]
s-seen-inv-maintained-perm-step-c2[*OF assms*]
s-pred-inv-maintained-perm-step-alt[*OF assms*(*1*,*2*)]
s-suc-inv-maintained-perm-step-c2[*OF assms*])

lemma *s-perm-inv-maintained-step-c3*:
assumes *s-perm-inv α T B SA0 SA i*
and *i* = *Suc n*
and *Suc n* < *length SA*
and *SA* ! *Suc n* = *Suc j*
and *suffix-type T j* = *L-type*
shows *s-perm-inv α T B SA0 SA n*
unfolding *s-perm-inv-def*
by (*clarsimp simp*: *s-perm-inv-elims*[*OF assms*(*1*)]
s-seen-inv-maintained-perm-step-c3[*OF assms*]
s-pred-inv-maintained-perm-step-alt[*OF assms*(*1*,*2*)]
s-suc-inv-maintained-perm-step-c3[*OF assms*])

lemma *s-perm-inv-maintained-step-c4*:
assumes *s-perm-inv α T B SA0 SA i*
and *i* = *Suc n*
and *Suc n* < *length SA*
and *SA* ! *Suc n* = *Suc j*
and *suffix-type T j* = *S-type*
and *b* = *α* (*T* ! *j*)
and *k* = *B* ! *b* − *Suc 0*

```
shows s-perm-inv α T (B[b := k]) SA0 (SA[k := j]) n
  unfolding s-perm-inv-def
  by (clarsimp simp: s-perm-inv-elims[OF assms(1)]
                     s-distinct-inv-maintained-perm-step[OF assms]
                     s-bucket-ptr-inv-maintained-perm-step[OF assms]
                     s-locations-inv-maintained-perm-step[OF assms]
                     s-unchanged-inv-maintained-perm-step[OF assms]
                     s-seen-inv-maintained-perm-step-c4[OF assms]
                     s-pred-inv-maintained-perm-step[OF assms]
                     s-suc-inv-maintained-perm-step-c4[OF assms])

theorem abs-induce-s-perm-step:
  assumes s-perm-inv α T B SA0 SA i
  and     abs-induce-s-step (B, SA, i) (α, T) = (B', SA', i')
shows s-perm-inv α T B' SA0 SA' i'
proof (cases i)
  case 0
  then show ?thesis
    using assms by force
next
  case (Suc n)
  assume i = Suc n
  have T ≠ []
    using s-perm-inv-elims(15)[OF assms(1)] by fastforce
  show ?thesis
  proof (cases Suc n < length SA ∧ SA ! Suc n < length T)
    assume Suc n < length SA ∧ SA ! Suc n < length T
    hence Suc n < length SA SA ! Suc n < length T
      by blast+
    show ?thesis
    proof (cases SA ! Suc n)
      case 0
      then show ?thesis
        using s-perm-inv-maintained-step-c2[OF assms(1) ‹i = Suc n› ‹Suc n <
length SA› 0] assms
        by (clarsimp simp: ‹i = Suc n› ‹Suc n < length SA› 0 ‹T ≠ []›)
    next
      case (Suc j)
      assume SA ! Suc n = Suc j
      hence Suc j < length T
        using ‹SA ! Suc n < length T› by auto
      show ?thesis
      proof (cases suffix-type T j)
        case S-type
        then show ?thesis
          using assms ‹i = Suc n› ‹Suc n < length SA› ‹SA ! Suc n = Suc j›
                s-perm-inv-maintained-step-c4[OF assms(1), of n j α (T ! j) B ! α
(T ! j) − Suc 0]
          by (clarsimp simp: Let-def ‹Suc j < length T›)
```

```
      next
        case L-type
        then show ?thesis
          using assms ‹i = Suc n› ‹Suc n < length SA› ‹SA ! Suc n = Suc j›
                s-perm-inv-maintained-step-c3[OF assms(1)]
          by (clarsimp simp: Let-def ‹Suc j < length T›)
      qed
    qed
  next
    assume ¬(Suc n < length SA ∧ SA ! Suc n < length T)
    hence ¬ Suc n < length SA ∨ ¬ SA ! Suc n < length T
      by blast
    then show ?thesis
    proof
      assume ¬ Suc n < length SA
      then show ?thesis
        using assms ‹i = Suc n› s-perm-inv-maintained-step-c1[OF assms(1)] by
force
    next
      assume ¬ SA ! Suc n < length T
      hence length T ≤ SA ! Suc n
        by simp
      then show ?thesis
        using assms ‹i = Suc n› s-perm-inv-maintained-step-c1-alt[OF assms(1)]
by simp
    qed
  qed
qed

corollary abs-induce-s-perm-step-alt:
 ⋀a. s-perm-inv-alt α T SA0 a ⟹ s-perm-inv-alt α T SA0 (abs-induce-s-step a
(α, T))
 by (metis abs-induce-s-perm-step s-perm-inv-alt.elims(2) s-perm-inv-alt.elims(3))

theorem abs-induce-s-perm-alt-maintained:
  assumes s-perm-inv-alt α T SA0 (B, SA, length T)
  shows s-perm-inv-alt α T SA0 (abs-induce-s-base α T B SA)
  unfolding abs-induce-s-base-def
  using repeat-maintain-inv[of s-perm-inv-alt α T SA0 abs-induce-s-step (α, T),
OF - assms(1)]
        abs-induce-s-perm-step-alt
  by blast

corollary abs-induce-s-perm-maintained:
  assumes abs-induce-s-base α T B SA = (B', SA', n)
  and     s-perm-inv α T B SA0 SA (length T)
shows s-perm-inv α T B' SA0 SA' n
  using assms abs-induce-s-perm-alt-maintained by force
```

lemma *s-perm-inv-0-B-val*:
assumes *s-perm-inv* α *T B SA SA' 0*
and $b \leq \alpha$ (*Max* (*set T*))
shows *B* ! *b* = *s-bucket-start* α *T b*
proof −
from *s-bucket-ptr-lower-bound*[*OF s-perm-inv-elims*(*2*)[*OF assms*(*1*)] *assms*(*2*)]
have *s-bucket-start* α *T b* ≤ *B* ! *b* **.**

have *s-bucket-start* α *T b* ≥ *0*
by *blast*
hence *s-bucket-start* α *T b* = *0* ∨ *0* < *s-bucket-start* α *T b*
by *blast*
with ‹*s-bucket-start* α *T b* ≤ *B* ! *b*›
have *B* ! *b* = *s-bucket-start* α *T b* ∨ *0* < *B* ! *b*
by *linarith*
then show *?thesis*
proof
assume *B* ! *b* = *s-bucket-start* α *T b*
then show *?thesis* **.**
next
assume *0* < *B* ! *b*
with *s-B-val*[*OF s-perm-inv-elims*(*1*−*6*,*8*,*10*−*13*,*15*)[*OF assms*(*1*)] *assms*(*2*)]
show *?thesis* **.**
qed
qed

lemma *s-perm-inv-0-list-slice-bucket*:
assumes *s-perm-inv* α *T B SA SA' 0*
and $b \leq \alpha$ (*Max* (*set T*))
shows *set* (*list-slice SA'* (*bucket-start* α *T b*) (*bucket-end* α *T b*)) = *bucket* α *T b*
by (*meson assms bucket-eq-list-slice s-perm-inv-0-B-val s-perm-inv-elims*(*1*−*4*,*10*−*12*))

lemma *s-perm-inv-0-distinct-list-slice*:
assumes *s-perm-inv* α *T B SA SA' 0*
and $b \leq \alpha$ (*Max* (*set T*))
shows *distinct* (*list-slice SA'* (*bucket-start* α *T b*) (*bucket-end* α *T b*))
(**is** *distinct ?xs*)
proof −
let *?ys* = *list-slice SA'* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)
and *?zs* = *list-slice SA'* (*s-bucket-start* α *T b*) (*bucket-end* α *T b*)

have *?xs* = *?ys* @ *?zs*
by (*metis list-slice-append bucket-start-le-s-bucket-start l-bucket-end-le-bucket-end*
s-bucket-start-eq-l-bucket-end)

from *l-types-initD*(*1*)[*OF l-types-init-maintained*[*OF s-perm-inv-elims*(*2*,*4*,*10*−*12*)[*OF*
assms(*1*)]]
assms(*2*)]

have *set ?ys = l-bucket α T b* **.**
moreover
from *s-bucket-eq-list-slice*[*OF s-perm-inv-elims*(*1*,*3*,*11*)[*OF assms*(*1*)] *assms*(*2*)
s-perm-inv-0-B-val[*OF assms*]]
have *set ?zs = s-bucket α T b* **.**
ultimately have *set ?ys ∩ set ?zs = {}*
using *disjoint-l-s-bucket* **by** *blast*
with *s-distinct-invD*[*OF s-perm-inv-elims*(*1*), *OF assms*, *simplified s-perm-inv-0-B-val*[*OF assms*]]
l-types-initD(*2*)[*OF l-types-init-maintained*[*OF s-perm-inv-elims*(*2*,*4*,*10−12*)[*OF assms*(*1*)]]
assms(*2*)]
‹*?xs = ?ys @ ?zs*›
show *?thesis*
by *auto*
qed

lemma *abs-induce-s-base-distinct*:
assumes *abs-induce-s-base α T B SA = (B′, SA′, n)*
and *s-perm-inv α T B′ SA SA′ n*
shows *distinct SA′*
proof(*intro distinct-conv-nth*[*THEN iffD2*] *allI impI*)
fix *i j*
assume *i < length SA′ j < length SA′ i ≠ j*
hence *i < length T j < length T*
using *assms*(*2*) *s-perm-inv-elims*(*11*) **by** *fastforce+*

from *abs-induce-s-base-index*[*of α T B SA*] *assms*(*1*)
have *n = 0*
by *simp*

from *index-in-bucket-interval-gen*[*OF* ‹*i < length T*› *s-perm-inv-elims*(*8*)[*OF assms*(*2*)]]
obtain *b0* **where**
b0 ≤ α (*Max* (*set T*))
bucket-start α T b0 ≤ i
i < bucket-end α T b0
by *blast*

have *bucket-end α T b0 ≤ length SA′*
using *assms*(*2*) *bucket-end-le-length s-perm-inv-elims*(*11*) **by** *fastforce*

let *?xs = list-slice SA′* (*bucket-start α T b0*) (*bucket-end α T b0*)

from *index-in-bucket-interval-gen*[*OF* ‹*j < length T*› *s-perm-inv-elims*(*8*)[*OF assms*(*2*)]]
obtain *b1* **where**
b1 ≤ α (*Max* (*set T*))
bucket-start α T b1 ≤ j

```
    j < bucket-end α T b1
    by blast

  have bucket-end α T b1 ≤ length SA'
    using assms(2) bucket-end-le-length s-perm-inv-elims(11) by fastforce

  have b0 ≠ b1 ⟹ SA' ! i ≠ SA' ! j
  proof −
    assume b0 ≠ b1
    hence bucket α T b0 ∩ bucket α T b1 = {}
      by (metis (mono-tags, lifting) Int-emptyI bucket-def mem-Collect-eq)
    moreover
    from s-perm-inv-0-list-slice-bucket[OF assms(2)[simplified ‹n = 0›] ‹b0 ≤ -›]
        list-slice-nth-mem[OF ‹bucket-start α T b0 ≤ i› ‹i < bucket-end α T b0›
          ‹bucket-end α T b0 ≤ -›]
    have SA' ! i ∈ bucket α T b0
      by blast
    moreover
    from s-perm-inv-0-list-slice-bucket[OF assms(2)[simplified ‹n = 0›] ‹b1 ≤ -›]
        list-slice-nth-mem[OF ‹bucket-start α T b1 ≤ j› ‹j < bucket-end α T b1›
          ‹bucket-end α T b1 ≤ -›]
    have SA' ! j ∈ bucket α T b1
      by blast
    ultimately show ?thesis
      by auto
  qed
  moreover
  have b0 = b1 ⟹ SA' ! i ≠ SA' ! j
  proof −
    assume b0 = b1
    with ‹bucket-start α T b1 ≤ j› ‹j < bucket-end α T b1›
    have bucket-start α T b0 ≤ j j < bucket-end α T b0
      by simp-all
    with list-slice-nth-eq-iff-index-eq[
        OF s-perm-inv-0-distinct-list-slice[OF assms(2)[simplified‹n = 0›] ‹b0 ≤
-›]
          ‹bucket-end - - b0 ≤ -› ‹bucket-start α T b0 ≤ i› ‹i < bucket-end α T
b0›, of j]
        ‹i ≠ j›
    show ?thesis
      by blast
  qed
  ultimately show SA' ! i ≠ SA' ! j
    by blast
qed

lemma abs-induce-s-base-subset-upt:
  assumes abs-induce-s-base α T B SA = (B', SA', n)
  and     s-perm-inv α T B' SA SA' n
```

shows *set SA′ ⊆ {0..<length T}*
proof
  fix *x*
  assume *x ∈ set SA′*
  from *in-set-conv-nth*[*THEN iffD1*, *OF ‹x ∈ set SA′›*]
  obtain *i* **where**
    i < length SA′
    SA′ ! i = x
    by *blast*
  hence *i < length T*
    using *assms*(*2*) *s-perm-inv-elims*(*11*) **by** *fastforce*
  with *index-in-bucket-interval-gen*[*OF - s-perm-inv-elims*(*8*)[*OF assms*(*2*)]]
  obtain *b* **where**
    b ≤ α (Max (set T))
    bucket-start α T b ≤ i
    i < bucket-end α T b
    by *blast*

  from *abs-induce-s-base-index*[*of α T B SA*] *assms*(*1*)
  have *n = 0*
    by *simp*

  have *bucket-end α T b ≤ length SA′*
    using *assms*(*2*) *bucket-end-le-length s-perm-inv-elims*(*11*) **by** *fastforce*
  with *s-perm-inv-0-list-slice-bucket*[*OF assms*(*2*)[*simplified ‹n = 0›*] *‹b ≤ -›*]
      ‹SA′ ! i = x› ‹bucket-start α T b ≤ i› ‹i < bucket-end α T b›
  have *x ∈ bucket α T b*
    using *list-slice-nth-mem* **by** *blast*
  hence *x < length T*
    using *bucket-def* **by** *blast*
  then show *x ∈ {0..< length T}*
    by *simp*
qed

corollary *abs-induce-s-base-eq-upt*:
  assumes *abs-induce-s-base α T B SA = (B′, SA′, n)*
  and *s-perm-inv α T B′ SA SA′ n*
shows *set SA′ = {0..<length T}*
  by (*rule card-subset-eq*[*OF finite-atLeastLessThan abs-induce-s-base-subset-upt*[*OF assms*]];
      clarsimp simp: *distinct-card*[*OF abs-induce-s-base-distinct*[*OF assms*]]
          s-perm-inv-elims(*11*)[*OF assms*(*2*)])

theorem *abs-induce-s-base-perm*:
  assumes *abs-induce-s-base α T B SA = (B′, SA′, n)*
  and *s-perm-inv α T B′ SA SA′ n*
shows *SA′ <~~> [0..< length T]*
  by (*rule perm-distinct-set-of-upt-iff*[*THEN iffD2*];
    clarsimp simp: *abs-induce-s-base-distinct*[*OF assms*] *abs-induce-s-base-eq-upt*[*OF*

assms])

83.3.9 Sorted

lemma *s-sorted-established*:
assumes *s-bucket-init* α *T B*
and *strict-mono* α
and *valid-list T*
and α *bot* = *0*
and *b* $\leq$ α (*Max* (*set T*))
shows *sorted-wrt R* (*list-slice SA* (*B* ! *b*) (*bucket-end* α *T b*))
(**is** *sorted-wrt R ?xs*)
proof −
have *b* = *0* ∨ *0* < *b*
by *blast*
moreover
have *0* < *b* ⟹ *?thesis*
proof −
assume *0* < *b*
hence *B* ! *b* = *bucket-end* α *T b*
by (*simp add*: ‹*b* ≤ α (*Max* (*set T*))› *assms*(*1*) *s-bucket-initD*)
then show *?thesis*
by *simp*
qed
moreover
have *b* = *0* ⟹ *?thesis*
proof −
assume *b* = *0*
hence *bucket-end* α *T b* = *Suc 0*
by (*simp add*: *assms*(*2*−*4*) *valid-list-bucket-end-0*)
moreover
from ‹*b* = *0*›
have *B* ! *b* = *0*
using *assms*(*1*) *s-bucket-initD*(*2*) **by** *auto*
ultimately show *?thesis*
by (*simp add*: *sorted-wrt01*)
qed
ultimately show *?thesis*
by *blast*
qed

lemma *s-sorted-inv-established*:
assumes *s-bucket-init* α *T B*
and *strict-mono* α
and *valid-list T*
and α *bot* = *0*
shows *s-sorted-inv* α *T B SA*
unfolding *s-sorted-inv-def*
using *assms ordlistns.sorted-map s-sorted-established* **by** *blast*

lemma *s-prefix-sorted-inv-established*:
assumes *s-bucket-init* α T B
and *strict-mono* α
and *valid-list* T
and α *bot* = *0*
shows *s-prefix-sorted-inv* α T B *SA*
unfolding *s-prefix-sorted-inv-def*
using *assms ordlistns.sorted-map s-sorted-established* **by** *blast*

lemma *s-sorted-maintained-unchanged-step*:
assumes *s-perm-inv* α T B *SA0 SA i*
and i = *Suc n*
and *Suc n* < *length SA*
and *SA* ! *Suc n* = *Suc j*
and *suffix-type T j* = *S-type*
and $b = \alpha$ (T ! j)
and k = B ! b − *Suc 0*
and $b' \leq \alpha$ (*Max* (*set T*))
and *sorted-wrt R* (*list-slice SA* (B ! b') (*bucket-end* α T b'))
and $b \neq b'$
shows *sorted-wrt R* (*list-slice* (*SA*[k := j]) ((*B*[b := k]) ! b') (*bucket-end* α T b'))
proof −
let *?xs* = *list-slice* (*SA*[k := j]) (*B*[b := k] ! b') (*bucket-end* α T b')

have *bucket-end* α T $b \leq$ *length T*
using *bucket-end-le-length* **by** *blast*
moreover
have B ! $b \leq$ *bucket-end* α T b
using *assms*(*1*,*5*,*6*) *s-bucket-ptr-upper-bound s-perm-inv-elims*(*2*,*8*) *strict-mono-less-eq*
suffix-type-s-bound **by** *fastforce*
ultimately have k < *length T*
using *assms*(*1*,*7*) *s-perm-inv-elims*(*15*) **by** *fastforce*
hence k < *length SA*
by (*metis assms*(*1*) *s-perm-inv-def*)

from *s-bucket-ptr-strict-lower-bound*[*OF s-perm-inv-elims*(*1*−*6*,*8*,*10*−*14*)[*OF assms*(*1*)] *assms*(*2*−*6*)]
have *s-bucket-start* α T b < B ! b .
hence k < B ! b
using *assms*(*7*) *diff-less gr-implies-not-zero* **by** *blast*

have *s-bucket-start* α T $b \leq k$
using *assms s-bucket-ptr-strict-lower-bound s-perm-inv-def* **by** *fastforce*
hence *bucket-start* α T $b \leq k$
using *bucket-start-le-s-bucket-start le-trans* **by** *blast*

from ‹$b \neq b'$›
have B[b := k] ! b' = B ! b'

```
    by simp

  have k < B ! b' ∨ bucket-end α T b' ≤ k
  proof −
    from ‹b ≠ b'›
    have b < b' ∨ b' < b
      using nat-neq-iff by blast
    moreover
    have b < b' ⟹ k < B ! b'
    proof −
      assume b < b'
      hence bucket-end α T b ≤ bucket-start α T b'
        by (simp add: less-bucket-end-le-start)
      hence k < bucket-start α T b'
        using ‹B ! b ≤ bucket-end α T b› ‹k < B ! b› by linarith
     with s-bucket-ptr-lower-bound[OF s-perm-inv-elims(2)[OF assms(1)] ‹b' ≤ -›]
      show ?thesis
        by (meson bucket-start-le-s-bucket-start order.strict-trans2)
    qed
    moreover
    have b' < b ⟹ bucket-end α T b' ≤ k
    proof −
      assume b' < b
      hence bucket-end α T b' ≤ bucket-start α T b
        by (simp add: less-bucket-end-le-start)
      then show ?thesis
        using ‹bucket-start α T b ≤ k› by linarith
    qed
    ultimately show ?thesis
      by blast
  qed
  with list-slice-update-unchanged-1
       list-slice-update-unchanged-2
  have ?xs = list-slice SA (B ! b') (bucket-end α T b')
    using ‹B[b := k] ! b' = B ! b'› by auto
  then show ?thesis
    using assms(9) by auto
qed

lemma s-sorted-inv-maintained-step:
  assumes s-perm-inv α T B SA0 SA i
  and     s-sorted-pre α T SA0
  and     s-sorted-inv α T B SA
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
```

```
shows s-sorted-inv α T (B[b := k]) (SA[k := j])
  unfolding s-sorted-inv-def
proof (safe)
  fix b'
  assume b' ≤ α (Max (set T))
  let ?xs = list-slice (SA[k := j]) (B[b := k] ! b') (bucket-end α T b')

  have bucket-end α T b ≤ length T
    using bucket-end-le-length by blast

  moreover
  have B ! b ≤ bucket-end α T b
    using assms(1,7,8) s-bucket-ptr-upper-bound suffix-type-s-bound
        s-perm-inv-elims(2,8) strict-mono-less-eq
    by fastforce

  ultimately have k < length T
    using assms(1,9) s-perm-inv-elims(15) by fastforce
  hence k < length SA
    by (metis assms(1) s-perm-inv-def)

  from s-bucket-ptr-strict-lower-bound
        [OF s-perm-inv-elims(1−6,8,10−14)
        [OF assms(1)] assms(4−8)]
  have s-bucket-start α T b < B ! b .
  hence k < B ! b
    using assms(9) diff-less gr-implies-not-zero by blast

  have s-bucket-start α T b ≤ k
    using assms s-bucket-ptr-strict-lower-bound s-perm-inv-def
    by fastforce
  hence bucket-start α T b ≤ k
    using bucket-start-le-s-bucket-start le-trans
    by blast
  hence b ≤ α (Max (set T))
    by (metis ‹k < length SA› assms(1) bucket-end-Max dual-order.trans
          less-bucket-end-le-start s-perm-inv-elims(8,11) leD leI)

  have b = b' ∨ b ≠ b'
    by blast
  moreover
  have b = b' ⟹ ordlistns.sorted (map (suffix T) ?xs)
  proof −
    assume b = b'
    hence B[b := k] ! b' = k
      by (meson ‹b' ≤ α (Max (set T))› assms(1) le-less-trans nth-list-update-eq
          s-perm-inv-elims(9))

    have SA[k := j] ! k = j
```

```
  by (simp add: ‹k < length SA›)

from list-slice-update-unchanged-1
    ‹k < B ! b›
    ‹SA[k := j] ! k = j›
    ‹B[b := k] ! b' = k›
    ‹B ! b ≤ bucket-end α T b›
    ‹b = b'› ‹k < length SA›
have ?xs = j # list-slice SA (B ! b) (bucket-end α T b)
  by (metis Suc-pred assms(9) length-list-update not-le
        less-nat-zero-code list-slice-Suc less-le-trans)
moreover
have ordlistns.sorted
       (map (suffix T) (j # list-slice SA (B ! b) (bucket-end α T b)))
proof −
  let ?ys = list-slice SA (B ! b) (bucket-end α T b)

  have A: map (suffix T) (j # ?ys) = (suffix T j) # map (suffix T) ?ys
    by simp

  from s-sorted-invD[OF assms(3) ‹b ≤ -›]
  have B: ordlistns.sorted (map (suffix T) ?ys) .

  have ?ys = [] ∨ ?ys ≠ []
    by blast
  hence map (suffix T) ?ys = [] ∨ map (suffix T) ?ys ≠ []
    by simp
  moreover
  have map (suffix T) ?ys = [] ⟹ ?thesis
    using ordlistns.sorted-cons-nil by fastforce
  moreover
  have map (suffix T) ?ys ≠ [] ⟹
         ordlistns.sorted ((suffix T j) # map (suffix T) ?ys)
  proof (rule ordlistns.sorted-consI[OF - B])
    assume map (suffix T) (list-slice SA (B ! b) (bucket-end α T b)) ≠ []
    then
    show map (suffix T) (list-slice SA (B ! b) (bucket-end α T b)) ≠ []
      by simp
  next
    assume map (suffix T) ?ys ≠ []
    hence map (suffix T) ?ys ! 0 = suffix T (?ys ! 0)
      by (metis length-greater-0-conv list.simps(8) nth-map)
    moreover
    have list-less-eq-ns (suffix T j) (suffix T (?ys ! 0))
    proof −
      have ?ys ! 0 ∈ s-bucket α T b
        by (metis assms(1) length-greater-0-conv s-perm-inv-elims(3)
              length-map nth-mem s-locations-inv-in-list-slice
              ‹b = b'›
```

```
        ‹b' ≤ α (Max (set T))›
        ‹map (suffix T) ?ys ≠ []› )
hence α (T ! (?ys ! 0)) = b suffix-type T (?ys ! 0) = S-type
  by (simp add: s-bucket-def bucket-def)+
hence T ! j = T ! (?ys ! 0)
  using assms(1,8) s-perm-inv-elims(8) strict-mono-eq by fastforce

have ?ys ! 0 = SA ! (B ! b)
  using ‹map (suffix T) ?ys ≠ []› nth-list-slice by fastforce

have b ≠ 0
  by (metis ‹s-bucket-start α T b < B ! b› assms(1)
        gr-implies-not-zero s-bucket-ptr-0 s-perm-inv-elims(2))

have in-s-current-bucket α T B b (B ! b)
  using ‹b = b'› ‹b' ≤ α (Max (set T))› ‹map (suffix T) ?ys ≠ []›
  by (metis ‹B ! b ≤ bucket-end α T b› in-s-current-bucket-def
        le-eq-less-or-eq list.map-disc-iff list-slice-n-n)
  with s-pred-invD
      [OF s-perm-inv-elims(6)[OF assms(1)] - ‹b ≠ 0›,
       of B ! b]
obtain i' where i'-assms:
  i' < length SA
  SA ! i' = Suc (SA ! (B ! b))
  B ! b < i'
  i < i'
  by blast

let ?b0 = α (T ! (Suc j))
and ?b1 = α (T ! (Suc (SA ! (B ! b))))

have i-less: i < length SA
  by (simp add: assms(4-5))

have ?b0 ≤ α (Max (set T))
  by (metis Max-greD Suc-leI assms(1,4-6) strict-mono-leD
        s-perm-inv-elims(5,8) s-seen-invD(1) lessI)

have ?b1 ≤ α (Max (set T))
  by (metis Max-greD i'-assms(1,2,4) assms(1)
      less-imp-le-nat s-perm-inv-elims(5,8)
      s-seen-invD(1) strict-mono-leD)

have S0: suffix T j = T ! j # suffix T (Suc j)
  using assms(7) suffix-cons-Suc suffix-type-s-bound
  by blast

have S1: suffix T (?ys ! 0) =
        T ! (?ys ! 0) # suffix T (Suc (SA ! (B ! b)))
```

```
    using ‹?ys ! 0 = SA ! (B ! b)›
          ‹suffix-type T (?ys ! 0) = S-type› suffix-cons-Suc
          suffix-type-s-bound by auto

  have ?b0 ≤ ?b1
  proof(rule ccontr)
    assume ¬?b0 ≤ ?b1
    hence ?b1 < ?b0
      by simp
    hence bucket-end α T ?b1 ≤ bucket-start α T ?b0
      by (simp add: less-bucket-end-le-start)
        with s-index-upper-bound[OF s-perm-inv-elims(2,5)[OF assms(1)]
i'-assms(1)]
          s-index-lower-bound[OF s-perm-inv-elims(2,5)[OF assms(1)] i-less,
                              simplified]
          order.strict-implies-order[OF i'-assms(4)]
    show False
      using i'-assms(2) assms(4,6) by auto
  qed
  hence ?b0 = ?b1 ∨ ?b0 < ?b1
    by linarith
  moreover
  have ?b0 < ?b1 ⟹
       list-less-eq-ns
         (suffix T (Suc j))
         (suffix T (Suc (SA ! (B ! b))))
  proof −
    assume ?b0 < ?b1
    hence T ! (Suc j) < T ! (Suc (SA ! (B ! b)))
      using assms(1) s-perm-inv-elims(8) strict-mono-less by blast
    then show ?thesis
      by (metis i'-assms(1,2,4) assms(1,4−6) s-perm-inv-def
                leD s-seen-invD(1) list-less-eq-ns-linear
                suffix-cons-Suc list-less-eq-ns-cons)
  qed
  moreover
  have ?b0 = ?b1 ⟹
        list-less-eq-ns
          (suffix T (Suc j))
          (suffix T (Suc (SA ! (B ! b))))
  proof −
    assume ?b0 = ?b1
    with s-index-upper-bound
          [OF s-perm-inv-elims(2,5)[OF assms(1)]
        ‹i' < -›] i'-assms(4)
    have i' < bucket-end α T ?b0
      by (simp add: ‹SA ! i' = Suc (SA ! (B ! b))›)

    have suffix-type T (SA ! i) = S-type ∨
```

```
        suffix-type T (SA ! i) = L-type
    using SL-types.exhaust by blast
  moreover
  have suffix-type T (SA ! i) = S-type ⟹ ?thesis
  proof −
    assume suffix-type T (SA ! i) = S-type
    with s-seen-invD(3)
        [OF s-perm-inv-elims(5)[OF assms(1)] i-less,
         simplified]
    have in-s-current-bucket α T B ?b0 i
      by (simp add: assms(4) assms(6))
    hence B ! ?b0 ≤ i
      using in-s-current-bucket-def by blast
    hence ∃ m. B ! ?b0 + m = i
      using less-eqE by blast
    then obtain m0 where m0-assm:
      B ! ?b0 + m0 = i
      by blast

    from ‹B ! ?b0 ≤ i› i'-assms(4)
    have ∃ m. B ! ?b0 + m = i'
      by presburger
    then obtain m1 where m1-assm:
      B ! ?b0 + m1 = i'
      by blast
    hence B ! ?b0 + m0 ≤ B ! ?b0 + m1
      by (simp add: m0-assm i'-assms(4) dual-order.order-iff-strict)
    hence m0 ≤ m1
      using add-le-imp-le-left by blast

    have (list-slice SA (B ! ?b0) (bucket-end α T ?b0)) ! m0 =
        Suc j
      using m0-assm i-less
          ‹in-s-current-bucket α T B ?b0 i› assms(4,6)
          in-s-current-bucketD(3)
      by (metis ‹B ! α (T ! Suc j) ≤ i› diff-add-inverse list-slice-nth)
    moreover
    have (list-slice SA (B ! ?b0) (bucket-end α T ?b0)) ! m1 =
        Suc (SA ! (B ! b))
      using m1-assm i'-assms
          ‹i' < bucket-end α T ?b0›
      by (metis diff-add-inverse le-add1 list-slice-nth)
    moreover
    have length (list-slice SA (B ! ?b0) (bucket-end α T ?b0)) =
        (bucket-end α T ?b0) − B ! ?b0
        by (metis assms(1) bucket-end-le-length length-list-slice min-def
s-perm-inv-def)
    with ‹B ! ?b0 + m1 = i'›
        ‹i' < bucket-end α T ?b0›
```

```
      have m1 < length (list-slice SA (B ! ?b0) (bucket-end α T ?b0))
        by linarith
      ultimately
      show ?thesis
        using ordlistns.sorted-nth-mono
              [OF s-sorted-invD[OF assms(3) ‹?b0 ≤ α (Max -)›]
                  ‹m0 ≤ m1›]
             ‹m0 ≤ m1›
        by auto
    qed
    moreover
    have suffix-type T (SA ! i) = L-type ⟹ ?thesis
    proof −
      assume suffix-type T (SA ! i) = L-type
      with s-seen-invD(2)
           [OF s-perm-inv-elims(5)[OF assms(1)] i-less,
            simplified]
      have in-l-bucket α T ?b0 i
        by (simp add: assms(4) assms(6))
      hence bucket-start α T ?b0 ≤ i
        by (simp add: in-l-bucket-def)
      hence ∃ m. bucket-start α T ?b0 + m = i
        using less-eqE by blast
      then obtain m0 where start-plus-m0-eq:
        bucket-start α T ?b0 + m0 = i
        by blast

      have suffix-type T (SA ! i') = L-type ∨
           suffix-type T (SA ! i') = S-type
        using SL-types.exhaust by blast
      moreover
      have suffix-type T (SA ! i') = S-type ⟹ ?thesis
      proof −
        assume suffix-type T (SA ! i') = S-type

        have SA ! i < length T
          by (meson ‹i < length SA› assms(1) order-refl s-perm-inv-elims(5)
s-seen-invD(1))

        have SA ! i' < length T
         by (simp add: ‹suffix-type T (SA ! i') = S-type› suffix-type-s-bound)

        from ‹?b0 = ?b1›
        have T ! (Suc j) = T ! Suc (SA ! (B ! b))
          using assms(1) s-perm-inv-def strict-mono-eq by blast
        hence hd (suffix T (SA ! i')) = hd (suffix T (SA ! i))
          by (metis assms(4,6) list.sel(1) suffix-cons-Suc
                ‹SA ! i < length T› ‹SA ! i' < length T›
                ‹SA ! i' = Suc (SA ! (B ! b))›)
```

```
  with l-less-than-s-type
       [OF s-perm-inv-elims(13)[OF assms(1)]
           ‹SA ! i' < length T›
           ‹SA ! i < length T› -
           ‹suffix-type T (SA ! i') = -›
           ‹suffix-type T (SA ! i) = -›]
  have list-less-ns (suffix T (SA ! i)) (suffix T (SA ! i')).
  then show ?thesis
    by (simp add: ‹SA ! i' = Suc (SA ! (B ! b))› assms(4,6))
qed
moreover
have suffix-type T (SA ! i') = L-type ⟹ ?thesis
proof −
  assume suffix-type T (SA ! i') = L-type
  with s-seen-invD(2)
       [OF s-perm-inv-elims(5)[OF assms(1)]
           ‹i' < length SA›,
        simplified
          ‹?b0 = ?b1›[symmetric]
          ‹SA ! i' = Suc (SA ! (B ! b))›]
          ‹SA ! i' = Suc (SA ! (B ! b))›
  have in-l-bucket α T ?b0 i'
    by (simp add: ‹i < i'› dual-order.order-iff-strict)
  hence i'-le-end: i' < l-bucket-end α T ?b0
    by (simp add: in-l-bucket-def)
  hence ∃ m. bucket-start α T ?b0 + m = i'
    by (metis ‹in-l-bucket α T ?b0 i'› in-l-bucket-def less-eqE)
  then obtain m1 where start-plus-m1-eq:
    bucket-start α T ?b0 + m1 = i'
    by blast

  let ?zs =
      list-slice SA
        (bucket-start α T ?b0)
        (l-bucket-end α T ?b0)

  have ?zs ! m0 = Suc j
    by (metis ‹bucket-start α T (α (T ! Suc j)) ≤ i›
              assms(4,6)  i'-assms(4) i'-le-end
              diff-add-inverse dual-order.order-iff-strict
              i-less list-slice-nth order.strict-trans1
              start-plus-m0-eq)
  moreover
  have ?zs ! m1 = Suc (SA ! (B ! b))
    using list-slice-nth dual-order.order-iff-strict i'-assms(4)
          le-trans [OF ‹bucket-start α T (α (T ! Suc j)) ≤ i›]
          ‹SA ! i' = Suc (SA ! (B ! b))›
          ‹bucket-start α T ?b0 + m1 = i'›
          ‹i' < l-bucket-end α T ?b0›
```

```
            ‹i' < length SA›
          by (metis diff-add-inverse)
        moreover
        have m0 ≤ m1
          using start-plus-m0-eq start-plus-m1-eq i'-assms(4)
          by linarith
        moreover
        have length ?zs = l-bucket-end α T ?b0 − bucket-start α T ?b0
          by (metis ‹?b0 ≤ α (Max (set T))› assms(1)
                    add-diff-cancel-left' distinct-card
                    l-bucket-end-def l-bucket-size-def
                    l-types-init-def s-perm-inv-def
                    l-types-init-maintained)
        hence m1 < length ?zs
          using start-plus-m1-eq i'-le-end by linarith
        moreover
        from s-sorted-pre-maintained
             [OF s-perm-inv-elims(2,4,10,11)[OF assms(1)]
                 assms(2)]
        have s-sorted-pre α T SA .
        ultimately show ?thesis
          using ordlistns.sorted-nth-mono
                [OF s-sorted-preD
                    [of α T SA,
                     OF - ‹?b0 ≤ α (Max -)›],
                 of m0 m1]
          by (simp add: ‹m1 < length ?zs› le-less-trans
                        ‹s-sorted-pre α T SA›)
      qed
      ultimately show ?thesis
        by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately
  have list-less-eq-ns
         (suffix T (Suc j))
         (suffix T (Suc (SA ! (B ! b))))
    by blast
  with S0 S1 ‹T ! j = T ! (?ys ! 0)›
  show ?thesis
    by (simp add: list-less-eq-ns-cons)
qed
ultimately
show list-less-eq-ns (suffix T j) (map (suffix T) ?ys ! 0)
  by simp
qed
ultimately show ?thesis
```

```
      using A by fastforce
    qed
    ultimately show ?thesis
      by simp
  qed
  moreover
  have b ≠ b' ⟹ ordlistns.sorted (map (suffix T) ?xs)
  proof −
    assume b ≠ b'
    with s-sorted-maintained-unchanged-step[OF assms(1,4−) ‹b' ≤ -›]
         s-sorted-invD[OF assms(3) ‹b' ≤ -›]
    show ?thesis
      using ordlistns.sorted-map by blast
  qed
  ultimately show ordlistns.sorted (map (suffix T) ?xs)
    by blast
qed

lemma s-prefix-sorted-inv-maintained-step:
  assumes s-perm-inv α T B SA0 SA i
  and     s-prefix-sorted-pre α T SA0
  and     s-prefix-sorted-inv α T B SA
  and     i = Suc n
  and     Suc n < length SA
  and     SA ! Suc n = Suc j
  and     suffix-type T j = S-type
  and     b = α (T ! j)
  and     k = B ! b − Suc 0
shows s-prefix-sorted-inv α T (B[b := k]) (SA[k := j])
  unfolding s-prefix-sorted-inv-def
proof (safe)
  fix b'
  assume b' ≤ α (Max (set T))
  let ?xs = list-slice (SA[k := j]) (B[b := k] ! b') (bucket-end α T b')

  have bucket-end α T b ≤ length T
    using bucket-end-le-length by blast
  moreover
  have B ! b ≤ bucket-end α T b
   using assms(1,7,8) s-bucket-ptr-upper-bound s-perm-inv-elims(2,8) strict-mono-less-eq
         suffix-type-s-bound by fastforce

  ultimately have k < length T
    using assms(1,9) s-perm-inv-elims(15) by fastforce
  hence k < length SA
    by (metis assms(1) s-perm-inv-def)

  from s-bucket-ptr-strict-lower-bound
       [OF s-perm-inv-elims(1−6,8,10−14)[OF assms(1)] assms(4−8)]
```

```
  have s-bucket-start α T b < B ! b .
  hence k < B ! b
    using assms(9) diff-less gr-implies-not-zero by blast

  have s-bucket-start α T b ≤ k
    using assms s-bucket-ptr-strict-lower-bound s-perm-inv-def by fastforce
  hence bucket-start α T b ≤ k
    using bucket-start-le-s-bucket-start le-trans by blast
  hence b ≤ α (Max (set T))
    by (metis ‹k < length SA› assms(1) bucket-end-Max dual-order.trans
          leD leI  s-perm-inv-elims(8,11) less-bucket-end-le-start)

  have b = b' ∨ b ≠ b'
    by blast
  moreover
  have b = b' ⟹ ordlistns.sorted (map (lms-slice T) ?xs)
  proof −
    assume b = b'
    hence B[b := k] ! b' = k
      by (meson ‹b' ≤ α (Max (set T))› assms(1) le-less-trans
            nth-list-update-eq s-perm-inv-elims(9))

    have SA[k := j] ! k = j
      by (simp add: ‹k < length SA›)

    from list-slice-update-unchanged-1[OF ‹k < B ! b›]
        ‹k < B ! b› ‹SA[k := j] ! k = j› ‹B[b := k] ! b' = k›
        ‹B ! b ≤ bucket-end α T b›
        ‹b = b'› ‹k < length SA›
    have ?xs = j # list-slice SA (B ! b) (bucket-end α T b)
      by (metis Suc-pred assms(9) length-list-update not-le
            less-nat-zero-code list-slice-Suc less-le-trans)
    moreover
    have ordlistns.sorted
            (map (lms-slice T)
                (j # list-slice SA (B ! b)
                (bucket-end α T b)))
    proof −
      let ?ys = list-slice SA (B ! b) (bucket-end α T b)

      have A: map (lms-slice T) (j # ?ys) = (lms-slice T j) # map (lms-slice T)
?ys
        by simp

      from s-prefix-sorted-invD[OF assms(3) ‹b ≤ -›]
      have B: ordlistns.sorted (map (lms-slice T) ?ys) .

      have ?ys = [] ∨ ?ys ≠ []
        by blast
```

```
    hence map (lms-slice T) ?ys = [] ∨ map (lms-slice T) ?ys ≠ []
      by simp
    moreover
    have map (lms-slice T) ?ys = [] ⟹ ?thesis
      using ordlistns.sorted-cons-nil by fastforce
    moreover
    have map (lms-slice T) ?ys ≠ [] ⟹
        ordlistns.sorted ((lms-slice T j) # map (lms-slice T) ?ys)
    proof (rule ordlistns.sorted-consI[OF - B])
      assume map (lms-slice T) ?ys ≠ []
      hence map (lms-slice T) ?ys ! 0 = lms-slice T (?ys ! 0)
        by (metis length-greater-0-conv list.simps(8) nth-map)
      moreover
      have list-less-eq-ns (lms-slice T j) (lms-slice T (?ys ! 0))
      proof −
        have ?ys ! 0 ∈ s-bucket α T b
          by (metis ‹b = b'› ‹b' ≤ α (Max (set T))› ‹map (lms-slice T) ?ys ≠ []›
assms(1)
              length-greater-0-conv length-map nth-mem s-locations-inv-in-list-slice
                s-perm-inv-elims(3))
        hence α (T ! (?ys ! 0)) = b suffix-type T (?ys ! 0) = S-type
          by (simp add: s-bucket-def bucket-def)+
        hence T ! j = T ! (?ys ! 0)
          using assms(1,8) s-perm-inv-elims(8) strict-mono-eq by fastforce

        have ?ys ! 0 = SA ! (B ! b)
          using ‹map (lms-slice T) ?ys ≠ []› nth-list-slice by fastforce

        have b ≠ 0
          by (metis ‹s-bucket-start α T b < B ! b› assms(1)
              gr-implies-not-zero s-bucket-ptr-0 s-perm-inv-elims(2))

        have in-s-current-bucket α T B b (B ! b)
          using ‹b = b'› ‹b' ≤ α (Max (set T))›
              ‹map (lms-slice T) ?ys ≠ []›
              in-s-current-bucket-def
          by (metis ‹B ! b ≤ bucket-end α T b›
                dual-order.order-iff-strict list.map-disc-iff
                list-slice-n-n)
          with s-pred-invD
              [OF s-perm-inv-elims(6)[OF assms(1)] - ‹b ≠ 0›,
               of B ! b]
        obtain i' where
          i' < length SA
          SA ! i' = Suc (SA ! (B ! b))
          B ! b < i'
          i < i'
          by blast
```

```
        let ?b0 = α (T ! (Suc j))
        and ?b1 = α (T ! (Suc (SA ! (B ! b))))

        have i < length SA
          by (simp add: assms(4-5))

        have ?b0 ≤ α (Max (set T))
          by (metis Max-greD Suc-leI assms(1,4-6) lessI s-perm-inv-elims(5,8)
s-seen-invD(1)
              strict-mono-leD)

        have ?b1 ≤ α (Max (set T))
          by (metis Max-greD ‹SA ! i' = Suc (SA ! (B ! b))› ‹i < i'› ‹i' < length
SA› assms(1)
            less-imp-le-nat s-perm-inv-elims(5) s-perm-inv-elims(8) s-seen-invD(1)
              strict-mono-leD)

        have S0: lms-slice T j = T ! j # lms-slice T (Suc j)
          using assms(7) lms-slice-cons suffix-type-s-bound by blast

        have S1:
          lms-slice T (?ys ! 0) = T ! (?ys ! 0) # lms-slice T (Suc (SA ! (B ! b)))
            using ‹?ys ! 0 = SA ! (B ! b)› ‹suffix-type T (?ys ! 0) = S-type›
lms-slice-cons
              suffix-type-s-bound by auto

        have ?b0 ≤ ?b1
        proof(rule ccontr)
          assume ¬?b0 ≤ ?b1
          hence ?b1 < ?b0
            by simp
          hence bucket-end α T ?b1 ≤ bucket-start α T ?b0
            by (simp add: less-bucket-end-le-start)
         with s-index-upper-bound[OF s-perm-inv-elims(2,5)[OF assms(1)] ‹i' <
length SA›]
              s-index-lower-bound[OF s-perm-inv-elims(2,5)[OF assms(1)] ‹i <
length SA›,
              simplified]
            order.strict-implies-order[OF ‹i < i'›]
          show False
            using ‹SA ! i' = Suc (SA ! (B ! b))› assms(4,6) by auto
        qed
        hence ?b0 = ?b1 ∨ ?b0 < ?b1
          by linarith
        moreover
        have
          ?b0 < ?b1 ⟹
          list-less-eq-ns (lms-slice T (Suc j)) (lms-slice T (Suc (SA ! (B ! b))))
        proof −
```

```
          assume ?b0 < ?b1
          hence T ! (Suc j) < T ! (Suc (SA ! (B ! b)))
            using assms(1) s-perm-inv-elims(8) strict-mono-less by blast
          moreover
          have Suc j < length T
          using ‹i < length SA› assms(1,4,6) s-perm-inv-elims(5) s-seen-invD(1)
by fastforce
          hence ∃ as. lms-slice T (Suc j) = T ! (Suc j) # as
            by (metis dual-order.strict-trans abs-find-next-lms-lower-bound-1 lessI
list-slice-Suc
                  lms-slice-def)
          then obtain as where
            lms-slice T (Suc j) = T ! (Suc j) # as
            by blast
          moreover
          have Suc (SA ! (B ! b)) < length T
           using ‹SA ! i' = Suc (SA ! (B ! b))› ‹i < i'› ‹i' < length SA› assms(1)
                s-perm-inv-elims(5) s-seen-invD(1) by fastforce
          hence ∃ bs. lms-slice T (Suc (SA ! (B ! b))) = T ! (Suc (SA ! (B ! b)))
# bs
            by (metis abs-find-next-lms-lower-bound-1 less-Suc-eq
                  list-slice-Suc lms-slice-def)
          then obtain bs where
            lms-slice T (Suc (SA ! (B ! b))) = T ! (Suc (SA ! (B ! b))) # bs
            by blast
          ultimately show ?thesis
            using list-less-eq-ns-cons by fastforce
        qed
        moreover
        have
          ?b0 = ?b1 ⟹
          list-less-eq-ns (lms-slice T (Suc j)) (lms-slice T (Suc (SA ! (B ! b))))
        proof −
          assume ?b0 = ?b1
         with s-index-upper-bound[OF s-perm-inv-elims(2,5)[OF assms(1)] ‹i' <
-›] ‹i < i'›
          have i' < bucket-end α T ?b0
            by (simp add: ‹SA ! i' = Suc (SA ! (B ! b))›)

          have suffix-type T (SA ! i) = S-type ∨ suffix-type T (SA ! i) = L-type
            using SL-types.exhaust by blast
          moreover
          have suffix-type T (SA ! i) = S-type ⟹ ?thesis
          proof −
            assume suffix-type T (SA ! i) = S-type
           with s-seen-invD(3)[OF s-perm-inv-elims(5)[OF assms(1)] ‹i < length
SA›, simplified]
            have in-s-current-bucket α T B ?b0 i
              by (simp add: assms(4) assms(6))
```

```
            hence B ! ?b0 ≤ i
              using in-s-current-bucket-def by blast
            hence ∃m. B ! ?b0 + m = i
              using less-eqE by blast
            then obtain m0 where
              B ! ?b0 + m0 = i
              by blast

            from ‹B ! ?b0 ≤ i› ‹i < i'›
            have ∃m. B ! ?b0 + m = i'
              by presburger
            then obtain m1 where
              B ! ?b0 + m1 = i'
              by blast
            hence B ! ?b0 + m0 ≤ B ! ?b0 + m1
              by (simp add: ‹B ! α (T ! Suc j) + m0 = i›
                  ‹i < i'› dual-order.order-iff-strict)
            hence m0 ≤ m1
              using add-le-imp-le-left by blast

            have (list-slice SA (B ! ?b0) (bucket-end α T ?b0)) ! m0 = Suc j
              using ‹B ! α (T ! Suc j) + m0 = i› ‹i < length SA›
                    ‹in-s-current-bucket α T B ?b0 i› assms(4,6)
                    in-s-current-bucketD(3)
              by (metis ‹B ! α (T ! Suc j) ≤ i› diff-add-inverse list-slice-nth)
            moreover
            have (list-slice SA (B ! ?b0) (bucket-end α T ?b0)) ! m1 = Suc (SA !
(B ! b))
                using ‹B ! ?b0 + m1 = i'› ‹SA ! i' = Suc (SA ! (B ! b))› ‹i' <
bucket-end α T ?b0›
                    ‹i' < length SA›
              by (metis diff-add-inverse le-add1 list-slice-nth)
            moreover
            have length (list-slice SA (B ! ?b0) (bucket-end α T ?b0))
                = (bucket-end α T ?b0) − B ! ?b0
                by (metis assms(1) bucket-end-le-length length-list-slice min-def
s-perm-inv-def)
            with ‹B ! ?b0 + m1 = i'› ‹i' < bucket-end α T ?b0›
            have m1 < length (list-slice SA (B ! ?b0) (bucket-end α T ?b0))
              by linarith
            ultimately
            show ?thesis
              using ordlistns.sorted-nth-mono[OF
                    s-prefix-sorted-invD[OF assms(3) ‹?b0 ≤ α (Max -)›] ‹m0 ≤
m1›]
                    ‹m0 ≤ m1› by auto
          qed
          moreover
          have suffix-type T (SA ! i) = L-type ⟹ ?thesis
```

```
          proof −
            assume suffix-type T (SA ! i) = L-type
           with s-seen-invD(2)[OF s-perm-inv-elims(5)[OF assms(1)] ‹i < length
SA›, simplified]
            have in-l-bucket α T ?b0 i
              by (simp add: assms(4) assms(6))
            hence bucket-start α T ?b0 ≤ i
              by (simp add: in-l-bucket-def)
            hence ∃ m. bucket-start α T ?b0 + m = i
              using less-eqE by blast
            then obtain m0 where
              bucket-start α T ?b0 + m0 = i
              by blast

           have suffix-type T (SA ! i') = L-type ∨ suffix-type T (SA ! i') = S-type
              using SL-types.exhaust by blast
            moreover
            have suffix-type T (SA ! i') = S-type ⟹ ?thesis
            proof −
              assume suffix-type T (SA ! i') = S-type

              have SA ! i < length T
               by (meson ‹i < length SA› assms(1) order-refl s-perm-inv-elims(5)
s-seen-invD(1))

              have SA ! i' < length T
               by (simp add: ‹suffix-type T (SA ! i') = S-type› suffix-type-s-bound)

              from ‹?b0 = ?b1›
              have T ! (Suc j) = T ! Suc (SA ! (B ! b))
                using assms(1) s-perm-inv-def strict-mono-eq by blast
              hence hd (suffix T (SA ! i')) = hd (suffix T (SA ! i))
                by (metis ‹SA ! i < length T› ‹SA ! i' < length T› ‹SA ! i' = Suc
(SA ! (B ! b))›
                          assms(4) assms(6) list.sel(1) suffix-cons-Suc)
              hence list-less-ns (lms-slice T (SA ! i)) (lms-slice T (SA ! i'))
               using ‹SA ! i < length T› ‹SA ! i' < length T› ‹SA ! i' = Suc (SA
! (B ! b))›
                     ‹T ! Suc j = T ! Suc (SA ! (B ! b))› ‹suffix-type T (SA ! i') =
S-type›
                ‹suffix-type T (SA ! i) = L-type› assms(1,4,6) s-perm-inv-elims(13)
                     lms-slice-l-less-than-s-type by fastforce
              then show ?thesis
                by (simp add: ‹SA ! i' = Suc (SA ! (B ! b))› assms(4,6))
            qed
            moreover
            have suffix-type T (SA ! i') = L-type ⟹ ?thesis
            proof −
              assume suffix-type T (SA ! i') = L-type
```

```
                with s-seen-invD(2)[OF s-perm-inv-elims(5)[OF assms(1)] ‹i' <
length SA›,
                  simplified ‹?b0 = ?b1›[symmetric] ‹SA ! i' = Suc (SA ! (B ! b))›]
                  ‹SA ! i' = Suc (SA ! (B ! b))›
              have in-l-bucket α T ?b0 i'
                by (simp add: ‹i < i'› dual-order.order-iff-strict)
              hence i' < l-bucket-end α T ?b0
                by (simp add: in-l-bucket-def)
              hence ∃ m. bucket-start α T ?b0 + m = i'
                by (metis ‹in-l-bucket α T ?b0 i'› in-l-bucket-def less-eqE)
              then obtain m1 where
                bucket-start α T ?b0 + m1 = i'
                by blast

              let ?zs = list-slice SA (bucket-start α T ?b0) (l-bucket-end α T ?b0)

              have ?zs ! m0 = Suc j
                using ‹bucket-start α T ?b0 + m0 = i› ‹i < i'› ‹i < length SA›
                    ‹i' < l-bucket-end α T ?b0› assms(4,6)
                by (metis ‹bucket-start α T (α (T ! Suc j)) ≤ i›
                      diff-add-inverse dual-order.order-iff-strict
                      list-slice-nth order.strict-trans1)
              moreover
              have ?zs ! m1 = Suc (SA ! (B ! b))
                using ‹SA ! i' = Suc (SA ! (B ! b))› ‹bucket-start α T ?b0 + m1
= i'›
                    ‹i' < l-bucket-end α T ?b0› ‹i' < length SA›
                by (metis ‹in-l-bucket α T (α (T ! Suc j)) i'›
                      diff-add-inverse in-l-bucket-def list-slice-nth)
              moreover
              have m0 ≤ m1
                using ‹bucket-start α T ?b0 + m0 = i› ‹bucket-start α T ?b0 +
m1 = i'› ‹i < i'›
                by linarith
              moreover
              have length ?zs = l-bucket-end α T ?b0 − bucket-start α T ?b0
                by (metis ‹?b0 ≤ α (Max (set T))› add-diff-cancel-left' assms(1)
distinct-card
                 l-bucket-end-def l-bucket-size-def l-types-init-def l-types-init-maintained
                    s-perm-inv-def)
              hence m1 < length ?zs
                using ‹bucket-start α T ?b0 + m1 = i'› ‹i' < l-bucket-end α T
?b0› by linarith
              moreover
            from s-prefix-sorted-pre-maintained[OF s-perm-inv-elims(2,4,10,11)[OF
assms(1)]
                  assms(2)]
              have s-prefix-sorted-pre α T SA .
              ultimately show ?thesis
```

```
            using ordlistns.sorted-nth-mono[OF s-prefix-sorted-preD[of α T SA,
                OF - ‹?b0 ≤ α (Max -)›], of m0 m1]
              by (simp add: ‹m1 < length ?zs› ‹s-prefix-sorted-pre α T SA›
le-less-trans)
          qed
          ultimately show ?thesis
            by blast
        qed
        ultimately show ?thesis
          by blast
      qed
      ultimately have
        list-less-eq-ns (lms-slice T (Suc j)) (lms-slice T (Suc (SA ! (B ! b))))
        by blast
      with S0 S1 ‹T ! j = T ! (?ys ! 0)›
      show ?thesis
        by (simp add: list-less-eq-ns-cons)
    qed
    ultimately show list-less-eq-ns (lms-slice T j) (map (lms-slice T) ?ys ! 0)
      by simp
  qed
  ultimately show ?thesis
    using A by fastforce
 qed
 ultimately show ?thesis
   by simp
qed
moreover
have b ≠ b' ⟹ ordlistns.sorted (map (lms-slice T) ?xs)
proof −
  assume b ≠ b'
  with s-sorted-maintained-unchanged-step[OF assms(1,4−) ‹b' ≤ -›]
       s-prefix-sorted-invD[OF assms(3) ‹b' ≤ -›]
  show ?thesis
    using ordlistns.sorted-map by blast
qed
ultimately show ordlistns.sorted (map (lms-slice T) ?xs)
  by blast
qed

theorem abs-induce-s-sorted-step:
  assumes s-perm-inv α T B SA0 SA i
  and     s-sorted-pre α T SA0
  and     s-sorted-inv α T B SA
  and     abs-induce-s-step (B, SA, i) (α, T) = (B', SA', i')
shows  s-sorted-inv α T B' SA'
proof (rule abs-induce-s-step.elims[OF assms(4)];
       clarsimp simp: assms(3,4) Let-def not-less
             split: if-splits SL-types.splits nat.splits[where ?nat = i] nat.splits)
```

assume $B = B'$ $SA' = SA$
with *assms*(*3*)
show *s-sorted-inv* α T B' SA
by *simp*
next
assume $B = B'$ $SA' = SA$
with *assms*(*3*)
show *s-sorted-inv* α T B' SA
by *simp*
next
assume $B = B'$ $SA' = SA$
with *assms*(*3*)
show *s-sorted-inv* α T B' SA
by *simp*
next
assume $B = B'$ $SA' = SA$
with *assms*(*3*)
show *s-sorted-inv* α T B' SA
by *simp*
next
fix j

let *?b* = α (T ! j)
let *?k* = B ! *?b* − *Suc 0*

assume *A*: i = *Suc* i' *Suc* i' < *length SA SA* ! *Suc* i' = *Suc* j *suffix-type* T j = *S-type*
$B' = B[?b := ?k]$ $SA' = SA[?k := j]$

from *s-sorted-inv-maintained-step*[*OF assms*(*1*−*3*) *A*(*1*−*4*), *of ?b ?k*, *simplified*]
show *s-sorted-inv* α T ($B[?b := ?k]$) ($SA[?k := j]$) .
qed

corollary *abs-induce-s-sorted-step-alt*:
$\bigwedge a$. *s-sorted-inv-alt* α T *SA0* $a \Longrightarrow$ *s-sorted-inv-alt* α T *SA0* (*abs-induce-s-step* a (α, T))
proof −
fix a
assume *s-sorted-inv-alt* α T *SA0* a

have $\exists B$ *SA* i. $a = (B, SA, i)$
by (*meson prod-cases3*)
then obtain B *SA* i **where**
$a = (B, SA, i)$
by *blast*
with ‹*s-sorted-inv-alt* α T *SA0* a›
have *P*: *s-perm-inv* α T B *SA0 SA* i *s-sorted-pre* α T *SA0 s-sorted-inv* α T B *SA*
by *simp-all*

```
  from abs-induce-s-step-ex[of (B, SA, i) (α, T)]
  obtain B' SA' i' where
    Q: abs-induce-s-step (B, SA, i) (α, T) = (B', SA', i')
    by blast

 from abs-induce-s-sorted-step[OF P Q] abs-induce-s-perm-step[OF P(1) Q] ‹s-sorted-pre
- - -›
  show s-sorted-inv-alt α T SA0 (abs-induce-s-step a (α, T))
    using Q ‹a = (B, SA, i)› by auto
qed

theorem abs-induce-s-sorted-alt-maintained:
  assumes s-sorted-inv-alt α T SA0 (B, SA, length T)
  shows s-sorted-inv-alt α T SA0 (abs-induce-s-base α T B SA)
  unfolding abs-induce-s-base-def
  using repeat-maintain-inv
        [of s-sorted-inv-alt α T SA0 abs-induce-s-step (α, T), OF - assms(1)]
            abs-induce-s-sorted-step-alt
  by blast

corollary abs-induce-s-sorted-maintained:
  assumes abs-induce-s-base α T B SA = (B', SA', n)
  and     s-perm-inv α T B SA0 SA (length T)
  and     s-sorted-pre α T SA0
  and     s-sorted-inv α T B SA
shows s-sorted-inv α T B' SA'
  using assms abs-induce-s-sorted-alt-maintained by force

theorem abs-induce-s-prefix-sorted-step:
  assumes s-perm-inv α T B SA0 SA i
  and     s-prefix-sorted-pre α T SA0
  and     s-prefix-sorted-inv α T B SA
  and     abs-induce-s-step (B, SA, i) (α, T) = (B', SA', i')
shows s-prefix-sorted-inv α T B' SA'
proof (rule abs-induce-s-step.elims[OF assms(4)];
       clarsimp simp: assms(3,4) Let-def not-less
                 split: if-splits SL-types.splits nat.splits[where ?nat = i] nat.splits)
  assume B = B'
  with assms(3)
  show s-prefix-sorted-inv α T B' SA
    by simp
next
  assume B = B'
  with assms(3)
  show s-prefix-sorted-inv α T B' SA
    by simp
next
  assume B = B'
```

with *assms*(*3*)
show *s-prefix-sorted-inv* α *T B' SA*
by *simp*
next
assume $B = B'$
with *assms*(*3*)
show *s-prefix-sorted-inv* α *T B' SA*
by *simp*
next
fix *j*

let *?b* = α (*T* ! *j*)
let *?k* = *B* ! *?b* − *Suc 0*

assume *A*: *i* = *Suc i' Suc i'* < *length SA SA* ! *Suc i'* = *Suc j suffix-type T j* = *S-type*
B' = *B*[*?b* := *?k*] *SA'* = *SA*[*?k* := *j*]

from *s-prefix-sorted-inv-maintained-step*[*OF assms*(*1*−*3*) *A*(*1*−*4*), *of ?b ?k*, *simplified*]
show *s-prefix-sorted-inv* α *T* (*B*[*?b* := *?k*]) (*SA*[*?k* := *j*]) **.**
qed

corollary *abs-induce-s-prefix-sorted-step-alt*:
⋀*a*. *s-prefix-sorted-inv-alt* α *T SA0 a* ⟹
s-prefix-sorted-inv-alt α *T SA0* (*abs-induce-s-step a* (α, *T*))
proof −
fix *a*
assume *s-prefix-sorted-inv-alt* α *T SA0 a*

have ∃ *B SA i*. *a* = (*B*, *SA*, *i*)
by (*meson prod-cases3*)
then obtain *B SA i* **where**
a = (*B*, *SA*, *i*)
by *blast*
with ‹*s-prefix-sorted-inv-alt* α *T SA0 a*›
have *P*: *s-perm-inv* α *T B SA0 SA i s-prefix-sorted-pre* α *T SA0 s-prefix-sorted-inv* α *T B SA*
by *simp-all*

from *abs-induce-s-step-ex*[*of* (*B*, *SA*, *i*) (α, *T*)]
obtain *B' SA' i'* **where**
Q: *abs-induce-s-step* (*B*, *SA*, *i*) (α, *T*) = (*B'*, *SA'*, *i'*)
by *blast*

from *abs-induce-s-prefix-sorted-step*[*OF P Q*] *abs-induce-s-perm-step*[*OF P*(*1*) *Q*] ‹*s-prefix-sorted-pre - - -*›
show *s-prefix-sorted-inv-alt* α *T SA0* (*abs-induce-s-step a* (α, *T*))
using *Q* ‹*a* = (*B*, *SA*, *i*)› **by** *auto*

qed

theorem *abs-induce-s-prefix-sorted-alt-maintained*:
assumes *s-prefix-sorted-inv-alt* α *T SA0* (*B*, *SA*, *length T*)
shows *s-prefix-sorted-inv-alt* α *T SA0* (*abs-induce-s-base* α *T B SA*)
unfolding *abs-induce-s-base-def*
using *repeat-maintain-inv*[*of s-prefix-sorted-inv-alt* α *T SA0 abs-induce-s-step* (α, *T*),
OF - assms(*1*)]
abs-induce-s-prefix-sorted-step-alt
by *blast*

corollary *abs-induce-s-prefix-sorted-maintained*:
assumes *abs-induce-s-base* α *T B SA* = (*B'*, *SA'*, *n*)
and *s-perm-inv* α *T B SA0 SA* (*length T*)
and *s-prefix-sorted-pre* α *T SA0*
and *s-prefix-sorted-inv* α *T B SA*
shows *s-prefix-sorted-inv* α *T B' SA'*
using *assms abs-induce-s-prefix-sorted-alt-maintained* **by** *force*

theorem *s-sorted-bucket*:
assumes *s-perm-inv* α *T B SA0 SA 0*
and *s-sorted-pre* α *T SA0*
and *s-sorted-inv* α *T B SA*
and *b* $\leq$ α (*Max* (*set T*))
shows *ordlistns.sorted* (*map* (*suffix T*) (*list-slice SA* (*bucket-start* α *T b*) (*bucket-end* α *T b*)))
(**is** *ordlistns.sorted* (*map* (*suffix T*) *?xs*))
proof −
let *?ys* = *list-slice SA* (*bucket-start* α *T b*) (*l-bucket-end* α *T b*)
and *?zs* = *list-slice SA* (*s-bucket-start* α *T b*) (*bucket-end* α *T b*)

from *s-perm-inv-0-B-val*[*OF assms*(*1*,*4*)]
have *B* ! *b* = *s-bucket-start* α *T b* **.**

from *s-sorted-pre-maintained*[*OF s-perm-inv-elims*(*2*,*4*,*10*,*11*)[*OF assms*(*1*)] *assms*(*2*)]
have *s-sorted-pre* α *T SA* **.**

have *?xs* = *?ys* @ *?zs*
by (*metis bucket-start-le-s-bucket-start l-bucket-end-le-bucket-end list-slice-append*
s-bucket-start-eq-l-bucket-end)
hence *map* (*suffix T*) *?xs* = *map* (*suffix T*) *?ys* @ *map* (*suffix T*) *?zs*
by *simp*
moreover
from *s-sorted-preD*[*OF* ‹*s-sorted-pre - - SA*› ‹*b* $\leq$ -›]
have *ordlistns.sorted* (*map* (*suffix T*) *?ys*) **.**
moreover
from *s-sorted-invD*[*OF assms*(*3*,*4*), *simplified* ‹- = *s-bucket-start* α - -›]
have *ordlistns.sorted* (*map* (*suffix T*) *?zs*) **.**

```
  moreover
  have ∀ y ∈ set (map (suffix T) ?ys). ∀ z ∈ set (map (suffix T) ?zs). list-less-eq-ns
y z
  proof(safe)
    fix y z
    assume y ∈ set (map (suffix T) ?ys) z ∈ set (map (suffix T) ?zs)
    with in-set-mapD[of y suffix T ?ys]
         in-set-mapD[of z suffix T ?zs]
    obtain i j where
      y = suffix T i i ∈ set ?ys
      z = suffix T j j ∈ set ?zs
      by blast

   from l-types-initD(1)[OF l-types-init-maintained[OF s-perm-inv-elims(2,4,10−12)[OF
assms(1)]]
          ‹b ≤ -›]
         ‹i ∈ set ?ys›
    have i ∈ l-bucket α T b
      by blast
    hence suffix-type T i = L-type α (T ! i) = b i < length T
      by (simp add: l-bucket-def bucket-def)+
    moreover
    from s-bucket-eq-list-slice[OF s-perm-inv-elims(1,3,11)[OF assms(1)] ‹b ≤ -›
         ‹B ! b = s-bucket-start α - -›]
        ‹j ∈ set ?zs›
    have j ∈ s-bucket α T b
      by blast
    hence suffix-type T j = S-type α (T ! j) = b j < length T
      by (simp add: s-bucket-def bucket-def)+
    moreover
    have T ! i = T ! j
      using calculation(2,5) s-perm-inv-elims(8)[OF assms(1)] strict-mono-eq by
fastforce
    ultimately have list-less-eq-ns (suffix T i) (suffix T j)
      using l-less-than-s-type-suffix[of j T i] by simp
    then show list-less-eq-ns y z
      using ‹y = suffix T i› ‹z = suffix T j› by blast
  qed
  ultimately show ?thesis
    by (simp add: ordlistns.sorted-append)
qed

theorem abs-induce-s-base-sorted:
  assumes abs-induce-s-base α T B SA = (B', SA', n)
  and     s-perm-inv α T B SA0 SA (length T)
  and     s-sorted-pre α T SA0
  and     s-sorted-inv α T B SA
shows ordlistns.sorted (map (suffix T) SA')
proof (intro sorted-wrt-mapI)
```

fix *i j*
assume *i* < *j j* < *length SA′*
hence *i* < *length T j* < *length T*
by (*metis* (*no-types*, *lifting*) *abs-induce-s-perm-maintained*
assms(*1*,*2*) *order.strict-trans s-perm-inv-elims*(*11*))+

let *?goal* = *list-less-eq-ns* (*suffix T* (*SA′* ! *i*)) (*suffix T* (*SA′* ! *j*))

from *index-in-bucket-interval-gen*[*OF* ‹*i* < *length* -› *s-perm-inv-elims*(*8*)[*OF assms*(*2*)]]
obtain *b0* **where**
b0 ≤ *α* (*Max* (*set T*))
bucket-start α T b0 ≤ *i*
i < *bucket-end α T b0*
by *blast*

from *index-in-bucket-interval-gen*[*OF* ‹*j* < *length T*› *s-perm-inv-elims*(*8*)[*OF assms*(*2*)]]
obtain *b1* **where**
b1 ≤ *α* (*Max* (*set T*))
bucket-start α T b1 ≤ *j*
j < *bucket-end α T b1*
by *blast*

from *abs-induce-s-perm-maintained*[*OF assms*(*1*,*2*)]
have *s-perm-inv α T B′ SA0 SA′ n* **.**
moreover
have *n* = *0*
by (*metis Pair-inject assms*(*1*) *abs-induce-s-base-index*)
ultimately have *s-perm-inv α T B′ SA0 SA′ 0*
by *simp*

let *?xs* = *list-slice SA′* (*bucket-start α T b0*) (*bucket-end α T b0*)
and *?ys* = *list-slice SA′* (*bucket-start α T b1*) (*bucket-end α T b1*)

have *b0* ≤ *b1*
proof (*rule ccontr*)
assume ¬ *b0* ≤ *b1*
hence *b1* < *b0*
by (*simp add*: *not-le*)
hence *bucket-end α T b1* ≤ *bucket-start α T b0*
by (*simp add*: *less-bucket-end-le-start*)
with ‹*j* < *bucket-end α T b1*› ‹*bucket-start α T b0* ≤ *i*› ‹*i* < *j*›
show *False*
by *linarith*
qed
hence *b0* = *b1* ∨ *b0* < *b1*
by (*simp add*: *nat-less-le*)
moreover
have *b0* < *b1* ⟹ *?goal*

```
  proof −
    assume b0 < b1
    moreover
    from s-perm-inv-0-list-slice-bucket[OF ‹s-perm-inv - - - - - 0› ‹b0 ≤ α -›]
    have set ?xs = bucket α T b0 .
    hence SA' ! i ∈ bucket α T b0
      by (metis ‹bucket-start α T b0 ≤ i› ‹i < bucket-end α T b0› ‹s-perm-inv α
T B' SA0 SA' 0›
          bucket-end-le-length list-slice-nth-mem s-perm-inv-elims(11))
    hence α (T ! (SA' ! i)) = b0 SA' ! i < length T
      by (simp add: bucket-def)+
    moreover
    from s-perm-inv-0-list-slice-bucket[OF ‹s-perm-inv - - - - - 0› ‹b1 ≤ α -›]
    have set ?ys = bucket α T b1 .
    hence SA' ! j ∈ bucket α T b1
      by (metis ‹bucket-start α T b1 ≤ j› ‹j < bucket-end α T b1› ‹s-perm-inv α
T B' SA0 SA' 0›
          bucket-end-le-length list-slice-nth-mem s-perm-inv-elims(11))
    hence α (T ! (SA' ! j)) = b1 SA' ! j < length T
      by (simp add: bucket-def)+
    ultimately have T ! (SA' ! i) < T ! (SA' ! j)
      using assms(2) s-perm-inv-elims(8) strict-mono-less by blast
    then show ?thesis
    by (metis ‹SA' ! i < length T› ‹SA' ! j < length T› less-imp-le list-less-eq-ns-cons
neq-iff
          suffix-cons-Suc)
  qed
  moreover
  have b0 = b1 ⟹ ?goal
  proof −
    assume b0 = b1
    with ‹j < bucket-end α T b1›
    have j < bucket-end α T b0
      by simp

    have ∃ i'. i = bucket-start α T b0 + i'
      using ‹bucket-start α T b0 ≤ i› nat-le-iff-add by blast
    then obtain i' where
      i = bucket-start α T b0 + i'
      by blast

    have ∃ j'. j = bucket-start α T b0 + j'
      by (simp add: ‹b0 = b1› ‹bucket-start α T b1 ≤ j› le-Suc-ex)
    then obtain j' where
      j = bucket-start α T b0 + j'
      by blast
    with ‹i < j› ‹i = bucket-start α T b0 + i'›
    have i' ≤ j'
      by linarith
```

```
    have j' < length ?xs
      by (metis ‹b0 = b1› ‹j < bucket-end α T b1› ‹j = bucket-start α T b0 + j'›
                ‹s-perm-inv α T B' SA0 SA' n› add.commute bucket-end-le-length
length-list-slice
              less-diff-conv min-def s-perm-inv-elims(11))
    with ordlistns.sorted-nth-mono[OF s-sorted-bucket[OF ‹s-perm-inv - - - - - 0›
assms(3)
          abs-induce-s-sorted-maintained[OF assms] ‹b0 ≤ α -›] ‹i' ≤ j'›]
    have list-less-eq-ns (suffix T (?xs ! i')) (suffix T (?xs ! j'))
      using ‹i' ≤ j'› nth-map by auto
    moreover
    have SA' ! i = ?xs ! i'
      using ‹i < bucket-end α T b0› ‹i < length T› ‹i = bucket-start α T b0 + i'›
            ‹s-perm-inv α T B' SA0 SA' n› s-perm-inv-elims(11)
      by (metis ‹bucket-start α T b0 ≤ i› diff-add-inverse list-slice-nth)
     moreover
    have SA' ! j = ?xs ! j'
      using ‹j < bucket-end α T b0› ‹j < length SA'›
            ‹j = bucket-start α T b0 + j'›
      by (simp add: nth-list-slice)
    ultimately show ?goal
      by simp
  qed
  ultimately show ?goal
    by blast
qed

theorem s-prefix-sorted-bucket:
  assumes s-perm-inv α T B SA0 SA 0
  and     s-prefix-sorted-pre α T SA0
  and     s-prefix-sorted-inv α T B SA
  and     b ≤ α (Max (set T))
shows ordlistns.sorted (map (lms-slice T) (list-slice SA (bucket-start α T b) (bucket-end
α T b)))
      (is ordlistns.sorted (map (lms-slice T) ?xs))
proof −
  let ?ys = list-slice SA (bucket-start α T b) (l-bucket-end α T b)
  and ?zs = list-slice SA (s-bucket-start α T b) (bucket-end α T b)

  from s-perm-inv-0-B-val[OF assms(1,4)]
  have B ! b = s-bucket-start α T b .

 from s-prefix-sorted-pre-maintained[OF s-perm-inv-elims(2,4,10,11)[OF assms(1)]
assms(2)]
  have s-prefix-sorted-pre α T SA .

  have ?xs = ?ys @ ?zs
   by (metis bucket-start-le-s-bucket-start l-bucket-end-le-bucket-end list-slice-append
```

```
        s-bucket-start-eq-l-bucket-end)
  hence map (lms-slice T) ?xs = map (lms-slice T) ?ys @ map (lms-slice T) ?zs
    by simp
  moreover
  from s-prefix-sorted-preD[OF ‹s-prefix-sorted-pre - - SA› ‹b ≤ -›]
  have ordlistns.sorted (map (lms-slice T) ?ys) .
  moreover
  from s-prefix-sorted-invD[OF assms(3,4), simplified ‹- = s-bucket-start α - -›]
  have ordlistns.sorted (map (lms-slice T) ?zs) .
  moreover
  have ∀ y ∈ set (map (lms-slice T) ?ys). ∀ z ∈ set (map (lms-slice T) ?zs).
        list-less-eq-ns y z
  proof safe
    fix y z
    assume y ∈ set (map (lms-slice T) ?ys)
           z ∈ set (map (lms-slice T) ?zs)
    with in-set-mapD[of y lms-slice T ?ys]
         in-set-mapD[of z lms-slice T ?zs]
    obtain i j where
      y = lms-slice T i
      i ∈ set ?ys
      z = lms-slice T j
      j ∈ set ?zs
      by blast

    from l-types-initD(1)[OF l-types-init-maintained[OF s-perm-inv-elims(2,4,10−12)[OF
assms(1)]]
          ‹b ≤ -›]
         ‹i ∈ set ?ys›
    have i ∈ l-bucket α T b
      by blast
    hence suffix-type T i = L-type α (T ! i) = b i < length T
      by (simp add: l-bucket-def bucket-def)+
    moreover
    from s-bucket-eq-list-slice[OF s-perm-inv-elims(1,3,11)[OF assms(1)] ‹b ≤ -›
         ‹B ! b = s-bucket-start α - -›]
        ‹j ∈ set ?zs›
    have j ∈ s-bucket α T b
      by blast
    hence suffix-type T j = S-type α (T ! j) = b j < length T
      by (simp add: s-bucket-def bucket-def)+
    moreover
    have T ! i = T ! j
      using assms(1) calculation(2,5) s-perm-inv-elims(8) strict-mono-eq by fast-
force
    ultimately have list-less-eq-ns (lms-slice T i) (lms-slice T j)
      using lms-slice-l-less-than-s-type[of i T j] by simp
    then show list-less-eq-ns y z
      by (simp add: ‹y = lms-slice T i› ‹z = lms-slice T j›)
```

```
  qed
  ultimately show ?thesis
    by (simp add: ordlistns.sorted-append)
qed

theorem abs-induce-s-base-prefix-sorted:
  assumes abs-induce-s-base α T B SA = (B', SA', n)
  and     s-perm-inv α T B SA0 SA (length T)
  and     s-prefix-sorted-pre α T SA0
  and     s-prefix-sorted-inv α T B SA
shows ordlistns.sorted (map (lms-slice T) SA')
proof (intro sorted-wrt-mapI)
  fix i j
  assume i < j j < length SA'
  hence i < length T j < length T
    by (metis (no-types, lifting) abs-induce-s-perm-maintained
          assms(1,2) order.strict-trans s-perm-inv-elims(11))+

  let ?goal = list-less-eq-ns (lms-slice T (SA' ! i)) (lms-slice T (SA' ! j))

 from index-in-bucket-interval-gen[OF ‹i < length -› s-perm-inv-elims(8)[OF assms(2)]]
  obtain b0 where
    b0 ≤ α (Max (set T))
    bucket-start α T b0 ≤ i
    i < bucket-end α T b0
    by blast

  from index-in-bucket-interval-gen[OF ‹j < length T› s-perm-inv-elims(8)[OF
assms(2)]]
  obtain b1 where
    b1 ≤ α (Max (set T))
    bucket-start α T b1 ≤ j
    j < bucket-end α T b1
    by blast

  from abs-induce-s-perm-maintained[OF assms(1,2)]
  have s-perm-inv α T B' SA0 SA' n .
  moreover
  have n = 0
    by (metis Pair-inject assms(1) abs-induce-s-base-index)
  ultimately have s-perm-inv α T B' SA0 SA' 0
    by simp

  let ?xs = list-slice SA' (bucket-start α T b0) (bucket-end α T b0)
  and ?ys = list-slice SA' (bucket-start α T b1) (bucket-end α T b1)

  have b0 ≤ b1
  proof (rule ccontr)
    assume ¬b0 ≤ b1
```

```
    hence b1 < b0
      by (simp add: not-le)
    hence bucket-end α T b1 ≤ bucket-start α T b0
      by (simp add: less-bucket-end-le-start)
    with ‹j < bucket-end α T b1› ‹bucket-start α T b0 ≤ i› ‹i < j›
    show False
      by linarith
  qed
  hence b0 = b1 ∨ b0 < b1
    by (simp add: nat-less-le)
  moreover
  have b0 < b1 ⟹ ?goal
  proof −
    assume b0 < b1
    moreover
    from s-perm-inv-0-list-slice-bucket[OF ‹s-perm-inv - - - - - 0› ‹b0 ≤ α -›]
    have set ?xs = bucket α T b0 .
    hence SA' ! i ∈ bucket α T b0
      by (metis ‹bucket-start α T b0 ≤ i› ‹i < bucket-end α T b0› ‹s-perm-inv α
T B' SA0 SA' 0›
          bucket-end-le-length list-slice-nth-mem s-perm-inv-elims(11))
    hence α (T ! (SA' ! i)) = b0 SA' ! i < length T
      by (simp add: bucket-def)+
    moreover
    from s-perm-inv-0-list-slice-bucket[OF ‹s-perm-inv - - - - - 0› ‹b1 ≤ α -›]
    have set ?ys = bucket α T b1 .
    hence SA' ! j ∈ bucket α T b1
      by (metis ‹bucket-start α T b1 ≤ j› ‹j < bucket-end α T b1› ‹s-perm-inv α
T B' SA0 SA' 0›
          bucket-end-le-length list-slice-nth-mem s-perm-inv-elims(11))
    hence α (T ! (SA' ! j)) = b1 SA' ! j < length T
      by (simp add: bucket-def)+
    ultimately have T ! (SA' ! i) < T ! (SA' ! j)
      using assms(2) s-perm-inv-elims(8) strict-mono-less by blast
    then show ?thesis
    by (metis ‹SA' ! i < length T› ‹SA' ! j < length T› abs-find-next-lms-lower-bound-1

          less-SucI less-imp-le list-less-eq-ns-cons list-slice-Suc neq-iff lms-slice-def)
  qed
  moreover
  have b0 = b1 ⟹ ?goal
  proof −
    assume b0 = b1
    with ‹j < bucket-end α T b1›
    have j < bucket-end α T b0
      by simp

    have ∃ i'. i = bucket-start α T b0 + i'
      using ‹bucket-start α T b0 ≤ i› nat-le-iff-add by blast
```

```
    then obtain i' where
      i = bucket-start α T b0 + i'
      by blast

    have ∃j'. j = bucket-start α T b0 + j'
      by (simp add: ‹b0 = b1› ‹bucket-start α T b1 ≤ j› le-Suc-ex)
    then obtain j' where
      j = bucket-start α T b0 + j'
      by blast
    with ‹i < j› ‹i = bucket-start α T b0 + i'›
    have i' ≤ j'
      by linarith

    have j' < length ?xs
      by (metis ‹b0 = b1›
            ‹j < bucket-end α T b1›
            ‹j = bucket-start α T b0 + j'›
            ‹s-perm-inv α T B' SA0 SA' n›
            add.commute bucket-end-le-length length-list-slice
            less-diff-conv min-def s-perm-inv-elims(11))
    with ordlistns.sorted-nth-mono
          [OF s-prefix-sorted-bucket
              [OF ‹s-perm-inv - - - - - 0› assms(3)
                  abs-induce-s-prefix-sorted-maintained
                    [OF assms] ‹b0 ≤ α -›] ‹i' ≤ j'›]
    have list-less-eq-ns (lms-slice T (?xs ! i')) (lms-slice T (?xs ! j'))
      using ‹i' ≤ j'› nth-map by auto
    moreover
    have SA' ! i = ?xs ! i'
      using ‹i < bucket-end α T b0›
            ‹i < length T›
            ‹i = bucket-start α T b0 + i'›
            ‹s-perm-inv α T B' SA0 SA' n›
            ‹j' < length (list-slice SA'
                    (bucket-start α T b0)
                    (bucket-end α T b0))›
      by (metis ‹i' ≤ j'› nth-list-slice order.strict-trans1)
    moreover
    have SA' ! j = ?xs ! j'
      using ‹j < bucket-end α T b0› ‹j < length SA'›
            ‹j = bucket-start α T b0 + j'›
      by (simp add: nth-list-slice)
    ultimately show ?goal
      by simp
  qed
  ultimately show ?goal
    by blast
qed
```

84 Induce S Correctness Theorems

theorem *abs-induce-s-perm*:
assumes *s-perm-pre* α *T B SA* (*length T*)
shows *abs-induce-s* α *T B SA* <~~> [*0..<* *length T*]
proof −
from *s-perm-inv-established assms*[*simplified s-perm-pre-def*]
have *s-perm-inv* α *T B SA SA* (*length T*)
by *blast*
moreover
from *abs-induce-s-base-index*[*of* α *T B SA*]
obtain *B′ SA′* **where**
abs-induce-s-base α *T B SA* = (*B′*, *SA′*, *0*)
by *blast*
ultimately have *s-perm-inv* α *T B′ SA SA′ 0*
using *abs-induce-s-perm-maintained*[*of* α *T B SA B′ SA′ 0 SA*]
by *blast*
hence *SA′* <~~> [*0..<* *length T*]
using *abs-induce-s-base-perm*[*OF* ‹*abs-induce-s-base* α *T B SA* = (*B′*, *SA′*, *0*)›]
by *blast*
with ‹*abs-induce-s-base* α *T B SA* = (*B′*, *SA′*, *0*)›
show *?thesis*
by (*simp add*: *abs-induce-s-def*)
qed

— Used in SAIS algorithm as part of inducing the suffix ordering based on LMS
theorem *abs-induce-s-sorted*:
assumes *s-perm-pre* α *T B SA* (*length T*)
and *s-sorted-pre* α *T SA*
shows *ordlistns.sorted* (*map* (*suffix T*) (*abs-induce-s* α *T B SA*))
proof −
from *s-perm-inv-established assms*(*1*)[*simplified s-perm-pre-def*]
have *s-perm-inv* α *T B SA SA* (*length T*)
by *blast*
moreover
from *abs-induce-s-base-index*[*of* α *T B SA*]
obtain *B′ SA′* **where**
abs-induce-s-base α *T B SA* = (*B′*, *SA′*, *0*)
by *blast*
moreover
from *s-sorted-inv-established assms*(*1*)[*simplified s-perm-pre-def*]
have *s-sorted-inv* α *T B SA*
by *blast*
ultimately have *ordlistns.sorted* (*map* (*suffix T*) *SA′*)
using *abs-induce-s-base-sorted*[*OF - - assms*(*2*), *of B SA B′ SA′ 0*]
by *blast*
then show *?thesis*
by (*simp add*: ‹*abs-induce-s-base* α *T B SA* = (*B′*, *SA′*, *0*)› *abs-induce-s-def*)

qed

— Used in SAIS algorithm as part of inducing the prefix ordering based on LMS
theorem *abs-induce-s-prefix-sorted*:
assumes *s-perm-pre* α *T B SA* (*length T*)
and *s-prefix-sorted-pre* α *T SA*
shows *ordlistns.sorted* (*map* (*lms-slice T*) (*abs-induce-s* α *T B SA*))
proof −

from *s-perm-inv-established assms*(*1*)[*simplified s-perm-pre-def*]
have *s-perm-inv* α *T B SA SA* (*length T*)
by *blast*
moreover
from *abs-induce-s-base-index*[*of* α *T B SA*]
obtain B' SA' **where**
abs-induce-s-base α *T B SA* = (B', SA', *0*)
by *blast*
moreover
from *s-prefix-sorted-inv-established assms*(*1*)[*simplified s-perm-pre-def*]
have *s-prefix-sorted-inv* α *T B SA*
by *blast*
ultimately have *ordlistns.sorted* (*map* (*lms-slice T*) SA')
using *abs-induce-s-base-prefix-sorted*[*OF - - assms*(*2*), *of B SA* B' SA' *0*]
by *blast*
then show *?thesis*
by (*simp add*: ‹*abs-induce-s-base* α *T B SA* = (B', SA', *0*)› *abs-induce-s-def*)
qed

end
theory *Abs-Induce-Verification*
imports
Abs-Induce-L-Verification
Abs-Induce-S-Verification
Abs-Bucket-Insert-Verification
begin

85 Bucket Initialisation Properties

lemma *l-bucket-init-map-bucket-start*:
l-bucket-init α *T* (*map* (*bucket-start* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))])
unfolding *l-bucket-init-def*
by (*metis add.left-neutral diff-zero le-imp-less-Suc length-map length-upt lessI nth-map-upt*)

lemma *lms-bucket-init-map-bucket-end*:
lms-bucket-init α *T* (*map* (*bucket-end* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))])
unfolding *lms-bucket-init-def*
by (*metis add.left-neutral diff-zero le-imp-less-Suc length-map length-upt lessI nth-map-upt*)

lemma *s-bucket-init-map-bucket-end*:
s-bucket-init α T ((*map* (*bucket-end* α T) [*0*..<*Suc* (α (*Max* (*set* T)))])[*0* := *0*])
unfolding *s-bucket-init-def*
by (*metis* (*no-types*, *lifting*) *length-greater-0-conv length-list-update list.size*(*3*)
nth-list-update lms-bucket-init-def lms-bucket-init-map-bucket-end)

abbreviation *bucket-starts* α $T \equiv$ *map* (*bucket-start* α T) [*0*..<*Suc* (α (*Max* (*set* T)))]

abbreviation *bucket-ends* α $T \equiv$ *map* (*bucket-end* α T) [*0*..<*Suc* (α (*Max* (*set* T)))]

86 Bucket Insert Precondition

lemma *lms-pre-established*:
assumes *set LMS* = {*i*. *abs-is-lms* T *i*}
and *distinct LMS*
and *strict-mono* α
shows *lms-pre* α T (*bucket-ends* α T) (*replicate* (*length* T) (*length* T)) (*rev LMS*)
(**is** *lms-pre* α T *?B ?SA* (*rev LMS*))
proof −
from *lms-bucket-init-map-bucket-end*[*of* α T]
have *lms-bucket-init* α T *?B* **.**
then show *lms-pre* α T *?B ?SA0* (*rev LMS*)
by (*clarsimp simp*: *lms-pre-def* ‹*lms-bucket-init* α T *?B*› *assms*
simp del: *upt-Suc*)
qed

87 Induce L Precondition

lemma *l-perm-pre-established*:
assumes *valid-list T*
and *strict-mono* α
and *lms-pre* α T *B SA* (*rev LMS*)
shows *l-perm-pre* α T (*bucket-starts* α T) (*abs-bucket-insert* α T *B SA* (*rev LMS*))
(**is** *l-perm-pre* α T *?B ?SA*)
unfolding *l-perm-pre-def*
proof *safe*
show *lms-init* α T *?SA*
by (*metis assms*(*3*) *abs-bucket-insert-vals lms-init-def lms-vals-postD*)
next
show *l-init* α T *?SA*
unfolding *l-init-def*
proof (*intro allI impI*; *elim conjE*)
fix *b i*
assume $b \leq \alpha$ (*Max* (*set* T)) $i <$ *length ?SA bucket-start* α T $b \leq i$
$i <$ *l-bucket-end* α T b

```
    hence i < lms-bucket-start α T b
      using l-bucket-end-le-lms-bucket-start less-le-trans by blast
    with lms-unknowns-postD[OF abs-bucket-insert-unknowns[OF assms(3)] ‹b ≤
-›
        ‹bucket-start - - - ≤ -›]
    show ?SA ! i = length T .
  qed
next
  show s-init α T ?SA
    unfolding s-init-def
  proof (intro allI impI; elim conjE)
    fix b i
    assume b ≤ α (Max (set T)) i < length ?SA l-bucket-end α T b ≤ i
        i < lms-bucket-start α T b
    hence bucket-start α T b ≤ i
      by (simp add: l-bucket-end-def)
    with lms-unknowns-postD[OF abs-bucket-insert-unknowns[OF assms(3)] ‹b ≤
-› -
        ‹i < lms-bucket-start - - -›]
    show ?SA ! i = length T .
  qed
next
  from l-bucket-init-map-bucket-start[of α T]
  show l-bucket-init α T ?B .
next
  show length ?SA = length T
    by (metis assms(3) abs-bucket-insert-length lms-pre-elims(2))
qed (force simp: valid-list-not-nil[OF assms(1)] assms(2))+
```

88 Induce S Precondition

```
lemma s-perm-pre-established:
  assumes valid-list T
  and     strict-mono α
  and     α bot = 0
  and     Suc 0 < length T
  and     lms-pre α T B0 SA0 (rev LMS)
  and     SA1 = abs-bucket-insert α T B0 SA0 (rev LMS)
  and     l-perm-pre α T B1 SA1
shows s-perm-pre α T ((bucket-ends α T)[0 := 0]) (abs-induce-l α T B1 SA1)
(length T)
    (is s-perm-pre α T ?B ?SA ?n)
  unfolding s-perm-pre-def
proof (intro conjI)
  from s-bucket-init-map-bucket-end[of α T]
  show s-bucket-init α T ?B .
next
  from valid-list-length-ex[OF assms(1)]
  obtain n where
```

length T = Suc n
by *blast*
hence ∃ *m*. *length T = Suc (Suc m)*
using *assms(4) old.nat.exhaust* **by** *auto*
then obtain *m* **where**
length T = Suc (Suc m)
by *blast*

from *abs-bucket-insert-bot-first*[*OF assms(5,1)* ‹*length T = Suc (Suc m)*› *assms(3)*]
have *SA1 ! 0 = n*
using ‹*length T = Suc (Suc m)*› ‹*length T = Suc n*› *assms(6)* **by** *auto*

have *0 ≤ α (Max (set T))*
by *simp*
moreover
have *s-bucket-start α T 0 ≤ 0*
by (*simp add*: *assms(1−3) valid-list-s-bucket-start-0*)
moreover
have *0 < bucket-end α T 0*
by (*simp add*: *assms(1−3) valid-list-bucket-end-0*)
ultimately have *?SA ! 0 = SA1 ! 0*
using *abs-induce-l-unchanged*[*OF* ‹*l-perm-pre - - - ->*›, *of 0 0*]
by *blast*
with ‹*SA1 ! 0 = n*›
have *?SA ! 0 = n*
by *simp*
with ‹*length T = Suc n*›
show *s-type-init T ?SA*
using *s-type-init-def* **by** *blast*
next
show *length ?SA = length T*
by (*metis abs-induce-l-length assms(7) l-perm-pre-elims(7)*)
next
from *abs-induce-l-distinct-l-bucket*[*OF assms(7)*]
abs-induce-l-list-slice-l-bucket[*OF assms(7)*]
show *l-types-init α T ?SA*
by (*simp add*: *l-types-init-def*)
qed(*force simp*: *assms*)+

89 Permutation

lemma *abs-sa-induce-permutation*:
assumes *set LMS* = {*i*. *abs-is-lms T i*}
and *distinct LMS*
and *valid-list T*
and *strict-mono α*
and *α bot = 0*
and *Suc 0 < length T*
shows *abs-sa-induce α T LMS* <~~> [*0..< length T*]

proof −
let *?B0* = *map* (*bucket-end* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))] **and**
?B1 = *map* (*bucket-start* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))] **and**
?SA0 = *replicate* (*length T*) (*length T*)

let *?B2* = *?B0*[*0* := *0*]
let *?SA1* = *abs-bucket-insert* α *T ?B0 ?SA0* (*rev LMS*)
let *?SA2* = *abs-induce-l* α *T ?B1 ?SA1*
let *?SA3* = *abs-induce-s* α *T* (*?B0*[*0* := *0*]) *?SA2*

from *lms-pre-established*[*OF assms*(*1*,*2*,*4*)]
have *lms-pre* α *T ?B0 ?SA0* (*rev LMS*) **.**

have *l-perm-pre* α *T ?B1 ?SA1*
using ‹*lms-pre* α *T ?B0 ?SA0* (*rev LMS*)› *assms*(*3*,*4*) *l-perm-pre-established*
by *blast*

have *s-perm-pre* α *T ?B2 ?SA2* (*length T*)
using ‹*l-perm-pre* α *T ?B1 ?SA1*› ‹*lms-pre* α *T ?B0 ?SA0* (*rev LMS*)›
assms(*3*−*6*)
s-perm-pre-established **by** *blast*
with *abs-induce-s-perm*[*of* α *T ?B0*[*0* := *0*] *?SA2*]
have *?SA3* <~~> [*0*..< *length T*]
by *blast*
then show *?thesis*
by (*metis abs-sa-induce-def*)
qed

90 Sorting

lemma *abs-sa-induce-suffix-sorted*:
assumes *set LMS* = {*i*. *abs-is-lms T i*}
and *distinct LMS*
and *valid-list T*
and *strict-mono* α
and α *bot* = *0*
and *Suc 0* < *length T*
and *ordlistns.sorted* (*map* (*suffix T*) *LMS*)
shows *ordlistns.sorted* (*map* (*suffix T*) (*abs-sa-induce* α *T LMS*))
proof −
let *?B0* = *map* (*bucket-end* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))] **and**
?B1 = *map* (*bucket-start* α *T*) [*0*..<*Suc* (α (*Max* (*set T*)))] **and**
?SA0 = *replicate* (*length T*) (*length T*)

let *?B2* = *?B0*[*0* := *0*]
let *?SA1* = *abs-bucket-insert* α *T ?B0 ?SA0* (*rev LMS*)
let *?SA2* = *abs-induce-l* α *T ?B1 ?SA1*
let *?SA3* = *abs-induce-s* α *T* (*?B0*[*0* := *0*]) *?SA2*

```
  from lms-pre-established[OF assms(1,2,4)]
  have lms-pre α T ?B0 ?SA0 (rev LMS) .

  have P0:
    ∀ b ≤ α (Max (set T)).
      ordlistns.sorted (map (suffix T)
        (list-slice ?SA1 (lms-bucket-start α T b) (bucket-end α T b)))
  proof(intro allI impI)
    fix b
    assume b ≤ α (Max (set T))
    from lms-suffix-sorted-bucket[OF ‹lms-pre - - - - -› - ‹b ≤ -›] assms(7)
    show ordlistns.sorted (map (suffix T)
          (list-slice ?SA1 (lms-bucket-start α T b) (bucket-end α T b)))
      by simp
  qed

  have l-perm-pre α T ?B1 ?SA1
    using ‹lms-pre α T ?B0 ?SA0 (rev LMS)› assms(3,4) l-perm-pre-established
by blast
  moreover
  have l-suffix-sorted-pre α T ?SA1
    using P0 l-suffix-sorted-pre-def by blast
  ultimately have P1:
    ∀ b ≤ α (Max (set T)).
    ordlistns.sorted (map (suffix T) (list-slice ?SA2 (bucket-start α T b) (l-bucket-end
α T b)))
    using abs-induce-l-suffix-sorted-l-bucket by blast

  have s-perm-pre α T ?B2 ?SA2 (length T)
     using ‹l-perm-pre α T ?B1 ?SA1› ‹lms-pre α T ?B0 ?SA0 (rev LMS)›
assms(3−6)
        s-perm-pre-established by blast
  moreover
  have s-sorted-pre α T ?SA2
    using P1 s-sorted-pre-def by blast
  ultimately show ?thesis
    by (metis abs-induce-s-sorted abs-sa-induce-def)
qed

— Used in SAIS algorithm to induce the prefix ordering based on LMS

theorem abs-sa-induce-prefix-sorted:
  assumes set LMS = {i. abs-is-lms T i}
  and     distinct LMS
  and     valid-list T
  and     strict-mono α
  and     α bot = 0
  and     Suc 0 < length T
shows ordlistns.sorted (map (lms-slice T) (abs-sa-induce α T LMS))
```

```
proof −
  let ?B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))] and
      ?B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))] and
      ?SA0 = replicate (length T) (length T)

  let ?B2 = ?B0[0 := 0]
  let ?SA1 = abs-bucket-insert α T ?B0 ?SA0 (rev LMS)
  let ?SA2 = abs-induce-l α T ?B1 ?SA1
  let ?SA3 = abs-induce-s α T (?B0[0 := 0]) ?SA2

  from lms-pre-established[OF assms(1,2,4)]
  have lms-pre α T ?B0 ?SA0 (rev LMS) .

  have P0:
    ∀ b ≤ α (Max (set T)).
      ordlistns.sorted (map (lms-prefix T)
        (list-slice ?SA1 (lms-bucket-start α T b) (bucket-end α T b)))
  proof(intro allI impI)
    fix b
    assume b ≤ α (Max (set T))
    from lms-prefix-sorted-bucket[OF ‹lms-pre - - - - -› ‹b ≤ -›]
    show ordlistns.sorted (map (lms-prefix T)
          (list-slice ?SA1 (lms-bucket-start α T b) (bucket-end α T b)))
      by simp
  qed

  have l-perm-pre α T ?B1 ?SA1
    using ‹lms-pre α T ?B0 ?SA0 (rev LMS)› assms(3,4) l-perm-pre-established
by blast
  moreover
  have l-prefix-sorted-pre α T ?SA1
    using P0 l-prefix-sorted-pre-def by blast
  ultimately have P1:
    ∀ b ≤ α (Max (set T)).
      ordlistns.sorted (map (lms-prefix T)
        (list-slice ?SA2 (bucket-start α T b) (l-bucket-end α T b)))
    using abs-induce-l-prefix-sorted-l-bucket by blast

  have P2:
    ∀ b ≤ α (Max (set T)).
      ordlistns.sorted (map (lms-slice T)
        (list-slice ?SA2 (bucket-start α T b) (l-bucket-end α T b)))
  proof (intro allI impI sorted-wrt-mapI)
    fix b i j

    let ?xs = list-slice ?SA2 (bucket-start α T b) (l-bucket-end α T b) and
        ?R1 = (λx y. list-less-eq-ns (lms-prefix T x) (lms-prefix T y)) and
        ?R2 = (λx y. list-less-eq-ns (lms-slice T x) (lms-slice T y))
```

assume $b \leq \alpha$ $(Max\ (set\ T))$ $i < j$ $j < length\ ?xs$
with *P1*
have *ordlistns.sorted* $(map\ (lms\text{-}prefix\ T)\ ?xs)$
by *blast*
with *sorted-wrt-mapD*[*OF* - ‹$i < j$› ‹$j < length\ ?xs$›]
have *list-less-eq-ns* $(lms\text{-}prefix\ T\ (?xs\ !\ i))$ $(lms\text{-}prefix\ T\ (?xs\ !\ j))$
by *blast*
moreover
from *abs-induce-l-list-slice-l-bucket*[*OF* ‹*l-perm-pre* - - - ->› ‹$b \leq$ ->›]
have $?xs\ !\ i \in l\text{-}bucket\ \alpha\ T\ b$
using *Suc-lessD* ‹$i < j$› ‹$j < length\ ?xs$› *less-trans-Suc nth-mem* **by** *blast*
hence $suffix\text{-}type\ T\ (?xs\ !\ i) = L\text{-}type$
using *l-bucket-def bucket-def* **by** *blast*
hence $lms\text{-}prefix\ T\ (?xs\ !\ i) = lms\text{-}slice\ T\ (?xs\ !\ i)$
by (*metis SL-types.distinct*(*1*) *abs-is-lms-def not-lms-imp-next-eq-lms-prefix*)
moreover
from *abs-induce-l-list-slice-l-bucket*[*OF* ‹*l-perm-pre* - - - ->› ‹$b \leq$ ->›]
have $?xs\ !\ j \in l\text{-}bucket\ \alpha\ T\ b$
using ‹$j < length\ ?xs$› *less-trans-Suc nth-mem* **by** *blast*
hence $suffix\text{-}type\ T\ (?xs\ !\ j) = L\text{-}type$
using *l-bucket-def bucket-def* **by** *blast*
hence $lms\text{-}prefix\ T\ (?xs\ !\ j) = lms\text{-}slice\ T\ (?xs\ !\ j)$
by (*metis SL-types.distinct*(*1*) *abs-is-lms-def not-lms-imp-next-eq-lms-prefix*)
ultimately show $list\text{-}less\text{-}eq\text{-}ns\ (lms\text{-}slice\ T\ (?xs\ !\ i))\ (lms\text{-}slice\ T\ (?xs\ !\ j))$
by *order*
qed

have $s\text{-}perm\text{-}pre\ \alpha\ T\ ?B2\ ?SA2\ (length\ T)$
using ‹*l-perm-pre* α *T ?B1 ?SA1*› ‹*lms-pre* α *T ?B0 ?SA0* (*rev LMS*)› *assms*(*3−6*)
s-perm-pre-established **by** *blast*
moreover
have $s\text{-}prefix\text{-}sorted\text{-}pre\ \alpha\ T\ ?SA2$
using *P2 s-prefix-sorted-pre-def* **by** *blast*
ultimately show *?thesis*
by (*metis abs-induce-s-prefix-sorted abs-sa-induce-def*)
qed

end
theory *Abs-Extract-LMS-Verification*
imports *../abs−def/Abs-SAIS Abs-Induce-Verification*
begin

91 Extract LMS types Proofs

lemma *abs-extract-lms-correct*:
$xs <^{\sim\sim}> [0..<length\ T] \Longrightarrow$
$distinct\ (abs\text{-}extract\text{-}lms\ T\ xs) \wedge set\ (abs\text{-}extract\text{-}lms\ T\ xs) = \{i.\ abs\text{-}is\text{-}lms\ T\ i\}$

by (*metis comp-apply distinct-filter distinct-upt filter-set get-lms-correct*
get-lms-set-n-gre-length order.refl perm-distinct-iff perm-set-eq set-rev)

lemma *set-abs-extract-lms-eq-all-lms*:
set (*abs-extract-lms T* [*0*..<*length T*]) = {*i*. *abs-is-lms T i*}
using *abs-extract-lms-correct* **by** *blast*

lemma *distinct-abs-extract-lms*:
distinct (*abs-extract-lms T* [*0*..<*length T*])
using *abs-extract-lms-correct* **by** *blast*

lemma *filter-abs-sa-induce-eq-all-lms*:
⟦*set LMS* = {*i*. *abs-is-lms T i*}; *distinct LMS*; *valid-list T*; *strict-mono* α; α *bot* = *0*;
Suc 0 < *length T*⟧ ⟹
set (*abs-extract-lms T* (*abs-sa-induce* α *T LMS*)) = {*i*. *abs-is-lms T i*}
using *abs-extract-lms-correct abs-sa-induce-permutation* **by** *blast*

lemma *distinct-filter-abs-sa-induce*:
⟦*set LMS* = {*i*. *abs-is-lms T i*}; *distinct LMS*; *valid-list T*; *strict-mono* α; α *bot* = *0*;
Suc 0 < *length T*⟧ ⟹
distinct (*abs-extract-lms T* (*abs-sa-induce* α *T LMS*))
using *abs-extract-lms-correct abs-sa-induce-permutation* **by** *blast*

end
theory *Abs-Order-LMS-Verification*
imports *../abs−def/Abs-SAIS*
begin

92 Order LMS-types Proofs

lemma *abs-order-lms-eq-map-nth*:
order-lms LMS xs = *map* (*nth LMS*) *xs*
by (*induct xs*; *simp*)

theorem *distinct-abs-order-lms*:
⟦*xs* <~~> [*0*..<*length LMS*]; *distinct LMS*⟧ ⟹
distinct (*order-lms LMS xs*)
apply (*subst abs-order-lms-eq-map-nth*)
apply (*erule distinct-map-nth-perm*[*of - length LMS*]; *simp*)

done

theorem *abs-order-lms-eq-all-lms*:
⟦*xs* <~~> [*0*..<*length LMS*]; *set LMS* = *S*⟧ ⟹
set (*order-lms LMS xs*) = *S*
apply (*subst abs-order-lms-eq-map-nth*)
apply (*frule set-map-nth-perm-eq*)
apply *simp*
done

end
theory *Abs-Rename-LMS-Verification*
imports *../abs−def/Abs-SAIS*

begin

93 Rename Mapping Proofs

lemma *abs-rename-mapping′-length*:
length (*abs-rename-mapping′ T LMS names i*) = *length names*
by (*induct rule*: *abs-rename-mapping′.induct*[*of - T LMS names i*]; *simp*)

lemma *abs-rename-mapping-length*:
length (*abs-rename-mapping T LMS*) = *length T*
by (*clarsimp simp*: *abs-rename-mapping-def abs-rename-mapping′-length*)

lemma *rename-mapping′-unchanged*:
⟦*x* ∉ *set LMS*; *x* < *length names*⟧ ⟹
(*abs-rename-mapping′ T LMS names i*) ! *x* = *names* ! *x*
by (*induct rule*: *abs-rename-mapping′.induct*[*of - T LMS names i*]; *simp*)

lemma *rename-mapping′-lms*:
assumes *distinct LMS*
and *ordlistns.sorted* (*map* (*lms-slice T*) *LMS*)
and *i* ∈ *set LMS*
and *i* < *length names*
shows (*abs-rename-mapping′ T LMS names j*) ! *i* =
j + (*ordlistns.elem-rank* ((*lms-slice T*) ‘ *set LMS*) (*lms-slice T i*))
using *assms*
proof (*induct arbitrary*: *i rule*: *abs-rename-mapping′.induct*[*of - T LMS names j*])
case (*1 uu names uv*)
then show *?case*
by *force*
next
case (*2 uw x names j*)
note *A* = *this*
hence *x* = *i*

```
    by force
  hence lms-slice uw ‘ set [x] = {lms-slice uw i}
    using A(1) by auto
  hence ordlistns.elem-rank (lms-slice uw ‘ set [x]) (lms-slice uw i) = 0
    by (simp add: ordlistns.elem-rank-def elm-rank-def)
  then show ?case
    by (simp add: ‹x = i› A(4))
next
  case (3 T a b xs names j)
  note IH = this

  have A1: distinct (b # xs)
    using IH(3) by fastforce

  have A2: ordlistns.sorted (map (lms-slice T) (b # xs))
    using IH(4) by fastforce

  have A3: i < length (names[a := j])
    by (simp add: IH(6))

  have A4: a ∉ set (b # xs)
    using IH(3) by auto

  have A5: ordlistns.elem-rank (lms-slice T ‘ set (a # b # xs)) (lms-slice T a) =
0
    unfolding ordlistns.elem-rank-def elm-rank-def using IH(4) by auto

  note IH1 = IH(1)[OF - A1 A2 - A3]
  note IH2 = IH(2)[OF - A1 A2 - A3]

  have P: i ∈ set (b # xs) ∨ i = a
    using IH(5) by force

  have lms-slice T a = lms-slice T b ∨
       lms-slice T a ≠ lms-slice T b
    by blast
  then show ?case
  proof
    assume B: lms-slice T a = lms-slice T b
    hence C: abs-rename-mapping' T (a # b # xs) names j =
             abs-rename-mapping' T (b # xs) (names[a := j]) j
      by simp

    from IH1[OF B] C
    have i ∈ set (b # xs) ⟹ ?thesis
      by (simp add: B list.set-map)
    moreover
     from rename-mapping'-unchanged[OF A4, of names[a := j] T j, simplified
C[symmetric]] IH(6) A5
```

```
    have i = a ⟹ ?thesis
      by simp
    moreover
    note P
    ultimately show ?thesis
      by blast
  next
    assume B: lms-slice T a ≠ lms-slice T b
    hence C: abs-rename-mapping′ T (a # b # xs) names j =
            abs-rename-mapping′ T (b # xs) (names[a := j]) (Suc j)
      by simp

    have D: lms-slice T a ∉ lms-slice T ‘ set (b # xs)
    proof
      assume lms-slice T a ∈ lms-slice T ‘ set (b # xs)
      moreover
       from IH(4) ordlistns.sorted-simps(2)[of lms-slice T a map (lms-slice T) (b
# xs)]
      have ∀ y ∈ set (map (lms-slice T) (b # xs)). list-less-eq-ns (lms-slice T a) y
        by auto
      ultimately show False
        using A2 B by auto
    qed

    from rename-mapping′-unchanged[OF A4, of names[a := j] T Suc j, simplified
C[symmetric]]
    have i = a ⟹ ?thesis
      using A5 IH(6) by auto
    moreover
    have i ∈ set (b # xs) ⟹ ?thesis
    proof −
      assume i ∈ set (b # xs)
      with IH2[OF B, simplified C[symmetric]] C
      have abs-rename-mapping′ T (a # b # xs) names j ! i =
           j + Suc (ordlistns.elem-rank (lms-slice T ‘ set (b # xs)) (lms-slice T i))
        by linarith
      moreover
      have lms-slice T ‘ set (a # b # xs) =
           insert (lms-slice T a) (lms-slice T ‘ set (b # xs))
        by simp
      moreover
      have ordlistns.elem-rank (insert (lms-slice T a) (lms-slice T ‘ set (b # xs)))
            (lms-slice T i) =
           Suc (ordlistns.elem-rank (lms-slice T ‘ set (b # xs)) (lms-slice T i))
      proof (intro ordlistns.elem-rank-insert-min)
        from D
        show lms-slice T a ∉ lms-slice T ‘ set (b # xs) .
      next
        show finite (lms-slice T ‘ set (b # xs))
```

```
      by blast
    next
      show ∀ y∈lms-slice T ‘ set (b # xs). list-less-ns (lms-slice T a) y
        using D IH(4) by fastforce
    next
      show lms-slice T i ∈ lms-slice T ‘ set (b # xs)
        using ‹i ∈ set (b # xs)› by blast
    qed
    ultimately show ?thesis
      by presburger
  qed
  moreover
  note P
  ultimately show ?thesis
    by blast
 qed
qed

lemma abs-rename-mapping-lms:
 assumes distinct LMS
 and     ordlistns.sorted (map (lms-slice T) LMS)
 and     i ∈ set LMS
 and     i < length T
shows    (abs-rename-mapping T LMS) ! i =
         ordlistns.elem-rank ((lms-slice T) ‘ set LMS) (lms-slice T i)
 unfolding abs-rename-mapping-def
 using rename-mapping'-lms[where j = 0, simplified, OF assms(1−3)] assms(4)
       abs-rename-mapping-length[of T LMS]
 by auto


lemma abs-rename-mapping-lms-all:
 assumes distinct LMS
 and     ordlistns.sorted (map (lms-slice T) LMS)
 and     ∀ x ∈ set LMS. x < length T
shows ∀ x ∈ set LMS. (!) (abs-rename-mapping T LMS) x =
                  ordlistns.elem-rank (lms-slice T ‘ set LMS) (lms-slice T x)
 using assms(1) assms(2) assms(3) abs-rename-mapping-lms by blast

lemma map-abs-rename-mapping:
 assumes distinct LMS
 and     ordlistns.sorted (map (lms-slice T) LMS)
 and     ∀ x ∈ set LMS. x < length T
 and     set xs ⊆ set LMS
shows map ((!) (abs-rename-mapping T LMS)) xs =
      map (ordlistns.elem-rank (lms-slice T ‘ set LMS)) (map (lms-slice T) xs)
  using assms(1) assms(2) assms(3) assms(4) abs-rename-mapping-lms-all by
fastforce
```

94 Rename String Proofs

lemma *rename-list-length*:
  length (*rename-string xs names*) = *length xs*
  by (*induct xs*; *simp*)

theorem *rename-list-correct*:
  rename-string T names = *map* (λ*x*. *names* ! *x*) *T*
  by (*induct T*; *simp*)

corollary *rename-list-nth*:
  i < *length T* ⟹ (*rename-string T names*) ! *i* = *names* ! (*T* ! *i*)
  by (*simp add*: *rename-list-correct*)

end
theory *Abs-SAIS-Verification-With-Valid-Precondition*
  imports
    Abs-Induce-Verification
    Abs-Rename-LMS-Verification
    Abs-Extract-LMS-Verification
    Abs-Order-LMS-Verification
begin

95 SAIS General Helpers

termination *abs-sais*
  by (*relation measure* (λ*xs*. *length xs*))
    (*simp* (*no-asm-simp*)
        del: *List.list.size*(*4*)
        add: *rename-list-length length-filter-lms*)+

lemma *abs-sais-reduced-string*:
  assumes *LMS1* = *lms0-seq T*
  and *distinct LMS2*
  and *set LMS2* = {*i*. *abs-is-lms T i*}
  and *ordlistns.sorted* (*map* (*lms-slice T*) *LMS2*)
  and *names* = *abs-rename-mapping T LMS2*
  and *T'* = *rename-string LMS1 names*
shows *T'* = *lms-map T* (*lms0-suffix T*)
proof −
  let *?T'* = *lms0-map T*

  have *set-LMS1*: *set LMS1* = {*i*. *abs-is-lms T i*}
    using *assms*(*1*) *lms0-seq-has-all-lms* **by** *blast*

  have *distinct LMS1*

by (*simp add*: *assms*(*1*) *lms-seq-distinct*)

have $T' = map\ (\lambda x.\ names\ !\ x)\ LMS1$
using *rename-list-correct assms*(*6*) **by** *auto*

have $\forall x \in set\ LMS2.\ x < length\ T$
using *assms*(*3*) *abs-is-lms-imp-less-length* **by** *blast*
with *map-abs-rename-mapping*[*OF assms*(*2*,*4*), *simplified assms*(*3*),
of LMS1, *simplified* ‹*LMS1* = *lms0-seq T*›]
‹$T' = map\ (\lambda x.\ names\ !\ x)\ LMS1$› ‹*set LMS2* = {*i*. *abs-is-lms T i*}›
have $T' = map\ (ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ T))\ (map\ (lms\text{-}slice\ T)\ (lms0\text{-}seq\ T))$
using *assms*(*1*) *assms*(*5*) *set-LMS1* **by** *blast*
moreover
have $map\ (ordlistns.elem\text{-}rank\ (lms\text{-}substrs\ T))\ (map\ (lms\text{-}slice\ T)\ (lms0\text{-}seq\ T)) = ?T'$
unfolding *lms-map-def*
by (*metis* (*no-types*, *opaque-lifting*) *comp-apply lms-substr-seq-def lms-subtrs-seq-id-suffix*)
ultimately show $T' = ?T'$
by (*simp only*:)
qed

96 SAIS cases simplifications

lemma *abs-sais-distinct-simp*:
assumes $T = a\ \#\ b\ \#\ xs$
and $LMS0 = abs\text{-}extract\text{-}lms\ T\ [0..<length\ T]$
and $SA = abs\text{-}sa\text{-}induce\ id\ T\ LMS0$
and $LMS = abs\text{-}extract\text{-}lms\ T\ SA$
and $names = abs\text{-}rename\text{-}mapping\ T\ LMS$
and $T' = rename\text{-}string\ LMS0\ names$
and $distinct\ T'$
shows $abs\text{-}sais\ T = abs\text{-}sa\text{-}induce\ id\ T\ LMS$
proof −
let $?P = \lambda ys.\ abs\text{-}sais\ (a\ \#\ b\ \#\ xs) = ys$
from *subst*[*OF abs-sais.simps*(*3*), *of ?P a b xs*,
simplified Let-def assms(*1*−*6*)[*symmetric*] *assms*(*7*),
simplified]
show *?thesis*
by *simp*
qed

lemma *abs-sais-not-distinct-simp*:
assumes $T = a\ \#\ b\ \#\ xs$
and $LMS0 = abs\text{-}extract\text{-}lms\ T\ [0..<length\ T]$
and $SA = abs\text{-}sa\text{-}induce\ id\ T\ LMS0$
and $LMS = abs\text{-}extract\text{-}lms\ T\ SA$
and $names = abs\text{-}rename\text{-}mapping\ T\ LMS$
and $T' = rename\text{-}string\ LMS0\ names$

and *LMS1 = order-lms LMS0 (abs-sais T′)*
and *¬ distinct T′*
shows *abs-sais T = abs-sa-induce id T LMS1*
proof −
let *?P = λys. abs-sais (a # b # xs) = ys*
from *subst[OF abs-sais.simps(3), of ?P a b xs, simplified Let-def assms(1−7)[symmetric] assms(8), simplified]*
show *?thesis*
by *simp*
qed

97 SAIS returns a permutation

theorem *abs-sais-permutation*:
valid-list T ⟹ abs-sais T <~~> [0..<length T]
proof(*induct rule*: *abs-sais.induct*[*of* - *T*])
case *1*
then show *?case* **by** *simp*
next
case (*2 x*)
then show *?case* **by** *simp*
next
case (*3 a b xs*)
note *IH = this*
let *?T = a # b # xs*
have *T*: *?T = a # b # xs*
by (*simp only*:)
let *?LMS1 = abs-extract-lms ?T [0..<length ?T]*
have *LMS1*: *?LMS1 = abs-extract-lms ?T [0..<length ?T]*
by (*simp only*:)
let *?SA1 = abs-sa-induce id ?T ?LMS1*
have *SA1*: *?SA1 = abs-sa-induce id ?T ?LMS1*
by (*simp only*:)
let *?LMS2 = abs-extract-lms ?T ?SA1*
have *LMS2*: *?LMS2 = abs-extract-lms ?T ?SA1*
by (*simp only*:)
let *?names = abs-rename-mapping ?T ?LMS2*
have *names*: *?names = abs-rename-mapping ?T ?LMS2*
by (*simp only*:)
let *?T′ = rename-string ?LMS1 ?names*
have *T′*: *?T′ = rename-string ?LMS1 ?names*
by (*simp only*:)
let *?LMS3 = order-lms ?LMS1 (abs-sais ?T′)*
have *LMS3*: *?LMS3 = order-lms ?LMS1 (abs-sais ?T′)*
by (*simp only*:)

from *IH(1)*[*OF T LMS1 SA1 LMS2 names T′*]
have *IH′*: ⟦*¬distinct ?T′*; *valid-list ?T′*⟧ ⟹ *abs-sais ?T′ <~~> [0..<length ?T′]*

by *assumption*

have *distinct ?LMS1*
using *distinct-abs-extract-lms*
by *fastforce*

have *set ?LMS1 = {i. abs-is-lms ?T i}*
using *set-abs-extract-lms-eq-all-lms*
by (*metis comp-apply*)

have *id*: *strict-mono* (*id* :: *nat* ⇒ *nat*) (*id* :: *nat* ⇒ *nat*) *bot = 0*
by (*simp add*: *strict-mono-def bot-nat-def*)+

have *len*: *Suc 0 < length ?T*
by *simp*

from *distinct-filter-abs-sa-induce*
[*OF* ‹*set ?LMS1 = {i. abs-is-lms ?T i}*› ‹*distinct ?LMS1*› *IH*(*2*) *id len*]
have *distinct-LMS2*: *distinct ?LMS2*
by (*metis comp-apply*)

from *filter-abs-sa-induce-eq-all-lms*
[*OF* ‹*set ?LMS1 = {i. abs-is-lms ?T i}*› ‹*distinct ?LMS1*› *IH*(*2*) *id len*]
have *set-LMS2*: *set ?LMS2 = {i. abs-is-lms ?T i}*
by *blast*

from *abs-sa-induce-permutation*
[*OF* ‹*set ?LMS2 = {i. abs-is-lms ?T i}*› ‹*distinct ?LMS2*› *IH*(*2*) *id len*]
have *abs-sa-induce id ?T ?LMS2* <~~> [*0..<length ?T*]
by *assumption*

from *rename-list-length*[*of ?LMS1 ?names*]
have *length ?T' = length ?LMS1*
by *assumption*

have *sorted-LMS2*: *ordlistns.sorted* (*map* (*lms-slice ?T*) *?LMS2*)
by (*metis 3.prems* ‹*distinct ?LMS1*› ‹*set ?LMS1 = {i. abs-is-lms ?T i}*›
comp-apply len id
ordlistns.sorted-filter abs-sa-induce-prefix-sorted)

have *?LMS1 = lms0-seq ?T*
by (*metis comp-apply lms-seq-0-zeroth-lms lms-seq-def*)
with *abs-sais-reduced-string*[*OF - distinct-LMS2 set-LMS2 sorted-LMS2 names T'*]
have *?T' = lms0-map ?T*
by *blast*

have *abs-is-lms ?T* (*lms0 ?T*)
by (*metis 3.prems abs-find-next-lms-less-length-abs-is-lms length-Cons*

abs-is-lms-last len no-lms-between-i-and-next not-less-eq)

from *valid-list-lms-map*[*OF IH*(*2*) ‹*abs-is-lms ?T* (*lms0 ?T*)›]
have *valid-list* (*lms0-map ?T*) **.**
hence *valid-list ?T′*
by (*simp only*: ‹*?T′* = (*lms0-map ?T*)›)

from ‹*length ?T′* = *length ?LMS1*› ‹*distinct ?LMS1*› ‹*set ?LMS1* = {*i*. *abs-is-lms ?T i*}›
have *R1*: *abs-sais ?T′* <~~> [*0*..<*length ?T′*] ⟹ *distinct ?LMS3*
by (*metis distinct-abs-order-lms*)

from ‹*length ?T′* = *length ?LMS1*› ‹*distinct ?LMS1*› ‹*set ?LMS1* = {*i*. *abs-is-lms ?T i*}›
have *R2*: *abs-sais ?T′* <~~> [*0*..<*length ?T′*] ⟹ *set ?LMS3* = {*i*. *abs-is-lms ?T i*}
by (*metis* (*no-types*, *lifting*) *abs-order-lms-eq-all-lms*)

from *abs-sa-induce-permutation*[*OF R2 R1 IH*(*2*) *id len*]
have *R3*: *abs-sais ?T′* <~~> [*0*..<*length ?T′*] ⟹
abs-sa-induce id ?T ?LMS3 <~~> [*0*..<*length ?T*]
by *blast*

from *IH′*[*OF* - ‹*valid-list ?T′*›]
have ¬*distinct ?T′* ⟹ *abs-sais ?T′* <~~> [*0*..<*length ?T′*]
by *assumption*
with *R3*
have ¬*distinct ?T′* ⟹ *abs-sa-induce id ?T ?LMS3* <~~> [*0*..<*length ?T*]
by *blast*

have *distinct ?T′* ∨ ¬*distinct ?T′*
by *blast*
then show *?case*
proof
assume *A*: *distinct ?T′*
from *abs-sais-distinct-simp*[*OF T LMS1 SA1 LMS2 names T′ A*]
‹*abs-sa-induce id ?T ?LMS2* <~~> [*0*..<*length ?T*]›
show *?thesis*
by *metis*
next
assume *A*: ¬*distinct ?T′*
from *abs-sais-not-distinct-simp*[*OF T LMS1 SA1 LMS2 names T′ LMS3 A*]
‹¬*distinct ?T′* ⟹ *abs-sa-induce id ?T ?LMS3* <~~> [*0*..<*length ?T*]›[*OF A*]
show *?thesis*
by *metis*
qed
qed

98 SAIS Sorted Helpers

lemma *abs-sais-subset-idx*:
  assumes *valid-list T*
  shows *set* (*abs-sais T*) ⊆ {*0*..<*length T*}
  using *assms perm-distinct-set-of-upt-iff abs-sais-permutation* **by** *auto*

99 SAIS sorts suffixes

theorem *abs-sais-sorted-alt*:
  valid-list T ⟹
   ordlistns.strict-sorted (*map* (*suffix T*) (*abs-sais T*))
proof(*induct rule*: *abs-sais.induct*[*of - T*])
  case *1*
  then show *?case* **by** *simp*
next
  case (*2 x*)
  then show *?case* **by** *simp*
next
  case (*3 a b xs*)
  note *IH* = *this*
  let *?T* = *a* # *b* # *xs*
  have *T*: *?T* = *a* # *b* # *xs*
    by (*simp only*:)
  let *?LMS1* = *abs-extract-lms ?T* [*0*..<*length ?T*]
  have *LMS1*: *?LMS1* = *abs-extract-lms ?T* [*0*..<*length ?T*]
    by (*simp only*:)
  let *?SA1* = *abs-sa-induce id ?T ?LMS1*
  have *SA1*: *?SA1* = *abs-sa-induce id ?T ?LMS1*
    by (*simp only*:)
  let *?LMS2* = *abs-extract-lms ?T ?SA1*
  have *LMS2*: *?LMS2* = *abs-extract-lms ?T ?SA1*
    by (*simp only*:)
  let *?names* = *abs-rename-mapping ?T ?LMS2*
  have *names*: *?names* = *abs-rename-mapping ?T ?LMS2*
    by (*simp only*:)
  let *?T′* = *rename-string ?LMS1 ?names*
  have *T′*: *?T′* = *rename-string ?LMS1 ?names*
    by (*simp only*:)
  let *?LMS3* = *order-lms ?LMS1* (*abs-sais ?T′*)
  have *LMS3*: *?LMS3* = *order-lms ?LMS1* (*abs-sais ?T′*)
    by (*simp only*:)

  from *IH*(*1*)[*OF T LMS1 SA1 LMS2 names T′*]
  have *IH′*: ⟦¬*distinct ?T′*; *valid-list ?T′*⟧ ⟹
          ordlistns.strict-sorted (*map* (*suffix ?T′*) (*abs-sais ?T′*))
    by *blast*

  from *set-abs-extract-lms-eq-all-lms*[*of ?T*]

have *set-LMS1*: *set ?LMS1 = {i. abs-is-lms ?T i}*
by *simp*

from *distinct-abs-extract-lms*[*of ?T*]
have *distinct-LMS1*: *distinct ?LMS1*
by *simp*

have *id*: *strict-mono* (*id* :: *nat* ⇒ *nat*) (*id* :: *nat* ⇒ *nat*) *bot = 0*
by (*simp add*: *strict-mono-def bot-nat-def*)+

have *len*: *Suc 0 < length ?T*
by *simp*

from *distinct-filter-abs-sa-induce*[*OF* ‹*set ?LMS1 = {i. abs-is-lms ?T i}*› ‹*distinct ?LMS1*› *IH*(*2*) *id len*]
have *distinct-LMS2*: *distinct ?LMS2*
by *blast*

from *filter-abs-sa-induce-eq-all-lms*[*OF* ‹*set ?LMS1 = {i. abs-is-lms ?T i}*› ‹*distinct ?LMS1*› *IH*(*2*) *id len*]
have *set-LMS2*: *set ?LMS2 = {i. abs-is-lms ?T i}*
by *blast*

from *distinct-set-imp-perm*[*OF* ‹*distinct ?LMS1*› ‹*distinct ?LMS2*›]
‹*set ?LMS1 = {i. abs-is-lms ?T i}*›
‹*set ?LMS2 = {i. abs-is-lms ?T i}*›
have *?LMS1 <~~> ?LMS2*
by *blast*

have *sorted-LMS2*: *ordlistns.sorted* (*map* (*lms-slice ?T*) *?LMS2*)
by (*metis 3.prems* ‹*distinct ?LMS1*› ‹*set ?LMS1 = {i. abs-is-lms ?T i}*› *comp-apply len id*
ordlistns.sorted-filter abs-sa-induce-prefix-sorted)

have *?LMS1 = lms0-seq ?T*
by (*metis comp-apply lms-seq-0-zeroth-lms lms-seq-def*)

with *abs-sais-reduced-string*[*OF - distinct-LMS2 set-LMS2 sorted-LMS2 names T'*]
have *?T' = lms0-map ?T*
by *blast*

have *abs-is-lms ?T* (*lms0 ?T*)
by (*metis 3.prems abs-find-next-lms-less-length-abs-is-lms abs-is-lms-last*
len length-Cons no-lms-between-i-and-next not-less-eq)

from *valid-list-lms-map*[*OF IH*(*2*) ‹*abs-is-lms ?T* (*lms0 ?T*)›]
have *valid-list* (*lms0-map ?T*) **.**
hence *valid-list ?T'*

```
    by (simp only: ‹?T' = (lms0-map ?T)›)

  have distinct ?T' ∨ ¬distinct ?T'
    by blast
  hence ordlistns.sorted (map (suffix ?T) (abs-sais ?T))
  proof
    assume A: distinct ?T'
    hence distinct (lms0-map ?T)
      by (simp only: ‹?T' = (lms0-map ?T)›)
    with sorted-distinct-lms-substr-perm[OF sorted-LMS2]
    have ordlistns.sorted (map (suffix ?T) ?LMS2)
      by (metis ‹?LMS1 = lms0-seq ?T› ‹?LMS1 <~~> ?LMS2›)
    with abs-sa-induce-suffix-sorted[OF set-LMS2 distinct-LMS2 IH(2) id len]
    have ordlistns.sorted (map (suffix ?T) (abs-sa-induce id ?T ?LMS2))
      using ordlistns.strict-sorted-imp-sorted by blast
    with abs-sais-distinct-simp[OF T LMS1 SA1 LMS2 names T' A]
    show ?thesis
      by presburger
  next
    assume A: ¬distinct ?T'
    with IH'[OF - ‹valid-list ?T'›]
    have ordlistns.strict-sorted (map (suffix ?T') (abs-sais ?T'))
      by blast
   hence C1: ordlistns.strict-sorted (map (suffix (lms0-map ?T)) (abs-sais (lms0-map
?T)))
      by (simp only: ‹?T' = (lms0-map ?T)›)

    from abs-order-lms-eq-map-nth[of ?LMS1 abs-sais ?T']
        ‹?LMS1 = lms0-seq ?T› ‹?T' = lms0-map ?T›
    have C2: order-lms ?LMS1 (abs-sais ?T') =
            map (nth (lms0-seq ?T)) (abs-sais (lms0-map ?T))
      by (simp only:)

    note perm = abs-sais-permutation[OF ‹valid-list ?T'›,
                                simplified rename-list-length]

    note set-LMS3 = abs-order-lms-eq-all-lms[OF perm set-LMS1]
    note distinct-LMS3 = distinct-abs-order-lms[OF perm distinct-LMS1]

    from abs-sais-permutation[OF ‹valid-list (lms0-map ?T)›]
    have ∀ y ∈ set (abs-sais (lms0-map ?T)). y < card {i. abs-is-lms ?T i}
      by (metis atLeastLessThan-iff card-lms-suffixes length-reduced-seq perm-set-eq
set-upt)
    with sorted-reduced-seq-imp-lms[OF C1] C2
    have ordlistns.strict-sorted (map (suffix ?T) ?LMS3)
      by presburger
    with abs-sa-induce-suffix-sorted[OF set-LMS3 distinct-LMS3 IH(2) id len]
    have ordlistns.sorted (map (suffix ?T) (abs-sa-induce id ?T ?LMS3))
      using ordlistns.strict-sorted-imp-sorted by blast
```

with *abs-sais-not-distinct-simp*[*OF T LMS1 SA1 LMS2 names T′ LMS3 A*]
show *?thesis*
by *presburger*
qed
moreover
from *perm-distinct-set-of-upt-iff*[*THEN iffD1*, *OF abs-sais-permutation*[*OF IH*(*2*)]]
have *distinct* (*map* (*suffix ?T*) (*abs-sais ?T*))
by (*metis atLeastLessThan-iff distinct-suffixes*)
ultimately show *?case*
using *ordlistns.strict-sorted-iff* **by** *blast*
qed

theorem *abs-sais-sorted*:
valid-list T ⟹
strict-sorted (*map* (*suffix T*) (*abs-sais T*))
using *abs-sais-sorted-alt abs-sais-subset-idx valid-list-ordlist-ordlistns-strict-sorted-eq*
by *blast*

100 Verification of a SAIS construction algorithm

interpretation *abs-sais*: *Suffix-Array-Restricted abs-sais*
using *Suffix-Array-Restricted.intro abs-sais-permutation abs-sais-sorted* **by** *blast*

end
theory *Abs-SAIS-Verification*
imports *Abs-SAIS-Verification-With-Valid-Precondition*
begin

101 Final Theorem: Verification of a generalised SAIS construction algorithm

The @term *abs-sais* implementation produces an output that is equivalent to that of a suffix array construction algorithm for lists of any type that can be linearly ordered. This lifts the restriction that the algorithm only operates on natural numbers terminated by a bottom element.

interpretation *abs-sais-gen*: *Suffix-Array-General sa-nat-wrapper map-to-nat abs-sais*
by (*simp add*: *Suffix-Array-Restricted-imp-General abs-sais.Suffix-Array-Restricted-axioms*)

theorem *abs-sais-gen-is-Suffix-Array-General*:
Suffix-Array-General sa ⟷ *sa* = *sa-nat-wrapper map-to-nat abs-sais*
using *Suffix-Array-General-determinism abs-sais-gen.Suffix-Array-General-axioms*
by *auto*

end
theory *Bucket-Insert*
imports

../../util/Repeat
begin

102 Bucket Insert

fun *bucket-insert-step* ::
nat list × *nat list* × *nat* ⇒
((*'a* :: {*linorder*, *order-bot*}) ⇒ *nat*) × *'a list* × *nat list* ⇒
nat list × *nat list* × *nat*
where
bucket-insert-step (*B*, *SA*, *i*) (*α*, *T*, *LMS*) =
(*let b* = *α* (*T* ! (*LMS* ! *i*));
k = *B* ! *b* − *Suc 0*
in (*B*[*b* := *k*], *SA*[*k* := *LMS* ! *i*], *Suc i*))

definition *bucket-insert-base* ::
((*'a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat list*
⇒
nat list × *nat list* × *nat*
where
bucket-insert-base α T B SA LMS = *repeat* (*length LMS*) *bucket-insert-step* (*B*, *SA*, *0*) (*α*, *T*, *LMS*)

definition *bucket-insert* ::
((*'a* :: {*linorder*, *order-bot*}) ⇒ *nat*) ⇒ *'a list* ⇒ *nat list* ⇒ *nat list* ⇒ *nat list*
⇒
nat list
where
bucket-insert α T B SA LMS =
(*let* (*B'*, *SA'*, *i*) = *bucket-insert-base α T B SA LMS*
in SA')

end
theory *Get-Types*
imports
../prop/List-Type
../prop/LMS-List-Slice-Util
../../util/Repeat
begin

103 Suffix Types

fun
get-suffix-types-step-r0 ::
SL-types list × *nat* ⇒ *'a* :: {*linorder*, *order-bot*} *list* ⇒ *SL-types list* × *nat*
where
get-suffix-types-step-r0 (*xs*, *i*) *ys* =
(*case i of*

```
    0 ⇒ (xs, 0)
    | Suc j ⇒
      (if Suc j < length xs ∧ Suc j < length ys then
        (if ys ! j < ys ! Suc j then
          (xs[j := S-type], j)
         else if ys ! j > ys ! Suc j then
          (xs[j := L-type], j)
         else
          (xs[j := xs ! Suc j], j))
       else
        (xs, j)))
```

definition *get-suffix-types-base*
where
get-suffix-types-base xs ≡
repeat (*length xs* − *Suc 0*) *get-suffix-types-step-r0*
(*replicate* (*length xs*) *S-type*, *length xs* − *Suc 0*) *xs*

definition *get-suffix-types*
where
get-suffix-types xs ≡ *fst* (*get-suffix-types-base xs*)

104 LMS types

fun *is-lms-ref*
where
is-lms-ref ST 0 = *False* |
is-lms-ref ST (*Suc i*) =
(*if Suc i* < *length ST then ST* ! *i* = *L-type* ∧ *ST* ! (*Suc i*) = *S-type else False*)

105 Extracting LMS types

abbreviation *extract-lms ST xs* ≡ *filter* (λ*i*. *is-lms-ref ST i*) *xs*

106 LMS Substrings

definition *find-next-lms* :: *SL-types list* ⇒ *nat* ⇒ *nat*
where
find-next-lms ST i =
(*case find* (λ*j*. *is-lms-ref ST j*) [*Suc i*..<*length ST*] *of*
Some j ⇒ *j*
| - ⇒ *length ST*)

definition
lms-slice-ref ::
('*a* :: {*linorder*, *order-bot*}) *list* ⇒ *SL-types list* ⇒ *nat* ⇒ '*a list*
where
lms-slice-ref T ST i =

list-slice T i (Suc (find-next-lms ST i))

107 Rename Mapping

fun *rename-mapping′* ::
  (′*a* :: {*linorder*, *order-bot*}) *list* ⇒ *SL-types list* ⇒
   nat list ⇒ *nat list* ⇒ *nat* ⇒ *nat list*
where
  rename-mapping′ - - [] names - = names |
  rename-mapping′ - - [x] names i = names[*x* := *i*] |
  rename-mapping′ T ST (*a* # *b* # *xs*) *names i* =
    (*if lms-slice-ref T ST a = lms-slice-ref T ST b*
      then
        rename-mapping′ T ST (*b* # *xs*) (*names*[*a* := *i*]) *i*
     else
        rename-mapping′ T ST (*b* # *xs*) (*names*[*a* := *i*]) (*Suc i*))

definition
  rename-mapping ::
    (′*a* :: {*linorder*, *order-bot*}) *list* ⇒ *SL-types list* ⇒ *nat list* ⇒ *nat list*
where
  rename-mapping T ST LMS =
    rename-mapping′ T ST LMS (*replicate* (*length T*) (*length T*)) *0*

end
theory *Induce-L*
  imports
    ../../util/Repeat
     ../prop/Buckets
begin

108 Induce L Refinement

fun *induce-l-step-r0* ::
  nat list × *nat list* × *nat* ⇒
   ((′*a* :: {*linorder*, *order-bot*}) ⇒ *nat*) × ′*a list* ⇒
   nat list × *nat list* × *nat*
  where
induce-l-step-r0 (*B*, *SA*, *i*) (α, *T*) =
  (*if SA* ! *i* < *length T*
   then
     (*case SA* ! *i of*
        Suc j ⇒
          (*case suffix-type T j of*
             L-type ⇒
               (*let k* = α (*T* ! *j*);
                    l = *B* ! *k*
                in (*B*[*k* := *Suc l*], *SA*[*l* := *j*], *Suc i*))

```
          | - ⇒ (B, SA, Suc i))
      | - ⇒ (B, SA, Suc i))
  else (B, SA, Suc i))

fun induce-l-step ::
  nat list × nat list × nat ⇒
   (('a :: {linorder, order-bot}) ⇒ nat) × 'a list × SL-types list⇒
   nat list × nat list × nat
  where
induce-l-step (B, SA, i) (α, T, ST) =
  (if SA ! i < length T
   then
    (case SA ! i of
      Suc j ⇒
       (case ST ! j of
          L-type ⇒
           (let k = α (T ! j);
                l = B ! k
            in (B[k := Suc (B ! k)], SA[l := j], Suc i))
        | - ⇒ (B, SA, Suc i))
    | - ⇒ (B, SA, Suc i))
  else (B, SA, Suc i))

definition induce-l-base ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list ⇒
   nat list × nat list × nat
  where
induce-l-base α T ST B SA = repeat (length T) induce-l-step (B, SA, 0) (α, T,
ST)

definition induce-l ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list ⇒
   nat list
  where
induce-l α T ST B SA = (let (B', SA', i) = induce-l-base α T ST B SA in SA')

end
theory Induce-S
  imports ../abs−proof/Abs-Induce-S-Verification
begin
```

109 Induce S Refinement

```
fun induce-s-step-r0 ::
  nat list × nat list × nat ⇒
   (('a :: {linorder, order-bot}) ⇒ nat) × 'a list ⇒
   nat list × nat list × nat
  where
induce-s-step-r0 (B, SA, i) (α, T) =
  (case i of
    Suc n ⇒
      (if Suc n < length SA ∧ SA ! Suc n < length T then
        (case SA ! Suc n of
          Suc j ⇒
            (case suffix-type T j of
              S-type ⇒
                (let b = α (T ! j);
                     k = B ! b − Suc 0
                 in (B[b := k], SA[k := j], n)
                )
              | - ⇒ (B, SA, n)
            )
          | - ⇒ (B, SA, n)
        )
        else
          (B, SA, n)
      )
   | -   ⇒ (B, SA, 0)
  )

fun induce-s-step-r1 ::
  nat list × nat list × nat ⇒
   (('a :: {linorder, order-bot}) ⇒ nat) × 'a list × SL-types list ⇒
   nat list × nat list × nat
  where
induce-s-step-r1 (B, SA, i) (α, T, ST) =
  (case i of
    Suc n ⇒
      (if Suc n < length SA ∧ SA ! Suc n < length T then
        (case SA ! Suc n of
          Suc j ⇒
            (case ST ! j of
              S-type ⇒
                (let b = α (T ! j);
                     k = B ! b − Suc 0
                 in (B[b := k], SA[k := j], n)
                )
              | - ⇒ (B, SA, n)
            )
          | - ⇒ (B, SA, n)
```

```
        )
        else
          (B, SA, n)
        )
    | - ⇒ (B, SA, 0)
  )

fun induce-s-step-r2 ::
  nat list × nat list × nat ⇒
   (('a :: {linorder, order-bot}) ⇒ nat) × 'a list × SL-types list ⇒
   nat list × nat list × nat
  where
induce-s-step-r2 (B, SA, i) (α, T, ST) =
  (case i of
    Suc n ⇒
      (if Suc n < length SA  then
        (case SA ! Suc n of
          Suc j ⇒
            (case ST ! j of
              S-type ⇒
                (let b = α (T ! j);
                     k = B ! b − Suc 0
                 in (B[b := k], SA[k := j], n)
                )
              | - ⇒ (B, SA, n)
            )
          | - ⇒ (B, SA, n)
        )
        else
          (B, SA, n)
      )
    | - ⇒ (B, SA, 0)
  )

fun induce-s-step ::
  nat list × nat list × nat ⇒
   (('a :: {linorder, order-bot}) ⇒ nat) × 'a list × SL-types list ⇒
   nat list × nat list × nat
  where
induce-s-step (B, SA, i) (α, T, ST) =
  (case i of
    Suc n ⇒
    (case SA ! Suc n of
        Suc j ⇒
          (case ST ! j of
            S-type ⇒
              (let b = α (T ! j);
                   k = B ! b − Suc 0
               in (B[b := k], SA[k := j], n)
```

```
            )
          | - ⇒ (B, SA, n)
        )
      | - ⇒ (B, SA, n)
    )
  | -   ⇒ (B, SA, 0)
 )

definition induce-s-base ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list ⇒
   nat list × nat list × nat
  where
induce-s-base α T ST B SA = repeat (length T − Suc 0) induce-s-step (B, SA,
length T − Suc 0) (α, T, ST)

definition induce-s ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list ⇒
   nat list
  where
induce-s α T ST B SA = (let (B', SA', i) = induce-s-base α T ST B SA in SA')

end
theory Induce
  imports Induce-S Induce-L Bucket-Insert
begin
```

110 Induce

```
definition sa-induce-r0 ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   nat list ⇒
   nat list
  where
sa-induce-r0 α T LMS =
  (let
      B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))];
      B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))];

      — Initialise SA
      SA = replicate (length T) (length T);
```

```
      — Get the suffix types
      ST = abs-get-suffix-types T;

      — Insert the LMS types into the suffix array
      SA = abs-bucket-insert α T B0 SA (rev LMS);

      — Insert the L types into the suffix array
      SA = induce-l α T ST B1 SA

    — Insert the S types into the suffix array
    in induce-s α T ST (B0[0 := 0]) SA)

definition sa-induce-r1 ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list
  where
sa-induce-r1 α T ST LMS =
  (let
      B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))];
      B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))];

      — Initialise SA
      SA = replicate (length T) (length T);

      — Insert the LMS types into the suffix array
      SA = abs-bucket-insert α T B0 SA (rev LMS);

      — Insert the L types into the suffix array
      SA = induce-l α T ST B1 SA

    — Insert the S types into the suffix array
    in induce-s α T ST (B0[0 := 0]) SA)

definition sa-induce-r2 ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list
  where
sa-induce-r2 α T ST LMS =
  (let
      B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))];
      B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))];
```

— Initialise SA
SA = *replicate* (*length T*) (*length T*);

— Insert the LMS types into the suffix array
SA = *bucket-insert* α *T B0 SA* (*rev LMS*);

— Insert the L types into the suffix array
SA = *induce-l* α *T ST B1 SA*

— Insert the S types into the suffix array
in induce-s α *T ST* (*B0*[*0* := *0*]) *SA*)

abbreviation *sa-induce* ≡ *sa-induce-r2*

end
theory *SAIS*
imports *Induce Get-Types*
begin

111 SAIS

function *sais-r0* ::
nat list ⇒
nat list
where
sais-r0 [] = [] |
sais-r0 [*x*] = [*0*] |
sais-r0 (*a* # *b* # *xs*) =
(*let*
T = *a* # *b* # *xs*;

— Compute the suffix types
ST = *abs-get-suffix-types T*;

— Extract the LMS types
LMS0 = *extract-lms ST* [*0*..<*length T*];

— Induce the prefix ordering based on LMS
SA = *sa-induce id T ST LMS0*;

— Extract the LMS types
LMS1 = *extract-lms ST SA*;

— Create a new alphabet
names = *rename-mapping T ST LMS1*;

— Make a reduced string (2 lines)
T' = *rename-string LMS0 names*;

```
    — Obtain the correct ordering of LMS-types
    LMS2 = (if distinct T' then LMS1 else order-lms LMS0 (sais-r0 T'))

  — Induce the suffix ordering based of LMS
  in sa-induce id T ST LMS2)
 by pat-completeness blast+

function sais-r1 ::
 nat list ⇒
  nat list
 where
sais-r1 [] = [] |
sais-r1 [x] = [0] |
sais-r1 (a # b # xs) =
 (let
    T = a # b # xs;

    — Compute the suffix types
    ST = get-suffix-types T;

    — Extract the LMS types
    LMS0 = extract-lms ST [0..<length T];

    — Induce the prefix ordering based on LMS
    SA = sa-induce id T ST LMS0;

    — Extract the LMS types
    LMS1 = extract-lms ST SA;

    — Create a new alphabet
    names = rename-mapping T ST LMS1;

    — Make a reduced string
    T' = rename-string LMS0 names;

    — Obtain the correct ordering of LMS-types
    LMS2 = (if distinct T' then LMS1 else order-lms LMS0 (sais-r1 T'))

  — Induce the suffix ordering based of LMS
  in sa-induce id T ST LMS2)
 by pat-completeness blast+

abbreviation sais ≡ sais-r1

end
theory Bucket-Insert-Verification
 imports
  ../abs−proof/Abs-Bucket-Insert-Verification
  ../def/Bucket-Insert
```

begin

112 Bucket Insert

lemma *abs-bucket-insert-step-cons*:
assumes *bucket-insert-step* (*B*, *SA*, *Suc i*) (*α*, *T*, *a # xs*) = (*B1*, *SA1*, *j1*)
and *bucket-insert-step* (*B*, *SA*, *i*) (*α*, *T*, *xs*) = (*B2*, *SA2*, *j2*)
shows *B1* = *B2* ∧ *SA1* = *SA2*
by (*metis assms*(*1*) *assms*(*2*) *bucket-insert-step.simps nth-Cons-Suc prod.sel*(*1*) *prod.sel*(*2*))

lemma *abs-bucket-insert-base-cons'*:
assumes *repeat n bucket-insert-step* (*B*, *SA*, *Suc i*) (*α*, *T*, *x # xs*) = (*B1*, *SA1*, *j1*)
and *repeat n bucket-insert-step* (*B*, *SA*, *i*) (*α*, *T*, *xs*) = (*B2*, *SA2*, *j2*)
shows *B1* = *B2* ∧ *SA1* = *SA2*
using *assms*
proof (*induct n arbitrary*: *B SA i*)
case *0*
then show *?case*
by (*simp add*: *repeat-0*)
next
case (*Suc n*)
note *IH* = *this*

let *?b* = *α* (*T* ! (*xs* ! *i*))
let *?k* = *B* ! *?b* − *Suc 0*

have *bucket-insert-step* (*B*, *SA*, *Suc i*) (*α*, *T*, *x # xs*)
= (*B*[*?b* := *?k*], *SA*[*?k* := *xs* ! *i*], *Suc* (*Suc i*))
by (*metis bucket-insert-step.simps nth-Cons-Suc*)
with *IH*(*2*) *repeat-step-forward*[*of n bucket-insert-step* (*B*, *SA*, *Suc i*) (*α*, *T*, *x # xs*)]
have *repeat n bucket-insert-step* (*B*[*?b* := *?k*], *SA*[*?k* := *xs* ! *i*], *Suc* (*Suc i*)) (*α*, *T*, *x # xs*)
= (*B1*, *SA1*, *j1*)
by *simp*
moreover
have *bucket-insert-step* (*B*, *SA*, *i*) (*α*, *T*, *xs*) = (*B*[*?b* := *?k*], *SA*[*?k* := *xs* ! *i*], *Suc i*)
by (*metis bucket-insert-step.simps*)
with *IH*(*3*) *repeat-step-forward*[*of n bucket-insert-step* (*B*, *SA*, *i*) (*α*, *T*, *xs*)]
have *repeat n bucket-insert-step* (*B*[*?b* := *?k*], *SA*[*?k* := *xs* ! *i*], *Suc i*) (*α*, *T*, *xs*)
= (*B2*, *SA2*, *j2*)
by *simp*
ultimately show *?case*
using *IH*(*1*)[*of B*[*?b* := *?k*] *SA*[*?k* := *xs* ! *i*] *Suc i*]
by *blast*
qed

lemma *bucket-insert-base-cons*:
assumes $b = \alpha\ (T\ !\ a)$
and $k = B\ !\ b - Suc\ 0$
and *bucket-insert-base* $\alpha\ T\ B\ SA\ (a\ \#\ xs) = (B1,\ SA1,\ j1)$
and *bucket-insert-base* $\alpha\ T\ (B[b := k])\ (SA[k := a])\ xs = (B2,\ SA2,\ j2)$
shows $B1 = B2 \wedge SA1 = SA2$
proof −
from *assms*(*1*,*2*)
have *bucket-insert-step* $(B,\ SA,\ 0)\ (\alpha,\ T,\ a\ \#\ xs) = (B[b := k],\ SA[k := a],\ Suc\ 0)$
by (*metis bucket-insert-step.simps nth-Cons-0*)
with *assms*(*3*)[*simplified bucket-insert-base-def*, *simplified*]
repeat-step-forward[*of length xs bucket-insert-step* $(B,\ SA,\ 0)\ (\alpha,\ T,\ a\ \#\ xs)$]
have *A*: *repeat* (*length xs*) *bucket-insert-step* $(B[b := k],\ SA[k := a],\ Suc\ 0)\ (\alpha,\ T,\ a\ \#\ xs)$
$= (B1,\ SA1,\ j1)$
by *simp*
with *abs-bucket-insert-base-cons'*[*of length xs* $B[b := k]\ SA[k := a]\ 0\ \alpha\ T\ a\ xs\ B1\ SA1\ j1\ B2\ SA2\ j2$]
assms(*4*)[*simplified bucket-insert-base-def*]
show *?thesis*
by *simp*
qed

lemma *bucket-insert-cons*:
assumes $b = \alpha\ (T\ !\ a)$
and $k = B\ !\ b - Suc\ 0$
shows *bucket-insert* $\alpha\ T\ B\ SA\ (a\ \#\ xs) =$ *bucket-insert* $\alpha\ T\ (B[b := k])\ (SA[k := a])\ xs$
by (*clarsimp simp*: *bucket-insert-def Let-def bucket-insert-base-cons*[*of* - α, *OF assms*]
split: *prod.splits*)

lemma *abs-bucket-insert-eq*:
abs-bucket-insert $\alpha\ T\ B\ SA\ xs =$ *bucket-insert* $\alpha\ T\ B\ SA\ xs$
proof (*induct xs arbitrary*: *B SA*)
case *Nil*
then show *?case*
unfolding *bucket-insert-def bucket-insert-base-def*
by (*simp add*: *repeat-0*)
next
case (*Cons a xs*)
note *IH* = *this*

let *?b* $= \alpha\ (T\ !\ a)$
let *?k* $= B\ !\ ?b - Suc\ 0$

have *abs-bucket-insert* $\alpha\ T\ B\ SA\ (a\ \#\ xs) =$ *abs-bucket-insert* $\alpha\ T\ (B[?b :=$

?k]) (*SA*[*?k* := *a*]) *xs*
by (*meson abs-bucket-insert.simps*(*2*))
moreover
from *bucket-insert-cons*[*of ?b* α *T a ?k B SA xs*, *simplified*]
have *bucket-insert* α *T B SA* (*a* # *xs*) = *bucket-insert* α *T* (*B*[*?b* := *?k*]) (*SA*[*?k* := *a*]) *xs* **.**
ultimately show *?case*
using *IH*[*of B*[*?b* := *?k*] *SA*[*?k* := *a*]]
by *simp*
qed

end
theory *Induce-L-Verification*
imports
../abs−proof/Abs-Induce-L-Verification
../def/Induce-L
begin

113 Induce L Refinement

lemma *abs-induce-l-step-to-r0*:
i < *length SA* ⟹ *abs-induce-l-step* (*B*, *SA*, *i*) (α, *T*) = *induce-l-step-r0* (*B*, *SA*, *i*) (α, *T*)
by (*clarsimp simp*: *Let-def split*: *prod.splits nat.splits SL-types.splits*)

lemma *induce-l-step-r0-to*:
⟦*length ST* = *length T*; ∀ *k* < *length ST*. *ST* ! *k* = *suffix-type T k*⟧ ⟹
induce-l-step-r0 (*B*, *SA*, *i*) (α, *T*) = *induce-l-step* (*B*, *SA*, *i*) (α, *T*, *ST*)
by (*clarsimp simp*: *Let-def split*: *prod.splits nat.splits SL-types.splits*)

lemma *abs-induce-l-step-to*:
assumes *i* < *length SA*
and *length ST* = *length T*
and ∀ *k* < *length ST*. *ST* ! *k* = *suffix-type T k*
shows *abs-induce-l-step* (*B*, *SA*, *i*) (α, *T*) = *induce-l-step* (*B*, *SA*, *i*) (α, *T*, *ST*)
by (*metis assms induce-l-step-r0-to abs-induce-l-step-to-r0*)

lemma *repeat-abs-induce-l-step-to*:
assumes *n* ≤ *length SA*
and *length ST* = *length T*
and ∀ *k* < *length ST*. *ST* ! *k* = *suffix-type T k*
shows *repeat n abs-induce-l-step* (*B*, *SA*, *0*) (α, *T*) = *repeat n induce-l-step* (*B*, *SA*, *0*) (α, *T*, *ST*)
using *assms*(*1*)
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *repeat-0*)
next

```
  case (Suc n)
  note IH = this

  from repeat-step[of n abs-induce-l-step (B, SA, 0) (α, T)]
  have A: repeat (Suc n) abs-induce-l-step (B, SA, 0) (α, T) =
          abs-induce-l-step (repeat n abs-induce-l-step (B, SA, 0) (α, T)) (α, T)
    by assumption

  from repeat-step[of n induce-l-step (B, SA, 0) (α, T, ST)]
  have B: repeat (Suc n) induce-l-step (B, SA, 0) (α, T, ST) =
          induce-l-step (repeat n induce-l-step (B, SA, 0) (α, T, ST)) (α, T, ST)
    by assumption

  from repeat-abs-induce-l-step-index[of n B SA 0 α T]
  obtain B' SA' where
    C: repeat n abs-induce-l-step (B, SA, 0) (α, T) = (B', SA', n)
    by auto
  with IH
  have D: repeat n induce-l-step (B, SA, 0) (α, T, ST) = (B', SA', n)
    by simp

  from IH(2)
  have n < length SA
    by simp
  with repeat-abs-induce-l-step-lengths[OF C]
  have n < length SA'
    by simp

  from abs-induce-l-step-to[OF ‹n < length SA'› assms(2–), of B']
      A B C D
  show ?case
    by simp
qed

lemma abs-induce-l-base-to:
  assumes length SA = length T
  and     length ST = length T
  and     ∀ i < length ST. ST ! i = suffix-type T i
shows abs-induce-l-base α T B SA = induce-l-base α T ST B SA
  unfolding induce-l-base-def abs-induce-l-base-def
  by (simp add: assms(1, 2,3) repeat-abs-induce-l-step-to)

lemma abs-induce-l-eq:
  assumes length SA = length T
  and     length ST = length T
  and     ∀ i < length ST. ST ! i = suffix-type T i
shows abs-induce-l α T B SA = induce-l α T ST B SA
  by (metis assms abs-induce-l-base-to abs-induce-l-def induce-l-def)
```

end
theory *Induce-S-Verification*
imports
../abs−proof/Abs-Induce-S-Verification
../def/Induce-S
begin

114 Induce S Refinement

lemma *abs-induce-s-step-to-r0*:
shows *induce-s-step-r0* (*B*, *SA*, *i*) (*α*, *T*) = *abs-induce-s-step* (*B*, *SA*, *i*) (*α*, *T*)
proof (*cases i*)
case *0*
then show *?thesis*
by *simp*
next
case (*Suc n*)
assume *i* = *Suc n*
then show *?thesis*
proof (*cases Suc n* < *length SA*)
assume *Suc n* < *length SA*
show *?thesis*
proof (*cases SA* ! *Suc n* < *length T*)
assume *SA* ! *Suc n* < *length T*
show *?thesis*
proof (*cases SA* ! *Suc n*)
case *0*
then show *?thesis*
by (*clarsimp simp*: ‹*i* = -› ‹*Suc n* < *length* -› ‹*SA* ! - < -›)
next
case (*Suc j*)
assume *SA* ! *Suc n* = *Suc j*
hence *Suc j* < *length T*
using ‹*SA* ! *Suc n* < *length T*› **by** *auto*
then show *?thesis*
by (*clarsimp simp*: ‹*i* = -› ‹*Suc n* < *length* -› ‹*SA* ! - < -›)
qed
next
assume ¬ *SA* ! *Suc n* < *length T*
then show *?thesis*
by *simp*
qed
next
assume ¬ *Suc n* < *length SA*
show *?thesis*
by (*clarsimp simp*: ‹*i* = -› ‹¬ -›)
qed
qed

lemma *induce-s-step-r0-to-r1*:
assumes *length ST = length T*
and ∀ *k < length ST. ST ! k = suffix-type T k*
shows *induce-s-step-r1 (B, SA, i) (α, T, ST) = induce-s-step-r0 (B, SA, i) (α, T)*
proof (*cases i*)
case *0*
then show *?thesis*
by *auto*
next
case (*Suc n*)
assume *i = Suc n*
then show *?thesis*
proof (*cases Suc n < length SA ∧ SA ! Suc n < length T*)
assume *Suc n < length SA ∧ SA ! Suc n < length T*
hence *Suc n < length SA SA ! Suc n < length T*
by *blast+*
then show *?thesis*
proof (*cases SA ! Suc n*)
case *0*
then show *?thesis*
by (*clarsimp simp*: ‹*i = -*› ‹*Suc n < length -*› ‹*SA ! - < -*›)
next
case (*Suc j*)
assume *SA ! Suc n = Suc j*
hence *ST ! j = suffix-type T j*
using ‹*SA ! Suc n < length T*› *assms*(*1*,*2*) **by** *force*
then show *?thesis*
by (*clarsimp simp*: ‹*i = -*› ‹*Suc n < length -*› ‹*SA ! - < -*› ‹*SA ! - = -*›)
qed
next
assume ¬ (*Suc n < length SA ∧ SA ! Suc n < length T*)
show *?thesis*
by (*simp add*: ‹¬ -›‹*i = Suc n*›)
qed
qed

lemma *abs-induce-s-step-to-r1*:
assumes *length ST = length T*
and ∀ *k < length ST. ST ! k = suffix-type T k*
shows *induce-s-step-r1 (B, SA, i) (α, T, ST) = abs-induce-s-step (B, SA, i) (α, T)*
by (*metis assms induce-s-step-r0-to-r1 abs-induce-s-step-to-r0*)

lemma *induce-s-step-r1-to-r2*:
assumes *s-perm-inv α T B SA0 SA i*
shows *induce-s-step-r2 (B, SA, i) (α, T, ST) = induce-s-step-r1 (B, SA, i) (α, T, ST)*
proof (*cases i*)

```
  case 0
  then show ?thesis
    by simp
next
  case (Suc n)
  then show ?thesis
  proof (cases Suc n < length SA)
    assume Suc n < length SA
    moreover
    have SA ! Suc n < length T
    by (metis Suc assms calculation dual-order.refl s-perm-inv-elims(5) s-seen-invD(1))
    ultimately show ?thesis
    proof (cases SA ! Suc n)
      case 0
      then show ?thesis
        using ‹i = Suc n› ‹Suc n < length SA› ‹SA ! Suc n < length T›
        by simp
    next
      case (Suc j)
      assume SA ! Suc n = Suc j
      then show ?thesis
      proof (cases ST ! j)
        assume ST ! j = S-type
        then show ?thesis
          using ‹i = Suc n› ‹Suc n < length SA› ‹SA ! Suc n < length T› ‹SA !
Suc n = Suc j›
          by (clarsimp simp: Let-def)
      next
        assume ST ! j = L-type
        then show ?thesis
          using ‹i = Suc n› ‹Suc n < length SA› ‹SA ! Suc n < length T› ‹SA !
Suc n = Suc j›
          by (clarsimp simp: Let-def)
        qed
    qed
  next
    assume i = Suc n ¬Suc n < length SA
    then show ?thesis
      by simp
  qed
qed

lemma abs-induce-s-step-to-r2:
  assumes s-perm-inv α T B SA0 SA i
  and     length ST = length T
  and     ∀ k < length ST. ST ! k = suffix-type T k
shows induce-s-step-r2 (B, SA, i) (α, T, ST) = abs-induce-s-step (B, SA, i) (α,
T)
 by (metis assms induce-s-step-r1-to-r2 induce-s-step-r0-to-r1 abs-induce-s-step-to-r0)
```

lemma *induce-s-step-r2-to*:
i < *length SA* ⟹ *induce-s-step* (*B*, *SA*, *i*) (*α*, *T*, *ST*) = *induce-s-step-r2* (*B*, *SA*, *i*) (*α*, *T*, *ST*)
by (*clarsimp simp*: *Let-def split*: *nat.splits*)

lemma *abs-induce-s-step-to*:
assumes *s-perm-inv α T B SA0 SA i*
and *length ST* = *length T*
and ∀ *k* < *length ST*. *ST* ! *k* = *suffix-type T k*
and *i* < *length SA*
shows *induce-s-step* (*B*, *SA*, *i*) (*α*, *T*, *ST*) = *abs-induce-s-step* (*B*, *SA*, *i*) (*α*, *T*)
by (*metis abs-induce-s-step-to-r2 assms induce-s-step-r2-to*)

lemma *abs-induce-s-base-to′*:
assumes *s-perm-inv α T B SA0 SA n*
and *length ST* = *length T*
and ∀ *k* < *length ST*. *ST* ! *k* = *suffix-type T k*
and *n* < *length SA*
shows *repeat m induce-s-step* (*B*, *SA*, *n*) (*α*, *T*, *ST*) = *repeat m abs-induce-s-step* (*B*, *SA*, *n*) (*α*, *T*)
using *assms*(*1*,*4*)
proof (*induct m arbitrary*: *B SA n*)
case *0*
then show *?case*
by (*simp add*: *repeat-0*)
next
case (*Suc m*)
note *IH* = *this* **and**
R0 = *repeat-step*[*of m abs-induce-s-step* (*B*, *SA*, *n*) (*α*, *T*)] **and**
R1 = *repeat-step*[*of m induce-s-step* (*B*, *SA*, *n*) (*α*, *T*, *ST*)]

from *repeat-abs-induce-s-step-index*[*of m B SA n α T*]
obtain *B′ SA′* **where** *S*:
repeat m abs-induce-s-step (*B*, *SA*, *n*) (*α*, *T*) = (*B′*, *SA′*, *n* − *m*)
length SA′ = *length SA*
length B′ = *length B*
by *blast*

have *n* − *m* < *length SA*
using *Suc.prems*(*2*) **by** *auto*
hence *n* − *m* < *length SA′*
using *S*(*2*) **by** *fastforce*

from *IH*(*1*)[*OF IH*(*2*,*3*)] *R1 S*
have *repeat* (*Suc m*) *induce-s-step* (*B*, *SA*, *n*) (*α*, *T*, *ST*)
= *induce-s-step* (*B′*, *SA′*, *n* − *m*) (*α*, *T*, *ST*)
by *simp*
moreover

```
  from IH(1)[OF IH(2)] R0 S
  have repeat (Suc m) abs-induce-s-step (B, SA, n) (α, T)
        = abs-induce-s-step (B', SA', n − m) (α, T)
    by simp
  moreover
  let ?P = λ(B, SA, i). s-perm-inv α T B SA0 SA i
  have s-perm-inv α T B' SA0 SA' (n − m)
    by (rule repeat-maintain-inv[of ?P abs-induce-s-step (α, T) (B, SA, n) m,
                                 simplified S, simplified, OF - IH(2)];
        clarsimp simp del: abs-induce-s-step.simps;
        erule (1) abs-induce-s-perm-step)
  with abs-induce-s-step-to[OF - assms(2,3) ‹n − m < length SA'›, of α B' SA0]
  have induce-s-step (B', SA', n − m) (α, T, ST) = abs-induce-s-step (B', SA', n
− m) (α, T)
    by blast
  ultimately show ?case
    by simp
qed

lemma repeat-abs-induce-step-gre-length:
  assumes length SA = length T
  shows
    length T ≤ Suc n ⟹
     repeat (Suc m) abs-induce-s-step (B, SA, Suc n) (α, T)
      = repeat m abs-induce-s-step (B, SA, n) (α, T)
proof (induct m arbitrary: n)
  case 0
  then show ?case
    by (simp add: repeat-0 repeat-step Let-def assms)
next
  case (Suc m)
  note IH = this

  from repeat-step[of Suc m abs-induce-s-step (B, SA, Suc n) (α, T)]
       IH(1)[OF IH(2)]
  have repeat (Suc (Suc m)) abs-induce-s-step (B, SA, Suc n) (α, T)
        = abs-induce-s-step (repeat m abs-induce-s-step (B, SA, n) (α, T)) (α, T)
    by presburger
  with repeat-step[of m abs-induce-s-step (B, SA, n) (α, T)]
  show ?case
    by presburger
qed

lemma abs-induce-s-base-to:
  assumes s-perm-pre α T B SA (length T)
  and     length ST = length T
  and     ∀ k < length ST. ST ! k = suffix-type T k
shows induce-s-base α T ST B SA = abs-induce-s-base α T B SA
proof −
```

note *A* = *assms*(*1*)[*simplified s-perm-pre-def*]

from *assms*(*1*)[*simplified s-perm-pre-def*]
have *s-perm-inv* α *T B SA SA* (*length T*)
by (*simp add*: *s-perm-inv-established*)
with *abs-induce-s-base-to'*[*OF - assms*(*2−*)]
have *repeat* (*length T − Suc 0*) *induce-s-step* (*B*, *SA*, *length T − Suc 0*) (α, *T*, *ST*)
= *repeat* (*length T − Suc 0*) *abs-induce-s-step* (*B*, *SA*, *length T − Suc 0*) (α, *T*)
by (*metis Suc-lessD Suc-pred A diff-Suc-less s-perm-inv-maintained-step-c1*)
moreover
have *repeat* (*length T*) *abs-induce-s-step* (*B*, *SA*, *length T*) (α, *T*)
= *repeat* (*length T − Suc 0*) *abs-induce-s-step* (*B*, *SA*, *length T − Suc 0*) (α, *T*)
by (*metis Suc-lessD Suc-pred A repeat-abs-induce-step-gre-length*)
ultimately show *?thesis*
by (*simp add*: *abs-induce-s-base-def induce-s-base-def*)
qed

lemma *abs-induce-s-eq*:
assumes *s-perm-pre* α *T B SA* (*length T*)
and *length ST* = *length T*
and $\forall k <$ *length ST*. *ST* ! *k* = *suffix-type T k*
shows *abs-induce-s* α *T B SA* = *induce-s* α *T ST B SA*
by (*simp add*: *assms abs-induce-s-base-to abs-induce-s-def induce-s-def*)

end
theory *Induce-Verification*
imports
../abs−proof/Abs-Induce-Verification
../def/Induce
Induce-S-Verification Induce-L-Verification Bucket-Insert-Verification
begin

115 Induce

lemma *sa-induce-to-r0*:
assumes *set LMS* = {*i*. *abs-is-lms T i*}
and *distinct LMS*
and *valid-list T*
and *strict-mono* α
and α *bot* = *0*
and *Suc 0* < *length T*
shows *abs-sa-induce* α *T LMS* = *sa-induce-r0* α *T LMS*
proof −

let *?ST* = *abs-get-suffix-types T*

```
  note A = length-abs-get-suffix-types[of T]

  from get-suffix-types-correct[of T] A
  have B: ∀ i < length ?ST. ?ST ! i = suffix-type T i
    by simp

  let ?B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))] and
      ?B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))] and
      ?SA0 = replicate (length T) (length T)

  let ?B2 = ?B0[0 := 0]
  let ?SA1 = abs-bucket-insert α T ?B0 ?SA0 (rev LMS)
  let ?SA2 = abs-induce-l α T ?B1 ?SA1
  let ?SA3 = abs-induce-s α T (?B0[0 := 0]) ?SA2

  let ?SA2' = induce-l α T ?ST ?B1 ?SA1
  let ?SA3' = induce-s α T ?ST (?B0[0 := 0]) ?SA2

  from lms-pre-established[OF assms(1,2,4)]
  have lms-pre α T ?B0 ?SA0 (rev LMS) .

  have l-perm-pre α T ?B1 ?SA1
    using ‹lms-pre α T ?B0 ?SA0 (rev LMS)›
assms(3,4) l-perm-pre-established by blast
  with A B
  have ?SA2 = ?SA2'
    using abs-induce-l-eq l-perm-pre-elims(7) by blast

  have s-perm-pre α T ?B2 ?SA2 (length T)
     using ‹l-perm-pre α T ?B1 ?SA1› ‹lms-pre α T ?B0 ?SA0 (rev LMS)›
assms(3−6)
       s-perm-pre-established by blast
  with A B
  have ?SA3 = ?SA3'
    using abs-induce-s-eq by blast
  then show ?thesis
    by (metis ‹?SA2 = ?SA2'› abs-sa-induce-def sa-induce-r0-def)
qed

definition sa-induce-r1 ::
  (('a :: {linorder, order-bot}) ⇒ nat) ⇒
   'a list ⇒
   SL-types list ⇒
   nat list ⇒
   nat list
  where
sa-induce-r1 α T ST LMS =
  (let
     B0 = map (bucket-end α T) [0..<Suc (α (Max (set T)))];
```

```
    B1 = map (bucket-start α T) [0..<Suc (α (Max (set T)))];

    — Initialise SA
    SA = replicate (length T) (length T);

    — Insert the LMS types into the suffix array
    SA = abs-bucket-insert α T B0 SA (rev LMS);

    — Insert the L types into the suffix array
    SA = induce-l α T ST B1 SA

  — Insert the S types into the suffix array
  in induce-s α T ST (B0[0 := 0]) SA)

lemma sa-induce-r0-to-r1:
  assumes length ST = length T
  and     ∀ i < length ST. ST ! i = suffix-type T i
shows sa-induce-r0 α T LMS = sa-induce-r1 α T ST LMS
proof −
  let ?ST = abs-get-suffix-types T

  note A = length-abs-get-suffix-types[of T]

  from get-suffix-types-correct[of T] A
  have B: ∀ i < length ?ST. ?ST ! i = suffix-type T i
    by simp
  with A
  have ?ST = ST
    by (simp add: assms nth-equalityI)
  then show ?thesis
    by (simp add: sa-induce-r0-def sa-induce-r1-def)
qed

lemma sa-induce-to-r1:
  assumes set LMS = {i. abs-is-lms T i}
  and     distinct LMS
  and     valid-list T
  and     strict-mono α
  and     α bot = 0
  and     Suc 0 < length T
  and     length ST = length T
  and     ∀ i < length ST. ST ! i = suffix-type T i
shows abs-sa-induce α T LMS = sa-induce-r1 α T ST LMS
  by (simp add: assms sa-induce-r0-to-r1 sa-induce-to-r0)

lemma sa-induce-r1-to-r2:
  sa-induce-r1 α T ST LMS = sa-induce-r2 α T ST LMS
  by (simp add: abs-bucket-insert-eq sa-induce-r1-def sa-induce-r2-def)
```

lemma *abs-sa-induce-to-r2*:
assumes *set LMS* = {*i*. *abs-is-lms T i*}
and *distinct LMS*
and *valid-list T*
and *strict-mono* α
and α *bot* = *0*
and *Suc 0* < *length T*
and *length ST* = *length T*
and ∀ *i* < *length ST*. *ST* ! *i* = *suffix-type T i*
shows *abs-sa-induce* α *T LMS* = *sa-induce-r2* α *T ST LMS*
by (*metis assms sa-induce-r1-to-r2 sa-induce-to-r1*)

end
theory *Get-Types-Verification*
imports
../abs−def/Abs-SAIS
../def/Get-Types
begin

116 Suffix Types

lemma *get-suffix-types-step-r0-ret*:
∃ *xs′ i′*. *get-suffix-types-step-r0* (*xs*, *i*) *ys* = (*xs′*, *i′*) ∧
length xs′ = *length xs* ∧ (*i* = *0* ⟶ *i′* = *0*) ∧ (∃ *j*. *i* = *Suc j* ⟶ *i′* = *j*)
by (*cases i*; *simp*)

lemma *get-suffix-types-step-r0-0*:
get-suffix-types-step-r0 (*xs*, *0*) *ys* = (*xs*, *0*)
by *simp*

lemma *get-suffix-types-step-r0-Suc*:
⟦*Suc i* < *length xs*; *length xs* = *length ys*; ∀ *k* < *length xs*. *i* < *k* ⟶ *xs* ! *k* = *suffix-type ys k*⟧ ⟹
get-suffix-types-step-r0 (*xs*, *Suc i*) *ys* = (*xs*[*i* := *suffix-type ys i*], *i*)
apply *simp*
by (*metis nth-gr-imp-l-type nth-less-imp-s-type suffix-type-neq*)

fun *get-suffix-types-inv*
where *get-suffix-types-inv ys* (*xs*, *i*) =
(*length xs* = *length ys* ∧ *i* < *length xs* ∧ (∀ *k* < *length xs*. *i* ≤ *k* ⟶ *xs* ! *k* = *suffix-type ys k*))

lemma *get-suffix-types-inv-maintained*:
assumes *get-suffix-types-inv ys* (*xs*, *i*)
shows *get-suffix-types-inv ys* (*get-suffix-types-step-r0* (*xs*, *i*) *ys*)
proof (*cases i*)
case *0*
hence *get-suffix-types-step-r0* (*xs*, *i*) *ys* = (*xs*, *0*)
using *get-suffix-types-step-r0-0* **by** *simp*

```
  moreover
  have get-suffix-types-inv ys (xs, 0)
    using 0 assms by auto
  ultimately show ?thesis
    by presburger
next
  case (Suc n)
  hence get-suffix-types-step-r0 (xs, i) ys = (xs[n := suffix-type ys n], n)
    by (metis Suc-leI assms get-suffix-types-inv.simps get-suffix-types-step-r0-Suc)
  moreover
  have get-suffix-types-inv ys (xs[n := suffix-type ys n], n)
    using Suc assms le-eq-less-or-eq by fastforce
  ultimately show ?thesis
    by simp
qed

lemma get-suffix-types-inv-established:
  xs ≠ [] ⟹ get-suffix-types-inv xs (replicate (length xs) S-type, length xs − Suc
0)
  by (simp add: suffix-type-last)

lemma get-suffix-types-base-prod':
  ∃ xs'. repeat n get-suffix-types-step-r0 (xs, m) ys = (xs', m − n)
proof (induct n arbitrary: xs m)
  case 0
  then show ?case
    by (simp add: repeat-0)
next
  case (Suc n)
  note IH = this

  from repeat-step[of n get-suffix-types-step-r0 (xs, m) ys]
  have repeat (Suc n) get-suffix-types-step-r0 (xs, m) ys
        = get-suffix-types-step-r0 (repeat n get-suffix-types-step-r0 (xs, m) ys) ys .
  moreover
  from IH[of xs m]
  obtain xs' where
    repeat n get-suffix-types-step-r0 (xs, m) ys = (xs', m − n)
    by blast
  moreover
  have ∃ xs''. get-suffix-types-step-r0 (xs', m − n) ys = (xs'', m − Suc n)
  proof (cases m−n)
    case 0
    hence get-suffix-types-step-r0 (xs', m − n) ys = (xs', 0)
      by auto
    moreover
    have m − Suc n = 0
      using 0 by auto
    ultimately show ?thesis
```

```
    by simp
  next
    case (Suc k)
    have (m − n < length xs′ ∧ m − n < length ys) ∨ ¬(m − n < length xs′ ∧ m
− n < length ys)
      by blast
    moreover
    have m − n < length xs′ ∧ m − n < length ys ⟹ ?thesis
      by (clarsimp simp add: Suc diff-Suc)
    moreover
    have ¬(m − n < length xs′ ∧ m − n < length ys) ⟹ ?thesis
     by (clarsimp simp add: Suc diff-Suc)
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?case
    by presburger
qed

lemma get-suffix-types-inv-holds:
  assumes xs ≠ []
  shows get-suffix-types-inv xs (get-suffix-types-base xs)
  unfolding get-suffix-types-base-def
  apply (rule repeat-maintain-inv)
   apply (metis get-suffix-types-inv-maintained prod.collapse)
  apply (rule get-suffix-types-inv-established[OF assms])
  done

lemma get-suffix-types-base-prod:
  ∃ xs′. get-suffix-types-base xs = (xs′, 0)
  unfolding get-suffix-types-base-def
  by (metis cancel-comm-monoid-add-class.diff-cancel get-suffix-types-base-prod′)

lemma get-suffix-types-base-ref:
  get-suffix-types-base xs = (abs-get-suffix-types xs, 0)
proof (cases xs ≠ [])
  assume ¬ xs ≠ []
  then show ?thesis
    by (clarsimp simp: get-suffix-types-base-def repeat-0 get-suffix-types-def)
next
  assume xs ≠ []
  with get-suffix-types-inv-holds
  have get-suffix-types-inv xs (get-suffix-types-base xs)
    by blast
  moreover
  from get-suffix-types-base-prod[of xs]
  obtain xs′ where
    get-suffix-types-base xs = (xs′, 0)
    by blast
```

ultimately have *get-suffix-types-inv xs* (*xs'*, *0*)
by *auto*
moreover
have *abs-get-suffix-types xs* = *xs'*
unfolding *list-eq-iff-nth-eq*
by (*metis bot-nat-0.extremum calculation get-suffix-types-correct*
get-suffix-types-inv.simps
length-abs-get-suffix-types)
ultimately show *?thesis*
by (*simp add*: ‹*get-suffix-types-base xs* = (*xs'*, *0*)›)
qed

lemma *get-suffix-types-eq*:
get-suffix-types xs = *abs-get-suffix-types xs*
by (*simp add*: *get-suffix-types-base-ref get-suffix-types-def*)

lemmas *length-get-suffix-types* =
length-abs-get-suffix-types[*simplified get-suffix-types-eq*]

117 LMS types

lemma *is-lms-refinement*:
assumes *length ST* = *length T* ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*
shows *is-lms-ref ST* = *abs-is-lms T*
proof
fix *i*
show *is-lms-ref ST i* = *abs-is-lms T i*
proof (*cases i*)
case *0*
then show *?thesis*
by (*simp add*: *abs-is-lms-0*)
next
case (*Suc n*)
then show *?thesis*
by (*metis Suc-lessD assms abs-is-lms-def abs-is-lms-imp-less-length is-lms-ref.simps*(*2*))
qed
qed

118 Extracting LMS types

lemma *extract-lms-eq*:
⟦*length ST* = *length T*; ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*⟧ ⟹
extract-lms ST = *abs-extract-lms T*
by (*clarsimp simp*: *fun-eq-iff is-lms-refinement*)

119 LMS Substrings

lemma *find-next-lms-refinement*:

⟦*length ST* = *length T*; ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*⟧ ⟹
find-next-lms ST= *abs-find-next-lms T*
unfolding *find-next-lms-def abs-find-next-lms-def*
apply (*clarsimp simp*: *is-lms-refinement fun-eq-iff*)
by *argo*

lemma *lms-slice-refinement*:
⟦*length ST* = *length T*; ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*⟧ ⟹
lms-slice-ref T ST = *lms-slice T*
unfolding *lms-slice-ref-def lms-slice-def*
by (*clarsimp simp*: *find-next-lms-refinement fun-eq-iff*)

120 Rename Mapping

lemma *rename-mapping′-refinement*:
assumes *length ST* = *length T* ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*
shows *rename-mapping′ T ST* = *abs-rename-mapping′ T*
proof (*intro fun-eq-iff*[*THEN iffD2*] *allI*)
fix *xs ns i*
show *rename-mapping′ T ST xs ns i* = *abs-rename-mapping′ T xs ns i*
using *assms*
proof (*induct rule*: *rename-mapping′.induct*[*of* - *T ST xs ns i*])
case (*1 T ST ns i*)
then show *?case*
by *simp*
next
case (*2 T ST x ns i*)
then show *?case*
by *simp*
next
case (*3 T ST a b xs ns i*)
then show *?case*
by (*simp add*: *lms-slice-refinement*)
qed
qed

lemma *rename-mapping-refinement*:
assumes *length ST* = *length T*
assumes ∀ *i* < *length T*. *ST* ! *i* = *suffix-type T i*
shows *rename-mapping T ST* = *abs-rename-mapping T*
by (*clarsimp simp*: *fun-eq-iff assms rename-mapping′-refinement abs-rename-mapping-def*
rename-mapping-def)

end
theory *SAIS-Verification*
imports
Get-Types-Verification
Induce-Verification
../abs−proof/Abs-SAIS-Verification-With-Valid-Precondition

../def/SAIS

begin

121 SAIS

termination *sais-r0*
apply (*relation measure* (λ*xs*. *length xs*))
apply *simp*
apply (*simp* (*no-asm-simp*)
del: *List.list.size*(*4*)
only: *extract-lms-eq*[*OF length-get-suffix-types get-suffix-types-correct*]
rename-mapping-refinement[*OF length-get-suffix-types*
get-suffix-types-correct] *get-suffix-types-eq*)
apply (*simp* (*no-asm-simp*)
del: *List.list.size*(*4*)
add: *rename-list-length length-filter-lms*)
done

lemma *abs-sais-r0-distinct-simp*:
assumes *T* = *a* # *b* # *xs*
and *ST* = *abs-get-suffix-types T*
and *LMS0* = *extract-lms ST* [*0*..<*length T*]
and *SA* = *sa-induce id T ST LMS0*
and *LMS* = *extract-lms ST SA*
and *names* = *rename-mapping T ST LMS*
and *T′* = *rename-string LMS0 names*
and *distinct T′*
shows *sais-r0 T* = *sa-induce id T ST LMS*
proof −
let *?P* = λ*ys*. *sais-r0* (*a* # *b* # *xs*) = *ys*
from *subst*[*OF sais-r0.simps*(*3*), *of ?P a b xs*, *simplified Let-def*
assms(*1*−*7*)[*symmetric*] *assms*(*8*), *simplified*]
show *?thesis*
by *simp*
qed

lemma *abs-sais-r0-not-distinct-simp*:
assumes *T* = *a* # *b* # *xs*
and *ST* = *abs-get-suffix-types T*
and *LMS0* = *extract-lms ST* [*0*..<*length T*]
and *SA* = *sa-induce id T ST LMS0*
and *LMS* = *extract-lms ST SA*
and *names* = *rename-mapping T ST LMS*
and *T′* = *rename-string LMS0 names*
and *LMS1* = *order-lms LMS0* (*sais-r0 T′*)
and ¬*distinct T′*
shows *sais-r0 T* = *sa-induce id T ST LMS1*
proof −

```
  let ?P = λys. sais-r0 (a # b # xs) = ys
  from subst[OF sais-r0.simps(3), of ?P a b xs, simplified Let-def
        assms(1−8)[symmetric] assms(9), simplified]
  show ?thesis
    by simp
qed

lemma abs-sais-to-r0:
  valid-list T ⟹ abs-sais T = sais-r0 T
proof(induct rule: abs-sais.induct[of - T])
  case 1
  then show ?case
    by simp
next
  case (2 x)
  then show ?case
    by simp
next
  case (3 a b xs)
  note IH = this

  let ?T = a # b # xs
  have T: ?T = a # b # xs
    by (simp only:)

 let ?ST = abs-get-suffix-types ?T
  have ST: ?ST = abs-get-suffix-types ?T
    by (simp only:)
  from get-suffix-types-correct[of ?T] length-abs-get-suffix-types[of ?T]
  have ∀ i < length ?ST. ?ST ! i = suffix-type ?T i
    by (simp add: get-suffix-types-eq)
  note st-thms = length-get-suffix-types[of ?T]
                 ‹∀ i < length ?ST. ?ST ! i = suffix-type ?T i›

  let ?LMS1 = abs-extract-lms ?T [0..<length ?T]
  let ?LMS1' = extract-lms ?ST [0..<length ?T]
  have LMS1: ?LMS1 = abs-extract-lms ?T [0..<length ?T]
    by (simp only:)
  have LMS1': ?LMS1' = extract-lms ?ST [0..<length ?T]
    by (simp only:)
  have distinct ?LMS1
    using distinct-abs-extract-lms
    by fastforce
  have set ?LMS1 = {i. abs-is-lms ?T i}
    using set-abs-extract-lms-eq-all-lms
    by (metis comp-apply)
  have ?LMS1 = ?LMS1'
    by (metis extract-lms-eq get-suffix-types-correct length-get-suffix-types)
 note lms1-thms = ‹set ?LMS1 = {i. abs-is-lms ?T i}› ‹distinct ?LMS1› ‹?LMS1
```

```
= ?LMS1'⟩

  have id: strict-mono (id :: nat ⇒ nat) (id :: nat ⇒ nat) bot = 0
    by (simp add: strict-mono-def bot-nat-def)+

  have len: Suc 0 < length ?T
    by simp

  let ?SA1 = abs-sa-induce id ?T ?LMS1
  have SA1: ?SA1 = abs-sa-induce id ?T ?LMS1
    by (simp only:)
  let ?SA1' = sa-induce id ?T ?ST ?LMS1'
  have SA1': ?SA1' = sa-induce id ?T ?ST ?LMS1'
    by (simp only:)
  have ?SA1 = ?SA1'
    by (metis 3.prems len lms1-thms id  st-thms abs-sa-induce-to-r2)

  let ?LMS2 = abs-extract-lms ?T ?SA1
  let ?LMS2' = extract-lms ?ST ?SA1'
  have LMS2: ?LMS2 = abs-extract-lms ?T ?SA1
    by (simp only:)
  have LMS2': ?LMS2' = extract-lms ?ST ?SA1'
    by (simp only:)
  have ?LMS2 = ?LMS2'
    using ⟨?SA1 = ?SA1'⟩ st-thms comp-apply is-lms-refinement
    by (metis (no-types, lifting) get-suffix-types-eq)
  have ?LMS1 <~~> ?LMS2
    by (metis 3.prems distinct-filter-abs-sa-induce lms1-thms len id
          distinct-set-imp-perm filter-abs-sa-induce-eq-all-lms)
  hence distinct ?LMS2 set ?LMS2 = {i. abs-is-lms ?T i}
    using lms1-thms perm-distinct-iff perm-set-eq by blast+
  have ordlistns.sorted (map (lms-slice ?T) ?LMS2)
     by (metis 3.prems ⟨distinct ?LMS1⟩ ⟨set ?LMS1 = {i. abs-is-lms ?T i}⟩
comp-apply len id
          ordlistns.sorted-filter abs-sa-induce-prefix-sorted)
  note lms2-thms = ⟨distinct ?LMS2⟩ ⟨set ?LMS2 = {i. abs-is-lms ?T i}⟩
                ⟨ordlistns.sorted (map (lms-slice ?T) ?LMS2)⟩
                ⟨?LMS2 = ?LMS2'⟩

  let ?names = abs-rename-mapping ?T ?LMS2
  let ?names' = rename-mapping ?T ?ST ?LMS2'
  have names: ?names = abs-rename-mapping ?T ?LMS2
    by (simp only:)
  have names': ?names' = rename-mapping ?T ?ST ?LMS2'
    by (simp only:)
  have ?names = ?names'
    by (metis st-thms lms2-thms(4) rename-mapping-refinement)

  let ?T' = rename-string ?LMS1 ?names
```

```
  let ?T'' = rename-string ?LMS1' ?names'
  have T': ?T' = rename-string ?LMS1 ?names
    by (simp only:)
  have T'': ?T'' = rename-string ?LMS1' ?names'
    by (simp only:)
  have ?T' = ?T''
    using ‹?names = ?names'› lms1-thms(3) by argo

  let ?LMS3 = order-lms ?LMS1 (abs-sais ?T')
  let ?LMS3' = order-lms ?LMS1' (sais-r0 ?T'')
  have LMS3: ?LMS3 = order-lms ?LMS1 (abs-sais ?T')
    by (simp only:)
  have LMS3': ?LMS3' = order-lms ?LMS1' (sais-r0 ?T'')
    by (simp only:)

  have ?LMS1 = lms0-seq ?T
    by (metis comp-apply lms-seq-0-zeroth-lms lms-seq-def)
  with abs-sais-reduced-string[OF - lms2-thms(1-3) names T']
  have ?T' = lms0-map ?T
    by blast

  have abs-is-lms ?T (lms0 ?T)
    by (metis 3.prems abs-find-next-lms-less-length-abs-is-lms abs-is-lms-last
          len length-Cons no-lms-between-i-and-next not-less-eq)

  from valid-list-lms-map[OF IH(2) ‹abs-is-lms ?T (lms0 ?T)›]
  have valid-list (lms0-map ?T) .
  hence valid-list ?T'
    by (simp only: ‹?T' = (lms0-map ?T)›)

  have distinct ?T' ⟹ ?case
  proof −
    assume distinct ?T'
    hence distinct ?T''
      by (simp only: ‹?T' = ?T''›)
    note lms2-thms = filter-abs-sa-induce-eq-all-lms[OF lms1-thms(1,2) IH(2) id
len]
                  distinct-filter-abs-sa-induce[OF lms1-thms(1,2) IH(2) id len]
    from abs-sa-induce-to-r2 lms2-thms(1,2) IH(2) id len st-thms
    have abs-sa-induce id ?T ?LMS2 = sa-induce id ?T ?ST ?LMS2'
      by (metis ‹?LMS2 = ?LMS2'› comp-apply)
    with abs-sais-distinct-simp[OF T LMS1 SA1 LMS2 names T' ‹distinct ?T'›]
      abs-sais-r0-distinct-simp[OF T ST LMS1' SA1' LMS2' names' T'' ‹distinct
?T''›]
    show ?thesis
      by presburger
  qed
  moreover
  have ¬distinct ?T' ⟹ ?case
```

```
  proof −
    assume ¬distinct ?T'
    hence ¬distinct ?T''
      using ‹?T' = ?T''› by argo

    from IH(1)[OF T LMS1 SA1 LMS2 names T', OF ‹¬distinct ?T'› ‹valid-list
?T'›]
    have abs-sais ?T' = sais-r0 ?T''
      using ‹?T' = ?T''› by argo
    hence ?LMS3' = ?LMS3
      using lms1-thms(3) by argo

    have abs-sais-perm: abs-sais ?T' <~~> [0..<length ?LMS1]
      using abs-sais-permutation[OF ‹valid-list ?T'›, simplified rename-list-length]
      by blast

    note lms3-thms = abs-order-lms-eq-all-lms[OF abs-sais-perm lms1-thms(1)]
                     distinct-abs-order-lms[OF abs-sais-perm lms1-thms(2)]
    from abs-sa-induce-to-r2[OF lms3-thms ‹valid-list ?T› id len st-thms]
    have abs-sa-induce id ?T ?LMS3 = sa-induce id ?T ?ST ?LMS3'
      using ‹abs-sais ?T' = sais-r0 ?T''› lms1-thms(3) by argo
   with abs-sais-not-distinct-simp[OF T LMS1 SA1 LMS2 names T' LMS3 ‹¬distinct
?T'›]
          abs-sais-r0-not-distinct-simp[OF T ST LMS1' SA1' LMS2' names' T''
LMS3' ‹¬distinct ?T''›]
        ‹abs-sais ?T' = sais-r0 ?T'›
    show ?thesis
     by presburger
  qed
  ultimately show ?case
    by blast
qed

termination sais-r1
  apply (relation measure (λxs. length xs))
   apply simp
  apply (simp (no-asm-simp)
              del: List.list.size(4)
             only: extract-lms-eq[OF length-abs-get-suffix-types get-suffix-types-correct]
                           rename-mapping-refinement[OF length-abs-get-suffix-types
get-suffix-types-correct])
  apply (simp (no-asm-simp)
              del: List.list.size(4)
              add: rename-list-length length-filter-lms)
  apply (metis get-suffix-types-correct get-suffix-types-eq is-lms-refinement
               length-filter-lms length-get-suffix-types list.discI)
  done

lemma abs-sais-r0-to-r1:
```

sais-r1 T = sais-r0 T
apply (*induct rule*: *sais-r0.induct*[*of* - *T*])
apply *simp*
apply *simp*
apply (*subst sais-r1.simps*)
apply (*subst get-suffix-types-eq*)
apply (*subst sais-r0.simps*)
apply (*clarsimp simp only*: *Let-def split*: *if-splits*)
by *presburger*

lemma *abs-sais-to-r1*:
valid-list T $\Longrightarrow$ *sais-r1 T = abs-sais T*
by (*simp add*: *abs-sais-r0-to-r1 abs-sais-to-r0*)

122 Correctness

interpretation *sais*: *Suffix-Array-Restricted sais*
by (*simp add*: *Suffix-Array-Restricted.intro Suffix-Array-Restricted.sa-r-permutation Suffix-Array-Restricted.sa-r-sorted abs-sais.Suffix-Array-Restricted-axioms abs-sais-to-r1*)

interpretation *abs-sais-ref-gen*: *Suffix-Array-General sa-nat-wrapper map-to-nat sais*
by (*simp add*: *Suffix-Array-Restricted-imp-General sais.Suffix-Array-Restricted-axioms*)

theorem *sais-gen-is-Suffix-Array-General*:
Suffix-Array-General sa $\longleftrightarrow$ *sa = sa-nat-wrapper map-to-nat sais*
using *Suffix-Array-General-determinism abs-sais-ref-gen.Suffix-Array-General-axioms*
by *auto*

end
theory *Code-Extraction*
imports *../abs−proof/Abs-SAIS-Verification ../proof/SAIS-Verification*
begin

lemma [*code*]:
abs-is-lms T i =
(*if i > 0 then*
if suffix-type T i = S-type $\wedge$ *suffix-type T (i − 1) = L-type*
then True
else False
else False)
by (*metis abs-is-lms-0 One-nat-def Suc-pred bot-nat-0.not-eq-extremum i-s-type-imp-Suc-i-not-lms abs-is-lms-def suffix-type-def*)

definition
bucket-upt-code :: ('*a* :: {*linorder*,*order-bot*} $\Rightarrow$ *nat*) $\Rightarrow$ '*a list* $\Rightarrow$ *nat* $\Rightarrow$ *nat set*
where

```
  bucket-upt-code α T b ≡
    set (filter (λx. α (T ! x) < b) [0..<length T])

lemma [code]:
  bucket-upt α T b = bucket-upt-code α T b
proof(safe)
  fix x
  assume x ∈ bucket-upt α T b
  hence x < length T α (T ! x) < b
    by (simp add: bucket-upt-def)+
  then show x ∈ bucket-upt-code α T b
    by (simp add: bucket-upt-code-def)
next
  fix x
  assume x ∈ bucket-upt-code α T b
  hence x < length T α (T ! x) < b
    by (simp add: bucket-upt-code-def)+
  then show x ∈ bucket-upt α T b
    by (simp add: bucket-upt-def)
qed


export-code abs-sais in Haskell
  module-name SAIS file-prefix abs-sais

export-code sais in Haskell
  module-name SAIS-REF file-prefix sais

end
theory SACA-Equiv
  imports sais/abs−proof/Abs-SAIS-Verification
          simple/Simple-SACA-Verification
          sais/proof/SAIS-Verification
begin

lemma Suffix-Array-General-imp-suffix-array:
  Suffix-Array-General sa ⟹
   sa s = simple-saca s
 using Suffix-Array-General-determinism simple-saca.Suffix-Array-General-axioms
by blast

theorem Suffix-Array-General-equiv-spec:
  Suffix-Array-General sa ⟷
   sa = simple-saca
 using Suffix-Array-General-imp-suffix-array simple-saca.Suffix-Array-General-axioms
by blast

corollary abs-sais-equiv-simple-saca:
  sa-nat-wrapper map-to-nat abs-sais = simple-saca
```

using *Suffix-Array-General-equiv-spec abs-sais-gen.Suffix-Array-General-axioms*
by *auto*

corollary *sais-equiv-simple-saca*:
sa-nat-wrapper map-to-nat sais = simple-saca
using *sais-gen-is-Suffix-Array-General*
Suffix-Array-General-equiv-spec
by *auto*

end

References

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