

# Stirling's formula

Manuel Eberl

June 20, 2024

## Abstract

This work contains a proof of Stirling's formula both for the factorial  $n! \sim \sqrt{2\pi n}(n/e)^n$  on natural numbers and the real Gamma function  $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$ . The proof is based on work by Graham Jameson [3].

This is then extended to the full asymptotic expansion

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{k+1}}{k(k+1)} z^{-k} - \frac{1}{n} \int_0^\infty B_n([t])(t+z)^{-n} dt$$

uniformly for all complex  $z \neq 0$  in the cone  $\arg(z) \leq \alpha$  for any  $\alpha \in (0, \pi)$ , with which the above asymptotic relation for  $\Gamma$  is also extended to complex arguments.

## Contents

<b>1</b>	<b>Stirling's Formula</b>	<b>1</b>
<b>2</b>	<b>Complete asymptotics of the logarithmic Gamma function</b>	<b>5</b>
2.1	Auxiliary Facts . . . . .	6
2.2	Cones in the complex plane . . . . .	6
2.3	Another integral representation of the Beta function . . . . .	7
2.4	Asymptotics of the real log $\Gamma$ function and its derivatives . . . . .	8
2.5	Asymptotics of the complex Gamma function . . . . .	15

## 1 Stirling's Formula

**theory** *Stirling-Formula*

**imports**

*HOL-Analysis.Analysis*

*Landau-Symbols.Landau-More*

*HOL-Real-Asymp.Real-Asymp*

**begin**

**context**  
**begin**

First, we define the  $S_n^*$  from Jameson's article:

**qualified definition**  $S' :: nat \Rightarrow real \Rightarrow real$  **where**

$$S' n x = 1/(2*x) + (\sum r=1..<n. 1/(of-nat r+x)) + 1/(2*(n+x))$$

Next, the trapezium (also called  $T$  in Jameson's article):

**qualified definition**  $T :: real \Rightarrow real$  **where**

$$T x = 1/(2*x) + 1/(2*(x+1))$$

Now we define The difference  $\Delta(x)$ :

**qualified definition**  $D :: real \Rightarrow real$  **where**

$$D x = T x - \ln(x+1) + \ln x$$

**qualified lemma**  $S'$ -telescope-trapezium:

**assumes**  $n > 0$

**shows**  $S' n x = (\sum r<n. T (of-nat r + x))$

*<proof>* **lemma**  $stirling$ -trapezium:

**assumes**  $x: (x::real) > 0$

**shows**  $D x \in \{0 .. 1/(12*x^2) - 1/(12 * (x+1)^2)\}$

*<proof>*

The following functions correspond to the  $p_n(x)$  from the article. The special case  $n = 0$  would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition  $n \neq 0$ :

**qualified definition**  $p :: nat \Rightarrow real \Rightarrow real$  **where**

$$p n x = (if n = 0 then 1/x else (\sum r<n. D (real r + x)))$$

We can write the Digamma function in terms of  $S'$ :

**qualified lemma**  $S'$ -LIMSEQ-Digamma:

**assumes**  $x: x \neq 0$

**shows**  $(\lambda n. \ln (real n) - S' n x - 1/(2*x)) \longrightarrow Digamma x$

*<proof>*

Moreover, we can give an expansion of  $S'$  with the  $p$  as variation terms.

**qualified lemma**  $S'$ -approx:

$$S' n x = \ln (real n + x) - \ln x + p n x$$

*<proof>*

We define the limit of the  $p$  (simply called  $p(x)$  in Jameson's article):

**qualified definition**  $P :: real \Rightarrow real$  **where**

$$P x = (\sum n. D (real n + x))$$

**qualified lemma**  $D$ -summable:

**assumes**  $x: x > 0$

**shows** *summable* ( $\lambda n. D (real\ n + x)$ )  
 ⟨*proof*⟩ **lemma** *p-LIMSEQ*:  
**assumes**  $x: x > 0$   
**shows**  $(\lambda n. p\ n\ x) \longrightarrow P\ x$   
 ⟨*proof*⟩

This gives us an expansion of the Digamma function:

**lemma** *Digamma-approx*:  
**assumes**  $x: (x :: real) > 0$   
**shows**  $Digamma\ x = \ln\ x - 1 / (2 * x) - P\ x$   
 ⟨*proof*⟩

Next, we derive some bounds on  $P$ :

**qualified lemma** *p-ge-0*:  $x > 0 \implies p\ n\ x \geq 0$   
 ⟨*proof*⟩ **lemma** *P-ge-0*:  $x > 0 \implies P\ x \geq 0$   
 ⟨*proof*⟩ **lemma** *p-upper-bound*:  
**assumes**  $x > 0\ n > 0$   
**shows**  $p\ n\ x \leq 1 / (12 * x^2)$   
 ⟨*proof*⟩ **lemma** *P-upper-bound*:  
**assumes**  $x > 0$   
**shows**  $P\ x \leq 1 / (12 * x^2)$   
 ⟨*proof*⟩

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function  $g$  from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

**qualified definition**  $g :: real \Rightarrow real$  **where**  
 $g\ x = \ln\text{-Gamma}\ x - (x - 1/2) * \ln\ x + x$

**qualified lemma** *DERIV-g*:  $x > 0 \implies (g\ \text{has-field-derivative}\ -P\ x)\ (at\ x)$   
 ⟨*proof*⟩ **lemma** *isCont-P*:  
**assumes**  $x > 0$   
**shows**  $isCont\ P\ x$   
 ⟨*proof*⟩ **lemma** *P-continuous-on* [*THEN* *continuous-on-subset*]:  $continuous\text{-on}\ \{0 < ..\}$   
 $P$   
 ⟨*proof*⟩ **lemma** *P-integrable*:  
**assumes**  $a: a > 0$   
**shows**  $P\ integrable\text{-on}\ \{a.. \}$   
 ⟨*proof*⟩ **definition**  $c :: real$  **where**  $c = \text{integral}\ \{1.. \}\ (\lambda x. -P\ x) + g\ 1$

We can now give bounds on  $g$ :

**qualified lemma** *g-bounds*:  
**assumes**  $x: x \geq 1$   
**shows**  $g\ x \in \{c..c + 1 / (12 * x)\}$   
 ⟨*proof*⟩

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

**qualified lemma** *ln-Gamma-bounds-aux:*

$x \geq 1 \implies \ln\text{-Gamma } x \geq c + (x - 1/2) * \ln x - x$

$x \geq 1 \implies \ln\text{-Gamma } x \leq c + (x - 1/2) * \ln x - x + 1/(12*x)$

*<proof>* **lemma** *Gamma-bounds-aux:*

**assumes**  $x: x \geq 1$

**shows**  $\text{Gamma } x \geq \exp c * x \text{ powr } (x - 1/2) / \exp x$

$\text{Gamma } x \leq \exp c * x \text{ powr } (x - 1/2) / \exp x * \exp (1/(12*x))$

*<proof>* **lemma** *Gamma-asymp-equiv-aux:*

$\text{Gamma} \sim_{[at-top]} (\lambda x. \exp c * x \text{ powr } (x - 1/2) / \exp x)$

*<proof>*

**include** *asymp-equiv-notation*

*<proof>* **lemma** *exp-1-powr-real [simp]:*  $\exp (1::\text{real}) \text{ powr } x = \exp x$

*<proof>* **lemma** *fact-asymp-equiv-aux:*

$\text{fact} \sim_{[at-top]} (\lambda x. \exp c * \text{sqrt } (\text{real } x) * (\text{real } x / \exp 1) \text{ powr } \text{real } x)$

*<proof>*

**include** *asymp-equiv-notation*

*<proof>*

We can also bound *Digamma* above and below.

**lemma** *Digamma-plus-1-gt-ln:*

**assumes**  $x: x > (0 :: \text{real})$

**shows**  $\text{Digamma } (x + 1) > \ln x$

*<proof>*

**lemma** *Digamma-less-ln:*

**assumes**  $x: x > (0 :: \text{real})$

**shows**  $\text{Digamma } x < \ln x$

*<proof>*

We still need to determine the constant term  $c$ , which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

**qualified lemma** *powr-mult-2:*  $(x::\text{real}) > 0 \implies x \text{ powr } (y * 2) = (x^2) \text{ powr } y$

*<proof>* **lemma** *exp-mult-2:*  $\exp (y * 2 :: \text{real}) = \exp y * \exp y$

*<proof>* **lemma** *exp-c:*  $\exp c = \text{sqrt } (2*\pi)$

*<proof>*

**include** *asymp-equiv-notation*

*<proof>* **lemma**  $c: c = \ln (2*\pi) / 2$

*<proof>*

This gives us the final bounds:

**theorem** *Gamma-bounds:*

**assumes**  $x \geq 1$

**shows**  $\text{Gamma } x \geq \text{sqrt } (2*\pi/x) * (x / \exp 1) \text{ powr } x$  (*is ?th1*)

$\text{Gamma } x \leq \text{sqrt } (2*\pi/x) * (x / \exp 1) \text{ powr } x * \exp (1 / (12 * x))$  (*is*

*?th2*)

*<proof>*

**theorem** *ln-Gamma-bounds:*

**assumes**  $x \geq 1$

**shows**  $\ln\text{-Gamma } x \geq \ln (2*\pi/x) / 2 + x * \ln x - x$  (**is** ?th1)

$\ln\text{-Gamma } x \leq \ln (2*\pi/x) / 2 + x * \ln x - x + 1/(12*x)$  (**is** ?th2)

*<proof>*

**theorem** *fact-bounds:*

**assumes**  $n > 0$

**shows**  $(\text{fact } n :: \text{real}) \geq \text{sqrt } (2*\pi*n) * (n / \text{exp } 1) ^ n$  (**is** ?th1)

$(\text{fact } n :: \text{real}) \leq \text{sqrt } (2*\pi*n) * (n / \text{exp } 1) ^ n * \text{exp } (1 / (12 * n))$  (**is** ?th2)

*<proof>*

**theorem** *ln-fact-bounds:*

**assumes**  $n > 0$

**shows**  $\ln (\text{fact } n :: \text{real}) \geq \ln (2*\pi*n)/2 + n * \ln n - n$  (**is** ?th1)

$\ln (\text{fact } n :: \text{real}) \leq \ln (2*\pi*n)/2 + n * \ln n - n + 1/(12*\text{real } n)$  (**is** ?th2)

*<proof>*

**theorem** *Gamma-asymp-equiv:*

$\text{Gamma} \sim_{[\text{at-top}]} (\lambda x. \text{sqrt } (2*\pi/x) * (x / \text{exp } 1) \text{ powr } x :: \text{real})$

*<proof>*

**theorem** *fact-asymp-equiv:*

$\text{fact} \sim_{[\text{at-top}]} (\lambda n. \text{sqrt } (2*\pi*n) * (n / \text{exp } 1) ^ n :: \text{real})$

*<proof>*

**corollary** *stirling-tendsto-sqrt-pi:*

$(\lambda n. \text{fact } n / (\text{sqrt } (2 * n) * (n / \text{exp } 1) ^ n)) \longrightarrow \text{sqrt } \pi$

*<proof>*

**end**

**end**

## 2 Complete asymptotics of the logarithmic Gamma function

**theory** *Gamma-Asymptotics*

**imports**

*HOL-Complex-Analysis.Complex-Analysis*

*Bernoulli.Bernoulli-FPS*

*Bernoulli.Periodic-Bernpoly*

*Stirling-Formula*

**begin**

## 2.1 Auxiliary Facts

**lemma** *stirling-limit-aux1*:

$((\lambda y. Ln (1 + z * \text{of-real } y) / \text{of-real } y) \longrightarrow z)$  (at-right 0) **for**  $z :: \text{complex}$   
*<proof>*

**lemma** *stirling-limit-aux2*:

$((\lambda y. y * Ln (1 + z / \text{of-real } y)) \longrightarrow z)$  at-top **for**  $z :: \text{complex}$   
*<proof>*

**lemma** *Union-atLeastAtMost*:

**assumes**  $N > 0$

**shows**  $(\bigcup n \in \{0..<N\}. \{\text{real } n.. \text{real } (n + 1)\}) = \{0.. \text{real } N\}$

*<proof>*

## 2.2 Cones in the complex plane

**definition** *complex-cone*  $:: \text{real} \Rightarrow \text{real} \Rightarrow \text{complex set}$  **where**

*complex-cone*  $a\ b = \{z. \exists y \in \{a..b\}. z = \text{rcis } (\text{norm } z)\ y\}$

**abbreviation** *complex-cone'*  $:: \text{real} \Rightarrow \text{complex set}$  **where**

*complex-cone'*  $a \equiv \text{complex-cone } (-a)\ a$

**lemma** *zero-in-complex-cone* [*simp, intro*]:  $a \leq b \implies 0 \in \text{complex-cone } a\ b$

*<proof>*

**lemma** *complex-coneE*:

**assumes**  $z \in \text{complex-cone } a\ b$

**obtains**  $r\ \alpha$  **where**  $r \geq 0\ \alpha \in \{a..b\}\ z = \text{rcis } r\ \alpha$

*<proof>*

**lemma** *arg-cis* [*simp*]:

**assumes**  $x \in \{-\pi <.. \pi\}$

**shows**  $\text{Arg } (\text{cis } x) = x$

*<proof>*

**lemma** *arg-mult-of-real-left* [*simp*]:

**assumes**  $r > 0$

**shows**  $\text{Arg } (\text{of-real } r * z) = \text{Arg } z$

*<proof>*

**lemma** *arg-mult-of-real-right* [*simp*]:

**assumes**  $r > 0$

**shows**  $\text{Arg } (z * \text{of-real } r) = \text{Arg } z$

*<proof>*

**lemma** *arg-rcis* [*simp*]:

**assumes**  $x \in \{-\pi <.. \pi\}\ r > 0$

**shows**  $\text{Arg } (\text{rcis } r\ x) = x$

*<proof>*

**lemma** *rcis-in-complex-cone* [*intro*]:  
**assumes**  $\alpha \in \{a..b\}$   $r \geq 0$   
**shows**  $rcis\ r\ \alpha \in complex-cone\ a\ b$   
 $\langle proof \rangle$

**lemma** *arg-imp-in-complex-cone*:  
**assumes**  $Arg\ z \in \{a..b\}$   
**shows**  $z \in complex-cone\ a\ b$   
 $\langle proof \rangle$

**lemma** *complex-cone-altdef*:  
**assumes**  $-\pi < a$   $a \leq b$   $b \leq \pi$   
**shows**  $complex-cone\ a\ b = insert\ 0\ \{z. Arg\ z \in \{a..b\}\}$   
 $\langle proof \rangle$

**lemma** *nonneg-of-real-in-complex-cone* [*simp, intro*]:  
**assumes**  $x \geq 0$   $a \leq 0$   $0 \leq b$   
**shows**  $of-real\ x \in complex-cone\ a\ b$   
 $\langle proof \rangle$

**lemma** *one-in-complex-cone* [*simp, intro*]:  $a \leq 0 \implies 0 \leq b \implies 1 \in complex-cone\ a\ b$   
 $\langle proof \rangle$

**lemma** *of-nat-in-complex-cone* [*simp, intro*]:  $a \leq 0 \implies 0 \leq b \implies of-nat\ n \in complex-cone\ a\ b$   
 $\langle proof \rangle$

## 2.3 Another integral representation of the Beta function

**lemma** *complex-cone-inter-nonpos-Reals*:  
**assumes**  $-\pi < a$   $a \leq b$   $b < \pi$   
**shows**  $complex-cone\ a\ b \cap \mathbb{R}_{\leq 0} = \{0\}$   
 $\langle proof \rangle$

**theorem**  
**assumes**  $a: a > 0$  **and**  $b: b > (0 :: real)$   
**shows** *has-integral-Beta-real'*:  
 $((\lambda u. u\ powr\ (b - 1) / (1 + u)\ powr\ (a + b))\ has-integral\ Beta\ a\ b)\ \{0 <..\}$   
**and** *Beta-conv-nn-integral*:  
 $Beta\ a\ b = (\int^{+} u. ennreal\ (indicator\ \{0 <..\}\ u * u\ powr\ (b - 1) / (1 + u)\ powr\ (a + b))\ \partial lborel)$   
 $\langle proof \rangle$

**lemma** *has-integral-Beta2*:  
**fixes**  $a :: real$   
**assumes**  $a < -1/2$   
**shows**  $((\lambda x. (1 + x ^ 2)\ powr\ a)\ has-integral\ Beta\ (- a - 1 / 2)\ (1 / 2) /$

2)  $\{0<..\}$   
 $\langle proof \rangle$

**lemma** *has-integral-Beta3*:

**fixes**  $a\ b :: real$

**assumes**  $a < -1/2$  **and**  $b > 0$

**shows**  $((\lambda x. (b + x \wedge 2) \text{ powr } a) \text{ has-integral}$

$Beta (-a - 1/2) (1/2) / 2 * b \text{ powr } (a + 1/2)) \{0<..\}$

$\langle proof \rangle$

## 2.4 Asymptotics of the real $\log \Gamma$ function and its derivatives

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order  $O(s^{-n})$ .

**definition** *stirling-integral*  $:: nat \Rightarrow 'a :: \{real-normed-div-algebra, banach\} \Rightarrow 'a$   
**where**

*stirling-integral*  $n\ s =$

$lim (\lambda N. \text{ integral } \{0..N\} (\lambda x. \text{ of-real } (pbernpoly\ n\ x) / (\text{ of-real } x + s) \wedge n))$

**context**

**fixes**  $s :: complex$  **assumes**  $s: s \notin \mathbb{R}_{\leq 0}$

**fixes** *approx*  $:: nat \Rightarrow complex$

**defines** *approx*  $\equiv (\lambda N.$

$(\sum n = 1..<N. s / \text{ of-nat } n - \ln (1 + s / \text{ of-nat } n)) - (\text{ euler-mascheroni } * s$   
 $+ \ln s) - \text{ -- } \longrightarrow \text{ ln-Gamma } s$

$(\text{ ln-Gamma } (\text{ of-nat } N) - \ln (2 * pi / \text{ of-nat } N) / 2 - \text{ of-nat } N * \ln (\text{ of-nat } N) + \text{ of-nat } N) - \text{ -- } \longrightarrow 0$

$s * (\text{ harm } (N - 1) - \ln (\text{ of-nat } (N - 1)) - \text{ euler-mascheroni}) + \text{ -- } \longrightarrow 0$

$s * (\ln (\text{ of-nat } N + s) - \ln (\text{ of-nat } (N - 1))) - \text{ -- } \longrightarrow 0$

$(1/2) * (\ln (\text{ of-nat } N + s) - \ln (\text{ of-nat } N)) + \text{ -- } \longrightarrow 0$

$\text{ of-nat } N * (\ln (\text{ of-nat } N + s) - \ln (\text{ of-nat } N)) - \text{ -- } \longrightarrow s$

$(s - 1/2) * \ln s - \ln (2 * pi) / 2)$

**begin**

**qualified lemma**

**assumes**  $N: N > 0$

**shows** *integrable-pbernpoly-1*:

$(\lambda x. \text{ of-real } (-pbernpoly\ 1\ x) / (\text{ of-real } x + s)) \text{ integrable-on } \{0..real\ N\}$

**and** *integral-pbernpoly-1-aux*:

$\text{ integral } \{0..real\ N\} (\lambda x. -\text{ of-real } (pbernpoly\ 1\ x) / (\text{ of-real } x + s)) =$

*approx*  $N$

**and** *has-integral-pbernpoly-1*:

$((\lambda x. pbernpoly\ 1\ x / (x + s)) \text{ has-integral}$

$(\sum m < N. (\text{ of-nat } m + 1 / 2 + s) * (\ln (\text{ of-nat } m + s) - \ln (\text{ of-nat } m + 1 + s)) + 1)) \{0..real\ N\}$

$\langle proof \rangle$

**lemma** *integrable-ln-Gamma-aux*:

**shows**  $(\lambda x. \text{ of-real } (pbernpoly\ n\ x) / (\text{ of-real } x + s) \wedge n) \text{ integrable-on } \{0..real\}$



$N\}$   
 $\langle proof \rangle$

This following proof is based on ‘‘Rudiments of the theory of the gamma function’’ by Bruce Berndt [1].

**lemma** *tendsto-of-real-0-I:*

$(f \longrightarrow 0) G \implies ((\lambda x. (of-real (f x))) \longrightarrow (0 :: 'a::real-normed-div-algebra)) G$

$\langle proof \rangle$  **lemma** *integral-pbernpoly-1:*

$(\lambda N. integral \{0..real N\} (\lambda x. pbernpoly 1 x / (x + s))) \longrightarrow -ln-Gamma s - s + (s - 1 / 2) * ln s + ln (2 * pi) / 2$

$\langle proof \rangle$  **lemma** *pbernpoly-integral-conv-pbernpoly-integral-Suc:*

**assumes**  $n \geq 1$

**shows**  $integral \{0..real N\} (\lambda x. pbernpoly n x / (x + s) ^ n) =$   
 $of-real (pbernpoly (Suc n) (real N)) / (of-nat (Suc n) * (s + of-nat N$   
 $^ n) -$   
 $of-real (bernoulli (Suc n)) / (of-nat (Suc n) * s ^ n) + of-nat n / of-nat$   
 $(Suc n) *$   
 $integral \{0..real N\} (\lambda x. of-real (pbernpoly (Suc n) x) / (of-real x +$   
 $s) ^ Suc n)$

$\langle proof \rangle$

**lemma** *pbernpoly-over-power-tendsto-0:*

**assumes**  $n > 0$

**shows**  $(\lambda x. of-real (pbernpoly (Suc n) (real x)) / (of-nat (Suc n) * (s + of-nat x) ^ n)) \longrightarrow 0$

$\langle proof \rangle$

**lemma** *convergent-stirling-integral:*

**assumes**  $n > 0$

**shows**  $convergent (\lambda N. integral \{0..real N\} (\lambda x. of-real (pbernpoly n x) / (of-real x + s) ^ n))$  (**is convergent** ( $?f n$ ))

$\langle proof \rangle$

**lemma** *stirling-integral-conv-stirling-integral-Suc:*

**assumes**  $n > 0$

**shows**  $stirling-integral n s =$   
 $of-nat n / of-nat (Suc n) * stirling-integral (Suc n) s -$   
 $of-real (bernoulli (Suc n)) / (of-nat (Suc n) * s ^ n)$

$\langle proof \rangle$

**lemma** *stirling-integral-1-unfold:*

**assumes**  $m > 0$

**shows**  $stirling-integral 1 s = stirling-integral m s / of-nat m -$   
 $(\sum k=1..<m. of-real (bernoulli (Suc k)) / (of-nat k * of-nat (Suc k) *$   
 $s ^ k))$

$\langle proof \rangle$

**lemma** *ln-Gamma-stirling-complex:*

**assumes**  $m > 0$   
**shows**  $\ln\text{-Gamma } s = (s - 1 / 2) * \ln s - s + \ln (2 * \pi) / 2 +$   
 $(\sum_{k=1..<m.} \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k)) -$   
 $\text{stirling-integral } m s / \text{of-nat } m$   
 $\langle \text{proof} \rangle$

**lemma** *LIMSEQ-stirling-integral*:  
 $n > 0 \implies (\lambda x. \text{integral } \{0..real\ } x) (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s)^{\wedge} n))$   
 $\longrightarrow \text{stirling-integral } n s \langle \text{proof} \rangle$

**end**

**lemmas** *has-integral-of-real = has-integral-linear*[*OF - bounded-linear-of-real, unfolded o-def*]

**lemmas** *integral-of-real = integral-linear*[*OF - bounded-linear-of-real, unfolded o-def*]

**lemma** *integrable-ln-Gamma-aux-real*:  
**assumes**  $0 < s$   
**shows**  $(\lambda x. \text{pbernpoly } n x / (x + s)^{\wedge} n) \text{ integrable-on } \{0..real\ } N$   
 $\langle \text{proof} \rangle$

**lemma**  
**assumes**  $x > 0 \ n > 0$   
**shows** *stirling-integral-complex-of-real*:  
 $\text{stirling-integral } n (\text{complex-of-real } x) = \text{of-real } (\text{stirling-integral } n x)$   
**and** *LIMSEQ-stirling-integral-real*:  
 $(\lambda N. \text{integral } \{0..real\ } N) (\lambda t. \text{pbernpoly } n t / (t + x)^{\wedge} n)$   
 $\longrightarrow \text{stirling-integral } n x$   
**and** *stirling-integral-real-convergent*:  
 $\text{convergent } (\lambda N. \text{integral } \{0..real\ } N) (\lambda t. \text{pbernpoly } n t / (t + x)^{\wedge} n)$   
 $\langle \text{proof} \rangle$

**lemma** *ln-Gamma-stirling-real*:  
**assumes**  $x > (0 :: real) \ m > (0 :: nat)$   
**shows**  $\ln\text{-Gamma } x = (x - 1 / 2) * \ln x - x + \ln (2 * \pi) / 2 +$   
 $(\sum_{k=1..<m.} \text{bernoulli } (\text{Suc } k) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * x^{\wedge} k))$   
 $-\text{stirling-integral } m x / \text{of-nat } m$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound-aux*:  
**assumes**  $n: n > (1 :: nat)$   
**obtains**  $c$  **where**  $\bigwedge s. \text{Re } s > 0 \implies \text{norm } (\text{stirling-integral } n s) \leq c / \text{Re } s^{\wedge} (n - 1)$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound-aux-integral1*:  
**fixes**  $a\ b\ c :: \text{real}$  **and**  $n :: \text{nat}$   
**assumes**  $a \geq 0\ b > 0\ c \geq 0\ n > 1\ l < a - b\ r > a + b$   
**shows**  $((\lambda x. c / \max b |x - a| ^ n) \text{ has-integral } 2 * c * (n / (n - 1)) / b ^ (n - 1) - c / (n - 1) * (1 / (a - l) ^ (n - 1) + 1 / (r - a) ^ (n - 1)))$   
 $\{l..r\}$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound-aux-integral2*:  
**fixes**  $a\ b\ c :: \text{real}$  **and**  $n :: \text{nat}$   
**assumes**  $a \geq 0\ b > 0\ c \geq 0\ n > 1$   
**obtains**  $I$  **where**  $((\lambda x. c / \max b |x - a| ^ n) \text{ has-integral } I) \{l..r\}$   
 $I \leq 2 * c * (n / (n - 1)) / b ^ (n - 1)$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound-aux'*:  
**assumes**  $n: n > (1::\text{nat})$  **and**  $\alpha: \alpha \in \{0 < .. < \pi i\}$   
**obtains**  $c$  **where**  $\bigwedge s::\text{complex}. s \in \text{complex-cone}'\ \alpha - \{0\} \implies$   
 $\text{norm} (\text{stirling-integral } n\ s) \leq c / \text{norm } s ^ (n - 1)$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound*:  
**assumes**  $n > 0$   
**obtains**  $c$  **where**  
 $\bigwedge s. \text{Re } s > 0 \implies \text{norm} (\text{stirling-integral } n\ s) \leq c / \text{Re } s ^ n$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-bound'*:  
**assumes**  $n > 0$  **and**  $\alpha \in \{0 < .. < \pi i\}$   
**obtains**  $c$  **where**  
 $\bigwedge s::\text{complex}. s \in \text{complex-cone}'\ \alpha - \{0\} \implies \text{norm} (\text{stirling-integral } n\ s) \leq c /$   
 $\text{norm } s ^ n$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-holomorphic* [*holomorphic-intros*]:  
**assumes**  $m: m > 0$  **and**  $A \cap \mathbb{R}_{\leq 0} = \{\}$   
**shows** *stirling-integral*  $m$  *holomorphic-on*  $A$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-integral-continuous-on-complex* [*continuous-intros*]:  
**assumes**  $m: m > 0$  **and**  $A \cap \mathbb{R}_{\leq 0} = \{\}$   
**shows** *continuous-on*  $A$  (*stirling-integral*  $m :: - \Rightarrow \text{complex}$ )  
 $\langle \text{proof} \rangle$

**lemma** *has-field-derivative-stirling-integral-complex*:  
**fixes**  $x :: \text{complex}$   
**assumes**  $x \notin \mathbb{R}_{\leq 0}\ n > 0$   
**shows** (*stirling-integral*  $n$  *has-field-derivative* *deriv* (*stirling-integral*  $n$ )  $x$ ) (*at*

$x$ )  
 ⟨proof⟩

**lemma**

**assumes**  $n: n > 0$  and  $x > 0$

**shows** *deriv-stirling-integral-complex-of-real*:

$(\text{deriv } \overset{\sim}{\sim} j) (\text{stirling-integral } n) (\text{complex-of-real } x) =$   
 $\text{complex-of-real } ((\text{deriv } \overset{\sim}{\sim} j) (\text{stirling-integral } n) x) \text{ (is ?lhs } x = \text{?rhs } x)$

**and** *differentiable-stirling-integral-real*:

$(\text{deriv } \overset{\sim}{\sim} j) (\text{stirling-integral } n) \text{ field-differentiable at } x \text{ (is ?thesis2)}$

⟨proof⟩

Unfortunately, asymptotic power series cannot, in general, be differentiated. However, since *ln-Gamma* is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the  $j$ -th derivative of the remainder term at some value  $x$  by applying Cauchy's integral formula along a circle centred at  $x$  with radius  $\frac{1}{2}x$ .

**lemma** *deriv-stirling-integral-real-bound*:

**assumes**  $m: m > 0$

**shows**  $(\text{deriv } \overset{\sim}{\sim} j) (\text{stirling-integral } m) \in O(\lambda x::\text{real. } 1 / x \wedge (m + j))$

⟨proof⟩

**definition** *stirling-sum* where

*stirling-sum*  $j$   $m$   $x =$

$(-1) \wedge j * (\sum k = 1..<m. (\text{of-real } (\text{bernoulli } (\text{Suc } k)) * \text{pochhammer } (\text{of-nat } k) j / (\text{of-nat } k * \text{of-nat } (\text{Suc } k))) * \text{inverse } x \wedge (k + j))$

**definition** *stirling-sum'* where

*stirling-sum'*  $j$   $m$   $x =$

$(-1) \wedge (\text{Suc } j) * (\sum k \leq m. (\text{of-real } (\text{bernoulli}' k) * \text{pochhammer } (\text{of-nat } (\text{Suc } k)) (j - 1) * \text{inverse } x \wedge (k + j)))$

**lemma** *stirling-sum-complex-of-real*:

*stirling-sum*  $j$   $m$   $(\text{complex-of-real } x) = \text{complex-of-real } (\text{stirling-sum } j$   $m$   $x)$

⟨proof⟩

**lemma** *stirling-sum'-complex-of-real*:

*stirling-sum'*  $j$   $m$   $(\text{complex-of-real } x) = \text{complex-of-real } (\text{stirling-sum}' j$   $m$   $x)$

⟨proof⟩

**lemma** *has-field-derivative-stirling-sum-complex* [derivative-intros]:

$\text{Re } x > 0 \implies (\text{stirling-sum } j$   $m \text{ has-field-derivative } \text{stirling-sum } (\text{Suc } j) m x) \text{ (at$

$x$ )  
 ⟨proof⟩

**lemma** *has-field-derivative-stirling-sum-real* [derivative-intros]:

$x > (0::real) \implies (\text{stirling-sum } j \ m \ \text{has-field-derivative } \text{stirling-sum } (Suc \ j) \ m \ x)$   
 (at  $x$ )  
 ⟨proof⟩

**lemma** *has-field-derivative-stirling-sum'-complex* [derivative-intros]:

**assumes**  $j > 0 \ \text{Re } x > 0$   
**shows**  $(\text{stirling-sum}' \ j \ m \ \text{has-field-derivative } \text{stirling-sum}' (Suc \ j) \ m \ x)$  (at  $x$ )  
 ⟨proof⟩

**lemma** *has-field-derivative-stirling-sum'-real* [derivative-intros]:

**assumes**  $j > 0 \ x > (0::real)$   
**shows**  $(\text{stirling-sum}' \ j \ m \ \text{has-field-derivative } \text{stirling-sum}' (Suc \ j) \ m \ x)$  (at  $x$ )  
 ⟨proof⟩

**lemma** *higher-deriv-stirling-sum-complex*:

$\text{Re } x > 0 \implies (\text{deriv } \hat{\hat{}} \ i) (\text{stirling-sum } j \ m) \ x = \text{stirling-sum } (i + j) \ m \ x$   
 ⟨proof⟩

**definition** *Polygamma-approx* ::  $\text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a :: \{\text{real-normed-field}, \text{ln}\}$   
**where**

$\text{Polygamma-approx } j \ m =$   
 $(\text{deriv } \hat{\hat{}} \ j) (\lambda x. (x - 1 / 2) * \ln x - x + \text{of-real } (\ln (2 * \text{pi})) / 2 +$   
 $\text{stirling-sum } 0 \ m \ x)$

**lemma** *Polygamma-approx-Suc*:  $\text{Polygamma-approx } (Suc \ j) \ m = \text{deriv } (\text{Polygamma-approx } j \ m)$

⟨proof⟩

**lemma** *Polygamma-approx-0*:

$\text{Polygamma-approx } 0 \ m \ x = (x - 1/2) * \ln x - x + \text{of-real } (\ln (2*\text{pi})) / 2 +$   
 $\text{stirling-sum } 0 \ m \ x$   
 ⟨proof⟩

**lemma** *Polygamma-approx-1-complex*:

$\text{Re } x > 0 \implies$   
 $\text{Polygamma-approx } (Suc \ 0) \ m \ x = \ln x - 1 / (2*x) + \text{stirling-sum } (Suc \ 0)$   
 $m \ x$   
 ⟨proof⟩

**lemma** *Polygamma-approx-1-real*:

$x > (0 :: real) \implies$   
 $\text{Polygamma-approx } (Suc \ 0) \ m \ x = \ln x - 1 / (2*x) + \text{stirling-sum } (Suc \ 0)$   
 $m \ x$   
 ⟨proof⟩

**lemma** *stirling-sum-2-conv-stirling-sum'-1*:  
**fixes**  $x :: 'a :: \{\text{real-div-algebra}, \text{field-char-0}\}$   
**assumes**  $m > 0 \ x \neq 0$   
**shows**  $\text{stirling-sum}'\ 1\ m\ x = 1 / x + 1 / (2 * x^2) + \text{stirling-sum}\ 2\ m\ x$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-2-real*:  
**assumes**  $x > (0::\text{real}) \ m > 0$   
**shows**  $\text{Polygamma-approx}\ (\text{Suc}\ (\text{Suc}\ 0))\ m\ x = \text{stirling-sum}'\ 1\ m\ x$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-2-complex*:  
**assumes**  $\text{Re}\ x > 0 \ m > 0$   
**shows**  $\text{Polygamma-approx}\ (\text{Suc}\ (\text{Suc}\ 0))\ m\ x = \text{stirling-sum}'\ 1\ m\ x$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-ge-2-real*:  
**assumes**  $x > (0::\text{real}) \ m > 0$   
**shows**  $\text{Polygamma-approx}\ (\text{Suc}\ (\text{Suc}\ j))\ m\ x = \text{stirling-sum}'\ (\text{Suc}\ j)\ m\ x$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-ge-2-complex*:  
**assumes**  $\text{Re}\ x > 0 \ m > 0$   
**shows**  $\text{Polygamma-approx}\ (\text{Suc}\ (\text{Suc}\ j))\ m\ x = \text{stirling-sum}'\ (\text{Suc}\ j)\ m\ x$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-complex-of-real*:  
**assumes**  $x > 0 \ m > 0$   
**shows**  $\text{Polygamma-approx}\ j\ m\ (\text{complex-of-real}\ x) = \text{of-real}\ (\text{Polygamma-approx}\ j\ m\ x)$   
 $\langle \text{proof} \rangle$

**lemma** *higher-deriv-Polygamma-approx [simp]*:  
 $(\text{deriv}\ \overset{\sim}{\sim} j)\ (\text{Polygamma-approx}\ i\ m) = \text{Polygamma-approx}\ (j + i)\ m$   
 $\langle \text{proof} \rangle$

**lemma** *stirling-sum-holomorphic [holomorphic-intros]*:  
 $0 \notin A \implies \text{stirling-sum}\ j\ m\ \text{holomorphic-on}\ A$   
 $\langle \text{proof} \rangle$

**lemma** *Polygamma-approx-holomorphic [holomorphic-intros]*:  
 $\text{Polygamma-approx}\ j\ m\ \text{holomorphic-on}\ \{s.\ \text{Re}\ s > 0\}$   
 $\langle \text{proof} \rangle$

**lemma** *higher-deriv-lnGamma-stirling*:  
**assumes**  $m: m > 0$   
**shows**  $(\lambda x::\text{real}.\ (\text{deriv}\ \overset{\sim}{\sim} j)\ \text{ln-Gamma}\ x - \text{Polygamma-approx}\ j\ m\ x) \in O(\lambda x.\ 1 / x^{(m + j)})$

*<proof>*

**lemma** *Polygamma-approx-1-real'*:

**assumes**  $x: (x::real) > 0$  **and**  $m: m > 0$

**shows**  $Polygamma\text{-}approx\ 1\ m\ x = \ln\ x - (\sum k = Suc\ 0..m. bernoulli'\ k * inverse\ x^k / real\ k)$

*<proof>*

**theorem**

**assumes**  $m: m > 0$

**shows** *ln-Gamma-real-asymptotics*:

$(\lambda x. \ln\text{-}Gamma\ x - ((x - 1 / 2) * \ln\ x - x + \ln\ (2 * pi) / 2 + (\sum k = 1..<m. bernoulli\ (Suc\ k) / (real\ k * real\ (Suc\ k)) / x^k))) \in O(\lambda x. 1 / x^m)$  (**is** *?th1*)

**and** *Digamma-real-asymptotics*:

$(\lambda x. Digamma\ x - (\ln\ x - (\sum k=1..m. bernoulli'\ k / real\ k / x^k))) \in O(\lambda x. 1 / (x^{Suc\ m}))$  (**is** *?th2*)

**and** *Polygamma-real-asymptotics*:  $j > 0 \implies$

$(\lambda x. Polygamma\ j\ x - (-1)^{Suc\ j} * (\sum k \leq m. bernoulli'\ k * pochhammer\ (real\ (Suc\ k))\ (j - 1) / x^{(k + j)})) \in O(\lambda x. 1 / x^{(m+j+1)})$  (**is**  $- \implies$  *?th3*)

*<proof>*

## 2.5 Asymptotics of the complex Gamma function

The  $m$ -th order remainder of Stirling's formula for  $\log \Gamma$  is  $O(s^{-m})$  uniformly over any complex cone  $\text{Arg}(z) \leq \alpha$ ,  $z \neq 0$  for any angle  $\alpha \in (0, \pi)$ . This means that there is bounded by  $cz^{-m}$  for some constant  $c$  for all  $z$  in this cone.

**context**

**fixes**  $F$  **and**  $\alpha$

**assumes**  $\alpha: \alpha \in \{0 < .. < pi\}$

**defines**  $F \equiv principal\ (complex\text{-}cone'\ \alpha - \{0\})$

**begin**

**lemma** *stirling-integral-bigo*:

**fixes**  $m :: nat$

**assumes**  $m: m > 0$

**shows** *stirling-integral*  $m \in O[F](\lambda s. 1 / s^m)$

*<proof>*

**end**

The following is a more explicit statement of this:

**theorem** *ln-Gamma-complex-asymptotics-explicit*:

**fixes**  $m :: nat$  **and**  $\alpha :: real$

**assumes**  $m > 0$  **and**  $\alpha \in \{0 < .. < pi\}$

**obtains**  $C :: real$  **and**  $R :: complex \Rightarrow complex$

**where**  $\forall s::\text{complex. } s \notin \mathbb{R}_{\leq 0} \longrightarrow$   
 $\text{ln-Gamma } s = (s - 1/2) * \ln s - s + \ln (2 * \pi) / 2 +$   
 $(\sum_{k=1..m.} \text{bernoulli } (k+1) / (k * (k+1) * s ^ k)) - R s$   
**and**  $\forall s. s \neq 0 \wedge |\text{Arg } s| \leq \alpha \longrightarrow \text{norm } (R s) \leq C / \text{norm } s ^ m$   
 <proof>

Lastly, we can also derive the asymptotics of  $\Gamma$  itself:

$$\Gamma(z) \sim \sqrt{2\pi/z} \left(\frac{z}{e}\right)^z$$

uniformly for  $|z| \rightarrow \infty$  within the cone  $\text{Arg}(z) \leq \alpha$  for  $\alpha \in (0, \pi)$ :

**context**

**fixes**  $F$  and  $\alpha$

**assumes**  $\alpha: \alpha \in \{0 < .. < \pi\}$

**defines**  $F \equiv \text{inf at-infinity (principal (complex-cone' } \alpha))$

**begin**

**lemma** *Gamma-complex-asymp-equiv:*

$\text{Gamma} \sim_{[F]} (\lambda s. \text{sqrt } (2 * \pi) * (s / \text{exp } 1) \text{ powr } s / s \text{ powr } (1 / 2))$   
 <proof>

**end**

**end**

## References

- [1] B. Berndt. *Rudiments of the Theory of the Gamma Function*. University of Chicago, 1976.
- [2] A. Erdélyi. *Asymptotic Expansions*. Dover Publications, 1956.
- [3] G. J. O. Jameson. A simple proof of Stirling's formula for the Gamma function. *The Mathematical Gazette*, 99:68–74, 3 2015.