

Stirling's formula

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Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n} (n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x} (x/e)^x$. The proof is based on work by Graham Jameson [3].

This is then extended to the full asymptotic expansion

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{k+1}}{k(k+1)} z^{-k} - \frac{1}{n} \int_0^\infty B_n([t])(t+z)^{-n} dt$$

uniformly for all complex $z \neq 0$ in the cone $\arg(z) \leq \alpha$ for any $\alpha \in (0, \pi)$, with which the above asymptotic relation for Γ is also extended to complex arguments.

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1 Stirling's Formula

theory *Stirling-Formula*

imports

HOL-Analysis.Analysis

Landau-Symbols.Landau-More

HOL-Real-Asymp.Real-Asymp

begin

context
begin

First, we define the S_n^* from Jameson's article:

qualified definition $S' :: nat \Rightarrow real \Rightarrow real$ **where**

$$S' n x = 1/(2*x) + (\sum r=1..<n. 1/(of-nat r+x)) + 1/(2*(n+x))$$

Next, the trapezium (also called T in Jameson's article):

qualified definition $T :: real \Rightarrow real$ **where**

$$T x = 1/(2*x) + 1/(2*(x+1))$$

Now we define The difference $\Delta(x)$:

qualified definition $D :: real \Rightarrow real$ **where**

$$D x = T x - \ln(x+1) + \ln x$$

qualified lemma S' -telescope-trapezium:

assumes $n > 0$

shows $S' n x = (\sum r < n. T (of-nat r + x))$

proof (cases n)

case (Suc m)

hence $m: Suc m = n$ **by** simp

have $(\sum r < n. T (of-nat r + x)) =$

$$(\sum r < Suc m. 1 / (2 * real r + 2 * x)) + (\sum r < n. 1 / (2 * real (Suc r) + 2 * x))$$

unfolding m **by** (simp add: T-def sum.distrib algebra-simps)

also have $(\sum r < Suc m. 1 / (2 * real r + 2 * x)) =$

$$1/(2*x) + (\sum r < m. 1 / (2 * real (Suc r) + 2 * x)) \text{ (is - = ?a + ?S)}$$

by (subst sum.lessThan-Suc-shift) simp

also have $(\sum r < n. 1 / (2 * real (Suc r) + 2 * x)) =$

$$?S + 1 / (2*(real m + x + 1)) \text{ (is - = - + ?b) by (simp add: Suc)}$$

also have $?a + ?S + (?S + ?b) = 2*?S + ?a + ?b$ **by** (simp add: add-ac)

also have $2 * ?S = (\sum r=0..<m. 1 / (real (Suc r) + x))$

unfolding sum-distrib-left **by** (intro sum.cong) (auto simp add: divide-simps)

also have $(\sum r=0..<m. 1 / (real (Suc r) + x)) = (\sum r=Suc 0..<Suc m. 1 / (real r + x))$

by (subst sum.atLeast-Suc-lessThan-Suc-shift) simp-all

also have $\dots = (\sum r=1..<n. 1 / (real r + x))$ **unfolding** m **by** simp

also have $\dots + ?a + ?b = S' n x$ **by** (simp add: S'-def Suc)

finally show ?thesis ..

qed (insert assms, simp-all)

qualified lemma $stirling$ -trapezium:

assumes $x: (x::real) > 0$

shows $D x \in \{0 .. 1/(12*x^2) - 1/(12 * (x+1)^2)\}$

proof -

define y **where** $y = 1 / (2*x + 1)$

from x **have** $y: y > 0 \ y < 1$ **by** (simp-all add: divide-simps y-def)

from x **have** $D x = T x - \ln ((x + 1) / x)$ **by** (*subst ln-div*) (*simp-all add: D-def*)
also from x **have** $(x + 1) / x = 1 + 1 / x$ **by** (*simp add: field-simps*)
finally have $D: D x = T x - \ln (1 + 1/x)$.

from y **have** $(\lambda n. y * y^{\wedge} n)$ *sums* $(y * (1 / (1 - y)))$
by (*intro geometric-sums sums-mult*) *simp-all*
hence $(\lambda n. y^{\wedge} \text{Suc } n)$ *sums* $(y / (1 - y))$ **by** *simp*
also from x **have** $y / (1 - y) = 1 / (2*x)$ **by** (*simp add: y-def divide-simps*)
finally have $*$: $(\lambda n. y^{\wedge} \text{Suc } n)$ *sums* $(1 / (2*x))$.

from y **have** $(\lambda n. (-y) * (-y)^{\wedge} n)$ *sums* $((-y) * (1 / (1 - (-y))))$
by (*intro geometric-sums sums-mult*) *simp-all*
hence $(\lambda n. (-y)^{\wedge} \text{Suc } n)$ *sums* $(-y / (1 + y))$ **by** *simp*
also from x **have** $y / (1 + y) = 1 / (2*(x+1))$ **by** (*simp add: y-def divide-simps*)
finally have $**$: $(\lambda n. (-y)^{\wedge} \text{Suc } n)$ *sums* $(-1 / (2*(x+1)))$.

from *sums-diff*[*OF * ***] **have** $\text{sum1}: (\lambda n. y^{\wedge} \text{Suc } n - (-y)^{\wedge} \text{Suc } n)$ *sums* $T x$
by (*simp add: T-def*)

from y **have** $\text{abs } y < 1$ $\text{abs } (-y) < 1$ **by** *simp-all*
from *sums-diff*[*OF this* [*THEN ln-series*]]
have $(\lambda n. y^{\wedge} n / \text{real } n - (-y)^{\wedge} n / \text{real } n)$ *sums* $(\ln (1 + y) - \ln (1 - y))$
by *simp*
also from y **have** $\ln (1 + y) - \ln (1 - y) = \ln ((1 + y) / (1 - y))$ **by** (*simp add: ln-div*)
also from x **have** $(1 + y) / (1 - y) = 1 + 1/x$ **by** (*simp add: divide-simps y-def*)
finally have $(\lambda n. y^{\wedge} n / \text{real } n - (-y)^{\wedge} n / \text{real } n)$ *sums* $\ln (1 + 1/x)$.
hence $\text{sum2}: (\lambda n. y^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n) - (-y)^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n))$ *sums* $\ln (1 + 1/x)$
by (*subst sums-Suc-iff*) *simp*

have $\ln (1 + 1/x) \leq T x$
proof (*rule sums-le* [*OF - sum2 sum1*])
fix $n :: \text{nat}$
show $y^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n) - (-y)^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n) \leq y^{\wedge} \text{Suc } n - (-y)^{\wedge} \text{Suc } n$
proof (*cases even n*)
case *True*
hence $\text{eq}: A - (-y)^{\wedge} \text{Suc } n / B = A + y^{\wedge} \text{Suc } n / B$ $A - (-y)^{\wedge} \text{Suc } n = A + y^{\wedge} \text{Suc } n$
for $A B$ **by** *simp-all*
from y **show** *?thesis* **unfolding** eq
by (*intro add-mono*) (*auto simp: divide-simps*)
qed *simp-all*
qed
hence $D x \geq 0$ **by** (*simp add: D*)

define c **where** $c = (\lambda n. \text{if even } n \text{ then } 2 * (1 - 1 / \text{real } (\text{Suc } n)) \text{ else } 0)$
note $\text{sums-diff}[OF \text{ sum1 sum2}]$
also have $(\lambda n. y^{\wedge} \text{Suc } n - (-y)^{\wedge} \text{Suc } n - (y^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n) - (-y)^{\wedge} \text{Suc } n / \text{real } (\text{Suc } n))) = (\lambda n. c \ n * y^{\wedge} \text{Suc } n)$
by $(\text{intro ext}) (\text{simp add: c-def algebra-simps})$
finally have $\text{sum3}: (\lambda n. c \ n * y^{\wedge} \text{Suc } n) \text{ sums } D \ x$ **by** $(\text{simp add: } D)$

from y **have** $(\lambda n. y^{\wedge} 2 * (\text{of-nat } (\text{Suc } n) * y^{\wedge} n)) \text{ sums } (y^{\wedge} 2 * (1 / (1 - y)^{\wedge} 2))$
by $(\text{intro sums-mult geometric-deriv-sums}) \text{ simp-all}$
hence $(\lambda n. \text{of-nat } (\text{Suc } n) * y^{\wedge} (n+2)) \text{ sums } (y^{\wedge} 2 / (1 - y)^{\wedge} 2)$
by $(\text{simp add: algebra-simps power2-eq-square})$
also from x **have** $y^{\wedge} 2 / (1 - y)^{\wedge} 2 = 1 / (4 * x^{\wedge} 2)$ **by** $(\text{simp add: y-def divide-simps})$
finally have $*$: $(\lambda n. \text{real } (\text{Suc } n) * y^{\wedge} (\text{Suc } (\text{Suc } n))) \text{ sums } (1 / (4 * x^2))$ **by** simp

from y **have** $(\lambda n. y^{\wedge} 2 * (\text{of-nat } (\text{Suc } n) * (-y)^{\wedge} n)) \text{ sums } (y^{\wedge} 2 * (1 / (1 - (-y)^{\wedge} 2)))$
by $(\text{intro sums-mult geometric-deriv-sums}) \text{ simp-all}$
hence $(\lambda n. \text{of-nat } (\text{Suc } n) * (-y)^{\wedge} (n+2)) \text{ sums } (y^{\wedge} 2 / (1 + y)^{\wedge} 2)$
by $(\text{simp add: algebra-simps power2-eq-square})$
also from x **have** $y^{\wedge} 2 / (1 + y)^{\wedge} 2 = 1 / (2^{\wedge} 2 * (x+1)^{\wedge} 2)$
unfolding $\text{power-mult-distrib [symmetric]}$ **by** $(\text{simp add: y-def divide-simps add-ac})$
finally have $**$: $(\lambda n. \text{real } (\text{Suc } n) * (-y)^{\wedge} (\text{Suc } (\text{Suc } n))) \text{ sums } (1 / (4 * (x + 1)^2))$ **by** simp

define d **where** $d = (\lambda n. \text{if even } n \text{ then } 2 * \text{real } n \text{ else } 0)$
note $\text{sums-diff}[OF * **]$
also have $(\lambda n. \text{real } (\text{Suc } n) * y^{\wedge} (\text{Suc } (\text{Suc } n)) - \text{real } (\text{Suc } n) * (-y)^{\wedge} (\text{Suc } (\text{Suc } n))) = (\lambda n. d (\text{Suc } n) * y^{\wedge} \text{Suc } (\text{Suc } n))$
by $(\text{intro ext}) (\text{simp-all add: d-def})$
finally have $(\lambda n. d \ n * y^{\wedge} \text{Suc } n) \text{ sums } (1 / (4 * x^2) - 1 / (4 * (x + 1)^2))$
by $(\text{subst (asm) sums-Suc-iff}) (\text{simp add: d-def})$
from $\text{sums-mult}[OF \text{ this, of } 1/3] \ x$
have $\text{sum4}: (\lambda n. d \ n / 3 * y^{\wedge} \text{Suc } n) \text{ sums } (1 / (12 * x^{\wedge} 2) - 1 / (12 * (x + 1)^{\wedge} 2))$
by $(\text{simp add: field-simps})$

have $D \ x \leq (1 / (12 * x^{\wedge} 2) - 1 / (12 * (x + 1)^{\wedge} 2))$
proof $(\text{intro sums-le [OF - sum3 sum4] allI})$
fix $n :: \text{nat}$
define $c' :: \text{nat} \Rightarrow \text{real}$
where $c' = (\lambda n. \text{if odd } n \vee n = 0 \text{ then } 0 \text{ else if } n = 2 \text{ then } 4/3 \text{ else } 2)$
show $c \ n * y^{\wedge} \text{Suc } n \leq d \ n / 3 * y^{\wedge} \text{Suc } n$
proof $(\text{intro mult-right-mono})$
have $c \ n \leq c' \ n$ **by** $(\text{simp add: c-def c'-def})$
also consider $n = 0 \mid n = 1 \mid n = 2 \mid n \geq 3$ **by** force

hence $c' n \leq d n / 3$ **by cases** (*simp-all add: c'-def d-def*)
finally show $c n \leq d n / 3$.
qed (*insert y, simp*)
qed

with $\langle D x \geq 0 \rangle$ **show** *?thesis by simp*
qed

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

qualified definition $p :: nat \Rightarrow real \Rightarrow real$ **where**
 $p n x = (if\ n = 0\ then\ 1/x\ else\ (\sum\ r < n.\ D\ (real\ r + x)))$

We can write the Digamma function in terms of S' :

qualified lemma *S'-LIMSEQ-Digamma*:

assumes $x: x \neq 0$
shows $(\lambda n. \ln (real\ n) - S' n x - 1/(2*x)) \longrightarrow Digamma\ x$
proof –
define c **where** $c = (\lambda n. \ln (real\ n) - (\sum\ r < n.\ inverse\ (x + real\ r)))$
have *eventually* $(\lambda n. 1 / (2 * (x + real\ n)) = c n - (\ln (real\ n) - S' n x - 1/(2*x)))$ *at-top*
using *eventually-gt-at-top[of 0::nat]*
proof *eventually-elim*
fix $n :: nat$
assume $n: n > 0$
have $c n - (\ln (real\ n) - S' n x - 1/(2*x)) =$
 $-(\sum\ r < n.\ inverse\ (real\ r + x)) + (1/x + (\sum\ r = 1..<n.\ inverse\ (real\ r + x))) + 1/(2*(real\ n + x))$
using x **by** (*simp add: S'-def c-def field-simps*)
also have $1/x + (\sum\ r = 1..<n.\ inverse\ (real\ r + x)) = (\sum\ r < n.\ inverse\ (real\ r + x))$
unfolding *lessThan-atLeast0* **using** n
by (*subst (2) sum.atLeast-Suc-lessThan*) (*simp-all add: field-simps*)
finally show $1 / (2 * (x + real\ n)) = c n - (\ln (real\ n) - S' n x - 1/(2*x))$
by *simp*
qed
moreover have $(\lambda n. 1 / (2 * (x + real\ n))) \longrightarrow 0$
by *real-asymp*
ultimately have $(\lambda n. c n - (\ln (real\ n) - S' n x - 1/(2*x))) \longrightarrow 0$
by (*blast intro: Lim-transform-eventually*)
from *tendsto-minus[OF this]* **have** $(\lambda n. (\ln (real\ n) - S' n x - 1/(2*x)) - c n) \longrightarrow 0$ **by** *simp*
moreover from *Digamma-LIMSEQ[OF x]* **have** $c \longrightarrow Digamma\ x$ **by** (*simp add: c-def*)
ultimately show $(\lambda n. \ln (real\ n) - S' n x - 1/(2*x)) \longrightarrow Digamma\ x$
by (*rule Lim-transform [rotated]*)
qed

Moreover, we can give an expansion of S' with the p as variation terms.

qualified lemma S' -*approx*:

$$S' n x = \ln (\text{real } n + x) - \ln x + p n x$$

proof (*cases* $n = 0$)

case *True*

thus *?thesis* **by** (*simp add: p-def S'-def*)

next

case *False*

hence $S' n x = (\sum r < n. T (\text{real } r + x))$

by (*subst S'-telescope-trapezium simp-all*)

also have $\dots = (\sum r < n. \ln (\text{real } r + x + 1) - \ln (\text{real } r + x) + D (\text{real } r + x))$

by (*simp add: D-def*)

also have $\dots = (\sum r < n. \ln (\text{real } (\text{Suc } r) + x) - \ln (\text{real } r + x)) + p n x$

using *False* **by** (*simp add: sum.distrib add-ac p-def*)

also have $(\sum r < n. \ln (\text{real } (\text{Suc } r) + x) - \ln (\text{real } r + x)) = \ln (\text{real } n + x) - \ln x$

by (*subst sum-lessThan-telescope simp-all*)

finally show *?thesis* .

qed

We define the limit of the p (simply called $p(x)$ in Jameson's article):

qualified definition $P :: \text{real} \Rightarrow \text{real}$ **where**

$$P x = (\sum n. D (\text{real } n + x))$$

qualified lemma D -*summable*:

assumes $x: x > 0$

shows *summable* $(\lambda n. D (\text{real } n + x))$

proof –

have $*$: *summable* $(\lambda n. 1 / (12 * (x + \text{real } n)^2) - 1 / (12 * (x + \text{real } (\text{Suc } n))^2))$

by (*rule telescope-summable'*) *real-asymp*

show *summable* $(\lambda n. D (\text{real } n + x))$

proof (*rule summable-comparison-test[OF - *], rule exI[of - 2], safe*)

fix $n :: \text{nat}$ **assume** $n \geq 2$

show *norm* $(D (\text{real } n + x)) \leq 1 / (12 * (x + \text{real } n)^2) - 1 / (12 * (x + \text{real } (\text{Suc } n))^2)$

using *stirling-trapezium[of real n + x]* x **by** (*auto simp: algebra-simps*)

qed

qed

qualified lemma p -*LIMSEQ*:

assumes $x: x > 0$

shows $(\lambda n. p n x) \longrightarrow P x$

proof (*rule Lim-transform-eventually*)

from D -*summable*[*OF* x] **have** $(\lambda n. D (\text{real } n + x))$ *sums* $P x$ **unfolding** P -*def*

by (*simp add: sums-iff*)

then show $(\lambda n. \sum r < n. D (\text{real } r + x)) \longrightarrow P x$ **by** (*simp add: sums-def*)

moreover from *eventually-gt-at-top*[*of* 1]

show *eventually* $(\lambda n. (\sum r < n. D (\text{real } r + x)) = p \ n \ x)$ *at-top*
by *eventually-elim* (*auto simp: p-def*)
qed

This gives us an expansion of the Digamma function:

lemma *Digamma-approx:*

assumes $x: (x :: \text{real}) > 0$

shows $\text{Digamma } x = \ln x - 1 / (2 * x) - P \ x$

proof –

have *eventually* $(\lambda n. \ln (\text{inverse } (1 + x / \text{real } n)) + \ln x - 1/(2*x) - p \ n \ x =$
 $\ln (\text{real } n) - S' \ n \ x - 1/(2*x))$ *at-top*

using *eventually-gt-at-top*[*of 1::nat*]

proof *eventually-elim*

fix $n :: \text{nat}$ **assume** $n: n > 1$

have $\ln (\text{real } n) - S' \ n \ x = \ln ((\text{real } n) / (\text{real } n + x)) + \ln x - p \ n \ x$

using *assms n unfolding S'-approx by (subst ln-div) (auto simp: algebra-simps)*

also from n **have** $\text{real } n / (\text{real } n + x) = \text{inverse } (1 + x / \text{real } n)$ **by** (*simp add: field-simps*)

finally show $\ln (\text{inverse } (1 + x / \text{real } n)) + \ln x - 1/(2*x) - p \ n \ x =$
 $\ln (\text{real } n) - S' \ n \ x - 1/(2*x)$ **by** *simp*

qed

moreover have $(\lambda n. \ln (\text{inverse } (1 + x / \text{real } n)) + \ln x - 1/(2*x) - p \ n \ x)$
 $\longrightarrow \ln (\text{inverse } (1 + 0)) + \ln x - 1/(2*x) - P \ x$

by (*rule tendsto-intros p-LIMSEQ x real-tendsto-divide-at-top*
filterlim-real-sequentially | simp)+

hence $(\lambda n. \ln (\text{inverse } (1 + x / \text{real } n)) + \ln x - 1/(2*x) - p \ n \ x)$
 $\longrightarrow \ln x - 1/(2*x) - P \ x$ **by** *simp*

ultimately have $(\lambda n. \ln (\text{real } n) - S' \ n \ x - 1 / (2 * x)) \longrightarrow \ln x - 1/(2*x)$
 $- P \ x$

by (*blast intro: Lim-transform-eventually*)

moreover from x **have** $(\lambda n. \ln (\text{real } n) - S' \ n \ x - 1 / (2 * x)) \longrightarrow \text{Digamma } x$

by (*intro S'-LIMSEQ-Digamma simp-all*)

ultimately show $\text{Digamma } x = \ln x - 1 / (2 * x) - P \ x$

by (*rule LIMSEQ-unique [rotated]*)

qed

Next, we derive some bounds on P :

qualified lemma *p-ge-0:* $x > 0 \implies p \ n \ x \geq 0$

using *stirling-trapezium*[*of real n + x for n*]

by (*auto simp add: p-def intro!: sum-nonneg*)

qualified lemma *P-ge-0:* $x > 0 \implies P \ x \geq 0$

by (*rule tendsto-lowerbound[OF p-LIMSEQ]*)

(*insert p-ge-0*[*of x*], *simp-all*)

qualified lemma *p-upper-bound:*

assumes $x > 0 \ n > 0$

shows $p\ n\ x \leq 1/(12*x^2)$
proof –
from *assms* **have** $p\ n\ x = (\sum\ r < n. D\ (real\ r + x))$
by (*simp add: p-def*)
also have $\dots \leq (\sum\ r < n. 1/(12*(real\ r + x)^2) - 1/(12 * (real\ (Suc\ r) + x)^2))$
using *stirling-trapezium[of real r + x for r] assms*
by (*intro sum-mono*) (*simp add: add-ac*)
also have $\dots = 1 / (12 * x^2) - 1 / (12 * (real\ n + x)^2)$
by (*subst sum-lessThan-telescope'*) *simp*
also have $\dots \leq 1 / (12 * x^2)$ **by** *simp*
finally show *?thesis* .
qed

qualified lemma *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12*x^2)$
proof (*rule tendsto-upperbound*)
show *eventually* $(\lambda n. p\ n\ x \leq 1 / (12 * x^2))$ *at-top*
using *eventually-gt-at-top[of 0]*
by *eventually-elim* (*use p-upper-bound[of x] assms in auto*)
show $(\lambda n. p\ n\ x) \longrightarrow P\ x$
by (*simp add: assms p-LIMSEQ*)
qed *auto*

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function *g* from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

qualified definition $g :: real \Rightarrow real$ **where**
 $g\ x = ln-Gamma\ x - (x - 1/2) * ln\ x + x$

qualified lemma *DERIV-g*: $x > 0 \implies (g\ \text{has-field-derivative}\ -P\ x)$ (*at x*)
unfolding *g-def* [*abs-def*] **using** *Digamma-approx*[*of x*]
by (*auto intro!: derivative-eq-intros simp: field-simps*)

qualified lemma *isCont-P*:
assumes $x > 0$
shows *isCont* $P\ x$
proof –
define $D' :: real \Rightarrow real$
where $D' = (\lambda x. - 1 / (2 * x^2 * (x+1)^2))$
have *DERIV-D*: (*D has-field-derivative D' x*) (*at x*) **if** $x > 0$ **for** x
unfolding *D-def* [*abs-def*] *D'-def* *T-def*
by (*insert that, (rule derivative-eq-intros refl | simp)+*)
(simp add: power2-eq-square divide-simps, (simp add: algebra-simps)?)
note *this* [*THEN DERIV-chain2, derivative-intros*]
have (*P has-field-derivative* $(\sum\ n. D' (real\ n + x))$) (*at x*)

```

    unfolding P-def [abs-def]
  proof (rule has-field-derivative-series')
    show convex {x/2<..} by simp
  next
    fix n :: nat and y :: real assume y: y ∈ {x/2<..}
    with assms have y > 0 by simp
    thus ((λa. D (real n + a)) has-real-derivative D' (real n + y)) (at y within
{x/2<..})
      by (auto intro!: derivative-eq-intros)
  next
    from assms D-summable[of x] show summable (λn. D (real n + x)) by simp
  next
    show uniformly-convergent-on {x/2<..} (λn x. ∑ i<n. D' (real i + x))
  proof (rule Weierstrass-m-test')
    fix n :: nat and y :: real
    assume y: y ∈ {x/2<..}
    with assms have y > 0 by auto
    have norm (D' (real n + y)) = (1 / (2 * (y + real n)^2)) * (1 / (y + real
(Suc n))^2)
      by (simp add: D'-def add-ac)
    also from y assms have ... ≤ (1 / (2 * (x/2)^2)) * (1 / (real (Suc n))^2)
      by (intro mult-mono divide-left-mono power-mono) simp-all
    also have 1 / (real (Suc n))^2 = inverse ((real (Suc n))^2) by (simp add:
field-simps)
    finally show norm (D' (real n + y)) ≤ (1 / (2 * (x/2)^2)) * inverse ((real
(Suc n))^2) .
  next
    show summable (λn. (1 / (2 * (x/2)^2)) * inverse ((real (Suc n))^2))
      by (subst summable-Suc-iff, intro summable-mult inverse-power-summable)
simp-all
  qed
  qed (insert assms, simp-all add: interior-open)
  thus ?thesis by (rule DERIV-isCont)
qed

```

```

qualified lemma P-continuous-on [THEN continuous-on-subset]: continuous-on
{0<..} P
  by (intro continuous-at-imp-continuous-on ballI isCont-P) auto

```

```

qualified lemma P-integrable:
  assumes a: a > 0
  shows P integrable-on {a..}
proof -
  define f where f = (λn x. if x ∈ {a..real n} then P x else 0)
  show P integrable-on {a..}
  proof (rule dominated-convergence)
    fix n :: nat
    from a have P integrable-on {a..real n}
      by (intro integrable-continuous-real P-continuous-on) auto
  qed

```

```

hence  $f\ n$  integrable-on  $\{a..real\ n\}$ 
  by (rule integrable-eq) (simp add: f-def)
thus  $f\ n$  integrable-on  $\{a..\}$ 
  by (rule integrable-on-superset) (auto simp: f-def)
next
  fix  $n :: nat$ 
  show  $norm\ (f\ n\ x) \leq of-real\ (1/12) * (1 / x^2)$  if  $x \in \{a..\}$  for  $x$ 
    using a  $P$ -ge-0  $P$ -upper-bound by (auto simp: f-def)
next
  show  $(\lambda x::real. of-real\ (1/12) * (1 / x^2))$  integrable-on  $\{a..\}$ 
    using has-integral-inverse-power-to-inf[of 2 a] a
    by (intro integrable-on-cmult-left) auto
next
  show  $(\lambda n. f\ n\ x) \longrightarrow P\ x$  if  $x \in \{a..\}$  for  $x$ 
  proof -
    have eventually  $(\lambda n. real\ n \geq x)$  at-top
      using filterlim-real-sequentially by (simp add: filterlim-at-top)
    with that not-frequently have eventually  $(\lambda n. f\ n\ x = P\ x)$  at-top
      by (force simp: f-def)
    thus  $(\lambda n. f\ n\ x) \longrightarrow P\ x$  by (simp add: tendsto-eventually)
  qed
qed
qed

```

qualified definition $c :: real$ **where** $c = integral\ \{1..\}\ (\lambda x. -P\ x) + g\ 1$

We can now give bounds on g :

qualified lemma g -bounds:

```

assumes  $x: x \geq 1$ 
shows  $g\ x \in \{c..c + 1/(12*x)\}$ 
proof -
  from assms have int-nonneg:  $integral\ \{x..\}\ P \geq 0$ 
    by (intro Henstock-Kurzweil-Integration.integral-nonneg  $P$ -integrable)
    (auto simp:  $P$ -ge-0)
  have int-upper-bound:  $integral\ \{x..\}\ P \leq 1/(12*x)$ 
  proof (rule has-integral-le)
    from  $x$  show  $(P\ has-integral\ integral\ \{x..\}\ P)\ \{x..\}$ 
      by (intro integrable-integral  $P$ -integrable) simp-all
    from  $x$  has-integral-mult-right[OF has-integral-inverse-power-to-inf[of 2 x], of 1/12]
      show  $((\lambda x. 1/(12*x^2))\ has-integral\ (1/(12*x)))\ \{x..\}$  by (simp add:
field-simps)
  qed (insert  $P$ -upper-bound  $x$ , simp-all)

```

note $DERIV$ - g [*THEN DERIV-chain2, derivative-intros*]

from *assms* **have** int1: $((\lambda x. -P\ x)\ has-integral\ (g\ x - g\ 1))\ \{1..x\}$

by (intro fundamental-theorem-of-calculus)

(auto simp: has-real-derivative-iff-has-vector-derivative [*symmetric*])

intro!: derivative-eq-intros)

from x **have** $int2: ((\lambda x. -P x) \text{ has-integral integral } \{x..\} (\lambda x. -P x)) \{x..\}$
by (*intro integrable-integral integrable-neg P-integrable*) *simp-all*
from *has-integral-Un[OF int1 int2]* x
have $((\lambda x. -P x) \text{ has-integral } g x - g 1 - \text{integral } \{x..\} P) (\{1..x\} \cup \{x..\})$
by (*simp add: max-def*)
also from x **have** $\{1..x\} \cup \{x..\} = \{1..\}$ **by** *auto*
finally have $((\lambda x. -P x) \text{ has-integral } g x - g 1 - \text{integral } \{x..\} P) \{1..\}$.
moreover have $((\lambda x. -P x) \text{ has-integral integral } \{1..\} (\lambda x. -P x)) \{1..\}$
by (*intro integrable-integral integrable-neg P-integrable*) *simp-all*
ultimately have $g x - g 1 - \text{integral } \{x..\} P = \text{integral } \{1..\} (\lambda x. -P x)$
by (*simp add: has-integral-unique*)
hence $g x = c + \text{integral } \{x..\} P$ **by** (*simp add: c-def algebra-simps*)
with *int-upper-bound int-nonneg* **show** $g x \in \{c..c + 1/(12*x)\}$ **by** *simp*
qed

Finally, we have bounds on \ln -Gamma, Gamma, and fact.

qualified lemma *ln-Gamma-bounds-aux*:

$x \geq 1 \implies \ln\text{-Gamma } x \geq c + (x - 1/2) * \ln x - x$
 $x \geq 1 \implies \ln\text{-Gamma } x \leq c + (x - 1/2) * \ln x - x + 1/(12*x)$
using *g-bounds[of x]* **by** (*simp-all add: g-def*)

qualified lemma *Gamma-bounds-aux*:

assumes $x: x \geq 1$
shows $\text{Gamma } x \geq \exp c * x \text{ powr } (x - 1/2) / \exp x$
 $\text{Gamma } x \leq \exp c * x \text{ powr } (x - 1/2) / \exp x * \exp (1/(12*x))$

proof –

have $\exp (\ln\text{-Gamma } x) \geq \exp (c + (x - 1/2) * \ln x - x)$
by (*subst exp-le-cancel-iff, rule ln-Gamma-bounds-aux*) (*simp add: x*)
with x **show** $\text{Gamma } x \geq \exp c * x \text{ powr } (x - 1/2) / \exp x$
by (*simp add: Gamma-real-pos-exp exp-add exp-diff powr-def del: exp-le-cancel-iff*)

next

have $\exp (\ln\text{-Gamma } x) \leq \exp (c + (x - 1/2) * \ln x - x + 1/(12*x))$
by (*subst exp-le-cancel-iff, rule ln-Gamma-bounds-aux*) (*simp add: x*)
with x **show** $\text{Gamma } x \leq \exp c * x \text{ powr } (x - 1/2) / \exp x * \exp (1/(12*x))$
by (*simp add: Gamma-real-pos-exp exp-add exp-diff powr-def del: exp-le-cancel-iff*)

qed

qualified lemma *Gamma-asymp-equiv-aux*:

$\text{Gamma} \sim_{[at\text{-top}]} (\lambda x. \exp c * x \text{ powr } (x - 1/2) / \exp x)$

proof (*rule asymp-equiv-sandwich*)

include *asymp-equiv-syntax*

show *eventually* $(\lambda x. \exp c * x \text{ powr } (x - 1/2) / \exp x \leq \text{Gamma } x)$ *at-top*
eventually $(\lambda x. \exp c * x \text{ powr } (x - 1/2) / \exp x * \exp (1/(12*x)) \geq \text{Gamma } x)$ *at-top*

using *eventually-ge-at-top[of 1::real]*

by (*eventually-elim; use Gamma-bounds-aux in force*)+

have $(\lambda x::\text{real}. \exp (1 / (12 * x))) \longrightarrow \exp 0$ *at-top*

by *real-asymp*

hence $(\lambda x. \exp (1 / (12 * x))) \sim (\lambda x. 1 :: \text{real})$

by (*intro asymp-equivI'*) *simp-all*
hence $(\lambda x. \exp c * x \text{ powr } (x - 1 / 2) / \exp x * 1) \sim$
 $(\lambda x. \exp c * x \text{ powr } (x - 1 / 2) / \exp x * \exp (1 / (12 * x)))$
by (*intro asymp-equiv-mult asymp-equiv-refl*) (*simp add: asymp-equiv-sym*)
thus $(\lambda x. \exp c * x \text{ powr } (x - 1 / 2) / \exp x) \sim$
 $(\lambda x. \exp c * x \text{ powr } (x - 1 / 2) / \exp x * \exp (1 / (12 * x)))$ **by** *simp*
qed *simp-all*

qualified lemma *exp-1-powr-real* [*simp*]: $\exp (1 :: \text{real}) \text{ powr } x = \exp x$
by (*simp add: powr-def*)

qualified lemma *fact-asymp-equiv-aux*:

fact \sim [*at-top*] $(\lambda x. \exp c * \text{sqrt } (\text{real } x) * (\text{real } x / \exp 1) \text{ powr } \text{real } x)$

proof –

include *asymp-equiv-syntax*

have *fact* \sim $(\lambda n. \text{Gamma } (\text{real } (\text{Suc } n)))$ **by** (*simp add: Gamma-fact*)

also have *eventually* $(\lambda n. \text{Gamma } (\text{real } (\text{Suc } n)) = \text{real } n * \text{Gamma } (\text{real } n))$

at-top

using *eventually-gt-at-top*[*of 0::nat*]

by *eventually-elim* (*insert Gamma-plus1*[*of real n for n*],

auto simp: add-ac of-nat-in-nonpos-Ints-iff)

also have $(\lambda n. \text{Gamma } (\text{real } n)) \sim (\lambda n. \exp c * \text{real } n \text{ powr } (\text{real } n - 1 / 2) / \exp (\text{real } n))$

by (*rule asymp-equiv-compose'*[*OF Gamma-asymp-equiv-aux*] *filterlim-real-sequentially*) +

also have *eventually* $(\lambda n. \text{real } n * (\exp c * \text{real } n \text{ powr } (\text{real } n - 1 / 2) / \exp (\text{real } n))) =$

$\exp c * \text{sqrt } (\text{real } n) * (\text{real } n / \exp 1) \text{ powr } \text{real } n$ *at-top*

using *eventually-gt-at-top*[*of 0::nat*]

proof *eventually-elim*

fix $n :: \text{nat}$ **assume** $n > 0$

thus $\text{real } n * (\exp c * \text{real } n \text{ powr } (\text{real } n - 1 / 2) / \exp (\text{real } n)) =$

$\exp c * \text{sqrt } (\text{real } n) * (\text{real } n / \exp 1) \text{ powr } \text{real } n$

by (*subst powr-diff*) (*simp-all add: powr-divide powr-half-sqrt field-simps*)

qed

finally show *?thesis* **by** – (*simp-all add: asymp-equiv-mult*)

qed

We can also bound *Digamma* above and below.

lemma *Digamma-plus-1-gt-ln*:

assumes $x > (0 :: \text{real})$

shows $\text{Digamma } (x + 1) > \ln x$

proof –

have $0 < (17 :: \text{real})$

by *simp*

also have $17 \leq 12 * x^2 + 28 * x + 17$

using x **by** *auto*

finally have $0 < (12 * x^2 + 28 * x + 17) / (12 * (x + 1)^2 * (1 + 2 * x))$

using x **by** (*intro divide-pos-pos mult-pos-pos zero-less-power*) *auto*

also have $\dots = 2 / (2 * x + 1) - 1 / (2 * (x + 1)) - 1 / (12 * (x + 1) ^ 2)$
using x **by** (*simp add: divide-simps*) (*auto simp: field-simps power2-eq-square add-eq-0-iff*)
also have $2 / (2 * x + 1) \leq \ln (x + 1) - \ln x$
using *ln-inverse-approx-ge*[*of x x + 1*] x **by** *simp*
also have $\dots - 1 / (2 * (x + 1)) - 1 / (12 * (x + 1) ^ 2) \leq$
 $\ln (x + 1) - \ln x - 1 / (2 * (x + 1)) - P (x + 1)$
using *P-upper-bound*[*of x + 1*] x **by** (*intro diff-mono*) *auto*
also have $\dots = \text{Digamma } (x + 1) - \ln x$
by (*subst Digamma-approx*) (*use x in auto*)
finally show $\text{Digamma } (x + 1) > \ln x$
by *simp*
qed

lemma *Digamma-less-ln*:
assumes $x: x > (0 :: \text{real})$
shows $\text{Digamma } x < \ln x$
proof –
have $\text{Digamma } x - \ln x = - (1 / (2 * x)) - P x$
by (*subst Digamma-approx*) (*use x in auto*)
also have $\dots < 0 - P x$
using x **by** (*intro diff-strict-right-mono*) *auto*
also have $\dots \leq 0$
using *P-ge-0*[*of x*] x **by** *simp*
finally show $\text{Digamma } x < \ln x$
by *simp*
qed

We still need to determine the constant term c , which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

qualified lemma *powr-mult-2*: $(x::\text{real}) > 0 \implies x \text{ powr } (y * 2) = (x ^ 2) \text{ powr } y$
by (*subst mult.commute*, *subst powr-powr [symmetric]*) (*simp add: powr-numeral*)

qualified lemma *exp-mult-2*: $\exp (y * 2 :: \text{real}) = \exp y * \exp y$
by (*subst exp-add [symmetric]*) *simp*

qualified lemma *exp-c*: $\exp c = \text{sqrt } (2 * \pi)$

proof –
include *asympt-equiv-syntax*
define p **where** $p = (\lambda n. \prod_{k=1..n} (4 * \text{real } k ^ 2) / (4 * \text{real } k ^ 2 - 1))$
have $p-0$ [*simp*]: $p 0 = 1$ **by** (*simp add: p-def*)
have $p\text{-Suc}$: $p (\text{Suc } n) = p n * (4 * \text{real } (\text{Suc } n) ^ 2) / (4 * \text{real } (\text{Suc } n) ^ 2 - 1)$
for n **unfolding** $p\text{-def}$ **by** (*subst prod.nat-ivl-Suc'*) *simp-all*
have p : $p = (\lambda n. 16 ^ n * \text{fact } n ^ 4 / (\text{fact } (2 * n) ^ 2 / (2 * \text{real } n + 1)))$
proof

```

fix n :: nat
have p n = (∏ k=1..n. (2*real k)2 / (2*real k - 1) / (2 * real k + 1))
unfolding p-def by (intro prod.cong refl) (simp add: field-simps power2-eq-square)
also have ... = (∏ k=1..n. (2*real k)2 / (2*real k - 1)) / (∏ k=1..n. (2 *
real (Suc k) - 1))
by (simp add: prod-dividef prod.distrib add-ac)
also have (∏ k=1..n. (2 * real (Suc k) - 1)) = (∏ k=Suc 1..Suc n. (2 * real
k - 1))
by (subst prod.atLeast-Suc-atMost-Suc-shift) simp-all
also have ... = (∏ k=1..n. (2 * real k - 1)) * (2 * real n + 1)
by (induction n) (simp-all add: prod.nat-ivl-Suc^)
also have (∏ k = 1..n. (2 * real k)2 / (2 * real k - 1)) / ... =
(∏ k = 1..n. (2 * real k)2 / (2 * real k - 1)2) / (2 * real n + 1)
unfolding power2-eq-square by (simp add: prod.distrib prod-dividef)
also have (∏ k = 1..n. (2 * real k)2 / (2 * real k - 1)2) =
(∏ k = 1..n. (2 * real k)4 / ((2*real k)*(2 * real k - 1))2)
by (rule prod.cong) (simp-all add: power2-eq-square eval-nat-numeral)
also have ... = 16^n * fact n4 / (∏ k=1..n. (2*real k) * (2*real k - 1))2
by (simp add: prod.distrib prod-dividef fact-prod
prod-power-distrib [symmetric] prod-constant)
also have (∏ k=1..n. (2*real k) * (2*real k - 1)) = fact (2*n)
by (induction n) (simp-all add: algebra-simps prod.nat-ivl-Suc^)
finally show p n = 16^n * fact n4 / (fact (2 * n))2 / (2 * real n + 1) .
qed

```

```

have p ~ (λn. 16^n * fact n4 / (fact (2 * n))2 / (2 * real n + 1))
by (simp add: p)
also have ... ~ (λn. 16^n * (exp c * sqrt (real n) * (real n / exp 1) powr real
n)4 /
(exp c * sqrt (real (2*n)) * (real (2*n) / exp 1) powr real
(2*n))2 /
(2 * real n + 1)) (is - ~ ?f)
by (intro asymp-equiv-mult asymp-equiv-divide asymp-equiv-refl mult-nat-left-at-top
fact-asymp-equiv-aux asymp-equiv-power asymp-equiv-compose^[OF
fact-asymp-equiv-aux])
simp-all
also have eventually (λn. ... n = exp c^2 / (4 + 2/n)) at-top
using eventually-gt-at-top[of 0::nat]
proof eventually-elim
fix n :: nat assume n: n > 0
have [simp]: 16^n = 4^n * (4^n :: real) by (simp add: power-mult-distrib
[symmetric])
from n have ?f n = exp c^2 * (n / (2*(2*n+1)))
by (simp add: power-mult-distrib divide-simps powr-mult real-sqrt-power-even)
(simp add: field-simps power2-eq-square eval-nat-numeral powr-mult-2
exp-mult-2 powr-realpow)
also from n have ... = exp c^2 / (4 + 2/n) by (simp add: field-simps)
finally show ?f n = ... .
qed

```

also have $(\lambda x. 4 + 2 / \text{real } x) \sim (\lambda x. 4)$
by *(subst asymp-equiv-add-right) auto*
finally have $p \longrightarrow \exp c \wedge 2 / 4$
by *(rule asymp-equivD-const) (simp-all add: asymp-equiv-divide)*
moreover have $p \longrightarrow \pi / 2$ **unfolding** *p-def* **by** *(rule wallis)*
ultimately have $\exp c \wedge 2 / 4 = \pi / 2$ **by** *(rule LIMSEQ-unique)*
hence $2 * \pi = \exp c \wedge 2$ **by** *simp*
also have $\text{sqrt} (\exp c \wedge 2) = \exp c$ **by** *simp*
finally show $\exp c = \text{sqrt} (2 * \pi)$..
qed

qualified lemma *c: c = ln (2*pi) / 2*
proof –
note *exp-c [symmetric]*
also have $\ln (\exp c) = c$ **by** *simp*
finally show *?thesis* **by** *(simp add: ln-sqrt)*
qed

This gives us the final bounds:

theorem *Gamma-bounds:*
assumes $x \geq 1$
shows $\text{Gamma } x \geq \text{sqrt} (2*\pi/x) * (x / \exp 1) \text{ powr } x$ (**is** *?th1*)
 $\text{Gamma } x \leq \text{sqrt} (2*\pi/x) * (x / \exp 1) \text{ powr } x * \exp (1 / (12 * x))$ (**is** *?th2*)
proof –
from *assms* **have** $\exp c * x \text{ powr } (x - 1/2) / \exp x = \text{sqrt} (2*\pi/x) * (x / \exp 1) \text{ powr } x$
by *(subst powr-diff)*
(simp add: exp-c real-sqrt-divide powr-divide powr-half-sqrt field-simps)
with *Gamma-bounds-aux[OF assms]* **show** *?th1 ?th2* **by** *simp-all*
qed

theorem *ln-Gamma-bounds:*
assumes $x \geq 1$
shows $\ln\text{-Gamma } x \geq \ln (2*\pi/x) / 2 + x * \ln x - x$ (**is** *?th1*)
 $\ln\text{-Gamma } x \leq \ln (2*\pi/x) / 2 + x * \ln x - x + 1/(12*x)$ (**is** *?th2*)
proof –
from *ln-Gamma-bounds-aux[OF assms]* *assms* **show** *?th1 ?th2*
by *(simp-all add: c field-simps ln-div)*
from *assms* **have** $\exp c * x \text{ powr } (x - 1/2) / \exp x = \text{sqrt} (2*\pi/x) * (x / \exp 1) \text{ powr } x$
by *(subst powr-diff)*
(simp add: exp-c real-sqrt-divide powr-divide powr-half-sqrt field-simps)
qed

theorem *fact-bounds:*
assumes $n > 0$
shows $(\text{fact } n :: \text{real}) \geq \text{sqrt} (2*\pi*n) * (n / \exp 1) \wedge n$ (**is** *?th1*)
 $(\text{fact } n :: \text{real}) \leq \text{sqrt} (2*\pi*n) * (n / \exp 1) \wedge n * \exp (1 / (12 * n))$ (**is** *?th2*)
qed

?th2)

proof –

from *assms* **have** $n: \text{real } n \geq 1$ **by** *simp*

from *assms* *Gamma-plus1* [*of real n*]

have $\text{real } n * \text{Gamma } (\text{real } n) = \text{Gamma } (\text{real } (\text{Suc } n))$

by (*simp add: of-nat-in-nonpos-Ints-iff add-ac*)

also have $\text{Gamma } (\text{real } (\text{Suc } n)) = \text{fact } n$ **by** (*subst Gamma-fact [symmetric]*)

simp

finally have $*$: $\text{fact } n = \text{real } n * \text{Gamma } (\text{real } n)$ **by** *simp*

have $2*\pi/n = 2*\pi*n / n^2$ **by** (*simp add: power2-eq-square*)

also have $\text{sqrt } \dots = \text{sqrt } (2*\pi*n) / n$ **by** (*subst real-sqrt-divide*) *simp-all*

also have $\text{real } n * \dots = \text{sqrt } (2*\pi*n)$ **by** *simp*

finally have $**$: $\text{real } n * \text{sqrt } (2*\pi/\text{real } n) = \text{sqrt } (2*\pi*\text{real } n)$.

note $*$

also note *Gamma-bounds(2)* [*OF n*]

also have $\text{real } n * (\text{sqrt } (2 * \pi / \text{real } n) * (\text{real } n / \text{exp } 1) \text{ powr } \text{real } n * \text{exp } (1 / (12 * \text{real } n))) =$
 $(\text{real } n * \text{sqrt } (2*\pi/n)) * (n / \text{exp } 1) \text{ powr } n * \text{exp } (1 / (12 * n))$

by (*simp add: algebra-simps*)

also from n **have** $(\text{real } n / \text{exp } 1) \text{ powr } \text{real } n = (\text{real } n / \text{exp } 1) ^ n$

by (*subst powr-realpow*) *simp-all*

also note $**$

finally show ?th2 **by** – (*insert assms, simp-all*)

have $\text{sqrt } (2*\pi*n) * (n / \text{exp } 1) \text{ powr } n = n * (\text{sqrt } (2*\pi/n) * (n / \text{exp } 1) \text{ powr } n)$

by (*subst ** [symmetric]*) (*simp add: field-simps*)

also from *assms* **have** $\dots \leq \text{real } n * \text{Gamma } (\text{real } n)$

by (*intro mult-left-mono Gamma-bounds(1)*) *simp-all*

also from n **have** $(\text{real } n / \text{exp } 1) \text{ powr } \text{real } n = (\text{real } n / \text{exp } 1) ^ n$

by (*subst powr-realpow*) *simp-all*

also note $*$ [*symmetric*]

finally show ?th1 .

qed

theorem *ln-fact-bounds*:

assumes $n > 0$

shows $\ln (\text{fact } n :: \text{real}) \geq \ln (2*\pi*n)/2 + n * \ln n - n$ (**is** ?th1)

$\ln (\text{fact } n :: \text{real}) \leq \ln (2*\pi*n)/2 + n * \ln n - n + 1/(12*\text{real } n)$ (**is** ?th2)

proof –

have $\ln (\text{fact } n :: \text{real}) \geq \ln (\text{sqrt } (2*\pi*\text{real } n) * (\text{real } n / \text{exp } 1) ^ n)$

using *fact-bounds(1)* [*OF assms*] *assms* **by** (*subst ln-le-cancel-iff*) *auto*

also from *assms* **have** $\ln (\text{sqrt } (2*\pi*\text{real } n) * (\text{real } n / \text{exp } 1) ^ n) = \ln (2*\pi*n)/2 + n * \ln n - n$

by (*simp add: ln-mult ln-div ln-realpow ln-sqrt field-simps*)

finally show ?th1 .

next
have $\ln (\text{fact } n :: \text{real}) \leq \ln (\text{sqrt } (2 * \text{pi} * \text{real } n) * (\text{real } n / \text{exp } 1) ^ n * \text{exp } (1 / (12 * \text{real } n)))$
using *fact-bounds(2)[OF assms] assms* **by** (*subst ln-le-cancel-iff*) *auto*
also from *assms* **have** $\dots = \ln (2 * \text{pi} * n) / 2 + n * \ln n - n + 1 / (12 * \text{real } n)$
by (*simp add: ln-mult ln-div ln-realpow ln-sqrt field-simps*)
finally show *?th2* .
qed

theorem *Gamma-asymp-equiv:*

$\text{Gamma} \sim_{[\text{at-top}]} (\lambda x. \text{sqrt } (2 * \text{pi} / x) * (x / \text{exp } 1) \text{ powr } x :: \text{real})$

proof –

note *Gamma-asymp-equiv-aux*

also have *eventually* $(\lambda x. \text{exp } c * x \text{ powr } (x - 1/2) / \text{exp } x = \text{sqrt } (2 * \text{pi} / x) * (x / \text{exp } 1) \text{ powr } x)$ *at-top*

using *eventually-gt-at-top[of 0::real]*

proof *eventually-elim*

fix $x :: \text{real}$ **assume** $x > 0$

thus $\text{exp } c * x \text{ powr } (x - 1/2) / \text{exp } x = \text{sqrt } (2 * \text{pi} / x) * (x / \text{exp } 1) \text{ powr } x$
by (*subst powr-diff*)

(*simp add: exp-c powr-half-sqrt powr-divide field-simps real-sqrt-divide*)

qed

finally show *?thesis* .

qed

theorem *fact-asymp-equiv:*

$\text{fact} \sim_{[\text{at-top}]} (\lambda n. \text{sqrt } (2 * \text{pi} * n) * (n / \text{exp } 1) ^ n :: \text{real})$

proof –

note *fact-asymp-equiv-aux*

also have *eventually* $(\lambda n. \text{exp } c * \text{sqrt } (\text{real } n) = \text{sqrt } (2 * \text{pi} * \text{real } n))$ *at-top*

using *eventually-gt-at-top[of 0::nat]* **by** *eventually-elim (simp add: exp-c real-sqrt-mult)*

also have *eventually* $(\lambda n. (n / \text{exp } 1) \text{ powr } n = (n / \text{exp } 1) ^ n)$ *at-top*

using *eventually-gt-at-top[of 0::nat]* **by** *eventually-elim (simp add: powr-realpow)*

finally show *?thesis* .

qed

corollary *stirling-tendsto-sqrt-pi:*

$(\lambda n. \text{fact } n / (\text{sqrt } (2 * n) * (n / \text{exp } 1) ^ n)) \longrightarrow \text{sqrt } \text{pi}$

proof –

have $*$: $(\lambda n. \text{fact } n / (\text{sqrt } (2 * \text{pi} * n) * (n / \text{exp } 1) ^ n)) \longrightarrow 1$

using *fact-asymp-equiv* **by** (*simp add: asymp-equiv-def*)

have $(\lambda n. \text{sqrt } \text{pi} * \text{fact } n / (\text{sqrt } (2 * \text{pi} * \text{real } n) * (\text{real } n / \text{exp } 1) ^ n))$
 $= (\lambda n. \text{fact } n / (\text{sqrt } (\text{real } (2 * n)) * (\text{real } n / \text{exp } 1) ^ n))$

by (*force simp add: divide-simps powr-realpow real-sqrt-mult*)

with *tendsto-mult-left[OF *, of sqrt pi]* **show** *?thesis* **by** *simp*

qed

end

end

2 Complete asymptotics of the logarithmic Gamma function

theory *Gamma-Asymptotics*

imports

HOL-Complex-Analysis.Complex-Analysis

Bernoulli.Bernoulli-FPS

Bernoulli.Periodic-Bernpoly

Stirling-Formula

begin

2.1 Auxiliary Facts

lemma *stirling-limit-aux1*:

$((\lambda y. \text{Ln } (1 + z * \text{of-real } y) / \text{of-real } y) \longrightarrow z) \text{ (at-right } 0) \text{ for } z :: \text{complex}$

proof (cases $z = 0$)

case *True*

then show ?thesis by simp

next

case *False*

have $((\lambda y. \ln (1 + z * \text{of-real } y)) \text{ has-vector-derivative } 1 * z) \text{ (at } 0)$

by (rule *has-vector-derivative-real-field*) (auto intro!: *derivative-eq-intros*)

then have $(\lambda y. (\text{Ln } (1 + z * \text{of-real } y) - \text{of-real } y * z) / \text{of-real } |y|) \rightarrow 0$

by (auto simp add: *has-vector-derivative-def has-derivative-def netlimit-at scaleR-conv-of-real field-simps*)

then have $((\lambda y. (\text{Ln } (1 + z * \text{of-real } y) - \text{of-real } y * z) / \text{of-real } |y|) \longrightarrow 0) \text{ (at-right } 0)$

by (rule *filterlim-mono[OF - - at-le]*) *simp-all*

also have ?this $\longleftrightarrow ((\lambda y. \text{Ln } (1 + z * \text{of-real } y) / (\text{of-real } y) - z) \longrightarrow 0) \text{ (at-right } 0)$

using *eventually-at-right-less[of 0::real]*

by (intro *filterlim-cong refl*) (auto elim!: *eventually-mono simp: field-simps*)

finally show ?thesis by (*simp only: LIM-zero-iff*)

qed

lemma *stirling-limit-aux2*:

$((\lambda y. y * \text{Ln } (1 + z / \text{of-real } y)) \longrightarrow z) \text{ at-top for } z :: \text{complex}$

using *stirling-limit-aux1* [of z] by (*subst filterlim-at-top-to-right*) (*simp add: field-simps*)

lemma *Union-atLeastAtMost*:

assumes $N > 0$

shows $(\bigcup n \in \{0..<N\}. \{\text{real } n.. \text{real } (n + 1)\}) = \{0.. \text{real } N\}$

proof (intro *equalityI subsetI*)

fix x assume $x: x \in \{0.. \text{real } N\}$

thus $x \in (\bigcup n \in \{0..<N\}. \{\text{real } n.. \text{real } (n + 1)\})$

proof (cases $x = \text{real } N$)

```

    case True
    with assms show ?thesis by (auto intro!: bexI[of - N - 1])
next
case False
with x have x: x ≥ 0 x < real N by simp-all
hence x ≥ real (nat ⌊x⌋) x ≤ real (nat ⌊x⌋ + 1) by linarith+
moreover from x have nat ⌊x⌋ < N by linarith
ultimately have ∃ n ∈ {0..<N}. x ∈ {real n..real (n + 1)}
  by (intro bexI[of - nat ⌊x⌋]) simp-all
thus ?thesis by blast
qed
qed auto

```

2.2 Cones in the complex plane

definition *complex-cone* :: $\text{real} \Rightarrow \text{real} \Rightarrow \text{complex set}$ **where**
complex-cone a b = $\{z. \exists y \in \{a..b\}. z = \text{rcis} (\text{norm } z) y\}$

abbreviation *complex-cone'* :: $\text{real} \Rightarrow \text{complex set}$ **where**
complex-cone' a \equiv *complex-cone* (-a) a

lemma *zero-in-complex-cone* [simp, intro]: $a \leq b \implies 0 \in \text{complex-cone } a b$
 by (auto simp: complex-cone-def)

lemma *complex-coneE*:

assumes $z \in \text{complex-cone } a b$
 obtains $r \alpha$ where $r \geq 0 \alpha \in \{a..b\} z = \text{rcis } r \alpha$

proof –

from assms obtain y where $y \in \{a..b\} z = \text{rcis} (\text{norm } z) y$
 unfolding *complex-cone-def* by auto
 thus ?thesis using that[of norm z y] by auto

qed

lemma *arg-cis* [simp]:

assumes $x \in \{-\pi <.. \pi\}$
 shows $\text{Arg} (\text{cis } x) = x$
 using assms by (intro cis-Arg-unique) auto

lemma *arg-mult-of-real-left* [simp]:

assumes $r > 0$
 shows $\text{Arg} (\text{of-real } r * z) = \text{Arg } z$

proof (cases $z = 0$)

case False

thus ?thesis

using *Arg-bounded*[of z] assms

by (intro cis-Arg-unique) (auto simp: sgn-mult sgn-of-real cis-Arg)

qed auto

lemma *arg-mult-of-real-right* [simp]:

```

assumes  $r > 0$ 
shows  $\text{Arg } (z * \text{of-real } r) = \text{Arg } z$ 
by (subst mult.commute, subst arg-mult-of-real-left) (simp-all add: assms)

lemma arg-rcis [simp]:
assumes  $x \in \{-\pi < .. \pi\}$   $r > 0$ 
shows  $\text{Arg } (\text{rcis } r x) = x$ 
using assms by (simp add: rcis-def)

lemma rcis-in-complex-cone [intro]:
assumes  $\alpha \in \{a..b\}$   $r \geq 0$ 
shows  $\text{rcis } r \alpha \in \text{complex-cone } a b$ 
using assms by (auto simp: complex-cone-def)

lemma arg-imp-in-complex-cone:
assumes  $\text{Arg } z \in \{a..b\}$ 
shows  $z \in \text{complex-cone } a b$ 
proof –
have  $z = \text{rcis } (\text{norm } z) (\text{Arg } z)$ 
by (simp add: rcis-cmod-Arg)
also have  $\dots \in \text{complex-cone } a b$ 
using assms by auto
finally show ?thesis .
qed

lemma complex-cone-altdef:
assumes  $-\pi < a \leq b \leq \pi$ 
shows  $\text{complex-cone } a b = \text{insert } 0 \{z. \text{Arg } z \in \{a..b\}\}$ 
proof (intro equalityI subsetI)
fix  $z$  assume  $z \in \text{complex-cone } a b$ 
then obtain  $r \alpha$  where  $*$ :  $r \geq 0 \alpha \in \{a..b\} z = \text{rcis } r \alpha$ 
by (auto elim: complex-coneE)
have  $\text{Arg } z \in \{a..b\}$  if [simp]:  $z \neq 0$ 
proof –
have  $r > 0$  using that  $*$  by (subst (asm) *) auto
hence  $\alpha \in \{a..b\}$ 
using  $*(1,2)$  assms by (auto simp: *(1))
moreover from assms  $*(2)$  have  $\alpha \in \{-\pi < .. \pi\}$ 
by auto
ultimately show ?thesis using  $*(3)$   $\langle r > 0 \rangle$ 
by (subst *) auto
qed
thus  $z \in \text{insert } 0 \{z. \text{Arg } z \in \{a..b\}\}$ 
by auto
qed (use assms in  $\langle \text{auto intro: arg-imp-in-complex-cone} \rangle$ )

lemma nonneg-of-real-in-complex-cone [simp, intro]:
assumes  $x \geq 0 \ a \leq 0 \ 0 \leq b$ 
shows  $\text{of-real } x \in \text{complex-cone } a b$ 

```

proof –

from *assms* **have** *rcis* $x \ 0 \in \text{complex-cone } a \ b$
by (*intro rcis-in-complex-cone*) *auto*
thus *?thesis* **by** *simp*
qed

lemma *one-in-complex-cone* [*simp, intro*]: $a \leq 0 \implies 0 \leq b \implies 1 \in \text{complex-cone } a \ b$

using *nonneg-of-real-in-complex-cone[of 1]* **by** (*simp del: nonneg-of-real-in-complex-cone*)

lemma *of-nat-in-complex-cone* [*simp, intro*]: $a \leq 0 \implies 0 \leq b \implies \text{of-nat } n \in \text{complex-cone } a \ b$

using *nonneg-of-real-in-complex-cone[of real n]* **by** (*simp del: nonneg-of-real-in-complex-cone*)

2.3 Another integral representation of the Beta function

lemma *complex-cone-inter-nonpos-Reals*:

assumes $-\pi < a \ a \leq b \ b < \pi$

shows $\text{complex-cone } a \ b \cap \mathbb{R}_{\leq 0} = \{0\}$

proof (*safe elim!: nonpos-Reals-cases*)

fix $x :: \text{real}$

assume *complex-of-real* $x \in \text{complex-cone } a \ b \ x \leq 0$

hence $\neg(x < 0)$

using *assms* **by** (*intro notI*) (*auto simp: complex-cone-altdef*)

with $\langle x \leq 0 \rangle$ **show** *complex-of-real* $x = 0$ **by** *auto*

qed (*use assms in auto*)

theorem

assumes $a > 0$ **and** $b > 0$ ($:: \text{real}$)

shows *has-integral-Beta-real*:

$((\lambda u. u \ \text{powr } (b - 1) / (1 + u) \ \text{powr } (a + b)) \ \text{has-integral } \text{Beta } a \ b) \ \{0 < ..\}$

and *Beta-conv-nn-integral*:

$\text{Beta } a \ b = (\int^{+} u. \ \text{ennreal } (\text{indicator } \{0 < ..\} \ u * u \ \text{powr } (b - 1) / (1 + u) \ \text{powr } (a + b)) \ \partial \text{lborel})$

proof –

define I **where**

$I = (\int^{+} u. \ \text{ennreal } (\text{indicator } \{0 < ..\} \ u * u \ \text{powr } (b - 1) / (1 + u) \ \text{powr } (a + b)) \ \partial \text{lborel})$

have $\text{Gamma } (a + b) > 0 \ \text{Beta } a \ b > 0$

using *assms* **by** (*simp-all add: add-pos-pos Beta-def*)

from $a \ b$ **have** $\text{ennreal } (\text{Gamma } a * \text{Gamma } b) =$

$(\int^{+} t. \ \text{ennreal } (\text{indicator } \{0 ..\} \ t * t \ \text{powr } (a - 1) / \exp t) \ \partial \text{lborel}) *$

$(\int^{+} t. \ \text{ennreal } (\text{indicator } \{0 ..\} \ t * t \ \text{powr } (b - 1) / \exp t) \ \partial \text{lborel})$

by (*subst ennreal-mult'*) (*simp-all add: Gamma-conv-nn-integral-real*)

also have $\dots = (\int^{+} t. \ \int^{+} u. \ \text{ennreal } (\text{indicator } \{0 ..\} \ t * t \ \text{powr } (a - 1) / \exp t) *$

$\text{ennreal } (\text{indicator } \{0 ..\} \ u * u \ \text{powr } (b - 1) / \exp u) \ \partial \text{lborel}$

$\partial \text{lborel})$

by (*simp add: nn-integral-cmult nn-integral-multc*)

also have ... = $(\int^{+t}. \text{indicator } \{0<..\} t * (\int^{+u}. \text{indicator } \{0..\} u * t \text{ powr } (a - 1) * u \text{ powr } (b - 1)) / \exp (t + u) \partial \text{lborel}) \partial \text{lborel}$
by (*intro nn-integral-cong-AE AE-I[of - - {0}]*)
(auto simp: indicator-def divide-ennreal ennreal-mult' [symmetric] exp-add mult-ac)
also have ... = $(\int^{+t}. \text{indicator } \{0<..\} t * (\int^{+u}. \text{indicator } \{0..\} u * t \text{ powr } (a - 1) * u \text{ powr } (b - 1)) / \exp (t + u) \partial(\text{density } (\text{distr lborel borel } ((* t)) (\lambda x. \text{ennreal } |t|))) \partial \text{lborel})$
by (*intro nn-integral-cong mult-indicator-cong, subst lborel-distr-mult' [symmetric]*)
auto
also have ... = $(\int^{+(t::\text{real})}. \text{indicator } \{0<..\} t * (\int^{+u}. \text{indicator } \{0..\} (u * t) * t \text{ powr } a * (u * t) \text{ powr } (b - 1) / \exp (t + t * u) \partial \text{lborel}) \partial \text{lborel})$
by (*intro nn-integral-cong mult-indicator-cong*)
(auto simp: nn-integral-density nn-integral-distr algebra-simps powr-diff simp flip: ennreal-mult)
also have ... = $(\int^{+(t::\text{real})}. \int^{+u}. \text{indicator } (\{0<..\} \times \{0..\}) (t, u) * t \text{ powr } a * (u * t) \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial \text{lborel}) \partial \text{lborel}$
by (*subst nn-integral-cmult [symmetric], simp, intro nn-integral-cong*)
(auto simp: indicator-def zero-le-mult-iff algebra-simps)
also have ... = $(\int^{+(t::\text{real})}. \int^{+u}. \text{indicator } (\{0<..\} \times \{0..\}) (t, u) * t \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial \text{lborel}) \partial \text{lborel}$
by (*intro nn-integral-cong*) *(auto simp: powr-add powr-diff indicator-def powr-mult field-simps)*
also have ... = $(\int^{+(u::\text{real})}. \int^{+t}. \text{indicator } (\{0<..\} \times \{0..\}) (t, u) * t \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial \text{lborel}) \partial \text{lborel}$
by (*rule lborel-pair.Fubini'*) *auto*
also have ... = $(\int^{+(u::\text{real})}. \text{indicator } \{0..\} u * (\int^{+t}. \text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial \text{lborel}) \partial \text{lborel}$
by (*intro nn-integral-cong mult-indicator-cong*) *(auto simp: indicator-def)*
also have ... = $(\int^{+(u::\text{real})}. \text{indicator } \{0<..\} u * (\int^{+t}. \text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial \text{lborel}) \partial \text{lborel}$
by (*intro nn-integral-cong-AE AE-I[of - - {0}]*) *(auto simp: indicator-def)*
also have ... = $(\int^{+(u::\text{real})}. \text{indicator } \{0<..\} u * (\int^{+t}. \text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp (t * (u + 1)) \partial(\text{density } (\text{distr lborel borel } ((* (1/(1+u)))) (\lambda x. \text{ennreal } |1/(1+u)|))) \partial \text{lborel}) \partial \text{lborel}$
by (*intro nn-integral-cong mult-indicator-cong, subst lborel-distr-mult' [symmetric]*)
auto
also have ... = $(\int^{+(u::\text{real})}. \text{indicator } \{0<..\} u * (\int^{+t}. \text{ennreal } (1 / (u + 1)) * \text{ennreal } (\text{indicator } \{0<..\} (t / (u + 1))) * (t / (1+u)) \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp t) \partial \text{lborel}) \partial \text{lborel}$

∂lborel) ∂lborel)
by (*intro nn-integral-cong mult-indicator-cong*)
(auto simp: nn-integral-distr nn-integral-density add-ac)
also have $\dots = (\int^{+u}. \int^{+t}. \text{indicator } (\{0<..\} \times \{0<..\}) (u, t) * 1/(u+1) * (t / (u+1)) \text{ powr } (a + b - 1) * u \text{ powr } (b - 1) / \exp t \partial\text{lborel } \partial\text{lborel})$
by (*subst nn-integral-cmult [symmetric], simp, intro nn-integral-cong*)
(auto simp: indicator-def field-simps divide-ennreal simp flip: ennreal-mult ennreal-mult')
also have $\dots = (\int^{+u}. \int^{+t}. \text{ennreal } (\text{indicator } \{0<..\} u * u \text{ powr } (b - 1) / (1 + u) \text{ powr } (a + b))) * \text{ennreal } (\text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) / \exp t) \partial\text{lborel } \partial\text{lborel})$
by (*intro nn-integral-cong*)
(auto simp: indicator-def powr-add powr-diff powr-divide powr-minus divide-simps add-ac simp flip: ennreal-mult)
also have $\dots = I * (\int^{+t}. \text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) / \exp t \partial\text{lborel})$
by (*simp add: nn-integral-cmult nn-integral-multc I-def*)
also have $(\int^{+t}. \text{indicator } \{0<..\} t * t \text{ powr } (a + b - 1) / \exp t \partial\text{lborel}) = \text{ennreal } (\text{Gamma } (a + b))$
using *assms*
by (*subst Gamma-conv-nn-integral-real*)
(auto intro!: nn-integral-cong-AE[OF AE-I[of - - {0}]] simp: indicator-def split: if-splits split-of-bool-asm)
finally have $\text{ennreal } (\text{Gamma } a * \text{Gamma } b) = I * \text{ennreal } (\text{Gamma } (a + b)) .$
hence $\text{ennreal } (\text{Gamma } a * \text{Gamma } b) / \text{ennreal } (\text{Gamma } (a + b)) = I * \text{ennreal } (\text{Gamma } (a + b)) / \text{ennreal } (\text{Gamma } (a + b))$ **by** *simp*
also have $\dots = I$
using $\langle \text{Gamma } (a + b) > 0 \rangle$ **by** (*intro ennreal-mult-divide-eq auto*)
also have $\text{ennreal } (\text{Gamma } a * \text{Gamma } b) / \text{ennreal } (\text{Gamma } (a + b)) = \text{ennreal } (\text{Gamma } a * \text{Gamma } b / \text{Gamma } (a + b))$
using *assms* **by** (*intro divide-ennreal auto*)
also have $\dots = \text{ennreal } (\text{Beta } a b)$
by (*simp add: Beta-def*)
finally show $*$: $\text{ennreal } (\text{Beta } a b) = I .$

define f **where** $f = (\lambda u. u \text{ powr } (b - 1) / (1 + u) \text{ powr } (a + b))$
have $(\lambda u. \text{indicator } \{0<..\} u * f u) \text{ has-integral } \text{Beta } a b$ *UNIV*
using $\langle \text{Beta } a b > 0 \rangle$
by (*subst has-integral-iff-nn-integral-lebesgue*)
(auto simp: f-def measurable-completion nn-integral-completion I-def mult-ac)
also have $(\lambda u. \text{indicator } \{0<..\} u * f u) = (\lambda u. \text{if } u \in \{0<..\} \text{ then } f u \text{ else } 0)$
by (*auto simp: fun-eq-iff*)
also have $(\dots \text{ has-integral } \text{Beta } a b) \text{ UNIV} \iff (f \text{ has-integral } \text{Beta } a b) \{0<..\}$
by (*rule has-integral-restrict-UNIV*)
finally show \dots **by** (*simp add: f-def*)
qed

lemma *has-integral-Beta2*:
fixes $a :: \text{real}$
assumes $a < -1/2$
shows $((\lambda x. (1 + x^2)^{\text{powr } a}) \text{ has-integral } \text{Beta } (-a - 1/2) (1/2) / 2) \{0 < ..\}$
proof –
define f **where** $f = (\lambda u. u^{\text{powr } (-1/2)} / (1 + u)^{\text{powr } (-a)})$
define C **where** $C = \text{Beta } (-a - 1/2) (1/2)$
have $I: (f \text{ has-integral } C) \{0 < ..\}$
using *has-integral-Beta-real* [of $-a - 1/2$ $1/2$] *assms*
by (*simp-all add: diff-divide-distrib f-def C-def*)

define g **where** $g = (\lambda x. x^2 :: \text{real})$
have $\text{bij}: \text{bij-betw } g \{0 < ..\} \{0 < ..\}$
by (*intro bij-betwI* [of $- - - \text{sqrt}$] (*auto simp: g-def*))

have $(f \text{ absolutely-integrable-on } g \{0 < ..\} \wedge \text{integral } (g \{0 < ..\}) f = C)$
using I bij **by** (*simp add: bij-betw-def has-integral-iff absolutely-integrable-on-def f-def*)
also have $?this \longleftrightarrow ((\lambda x. |2 * x| * f (g x)) \text{ absolutely-integrable-on } \{0 < ..\} \wedge \text{integral } \{0 < ..\} (\lambda x. |2 * x| * f (g x)) = C)$
using bij **by** (*intro has-absolute-integral-change-of-variables-1'* [*symmetric*])
(*auto intro!: derivative-eq-intros simp: g-def bij-betw-def*)
finally have $((\lambda x. |2 * x| * f (g x)) \text{ has-integral } C) \{0 < ..\}$
by (*simp add: absolutely-integrable-on-def f-def has-integral-iff*)
also have $?this \longleftrightarrow ((\lambda x :: \text{real}. 2 * (1 + x^2)^{\text{powr } a}) \text{ has-integral } C) \{0 < ..\}$
by (*intro has-integral-cong*) (*auto simp: f-def g-def powr-def exp-minus ln-realpow field-simps*)
finally have $((\lambda x :: \text{real}. 1/2 * (2 * (1 + x^2)^{\text{powr } a})) \text{ has-integral } 1/2 * C) \{0 < ..\}$
by (*intro has-integral-mult-right*)
thus $?thesis$ **by** (*simp add: C-def*)
qed

lemma *has-integral-Beta3*:
fixes $a b :: \text{real}$
assumes $a < -1/2$ **and** $b > 0$
shows $((\lambda x. (b + x^2)^{\text{powr } a}) \text{ has-integral } \text{Beta } (-a - 1/2) (1/2) / 2 * b^{\text{powr } (a + 1/2)}) \{0 < ..\}$
proof –
define C **where** $C = \text{Beta } (-a - 1/2) (1/2) / 2$
have $\text{int}: \text{nn-integral lborel } (\lambda x. \text{indicator } \{0 < ..\} x * (1 + x^2)^{\text{powr } a}) = C$
using *nn-integral-has-integral-lebesgue* [*OF - has-integral-Beta2* [*OF assms(1)*]]
by (*auto simp: C-def*)
have $\text{nn-integral lborel } (\lambda x. \text{indicator } \{0 < ..\} x * (b + x^2)^{\text{powr } a}) =$
 $(\int^+ x. \text{ennreal } (\text{indicat-real } \{0 < ..\} (x * \text{sqrt } b)) * (b + (x * \text{sqrt } b)^2)^{\text{powr } a}$
 $* \text{sqrt } b) \partial \text{lborel})$
using *assms*

by (*subst lborel-distr-mult'[of sqrt b]*)
(auto simp: nn-integral-density nn-integral-distr mult-ac simp flip: ennreal-mult)
also have $\dots = (\int^{+x}. \text{ennreal } (\text{indicat-real } \{0<..\} x * (b * (1 + x^2))) \text{ powr } a * \text{sqrt } b) \partial \text{lborel}$
using *assms*
by (*intro nn-integral-cong*) (*auto simp: indicator-def field-simps zero-less-mult-iff*)
also have $\dots = (\int^{+x}. \text{ennreal } (\text{indicat-real } \{0<..\} x * b \text{ powr } (a + 1/2) * (1 + x^2) \text{ powr } a) \partial \text{lborel})$
using *assms*
by (*intro nn-integral-cong*) (*auto simp: indicator-def powr-add powr-half-sqrt powr-mult*)
also have $\dots = b \text{ powr } (a + 1/2) * (\int^{+x}. \text{ennreal } (\text{indicat-real } \{0<..\} x * (1 + x^2) \text{ powr } a) \partial \text{lborel})$
using *assms* **by** (*subst nn-integral-cmult [symmetric]*) (*simp-all add: mult-ac flip: ennreal-mult*)
also have $(\int^{+x}. \text{ennreal } (\text{indicat-real } \{0<..\} x * (1 + x^2) \text{ powr } a) \partial \text{lborel}) = C$
using *int by simp*
also have $\text{ennreal } (b \text{ powr } (a + 1/2)) * \text{ennreal } C = \text{ennreal } (C * b \text{ powr } (a + 1/2))$
using *assms* **by** (*subst ennreal-mult*) (*auto simp: C-def mult-ac Beta-def*)
finally have $*$: $(\int^{+x}. \text{ennreal } (\text{indicat-real } \{0<..\} x * (b + x^2) \text{ powr } a) \partial \text{lborel}) = \dots$
hence $((\lambda x. \text{indicator } \{0<..\} x * (b + x^2) \text{ powr } a) \text{ has-integral } C * b \text{ powr } (a + 1/2)) \text{ UNIV}$
using *assms*
by (*subst has-integral-iff-nn-integral-lebesgue*)
(auto simp: C-def measurable-completion nn-integral-completion Beta-def)
also have $(\lambda x. \text{indicator } \{0<..\} x * (b + x^2) \text{ powr } a) =$
 $(\lambda x. \text{if } x \in \{0<..\} \text{ then } (b + x^2) \text{ powr } a \text{ else } 0)$
by (*auto simp: fun-eq-iff*)
finally show *?thesis*
by (*subst (asm) has-integral-restrict-UNIV*) (*auto simp: C-def*)
qed

2.4 Asymptotics of the real $\log \Gamma$ function and its derivatives

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order $O(s^{-n})$.

definition *stirling-integral* :: $\text{nat} \Rightarrow 'a :: \{\text{real-normed-div-algebra, banach}\} \Rightarrow 'a$
where

$$\text{stirling-integral } n \ s = \lim (\lambda N. \text{integral } \{0..N\} (\lambda x. \text{of-real } (\text{pbernpoly } n \ x) / (\text{of-real } x + s) ^ n))$$

context

fixes $s :: \text{complex}$ **assumes** $s: s \notin \mathbb{R}_{\leq 0}$

fixes *approx* :: $\text{nat} \Rightarrow \text{complex}$

defines *approx* $\equiv (\lambda N.$

$$(\sum n = 1..<N. s / \text{of-nat } n - \ln (1 + s / \text{of-nat } n)) - (\text{euler-mascheroni} * s$$

$+ \ln s) - \dots \longrightarrow \ln\text{-Gamma } s$
 $(\ln\text{-Gamma } (\text{of-nat } N) - \ln (2 * \pi / \text{of-nat } N) / 2 - \text{of-nat } N * \ln (\text{of-nat } N) + \text{of-nat } N) - \dots \longrightarrow 0$
 $s * (\text{harm } (N - 1) - \ln (\text{of-nat } (N - 1))) - \text{euler-mascheroni} + \dots \longrightarrow 0$
 $s * (\ln (\text{of-nat } N + s) - \ln (\text{of-nat } (N - 1))) - \dots \longrightarrow 0$
 $(1/2) * (\ln (\text{of-nat } N + s) - \ln (\text{of-nat } N)) + \dots \longrightarrow 0$
 $\text{of-nat } N * (\ln (\text{of-nat } N + s) - \ln (\text{of-nat } N)) - \dots \longrightarrow s$
 $(s - 1/2) * \ln s - \ln (2 * \pi) / 2$

begin

qualified lemma

assumes $N: N > 0$

shows *integrable-pbernpoly-1*:

$(\lambda x. \text{of-real } (-\text{pbernpoly } 1 x) / (\text{of-real } x + s)) \text{ integrable-on } \{0..real N\}$

and *integral-pbernpoly-1-aux*:

$\text{integral } \{0..real N\} (\lambda x. -\text{of-real } (\text{pbernpoly } 1 x) / (\text{of-real } x + s)) =$

approx N

and *has-integral-pbernpoly-1*:

$((\lambda x. \text{pbernpoly } 1 x / (x + s)) \text{ has-integral } (\sum m < N. (\text{of-nat } m + 1 / 2 + s) * (\ln (\text{of-nat } m + s) - \ln (\text{of-nat } m + 1 + s)) + 1)) \{0..real N\}$

proof –

let $?A = (\lambda n. \{ \text{of-nat } n.. \text{of-nat } (n+1) \}) ' \{0..<N\}$

have *has-integral*:

$((\lambda x. -\text{pbernpoly } 1 x / (x + s)) \text{ has-integral } (\text{of-nat } n + 1/2 + s) * (\ln (\text{of-nat } (n + 1) + s) - \ln (\text{of-nat } n + s))$

– 1)

$\{ \text{of-nat } n.. \text{of-nat } (n + 1) \}$ **for** n

proof (*rule has-integral-spike*)

have $((\lambda x. (\text{of-nat } n + 1/2 + s) * (1 / (\text{of-real } x + s)) - 1) \text{ has-integral } (\text{of-nat } n + 1/2 + s) * (\ln (\text{of-real } (\text{real } (n + 1)) + s) - \ln (\text{of-real } (\text{real } n) + s)) - 1)$

$\{ \text{of-nat } n.. \text{of-nat } (n + 1) \}$

using $s \text{ has-integral-const-real}[\text{of } 1 \text{ of-nat } n \text{ of-nat } (n + 1)]$

by (*intro has-integral-diff has-integral-mult-right fundamental-theorem-of-calculus*)

(*auto intro! derivative-eq-intros has-vector-derivative-real-field*

simp: has-real-derivative-iff-has-vector-derivative [symmetric] field-simps complex-nonpos-Reals-iff)

thus $((\lambda x. (\text{of-nat } n + 1/2 + s) * (1 / (\text{of-real } x + s)) - 1) \text{ has-integral } (\text{of-nat } n + 1/2 + s) * (\ln (\text{of-nat } (n + 1) + s) - \ln (\text{of-nat } n + s))$

– 1)

$\{ \text{of-nat } n.. \text{of-nat } (n + 1) \}$ **by** *simp*

show $-\text{pbernpoly } 1 x / (x + s) = (\text{of-nat } n + 1/2 + s) * (1 / (x + s)) - 1$

if $x \in \{ \text{of-nat } n.. \text{of-nat } (n + 1) \} - \{ \text{of-nat } (n + 1) \}$ **for** x

proof –

have $x: x \geq \text{real } n \ x < \text{real } (n + 1)$ **using** *that by simp-all*

hence $\text{floor } x = \text{int } n$ **by** *linarith*

moreover from s **have** *complex-of-real* $x \neq -s$

by (*auto simp add: complex-eq-iff complex-nonpos-Reals-iff simp del: of-nat-Suc*)
ultimately show $-pbernpoly\ 1\ x / (x + s) = (of\ nat\ n + 1/2 + s) * (1 / (x + s)) - 1$
by (*auto simp: pbernpoly-def bernpoly-def frac-def divide-simps add-eq-0-iff2*)
qed
qed simp-all
hence *: $\bigwedge I. I \in ?A \implies ((\lambda x. -pbernpoly\ 1\ x / (x + s))\ has_integral\ (Inf\ I + 1/2 + s) * (\ln\ (Inf\ I + 1 + s) - \ln\ (Inf\ I + s)) - 1)\ I$
by (*auto simp: add-ac*)
have $((\lambda x. -pbernpoly\ 1\ x / (x + s))\ has_integral\ (\sum I \in ?A. (Inf\ I + 1 / 2 + s) * (\ln\ (Inf\ I + 1 + s) - \ln\ (Inf\ I + s)) - 1))$
 $(\bigcup n \in \{0..<N\}. \{real\ n..real\ (n + 1)\})\ (is\ (-\ has_integral\ ?i)\ -)$
apply (*intro has-integral-Union * finite-imageI*)
apply (*force intro!: negligible-atLeastAtMostI pairwiseI*)
done
hence *has-integral*: $((\lambda x. -pbernpoly\ 1\ x / (x + s))\ has_integral\ ?i)\ \{0..real\ N\}$
by (*subst has-integral-spike-set-eq*)
(use Union-atLeastAtMost assms in <auto simp: intro!: empty-imp-negligible>)
hence $(\lambda x. -pbernpoly\ 1\ x / (x + s))\ integrable_on\ \{0..real\ N\}$
and *integral*: $integral\ \{0..real\ N\}\ (\lambda x. -pbernpoly\ 1\ x / (x + s)) = ?i$
by (*simp-all add: has-integral-iff*)
show $(\lambda x. -pbernpoly\ 1\ x / (x + s))\ integrable_on\ \{0..real\ N\}$ **by fact**

note *has-integral-neg[OF has-integral]*
also have $-?i = (\sum x < N. (of\ nat\ x + 1 / 2 + s) * (\ln\ (of\ nat\ x + s) - \ln\ (of\ nat\ x + 1 + s)) + 1)$
by (*subst sum.reindex*)
(simp-all add: inj-on-def atLeast0LessThan algebra-simps sum-negf [symmetric])
finally show *has-integral*:
 $((\lambda x. of\ real\ (pbernpoly\ 1\ x) / (of\ real\ x + s))\ has_integral\ \dots)\ \{0..real\ N\}$ **by**
simp

note *integral*
also have $?i = (\sum n < N. (of\ nat\ n + 1 / 2 + s) * (\ln\ (of\ nat\ n + 1 + s) - \ln\ (of\ nat\ n + s))) - N\ (is\ - = ?S - -)$
by (*subst sum.reindex*) (*simp-all add: inj-on-def sum-subtractf atLeast0LessThan*)
also have $?S = (\sum n < N. of\ nat\ n * (\ln\ (of\ nat\ n + 1 + s) - \ln\ (of\ nat\ n + s))) + (s + 1 / 2) * (\sum n < N. \ln\ (of\ nat\ (Suc\ n) + s) - \ln\ (of\ nat\ n + s))$
 $(is\ - = ?S1 + - * ?S2)$ **by** (*simp add: algebra-simps sum.distrib sum-subtractf sum-distrib-left*)
also have $?S2 = \ln\ (of\ nat\ N + s) - \ln\ s$ **by** (*subst sum-lessThan-telescope*)
simp
also have $?S1 = (\sum n = 1..<N. of\ nat\ n * (\ln\ (of\ nat\ n + 1 + s) - \ln\ (of\ nat\ n + s)))$
by (*intro sum.mono-neutral-right*) *auto*
also have $\dots = (\sum n = 1..<N. of\ nat\ n * \ln\ (of\ nat\ n + 1 + s)) - (\sum n = 1..<N.$

$of\text{-}nat\ n * ln\ (of\text{-}nat\ n + s)$
by (*simp add: algebra-simps sum-subtractf*)
also have $(\sum_{n=1..<N}. of\text{-}nat\ n * ln\ (of\text{-}nat\ n + 1 + s)) =$
 $(\sum_{n=1..<N}. (of\text{-}nat\ n - 1) * ln\ (of\text{-}nat\ n + s)) + (N - 1) * ln$
 $(of\text{-}nat\ N + s)$
by (*induction N (simp-all add: add-ac of-nat-diff)*)
also have $\dots - (\sum_{n=1..<N}. of\text{-}nat\ n * ln\ (of\text{-}nat\ n + s)) =$
 $-(\sum_{n=1..<N}. ln\ (of\text{-}nat\ n + s)) + (N - 1) * ln\ (of\text{-}nat\ N + s)$
by (*induction N (simp-all add: algebra-simps)*)
also from s **have** *neg: s + of-nat x ≠ 0 for x*
by (*auto simp: complex-nonpos-Reals-iff complex-eq-iff*)
hence $(\sum_{n=1..<N}. ln\ (of\text{-}nat\ n + s)) = (\sum_{n=1..<N}. ln\ (of\text{-}nat\ n) + ln\ (1$
 $+ s/n))$
by (*intro sum.cong refl, subst Ln-times-of-nat [symmetric] (auto simp: di-*
vide-simps add-ac))
also have $\dots = ln\ (fact\ (N - 1)) + (\sum_{n=1..<N}. ln\ (1 + s/n))$
by (*induction N (simp-all add: Ln-times-of-nat fact-reduce add-ac)*)
also have $(\sum_{n=1..<N}. ln\ (1 + s/n)) = -(\sum_{n=1..<N}. s / n - ln\ (1 + s/n))$
 $+ s * (\sum_{n=1..<N}. 1 / of\text{-}nat\ n)$
by (*simp add: sum-distrib-left sum-subtractf*)
also from N **have** $ln\ (fact\ (N - 1)) = ln\text{-Gamma}\ (of\text{-}nat\ N :: complex)$
by (*simp add: ln-Gamma-complex-conv-fact*)
also have $\{1..<N\} = \{1..N - 1\}$ **by** *auto*
hence $(\sum_{n=1..<N}. 1 / of\text{-}nat\ n) = (harm\ (N - 1) :: complex)$
by (*simp add: harm-def divide-simps*)
also have $-(ln\text{-Gamma}\ (of\text{-}nat\ N) + (- (\sum_{n=1..<N}. s / of\text{-}nat\ n - ln\ (1$
 $+ s / of\text{-}nat\ n)) +$
 $s * harm\ (N - 1))) + of\text{-}nat\ (N - 1) * ln\ (of\text{-}nat\ N + s) +$
 $(s + 1 / 2) * (ln\ (of\text{-}nat\ N + s) - ln\ s) - of\text{-}nat\ N = approx\ N$
using N **by** (*simp add: field-simps of-nat-diff ln-div approx-def Ln-of-nat*
ln-Gamma-complex-of-real [symmetric])
finally show $integral\ \{0..of\text{-}nat\ N\}\ (\lambda x. -of\text{-}real\ (pbernpoly\ 1\ x) / (of\text{-}real\ x +$
 $s)) = \dots$
by *simp*
qed

lemma *integrable-ln-Gamma-aux:*

shows $(\lambda x. of\text{-}real\ (pbernpoly\ n\ x) / (of\text{-}real\ x + s) ^ n)$ *integrable-on* $\{0..real\ N\}$

proof (*cases n = 1*)

case *True*

with s **show** *?thesis using integrable-neg[OF integrable-pbernpoly-1[of N]]*

by (*cases N = 0*) (*simp-all add: integrable-negligible*)

next

case *False*

from s **have** *of-real x + s ≠ 0 if x ≥ 0 for x using that*

by (*auto simp: complex-eq-iff add-eq-0-iff2 complex-nonpos-Reals-iff*)

with *False s show ?thesis*

by (*auto intro!: integrable-continuous-real continuous-intros*)

qed

This following proof is based on “Rudiments of the theory of the gamma function” by Bruce Berndt [1].

lemma *tendsto-of-real-0-I*:

$(f \longrightarrow 0) \ G \implies ((\lambda x. (of\text{-}real\ (f\ x))) \longrightarrow (0 :: 'a::real\text{-}normed\text{-}div\text{-}algebra))$
 G
using *tendsto-of-real-iff* **by** *force*

qualified lemma *integral-pbernpoly-1*:

$(\lambda N. \ integral\ \{0..real\ N\}\ (\lambda x. \ pbernpoly\ 1\ x\ /\ (x + s)))$
 $\longrightarrow -ln\text{-}Gamma\ s - s + (s - 1 / 2) * ln\ s + ln\ (2 * pi) / 2$

proof –

have *neq: s + of-real x ≠ 0 if x ≥ 0 for x :: real*

using *that s by (auto simp: complex-eq-iff complex-nonpos-Reals-iff)*

have $(approx \longrightarrow ln\text{-}Gamma\ s - 0 - 0 + 0 - 0 + s - (s - 1/2) * ln\ s - ln\ (2 * pi) / 2)$ *at-top*

unfolding *approx-def*

proof *(intro tendsto-add tendsto-diff)*

from *s have s': s ∉ ℤ_{≤0} by (auto simp: complex-nonpos-Reals-iff elim!: non-pos-Ints-cases)*

have $(\lambda n. \ \sum\ i=1..<n. \ s / of\text{-}nat\ i - ln\ (1 + s / of\text{-}nat\ i)) \longrightarrow$
 $ln\text{-}Gamma\ s + euler\text{-}mascheroni * s + ln\ s$ **(is ?f → -)**

using *ln-Gamma-series'-aux[OF s']* **unfolding** *sums-def*

by *(subst filterlim-sequentially-Suc [symmetric], subst (asm) sum.atLeast1-atMost-eq [symmetric])*

(simp add: atLeastLessThanSuc-atLeastAtMost)

thus $(\lambda n. \ ?f\ n - (euler\text{-}mascheroni * s + ln\ s)) \longrightarrow ln\text{-}Gamma\ s$ *at-top*

by *(auto intro: tendsto-eq-intros)*

next

show $(\lambda x. \ complex\text{-}of\text{-}real\ (ln\text{-}Gamma\ (real\ x) - ln\ (2 * pi / real\ x) / 2 - real\ x * ln\ (real\ x) + real\ x)) \longrightarrow 0$

proof *(intro tendsto-of-real-0-I*

filterlim-compose[OF tendsto-sandwich filterlim-real-sequentially])

show *eventually* $(\lambda x::real. \ ln\text{-}Gamma\ x - ln\ (2 * pi / x) / 2 - x * ln\ x + x \geq 0)$ *at-top*

using *eventually-ge-at-top[of 1::real]*

by *eventually-elim (insert ln-Gamma-bounds(1), simp add: algebra-simps)*

show *eventually* $(\lambda x::real. \ ln\text{-}Gamma\ x - ln\ (2 * pi / x) / 2 - x * ln\ x + x$

\leq

$1 / 12 * inverse\ x)$ *at-top*

using *eventually-ge-at-top[of 1::real]*

by *eventually-elim (insert ln-Gamma-bounds(2), simp add: field-simps)*

show $(\lambda x::real. \ 1 / 12 * inverse\ x) \longrightarrow 0)$ *at-top*

by *real-asymp*

qed *simp-all*

next

have $(\lambda x. \ s * of\text{-}real\ (harm\ (x - 1) - ln\ (real\ (x - 1)) - euler\text{-}mascheroni))$

\longrightarrow

$s * \text{of-real } (\text{euler-mascheroni} - \text{euler-mascheroni})$
by (*subst filterlim-sequentially-Suc [symmetric], intro tendsto-intros*)
(insert euler-mascheroni-LIMSEQ, simp-all)
also have $?this \longleftrightarrow (\lambda x. s * (\text{harm } (x - 1) - \ln (\text{of-nat } (x - 1))) - \text{euler-mascheroni}) \longrightarrow 0$
by (*intro filterlim-cong refl eventually-mono[OF eventually-gt-at-top[of 1::nat]]*)
(auto simp: of-real-harm simp del: of-nat-diff)
finally show $(\lambda x. s * (\text{harm } (x - 1) - \ln (\text{of-nat } (x - 1))) - \text{euler-mascheroni}) \longrightarrow 0$.
next
have $((\lambda x. \ln (1 + (s + 1) / \text{of-real } x)) \longrightarrow \ln (1 + 0)) \text{ at-top (is ?P)}$
by (*intro tendsto-intros tendsto-divide-0[OF tendsto-const]*)
(simp-all add: filterlim-ident filterlim-at-infinity-conv-norm-at-top filterlim-abs-real)
also have $\ln (\text{of-real } (x + 1) + s) - \ln (\text{complex-of-real } x) = \ln (1 + (s + 1) / \text{of-real } x)$
if $x > 1$ **for** x **using** *that s*
using *Ln-divide-of-real[of x of-real (x + 1) + s, symmetric] neq[of x+1]*
by (*simp add: field-simps Ln-of-real*)
hence $?P \longleftrightarrow ((\lambda x. \ln (\text{of-real } (x + 1) + s) - \ln (\text{of-real } x)) \longrightarrow 0) \text{ at-top}$
by (*intro filterlim-cong refl*)
(auto intro: eventually-mono[OF eventually-gt-at-top[of 1::real]])
finally have $((\lambda n. \ln (\text{of-real } (\text{real } n + 1) + s) - \ln (\text{of-real } (\text{real } n))) \longrightarrow 0) \text{ at-top}$
by (*rule filterlim-compose[OF - filterlim-real-sequentially]*)
hence $((\lambda n. \ln (\text{of-nat } n + s) - \ln (\text{of-nat } (n - 1))) \longrightarrow 0) \text{ at-top}$
by (*subst filterlim-sequentially-Suc [symmetric] (simp add: add-ac)*)
thus $(\lambda x. s * (\ln (\text{of-nat } x + s) - \ln (\text{of-nat } (x - 1)))) \longrightarrow 0$
by (*rule tendsto-mult-right-zero*)
next
have $((\lambda x. \ln (1 + s / \text{of-real } x)) \longrightarrow \ln (1 + 0)) \text{ at-top (is ?P)}$
by (*intro tendsto-intros tendsto-divide-0[OF tendsto-const]*)
(simp-all add: filterlim-ident filterlim-at-infinity-conv-norm-at-top filterlim-abs-real)
also have $\ln (\text{of-real } x + s) - \ln (\text{of-real } x) = \ln (1 + s / \text{of-real } x)$ **if** $x > 0$
for x
using *Ln-divide-of-real[of x of-real x + s] neq[of x] that*
by (*auto simp: field-simps Ln-of-real*)
hence $?P \longleftrightarrow ((\lambda x. \ln (\text{of-real } x + s) - \ln (\text{of-real } x)) \longrightarrow 0) \text{ at-top}$
using s **by** (*intro filterlim-cong refl*)
(auto intro: eventually-mono [OF eventually-gt-at-top[of 1::real]])
finally have $(\lambda x. (1/2) * (\ln (\text{of-real } (\text{real } x) + s) - \ln (\text{of-real } (\text{real } x)))) \longrightarrow 0$
by (*rule tendsto-mult-right-zero[OF filterlim-compose[OF - filterlim-real-sequentially]]*)
thus $(\lambda x. (1/2) * (\ln (\text{of-nat } x + s) - \ln (\text{of-nat } x))) \longrightarrow 0$ **by** *simp*
next
have $((\lambda x. x * (\ln (1 + s / \text{of-real } x))) \longrightarrow s) \text{ at-top (is ?P)}$
by (*rule stirling-limit-aux2*)
also have $\ln (1 + s / \text{of-real } x) = \ln (\text{of-real } x + s) - \ln (\text{of-real } x)$ **if** $x > 1$

for x
using *that* s *Ln-divide-of-real* [*of* x *of-real* $x + s$, *symmetric*] *neq*[*of* x]
by (*auto simp: Ln-of-real field-simps*)
hence $?P \iff ((\lambda x. \text{of-real } x * (\ln (\text{of-real } x + s) - \ln (\text{of-real } x))) \longrightarrow s)$
at-top
by (*intro filterlim-cong refl*)
*(auto intro: eventually-mono[OF eventually-gt-at-top[*of* 1::real]])*
finally have $(\lambda n. \text{of-real } (\text{real } n) * (\ln (\text{of-real } (\text{real } n) + s) - \ln (\text{of-real } (\text{real } n)))) \longrightarrow s$
by (*rule filterlim-compose[OF - filterlim-real-sequentially]*)
thus $(\lambda n. \text{of-nat } n * (\ln (\text{of-nat } n + s) - \ln (\text{of-nat } n))) \longrightarrow s$ **by** *simp*
qed *simp-all*
also have $?this \iff ((\lambda N. \text{integral } \{0.. \text{real } N\} (\lambda x. -\text{pbernpoly } 1 \ x / (x + s))) \longrightarrow$
 $\text{ln-Gamma } s + s - (s - 1/2) * \ln s - \ln (2 * \text{pi}) / 2)$ *at-top*
using *integral-pbernpoly-1-aux*
by (*intro filterlim-cong refl*)
*(auto intro: eventually-mono[OF eventually-gt-at-top[*of* 0::nat]])*
also have $(\lambda N. \text{integral } \{0.. \text{real } N\} (\lambda x. -\text{pbernpoly } 1 \ x / (x + s))) =$
 $(\lambda N. -\text{integral } \{0.. \text{real } N\} (\lambda x. \text{pbernpoly } 1 \ x / (x + s)))$
by (*simp add: fun-eq-iff*)
finally show $?thesis$ **by** (*simp add: tendsto-minus-cancel-left [symmetric] algebra-simps*)
qed

qualified lemma *pbernpoly-integral-conv-pbernpoly-integral-Suc:*

assumes $n \geq 1$
shows $\text{integral } \{0.. \text{real } N\} (\lambda x. \text{pbernpoly } n \ x / (x + s) ^ n) =$
 $\text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{real } N)) / (\text{of-nat } (\text{Suc } n) * (s + \text{of-nat } N) ^ n) -$
 $\text{of-real } (\text{bernoulli } (\text{Suc } n)) / (\text{of-nat } (\text{Suc } n) * s ^ n) + \text{of-nat } n / \text{of-nat } (\text{Suc } n) *$
 $\text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } (\text{Suc } n) \ x) / (\text{of-real } x + s) ^ \text{Suc } n)$
proof –
note [*derivative-intros*] = *has-field-derivative-pbernpoly-Suc'*
define I **where** $I = -\text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{of-nat } N)) / (\text{of-nat } (\text{Suc } n) * (\text{of-nat } N + s) ^ n) +$
 $\text{of-real } (\text{bernoulli } (\text{Suc } n) / \text{real } (\text{Suc } n)) / s ^ n +$
 $\text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n \ x) / (\text{of-real } x + s) ^ n)$
have $((\lambda x. (-\text{of-nat } n * \text{inverse } (\text{of-real } x + s) ^ \text{Suc } n) * (\text{of-real } (\text{pbernpoly } (\text{Suc } n) \ x) / (\text{of-nat } (\text{Suc } n))))$
 $\text{has-integral } -I) \{0.. \text{real } N\}$
proof (*rule integration-by-parts-interior-strong[OF bounded-bilinear-mult]*)
fix $x :: \text{real}$ **assume** $x \in \{0 < .. < \text{real } N\} - \text{real ' } \{0..N\}$
have $x \notin \mathbb{Z}$
proof
assume $x \in \mathbb{Z}$

then obtain n **where** $x = \text{of-int } n$ **by** (*auto elim!: Ints-cases*)
with x **have** x' : $x = \text{of-nat } (\text{nat } n)$ **by** *simp*
from x **show** *False* **by** (*auto simp: x'*)
qed
hence $((\lambda x. \text{of-real } (\text{pbernpoly } (\text{Suc } n) x / \text{of-nat } (\text{Suc } n))) \text{ has-vector-derivative } \text{complex-of-real } (\text{pbernpoly } n x)) \text{ (at } x)$
by (*intro has-vector-derivative-of-real*) (*auto intro!: derivative-eq-intros*)
thus $((\lambda x. \text{of-real } (\text{pbernpoly } (\text{Suc } n) x) / \text{of-nat } (\text{Suc } n)) \text{ has-vector-derivative } \text{complex-of-real } (\text{pbernpoly } n x)) \text{ (at } x)$ **by** *simp*
from x s **have** $\text{complex-of-real } x + s \neq 0$
by (*auto simp: complex-eq-iff complex-nonpos-Reals-iff*)
thus $((\lambda x. \text{inverse } (\text{of-real } x + s) ^ n) \text{ has-vector-derivative } - \text{of-nat } n * \text{inverse } (\text{of-real } x + s) ^ (\text{Suc } n)) \text{ (at } x)$ **using** x s *assms*
by (*auto intro!: derivative-eq-intros has-vector-derivative-real-field simp: divide-simps power-add [symmetric]*)
simp del: power-Suc
next
have $\text{complex-of-real } x + s \neq 0$ **if** $x \geq 0$ **for** x
using *that* s **by** (*auto simp: complex-eq-iff complex-nonpos-Reals-iff*)
thus $\text{continuous-on } \{0.. \text{real } N\} (\lambda x. \text{inverse } (\text{of-real } x + s) ^ n)$
 $\text{continuous-on } \{0.. \text{real } N\} (\lambda x. \text{complex-of-real } (\text{pbernpoly } (\text{Suc } n) x) / \text{of-nat } (\text{Suc } n))$
using *assms* s **by** (*auto intro!: continuous-intros simp del: of-nat-Suc*)
next
have $((\lambda x. \text{inverse } (\text{of-real } x + s) ^ n * \text{of-real } (\text{pbernpoly } n x)) \text{ has-integral } \text{pbernpoly } (\text{Suc } n) (\text{of-nat } N) / (\text{of-nat } (\text{Suc } n) * (\text{of-nat } N + s) ^ n) - \text{of-real } (\text{bernoulli } (\text{Suc } n) / \text{real } (\text{Suc } n)) / s ^ n - I) \{0.. \text{real } N\}$
using *integrable-ln-Gamma-aux*[*of n N*] *assms*
by (*auto simp: I-def has-integral-integral divide-simps*)
thus $((\lambda x. \text{inverse } (\text{of-real } x + s) ^ n * \text{of-real } (\text{pbernpoly } n x)) \text{ has-integral } \text{inverse } (\text{of-real } (\text{real } N) + s) ^ n * (\text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{real } N)) / \text{of-nat } (\text{Suc } n)) - \text{inverse } (\text{of-real } 0 + s) ^ n * (\text{of-real } (\text{pbernpoly } (\text{Suc } n) 0) / \text{of-nat } (\text{Suc } n)) - I) \{0.. \text{real } N\}$ **by** (*simp-all add: field-simps*)
qed *simp-all*
also have $(\lambda x. - \text{of-nat } n * \text{inverse } (\text{of-real } x + s) ^ (\text{Suc } n) * (\text{of-real } (\text{pbernpoly } (\text{Suc } n) x) / \text{of-nat } (\text{Suc } n))) =$
 $(\lambda x. - (\text{of-nat } n / \text{of-nat } (\text{Suc } n)) * \text{of-real } (\text{pbernpoly } (\text{Suc } n) x) / (\text{of-real } x + s) ^ (\text{Suc } n))$
by (*simp add: divide-simps fun-eq-iff*)
finally have $((\lambda x. - (\text{of-nat } n / \text{of-nat } (\text{Suc } n)) * \text{of-real } (\text{pbernpoly } (\text{Suc } n) x) / (\text{of-real } x + s) ^ (\text{Suc } n)) \text{ has-integral } - I) \{0.. \text{real } N\}$.
from *has-integral-neg*[*OF this*] **show** *?thesis*
by (*auto simp add: I-def has-integral-iff algebra-simps integral-mult-right [symmetric]*)
simp del: power-Suc of-nat-Suc)

qed

lemma *pbernpoly-over-power-tendsto-0*:

assumes $n > 0$

shows $(\lambda x. \text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{real } x)) / (\text{of-nat } (\text{Suc } n) * (s + \text{of-nat } x) ^ n)) \longrightarrow 0$

proof –

from s **have** $\text{neg: } s + \text{of-nat } n \neq 0$ **for** n

by (*auto simp: complex-eq-iff complex-nonpos-Reals-iff*)

obtain c **where** $c: \bigwedge x. \text{norm } (\text{pbernpoly } (\text{Suc } n) x) \leq c$

using *bounded-pbernpoly* **by** *auto*

have *eventually* $(\lambda x. \text{real } x + \text{Re } s > 0)$ *at-top*

by *real-asymp*

hence *eventually* $(\lambda x. \text{norm } (\text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{real } x)) / (\text{of-nat } (\text{Suc } n) * (s + \text{of-nat } x) ^ n)) \leq (c / \text{real } (\text{Suc } n)) / (\text{real } x + \text{Re } s) ^ n)$ *at-top*

using *eventually-gt-at-top*[*of 0::nat*]

proof *eventually-elim*

case (*elim* x)

have $\text{norm } (\text{of-real } (\text{pbernpoly } (\text{Suc } n) (\text{real } x)) / (\text{of-nat } (\text{Suc } n) * (s + \text{of-nat } x) ^ n)) \leq (c / \text{real } (\text{Suc } n)) / \text{norm } (s + \text{of-nat } x) ^ n$ (**is** \leq *?rhs*) **using** c [*of* x]

by (*auto simp: norm-divide norm-mult norm-power neg field-simps simp del: of-nat-Suc*)

also **have** $\text{real } x + \text{Re } s \leq \text{cmod } (s + \text{of-nat } x)$

using *complex-Re-le-cmod*[*of* $s + \text{of-nat } x$] s **by** (*auto simp add: complex-nonpos-Reals-iff*)

hence $\text{?rhs} \leq (c / \text{real } (\text{Suc } n)) / (\text{real } x + \text{Re } s) ^ n$ **using** s *elim* c [*of* 0] *neg*[*of* x]

by (*intro divide-left-mono power-mono mult-pos-pos divide-nonneg-pos zero-less-power*) *auto*

finally **show** *?case* .

qed

moreover **have** $(\lambda x. (c / \text{real } (\text{Suc } n)) / (\text{real } x + \text{Re } s) ^ n) \longrightarrow 0$

using $\langle n > 0 \rangle$ **by** *real-asymp*

ultimately **show** *?thesis* **by** (*rule Lim-null-comparison*)

qed

lemma *convergent-stirling-integral*:

assumes $n > 0$

shows *convergent* $(\lambda N. \text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^ n))$ (**is** *convergent* (*?f* n))

proof –

have *convergent* (*?f* (*Suc* n)) **for** n

proof (*induction* n)

case 0

thus *?case* **using** *integral-pbernpoly-1* **by** (*auto intro!: convergentI*)

next

case (*Suc* n)

have *convergent* $(\lambda N. ?f (Suc\ n)\ N -$
of-real (pbernpoly (Suc (Suc n)) (real N)) /
*(of-nat (Suc (Suc n)) * (s + of-nat N) ^ Suc n) +*
of-real (bernoulli (Suc (Suc n)) / (real (Suc (Suc n)))) / s ^ Suc n)
(is convergent ?g)
by (*intro convergent-add convergent-diff Suc*
convergent-const convergentI[OF pbernpoly-over-power-tendsto-0]) simp-all
also have $?g = (\lambda N. of-nat (Suc\ n) / of-nat (Suc\ (Suc\ n)) * ?f (Suc\ (Suc\ n)))$
N) using s
by (*subst pbernpoly-integral-conv-pbernpoly-integral-Suc*
(auto simp: fun-eq-iff field-simps simp del: of-nat-Suc power-Suc)
also have *convergent ... \longleftrightarrow convergent (?f (Suc (Suc n)))*
by (*intro convergent-mult-const-iff (simp-all del: of-nat-Suc)*
finally show *?case .*
qed
from *this[of n - 1] assms show ?thesis by simp*
qed

lemma *stirling-integral-conv-stirling-integral-Suc:*
assumes $n > 0$
shows *stirling-integral n s =*
*of-nat n / of-nat (Suc n) * stirling-integral (Suc n) s -*
*of-real (bernoulli (Suc n)) / (of-nat (Suc n) * s ^ n)*

proof -
have $(\lambda N. of-real (pbernpoly (Suc\ n)\ (real\ N)) / (of-nat (Suc\ n) * (s + of-nat$
N) ^ n) -
*of-real (bernoulli (Suc n)) / (real (Suc n) * s ^ n) +*
*integral {0..real N} ($\lambda x. of-nat n / of-nat (Suc n) *$*
(of-real (pbernpoly (Suc n) x) / (of-real x + s) ^ Suc n)))
 $\longrightarrow 0 - of-real (bernoulli (Suc n)) / (of-nat (Suc n) * s ^ n) +$
 $of-nat n / of-nat (Suc n) * stirling-integral (Suc n) s$ **(is ?f \longrightarrow**
-)
unfolding *stirling-integral-def integral-mult-right*
using *convergent-stirling-integral[of Suc n] assms s*
by (*intro tendsto-intros pbernpoly-over-power-tendsto-0*
(auto simp: convergent-LIMSEQ-iff simp del: of-nat-Suc)
also have $?this \longleftrightarrow (\lambda N. integral \{0..real\ N\}$
 $(\lambda x. of-real (pbernpoly\ n\ x) / (of-real\ x + s) ^ n)) \longrightarrow$
 $of-nat\ n / of-nat (Suc\ n) * stirling-integral (Suc\ n) s -$
 $of-real (bernoulli (Suc\ n)) / (of-nat (Suc\ n) * s ^ n)$
using *eventually-gt-at-top[of 0::nat] pbernpoly-integral-conv-pbernpoly-integral-Suc[of*
n]
assms unfolding integral-mult-right
by (*intro filterlim-cong refl (auto elim!: eventually-mono simp del: power-Suc)*
finally show *?thesis unfolding stirling-integral-def[of n] by (rule limI)*
qed

lemma *stirling-integral-1-unfold:*
assumes $m > 0$

shows $\text{stirling-integral } 1 \ s = \text{stirling-integral } m \ s / \text{of-nat } m -$
 $(\sum k=1..<m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k))$
proof –
have $\text{stirling-integral } 1 \ s = \text{stirling-integral } (\text{Suc } m) \ s / \text{of-nat } (\text{Suc } m) -$
 $(\sum k=1..<\text{Suc } m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k))$ **for** m
proof (*induction* m)
case ($\text{Suc } m$)
let $?C = (\sum k = 1..<\text{Suc } m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k))$
note Suc.IH
also have $\text{stirling-integral } (\text{Suc } m) \ s / \text{of-nat } (\text{Suc } m) =$
 $\text{stirling-integral } (\text{Suc } (\text{Suc } m)) \ s / \text{of-nat } (\text{Suc } (\text{Suc } m)) -$
 $\text{of-real } (\text{bernoulli } (\text{Suc } (\text{Suc } m))) /$
 $(\text{of-nat } (\text{Suc } m) * \text{of-nat } (\text{Suc } (\text{Suc } m)) * s^{\wedge} \text{Suc } m)$
(is $- = ?A - ?B$) **by** (*subst* *stirling-integral-conv-stirling-integral-Suc*)
(simp-all del: of-nat-Suc power-Suc add: divide-simps)
also have $?A - ?B - ?C = ?A - (?B + ?C)$ **by** (*rule* *diff-diff-eq*)
also have $?B + ?C = (\sum k = 1..<\text{Suc } (\text{Suc } m). \text{of-real } (\text{bernoulli } (\text{Suc } k)) /$
 $(\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k))$
using s **by** (*simp* *add: divide-simps*)
finally show $?case$.
qed *simp-all*
note $\text{this}[\text{of } m - 1]$
also from *assms* **have** $\text{Suc } (m - 1) = m$ **by** *simp*
finally show $?thesis$.
qed

lemma *ln-Gamma-stirling-complex*:

assumes $m > 0$
shows $\text{ln-Gamma } s = (s - 1 / 2) * \ln s - s + \ln (2 * \text{pi}) / 2 +$
 $(\sum k=1..<m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k)) -$
 $\text{stirling-integral } m \ s / \text{of-nat } m$

proof –
have $\text{ln-Gamma } s = (s - 1 / 2) * \ln s - s + \ln (2 * \text{pi}) / 2 - \text{stirling-integral } 1 \ s$
using *limI[OF integral-pbernpoly-1]* **by** (*simp* *add: stirling-integral-def algebra-simps*)
also have $\text{stirling-integral } 1 \ s = \text{stirling-integral } m \ s / \text{of-nat } m -$
 $(\sum k = 1..<m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s^{\wedge} k))$
using *assms* **by** (*rule* *stirling-integral-1-unfold*)
finally show $?thesis$ **by** *simp*
qed

lemma *LIMSEQ-stirling-integral*:

$n > 0 \implies (\lambda x. \text{integral } \{0.. \text{real } x\} (\lambda x. \text{of-real } (\text{pbernpoly } n \ x) / (\text{of-real } x + s)))$

$\hat{\ }n))$
 \longrightarrow *stirling-integral n s unfolding* *stirling-integral-def*
using *convergent-stirling-integral[of n]* **by** (*simp only: convergent-LIMSEQ-iff*)

end

lemmas *has-integral-of-real = has-integral-linear[OF - bounded-linear-of-real, unfolded o-def]*
lemmas *integral-of-real = integral-linear[OF - bounded-linear-of-real, unfolded o-def]*

lemma *integrable-ln-Gamma-aux-real:*
assumes $0 < s$
shows $(\lambda x. \text{pbernpoly } n \ x / (x + s) ^ n)$ *integrable-on* $\{0..real \ N\}$
proof –
have $(\lambda x. \text{complex-of-real } (\text{pbernpoly } n \ x / (x + s) ^ n))$ *integrable-on* $\{0..real \ N\}$
using *integrable-ln-Gamma-aux[of of-real s n N] assms by simp*
from *integrable-linear[OF this bounded-linear-Re] show ?thesis*
by (*simp only: o-def Re-complex-of-real*)
qed

lemma
assumes $x > 0 \ n > 0$
shows *stirling-integral-complex-of-real:*
 $\text{stirling-integral } n \ (\text{complex-of-real } x) = \text{of-real } (\text{stirling-integral } n \ x)$
and *LIMSEQ-stirling-integral-real:*
 $(\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n))$
 $\longrightarrow \text{stirling-integral } n \ x$
and *stirling-integral-real-convergent:*
 $\text{convergent } (\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n))$
proof –
have $(\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{of-real } (\text{pbernpoly } n \ t / (t + x) ^ n)))$
 $\longrightarrow \text{stirling-integral } n \ (\text{complex-of-real } x)$
using *LIMSEQ-stirling-integral[of complex-of-real x n] assms by simp*
hence $(\lambda N. \text{of-real } (\text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n)))$
 $\longrightarrow \text{stirling-integral } n \ (\text{complex-of-real } x)$
using *integrable-ln-Gamma-aux-real[OF assms(1), of n]*
by (*subst (asm) integral-of-real) simp*
from *tendsto-Re[OF this]*
have $(\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n))$
 $\longrightarrow \text{Re } (\text{stirling-integral } n \ (\text{complex-of-real } x))$ **by** *simp*
thus *convergent* $(\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n))$
by (*rule convergentI*)
thus $(\lambda N. \text{integral } \{0..real \ N\} \ (\lambda t. \text{pbernpoly } n \ t / (t + x) ^ n))$
 $\longrightarrow \text{stirling-integral } n \ x$ **unfolding** *stirling-integral-def*
by (*simp add: convergent-LIMSEQ-iff*)
from *tendsto-of-real[OF this, where 'a = complex]*
integrable-ln-Gamma-aux-real[OF assms(1), of n]
have $(\lambda xa. \text{integral } \{0..real \ xa\})$

$(\lambda x a. \text{complex-of-real } (p\text{bernpoly } n \ x a) / (\text{complex-of-real } x a + x$
 $\hat{\ } n))$
 $\longrightarrow \text{complex-of-real } (\text{stirling-integral } n \ x)$
by $(\text{subst } (asm) \ \text{integral-of-real } [\text{symmetric}]) \ \text{simp-all}$
from $LIMSEQ\text{-unique}[OF \ \text{this } LIMSEQ\text{-stirling-integral}[of \ \text{complex-of-real } x \ n]]$
 $assms$
show $\text{stirling-integral } n \ (\text{complex-of-real } x) = \text{of-real } (\text{stirling-integral } n \ x)$ **by**
 simp
qed

lemma $\text{ln-Gamma-stirling-real}$:

assumes $x > (0 :: \text{real}) \ m > (0 :: \text{nat})$

shows $\text{ln-Gamma } x = (x - 1 / 2) * \ln \ x - x + \ln (2 * \pi) / 2 +$
 $(\sum k = 1..<m. \text{bernoulli } (\text{Suc } k) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * x \hat{\ } k))$

–

$\text{stirling-integral } m \ x / \text{of-nat } m$

proof –

from $assms$ **have** $\text{complex-of-real } (\text{ln-Gamma } x) = \text{ln-Gamma } (\text{complex-of-real } x)$

by $(\text{simp add: } \text{ln-Gamma-complex-of-real})$

also have $\text{ln-Gamma } (\text{complex-of-real } x) = \text{complex-of-real } ($

$(x - 1 / 2) * \ln \ x - x + \ln (2 * \pi) / 2 +$

$(\sum k = 1..<m. \text{bernoulli } (\text{Suc } k) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * x \hat{\ } k))$

$k))$ –

$\text{stirling-integral } m \ x / \text{of-nat } m)$ **using** $assms$

by $(\text{subst } \text{ln-Gamma-stirling-complex}[of \ - \ m])$

$(\text{simp-all add: } \text{Ln-of-real } \text{stirling-integral-complex-of-real})$

finally show $?thesis$ **by** $(\text{subst } (asm) \ \text{of-real-eq-iff})$

qed

lemma $\text{stirling-integral-bound-aux}$:

assumes $n: n > (1 :: \text{nat})$

obtains c **where** $\bigwedge s. \text{Re } s > 0 \implies \text{norm } (\text{stirling-integral } n \ s) \leq c / \text{Re } s \hat{\ } (n - 1)$

proof –

obtain c **where** $c: \text{norm } (p\text{bernpoly } n \ x) \leq c$ **for** x **by** $(\text{rule } \text{bounded-pbernpoly}[of \ n]) \ \text{blast}$

have $c': p\text{bernpoly } n \ x \leq c$ **for** x **using** $c[of \ x]$ **by** $(\text{simp add: } \text{abs-real-def } \text{split: } \text{if-splits})$

from $c[of \ 0]$ **have** $c\text{-nonneg: } c \geq 0$ **by** simp

have $\text{norm } (\text{stirling-integral } n \ s) \leq c / (\text{real } n - 1) / \text{Re } s \hat{\ } (n - 1)$ **if** $s: \text{Re } s > 0$ **for** s

proof $(\text{rule } \text{Lim-norm-ubound}[OF \ - \ LIMSEQ\text{-stirling-integral}])$

have $\text{pos: } x + \text{norm } s > 0$ **if** $x \geq 0$ **for** x **using** s **that** **by** $(\text{intro } \text{add-nonneg-pos})$
 auto

have $\text{nz: } \text{of-real } x + s \neq 0$ **if** $x \geq 0$ **for** x **using** s **that** **by** $(\text{auto } \text{simp: } \text{complex-eq-iff})$

let $?bound = \lambda N. c / (\text{Re } s \hat{\ } (n - 1) * (\text{real } n - 1))$ –

$c / ((\text{real } N + \text{Re } s) ^{(n-1)} * (\text{real } n - 1))$

show *eventually* $(\lambda N. \text{norm } (\text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^n)) \leq c / (\text{real } n - 1) / \text{Re } s ^{(n-1)})$ *at-top*

using *eventually-gt-at-top*[*of 0::nat*]

proof *eventually-elim*

case (*elim N*)

let $?F = \lambda x. -c / ((x + \text{Re } s) ^{(n-1)} * (\text{real } n - 1))$

from $n s$ **have** $((\lambda x. c / (x + \text{Re } s) ^n) \text{has-integral } (?F (\text{real } N) - ?F 0))$ $\{0.. \text{real } N\}$

by (*intro fundamental-theorem-of-calculus*)

(auto intro!: derivative-eq-intros simp: divide-simps power-diff add-eq-0-iff2 has-real-derivative-iff-has-vector-derivative [symmetric])

also have $?F (\text{real } N) - ?F 0 = ?\text{bound } N$ **by** *simp*

finally have $*$: $((\lambda x. c / (x + \text{Re } s) ^n) \text{has-integral } ?\text{bound } N) \{0.. \text{real } N\}$

.

have $\text{norm } (\text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^n)) \leq \text{integral } \{0.. \text{real } N\} (\lambda x. c / (x + \text{Re } s) ^n)$

proof (*intro integral-norm-bound-integral integrable-ln-Gamma-aux s ballI*)

fix x **assume** $x: x \in \{0.. \text{real } N\}$

have $\text{norm } (\text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^n) \leq c / \text{norm } (\text{of-real } x + s) ^n$

unfolding *norm-divide norm-power* **using** c **by** (*intro divide-right-mono*)

simp-all

also have $\dots \leq c / (x + \text{Re } s) ^n$

using $x c$ *c-nonneg s nz*[*of x*] *complex-Re-le-cmod*[*of of-real x + s*]

by (*intro divide-left-mono power-mono mult-pos-pos zero-less-power add-nonneg-pos*) *auto*

finally show $\text{norm } (\text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^n) \leq \dots$

qed (*insert n s * pos nz c, auto simp: complex-nonpos-Reals-iff*)

also have $\dots = ?\text{bound } N$ **using** $*$ **by** (*simp add: has-integral-iff*)

also have $\dots \leq c / (\text{Re } s ^{(n-1)} * (\text{real } n - 1))$ **using** *c-nonneg elim s n* **by** *simp*

also have $\dots = c / (\text{real } n - 1) / (\text{Re } s ^{(n-1)})$ **by** *simp*

finally show $\text{norm } (\text{integral } \{0.. \text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^n)) \leq c / (\text{real } n - 1) / \text{Re } s ^{(n-1)}$.

qed

qed (*insert s n, simp-all add: complex-nonpos-Reals-iff*)

thus *?thesis* **by** (*rule that*)

qed

lemma *stirling-integral-bound-aux-integral1*:

fixes $a b c :: \text{real}$ **and** $n :: \text{nat}$

assumes $a \geq 0 b > 0 c \geq 0 n > 1 l < a - b r > a + b$

shows $((\lambda x. c / \max b |x - a| ^n) \text{has-integral}$

$2*c*(n / (n - 1))/b^{(n-1)} - c/(n-1) * (1/(a-l)^{(n-1)} + 1/(r-a)^{(n-1)}))$

$\{l..r\}$

proof –

define $x1\ x2$ **where** $x1 = a - b$ **and** $x2 = a + b$
define $F1$ **where** $F1 = (\lambda x::real. c / (a - x) ^ (n - 1) / (n - 1))$
define $F3$ **where** $F3 = (\lambda x::real. -c / (x - a) ^ (n - 1) / (n - 1))$
have $deriv$: $(F1\ has\ vector\ derivative\ (c / (a - x) ^ n))\ (at\ x\ within\ A)$
 $(F3\ has\ vector\ derivative\ (c / (x - a) ^ n))\ (at\ x\ within\ A)$
if $x \neq a$ **for** $x :: real$ **and** A
unfolding $F1\text{-def}\ F3\text{-def}$ **using** $assms$ **that**
by $(auto\ intro!$: $derivative\ eq\ intros\ simp$: $divide_simps\ power\ diff\ add\ eq\ 0\ iff2$
 $simp\ flip$: $has\ real\ derivative\ iff\ has\ vector\ derivative)$

from $assms$ **have** $((\lambda x. c / (a - x) ^ n)\ has\ integral\ (F1\ x1 - F1\ l))\ \{l..x1\}$
by $(intro\ fundamental\ theorem\ of\ calculus\ deriv)$ $(auto\ simp$: $x1\text{-def}\ max\text{-def}$
 $split$: $if\ splits)$
also **have** $?this \longleftrightarrow ((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ (F1\ x1 - F1\ l))$
 $\{l..x1\}$
using $assms$
by $(intro\ has\ integral\ spike\ finite\ eq\ [of\ \{l\}])$ $(auto\ simp$: $x1\text{-def}\ max\text{-def}\ split$:
 $if\ splits)$
finally **have** $I1$: $((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ (F1\ x1 - F1\ l))$
 $\{l..x1\}$.

have $((\lambda x. c / b ^ n)\ has\ integral\ (x2 - x1) * c / b ^ n)\ \{x1..x2\}$
using $has\ integral\ const\ real$ [$of\ c / b ^ n\ x1\ x2$] $assms$ **by** $(simp\ add$: $x1\text{-def}$
 $x2\text{-def})$
also **have** $?this \longleftrightarrow ((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ ((x2 - x1) * c /$
 $b ^ n))\ \{x1..x2\}$
by $(intro\ has\ integral\ spike\ finite\ eq\ [of\ \{x1,\ x2\}])$
 $(auto\ simp$: $x1\text{-def}\ x2\text{-def}\ split$: $if\ splits)$
finally **have** $I2$: $((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ ((x2 - x1) * c / b ^$
 $n))\ \{x1..x2\}$.

from $assms$ **have** $I3$: $((\lambda x. c / (x - a) ^ n)\ has\ integral\ (F3\ r - F3\ x2))\ \{x2..r\}$
by $(intro\ fundamental\ theorem\ of\ calculus\ deriv)$ $(auto\ simp$: $x2\text{-def}\ min\text{-def}$
 $split$: $if\ splits)$
also **have** $?this \longleftrightarrow ((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ (F3\ r - F3\ x2))$
 $\{x2..r\}$
using $assms$
by $(intro\ has\ integral\ spike\ finite\ eq\ [of\ \{r\}])$ $(auto\ simp$: $x2\text{-def}\ min\text{-def}\ split$:
 $if\ splits)$
finally **have** $I3$: $((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ (F3\ r - F3\ x2))$
 $\{x2..r\}$.

have $((\lambda x. c / max\ b\ |x - a| ^ n)\ has\ integral\ (F1\ x1 - F1\ l) + ((x2 - x1) * c / b ^ n + (F3\ r - F3\ x2))\ \{l..r\}$
using $assms$
by $(intro\ has\ integral\ combine$ [$OF\ - -\ has\ integral\ combine$ [$OF\ - -\ I1\ I2$] $I3$])
 $(auto\ simp$: $x1\text{-def}\ x2\text{-def})$
also **have** $(F1\ x1 - F1\ l) + ((x2 - x1) * c / b ^ n) + (F3\ r - F3\ x2) =$
 $F1\ x1 - F1\ l + F3\ r - F3\ x2 + (x2 - x1) * c / b ^ n$

by (simp add: algebra-simps)
 also have $x2 - x1 = 2 * b$
 using assms by (simp add: x2-def x1-def min-def max-def)
 also have $2 * b * c / b ^ n = 2 * c / b ^ (n - 1)$
 using assms by (simp add: power-diff field-simps)
 also have $F1 x1 - F1 l + F3 r - F3 x2 =$
 $c / (n - 1) * (2 / b ^ (n - 1) - 1 / (a - l) ^ (n - 1) - 1 / (r - a) ^ (n - 1))$
 using assms by (simp add: x1-def x2-def F1-def F3-def field-simps del: of-nat-diff)
 also have $\dots + 2 * c / b ^ (n - 1) =$
 $2 * c * (1 + 1 / (n - 1)) / b ^ (n - 1) - c / (n - 1) * (1 / (a - l) ^ (n - 1) +$
 $1 / (r - a) ^ (n - 1))$
 using assms by (simp add: field-simps del: of-nat-diff)
 also have $1 + 1 / (n - 1) = n / (n - 1)$
 using assms by (simp add: field-simps)
 finally show ?thesis .
 qed

lemma *stirling-integral-bound-aux-integral2*:

fixes $a b c :: real$ and $n :: nat$
 assumes $a \geq 0 b > 0 c \geq 0 n > 1$
 obtains I where $((\lambda x. c / \max b |x - a| ^ n) \text{ has-integral } I) \{l..r\}$
 $I \leq 2 * c * (n / (n - 1)) / b ^ (n - 1)$

proof –

define l' where $l' = \min l (a - b - 1)$
 define r' where $r' = \max r (a + b + 1)$

define A where $A = 2 * c * (n / (n - 1)) / b ^ (n - 1)$

define B where $B = c / real (n - 1) * (1 / (a - l') ^ (n - 1) + 1 / (r' - a) ^ (n - 1))$

have *has-int*: $((\lambda x. c / \max b |x - a| ^ n) \text{ has-integral } (A - B)) \{l'..r'\}$

using assms **unfolding** A -def B -def

by (intro *stirling-integral-bound-aux-integral1*) (auto simp: l' -def r' -def)

have $(\lambda x. c / \max b |x - a| ^ n) \text{ integrable-on } \{l..r\}$

by (rule *integrable-on-subinterval*[OF *has-integral-integrable*[OF *has-int*]])
 (auto simp: l' -def r' -def)

then obtain I where *has-int'*: $((\lambda x. c / \max b |x - a| ^ n) \text{ has-integral } I) \{l..r\}$

by (auto simp: *integrable-on-def*)

from assms have $I \leq A - B$

by (intro *has-integral-subset-le*[OF *has-int' has-int*]) (auto simp: l' -def r' -def)

also have $\dots \leq A$

using assms by (simp add: B -def l' -def r' -def)

finally show ?thesis using *that*[of I] *has-int'* **unfolding** A -def by *blast*

qed

lemma *stirling-integral-bound-aux'*:

assumes $n: n > (1::nat)$ and $\alpha: \alpha \in \{0 < .. < \pi\}$

obtains c where $\bigwedge s::complex. s \in \text{complex-cone}' \alpha - \{0\} \implies$

$$\text{norm (stirling-integral } n \text{ } s) \leq c / \text{norm } s \wedge (n - 1)$$

proof –

obtain c **where** $c: \text{norm (pbernpoly } n \text{ } x) \leq c$ **for** x **by** (rule bounded-pbernpoly[of n]) *blast*

have $c': \text{pbernpoly } n \text{ } x \leq c$ **for** x **using** c [of x] **by** (simp add: abs-real-def split: if-splits)

from c [of 0] **have** $c\text{-nonneg}: c \geq 0$ **by** simp

define D **where** $D = c * \text{Beta} (- (\text{real-of-int} (- \text{int } n) / 2) - 1 / 2) (1 / 2)$
/ 2

define C **where** $C = \max D (2 * c * (n / (n - 1))) / \sin \alpha \wedge (n - 1)$

have $*$: $\text{norm (stirling-integral } n \text{ } s) \leq C / \text{norm } s \wedge (n - 1)$

if $s: s \in \text{complex-cone}' \alpha - \{0\}$ **for** $s :: \text{complex}$

proof (rule Lim-norm-ubound[OF - LIMSEQ-stirling-integral])

from $s \alpha$ **have** $\text{Arg}: |\text{Arg } s| \leq \alpha$ **by** (auto simp: complex-cone-altdef)

have $s': s \notin \mathbb{R}_{\leq 0}$

using complex-cone-inter-nonpos-Reals[of $-\alpha \alpha$] α s **by** auto

from s **have** [simp]: $s \neq 0$ **by** auto

show eventually $(\lambda N. \text{norm (integral } \{0.. \text{real } N\}$

$$(\lambda x. \text{of-real (pbernpoly } n \text{ } x) / (\text{of-real } x + s) \wedge n)) \leq$$

$$C / \text{norm } s \wedge (n - 1)$$
 at-top

using eventually-gt-at-top[of $0::\text{nat}$]

proof eventually-elim

case (elim N)

show ?case

proof (cases $\text{Re } s > 0$)

case True

have $\text{int}' : ((\lambda x. c * (x \wedge 2 + \text{norm } s \wedge 2) \text{powr } (-n / 2)) \text{has-integral}$

$$D * (\text{norm } s \wedge 2) \text{powr } (-n / 2 + 1 / 2)) \{0 < ..\}$$

using has-integral-mult-left[OF has-integral-Beta3[of $-n/2 \text{norm } s \wedge 2$],

of c] *assms* True

unfolding D -def **by** (simp add: algebra-simps)

hence $\text{int}' : ((\lambda x. c * (x \wedge 2 + \text{norm } s \wedge 2) \text{powr } (-n / 2)) \text{has-integral}$

$$D * (\text{norm } s \wedge 2) \text{powr } (-n / 2 + 1 / 2)) \{0.. \}$$

by (subst has-integral-interior [symmetric]) simp-all

hence integrable: $(\lambda x. c * (x \wedge 2 + \text{norm } s \wedge 2) \text{powr } (-n / 2))$ integrable-on $\{0.. \}$

by (simp add: has-integral-iff)

have $\text{norm (integral } \{0.. \text{real } N\} (\lambda x. \text{of-real (pbernpoly } n \text{ } x) / (\text{of-real } x + s) \wedge n)) \leq$

$$\text{integral } \{0.. \text{real } N\} (\lambda x. c * (x \wedge 2 + \text{norm } s \wedge 2) \text{powr } (-n / 2))$$

proof (intro integral-norm-bound-integral s ballI integrable-ln-Gamma-aux)

have [simp]: $\{0 < ..\} - \{0::\text{real}.. \} = \{\}$ $\{0.. \} - \{0 < ..\} = \{0::\text{real}\}$

by auto

have $(\lambda x. c * (x^2 + (c \bmod s)^2) \text{powr } (\text{real-of-int } (- \text{int } n) / 2))$ integrable-on $\{0 < ..\}$

```

    using int by (simp add: has-integral-iff)
    also have ?this  $\longleftrightarrow$   $(\lambda x. c * (x^2 + (c \bmod s)^2) \text{ powr } (\text{real-of-int } (- \text{int } n) / 2)) \text{ integrable-on } \{0..\}$ 
      by (intro integrable-spike-set-eq) auto
    finally show  $(\lambda x. c * (x^2 + (c \bmod s)^2) \text{ powr } (\text{real-of-int } (- \text{int } n) / 2)) \text{ integrable-on } \{0..\text{real } N\}$  by (rule integrable-on-subinterval) auto
  next
    fix x assume x:  $x \in \{0..\text{real } N\}$ 
    have nz: complex-of-real  $x + s \neq 0$ 
      using True x by (auto simp: complex-eq-iff)
    have norm  $(\text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^ n) \leq c / \text{norm } (\text{of-real } x + s) ^ n$ 
      unfolding norm-divide norm-power using c by (intro divide-right-mono) simp-all
    also have  $\dots \leq c / \text{sqrt } (x ^ 2 + \text{norm } s ^ 2) ^ n$ 
      proof (intro divide-left-mono mult-pos-pos zero-less-power power-mono)
        show  $\text{sqrt } (x^2 + (c \bmod s)^2) \leq c \bmod (\text{complex-of-real } x + s)$ 
          using x True by (simp add: cmod-def algebra-simps power2-eq-square)
        qed (use x True c-nonneg assms nz in <auto simp: add-nonneg-pos>)
    also have  $\text{sqrt } (x ^ 2 + \text{norm } s ^ 2) ^ n = (x ^ 2 + \text{norm } s ^ 2) \text{ powr } (1/2 * n)$ 
      by (subst powr-powr [symmetric], subst powr-realpow)
      (auto simp: power-half-sqrt add-nonneg-pos)
    also have  $c / \dots = c * (x^2 + \text{norm } s ^ 2) \text{ powr } (-n / 2)$ 
      by (simp add: powr-minus field-simps)
    finally show  $\text{norm } (\text{complex-of-real } (\text{pbernpoly } n x) / (\text{complex-of-real } x + s) ^ n) \leq \dots$ 
      qed fact+
    also have  $\dots \leq \text{integral } \{0..\} (\lambda x. c * (x^2 + \text{norm } s ^ 2) \text{ powr } (-n / 2))$ 
      using c-nonneg
      by (intro integral-subset-le integrable integrable-on-subinterval[OF integrable]) auto
    also have  $\dots = D * (\text{norm } s ^ 2) \text{ powr } (-n / 2 + 1 / 2)$ 
      using int' by (simp add: has-integral-iff)
    also have  $(\text{norm } s ^ 2) \text{ powr } (-n / 2 + 1 / 2) = \text{norm } s \text{ powr } (2 * (-n / 2 + 1 / 2))$ 
      by (subst powr-powr [symmetric]) auto
    also have  $\dots = \text{norm } s \text{ powr } (-\text{real } (n - 1))$ 
      using assms by (simp add: of-nat-diff)
    also have  $D * \dots = D / \text{norm } s ^ (n - 1)$ 
      by (auto simp: powr-minus powr-realpow field-simps)
    also have  $\dots \leq C / \text{norm } s ^ (n - 1)$ 
      by (intro divide-right-mono) (auto simp: C-def)
    finally show  $\text{norm } (\text{integral } \{0..\text{real } N\} (\lambda x. \text{of-real } (\text{pbernpoly } n x) / (\text{of-real } x + s) ^ n)) \leq \dots$ 
  next

```

```

case False
have  $\cos |Arg\ s| = \cos (Arg\ s)$ 
  by (simp add: abs-if)
also have  $\cos (Arg\ s) = Re (rcis (norm\ s) (Arg\ s)) / norm\ s$ 
  by (subst Re-rcis) auto
also have  $\dots = Re\ s / norm\ s$ 
  by (subst rcis-cmod-Arg) auto
also have  $\dots \leq \cos (pi / 2)$ 
  using False by (auto simp: field-simps)
finally have  $|Arg\ s| \geq pi / 2$ 
  using Arg  $\alpha$  by (subst (asm) cos-mono-le-eq) auto

have  $\sin \alpha * norm\ s = \sin (pi - \alpha) * norm\ s$ 
  by simp
also have  $\dots \leq \sin (pi - |Arg\ s|) * norm\ s$ 
  using  $\alpha$  Arg  $\langle |Arg\ s| \geq pi / 2 \rangle$ 
  by (intro mult-right-mono sin-monotone-2pi-le) auto
also have  $\sin |Arg\ s| \geq 0$ 
  using Arg-bounded[of s] by (intro sin-ge-zero) auto
hence  $\sin (pi - |Arg\ s|) = |\sin |Arg\ s||$ 
  by simp
also have  $\dots = |\sin (Arg\ s)|$ 
  by (simp add: abs-if)
also have  $\dots * norm\ s = |Im (rcis (norm\ s) (Arg\ s))|$ 
  by (simp add: abs-mult)
also have  $\dots = |Im\ s|$ 
  by (subst rcis-cmod-Arg) auto
finally have abs-Im-ge:  $|Im\ s| \geq \sin \alpha * norm\ s$  .

have [simp]:  $Im\ s \neq 0 \ s \neq 0$ 
  using  $s \notin \mathbb{R}_{\leq 0}$  False
  by (auto simp: cmod-def zero-le-mult-iff complex-nonpos-Reals-iff)
have  $\sin \alpha > 0$ 
  using assms by (intro sin-gt-zero) auto

obtain I where I:  $((\lambda x. c / \max |Im\ s| |x + Re\ s| ^ n)$  has-integral I)
{0..real N}
       $I \leq 2 * c * (n / (n - 1)) / |Im\ s| ^ (n - 1)$ 
using  $s$  c-nonneg assms False
      stirling-integral-bound-aux-integral2[of -Re s |Im s| c n 0 real N] by
auto

have  $norm (integral \{0..real N\} (\lambda x. of-real (pbernpoly\ n\ x) / (of-real\ x + s) ^ n)) \leq$ 
       $integral \{0..real N\} (\lambda x. c / \max |Im\ s| |x + Re\ s| ^ n)$ 
proof (intro integral-norm-bound-integral integrable-ln-Gamma-aux s ballI)
  show  $(\lambda x. c / \max |Im\ s| |x + Re\ s| ^ n)$  integrable-on {0..real N}
    using I(1) by (simp add: has-integral-iff)
next

```

fix x **assume** $x: x \in \{0..real\ N\}$
have $nz: complex-of-real\ x + s \neq 0$
by (*auto simp: complex-eq-iff*)
have $norm\ (complex-of-real\ (pbernpoly\ n\ x) / (complex-of-real\ x + s) ^ n)$
 \leq
 $c / norm\ (complex-of-real\ x + s) ^ n$
unfolding *norm-divide norm-power* **using** $c[of\ x]$ **by** (*intro divide-right-mono*) *simp-all*
also have $\dots \leq c / \max\ |Im\ s|\ |x + Re\ s| ^ n$
using $c\text{-nonneg}\ nz\ abs\ Re\ le\ cmod[of\ of\ real\ x + s]\ abs\ Im\ le\ cmod[of\ of\ real\ x + s]$
by (*intro divide-left-mono power-mono mult-pos-pos zero-less-power*)
(auto simp: less-max-iff-disj)
finally show $norm\ (complex-of-real\ (pbernpoly\ n\ x) / (complex-of-real\ x + s) ^ n) \leq \dots$
qed (*auto simp: complex-nonpos-Reals-iff*)
also have $\dots \leq 2 * c * (n / (n - 1)) / |Im\ s| ^ (n - 1)$
using I **by** (*simp add: has-integral-iff*)
also have $\dots \leq 2 * c * (n / (n - 1)) / (\sin\ \alpha * norm\ s) ^ (n - 1)$
using $\langle \sin\ \alpha > 0 \rangle\ s\ c\text{-nonneg}\ abs\ Im\ ge$
by (*intro divide-left-mono mult-pos-pos zero-less-power power-mono mult-nonneg-nonneg*) *auto*
also have $\dots = 2 * c * (n / (n - 1)) / \sin\ \alpha ^ (n - 1) / norm\ s ^ (n - 1)$
by (*simp add: field-simps*)
also have $\dots \leq C / norm\ s ^ (n - 1)$
by (*intro divide-right-mono*) (*auto simp: C-def*)
finally show *?thesis* .
qed
qed
qed (*use that assms complex-cone-inter-nonpos-Reals[of $-\alpha$ α] α in auto*)
thus *?thesis* **by** (*rule that*)
qed

lemma *stirling-integral-bound:*

assumes $n > 0$

obtains c **where**

$\bigwedge s. Re\ s > 0 \implies norm\ (stirling-integral\ n\ s) \leq c / Re\ s ^ n$

proof –

let $?f = \lambda s. of\ nat\ n / of\ nat\ (Suc\ n) * stirling-integral\ (Suc\ n)\ s - of-real\ (bernoulli\ (Suc\ n)) / (of\ nat\ (Suc\ n) * s ^ n)$

from *stirling-integral-bound-aux[of $Suc\ n]$ assms* **obtain** c **where**

$c: \bigwedge s. Re\ s > 0 \implies norm\ (stirling-integral\ (Suc\ n)\ s) \leq c / Re\ s ^ n$ **by** *auto*

define $c1$ **where** $c1 = real\ n / real\ (Suc\ n) * c$

define $c2$ **where** $c2 = |bernoulli\ (Suc\ n)| / real\ (Suc\ n)$

have $c2\text{-nonneg}: c2 \geq 0$ **by** (*simp add: c2-def*)

show *?thesis*

proof (*rule that*)

fix $s :: complex$ **assume** $s: Re\ s > 0$

hence $s': s \notin \mathbb{R}_{\leq 0}$ **by** (*auto simp: complex-nonpos-Reals-iff*)

have *stirling-integral* $n\ s = ?f\ s$ **using** s' *assms*
by (*rule* *stirling-integral-conv-stirling-integral-Suc*)
also have $\text{norm } \dots \leq \text{norm } (\text{of-nat } n / \text{of-nat } (\text{Suc } n) * \text{stirling-integral } (\text{Suc } n)\ s) +$
 $\text{norm } (\text{of-real } (\text{bernoulli } (\text{Suc } n)) / (\text{of-nat } (\text{Suc } n) * s^{\wedge} n))$
by (*rule* *norm-triangle-ineq4*)
also have $\dots = \text{real } n / \text{real } (\text{Suc } n) * \text{norm } (\text{stirling-integral } (\text{Suc } n)\ s) +$
 $c2 / \text{norm } s^{\wedge} n$ (**is** $= ?A + ?B$)
by (*simp* *add: norm-divide norm-mult norm-power c2-def field-simps del: of-nat-Suc*)
also have $?A \leq \text{real } n / \text{real } (\text{Suc } n) * (c / \text{Re } s^{\wedge} n)$
by (*intro* *mult-left-mono c s*) *simp-all*
also have $\dots = c1 / \text{Re } s^{\wedge} n$ **by** (*simp* *add: c1-def*)
also have $c2 / \text{norm } s^{\wedge} n \leq c2 / \text{Re } s^{\wedge} n$ **using** s *c2-nonneg*
by (*intro* *divide-left-mono power-mono complex-Re-le-cmod mult-pos-pos zero-less-power*) *auto*
also have $c1 / \text{Re } s^{\wedge} n + c2 / \text{Re } s^{\wedge} n = (c1 + c2) / \text{Re } s^{\wedge} n$
using s **by** (*simp* *add: field-simps*)
finally show $\text{norm } (\text{stirling-integral } n\ s) \leq (c1 + c2) / \text{Re } s^{\wedge} n$ **by** $-$ *simp-all*
qed
qed

lemma *stirling-integral-bound'*:

assumes $n > 0$ **and** $\alpha \in \{0 < \dots < \pi\}$

obtains c **where**

$\bigwedge s :: \text{complex. } s \in \text{complex-cone}'\ \alpha - \{0\} \implies \text{norm } (\text{stirling-integral } n\ s) \leq c / \text{norm } s^{\wedge} n$

proof $-$

let $?f = \lambda s. \text{of-nat } n / \text{of-nat } (\text{Suc } n) * \text{stirling-integral } (\text{Suc } n)\ s -$
 $\text{of-real } (\text{bernoulli } (\text{Suc } n)) / (\text{of-nat } (\text{Suc } n) * s^{\wedge} n)$

from *stirling-integral-bound-aux'*[*of* $\text{Suc } n$] *assms* **obtain** c **where**

$c: \bigwedge s :: \text{complex. } s \in \text{complex-cone}'\ \alpha - \{0\} \implies$
 $\text{norm } (\text{stirling-integral } (\text{Suc } n)\ s) \leq c / \text{norm } s^{\wedge} n$ **by** *auto*

define $c1$ **where** $c1 = \text{real } n / \text{real } (\text{Suc } n) * c$

define $c2$ **where** $c2 = |\text{bernoulli } (\text{Suc } n)| / \text{real } (\text{Suc } n)$

have *c2-nonneg*: $c2 \geq 0$ **by** (*simp* *add: c2-def*)

show *?thesis*

proof (*rule* *that*)

fix $s :: \text{complex}$ **assume** $s: s \in \text{complex-cone}'\ \alpha - \{0\}$

have $s': s \notin \mathbb{R}_{\leq 0}$

using *complex-cone-inter-nonpos-Reals*[*of* $-\alpha$] *assms* s **by** *auto*

have *stirling-integral* $n\ s = ?f\ s$ **using** s' *assms*

by (*intro* *stirling-integral-conv-stirling-integral-Suc*) *auto*

also have $\text{norm } \dots \leq \text{norm } (\text{of-nat } n / \text{of-nat } (\text{Suc } n) * \text{stirling-integral } (\text{Suc } n)\ s) +$

$\text{norm } (\text{of-real } (\text{bernoulli } (\text{Suc } n)) / (\text{of-nat } (\text{Suc } n) * s^{\wedge} n))$

by (*rule* *norm-triangle-ineq4*)

also have $\dots = \text{real } n / \text{real } (\text{Suc } n) * \text{norm } (\text{stirling-integral } (\text{Suc } n)\ s) +$

$c2 / \text{norm } s \wedge n$ (is - = ?A + ?B)

by (*simp add: norm-divide norm-mult norm-power c2-def field-simps del: of-nat-Suc*)

also have $?A \leq \text{real } n / \text{real } (\text{Suc } n) * (c / \text{norm } s \wedge n)$

by (*intro mult-left-mono c s*) *simp-all*

also have $\dots = c1 / \text{norm } s \wedge n$ **by** (*simp add: c1-def*)

also have $c1 / \text{norm } s \wedge n + c2 / \text{norm } s \wedge n = (c1 + c2) / \text{norm } s \wedge n$

using *s* **by** (*simp add: divide-simps*)

finally show $\text{norm } (\text{stirling-integral } n \ s) \leq (c1 + c2) / \text{norm } s \wedge n$ **by** *simp-all*

qed

qed

lemma *stirling-integral-holomorphic* [*holomorphic-intros*]:

assumes $m: m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$

shows *stirling-integral* m *holomorphic-on* A

proof –

from *assms* **have** [*simp*]: $z \notin \mathbb{R}_{\leq 0}$ **if** $z \in A$ **for** z

using *that* **by** *auto*

let $?f = \lambda s::\text{complex. of-nat } m * ((s - 1 / 2) * \text{Ln } s - s + \text{of-real } (\ln (2 * \text{pi}) / 2) +$

$(\sum k=1..<m. \text{of-real } (\text{bernoulli } (\text{Suc } k)) / (\text{of-nat } k * \text{of-nat } (\text{Suc } k) * s$

$\wedge k)) -$

$\text{ln-Gamma } s)$

have $?f$ *holomorphic-on* A **using** *assms*

by (*auto intro!: holomorphic-intros simp del: of-nat-Suc elim!: nonpos-Reals-cases*)

also have $?this \longleftrightarrow \text{stirling-integral } m$ *holomorphic-on* A

using *assms* **by** (*intro holomorphic-cong refl*)

(*simp-all add: field-simps ln-Gamma-stirling-complex*)

finally show *stirling-integral* m *holomorphic-on* A .

qed

lemma *stirling-integral-continuous-on-complex* [*continuous-intros*]:

assumes $m: m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$

shows *continuous-on* A (*stirling-integral* m :: - \Rightarrow *complex*)

by (*intro holomorphic-on-imp-continuous-on* *stirling-integral-holomorphic* *assms*)

lemma *has-field-derivative-stirling-integral-complex*:

fixes $x :: \text{complex}$

assumes $x \notin \mathbb{R}_{\leq 0}$ $n > 0$

shows (*stirling-integral* n *has-field-derivative* *deriv* (*stirling-integral* n) x) (at x)

using *assms*

by (*intro holomorphic-derivI[OF* *stirling-integral-holomorphic*, of $n - \mathbb{R}_{\leq 0}$]) *auto*

lemma

assumes $n: n > 0$ **and** $x > 0$
shows *deriv-stirling-integral-complex-of-real*:
 $(\text{deriv } \tilde{j}) (\text{stirling-integral } n) (\text{complex-of-real } x) =$
 $\text{complex-of-real } ((\text{deriv } \tilde{j}) (\text{stirling-integral } n) x) \text{ (is } ?\text{lhs } x = ?\text{rhs } x)$
and *differentiable-stirling-integral-real*:
 $(\text{deriv } \tilde{j}) (\text{stirling-integral } n) \text{ field-differentiable at } x \text{ (is } ?\text{thesis2})$

proof –

let $?A = \{s. \text{Re } s > 0\}$
let $?f = \lambda j x. (\text{deriv } \tilde{j}) (\text{stirling-integral } n) (\text{complex-of-real } x)$
let $?f' = \lambda j x. \text{complex-of-real } ((\text{deriv } \tilde{j}) (\text{stirling-integral } n) x)$

have [*simp*]: *open* $?A$ **by** (*simp add: open-halfspace-Re-gt*)

have $?lhs x = ?rhs x \wedge (\text{deriv } \tilde{j}) (\text{stirling-integral } n) \text{ field-differentiable at } x$
if $x > 0$ **for** x **using** *that*
proof (*induction j arbitrary: x*)
case 0
have $((\lambda x. \text{Re } (\text{stirling-integral } n (\text{of-real } x))) \text{ has-field-derivative}$
 $\text{Re } (\text{deriv } (\lambda x. \text{stirling-integral } n x) (\text{of-real } x))) (\text{at } x) \text{ using } 0 n$
by (*auto intro!: derivative-intros has-vector-derivative-real-field*
field-differentiable-derivI holomorphic-on-imp-differentiable-at[of - ?A]
stirling-integral-holomorphic simp: complex-nonpos-Reals-iff)
also have $?this \longleftrightarrow (\text{stirling-integral } n \text{ has-field-derivative}$
 $\text{Re } (\text{deriv } (\lambda x. \text{stirling-integral } n x) (\text{of-real } x))) (\text{at } x)$
using *eventually-nhds-in-open[of \{0<..\} x] 0 n*
by (*intro has-field-derivative-cong-ev refl*)
(auto elim!: eventually-mono simp: stirling-integral-complex-of-real)
finally have *stirling-integral n field-differentiable at x*
by (*auto simp: field-differentiable-def*)
with $0 n$ **show** $?case$ **by** (*auto simp: stirling-integral-complex-of-real*)
next
case (*Suc j x*)
note $IH = \text{conjunct1}[OF \text{Suc.IH}] \text{conjunct2}[OF \text{Suc.IH}]$
have $*: (\text{deriv } \overset{\sim}{\text{Suc } j}) (\text{stirling-integral } n) (\text{complex-of-real } x) =$
 $\text{of-real } ((\text{deriv } \overset{\sim}{\text{Suc } j}) (\text{stirling-integral } n) x) \text{ if } x: x > 0 \text{ for } x$
proof –
have $\text{deriv } ((\text{deriv } \tilde{j}) (\text{stirling-integral } n)) (\text{complex-of-real } x) =$
 $\text{vector-derivative } (\lambda x. (\text{deriv } \tilde{j}) (\text{stirling-integral } n) (\text{of-real } x)) (\text{at } x)$
using $n x$
by (*intro vector-derivative-of-real-right [symmetric]*
holomorphic-on-imp-differentiable-at[of - ?A] holomorphic-higher-deriv
stirling-integral-holomorphic) (auto simp: complex-nonpos-Reals-iff)
also have $\dots = \text{vector-derivative } (\lambda x. \text{of-real } ((\text{deriv } \tilde{j}) (\text{stirling-integral } n) x)) (\text{at } x)$
using *eventually-nhds-in-open[of \{0<..\} x] x*
by (*intro vector-derivative-cong-eq) (auto elim!: eventually-mono simp:*
IH(1))
also have $\dots = \text{of-real } (\text{deriv } ((\text{deriv } \tilde{j}) (\text{stirling-integral } n)) x)$
by (*intro vector-derivative-of-real-left holomorphic-on-imp-differentiable-at[of*

```

- ?A]
  field-differentiable-imp-differentiable IH(2) x)
  finally show ?thesis by simp
qed
have (( $\lambda x. \operatorname{Re} ((\operatorname{deriv} \widehat{\sim} \operatorname{Suc} j) (\operatorname{stirling-integral} n) (\operatorname{of-real} x)))$ ) has-field-derivative

   $\operatorname{Re} (\operatorname{deriv} ((\operatorname{deriv} \widehat{\sim} \operatorname{Suc} j) (\operatorname{stirling-integral} n) (\operatorname{of-real} x)))$  (at x)
  using Suc.prem1 n
by (intro derivative-intros has-vector-derivative-real-field field-differentiable-derivI
  holomorphic-on-imp-differentiable-at[of - ?A] stirling-integral-holomorphic
  holomorphic-higher-deriv) (auto simp: complex-nonpos-Reals-iff)
also have ?this  $\longleftrightarrow$  (( $\operatorname{deriv} \widehat{\sim} \operatorname{Suc} j$ ) (stirling-integral n) has-field-derivative
   $\operatorname{Re} (\operatorname{deriv} ((\operatorname{deriv} \widehat{\sim} \operatorname{Suc} j) (\operatorname{stirling-integral} n) (\operatorname{of-real} x)))$  (at x)
  using eventually-nhds-in-open[of {0<..} x] Suc.prem1 *)
  by (intro has-field-derivative-cong-ev refl) (auto elim!: eventually-mono)
finally have ( $\operatorname{deriv} \widehat{\sim} \operatorname{Suc} j$ ) (stirling-integral n) field-differentiable at x
  by (auto simp: field-differentiable-def)
with *[OF Suc.prem1] show ?case by blast
qed
from this[OF assms(2)] show ?lhs x = ?rhs x ?thesis2 by blast+
qed

```

Unfortunately, asymptotic power series cannot, in general, be differentiated. However, since $\ln\text{-Gamma}$ is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the j -th derivative of the remainder term at some value x by applying Cauchy's integral formula along a circle centred at x with radius $\frac{1}{2}x$.

lemma *deriv-stirling-integral-real-bound:*

```

  assumes m: m > 0
  shows ( $\operatorname{deriv} \widehat{\sim} j$ ) (stirling-integral m)  $\in O(\lambda x::\operatorname{real}. 1 / x^{(m+j)})$ 
proof -
  obtain c where c:  $\bigwedge s. 0 < \operatorname{Re} s \implies c \operatorname{mod} (\operatorname{stirling-integral} m s) \leq c / \operatorname{Re} s^{m+1}$ 
  using stirling-integral-bound[OF m] by auto
  have  $0 \leq c \operatorname{mod} (\operatorname{stirling-integral} m 1)$  by simp
  also have  $\dots \leq c$  using c[of 1] by simp
  finally have c-nonneg:  $c \geq 0$  .
  define B where B = c * 2m+Suc j
  define B' where B' = B * fact j / 2

  have eventually ( $\lambda x::\operatorname{real}. \operatorname{norm} ((\operatorname{deriv} \widehat{\sim} j) (\operatorname{stirling-integral} m) x) \leq$ 
    B' *  $\operatorname{norm} (1 / x^{(m+j)})$ ) at-top
  using eventually-gt-at-top[of 0::real]
proof eventually-elim
  case (elim x)

```

have $s \notin \mathbb{R}_{\leq 0}$ **if** $s \in \text{cball } (\text{of-real } x) (x/2)$ **for** $s :: \text{complex}$
proof –
have $x - \text{Re } s \leq \text{norm } (\text{of-real } x - s)$ **using** $\text{complex-Re-le-cmod}[\text{of of-real } x - s]$ **by** simp
also from that have $\dots \leq x/2$ **by** $(\text{simp add: dist-complex-def})$
finally show $?thesis$ **using** elim **by** $(\text{auto simp: complex-nonpos-Reals-iff})$
qed
hence $(\lambda u. \text{stirling-integral } m \ u / (u - \text{of-real } x) \wedge \text{Suc } j)$ **has-contour-integral**
 $\text{complex-of-real } (2 * \text{pi}) * \text{i} / \text{fact } j *$
 $(\text{deriv } \wedge \wedge j) (\text{stirling-integral } m) (\text{of-real } x) (\text{circlepath } (\text{of-real } x) (x/2))$
using $m \ \text{elim}$
by $(\text{intro Cauchy-has-contour-integral-higher-derivative-circlepath}$
 $\text{stirling-integral-continuous-on-complex } \text{stirling-integral-holomorphic})$
auto
hence $\text{norm } (\text{of-real } (2 * \text{pi}) * \text{i} / \text{fact } j * (\text{deriv } \wedge \wedge j) (\text{stirling-integral } m)$
 $(\text{of-real } x)) \leq$
 $B / x \wedge (m + \text{Suc } j) * (2 * \text{pi} * (x / 2))$
proof $(\text{rule has-contour-integral-bound-circlepath})$
fix $u :: \text{complex}$ **assume** $\text{dist: norm } (u - \text{of-real } x) = x / 2$
have $\text{Re } (\text{of-real } x - u) \leq \text{norm } (\text{of-real } x - u)$ **by** $(\text{rule complex-Re-le-cmod})$
also have $\dots = x / 2$ **using** dist **by** $(\text{simp add: norm-minus-commute})$
finally have $\text{Re-}u: \text{Re } u \geq x/2$ **using** elim **by** simp
have $\text{norm } (\text{stirling-integral } m \ u / (u - \text{of-real } x) \wedge \text{Suc } j) \leq$
 $c / \text{Re } u \wedge m / (x / 2) \wedge \text{Suc } j$ **using** $\text{Re-}u \ \text{elim}$
unfolding $\text{norm-divide } \text{norm-power } \text{dist}$
by $(\text{intro divide-right-mono } \text{zero-le-power } c) \ \text{simp-all}$
also have $\dots \leq c / (x/2) \wedge m / (x / 2) \wedge \text{Suc } j$ **using** $c\text{-nonneg } \text{elim } \text{Re-}u$
by $(\text{intro divide-right-mono } \text{divide-left-mono } \text{power-mono}) \ \text{simp-all}$
also have $\dots = B / x \wedge (m + \text{Suc } j)$ **using** elim **by** $(\text{simp add: B-def}$
 $\text{field-simps } \text{power-add})$
finally show $\text{norm } (\text{stirling-integral } m \ u / (u - \text{of-real } x) \wedge \text{Suc } j) \leq B / x$
 $\wedge (m + \text{Suc } j)$.
qed $(\text{insert } \text{elim } c\text{-nonneg, auto simp: B-def simp del: power-Suc})$
hence $\text{cmod } ((\text{deriv } \wedge \wedge j) (\text{stirling-integral } m) (\text{of-real } x)) \leq B' / x \wedge (j + m)$
using elim **by** $(\text{simp add: field-simps } \text{norm-divide } \text{norm-mult } \text{norm-power}$
 $B'\text{-def})$
with $\text{elim } m$ **show** $?case$ **by** $(\text{simp-all add: add-ac } \text{deriv-stirling-integral-complex-of-real})$
qed
thus $?thesis$ **by** (rule bigO1)
qed

definition $\text{stirling-sum} :: \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a :: \{\text{real-div-algebra, comm-ring-1}\}$
where

$$\begin{aligned}
 \text{stirling-sum } j \ m \ x = & \\
 & (-1) \wedge j * (\sum k = 1..<m. (\text{of-real } (\text{bernoulli } (\text{Suc } k)) * \text{pochhammer } (\text{of-nat } \\
 & k) \ j / (\text{of-nat } k * \\
 & \text{of-nat } (\text{Suc } k))) * \text{inverse } x \wedge (k + j))
 \end{aligned}$$

definition $\text{stirling-sum}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a :: \{\text{real-div-algebra, comm-ring-1}\}$

where

$$\begin{aligned} \text{stirling-sum}' j m x = \\ (-1) \wedge (\text{Suc } j) * (\sum k \leq m. (\text{of-real } (\text{bernoulli}' k) * \\ \text{pochhammer } (\text{of-nat } (\text{Suc } k)) (j - 1) * \text{inverse } x \wedge (k + j))) \end{aligned}$$

lemma *stirling-sum-complex-of-real*:

$$\begin{aligned} \text{stirling-sum } j m (\text{complex-of-real } x) = \text{complex-of-real } (\text{stirling-sum } j m x) \\ \text{by } (\text{simp add: } \text{stirling-sum-def pochhammer-of-real [symmetric]} \text{ del: of-nat-Suc}) \end{aligned}$$

lemma *stirling-sum'-complex-of-real*:

$$\begin{aligned} \text{stirling-sum}' j m (\text{complex-of-real } x) = \text{complex-of-real } (\text{stirling-sum}' j m x) \\ \text{by } (\text{simp add: } \text{stirling-sum}'\text{-def pochhammer-of-real [symmetric]} \text{ del: of-nat-Suc}) \end{aligned}$$

lemma *has-field-derivative-stirling-sum-complex* [derivative-intros]:

$\text{Re } x > 0 \implies (\text{stirling-sum } j m \text{ has-field-derivative } \text{stirling-sum } (\text{Suc } j) m x) \text{ (at } x)$

$$\begin{aligned} \text{unfolding } \text{stirling-sum-def [abs-def]} \text{ sum-distrib-left} \\ \text{by } (\text{rule DERIV-sum}) (\text{auto intro!: derivative-eq-intros simp del: of-nat-Suc} \\ \text{simp: pochhammer-Suc power-diff}) \end{aligned}$$

lemma *has-field-derivative-stirling-sum-real* [derivative-intros]:

$x > (0::\text{real}) \implies (\text{stirling-sum } j m \text{ has-field-derivative } \text{stirling-sum } (\text{Suc } j) m x) \text{ (at } x)$

$$\begin{aligned} \text{unfolding } \text{stirling-sum-def [abs-def]} \text{ sum-distrib-left} \\ \text{by } (\text{rule DERIV-sum}) (\text{auto intro!: derivative-eq-intros simp del: of-nat-Suc} \\ \text{simp: pochhammer-Suc power-diff}) \end{aligned}$$

lemma *has-field-derivative-stirling-sum'-complex* [derivative-intros]:

assumes $j > 0 \text{ Re } x > 0$
shows $(\text{stirling-sum}' j m \text{ has-field-derivative } \text{stirling-sum}' (\text{Suc } j) m x) \text{ (at } x)$

proof (cases j)

case $(\text{Suc } j')$

from *assms* **have** [simp]: $x \neq 0$ **by** *auto*

define c **where** $c = (\lambda n. (-1) \wedge \text{Suc } j * \text{complex-of-real } (\text{bernoulli}' n) * \\ \text{pochhammer } (\text{of-nat } (\text{Suc } n)) j')$

define T **where** $T = (\lambda n x. c n * \text{inverse } x \wedge (j + n))$

define T' **where** $T' = (\lambda n x. - (\text{of-nat } (j + n)) * c n * \text{inverse } x \wedge (\text{Suc } (j + n)))$

have $((\lambda x. \sum k \leq m. T k x) \text{ has-field-derivative } (\sum k \leq m. T' k x)) \text{ (at } x) \text{ using } \text{assms Suc}$

by (*intro DERIV-sum*)

(*auto simp: T-def T'-def intro!: derivative-eq-intros*

simp: field-simps power-add [symmetric]} \text{ simp del: of-nat-Suc power-Suc}

of-nat-add)

also have $(\lambda x. \sum k \leq m. T k x) = \text{stirling-sum}' j m$

by (*simp add: Suc T-def c-def stirling-sum'-def fun-eq-iff add-ac mult.assoc sum-distrib-left*)

also have $(\sum k \leq m. T' k x) = \text{stirling-sum}' (\text{Suc } j) m x$

by (*simp add: T'-def c-def Suc stirling-sum'-def sum-distrib-left*)

sum-distrib-right algebra-simps pochhammer-Suc)

finally show *?thesis* .

qed (*insert assms, simp-all*)

lemma *has-field-derivative-stirling-sum'-real* [*derivative-intros*]:

assumes $j > 0$ $x > (0::real)$

shows (*stirling-sum' j m has-field-derivative* *stirling-sum' (Suc j) m x*) (*at x*)

proof (*cases j*)

case (*Suc j'*)

from *assms* **have** [*simp*]: $x \neq 0$ **by** *auto*

define *c* **where** $c = (\lambda n. (-1) ^ Suc j * (bernoulli' n) * pochhammer (of-nat (Suc n)) j' :: real)$

define *T* **where** $T = (\lambda n x. c n * inverse x ^ (j + n))$

define *T'* **where** $T' = (\lambda n x. - (of-nat (j + n)) * c n * inverse x ^ (Suc (j + n)))$

have ($(\lambda x. \sum k \leq m. T k x)$ *has-field-derivative* ($\sum k \leq m. T' k x$)) (*at x*) **using** *assms Suc*

by (*intro DERIV-sum*)

(*auto simp: T-def T'-def intro!: derivative-eq-intros*

simp: field-simps power-add [symmetric] simp del: of-nat-Suc power-Suc of-nat-add)

also have ($\lambda x. (\sum k \leq m. T k x) = \text{stirling-sum}' j m$

by (*simp add: Suc T-def c-def* *stirling-sum'-def fun-eq-iff add-ac mult.assoc sum-distrib-left*)

also have ($\sum k \leq m. T' k x = \text{stirling-sum}' (Suc j) m x$

by (*simp add: T'-def c-def Suc* *stirling-sum'-def sum-distrib-left sum-distrib-right algebra-simps pochhammer-Suc*)

finally show *?thesis* .

qed (*insert assms, simp-all*)

lemma *higher-deriv-stirling-sum-complex*:

$Re\ x > 0 \implies (\text{deriv } \widetilde{\sim} i) (\text{stirling-sum } j\ m)\ x = \text{stirling-sum } (i + j)\ m\ x$

proof (*induction i arbitrary: x*)

case (*Suc i*)

have $\text{deriv } ((\text{deriv } \widetilde{\sim} i) (\text{stirling-sum } j\ m))\ x = \text{deriv } (\text{stirling-sum } (i + j)\ m)\ x$

using *eventually-nhds-in-open*[*of {x. Re x > 0} x*] *Suc.prem*s

by (*intro deriv-cong-ev refl*) (*auto elim!: eventually-mono simp: open-halfspace-Re-gt Suc.IH*)

also from *Suc.prem*s **have** $\dots = \text{stirling-sum } (Suc\ (i + j))\ m\ x$

by (*intro DERIV-imp-deriv has-field-derivative-stirling-sum-complex*)

finally show *?case by simp*

qed *simp-all*

definition *Polygamma-approx* :: $nat \Rightarrow nat \Rightarrow 'a \Rightarrow 'a :: \{real-normed-field, ln\}$

where

$Polygamma-approx\ j\ m =$

$(\text{deriv } \widetilde{\sim} j) (\lambda x. (x - 1 / 2) * \ln x - x + \text{of-real } (\ln (2 * pi)) / 2 + \text{stirling-sum } 0\ m\ x)$

lemma *Polygamma-approx-Suc*: $\text{Polygamma-approx } (\text{Suc } j) \ m = \text{deriv } (\text{Polygamma-approx } j \ m)$

by (*simp add: Polygamma-approx-def*)

lemma *Polygamma-approx-0*:

$\text{Polygamma-approx } 0 \ m \ x = (x - 1/2) * \ln x - x + \text{of-real } (\ln (2*\pi)) / 2 + \text{stirling-sum } 0 \ m \ x$

by (*simp add: Polygamma-approx-def*)

lemma *Polygamma-approx-1-complex*:

$\text{Re } x > 0 \implies$

$\text{Polygamma-approx } (\text{Suc } 0) \ m \ x = \ln x - 1 / (2*x) + \text{stirling-sum } (\text{Suc } 0) \ m \ x$

unfolding *Polygamma-approx-Suc Polygamma-approx-0*

by (*intro DERIV-imp-deriv*)

(*auto intro!: derivative-eq-intros elim!: nonpos-Reals-cases simp: field-simps*)

lemma *Polygamma-approx-1-real*:

$x > (0 :: \text{real}) \implies$

$\text{Polygamma-approx } (\text{Suc } 0) \ m \ x = \ln x - 1 / (2*x) + \text{stirling-sum } (\text{Suc } 0) \ m \ x$

unfolding *Polygamma-approx-Suc Polygamma-approx-0*

by (*intro DERIV-imp-deriv*)

(*auto intro!: derivative-eq-intros elim!: nonpos-Reals-cases simp: field-simps*)

lemma *stirling-sum-2-conv-stirling-sum'-1*:

fixes $x :: 'a :: \{\text{real-div-algebra, field-char-0}\}$

assumes $m > 0 \ x \neq 0$

shows $\text{stirling-sum}' \ 1 \ m \ x = 1 / x + 1 / (2 * x^2) + \text{stirling-sum } 2 \ m \ x$

proof –

have *pochhammer-2*: $\text{pochhammer } (\text{of-nat } k) \ 2 = \text{of-nat } k * \text{of-nat } (\text{Suc } k)$ **for** k

by (*simp add: pochhammer-Suc eval-nat-numeral add-ac*)

have *stirling-sum 2 m x* =

$(\sum k = \text{Suc } 0..<m. \text{of-real } (\text{bernoulli}' (\text{Suc } k)) * \text{inverse } x ^ \text{Suc } (\text{Suc } k))$

unfolding *stirling-sum-def pochhammer-2 power2-minus power-one mult-1-left*

by (*intro sum.cong refl*)

(*simp-all add: stirling-sum-def pochhammer-2 power2-eq-square divide-simps bernoulli'-def*

del: of-nat-Suc power-Suc)

also have $1 / (2 * x^2) + \dots =$

$(\sum k=0..<m. \text{of-real } (\text{bernoulli}' (\text{Suc } k)) * \text{inverse } x ^ \text{Suc } (\text{Suc } k))$

using *assms*

by (*subst (2) sum.atLeast-Suc-lessThan*) (*simp-all add: power2-eq-square field-simps*)

also have $1 / x + \dots = (\sum k=0..<\text{Suc } m. \text{of-real } (\text{bernoulli}' k) * \text{inverse } x ^ \text{Suc } k)$

by (*subst sum.atLeast0-lessThan-Suc-shift*) (*simp-all add: bernoulli'-def divide-simps*)

also have $\dots = (\sum k \leq m. \text{of-real } (\text{bernoulli}' k) * \text{inverse } x \wedge \text{Suc } k)$
by *(intro sum.cong) auto*
also have $\dots = \text{stirling-sum}' 1 m x$ **by** *(simp add: stirling-sum'-def)*
finally show *?thesis* **by** *(simp add: add-ac)*
qed

lemma *Polygamma-approx-2-real:*

assumes $x > (0::\text{real})$ $m > 0$
shows $\text{Polygamma-approx } (\text{Suc } (\text{Suc } 0)) m x = \text{stirling-sum}' 1 m x$
proof –
have $\text{Polygamma-approx } (\text{Suc } (\text{Suc } 0)) m x = \text{deriv } (\text{Polygamma-approx } (\text{Suc } 0) m) x$
by *(simp add: Polygamma-approx-Suc)*
also have $\dots = \text{deriv } (\lambda x. \ln x - 1 / (2*x) + \text{stirling-sum } (\text{Suc } 0) m x) x$
using *eventually-nhds-in-open[of {0<..} x] assms*
by *(intro deriv-cong-ev) (auto elim!: eventually-mono simp: Polygamma-approx-1-real)*
also have $\dots = 1 / x + 1 / (2*x^2) + \text{stirling-sum } (\text{Suc } (\text{Suc } 0)) m x$ **using**
assms
by *(intro DERIV-imp-deriv) (auto intro!: derivative-eq-intros*
elim!: nonpos-Reals-cases simp: field-simps power2-eq-square)
also have $\dots = \text{stirling-sum}' 1 m x$ **using** *stirling-sum-2-conv-stirling-sum'-1[of*
m x] assms
by *(simp add: eval-nat-numeral)*
finally show *?thesis* .
qed

lemma *Polygamma-approx-2-complex:*

assumes $\text{Re } x > 0$ $m > 0$
shows $\text{Polygamma-approx } (\text{Suc } (\text{Suc } 0)) m x = \text{stirling-sum}' 1 m x$
proof –
have $\text{Polygamma-approx } (\text{Suc } (\text{Suc } 0)) m x = \text{deriv } (\text{Polygamma-approx } (\text{Suc } 0) m) x$
by *(simp add: Polygamma-approx-Suc)*
also have $\dots = \text{deriv } (\lambda x. \ln x - 1 / (2*x) + \text{stirling-sum } (\text{Suc } 0) m x) x$
using *eventually-nhds-in-open[of {s. Re s > 0} x] assms*
by *(intro deriv-cong-ev)*
(auto simp: open-halfspace-Re-gt elim!: eventually-mono simp: Polygamma-approx-1-complex)
also have $\dots = 1 / x + 1 / (2*x^2) + \text{stirling-sum } (\text{Suc } (\text{Suc } 0)) m x$ **using**
assms
by *(intro DERIV-imp-deriv) (auto intro!: derivative-eq-intros*
elim!: nonpos-Reals-cases simp: field-simps power2-eq-square)
also have $\dots = \text{stirling-sum}' 1 m x$ **using** *stirling-sum-2-conv-stirling-sum'-1[of*
m x] assms
by *(subst stirling-sum-2-conv-stirling-sum'-1) (auto simp: eval-nat-numeral)*
finally show *?thesis* .
qed

lemma *Polygamma-approx-ge-2-real:*

assumes $x > (0::\text{real})$ $m > 0$

shows $\text{Polygamma-approx } (\text{Suc } (\text{Suc } j)) m x = \text{stirling-sum}' (\text{Suc } j) m x$
using $\text{assms}(1)$
proof (*induction j arbitrary: x*)
 case ($0 x$)
 with assms **show** $?case$ **by** ($\text{simp add: Polygamma-approx-2-real}$)
next
 case ($\text{Suc } j x$)
 have $\text{Polygamma-approx } (\text{Suc } (\text{Suc } (\text{Suc } j))) m x = \text{deriv } (\text{Polygamma-approx } (\text{Suc } (\text{Suc } j)) m) x$
 by ($\text{simp add: Polygamma-approx-Suc}$)
 also have $\dots = \text{deriv } (\text{stirling-sum}' (\text{Suc } j) m) x$
 using $\text{eventually-nhds-in-open}[of \{0<.\} x] \text{Suc.prem}$
 by ($\text{intro deriv-cong-ev refl}$) ($\text{auto elim!: eventually-mono simp: Suc.IH}$)
 also have $\dots = \text{stirling-sum}' (\text{Suc } (\text{Suc } j)) m x$ **using** Suc.prem
 by ($\text{intro DERIV-imp-deriv derivative-intros}$) simp-all
 finally show $?case$.
qed

lemma $\text{Polygamma-approx-ge-2-complex}$:
 assumes $\text{Re } x > 0 m > 0$
 shows $\text{Polygamma-approx } (\text{Suc } (\text{Suc } j)) m x = \text{stirling-sum}' (\text{Suc } j) m x$
using $\text{assms}(1)$
proof (*induction j arbitrary: x*)
 case ($0 x$)
 with assms **show** $?case$ **by** ($\text{simp add: Polygamma-approx-2-complex}$)
next
 case ($\text{Suc } j x$)
 have $\text{Polygamma-approx } (\text{Suc } (\text{Suc } (\text{Suc } j))) m x = \text{deriv } (\text{Polygamma-approx } (\text{Suc } (\text{Suc } j)) m) x$
 by ($\text{simp add: Polygamma-approx-Suc}$)
 also have $\dots = \text{deriv } (\text{stirling-sum}' (\text{Suc } j) m) x$
 using $\text{eventually-nhds-in-open}[of \{x. \text{Re } x > 0\} x] \text{Suc.prem}$
 by ($\text{intro deriv-cong-ev refl}$) ($\text{auto elim!: eventually-mono simp: Suc.IH open-halfspace-Re-gt}$)
 also have $\dots = \text{stirling-sum}' (\text{Suc } (\text{Suc } j)) m x$ **using** Suc.prem
 by ($\text{intro DERIV-imp-deriv derivative-intros}$) simp-all
 finally show $?case$.
qed

lemma $\text{Polygamma-approx-complex-of-real}$:
 assumes $x > 0 m > 0$
 shows $\text{Polygamma-approx } j m (\text{complex-of-real } x) = \text{of-real } (\text{Polygamma-approx } j m x)$
proof (*cases j*)
 case 0
 with assms **show** $?thesis$ **by** ($\text{simp add: Polygamma-approx-0 Ln-of-real stirling-sum-complex-of-real}$)
next
 case [simp]: ($\text{Suc } j'$)
 thus $?thesis$

```

proof (cases j')
  case 0
    with assms show ?thesis
      by (simp add: Polygamma-approx-1-complex
            Polygamma-approx-1-real stirling-sum-complex-of-real Ln-of-real)
  next
    case (Suc j'')
      with assms show ?thesis
        by (simp add: Polygamma-approx-ge-2-complex Polygamma-approx-ge-2-real
              stirling-sum'-complex-of-real)
qed
qed

lemma higher-deriv-Polygamma-approx [simp]:
  (deriv  $\hat{\hat{}}$  j) (Polygamma-approx i m) = Polygamma-approx (j + i) m
  by (simp add: Polygamma-approx-def funpow-add)

lemma stirling-sum-holomorphic [holomorphic-intros]:
   $0 \notin A \implies$  stirling-sum j m holomorphic-on A
  unfolding stirling-sum-def by (intro holomorphic-intros) auto

lemma Polygamma-approx-holomorphic [holomorphic-intros]:
  Polygamma-approx j m holomorphic-on {s. Re s > 0}
  unfolding Polygamma-approx-def
  by (intro holomorphic-intros) (auto simp: open-halfspace-Re-gt elim!: nonpos-Reals-cases)

lemma higher-deriv-lnGamma-stirling:
  assumes m: m > 0
  shows ( $\lambda x::\text{real}.$  (deriv  $\hat{\hat{}}$  j) ln-Gamma x - Polygamma-approx j m x)  $\in O(\lambda x.$ 
  1 / x  $^{\wedge}$  (m + j))
  proof -
    have eventually ( $\lambda x.$  |(deriv  $\hat{\hat{}}$  j) ln-Gamma x - Polygamma-approx j m x| =
      inverse (real m) * |(deriv  $\hat{\hat{}}$  j) (stirling-integral m) x|) at-top
      using eventually-gt-at-top[of 0::real]
    proof eventually-elim
      case (elim x)
        note x = this
        have  $\forall_F y$  in nhds (complex-of-real x). y  $\in -\mathbb{R}_{\leq 0}$ 
          using elim by (intro eventually-nhds-in-open) auto
        hence (deriv  $\hat{\hat{}}$  j) ( $\lambda x.$  ln-Gamma x - Polygamma-approx 0 m x) (complex-of-real
  x) =
      (deriv  $\hat{\hat{}}$  j) ( $\lambda x.$  (-inverse (of-nat m)) * stirling-integral m x)
    (complex-of-real x)
      using x m
      by (intro higher-deriv-cong-ev refl)
      (auto elim!: eventually-mono simp: ln-Gamma-stirling-complex Polygamma-approx-def

      field-simps open-halfspace-Re-gt stirling-sum-def)
    also have ... = - inverse (of-nat m) * (deriv  $\hat{\hat{}}$  j) (stirling-integral m) (of-real

```

x) **using** $x m$
by (*intro higher-deriv-cmult*[*of - $\mathbf{R}_{\leq 0}$*] *stirling-integral-holomorphic*)
(*auto simp: open-halfspace-Re-gt*)
also have (*deriv $\sim j$*) ($\lambda x. \ln\text{-Gamma } x - \text{Polygamma-approx } 0 m x$) (*complex-of-real*
 x) =
(*deriv $\sim j$*) $\ln\text{-Gamma } (\text{of-real } x) - (\text{deriv } \sim j) (\text{Polygamma-approx}$
 $0 m) (\text{of-real } x)$
using x
by (*intro higher-deriv-diff*[*of - $\{s. \text{Re } s > 0\}$*])
(*auto intro!: holomorphic-intros elim!: nonpos-Reals-cases simp: open-halfspace-Re-gt*)
also have (*deriv $\sim j$*) ($\text{Polygamma-approx } 0 m$) (*complex-of-real* x) =
of-real ($\text{Polygamma-approx } j m x$) **using** $x m$
by (*simp add: Polygamma-approx-complex-of-real*)
also have *norm* ($- \text{inverse } (\text{of-nat } m) * (\text{deriv } \sim j) (\text{stirling-integral } m)$)
(*complex-of-real* x) =
inverse ($\text{real } m$) * $|(\text{deriv } \sim j) (\text{stirling-integral } m) x|$
using $x m$ **by** (*simp add: norm-mult norm-inverse deriv-stirling-integral-complex-of-real*)
also have (*deriv $\sim j$*) $\ln\text{-Gamma } (\text{complex-of-real } x) = \text{of-real } ((\text{deriv } \sim j)$
 $\ln\text{-Gamma } x)$ **using** x
by (*simp add: higher-deriv-ln-Gamma-complex-of-real*)
also have *norm* ($\dots - \text{of-real } (\text{Polygamma-approx } j m x)$) =
 $|(\text{deriv } \sim j) \ln\text{-Gamma } x - \text{Polygamma-approx } j m x|$
by (*simp only: of-real-diff [symmetric] norm-of-real*)
finally show *?case* .
qed
from *bigthetaI-cong*[*OF this*] m
have ($\lambda x::\text{real}. (\text{deriv } \sim j) \ln\text{-Gamma } x - \text{Polygamma-approx } j m x$) \in
 $\Theta(\lambda x. (\text{deriv } \sim j) (\text{stirling-integral } m) x)$ **by** *simp*
also have ($\lambda x::\text{real}. (\text{deriv } \sim j) (\text{stirling-integral } m) x$) $\in O(\lambda x. 1 / x^{(m +$
 $j)})$ **using** m
by (*rule deriv-stirling-integral-real-bound*)
finally show *?thesis* .
qed

lemma *Polygamma-approx-1-real'*:

assumes $x: (\text{real}) > 0$ **and** $m: m > 0$
shows $\text{Polygamma-approx } 1 m x = \ln x - (\sum k = \text{Suc } 0..m. \text{bernoulli}' k * \text{inverse } x^k / \text{real } k)$
proof -
have $\text{Polygamma-approx } 1 m x = \ln x - (1 / (2 * x) +$
 $(\sum k=\text{Suc } 0..<m. \text{bernoulli } (\text{Suc } k) * \text{inverse } x^{\text{Suc } k} / \text{real } (\text{Suc } k)))$
(*is - = - - (- + ?S)*) **using** x **by** (*simp add: Polygamma-approx-1-real stir-*
ling-sum-def)
also have $?S = (\sum k=\text{Suc } 0..<m. \text{bernoulli}' (\text{Suc } k) * \text{inverse } x^{\text{Suc } k} / \text{real}$
 $(\text{Suc } k))$
by (*intro sum.cong refl*) (*simp-all add: bernoulli'-def*)
also have $1 / (2 * x) + \dots =$
 $(\sum k=0..<m. \text{bernoulli}' (\text{Suc } k) * \text{inverse } x^{\text{Suc } k} / \text{real } (\text{Suc } k))$
using m

by (subst (2) sum.atLeast-Suc-lessThan) (simp-all add: field-simps)
 also have ... = $(\sum k = \text{Suc } 0..m. \text{bernoulli}' k * \text{inverse } x^k / \text{real } k)$ using
assms
 by (subst sum.shift-bounds-Suc-ivl [symmetric]) (simp add: atLeastLessThanSuc-atLeastAtMost)
 finally show ?thesis .
 qed

theorem

assumes $m: m > 0$

shows *ln-Gamma-real-asymptotics*:

$(\lambda x. \text{ln-Gamma } x - ((x - 1 / 2) * \text{ln } x - x + \text{ln } (2 * \text{pi}) / 2 +$
 $(\sum k = 1..<m. \text{bernoulli } (\text{Suc } k) / (\text{real } k * \text{real } (\text{Suc } k)) / x^k))$
 $\in O(\lambda x. 1 / x^m)$ (is ?th1)

and *Digamma-real-asymptotics*:

$(\lambda x. \text{Digamma } x - (\text{ln } x - (\sum k=1..m. \text{bernoulli}' k / \text{real } k / x^k))$
 $\in O(\lambda x. 1 / (x^{\text{Suc } m}))$ (is ?th2)

and *Polygamma-real-asymptotics*: $j > 0 \implies$

$(\lambda x. \text{Polygamma } j x - (-1)^{\text{Suc } j} * (\sum k \leq m. \text{bernoulli}' k *$
 $\text{pochhammer } (\text{real } (\text{Suc } k)) (j - 1) / x^{(k + j)})$
 $\in O(\lambda x. 1 / x^{(m+j+1)})$ (is - \implies ?th3)

proof –

define $G :: \text{nat} \Rightarrow \text{real} \Rightarrow \text{real}$ where

$G = (\lambda m. \text{if } m = 0 \text{ then } \text{ln-Gamma} \text{ else } \text{Polygamma } (m - 1))$

have *: $(\lambda x. G j x - h x) \in O(\lambda x. 1 / x^{(m + j)})$

if $\bigwedge x::\text{real}. x > 0 \implies \text{Polygamma-approx } j m x = h x$ for $j h$

proof –

have $(\lambda x. G j x - h x) \in$

$\Theta(\lambda x. (\text{deriv } ^{\wedge} j) \text{ln-Gamma } x - \text{Polygamma-approx } j m x)$ (is - \in

$\Theta(?f)$)

using that

by (intro bighetaI-cong) (auto intro: eventually-mono[OF eventually-gt-at-top[of 0::real]])

simp del: funpow.simps simp: higher-deriv-ln-Gamma-real G-def)

also have ?f $\in O(\lambda x::\text{real}. 1 / x^{(m + j)})$ using m

by (rule higher-deriv-lnGamma-stirling)

finally show ?thesis .

qed

note [[simproc del: simplify-landau-sum]]

from *[OF Polygamma-approx-0] assms show ?th1

by (simp add: G-def Polygamma-approx-0 stirling-sum-def field-simps)

from *[OF Polygamma-approx-1-real'] assms show ?th2 by (simp add: G-def field-simps)

assume $j: j > 0$

from *[OF Polygamma-approx-ge-2-real, of j - 1] assms j show ?th3

by (simp add: G-def stirling-sum'-def power-add power-diff field-simps)

qed

2.5 Asymptotics of the complex Gamma function

The m -th order remainder of Stirling's formula for $\log \Gamma$ is $O(s^{-m})$ uniformly over any complex cone $\text{Arg}(z) \leq \alpha$, $z \neq 0$ for any angle $\alpha \in (0, \pi)$. This means that there is bounded by cz^{-m} for some constant c for all z in this cone.

context

fixes F and α

assumes $\alpha: \alpha \in \{0 < .. < \pi\}$

defines $F \equiv \text{principal}(\text{complex-cone}' \alpha - \{0\})$

begin

lemma *stirling-integral-bigo*:

fixes $m :: \text{nat}$

assumes $m: m > 0$

shows *stirling-integral* $m \in O[F](\lambda s. 1 / s \wedge m)$

proof –

obtain c **where** $c: \bigwedge s. s \in \text{complex-cone}' \alpha - \{0\} \implies \text{norm}(\text{stirling-integral } m s) \leq c / \text{norm } s \wedge m$

using *stirling-integral-bound* $[OF \langle m > 0 \rangle \alpha]$ **by** *blast*

have $0 \leq \text{norm}(\text{stirling-integral } m 1 :: \text{complex})$

by *simp*

also have $\dots \leq c$

using $c[\text{of } 1] \alpha$ **by** *simp*

finally have $c \geq 0$.

have *eventually* $(\lambda s. s \in \text{complex-cone}' \alpha - \{0\}) F$

unfolding $F\text{-def}$ **by** *(auto simp: eventually-principal)*

hence *eventually* $(\lambda s. \text{norm}(\text{stirling-integral } m s) \leq c * \text{norm}(1 / s \wedge m)) F$

by *eventually-elim (use c in <simp add: norm-divide norm-power>)*

thus *stirling-integral* $m \in O[F](\lambda s. 1 / s \wedge m)$

by *(intro bigoI[of - c]) auto*

qed

end

The following is a more explicit statement of this:

theorem *ln-Gamma-complex-asymptotics-explicit*:

fixes $m :: \text{nat}$ and $\alpha :: \text{real}$

assumes $m > 0$ and $\alpha \in \{0 < .. < \pi\}$

obtains $C :: \text{real}$ and $R :: \text{complex} \Rightarrow \text{complex}$

where $\forall s :: \text{complex}. s \notin \mathbb{R}_{\leq 0} \longrightarrow$

$$\text{ln-Gamma } s = (s - 1/2) * \text{ln } s - s + \text{ln}(2 * \pi) / 2 +$$

$$(\sum k=1..m. \text{bernoulli}(k+1) / (k * (k+1) * s \wedge k)) - R s$$

and $\forall s. s \neq 0 \wedge |\text{Arg } s| \leq \alpha \longrightarrow \text{norm}(R s) \leq C / \text{norm } s \wedge m$

proof –

obtain c **where** $c: \bigwedge s. s \in \text{complex-cone}' \alpha - \{0\} \implies \text{norm}(\text{stirling-integral } m s) \leq c / \text{norm } s \wedge m$

```

    using stirling-integral-bound [OF assms] by blast
  have  $0 \leq \text{norm } (\text{stirling-integral } m \ 1 \ :: \text{ complex})$ 
    by simp
  also have  $\dots \leq c$ 
    using c[of 1] assms by simp
  finally have  $c \geq 0$  .
  define R where  $R = (\lambda s :: \text{complex. } \text{stirling-integral } m \ s \ / \ \text{of-nat } m)$ 
  show ?thesis
  proof (rule that)
    from ln-Gamma-stirling-complex [of - m] assms show
       $\forall s :: \text{complex. } s \notin \mathbb{R}_{\leq 0} \longrightarrow$ 
         $\text{ln-Gamma } s = (s - 1 / 2) * \text{ln } s - s + \text{ln } (2 * \text{pi}) / 2 +$ 
         $(\sum_{k=1..<m.} \text{bernoulli } (k+1) / (k * (k+1) * s^{\wedge} k)) - R \ s$ 
      by (auto simp add: R-def algebra-simps of-real-bernoulli)
    show  $\forall s. s \neq 0 \wedge |\text{Arg } s| \leq \alpha \longrightarrow \text{cmod } (R \ s) \leq c / \text{real } m / \text{cmod } s^{\wedge} m$ 
    proof (safe, goal-cases)
      case (1 s)
      show ?case
        using 1 c[of s] assms
        by (auto simp: complex-cone-altdef abs-le-iff R-def norm-divide field-simps)
    qed
  qed
qed

```

Lastly, we can also derive the asymptotics of Γ itself:

$$\Gamma(z) \sim \sqrt{2\pi/z} \left(\frac{z}{e}\right)^z$$

uniformly for $|z| \rightarrow \infty$ within the cone $\text{Arg}(z) \leq \alpha$ for $\alpha \in (0, \pi)$:

```

context
  fixes F and  $\alpha$ 
  assumes  $\alpha: \alpha \in \{0 < .. < \text{pi}\}$ 
  defines  $F \equiv \text{inf at-infinity } (\text{principal } (\text{complex-cone}' \ \alpha))$ 
begin

```

lemma *Gamma-complex-asymp-equiv:*

Gamma \sim [*F*] $(\lambda s. \text{sqrt } (2 * \text{pi}) * (s / \text{exp } 1) \text{ powr } s / s \text{ powr } (1 / 2))$

proof –

define *I* :: *complex* \Rightarrow *complex* **where** *I* = *stirling-integral 1*

have *eventually* $(\lambda s. s \in \text{complex-cone}' \ \alpha) \ F$

by (auto *simp: eventually-inf-principal F-def*)

moreover **have** *eventually* $(\lambda s. s \neq 0) \ F$

unfolding *F-def* *eventually-inf-principal*

using *eventually-not-equal-at-infinity* **by** *eventually-elim auto*

ultimately **have** *eventually* $(\lambda s. \text{Gamma } s =$

$\text{sqrt } (2 * \text{pi}) * (s / \text{exp } 1) \text{ powr } s / s \text{ powr } (1 / 2) / \text{exp } (I \ s)) \ F$

proof *eventually-elim*

case (*elim s*)

from *elim* **have** *s'*: $s \notin \mathbb{R}_{\leq 0}$

using *complex-cone-inter-nonpos-Reals*[of $-\alpha$] α **by** *auto*
from *elim* **have** [simp]: $s \neq 0$ **by** *auto*
from s' **have** $\Gamma s = \exp (\ln \Gamma s)$
unfolding *Gamma-complex-altdef* **using** *nonpos-Ints-subset-nonpos-Reals* **by**
auto
also from s' **have** $\ln \Gamma s = (s-1/2) * \text{Ln } s - s + \text{complex-of-real } (\ln$
 $(2 * \pi) / 2) - I s$
by (*subst ln-Gamma-stirling-complex*[of - 1]) (*simp-all add: exp-add exp-diff*
I-def)
also have $\exp \dots = \exp ((s - 1 / 2) * \text{Ln } s) / \exp s *$
 $\exp (\text{complex-of-real } (\ln (2 * \pi) / 2)) / \exp (I s)$
unfolding *exp-diff exp-add* **by** (*simp add: exp-diff exp-add*)
also have $\exp ((s - 1 / 2) * \text{Ln } s) = s \text{ powr } (s - 1 / 2)$
by (*simp add: powr-def*)
also have $\exp (\text{complex-of-real } (\ln (2 * \pi) / 2)) = \text{sqrt } (2 * \pi)$
by (*subst exp-of-real*) (*auto simp: powr-def simp flip: powr-half-sqrt*)
also have $\exp s = \exp 1 \text{ powr } s$
by (*simp add: powr-def*)
also have $s \text{ powr } (s - 1 / 2) / \exp 1 \text{ powr } s = (s \text{ powr } s / \exp 1 \text{ powr } s) / s$
 $\text{powr } (1/2)$
by (*subst powr-diff*) *auto*
also have $*$: $\text{Ln } (s / \exp 1) = \text{Ln } s - 1$
using *Ln-divide-of-real*[of $\exp 1$ s] **by** (*simp flip: exp-of-real*)
hence $s \text{ powr } s / \exp 1 \text{ powr } s = (s / \exp 1) \text{ powr } s$
unfolding *powr-def* **by** (*subst **) (*auto simp: exp-diff field-simps*)
finally show $\Gamma s = \text{sqrt } (2 * \pi) * (s / \exp 1) \text{ powr } s / s \text{ powr } (1 / 2)$
 $/ \exp (I s)$
by (*simp add: algebra-simps*)
qed
hence $\Gamma s \sim[F] (\lambda s. \text{sqrt } (2 * \pi) * (s / \exp 1) \text{ powr } s / s \text{ powr } (1 / 2) /$
 $\exp (I s))$
by (*rule asymp-equiv-refl-ev*)
also have $\dots \sim[F] (\lambda s. \text{sqrt } (2 * \pi) * (s / \exp 1) \text{ powr } s / s \text{ powr } (1 / 2) / 1)$
proof (*intro asymp-equiv-intros*)
have $F \leq \text{principal } (\text{complex-cone}' \alpha - \{0\})$
unfolding *le-principal F-def eventually-inf-principal*
using *eventually-not-equal-at-infinity* **by** *eventually-elim auto*
moreover have $I \in O[\text{principal } (\text{complex-cone}' \alpha - \{0\})](\lambda s. 1 / s)$
using *stirling-integral-bigo*[of α 1] α **unfolding** *F-def* **by** (*simp add: I-def*)
ultimately have $I \in O[F](\lambda s. 1 / s)$
by (*rule landau-o.big.filter-mono*)
also have $(\lambda s. 1 / s) \in o[F](\lambda s. 1)$
proof (*rule landau-o.smallI*)
fix $c :: \text{real}$
assume $c: c > 0$
hence *eventually* $(\lambda z::\text{complex}. \text{norm } z \geq 1 / c)$ *at-infinity*
by (*auto simp: eventually-at-infinity*)
moreover have *eventually* $(\lambda z::\text{complex}. z \neq 0)$ *at-infinity*
by (*rule eventually-not-equal-at-infinity*)

ultimately show *eventually* $(\lambda z :: \text{complex. norm } (1 / z) \leq c * \text{norm } (1 :: \text{complex})) F$
unfolding *F-def eventually-inf-principal*
by *eventually-elim (use <c > 0> in <auto simp: norm-divide field-simps>)*
qed
finally have $I \in o[F](\lambda s. 1)$.
from *smalloD-tendsto[OF this]* **have** *[tendsto-intros]:* $(I \longrightarrow 0) F$
by *simp*
show $(\lambda x. \text{exp } (I x)) \sim[F] (\lambda x. 1)$
by *(rule asymp-equivI' tendsto-eq-intros refl | simp)+*
qed
finally show *?thesis* **by** *simp*
qed
end
end

References

- [1] B. Berndt. *Rudiments of the Theory of the Gamma Function*. University of Chicago, 1976.
- [2] A. Erdélyi. *Asymptotic Expansions*. Dover Publications, 1956.
- [3] G. J. O. Jameson. A simple proof of Stirling’s formula for the Gamma function. *The Mathematical Gazette*, 99:68–74, 3 2015.