

Stewart's Theorem and Apollonius' Theorem

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December 9, 2023

Abstract

This entry formalizes the two geometric theorems, Stewart's and Apollonius' theorem. Stewart's Theorem [3] relates the length of a triangle's cevian to the lengths of the triangle's two sides. Apollonius' Theorem [2] is a specialisation of Stewart's theorem, restricting the cevian to be the median. The proof applies the law of cosines, some basic geometric facts about triangles and then simply transforms the terms algebraically to yield the conjectured relation. The formalization in Isabelle can closely follow the informal proofs described in the Wikipedia articles of those two theorems.

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1 Stewart's Theorem and Apollonius' Theorem

theory *Stewart-Apollonius*

imports

Triangle.Triangle

begin

1.1 Stewart's Theorem

theorem *Stewart*:

fixes $A B C D :: 'a::euclidean-space$

assumes *between* $(B, C) D$

assumes $a = \text{dist } B C$

assumes $b = \text{dist } A C$

assumes $c = \text{dist } B A$

assumes $d = \text{dist } A D$

assumes $m = \text{dist } B D$

assumes $n = \text{dist } C D$

shows $b^2 * m + c^2 * n = a * (d^2 + m * n)$

```

proof (cases)
  assume  $B \neq D \wedge C \neq D$ 
  let  $?\vartheta = \text{angle } B D A$ 
  let  $?\vartheta' = \text{angle } A D C$ 
  from  $\langle B \neq D \wedge C \neq D \rangle$  between  $- \rightarrow$  have  $\text{cos: } \text{cos } ?\vartheta' = - \text{cos } ?\vartheta$ 
    by (auto simp add: angle-inverse[of B C D] angle-commute[of A D C])
  from  $\langle \text{between } - \rightarrow \rangle$  have  $m + n = a$ 
    unfolding  $\langle a = - \rangle \langle m = - \rangle \langle n = - \rangle$ 
    by (metis (no-types) between dist-commute)
  have  $c^2 = m^2 + d^2 - 2 * d * m * \text{cos } ?\vartheta$ 
    unfolding  $\langle c = - \rangle \langle m = - \rangle \langle d = - \rangle$ 
    by (simp add: cosine-law-triangle[of B A D] dist-commute[of D A] dist-commute[of
D B])
  moreover have  $b^2 = n^2 + d^2 + 2 * d * n * \text{cos } ?\vartheta$ 
    unfolding  $\langle b = - \rangle \langle n = - \rangle \langle d = - \rangle$ 
    by (simp add: cosine-law-triangle[of A C D] cos dist-commute[of D A] dist-commute[of
D C])
  ultimately have  $b^2 * m + c^2 * n = n * m^2 + n^2 * m + (m + n) * d^2$  by
algebra
  also have  $\dots = (m + n) * (m * n + d^2)$  by algebra
  also from  $\langle m + n = a \rangle$  have  $\dots = a * (d^2 + m * n)$  by simp
  finally show ?thesis .
next
  assume  $\neg (B \neq D \wedge C \neq D)$ 
  from this assms show ?thesis by (auto simp add: dist-commute)
qed

```

Here is an equivalent formulation that is probably more suitable for further use in other geometry theories in Isabelle.

```

theorem Stewart':
  fixes  $A B C D :: 'a::\text{euclidean-space}$ 
  assumes between (B, C) D
  shows  $(\text{dist } A C)^2 * \text{dist } B D + (\text{dist } B A)^2 * \text{dist } C D = \text{dist } B C * ((\text{dist } A
D)^2 + \text{dist } B D * \text{dist } C D)$ 
  using assms by (auto intro: Stewart)

```

1.2 Apollonius' Theorem

Apollonius' theorem is a simple specialisation of Stewart's theorem, but historically predated Stewart's theorem by many centuries.

```

lemma Apollonius:
  fixes  $A B C :: 'a::\text{euclidean-space}$ 
  assumes  $B \neq C$ 
  assumes  $b = \text{dist } A C$ 
  assumes  $c = \text{dist } B A$ 
  assumes  $d = \text{dist } A$  (midpoint B C)
  assumes  $m = \text{dist } B$  (midpoint B C)
  shows  $b^2 + c^2 = 2 * (m^2 + d^2)$ 

```

```

proof –
  from  $\langle B \neq C \rangle$  have  $m \neq 0$ 
    unfolding  $\langle m = - \rangle$  using midpoint-eq-endpoint(1) by fastforce
  have between  $(B, C)$  (midpoint B C)
    by (simp add: between-midpoint)
  moreover have  $\text{dist } C \text{ (midpoint } B \ C) = \text{dist } B \text{ (midpoint } B \ C)$ 
    by (simp add: dist-midpoint)
  moreover have  $\text{dist } B \ C = 2 * \text{dist } B \text{ (midpoint } B \ C)$ 
    by (simp add: dist-midpoint)
  moreover note assms(2-5)
  ultimately have  $b^2 * m + c^2 * m = (2 * m) * (m^2 + d^2)$ 
    by (auto dest!: Stewart[where a=2 * m] simp add: power2-eq-square)
  from this have  $m * (b^2 + c^2) = m * (2 * (m^2 + d^2))$ 
    by (simp add: distrib-left semiring-normalization-rules(7))
  from this  $\langle m \neq 0 \rangle$  show ?thesis by auto
qed

```

Here is the equivalent formulation that is probably more suitable for further use in other geometry theories in Isabelle.

```

lemma Apollonius':
  fixes  $A \ B \ C :: 'a::euclidean-space$ 
  assumes  $B \neq C$ 
  shows  $(\text{dist } A \ C)^2 + (\text{dist } B \ A)^2 = 2 * ((\text{dist } B \ \text{midpoint } B \ C))^2 + (\text{dist } A \ \text{midpoint } B \ C)^2$ 
  using assms by (rule Apollonius) auto

end

```

References

- [1] D. B. Surowski. Advanced high-school mathematics, 2011. <https://www.math.ksu.edu/~dbski/writings/further.pdf> [Online; accessed 30-July-2017].
- [2] Wikipedia. Apollonius' theorem — wikipedia, the free encyclopedia, 2017. https://en.wikipedia.org/w/index.php?title=Apollonius%27_theorem&oldid=790659235 [Online; accessed 30-July-2017].
- [3] Wikipedia. Stewart's theorem — wikipedia, the free encyclopedia, 2017. https://en.wikipedia.org/w/index.php?title=Stewart%27s_theorem&oldid=790777285 [Online; accessed 30-July-2017].