Stewart’s Theorem and Apollonius’ Theorem

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Abstract

This entry formalizes the two geometric theorems, Stewart’s and Apollonius’ theorem. Stewart’s Theorem [3] relates the length of a triangle’s cevian to the lengths of the triangle’s two sides. Apollonius’ Theorem [2] is a specialization of Stewart’s theorem, restricting the cevian to be the median. The proof applies the law of cosines, some basic geometric facts about triangles and then simply transforms the terms algebraically to yield the conjectured relation. The formalization in Isabelle can closely follow the informal proofs described in the Wikipedia articles of those two theorems.

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1 Stewart’s Theorem and Apollonius’ Theorem

theory Stewart-Apollonius

imports Triangle.Triangle

begin

1.1 Stewart’s Theorem

theorem Stewart:
  fixes A B C D :: 'a::euclidean-space
  assumes between (B, C) D
  assumes a = dist B C
  assumes b = dist A C
  assumes c = dist B A
  assumes d = dist A D
  assumes m = dist B D
  assumes n = dist C D
  shows b^2 * m + c^2 * n = a * (d^2 + m * n)
proof (cases)
 assume $B \neq D \land C \neq D$
 let $?\theta = \text{angle } B \, D \, A$
 let $?\theta' = \text{angle } A \, D \, C$
 from $\langle B \neq D \land C \neq D \rangle \, \langle \text{between } - - \rangle$ have $\cos ?\theta' = - \cos ?\theta$
  by (auto simp add: angle-inverse[of $B \, C \, D$] angle-commute[of $A \, D \, C$])
 from $\langle \text{between } - - \rangle$ have $m + n = a$
  unfolding $\langle a = - \rangle \, \langle m = - \rangle \, \langle n = - \rangle$
  by (metis (no-types) between dist-commute)
 have $c^2 = m^2 + d^2 - 2 * d * m * \cos ?\theta$
  unfolding $\langle c = - \rangle \, \langle m = - \rangle \, \langle d = - \rangle$
 moreover have $b^2 = n^2 + d^2 + 2 * d * n * \cos ?\theta$
  unfolding $\langle b = - \rangle \, \langle n = - \rangle \, \langle d = - \rangle$
 ultimately have $b^2 * m + c^2 * n = n * m^2 + n^2 * m + (m + n) * d^2$ by algebra
 also have $\ldots = (m + n) * (m * n + d^2)$ by algebra
 also from $\langle m + n = a \rangle$ have $\ldots = a * (d^2 + m * n)$ by simp
 finally show $?\text{thesis }$.

next
 assume $\neg (B \neq D \land C \neq D)$
 from this assms show $?\text{thesis}$ by (auto simp add: dist-commute)
qed

Here is an equivalent formulation that is probably more suitable for further use in other geometry theories in Isabelle.

**Theorem Stewart'**:
 fixes $A \, B \, C \, D :: 'a::euclidean-space$
 assumes $\langle \text{between } (B, \, C) \, D \rangle$
 shows $\langle (\text{dist } A \, C)^2 * \text{dist } B \, D + (\text{dist } B \, A)^2 * \text{dist } C \, D = \text{dist } B \, C * ((\text{dist } A \, D)^2 + \text{dist } B \, D * \text{dist } C \, D) \rangle$
 using assms by (auto intro: Stewart)

### 1.2 Apollonius’ Theorem

Apollonius’ theorem is a simple specialisation of Stewart’s theorem, but historically predated Stewart’s theorem by many centuries.

**Lemma Apollonius**:
 fixes $A \, B \, C :: 'a::euclidean-space$
 assumes $B \neq C$
 assumes $b = \text{dist } A \, C$
 assumes $c = \text{dist } B \, A$
 assumes $d = \text{dist } A \, (\text{midpoint } B \, C)$
 assumes $m = \text{dist } B \, (\text{midpoint } B \, C)$
 shows $b^2 + c^2 = 2 * (m^2 + d^2)$
proof  
  from \( B \neq C \) have \( m \neq 0 \)
  unfolding \( \langle m = \cdot \rangle \) using midpoint-eq-endpoint(1) by fastforce
  have between \((B, C)\) (midpoint \( B \ C \))
    by (simp add: between-midpoint)
  moreover have dist \( C \) (midpoint \( B \ C \)) = dist \( B \) (midpoint \( B \ C \))
    by (simp add: dist-midpoint)
  moreover have dist \( B \ C \) = 2 * dist (midpoint \( B \ C \))
    by (simp add: dist-midpoint)
  moreover note assms(2−5)
  ultimately have \( b^2 \ast m + c^2 \ast m = (2 \ast m) \ast (m^2 + d^2) \)
    by (auto dest!: Stewart[where \( a=2 \ast m \)] simp add: power2_eq_square)
  from this have \( m \ast (b^2 + c^2) = m \ast (2 \ast (m^2 + d^2)) \)
    by (simp add: distrib-left semiring-normalization-rules(7))
  from this \( m \neq 0 \) show \(?thesis\) by auto
qed

Here is the equivalent formulation that is probably more suitable for further
use in other geometry theories in Isabelle.

lemma Apollonius':
  fixes \( A \ B \ C \) :: 'a::euclidean-space
  assumes \( B \neq C \)
  shows \((dist \( A \ C \))^2 + (dist \( B \ A \))^2 = 2 \ast ((dist \( B \) (midpoint \( B \ C \)))^2 + (dist \( A \) (midpoint \( B \ C \)))^2)\)
  using assms by (rule Apollonius) auto

end

References

