

The Steiner Deltoid as the Tangent Envelope of Wallace–Simson Lines in Isabelle/HOL

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Abstract

We formalize the classical Steiner deltoid construction in Isabelle/HOL using only the published Archive of Formal Proofs entry for the Wallace–Simson line. In unit-circumcircle coordinates, the deltoid has an explicit complex parametrisation. We prove its derivative formula, characterize the stationary parameters by $m^3 = abc$, and certify them as ordinary cusps using the second and third derivatives. We then prove an equality of sets between the appropriate Wallace–Simson line and the tangent line, for every parameter, including the cusp cases.

The development uses complex coordinates and handles a vertex parameter by cycling to the alternate pair of feet.

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1 Introduction

The Steiner deltoid is the envelope of the Wallace–Simson lines of a fixed triangle as the point on its circumcircle varies. This classical relationship was treated early by Butchart [2] and given a later elementary proof by de Guzmán [4]; computational treatments of envelopes include [1]. The construction is especially transparent after placing the circumcentre at the origin and scaling the circumcircle to the unit circle. If the vertex coordinates are a, b, c and the moving point is m , the deltoid parametrisation

is

$$d(m) = \frac{a + b + c}{2} + m + \frac{abc \overline{m}^2}{2}.$$

The underlying Wallace–Simson theorem is the classical line theorem for the three perpendicular feet [6, 3]; its Isabelle/HOL formalization is available as the published AFP entry [5].

This entry deliberately depends only on the published AFP entry *The Wallace–Simson Line Theorem in Isabelle/HOL*. In particular, the perpendicular feet, circle identities, and line interface are imported from that entry rather than copied from a local development.

2 Formal setup

The definition *steiner_deltoid_point* gives the displayed parametrisation, while *steiner_deltoid* collects its values as m runs over the unit circle. For differential statements, the theory introduces *unit_circle_param* and the real curve *steiner_deltoid_curve*. Its derivative is

$$\gamma'(t) = i N(e^{it}), \quad N(m) = m - abc \overline{m}^2.$$

The tangent direction is defined piecewise: it is $iN(m)$ when $N(m) \neq 0$, and m when $N(m) = 0$. At a stationary parameter the second derivative is $-3m$, so this chosen direction is the same tangent direction. The proof uses the closed form for a perpendicular foot and the published identity for the difference of two feet on a common circle.

3 Main result

The derivative theorem gives $\gamma'(t) = iN(e^{it})$. On the unit circle, the normal vanishes exactly when $m^3 = abc$, and *steiner_deltoid_stationary_parameter_is_ordinary_cusp* proves that the stationary parameters have nonzero, real-linearly independent second and third derivatives. The development separately proves that these two derivatives are perpendicular, a stronger property in this particular curve. Thus the formal cusp claim uses the standard independence criterion rather than merely identifying zeros of the first derivative.

The central theorem *simson_line_eq_steiner_deltoid_tangent_at_all* states the result as an equality of sets. If $m \neq b$, the line through the AB and BC feet is exactly the tangent at $d(m)$; if $m = b$, the line through the BC and CA feet is exactly that tangent. The proof has separate regular and stationary branches. In the regular branch the Simson direction is perpendicular to $N(m)$; in the stationary branch the real ratio of the two feet differences

shows that the line has direction m , the direction of the first nonzero derivative. The parameter-level theorem *simson_line_eq_steiner_deltoid_tangent_all* transports this equality to the real parameter t .

At a stationary parameter, the theorem *steiner_deltoid_cusp_tangent_is_second_derivative* explicitly identifies the piecewise-defined cusp tangent with the line in the direction of $\gamma''(t) = -3m$. The older foot-membership results are named *simson_line_is_tangent_to_steiner_deltoid_at_regular* and *simson_line_is_tangent_to_steiner_deltoid_regular* to make their non-stationary hypotheses visible in the public API.

```
theory Steiner-Deltoid
imports Simson.Simson HOL-Analysis.Derivative
begin

hide-const (open) Linear-Algebra.collinear
```

4 The deltoid parametrisation

We use circumcentre coordinates with the circumcircle as the unit circle. In these coordinates the classical parametrisation of the Steiner deltoid is

$$d(m) = (a + b + c)/2 + m + abc\overline{m}^2/2.$$

The unit-circle hypothesis turns the last term into the more familiar $abc/(2m^2)$.

```
definition steiner-deltoid-point ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
where steiner-deltoid-point a b c m =
  (a + b + c) / 2 + m + a * b * c * (cnj m)2 / 2
```

```
definition steiner-deltoid ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex set
where steiner-deltoid a b c =
  {steiner-deltoid-point a b c m | m. oncircle 0 1 m}
```

```
definition steiner-deltoid-normal ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
where steiner-deltoid-normal a b c m = m - a * b * c * (cnj m)2
```

For the differential-geometric statements we use the standard real parameter on the unit circle. Written without conjugation, the curve is

$$\gamma(t) = (a + b + c)/2 + e^{it} + abc e^{-2it}/2.$$

At a regular parameter the tangent direction is $iN(m)$. At a stationary parameter we use the first nonzero direction, namely the second derivative $-3m$.

definition *unit-circle-param* :: *real* $\Rightarrow$ *complex*
where *unit-circle-param* *t* = *exp* (*t* *_R *i*)

definition *steiner-deltoid-curve* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *real* $\Rightarrow$ *complex*
where *steiner-deltoid-curve* *a b c t* =
 $(a + b + c) / 2 + \exp (t *_{\mathbb{R}} i) +$
 $a * b * c * (\text{conj} (\exp (t *_{\mathbb{R}} i)))^2 / 2$

definition *steiner-deltoid-tangent-direction-at* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex*
where *steiner-deltoid-tangent-direction-at* *a b c m* =
(if steiner-deltoid-normal a b c m = 0 then m
*else i * steiner-deltoid-normal a b c m)*

definition *steiner-deltoid-tangent* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *real* $\Rightarrow$ *complex set*
where *steiner-deltoid-tangent* *a b c t* =
 $\{z. \text{on-line } (\text{steiner-deltoid-curve } a b c t)$
 $(\text{steiner-deltoid-curve } a b c t +$
 $\text{steiner-deltoid-tangent-direction-at } a b c$
 $(\text{unit-circle-param } t)) z\}$

definition *steiner-deltoid-tangent-at* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex set*
where *steiner-deltoid-tangent-at* *a b c m* =
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a b c m)$
 $(\text{steiner-deltoid-point } a b c m +$
 $\text{steiner-deltoid-tangent-direction-at } a b c m) z\}$

lemma *unit-circle-param-oncircle*:
shows *oncircle* 0 1 (*unit-circle-param t*)
 $\langle \text{proof} \rangle$

lemma *steiner-deltoid-curve-eq-point*:
shows *steiner-deltoid-curve* *a b c t* =
 $\text{steiner-deltoid-point } a b c (\text{unit-circle-param } t)$
 $\langle \text{proof} \rangle$

lemma *steiner-deltoid-tangent-eq-tangent-at*:
shows *steiner-deltoid-tangent* *a b c t* =
 $\text{steiner-deltoid-tangent-at } a b c (\text{unit-circle-param } t)$
 $\langle \text{proof} \rangle$

lemma *steiner-deltoid-curve-has-vector-derivative*:
shows $((\lambda t. \text{steiner-deltoid-curve } a b c t)$
 $\text{has-vector-derivative}$
 $i * \text{steiner-deltoid-normal } a b c (\text{unit-circle-param } t)) (\text{at } t)$
 $\langle \text{proof} \rangle$

definition *steiner-deltoid-curve-second-derivative* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{complex}$
where *steiner-deltoid-curve-second-derivative* $a\ b\ c\ t =$
 $- \text{unit-circle-param } t -$
 $2 * a * b * c * (\text{cnj } (\text{unit-circle-param } t))^2$

definition *steiner-deltoid-curve-third-derivative* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{complex}$
where *steiner-deltoid-curve-third-derivative* $a\ b\ c\ t =$
 $- i * \text{unit-circle-param } t +$
 $4 * i * a * b * c * (\text{cnj } (\text{unit-circle-param } t))^2$

definition *complex-real-linearly-independent* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{bool}$
where *complex-real-linearly-independent* $u\ v \longleftrightarrow$
 $\neg \text{collinear } 0\ u\ v$

definition *ordinary-cusp-parameter* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{bool}$
where *ordinary-cusp-parameter* $a\ b\ c\ t \longleftrightarrow$
 $\text{vector-derivative } (\lambda u. \text{steiner-deltoid-curve } a\ b\ c\ u) \text{ (at } t) = 0$
 $\wedge \text{steiner-deltoid-curve-second-derivative } a\ b\ c\ t \neq 0$
 $\wedge \text{steiner-deltoid-curve-third-derivative } a\ b\ c\ t \neq 0$
 $\wedge \text{complex-real-linearly-independent}$
 $(\text{steiner-deltoid-curve-second-derivative } a\ b\ c\ t)$
 $(\text{steiner-deltoid-curve-third-derivative } a\ b\ c\ t)$

lemma *steiner-deltoid-curve-first-derivative-has-vector-derivative*:
shows $((\lambda t. i * \text{steiner-deltoid-normal } a\ b\ c\ (\text{unit-circle-param } t))$
 $\text{has-vector-derivative}$
 $\text{steiner-deltoid-curve-second-derivative } a\ b\ c\ t) \text{ (at } t)$
 $\langle \text{proof} \rangle$

lemma *steiner-deltoid-curve-second-derivative-has-vector-derivative*:
shows $((\lambda t. \text{steiner-deltoid-curve-second-derivative } a\ b\ c\ t)$
 $\text{has-vector-derivative}$
 $\text{steiner-deltoid-curve-third-derivative } a\ b\ c\ t) \text{ (at } t)$
 $\langle \text{proof} \rangle$

lemma *steiner-deltoid-point-mem*:
assumes $\text{oncircle } 0\ 1\ m$
shows $\text{steiner-deltoid-point } a\ b\ c\ m \in \text{steiner-deltoid } a\ b\ c$
 $\langle \text{proof} \rangle$

lemma *unit-cn timer*:
assumes $\text{oncircle } 0\ 1\ z$
shows $\text{cnj } z = 1 / z$

$\langle \text{proof} \rangle$

lemma *unit-nonzero*:

assumes *oncircle* 0 1 *z*

shows $z \neq 0$

$\langle \text{proof} \rangle$

lemma *steiner-deltoid-normal-eq-zero-iff-cube*:

assumes *hm*: *oncircle* 0 1 *m*

shows *steiner-deltoid-normal* *a b c m* = 0 $\longleftrightarrow$

$m^3 = a * b * c$

$\langle \text{proof} \rangle$

lemma *steiner-deltoid-normal-nonzero-iff-cube-neg*:

assumes *hm*: *oncircle* 0 1 *m*

shows *steiner-deltoid-normal* *a b c m* $\neq 0 \longleftrightarrow$

$m^3 \neq a * b * c$

$\langle \text{proof} \rangle$

theorem *steiner-deltoid-curve-derivative-zero-iff*:

shows *vector-derivative* (λt . *steiner-deltoid-curve* *a b c t*) (*at t*) = 0

$\longleftrightarrow$ (*unit-circle-param t*)³ = *a * b * c*

$\langle \text{proof} \rangle$

lemma *steiner-deltoid-curve-second-derivative-stationary*:

fixes *a b c* :: complex **and** *t* :: real

assumes *hcube*: (*unit-circle-param t*)³ = *a * b * c*

shows *steiner-deltoid-curve-second-derivative* *a b c t* =
 $-3 * \text{unit-circle-param } t$

$\langle \text{proof} \rangle$

lemma *steiner-deltoid-curve-third-derivative-stationary*:

fixes *a b c* :: complex **and** *t* :: real

assumes *hcube*: (*unit-circle-param t*)³ = *a * b * c*

shows *steiner-deltoid-curve-third-derivative* *a b c t* =
 $3 * i * \text{unit-circle-param } t$

$\langle \text{proof} \rangle$

lemma *complex-perpendicular-cusp-directions*:

assumes *hm*: *oncircle* 0 1 *m*

shows *complex-perpendicular* ($-3 * m$) ($3 * i * m$)

$\langle \text{proof} \rangle$

lemma *complex-perpendicular-imp-real-linearly-independent*:

fixes *u v* :: complex

assumes *hperp*: *complex-perpendicular* *u v*

and *hu*: *u* $\neq 0$ **and** *hv*: *v* $\neq 0$

shows *complex-real-linearly-independent* *u v*

$\langle \text{proof} \rangle$

theorem *steiner-deltoid-stationary-parameter-is-ordinary-cusp:*

fixes $a\ b\ c :: \text{complex}$ **and** $t :: \text{real}$
assumes $hcube: (\text{unit-circle-param } t) ^ 3 = a * b * c$
shows *ordinary-cusp-parameter* $a\ b\ c\ t$
 $\langle \text{proof} \rangle$

lemma *foot-unit:*

fixes $a\ b\ m :: \text{complex}$
assumes $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $ab: a \neq b$
shows $\text{foot } a\ b\ m = (a + b + m - a * b * \text{cnj } m) / 2$
 $\langle \text{proof} \rangle$

lemma *deltoid-point-minus-foot:*

fixes $a\ b\ c\ m :: \text{complex}$
assumes $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $hc: \text{oncircle } 0\ 1\ c$ **and** $hm: \text{oncircle } 0\ 1\ m$
and $ab: a \neq b$
shows $\text{steiner-deltoid-point } a\ b\ c\ m - \text{foot } a\ b\ m =$
 $(c + m + a * b * \text{cnj } m + a * b * c * (\text{cnj } m)^2) / 2$
 $\langle \text{proof} \rangle$

lemma *feet-difference-unit:*

fixes $a\ b\ c\ m :: \text{complex}$
assumes $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $hc: \text{oncircle } 0\ 1\ c$ **and** $hm: \text{oncircle } 0\ 1\ m$
and $ab: a \neq b$ **and** $bc: b \neq c$
shows $\text{foot } b\ c\ m - \text{foot } a\ b\ m =$
 $(c - a) * (1 - b * \text{cnj } m) / 2$
 $\langle \text{proof} \rangle$

lemma *cusp-feet-difference-ratio-real:*

fixes $a\ b\ c\ m :: \text{complex}$
assumes $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $hc: \text{oncircle } 0\ 1\ c$ **and** $hm: \text{oncircle } 0\ 1\ m$
and $ab: a \neq b$ **and** $bc: b \neq c$
and $hcube: m ^ 3 = a * b * c$
shows $(\text{foot } b\ c\ m - \text{foot } a\ b\ m) / m \in \mathbb{R}$
 $\langle \text{proof} \rangle$

lemma *deltoid-cross-identity:*

fixes $a\ b\ c\ m :: \text{complex}$
assumes $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $hc: \text{oncircle } 0\ 1\ c$ **and** $hm: \text{oncircle } 0\ 1\ m$
shows
 $\text{cnj } ((c + m + a * b * \text{cnj } m +$
 $a * b * c * (\text{cnj } m)^2) / 2) *$
 $((c - a) * (1 - b * \text{cnj } m) / 2) =$

$$((c + m + a * b * \text{cnj } m + a * b * c * (\text{cnj } m)^2) / 2) * \text{cnj } ((c - a) * (1 - b * \text{cnj } m) / 2)$$
 <proof>

lemma *deltoid-point-collinear*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$ **and** $\text{hm}: \text{oncircle } 0 \ 1 \ m$
shows $\text{collinear } (\text{foot } a \ b \ m)$
 $(\text{steiner-deltoid-point } a \ b \ c \ m) (\text{foot } b \ c \ m)$
 <proof>

theorem *simson-line-contains-steiner-deltoid-point-pair*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$ **and** $\text{hm}: \text{oncircle } 0 \ 1 \ m$
and $\text{mb}: m \neq b$
shows $\text{on-line } (\text{foot } a \ b \ m) (\text{foot } b \ c \ m)$
 $(\text{steiner-deltoid-point } a \ b \ c \ m)$
 <proof>

theorem *simson-line-contains-steiner-deltoid-point*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$ **and** $\text{hm}: \text{oncircle } 0 \ 1 \ m$
shows $(m \neq b \longrightarrow \text{on-line } (\text{foot } a \ b \ m) (\text{foot } b \ c \ m) (\text{steiner-deltoid-point } a \ b \ c \ m)) \wedge$
 $(m \neq c \longrightarrow \text{on-line } (\text{foot } b \ c \ m) (\text{foot } c \ a \ m) (\text{steiner-deltoid-point } a \ b \ c \ m)) \wedge$
 $(m \neq a \longrightarrow \text{on-line } (\text{foot } c \ a \ m) (\text{foot } a \ b \ m) (\text{steiner-deltoid-point } a \ b \ c \ m))$
 <proof>

lemma *deltoid-normal-cross-identity*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$ **and** $\text{hm}: \text{oncircle } 0 \ 1 \ m$
shows

$$\text{cnj } (\text{cnj } ((c - a) * (1 - b * \text{cnj } m) / 2) * (m - a * b * c * (\text{cnj } m)^2)) =$$

$$-(\text{cnj } ((c - a) * (1 - b * \text{cnj } m) / 2) * (m - a * b * c * (\text{cnj } m)^2))$$

$\langle proof \rangle$

lemma *real-complex-mult*:

fixes $p\ q :: complex$

assumes $p \in \mathbb{R}$ **and** $q \in \mathbb{R}$

shows $p * q \in \mathbb{R}$

$\langle proof \rangle$

lemma *real-ratio-of-perpendicular*:

fixes $u\ n :: complex$

assumes $hperp$: *complex-perpendicular* $u\ n$ **and** hn : $n \neq 0$

shows $u / (i * n) \in \mathbb{R}$

$\langle proof \rangle$

lemma *on-line-of-collinear-perpendicular*:

fixes $x\ y\ z\ n :: complex$

assumes $hcol$: *collinear* $x\ z\ y$

and hxy : $x \neq y$

and $hperp$: *complex-perpendicular* $(y - x)\ n$

and hn : $n \neq 0$

shows *on-line* $x\ (x + i * n)\ z$

$\langle proof \rangle$

lemma *on-line-of-collinear-real-direction*:

fixes $x\ y\ z\ d :: complex$

assumes $hcol$: *collinear* $x\ z\ y$

and hxy : $x \neq y$

and $hratio2$: $(y - x) / d \in \mathbb{R}$

and hd : $d \neq 0$

shows *on-line* $x\ (x + d)\ z$

$\langle proof \rangle$

lemma *on-line-change-base*:

fixes $x\ y\ z\ d :: complex$

assumes hy : *on-line* $x\ (x + d)\ y$

and hz : *on-line* $x\ (x + d)\ z$

shows *on-line* $z\ (z + d)\ y$

$\langle proof \rangle$

lemma *on-line-set-eq-of-distinct*:

fixes $a\ b\ c\ d :: complex$

assumes hab : $a \neq b$ **and** hcd : $c \neq d$

and hc : *on-line* $a\ b\ c$ **and** hd : *on-line* $a\ b\ d$

shows $\{z. \text{on-line } a\ b\ z\} = \{z. \text{on-line } c\ d\ z\}$

$\langle proof \rangle$

lemma *on-line-eq-of-real-scaled-direction*:

fixes $x\ d :: complex$ **and** $r :: real$

assumes hd : $d \neq 0$ **and** hr : $r \neq 0$

shows $\{z. \text{on-line } x (x + d) z\} =$
 $\{z. \text{on-line } x (x + \text{complex-of-real } r * d) z\}$
 <proof>

theorem *steiner-deltoid-cusp-tangent-is-second-derivative*:
fixes $a \ b \ c :: \text{complex}$ **and** $t :: \text{real}$
assumes $hcube: (\text{unit-circle-param } t) ^ 3 = a * b * c$
shows $\text{steiner-deltoid-tangent-at } a \ b \ c (\text{unit-circle-param } t) =$
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a \ b \ c (\text{unit-circle-param } t))$
 $(\text{steiner-deltoid-point } a \ b \ c (\text{unit-circle-param } t) +$
 $\text{steiner-deltoid-curve-second-derivative } a \ b \ c \ t) z\}$
 <proof>

theorem *simson-line-perpendicular-to-steiner-deltoid-normal*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $hncol: \neg \text{collinear } a \ b \ c$
and $ha: \text{oncircle } 0 \ 1 \ a$ **and** $hb: \text{oncircle } 0 \ 1 \ b$
and $hc: \text{oncircle } 0 \ 1 \ c$ **and** $hm: \text{oncircle } 0 \ 1 \ m$
shows $\text{complex-perpendicular}$
 $(\text{foot } b \ c \ m - \text{foot } a \ b \ m)$
 $(\text{steiner-deltoid-normal } a \ b \ c \ m)$
 <proof>

theorem *simson-line-is-tangent-to-steiner-deltoid-at-regular*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $hncol: \neg \text{collinear } a \ b \ c$
and $ha: \text{oncircle } 0 \ 1 \ a$ **and** $hb: \text{oncircle } 0 \ 1 \ b$
and $hc: \text{oncircle } 0 \ 1 \ c$ **and** $hm: \text{oncircle } 0 \ 1 \ m$
and $hnormal: \text{steiner-deltoid-normal } a \ b \ c \ m \neq 0$
shows $(m \neq b \longrightarrow$
 $\text{foot } a \ b \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$
 $\text{foot } b \ c \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$
 $\text{foot } a \ b \ m \neq \text{foot } b \ c \ m) \wedge$
 $(m = b \longrightarrow$
 $\text{foot } b \ c \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$
 $\text{foot } c \ a \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$
 $\text{foot } b \ c \ m \neq \text{foot } c \ a \ m)$
 <proof>

theorem *simson-line-eq-steiner-deltoid-tangent-at*:
fixes $a \ b \ c \ m :: \text{complex}$
assumes $hncol: \neg \text{collinear } a \ b \ c$
and $ha: \text{oncircle } 0 \ 1 \ a$ **and** $hb: \text{oncircle } 0 \ 1 \ b$
and $hc: \text{oncircle } 0 \ 1 \ c$ **and** $hm: \text{oncircle } 0 \ 1 \ m$
and $hnormal: \text{steiner-deltoid-normal } a \ b \ c \ m \neq 0$
shows $(m \neq b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } a \ b \ m) (\text{foot } b \ c \ m) z\} =$
 $\text{steiner-deltoid-tangent-at } a \ b \ c \ m) \wedge$
 $(m = b \longrightarrow$

$\{z. \text{on-line } (\text{foot } b \ c \ m) (\text{foot } c \ a \ m) \ z\} =$
 $\text{steiner-deltoid-tangent-at } a \ b \ c \ m)$
 <proof>

theorem *simson-line-eq-steiner-deltoid-tangent-at-all:*

fixes $a \ b \ c \ m :: \text{complex}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$ **and** $\text{hm}: \text{oncircle } 0 \ 1 \ m$
shows $(m \neq b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } a \ b \ m) (\text{foot } b \ c \ m) \ z\} =$
 $\text{steiner-deltoid-tangent-at } a \ b \ c \ m) \wedge$
 $(m = b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } b \ c \ m) (\text{foot } c \ a \ m) \ z\} =$
 $\text{steiner-deltoid-tangent-at } a \ b \ c \ m)$
 <proof>

theorem *simson-line-eq-steiner-deltoid-tangent-all:*

fixes $a \ b \ c :: \text{complex}$ **and** $t :: \text{real}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$
shows $(\text{unit-circle-param } t \neq b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } a \ b \ (\text{unit-circle-param } t))$
 $(\text{foot } b \ c \ (\text{unit-circle-param } t)) \ z\} =$
 $\text{steiner-deltoid-tangent } a \ b \ c \ t) \wedge$
 $(\text{unit-circle-param } t = b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } b \ c \ (\text{unit-circle-param } t))$
 $(\text{foot } c \ a \ (\text{unit-circle-param } t)) \ z\} =$
 $\text{steiner-deltoid-tangent } a \ b \ c \ t)$
 <proof>

theorem *simson-line-is-tangent-to-steiner-deltoid-regular:*

fixes $a \ b \ c :: \text{complex}$ **and** $t :: \text{real}$
assumes $\text{hncol}: \neg \text{collinear } a \ b \ c$
and $\text{ha}: \text{oncircle } 0 \ 1 \ a$ **and** $\text{hb}: \text{oncircle } 0 \ 1 \ b$
and $\text{hc}: \text{oncircle } 0 \ 1 \ c$
and $\text{hregular}: (\text{unit-circle-param } t)^\wedge 3 \neq a * b * c$
shows $(\text{unit-circle-param } t \neq b \longrightarrow$
 $\text{foot } a \ b \ (\text{unit-circle-param } t) \in \text{steiner-deltoid-tangent } a \ b \ c \ t \wedge$
 $\text{foot } b \ c \ (\text{unit-circle-param } t) \in \text{steiner-deltoid-tangent } a \ b \ c \ t \wedge$
 $\text{foot } a \ b \ (\text{unit-circle-param } t) \neq \text{foot } b \ c \ (\text{unit-circle-param } t)) \wedge$
 $(\text{unit-circle-param } t = b \longrightarrow$
 $\text{foot } b \ c \ (\text{unit-circle-param } t) \in \text{steiner-deltoid-tangent } a \ b \ c \ t \wedge$
 $\text{foot } c \ a \ (\text{unit-circle-param } t) \in \text{steiner-deltoid-tangent } a \ b \ c \ t \wedge$
 $\text{foot } b \ c \ (\text{unit-circle-param } t) \neq \text{foot } c \ a \ (\text{unit-circle-param } t))$
 <proof>

theorem *simson-line-meets-steiner-deltoid:*

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fixes  $a\ b\ c\ m :: \text{complex}$ 
assumes  $hncol: \neg \text{collinear } a\ b\ c$ 
  and  $ha: \text{oncircle } 0\ 1\ a$  and  $hb: \text{oncircle } 0\ 1\ b$ 
  and  $hc: \text{oncircle } 0\ 1\ c$  and  $hm: \text{oncircle } 0\ 1\ m$ 
shows  $\text{steiner-deltoid-point } a\ b\ c\ m \in \text{steiner-deltoid } a\ b\ c \wedge$ 
   $((m \neq b \longrightarrow$ 
     $\text{on-line } (\text{foot } a\ b\ m)\ (\text{foot } b\ c\ m)$ 
     $(\text{steiner-deltoid-point } a\ b\ c\ m)) \wedge$ 
   $(m \neq c \longrightarrow$ 
     $\text{on-line } (\text{foot } b\ c\ m)\ (\text{foot } c\ a\ m)$ 
     $(\text{steiner-deltoid-point } a\ b\ c\ m)) \wedge$ 
   $(m \neq a \longrightarrow$ 
     $\text{on-line } (\text{foot } c\ a\ m)\ (\text{foot } a\ b\ m)$ 
     $(\text{steiner-deltoid-point } a\ b\ c\ m)))$ 
   $\langle \text{proof} \rangle$ 

end

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