

Standard Borel Spaces

Michikazu Hirata

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Abstract

This entry includes a formalization of standard Borel spaces and (a variant of) the Borel isomorphism theorem. A separable complete metrizable topological space is called a polish space and a measurable space generated from a polish space is called a standard Borel space. We formalize the notion of standard Borel spaces by establishing set-based metric spaces, and then prove (a variant of) the Borel isomorphism theorem. The theorem states that a standard Borel space is either a countable discrete space or isomorphic to \mathbb{R} .

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We refer to the HOL-Analysis library, the textbooks by Matsuzaka [2] and Srivastava [3], and the lecture note by Biskup [1].

1 Lemmas

```
theory Lemmas-StandardBorel
  imports HOL-Probability.Probability
begin
```

1.1 Lemmas for Abstract Topology

1.1.1 Generated By

```
lemma topology-generated-by-sub:
  assumes  $\bigwedge U. U \in \mathcal{U} \implies (\text{openin } X \ U)$ 
  and  $\text{openin } (\text{topology-generated-by } \mathcal{U}) \ U$ 
  shows  $\text{openin } X \ U$ 
<proof>
```

```
lemma topology-generated-by-open:
   $S = \text{topology-generated-by } \{U \mid U . \text{openin } S \ U\}$ 
<proof>
```

```
lemma topology-generated-by-eq:
  assumes  $\bigwedge U. U \in \mathcal{U} \implies (\text{openin } (\text{topology-generated-by } \mathcal{O}) \ U)$ 
  and  $\bigwedge U. U \in \mathcal{O} \implies (\text{openin } (\text{topology-generated-by } \mathcal{U}) \ U)$ 
  shows  $\text{topology-generated-by } \mathcal{O} = \text{topology-generated-by } \mathcal{U}$ 
<proof>
```

```
lemma topology-generated-by-homeomorphic-spaces:
  assumes  $\text{homeomorphic-map } X \ Y \ f \ X = \text{topology-generated-by } \mathcal{O}$ 
  shows  $Y = \text{topology-generated-by } ((\cdot) \ f \ \mathcal{O})$ 
```

<proof>

lemma *open-map-generated-topo:*

assumes $\bigwedge u. u \in U \implies \text{openin } S (f^{-1} u) \text{ inj-on } f (\text{topspace } (\text{topology-generated-by } U))$

shows *open-map* (*topology-generated-by* U) S f

<proof>

lemma *subtopology-generated-by:*

subtopology (*topology-generated-by* \mathcal{O}) $T = \text{topology-generated-by } \{T \cap U \mid U. U \in \mathcal{O}\}$

<proof>

lemma *prod-topology-generated-by:*

topology-generated-by $\{U \times V \mid U \in \mathcal{O} \wedge V \in \mathcal{U}\} = \text{prod-topology} (\text{topology-generated-by } \mathcal{O}) (\text{topology-generated-by } \mathcal{U})$

<proof>

lemma *prod-topology-generated-by-open:*

prod-topology S $S' = \text{topology-generated-by } \{U \times V \mid U \in \mathcal{O} \wedge V \in \mathcal{U}. \text{openin } S U \wedge \text{openin } S' V\}$

<proof>

lemma *product-topology-cong:*

assumes $\bigwedge i. i \in I \implies S_i = K_i$

shows *product-topology* S $I = \text{product-topology } K$ I

<proof>

lemma *topology-generated-by-without-empty:*

topology-generated-by $\mathcal{O} = \text{topology-generated-by } \{U \in \mathcal{O}. U \neq \{\}\}$

<proof>

lemma *topology-from-bij:*

assumes *bij-betw* f A (*topspace* S)

shows *homeomorphic-map* (*pullback-topology* A f S) S f *topspace* (*pullback-topology* A f S) = A

<proof>

lemma *openin-pullback-topology':*

assumes *bij-betw* f A (*topspace* S)

shows *openin* (*pullback-topology* A f S) $u \iff (\text{openin } S (f^{-1} u)) \wedge u \subseteq A$

<proof>

1.1.2 Isolated Point

definition *isolated-points-of* :: 'a topology \Rightarrow 'a set \Rightarrow 'a set (**infixr** *isolated'-points'-of* 80) **where**

$X \text{ isolated-points-of } A \equiv \{x \in \text{topspace } X \cap A. x \notin X \text{ derived-set-of } A\}$

lemma *isolated-points-of-eq*:

$X \text{ isolated-points-of } A = \{x \in \text{topspace } X \cap A. \exists U. x \in U \wedge \text{openin } X \ U \wedge U \cap (A - \{x\}) = \{\}\}$
(*proof*)

lemma *in-isolated-points-of*:

$x \in X \text{ isolated-points-of } A \longleftrightarrow x \in \text{topspace } X \wedge x \in A \wedge (\exists U. x \in U \wedge \text{openin } X \ U \wedge U \cap (A - \{x\}) = \{\})$
(*proof*)

lemma *derived-set-of-eq*:

$x \in X \text{ derived-set-of } A \longleftrightarrow x \in X \text{ closure-of } (A - \{x\})$
(*proof*)

1.1.3 Perfect Set

definition *perfect-set* :: 'a topology \Rightarrow 'a set \Rightarrow bool **where**

perfect-set $X \ A \longleftrightarrow \text{closedin } X \ A \wedge X \text{ isolated-points-of } A = \{\}$

abbreviation *perfect-space* $X \equiv \text{perfect-set } X \ (\text{topspace } X)$

lemma *perfect-space-euclidean*: *perfect-space* (*euclidean* :: 'a :: *perfect-space topology*)

(*proof*)

lemma *perfect-setI*:

assumes *closedin* $X \ A$

and $\bigwedge x \ T. \llbracket x \in A; x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T \wedge y \in A$

shows *perfect-set* $X \ A$

(*proof*)

lemma *perfect-spaceI*:

assumes $\bigwedge x \ T. \llbracket x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T$

shows *perfect-space* X

(*proof*)

lemma *perfect-setD*:

assumes *perfect-set* $X \ A$

shows $\text{closedin } X \ A \subseteq \text{topspace } X \ \bigwedge x \ T. \llbracket x \in A; x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T \wedge y \in A$

(*proof*)

lemma *perfect-space-perfect*:

perfect-set euclidean (*UNIV* :: 'a :: *perfect-space set*)

(*proof*)

lemma *perfect-set-subtopology*:

assumes *perfect-set* $X \ A$

shows *perfect-space* (*subtopology* $X \ A$)

<proof>

1.1.4 Bases and Sub-Bases in Abstract Topology

definition *subbase-in* :: [*'a topology, 'a set set*] \Rightarrow *bool* **where**
subbase-in *S* $\mathcal{O} \longleftrightarrow S = \text{topology-generated-by } \mathcal{O}$

definition *base-in* :: [*'a topology, 'a set set*] \Rightarrow *bool* **where**
base-in *S* $\mathcal{O} \longleftrightarrow (\forall U. \text{openin } S U \longleftrightarrow (\exists \mathcal{U}. U = \bigcup \mathcal{U} \wedge \mathcal{U} \subseteq \mathcal{O}))$

lemma *second-countable-base-in*: *second-countable* *S* $\longleftrightarrow (\exists \mathcal{O}. \text{countable } \mathcal{O} \wedge \text{base-in } S \mathcal{O})$
<proof>

definition *zero-dimensional* :: [*'a topology*] \Rightarrow *bool* **where**
zero-dimensional *S* $\longleftrightarrow (\exists \mathcal{O}. \text{base-in } S \mathcal{O} \wedge (\forall u \in \mathcal{O}. \text{openin } S u \wedge \text{closedin } S u))$

lemma *openin-base*:
assumes *base-in* *S* \mathcal{O} $U = \bigcup \mathcal{U}$ **and** $\mathcal{U} \subseteq \mathcal{O}$
shows *openin* *S* *U*
<proof>

lemma *base-is-subbase*:
assumes *base-in* *S* \mathcal{O}
shows *subbase-in* *S* \mathcal{O}
<proof>

lemma *subbase-in-subset*:
assumes *subbase-in* *S* \mathcal{O} **and** $U \in \mathcal{O}$
shows $U \subseteq \text{topspace } S$
<proof>

lemma *subbase-in-openin*:
assumes *subbase-in* *S* \mathcal{O} **and** $U \in \mathcal{O}$
shows *openin* *S* *U*
<proof>

lemma *base-in-subset*:
assumes *base-in* *S* \mathcal{O} **and** $U \in \mathcal{O}$
shows $U \subseteq \text{topspace } S$
<proof>

lemma *base-in-openin*:
assumes *base-in* *S* \mathcal{O} **and** $U \in \mathcal{O}$
shows *openin* *S* *U*
<proof>

lemma *base-in-def2*:
assumes $\bigwedge U. U \in \mathcal{O} \implies \text{openin } S U$

shows $\text{base-in } S \mathcal{O} \longleftrightarrow (\forall U. \text{openin } S U \longrightarrow (\forall x \in U. \exists W \in \mathcal{O}. x \in W \wedge W \subseteq U))$
 ⟨proof⟩

lemma *base-in-def2'*:
 $\text{base-in } S \mathcal{O} \longleftrightarrow (\forall b \in \mathcal{O}. \text{openin } S b) \wedge (\forall x. \text{openin } S x \longrightarrow (\exists B' \subseteq \mathcal{O}. \bigcup B' = x))$
 ⟨proof⟩

corollary *base-in-in-subset*:
assumes $\text{base-in } S \mathcal{O} \text{ openin } S u \ x \in u$
shows $\exists v \in \mathcal{O}. x \in v \wedge v \subseteq u$
 ⟨proof⟩

lemma *base-in-without-empty*:
assumes $\text{base-in } S \mathcal{O}$
shows $\text{base-in } S \{U \in \mathcal{O}. U \neq \{\}\}$
 ⟨proof⟩

lemma *second-countable-ex-without-empty*:
assumes $\text{second-countable } S$
shows $\exists \mathcal{O}. \text{countable } \mathcal{O} \wedge \text{base-in } S \mathcal{O} \wedge (\forall U \in \mathcal{O}. U \neq \{\})$
 ⟨proof⟩

lemma *subtopology-subbase-in*:
assumes $\text{subbase-in } S \mathcal{O}$
shows $\text{subbase-in } (\text{subtopology } S T) \{T \cap U \mid U. U \in \mathcal{O}\}$
 ⟨proof⟩

lemma *subtopology-base-in*:
assumes $\text{base-in } S \mathcal{O}$
shows $\text{base-in } (\text{subtopology } S T) \{T \cap U \mid U. U \in \mathcal{O}\}$
 ⟨proof⟩

lemma *second-countable-subtopology*:
assumes $\text{second-countable } S$
shows $\text{second-countable } (\text{subtopology } S T)$
 ⟨proof⟩

lemma *open-map-with-base*:
assumes $\text{base-in } S \mathcal{O} \wedge A. A \in \mathcal{O} \implies \text{openin } S' (f \text{ ` } A)$
shows $\text{open-map } S S' f$
 ⟨proof⟩

Construct a base from a subbase.

lemma *finite'-intersection-of-idempot* [simp]:
 $\text{finite' intersection-of finite' intersection-of } P = \text{finite' intersection-of } P$
 ⟨proof⟩

lemma *finite'-intersection-of-countable*:

assumes *countable* \mathcal{O}

shows *countable* (*Collect* (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)))

<proof>

lemma *finite'-intersection-of-openin*:

assumes (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)) *U*

shows *openin* (*topology-generated-by* \mathcal{O}) *U*

<proof>

lemma *topology-generated-by-finite-intersections*:

topology-generated-by $\mathcal{O} = \textit{topology-generated-by}$ (*Collect* (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)))

<proof>

lemma *base-from-subbase*:

assumes *subbase-in* S \mathcal{O}

shows *base-in* S (*Collect* (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)))

<proof>

lemma *countable-base-from-countable-subbase*:

assumes *countable* \mathcal{O} **and** *subbase-in* S \mathcal{O}

shows *second-countable* S

<proof>

lemma *prod-topology-second-countable*:

assumes *second-countable* S **and** *second-countable* S'

shows *second-countable* (*prod-topology* S S')

<proof>

Abstract version of the theorem $\exists K. \textit{topological-basis } K \wedge \textit{countable } K \wedge (\forall k \in K. \exists X. k = \textit{Pi}_E \textit{ UNIV } X \wedge (\forall i. \textit{open} (X i)) \wedge \textit{finite} \{i. X i \neq \textit{UNIV}\})$.

lemma *product-topology-countable-base-in*:

assumes *countable* I **and** $\bigwedge i. i \in I \implies \textit{second-countable} (S i)$

shows $\exists \mathcal{O}'. \textit{countable } \mathcal{O}' \wedge \textit{base-in} (\textit{product-topology } S I) \mathcal{O}' \wedge$

$(\forall k \in \mathcal{O}'. \exists X. k = (\prod_{E i \in I. X i}) \wedge (\forall i. \textit{openin} (S i) (X i)) \wedge \textit{finite} \{i. X i \neq \textit{topspace} (S i)\} \wedge \{i. X i \neq \textit{topspace} (S i)\} \subseteq I)$

<proof>

lemma *product-topology-second-countable*:

assumes *countable* I **and** $\bigwedge i. i \in I \implies \textit{second-countable} (S i)$

shows *second-countable* (*product-topology* $S I$)

<proof>

lemma *second-countable-euclidean[simp]*:

second-countable (*euclidean* :: 'a :: *second-countable-topology topology*)

<proof>

lemma *Cantor-Bendixon*:

assumes *second-countable* X

shows $\exists U P. \text{countable } U \wedge \text{openin } X U \wedge \text{perfect-set } X P \wedge U \cup P = \text{topspace } X \wedge U \cap P = \{\} \wedge (\forall a \neq \{\}. \text{openin } (\text{subtopology } X P) a \longrightarrow \text{uncountable } a)$
<proof>

1.1.5 Separable Spaces

definition *dense-in* :: [*'a topology, 'a set*] \Rightarrow *bool* **where**

dense-in $S U \longleftrightarrow (U \subseteq \text{topspace } S \wedge (\forall V. \text{openin } S V \longrightarrow V \neq \{\} \longrightarrow U \cap V \neq \{\}))$

lemma *dense-in-def2*:

dense-in $S U \longleftrightarrow (U \subseteq \text{topspace } S \wedge (S \text{ closure-of } U) = \text{topspace } S)$
<proof>

lemma *dense-in-topospace[simp]*: *dense-in* $S (\text{topspace } S)$

<proof>

lemma *dense-in-subset*:

assumes *dense-in* $S U$

shows $U \subseteq \text{topspace } S$

<proof>

lemma *dense-in-nonempty*:

assumes *topspace* $S \neq \{\}$ *dense-in* $S U$

shows $U \neq \{\}$

<proof>

lemma *dense-inI*:

assumes $U \subseteq \text{topspace } S$

and $\bigwedge V. \text{openin } S V \Longrightarrow V \neq \{\} \Longrightarrow U \cap V \neq \{\}$

shows *dense-in* $S U$

<proof>

lemma *dense-in-infinite*:

assumes *t1-space* X *infinite* (*topspace* X) *dense-in* $X U$

shows *infinite* U

<proof>

lemma *dense-in-prod*:

assumes *dense-in* $S U$ **and** *dense-in* $S' U'$

shows *dense-in* (*prod-topology* $S S'$) ($U \times U'$)

<proof>

lemma *separable-space-def2*: *separable-space* $S \longleftrightarrow (\exists U. \text{countable } U \wedge \text{dense-in } S U)$

<proof>

lemma *countable-space-separable-space:*

assumes *countable* (*topspace S*)

shows *separable-space S*

<proof>

lemma *separable-space-prod:*

assumes *separable-space S and separable-space S'*

shows *separable-space (prod-topology S S')*

<proof>

lemma *dense-in-product:*

assumes $\bigwedge i. i \in I \implies \text{dense-in } (T i) (U i)$

shows *dense-in (product-topology T I) ($\prod_{E i \in I. U i}$)*

<proof>

lemma *separable-countable-product:*

assumes *countable I and* $\bigwedge i. i \in I \implies \text{separable-space } (T i)$

shows *separable-space (product-topology T I)*

<proof>

lemma *separable-finite-product:*

assumes *finite I and* $\bigwedge i. i \in I \implies \text{separable-space } (T i)$

shows *separable-space (product-topology T I)*

<proof>

1.1.6 G_δ Set

lemma *gdelta-inD:*

assumes *gdelta-in S A*

shows $\exists \mathcal{U}. \mathcal{U} \neq \{\}$ \wedge *countable* $\mathcal{U} \wedge (\forall b \in \mathcal{U}. \text{open-in } S b) \wedge A = \bigcap \mathcal{U}$

<proof>

lemma *gdelta-inD':*

assumes *gdelta-in S A*

shows $\exists U. (\forall n :: \text{nat}. \text{open-in } S (U n)) \wedge A = \bigcap (\text{range } U)$

<proof>

lemma *gdelta-in-continuous-map:*

assumes *continuous-map X Y f gdelta-in Y a*

shows *gdelta-in X (f $^{-1}$ a \cap topspace X)*

<proof>

lemma *g-delta-of-inj-open-map:*

assumes *open-map X Y f inj-on f (topspace X) gdelta-in X a*

shows *gdelta-in Y (f $^{-1}$ a)*

<proof>

lemma *gdelta-in-prod:*

assumes *gdelta-in X A gdelta-in Y B*

shows *gdelta-in* (*prod-topology* $X Y$) ($A \times B$)
 ⟨*proof*⟩

corollary *gdelta-in-prod1*:
assumes *gdelta-in* $X A$
shows *gdelta-in* (*prod-topology* $X Y$) ($A \times \text{topspace } Y$)
 ⟨*proof*⟩

corollary *gdelta-in-prod2*:
assumes *gdelta-in* $Y B$
shows *gdelta-in* (*prod-topology* $X Y$) ($\text{topspace } X \times B$)
 ⟨*proof*⟩

lemma *continuous-map-imp-closed-graph'*:
assumes *continuous-map* $X Y f$ *Hausdorff-space* Y
shows *closedin* (*prod-topology* $Y X$) ($(\lambda x. (f x, x)) \text{ 'topspace } X$)
 ⟨*proof*⟩

1.1.7 Continuous Maps on First Countable Topology

Generalized version of *Metric-space* $?M ?d \implies \text{eventually } ?P$ (*atin* (*Metric-space.mtopology* $?M ?d$) $?a$) = $(\forall \sigma. \text{range } \sigma \subseteq ?M - \{?a\} \wedge \text{limitin} (\text{Metric-space.mtopology } ?M ?d) \sigma ?a \text{ sequentially} \longrightarrow (\forall_F n \text{ in sequentially. } ?P (\sigma n)))$

lemma *eventually-atin-sequentially*:
assumes *first-countable* X
shows *eventually* P (*atin* $X a$) $\longleftrightarrow (\forall \sigma. \text{range } \sigma \subseteq \text{topspace } X - \{a\} \wedge \text{limitin } X \sigma a \text{ sequentially} \longrightarrow \text{eventually } (\lambda n. P (\sigma n)) \text{ sequentially})$
 ⟨*proof*⟩

lemma *continuous-map-iff-limit-seq*:
assumes *first-countable* X
shows *continuous-map* $X Y f \longleftrightarrow (\forall xn x. \text{limitin } X xn x \text{ sequentially} \longrightarrow \text{limitin } Y (\lambda n. f (xn n)) (f x) \text{ sequentially})$
 ⟨*proof*⟩

1.1.8 Upper-Semicontinuous Functions

definition *upper-semicontinuous-map* :: [$'a \text{ topology, 'b} \implies 'a :: \text{linorder-topology}$] $\implies \text{bool}$ **where**
upper-semicontinuous-map $X f \longleftrightarrow (\forall a. \text{openin } X \{x \in \text{topspace } X. f x < a\})$

lemma *continuous-upper-semicontinuous*:
assumes *continuous-map* X (*euclidean* :: $'b :: \text{linorder-topology}$) *topology*) f
shows *upper-semicontinuous-map* $X f$
 ⟨*proof*⟩

lemma *upper-semicontinuous-map-iff-closed*:
upper-semicontinuous-map $X f \longleftrightarrow (\forall a. \text{closedin } X \{x \in \text{topspace } X. f x \geq a\})$
 ⟨*proof*⟩

lemma *upper-semicontinuous-map-real-iff*:
fixes $f :: 'a \Rightarrow \text{real}$
shows $\text{upper-semicontinuous-map } X f \longleftrightarrow \text{upper-semicontinuous-map } X (\lambda x. \text{ereal } (f x))$
 $\langle \text{proof} \rangle$

1.1.9 Lower-Semicontinuous Functions

definition *lower-semicontinuous-map* :: $['a \text{ topology}, 'a \Rightarrow 'b :: \text{linorder-topology}] \Rightarrow \text{bool}$ **where**
 $\text{lower-semicontinuous-map } X f \longleftrightarrow (\forall a. \text{openin } X \{x \in \text{topspace } X. a < f x\})$

lemma *continuous-lower-semicontinuous*:
assumes $\text{continuous-map } X (\text{euclidean} :: ('b :: \text{linorder-topology}) \text{ topology}) f$
shows $\text{lower-semicontinuous-map } X f$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-iff-closed*:
 $\text{lower-semicontinuous-map } X f \longleftrightarrow (\forall a. \text{closedin } X \{x \in \text{topspace } X. f x \leq a\})$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-real-iff*:
fixes $f :: 'a \Rightarrow \text{real}$
shows $\text{lower-semicontinuous-map } X f \longleftrightarrow \text{lower-semicontinuous-map } X (\lambda x. \text{ereal } (f x))$
 $\langle \text{proof} \rangle$

1.2 Lemmas for Measure Theory

1.2.1 Lemmas for Measurable Sets

lemma *measurable-preserve-sigma-sets*:
assumes $\text{sets } M = \text{sigma-sets } \Omega S S \subseteq \text{Pow } \Omega$
 $\bigwedge a. a \in S \implies f ' a \in \text{sets } N \text{ inj-on } f (\text{space } M) f ' \text{space } M \in \text{sets } N$
and $b \in \text{sets } M$
shows $f ' b \in \text{sets } N$
 $\langle \text{proof} \rangle$

inductive-set *sigma-sets-cinter* :: $'a \text{ set} \Rightarrow 'a \text{ set set} \Rightarrow 'a \text{ set set}$
for $sp :: 'a \text{ set}$ **and** $A :: 'a \text{ set set}$
where
 $\text{Basic-c}[\text{intro}, \text{simp}]: a \in A \implies a \in \text{sigma-sets-cinter } sp A$
 $|\text{Top-c}[\text{simp}]: sp \in \text{sigma-sets-cinter } sp A$
 $|\text{Inter-c}: (\bigwedge i::\text{nat}. a i \in \text{sigma-sets-cinter } sp A) \implies (\bigcap i. a i) \in \text{sigma-sets-cinter } sp A$
 $|\text{Union-c}: (\bigwedge i::\text{nat}. a i \in \text{sigma-sets-cinter } sp A) \implies (\bigcup i. a i) \in \text{sigma-sets-cinter } sp A$

inductive-set *sigma-sets-cinter-dunion* :: $'a \text{ set} \Rightarrow 'a \text{ set set} \Rightarrow 'a \text{ set set}$

for $sp :: 'a \text{ set}$ **and** $A :: 'a \text{ set set}$
where
 $Basic\text{-}cd[intro, simp]: a \in A \implies a \in \text{sigma-sets-cinter-dunion } sp \ A$
 | $Top\text{-}cd[simp]: sp \in \text{sigma-sets-cinter-dunion } sp \ A$
 | $Inter\text{-}cd: (\bigwedge i::nat. a \ i \in \text{sigma-sets-cinter-dunion } sp \ A) \implies (\bigcap i. a \ i) \in \text{sigma-sets-cinter-dunion } sp \ A$
 | $Union\text{-}cd: (\bigwedge i::nat. a \ i \in \text{sigma-sets-cinter-dunion } sp \ A) \implies \text{disjoint-family } a \implies (\bigcup i. a \ i) \in \text{sigma-sets-cinter-dunion } sp \ A$

lemma $\text{sigma-sets-cinter-dunion-subset}: \text{sigma-sets-cinter-dunion } sp \ A \subseteq \text{sigma-sets-cinter } sp \ A$
 <proof>

lemma $\text{sigma-sets-cinter-into-sp}$:
assumes $A \subseteq \text{Pow } sp \ x \in \text{sigma-sets-cinter } sp \ A$
shows $x \subseteq sp$
 <proof>

lemma $\text{sigma-sets-cinter-dunion-into-sp}$:
assumes $A \subseteq \text{Pow } sp \ x \in \text{sigma-sets-cinter-dunion } sp \ A$
shows $x \subseteq sp$
 <proof>

lemma $\text{sigma-sets-cinter-int}$:
assumes $a \in \text{sigma-sets-cinter } sp \ A \ b \in \text{sigma-sets-cinter } sp \ A$
shows $a \cap b \in \text{sigma-sets-cinter } sp \ A$
 <proof>

lemma $\text{sigma-sets-cinter-dunion-int}$:
assumes $a \in \text{sigma-sets-cinter-dunion } sp \ A \ b \in \text{sigma-sets-cinter-dunion } sp \ A$
shows $a \cap b \in \text{sigma-sets-cinter-dunion } sp \ A$
 <proof>

lemma $\text{sigma-sets-cinter-un}$:
assumes $a \in \text{sigma-sets-cinter } sp \ A \ b \in \text{sigma-sets-cinter } sp \ A$
shows $a \cup b \in \text{sigma-sets-cinter } sp \ A$
 <proof>

lemma $\text{sigma-sets-eq-cinter-dunion}$:
assumes $\text{metrizable-space } X$
shows $\text{sigma-sets } (\text{topspace } X) \ \{U. \text{openin } X \ U\} = \text{sigma-sets-cinter-dunion } (\text{topspace } X) \ \{U. \text{openin } X \ U\}$
 <proof>

lemma $\text{sigma-sets-eq-cinter}$:
assumes $\text{metrizable-space } X$
shows $\text{sigma-sets } (\text{topspace } X) \ \{U. \text{openin } X \ U\} = \text{sigma-sets-cinter } (\text{topspace } X) \ \{U. \text{openin } X \ U\}$
 <proof>

1.2.2 Measurable Isomorphisms

definition *measurable-isomorphic-map*: $['a \text{ measure, } 'b \text{ measure, } 'a \Rightarrow 'b] \Rightarrow \text{bool}$
where

measurable-isomorphic-map $M N f \longleftrightarrow \text{bij-betw } f \text{ (space } M \text{) (space } N \text{)} \wedge f \in M \rightarrow_M N \wedge \text{the-inv-into (space } M \text{) } f \in N \rightarrow_M M$

lemma *measurable-isomorphic-map-sets-cong*:

assumes *sets* $M = \text{sets } M' \text{ sets } N = \text{sets } N'$

shows *measurable-isomorphic-map* $M N f \longleftrightarrow \text{measurable-isomorphic-map } M' N' f$

<proof>

lemma *measurable-isomorphic-map-surj*:

assumes *measurable-isomorphic-map* $M N f$

shows $f ' \text{space } M = \text{space } N$

<proof>

lemma *measurable-isomorphic-mapI*:

assumes *bij-betw* $f \text{ (space } M \text{) (space } N \text{)} f \in M \rightarrow_M N \text{the-inv-into (space } M \text{) } f \in N \rightarrow_M M$

shows *measurable-isomorphic-map* $M N f$

<proof>

lemma *measurable-isomorphic-map-byWitness*:

assumes $f \in M \rightarrow_M N g \in N \rightarrow_M M \wedge x. x \in \text{space } M \Longrightarrow g (f x) = x \wedge x. x \in \text{space } N \Longrightarrow f (g x) = x$

shows *measurable-isomorphic-map* $M N f$

<proof>

lemma *measurable-isomorphic-map-restrict-space*:

assumes $f \in M \rightarrow_M N \wedge A. A \in \text{sets } M \Longrightarrow f ' A \in \text{sets } N \text{inj-on } f \text{ (space } M \text{)}$

shows *measurable-isomorphic-map* $M \text{ (restrict-space } N \text{ (} f ' \text{space } M \text{)) } f$

<proof>

lemma *measurable-isomorphic-mapD'*:

assumes *measurable-isomorphic-map* $M N f$

shows $\wedge A. A \in \text{sets } M \Longrightarrow f ' A \in \text{sets } N f \in M \rightarrow_M N$

$\exists g. \text{bij-betw } g \text{ (space } N \text{) (space } M \text{)} \wedge g \in N \rightarrow_M M \wedge (\forall x \in \text{space } M. g (f x) = x) \wedge (\forall x \in \text{space } N. f (g x) = x) \wedge (\forall A \in \text{sets } N. g ' A \in \text{sets } M)$

<proof>

lemma *measurable-isomorphic-map-inv*:

assumes *measurable-isomorphic-map* $M N f$

shows *measurable-isomorphic-map* $N M \text{ (the-inv-into (space } M \text{) } f \text{)}$

<proof>

lemma *measurable-isomorphic-map-comp*:

assumes *measurable-isomorphic-map* $M N f$ **and** *measurable-isomorphic-map* $N L g$

shows *measurable-isomorphic-map* $M L (g \circ f)$
(*proof*)

definition *measurable-isomorphic*::['a *measure*, 'b *measure*] \Rightarrow *bool* (**infixr** *measurable'-isomorphic* 50) **where**
 M *measurable-isomorphic* $N \longleftrightarrow (\exists f. \text{measurable-isomorphic-map } M N f)$

lemma *measurable-isomorphic-sets-cong*:
assumes *sets* $M = \text{sets } M'$ *sets* $N = \text{sets } N'$
shows M *measurable-isomorphic* $N \longleftrightarrow M'$ *measurable-isomorphic* N'
(*proof*)

lemma *measurable-isomorphicD*:
assumes M *measurable-isomorphic* N
shows $\exists f g. f \in M \rightarrow_M N \wedge g \in N \rightarrow_M M \wedge (\forall x \in \text{space } M. g (f x) = x) \wedge$
 $(\forall y \in \text{space } N. f (g y) = y) \wedge (\forall A \in \text{sets } M. f ' A \in \text{sets } N) \wedge (\forall A \in \text{sets } N. g ' A$
 $\in \text{sets } M)$
(*proof*)

lemma *measurable-isomorphic-cardinality-eq*:
assumes M *measurable-isomorphic* N
shows *space* $M \approx \text{space } N$
(*proof*)

lemma *measurable-isomorphic-count-spaces*: *count-space* A *measurable-isomorphic*
count-space $B \longleftrightarrow A \approx B$
(*proof*)

lemma *measurable-isomorphic-byWitness*:
assumes $f \in M \rightarrow_M N \wedge x. x \in \text{space } M \Longrightarrow g (f x) = x$
and $g \in N \rightarrow_M M \wedge y. y \in \text{space } N \Longrightarrow f (g y) = y$
shows M *measurable-isomorphic* N
(*proof*)

lemma *measurable-isomorphic-refl*:
 M *measurable-isomorphic* M
(*proof*)

lemma *measurable-isomorphic-sym*:
assumes M *measurable-isomorphic* N
shows N *measurable-isomorphic* M
(*proof*)

lemma *measurable-isomorphic-trans*:
assumes M *measurable-isomorphic* N **and** N *measurable-isomorphic* L
shows M *measurable-isomorphic* L
(*proof*)

lemma *measurable-isomorphic-empty*:

assumes $space\ M = \{\}$ $space\ N = \{\}$
shows $M\ measurable-isomorphic\ N$
 $\langle proof \rangle$

lemma *measurable-isomorphic-empty1*:
assumes $space\ M = \{\}$ $M\ measurable-isomorphic\ N$
shows $space\ N = \{\}$
 $\langle proof \rangle$

lemma *measurable-isomorphic-empty2*:
assumes $space\ N = \{\}$ $M\ measurable-isomorphic\ N$
shows $space\ M = \{\}$
 $\langle proof \rangle$

lemma *measurable-lift-product*:
assumes $\bigwedge i. i \in I \implies f\ i \in (M\ i) \rightarrow_M (N\ i)$
shows $(\lambda x\ i. if\ i \in I\ then\ f\ i\ (x\ i)\ else\ undefined) \in (\prod_{M\ i \in I. M\ i}) \rightarrow_M (\prod_{i \in I. N\ i})$
 $\langle proof \rangle$

lemma *measurable-isomorphic-map-lift-product*:
assumes $\bigwedge i. i \in I \implies measurable-isomorphic-map\ (M\ i)\ (N\ i)\ (h\ i)$
shows $measurable-isomorphic-map\ (\prod_{M\ i \in I. M\ i})\ (\prod_{M\ i \in I. N\ i})\ (\lambda x\ i. if\ i \in I\ then\ h\ i\ (x\ i)\ else\ undefined)$
 $\langle proof \rangle$

lemma *measurable-isomorphic-lift-product*:
assumes $\bigwedge i. i \in I \implies (M\ i)\ measurable-isomorphic\ (N\ i)$
shows $(\prod_{M\ i \in I. M\ i})\ measurable-isomorphic\ (\prod_{M\ i \in I. N\ i})$
 $\langle proof \rangle$

<https://math24.net/cantor-schroeder-bernstein-theorem.html>

lemma *Schroeder-Bernstein-measurable'*:
assumes $f' \in (space\ M) \in sets\ N\ g' \in (space\ N) \in sets\ M$
and $measurable-isomorphic-map\ M\ (restrict-space\ N\ (f' \in (space\ M)))\ f\ and\ measurable-isomorphic-map\ N\ (restrict-space\ M\ (g' \in (space\ N)))\ g$
shows $\exists h. measurable-isomorphic-map\ M\ N\ h$
 $\langle proof \rangle$

lemma *Schroeder-Bernstein-measurable*:
assumes $f \in M \rightarrow_M N \wedge A. A \in sets\ M \implies f' \in A \in sets\ N\ inj-on\ f\ (space\ M)$
and $g \in N \rightarrow_M M \wedge A. A \in sets\ N \implies g' \in A \in sets\ M\ inj-on\ g\ (space\ N)$
shows $\exists h. measurable-isomorphic-map\ M\ N\ h$
 $\langle proof \rangle$

lemma *measurable-isomorphic-from-embeddings*:
assumes $M\ measurable-isomorphic\ (restrict-space\ N\ B)\ N\ measurable-isomorphic\ (restrict-space\ M\ A)$
and $A \in sets\ M\ B \in sets\ N$

shows M measurable-isomorphic N
<proof>

lemma measurable-isomorphic-antisym:

assumes B measurable-isomorphic (restrict-space C c) A measurable-isomorphic
(restrict-space B b)

and $c \in$ sets C $b \in$ sets B C measurable-isomorphic A

shows C measurable-isomorphic B
<proof>

lemma countable-infinite-isomorphisc-to-nat-index:

assumes countable I **and** infinite I

shows $(\prod_M x \in I. M)$ measurable-isomorphic $(\prod_M (x :: \text{nat}) \in \text{UNIV}. M)$
<proof>

lemma PiM-PiM-isomorphic-to-PiM:

$(\prod_M i \in I. \prod_M j \in J. M i j)$ measurable-isomorphic $(\prod_M (i, j) \in I \times J. M i j)$
<proof>

lemma measurable-isomorphic-map-sigma-sets:

assumes sets $M =$ sigma-sets (space M) U measurable-isomorphic-map M N f

shows sets $N =$ sigma-sets (space N) $((\cdot) f ' U)$
<proof>

1.2.3 Borel Spaces Generated from Abstract Topologies

definition borel-of :: 'a topology \Rightarrow 'a measure **where**

borel-of $X \equiv$ sigma (topspace X) $\{U. \text{openin } X U\}$

lemma emeasure-borel-of: emeasure (borel-of X) $A = 0$

<proof>

lemma borel-of-euclidean: borel-of euclidean = borel

<proof>

lemma space-borel-of: space (borel-of X) = topspace X

<proof>

lemma sets-borel-of: sets (borel-of X) = sigma-sets (topspace X) $\{U. \text{openin } X U\}$

<proof>

lemma sets-borel-of-closed: sets (borel-of X) = sigma-sets (topspace X) $\{U. \text{closedin } X U\}$

<proof>

lemma borel-of-open:

assumes openin X U

shows $U \in$ sets (borel-of X)

<proof>

lemma *borel-of-closed:*

assumes *closedin X U*

shows $U \in \text{sets } (\text{borel-of } X)$

<proof>

lemma(**in** *Metric-space*) *nbh-sets[measurable]:* $(\bigcup a \in A. \text{mball } a \ e) \in \text{sets } (\text{borel-of } \text{mtopology})$

<proof>

lemma *borel-of-gdelta-in:*

assumes *gdelta-in X U*

shows $U \in \text{sets } (\text{borel-of } X)$

<proof>

lemma *borel-of-subtopology:*

$\text{borel-of } (\text{subtopology } X \ U) = \text{restrict-space } (\text{borel-of } X) \ U$

<proof>

lemma *sets-borel-of-discrete-topology:* $\text{sets } (\text{borel-of } (\text{discrete-topology } I)) = \text{sets } (\text{count-space } I)$

<proof>

lemma *continuous-map-measurable:*

assumes *continuous-map X Y f*

shows $f \in \text{borel-of } X \rightarrow_M \text{borel-of } Y$

<proof>

lemma *upper-semicontinuous-map-measurable:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{linorder-topology, second-countable-topology}\}$

assumes *upper-semicontinuous-map X f*

shows $f \in \text{borel-measurable } (\text{borel-of } X)$

<proof>

lemma *lower-semicontinuous-map-measurable:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{linorder-topology, second-countable-topology}\}$

assumes *lower-semicontinuous-map X f*

shows $f \in \text{borel-measurable } (\text{borel-of } X)$

<proof>

lemma *open-map-preserves-sets:*

assumes *open-map S T f inj-on f (topspace S) A \in sets (borel-of S)*

shows $f \text{ ` } A \in \text{sets } (\text{borel-of } T)$

<proof>

lemma *open-map-preserves-sets':*

assumes *open-map S (subtopology T (f ` (topspace S))) f inj-on f (topspace S) f ` (topspace S) \in sets (borel-of T) A \in sets (borel-of S)*

shows $f' A \in \text{sets} (\text{borel-of } T)$
<proof>

Abstract topology version of $\text{open} = \text{generate-topology } ?X \implies \text{borel} = \text{sigma UNIV } ?X$.

lemma *borel-of-second-countable'*:
assumes *second-countable* S **and** *subbase-in* $S \mathcal{U}$
shows $\text{borel-of } S = \text{sigma} (\text{topspace } S) \mathcal{U}$
<proof>

Abstract topology version $\text{borel} \otimes_M \text{borel} = \text{borel}$.

lemma *borel-of-prod*:
assumes *second-countable* S **and** *second-countable* S'
shows $\text{borel-of } S \otimes_M \text{borel-of } S' = \text{borel-of} (\text{prod-topology } S S')$
<proof>

lemma *product-borel-of-measurable*:
assumes $i \in I$
shows $(\lambda x. x i) \in (\text{borel-of} (\text{product-topology } S I)) \rightarrow_M \text{borel-of} (S i)$
<proof>

Abstract topology version of $\text{sets} (\text{Pi}_M \text{ UNIV } (\lambda \cdot. \text{borel})) \subseteq \text{sets borel}$

lemma *sets-PiM-subset-borel-of*:
 $\text{sets} (\text{Pi}_M i \in I. \text{borel-of} (S i)) \subseteq \text{sets} (\text{borel-of} (\text{product-topology } S I))$
<proof>

Abstract topology version of $\text{sets} (\text{Pi}_M \text{ UNIV } (\lambda i. \text{borel})) = \text{sets borel}$.

lemma *sets-PiM-equal-borel-of*:
assumes *countable* I **and** $\bigwedge i. i \in I \implies \text{second-countable} (S i)$
shows $\text{sets} (\text{Pi}_M i \in I. \text{borel-of} (S i)) = \text{sets} (\text{borel-of} (\text{product-topology } S I))$
<proof>

lemma *homeomorphic-map-borel-isomorphic*:
assumes *homeomorphic-map* $X Y f$
shows *measurable-isomorphic-map* $(\text{borel-of } X) (\text{borel-of } Y) f$
<proof>

lemma *homeomorphic-space-measurable-isomorphic*:
assumes S *homeomorphic-space* T
shows $\text{borel-of } S$ *measurable-isomorphic* $\text{borel-of } T$
<proof>

lemma *measurable-isomorphic-borel-map*:
assumes $\text{sets } M = \text{sets} (\text{borel-of } S)$ **and** $f: \text{measurable-isomorphic-map } M N f$
shows $\exists S'. \text{homeomorphic-map } S S' f \wedge \text{sets } N = \text{sets} (\text{borel-of } S')$
<proof>

lemma *measurable-isomorphic-borels*:

assumes *sets* $M = \text{sets (borel-of } S) \text{ } M \text{ measurable-isomorphic } N$
shows $\exists S'. S \text{ homeomorphic-space } S' \wedge \text{sets } N = \text{sets (borel-of } S')$
 $\langle \text{proof} \rangle$

end

1.3 Lemmas for Abstract Metric Spaces

theory *Set-Based-Metric-Space*
imports *Lemmas-StandardBorel*
begin

We prove additional lemmas related to set-based metric spaces.

1.3.1 Basic Lemmas

lemma

assumes *Metric-space* $M \ d \ \wedge x \ y. x \in M \implies y \in M \implies d \ x \ y = d' \ x \ y$
and $\wedge x \ y. d' \ x \ y = d' \ y \ x \ \wedge x \ y. d' \ x \ y \geq 0$
shows *Metric-space-eq: Metric-space* $M \ d'$
and *Metric-space-eq-mtopology: Metric-space.mtopology* $M \ d = \text{Metric-space.mtopology } M \ d'$
and *Metric-space-eq-mcomplete: Metric-space.mcomplete* $M \ d \longleftrightarrow \text{Metric-space.mcomplete } M \ d'$
 $\langle \text{proof} \rangle$

context *Metric-space*
begin

lemma *mtopology-base-in-balls: base-in mtopology* $\{ \text{mball } a \ \varepsilon \mid a \ \varepsilon. a \in M \wedge \varepsilon > 0 \}$
 $\langle \text{proof} \rangle$

lemma *closedin-metric2: closedin mtopology* $C \longleftrightarrow C \subseteq M \wedge (\forall x. x \in C \longleftrightarrow (\forall \varepsilon > 0. \text{mball } x \ \varepsilon \cap C \neq \{ \}))$
 $\langle \text{proof} \rangle$

lemma *openin-mtopology2:*
openin mtopology $U \longleftrightarrow U \subseteq M \wedge (\forall x \ n \ x. \text{limitin mtopology } x \ n \ x \text{ sequentially} \wedge x \in U \longrightarrow (\exists N. \forall n \geq N. x \ n \ n \in U))$
 $\langle \text{proof} \rangle$

lemma *closure-of-mball: mtopology closure-of mball* $a \ e \subseteq \text{mcball } a \ e$
 $\langle \text{proof} \rangle$

lemma *interior-of-mcball: mball* $a \ e \subseteq \text{mtopology interior-of mcball } a \ e$
 $\langle \text{proof} \rangle$

lemma *isolated-points-of-mtopology:*

mtopology isolated-points-of $A = \{x \in M \cap A. \forall xn. \text{range } xn \subseteq A \wedge \text{limitin } mtopology \text{ } xn \text{ } x \text{ sequentially} \longrightarrow (\exists no. \forall n \geq no. xn \ n = x)\}$
 ⟨proof⟩

lemma *perfect-set-mball-infinite*:
 assumes *perfect-set mtopology* A $a \in A$ $e > 0$
 shows *infinite* ($\text{mball } a \ e$)
 ⟨proof⟩

lemma *MCauchy-dist-Cauchy*:
 assumes *MCauchy* xn *MCauchy* yn
 shows *Cauchy* $(\lambda n. d (xn \ n) (yn \ n))$
 ⟨proof⟩

1.3.2 Dense in Metric Spaces

abbreviation *mdense* \equiv *dense-in mtopology*

<https://people.bath.ac.uk/mw2319/ma30252/sec-dense.html>

lemma *mdense-def*:
 $mdense \ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \forall \varepsilon > 0. \text{mball } x \ \varepsilon \cap U \neq \{\})$
 ⟨proof⟩

corollary *mdense-balls-cover*:
 assumes *mdense* U **and** $e > 0$
 shows $(\bigcup u \in U. \text{mball } u \ e) = M$
 ⟨proof⟩

lemma *mdense-empty-iff*: $mdense \ \{\} \longleftrightarrow M = \{\}$
 ⟨proof⟩

lemma *mdense-M*: $mdense \ M$
 ⟨proof⟩

lemma *mdense-def2*:
 $mdense \ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \forall \varepsilon > 0. \exists y \in U. d \ x \ y < \varepsilon)$
 ⟨proof⟩

lemma *mdense-def3*:
 $mdense \ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \exists xn. \text{range } xn \subseteq U \wedge \text{limitin } mtopology \text{ } xn \ x \text{ sequentially})$
 ⟨proof⟩

Diameter

definition *mdiameter* $:: 'a \ set \Rightarrow \text{ennreal}$ **where**
 $mdiameter \ A \equiv \bigsqcup \{\text{ennreal } (d \ x \ y) \mid x \ y. x \in A \cap M \wedge y \in A \cap M\}$

lemma *mdiameter-empty[simp]*:
 $mdiameter \ \{\} = 0$

<proof>

lemma *mdiameter-def2*:

assumes $A \subseteq M$

shows $\text{mdiameter } A = \bigsqcup \{ \text{ennreal } (d \ x \ y) \mid x \ y. \ x \in A \wedge y \in A \}$

<proof>

lemma *mdiameter-subset*:

assumes $A \subseteq B$

shows $\text{mdiameter } A \leq \text{mdiameter } B$

<proof>

lemma *mdiameter-cball-leq*: $\text{mdiameter } (\text{mcball } a \ \varepsilon) \leq \text{ennreal } (2 * \varepsilon)$

<proof>

lemma *mdiameter-ball-leq*:

$\text{mdiameter } (\text{mball } a \ \varepsilon) \leq \text{ennreal } (2 * \varepsilon)$

<proof>

lemma *mdiameter-is-sup*:

assumes $x \in A \cap M \ y \in A \cap M$

shows $d \ x \ y \leq \text{mdiameter } A$

<proof>

lemma *mdiameter-is-sup'*:

assumes $x \in A \cap M \ y \in A \cap M \ \text{mdiameter } A \leq \text{ennreal } r \ r \geq 0$

shows $d \ x \ y \leq r$

<proof>

lemma *mdiameter-le*:

assumes $\bigwedge x \ y. \ x \in A \implies y \in A \implies d \ x \ y \leq r$

shows $\text{mdiameter } A \leq r$

<proof>

lemma *mdiameter-eq-closure*: $\text{mdiameter } (\text{mtopology closure-of } A) = \text{mdiameter } A$

<proof>

lemma *mbounded-finite-mdiameter*: $\text{mbounded } A \iff A \subseteq M \wedge \text{mdiameter } A < \infty$

<proof>

Distance between a point and a set.

definition *d-set* :: 'a set \Rightarrow 'a \Rightarrow real **where**

d-set $A \equiv (\lambda x. \text{if } A \neq \{\} \wedge A \subseteq M \wedge x \in M \text{ then } \text{Inf } \{d \ x \ y \mid y. \ y \in A\} \text{ else } 0)$

lemma *d-set-nonneg[simp]*:

d-set $A \ x \geq 0$

<proof>

lemma *d-set-bdd-below*[simp]:

bdd-below $\{d\ x\ y\ \mid\ y.\ y \in A\}$
<proof>

lemma *d-set-singleton*[simp]:

$x \in M \implies y \in M \implies d\text{-set}\ \{y\}\ x = d\ x\ y$
<proof>

lemma *d-set-empty*[simp]:

d-set $\{\}$ $x = 0$
<proof>

lemma *d-set-notin*:

$x \notin M \implies d\text{-set}\ A\ x = 0$
<proof>

lemma *d-set-inA*:

assumes $x \in A$
shows *d-set* $A\ x = 0$
<proof>

lemma *d-set-nzeroD*:

assumes *d-set* $A\ x \neq 0$
shows $A \subseteq M\ x \notin A\ A \neq \{\}$
<proof>

lemma *d-set-antimono*:

assumes $A \subseteq B\ A \neq \{\}\ B \subseteq M$
shows *d-set* $B\ x \leq d\text{-set}\ A\ x$
<proof>

lemma *d-set-bounded*:

assumes $\bigwedge y.\ y \in A \implies d\ x\ y < K\ K > 0$
shows *d-set* $A\ x < K$
<proof>

lemma *d-set-tr*:

assumes $x \in M\ y \in M$
shows *d-set* $A\ x \leq d\ x\ y + d\text{-set}\ A\ y$
<proof>

lemma *d-set-abs-le*:

assumes $x \in M\ y \in M$
shows $|d\text{-set}\ A\ x - d\text{-set}\ A\ y| \leq d\ x\ y$
<proof>

lemma *d-set-inA-le*:

assumes $y \in A$

shows $d\text{-set } A \ x \leq d \ x \ y$
 $\langle \text{proof} \rangle$

lemma $d\text{-set-ball-empty}$:
assumes $A \neq \{\}$ $A \subseteq M$ $e > 0$ $x \in M$ $m\text{ball } x \ e \cap A = \{\}$
shows $d\text{-set } A \ x \geq e$
 $\langle \text{proof} \rangle$

lemma $d\text{-set-closed-pos}$:
assumes $\text{closedin } m\text{topology } A$ $A \neq \{\}$ $x \in M$ $x \notin A$
shows $d\text{-set } A \ x > 0$
 $\langle \text{proof} \rangle$

lemma $g\text{delta-in-closed}$:
assumes $\text{closedin } m\text{topology } M$
shows $g\text{delta-in } m\text{topology } M$
 $\langle \text{proof} \rangle$

Oscillation

definition $\text{osc-on} :: ['b \text{ set}, 'b \text{ topology}, 'b \Rightarrow 'a, 'b] \Rightarrow \text{ennreal}$ **where**
 $\text{osc-on } A \ X \ f \equiv (\lambda y. \sqcap \{m\text{diameter } (f \ ' (A \cap U)) \mid U. y \in U \wedge \text{openin } X \ U\})$

abbreviation $\text{osc } X \equiv \text{osc-on } (\text{topspace } X) \ X$

lemma osc-def : $\text{osc } X \ f = (\lambda y. \sqcap \{m\text{diameter } (f \ ' U) \mid U. y \in U \wedge \text{openin } X \ U\})$
 $\langle \text{proof} \rangle$

lemma osc-on-less-iff :
 $\text{osc-on } A \ X \ f \ x < t \iff (\exists v. x \in v \wedge \text{openin } X \ v \wedge m\text{diameter } (f \ ' (A \cap v)) < t)$
 $\langle \text{proof} \rangle$

lemma osc-less-iff :
 $\text{osc } X \ f \ x < t \iff (\exists v. x \in v \wedge \text{openin } X \ v \wedge m\text{diameter } (f \ ' v) < t)$
 $\langle \text{proof} \rangle$

end

definition $m\text{dist-set} :: 'a \text{ metric} \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{real}$ **where**
 $m\text{dist-set } m \equiv \text{Metric-space.d-set } (m\text{space } m) (m\text{dist } m)$

lemma(in Metric-space) $m\text{dist-set-Self}$: $m\text{dist-set } \text{Self} = d\text{-set}$
 $\langle \text{proof} \rangle$

lemma $m\text{dist-set-nonneg[simp]}$: $m\text{dist-set } m \ A \ x \geq 0$
 $\langle \text{proof} \rangle$

lemma $m\text{dist-set-singleton[simp]}$:
 $x \in m\text{space } m \implies y \in m\text{space } m \implies m\text{dist-set } m \ \{y\} \ x = m\text{dist } m \ x \ y$
 $\langle \text{proof} \rangle$

lemma *mdist-set-empty[simp]*: $mdist\text{-}set\ m\ \{\} x = 0$
<proof>

lemma *mdist-set-inA*:
assumes $x \in A$
shows $mdist\text{-}set\ m\ A\ x = 0$
<proof>

lemma *mdist-set-nzeroD*:
assumes $mdist\text{-}set\ m\ A\ x \neq 0$
shows $A \subseteq mspace\ m\ x \notin A\ A \neq \{\}$
<proof>

lemma *mdist-set-antimono*:
assumes $A \subseteq B\ A \neq \{\}\ B \subseteq mspace\ m$
shows $mdist\text{-}set\ m\ B\ x \leq mdist\text{-}set\ m\ A\ x$
<proof>

lemma *mdist-set-bounded*:
assumes $\bigwedge y. y \in A \implies mdist\ m\ x\ y < K\ K > 0$
shows $mdist\text{-}set\ m\ A\ x < K$
<proof>

lemma *mdist-set-tr*:
assumes $x \in mspace\ m\ y \in mspace\ m$
shows $mdist\text{-}set\ m\ A\ x \leq mdist\ m\ x\ y + mdist\text{-}set\ m\ A\ y$
<proof>

lemma *mdist-set-abs-le*:
assumes $x \in mspace\ m\ y \in mspace\ m$
shows $|mdist\text{-}set\ m\ A\ x - mdist\text{-}set\ m\ A\ y| \leq mdist\ m\ x\ y$
<proof>

lemma *mdist-set-inA-le*:
assumes $y \in A$
shows $mdist\text{-}set\ m\ A\ x \leq mdist\ m\ x\ y$
<proof>

lemma *mdist-set-ball-empty*:
assumes $A \neq \{\}\ A \subseteq mspace\ m\ e > 0\ x \in mspace\ m\ mball\text{-}of\ m\ x\ e \cap A = \{\}$
shows $mdist\text{-}set\ m\ A\ x \geq e$
<proof>

lemma *mdist-set-closed-pos*:
assumes *closedin* (*mtopology-of* m) $A\ A \neq \{\}\ x \in mspace\ m\ x \notin A$
shows $mdist\text{-}set\ m\ A\ x > 0$
<proof>

lemma *mdist-set-uniformly-continuous: uniformly-continuous-map m euclidean-metric*
(mdist-set m A)

<proof>

lemma *uniformly-continuous-map-add:*

fixes $f :: 'a \Rightarrow 'b::\text{real-normed-vector}$

assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map*
m euclidean-metric g

shows *uniformly-continuous-map m euclidean-metric* $(\lambda x. f x + g x)$

<proof>

lemma *uniformly-continuous-map-real-divide:*

fixes $f :: 'a \Rightarrow \text{real}$

assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map*
m euclidean-metric g

and $\bigwedge x. x \in \text{mspace } m \implies g x \neq 0$ $\bigwedge x. x \in \text{mspace } m \implies |g x| \geq a$ $a > 0$

$\bigwedge x. x \in \text{mspace } m \implies |g x| < Kg$

and $\bigwedge x. x \in \text{mspace } m \implies |f x| < Kf$

shows *uniformly-continuous-map m euclidean-metric* $(\lambda x. f x / g x)$

<proof>

lemma

assumes $e > 0$

shows *uniformly-continuous-map-from-capped-metric:uniformly-continuous-map*
(capped-metric e m1) m2 f \longleftrightarrow uniformly-continuous-map m1 m2 f (is ?g1)

and *uniformly-continuous-map-to-capped-metric:uniformly-continuous-map m1*
(capped-metric e m2) f \longleftrightarrow uniformly-continuous-map m1 m2 f (is ?g2)

<proof>

lemma *Urysohn-lemma-uniform:*

assumes *closedin (mtopology-of m) T closedin (mtopology-of m) U* $T \cap U = \{\}$
 $\bigwedge x y. x \in T \implies y \in U \implies \text{mdist } m x y \geq e$ $e > 0$

obtains $f :: 'a \Rightarrow \text{real}$

where *uniformly-continuous-map m euclidean-metric f*

$\bigwedge x. f x \geq 0$ $\bigwedge x. f x \leq 1$ $\bigwedge x. x \in T \implies f x = 1$ $\bigwedge x. x \in U \implies f x = 0$

<proof>

Open maps

lemma *Metric-space-open-map-from-dist:*

assumes $f \in \text{mspace } m1 \rightarrow \text{mspace } m2$

and $\bigwedge x \varepsilon. x \in \text{mspace } m1 \implies \varepsilon > 0 \implies \exists \delta > 0. \forall y \in \text{mspace } m1. \text{mdist } m2$
 $(f x) (f y) < \delta \implies \text{mdist } m1 x y < \varepsilon$

shows *open-map (mtopology-of m1) (subtopology (mtopology-of m2) (f 'mspace*
m1)) f

<proof>

1.3.3 Separability in Metric Spaces

context *Metric-space*

begin

For a metric space M , M is separable iff M is second countable.

lemma *generated-by-countable-balls:*

assumes *countable* U **and** *mdense* U

shows $m\text{topology} = \text{topology-generated-by } \{\text{mball } y (1 / \text{real } n) \mid y \text{ n. } y \in U\}$

<proof>

lemma *separable-space-imp-second-countable:*

assumes *separable-space* $m\text{topology}$

shows *second-countable* $m\text{topology}$

<proof>

corollary *separable-space-iff-second-countable:*

separable-space $m\text{topology} \longleftrightarrow \text{second-countable } m\text{topology}$

<proof>

lemma *Lindelof-mdiameter:*

assumes *separable-space* $m\text{topology}$ $0 < e$

shows $\exists U. \text{countable } U \wedge \bigcup U = M \wedge (\forall u \in U. \text{mdiameter } u < \text{ennreal } e)$

<proof>

end

lemma *metrizable-space-separable-iff-second-countable:*

assumes *metrizable-space* X

shows *separable-space* $X \longleftrightarrow \text{second-countable } X$

<proof>

abbreviation *mdense-of* m $U \equiv \text{dense-in } (m\text{topology-of } m) U$

lemma *mdense-of-def:* $\text{mdense-of } m U \longleftrightarrow (U \subseteq m\text{space } m \wedge (\forall x \in m\text{space } m. \forall \varepsilon > 0. \text{mball-of } m x \varepsilon \cap U \neq \{\}))$

<proof>

lemma *mdense-of-def2:* $\text{mdense-of } m U \longleftrightarrow (U \subseteq m\text{space } m \wedge (\forall x \in m\text{space } m. \forall \varepsilon > 0. \exists y \in U. \text{mdist } m x y < \varepsilon))$

<proof>

lemma *mdense-of-def3:* $\text{mdense-of } m U \longleftrightarrow (U \subseteq m\text{space } m \wedge (\forall x \in m\text{space } m. \exists xn. \text{range } xn \subseteq U \wedge \text{limitin } (m\text{topology-of } m) xn x \text{ sequentially}))$

<proof>

1.3.4 Compact Metric Spaces

context *Metric-space*

begin

lemma *mtotally-bounded-eq-compact-closedin:*

assumes *mcomplete closedin mtopology S*
shows *mtotally-bounded S* \longleftrightarrow *S* \subseteq *M* \wedge *compactin mtopology S*
 \langle *proof* \rangle

lemma *mtotally-bounded-def2*: *mtotally-bounded S* \longleftrightarrow $(\forall \varepsilon > 0. \exists K. \text{finite } K \wedge K \subseteq M \wedge S \subseteq (\bigcup_{x \in K}. \text{mball } x \ \varepsilon))$
 \langle *proof* \rangle

lemma *compact-space-imp-separable*:
assumes *compact-space mtopology*
shows *separable-space mtopology*
 \langle *proof* \rangle

lemma *separable-space-cfunspace*:
assumes *separable-space mtopology mcomplete*
and *metrizable-space X compact-space X*
shows *separable-space (mtopology-of (cfunspace X Self))*
 \langle *proof* \rangle

end

context *Submetric*
begin

lemma *separable-sub*:
assumes *separable-space mtopology*
shows *separable-space sub.mtopology*
 \langle *proof* \rangle

end

1.3.5 Discrete Distance

lemma(*in discrete-metric*) *separable-space-iff*: *separable-space disc.mtopology* \longleftrightarrow *countable M*
 \langle *proof* \rangle

1.3.6 Binary Product Metric Spaces

We define the L^1 -distance. L^1 -distance and L^2 distance (Euclid distance) generate the same topological space.

definition *prod-dist-L1* $\equiv \lambda d1 \ d2 \ (x,y) \ (x',y'). \ d1 \ x \ x' + d2 \ y \ y'$

context *Metric-space12*
begin

lemma *prod-L1-metric*: *Metric-space (M1 \times M2) (prod-dist-L1 d1 d2)*
 \langle *proof* \rangle

sublocale *Prod-metric-L1*: Metric-space $M1 \times M2$ *prod-dist-L1* $d1$ $d2$
{proof}

lemma *prod-dist-L1-geq*:
shows $d1\ x\ y \leq \text{prod-dist-L1}\ d1\ d2\ (x,x')\ (y,y')$
 $d2\ x'\ y' \leq \text{prod-dist-L1}\ d1\ d2\ (x,x')\ (y,y')$
{proof}

lemma *prod-dist-L1-ball*:
assumes $(x,x') \in \text{Prod-metric-L1.mball}\ (a,a')\ \varepsilon$
shows $x \in M1.mball\ a\ \varepsilon$
and $x' \in M2.mball\ a'\ \varepsilon$
{proof}

lemma *prod-dist-L1-ball'*:
assumes $z \in \text{Prod-metric-L1.mball}\ a\ \varepsilon$
shows $\text{fst}\ z \in M1.mball\ (\text{fst}\ a)\ \varepsilon$
and $\text{snd}\ z \in M2.mball\ (\text{snd}\ a)\ \varepsilon$
{proof}

lemma *prod-dist-L1-ball1'*: $\text{Prod-metric-L1.mball}\ (a1,a2)\ (\text{min}\ e1\ e2) \subseteq M1.mball\ a1\ e1 \times M2.mball\ a2\ e2$
{proof}

lemma *prod-dist-L1-ball1*:
assumes $b1 \in M1.mball\ a1\ e1$ $b2 \in M2.mball\ a2\ e2$
shows $\exists e12 > 0. \text{Prod-metric-L1.mball}\ (b1,b2)\ e12 \subseteq M1.mball\ a1\ e1 \times M2.mball\ a2\ e2$
{proof}

lemma *prod-dist-L1-ball2'*:
 $M1.mball\ a1\ e1 \times M2.mball\ a2\ e2 \subseteq \text{Prod-metric-L1.mball}\ (a1,a2)\ (e1 + e2)$
{proof}

lemma *prod-dist-L1-ball2*:
assumes $(b1,b2) \in \text{Prod-metric-L1.mball}\ (a1,a2)\ e12$
shows $\exists e1 > 0. \exists e2 > 0. M1.mball\ b1\ e1 \times M2.mball\ b2\ e2 \subseteq \text{Prod-metric-L1.mball}\ (a1,a2)\ e12$
{proof}

lemma *prod-dist-L1-mtopology*:
 $\text{Prod-metric-L1.mtopology} = \text{prod-topology}\ M1.mtopology\ M2.mtopology$
{proof}

lemma *prod-dist-L1-limitin-iff*: $\text{limitin}\ \text{Prod-metric-L1.mtopology}\ zn\ z\ \text{sequentially} \iff \text{limitin}\ M1.mtopology\ (\lambda n. \text{fst}\ (zn\ n))\ (\text{fst}\ z)\ \text{sequentially} \wedge \text{limitin}\ M2.mtopology\ (\lambda n. \text{snd}\ (zn\ n))\ (\text{snd}\ z)\ \text{sequentially}$
{proof}

lemma *prod-dist-L1-MCauchy-iff*: *Prod-metric-L1.MCauchy zn* \longleftrightarrow *M1.MCauchy*
 $(\lambda n. \text{fst } (zn \ n)) \wedge \text{M2.MCauchy } (\lambda n. \text{snd } (zn \ n))$
 $\langle \text{proof} \rangle$

end

1.3.7 Sum Metric Spaces

locale *Sum-metric* =
fixes $I :: 'i \text{ set}$
and $Mi :: 'i \Rightarrow 'a \text{ set}$
and $di :: 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{real}$
assumes *Mi-disj*: *disjoint-family-on* $Mi \ I$
and *d-nonneg*: $\bigwedge i \ x \ y. 0 \leq di \ i \ x \ y$
and *d-bounded*: $\bigwedge i \ x \ y. di \ i \ x \ y < 1$
and *Md-metric*: $\bigwedge i. i \in I \Longrightarrow \text{Metric-space } (Mi \ i) \ (di \ i)$
begin

abbreviation $M \equiv \bigcup_{i \in I}. Mi \ i$

lemma *Mi-inj-on*:
assumes $i \in I \ j \in I \ a \in Mi \ i \ a \in Mi \ j$
shows $i = j$
 $\langle \text{proof} \rangle$

definition *sum-dist* :: $['a, 'a] \Rightarrow \text{real}$ **where**
 $\text{sum-dist } x \ y \equiv (\text{if } x \in M \wedge y \in M \text{ then } (\text{if } \exists i \in I. x \in Mi \ i \wedge y \in Mi \ i \text{ then } di$
 $(THE \ i. i \in I \wedge x \in Mi \ i \wedge y \in Mi \ i) \ x \ y \text{ else } 1) \text{ else } 0)$

lemma *sum-dist-simps*:
shows $\bigwedge i. \llbracket i \in I; x \in Mi \ i; y \in Mi \ i \rrbracket \Longrightarrow \text{sum-dist } x \ y = di \ i \ x \ y$
and $\bigwedge i \ j. \llbracket i \in I; j \in I; i \neq j; x \in Mi \ i; y \in Mi \ j \rrbracket \Longrightarrow \text{sum-dist } x \ y = 1$
and $\bigwedge i. \llbracket i \in I; y \in M; x \in Mi \ i; y \notin Mi \ i \rrbracket \Longrightarrow \text{sum-dist } x \ y = 1$
and $\bigwedge i. \llbracket i \in I; x \in M; y \in Mi \ i; x \notin Mi \ i \rrbracket \Longrightarrow \text{sum-dist } x \ y = 1$
and $x \notin M \Longrightarrow \text{sum-dist } x \ y = 0 \ y \notin M \Longrightarrow \text{sum-dist } x \ y = 0$
 $\langle \text{proof} \rangle$

lemma *sum-dist-if-less1*:
assumes $i \in I \ x \in Mi \ i \ y \in M \ \text{sum-dist } x \ y < 1$
shows $y \in Mi \ i$
 $\langle \text{proof} \rangle$

lemma *inM-cases*:
assumes $x \in M \ y \in M$
and $\bigwedge i. \llbracket i \in I; x \in Mi \ i; y \in Mi \ i \rrbracket \Longrightarrow P \ x \ y$
and $\bigwedge i \ j. \llbracket i \in I; j \in I; i \neq j; x \in Mi \ i; y \in Mi \ j; x \neq y \rrbracket \Longrightarrow P \ x \ y$
shows $P \ x \ y \ \langle \text{proof} \rangle$

sublocale *Sum-metric*: *Metric-space* $M \ \text{sum-dist}$

<proof>

lemma *sum-dist-le1*: *sum-dist x y ≤ 1*
<proof>

lemma *sum-dist-ball-eq-ball*:
assumes $i \in I$ $e \leq 1$ $x \in M_i$
shows $Metric\text{-space.mball } (M_i) (d_i) x e = Sum\text{-metric.mball } x e$
<proof>

lemma *ball-le-sum-dist-ball*:
assumes $i \in I$
shows $Metric\text{-space.mball } (M_i) (d_i) x e \subseteq Sum\text{-metric.mball } x e$
<proof>

lemma *openin-mtopology-iff*:
 $openin\ Sum\text{-metric.mtopology } U \longleftrightarrow U \subseteq M \wedge (\forall i \in I. openin (Metric\text{-space.mtopology } (M_i) (d_i)) (U \cap M_i))$
<proof>

corollary *openin-mtopology-Mi*:
assumes $i \in I$
shows $openin\ Sum\text{-metric.mtopology } (M_i)$
<proof>

corollary *subtopology-mtopology-Mi*:
assumes $i \in I$
shows $subtopology\ Sum\text{-metric.mtopology } (M_i) = Metric\text{-space.mtopology } (M_i) (d_i)$
<proof>

lemma *limitin-Mi-limitin-M*:
assumes $i \in I$ *limitin (Metric-space.mtopology (M_i) (d_i)) xn x sequentially*
shows *limitin Sum-metric.mtopology xn x sequentially*
<proof>

lemma *limitin-M-limitin-Mi*:
assumes *limitin Sum-metric.mtopology xn x sequentially*
shows $\exists i \in I. limitin (Metric\text{-space.mtopology } (M_i) (d_i)) xn x sequentially$
<proof>

lemma *MCauchy-Mi-MCauchy-M*:
assumes $i \in I$ $Metric\text{-space.MCauchy } (M_i) (d_i) xn$
shows $Sum\text{-metric.MCauchy } xn$
<proof>

lemma *MCauchy-M-MCauchy-Mi*:
assumes $Sum\text{-metric.MCauchy } xn$

shows $\exists m. \exists i \in I. \text{Metric-space.MCauchy } (M\ i) \ (d\ i) \ (\lambda n. x\ n \ (n + m))$
 <proof>

lemma *separable-Mi-separable-M*:

assumes *countable I* $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (M\ i) \ (d\ i))$

shows *separable-space Sum-metric.mtopology*
 <proof>

lemma *separable-M-separable-Mi*:

assumes *separable-space Sum-metric.mtopology* $\bigwedge i. i \in I$

shows *separable-space* $(\text{Metric-space.mtopology } (M\ i) \ (d\ i))$
 <proof>

lemma *mcomplete-Mi-mcomplete-M*:

assumes $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (M\ i) \ (d\ i)$

shows *Sum-metric.mcomplete*
 <proof>

lemma *mcomplete-M-mcomplete-Mi*:

assumes *Sum-metric.mcomplete* $i \in I$

shows *Metric-space.mcomplete* $(M\ i) \ (d\ i)$
 <proof>

end

lemma *sum-metricI*:

fixes *Si*

assumes *disjoint-family-on Si I*

and $\bigwedge i\ x\ y. i \notin I \implies 0 \leq d\ i\ x\ y$

and $\bigwedge i\ x\ y. d\ i\ x\ y < 1$

and $\bigwedge i. i \in I \implies \text{Metric-space } (S\ i) \ (d\ i)$

shows *Sum-metric I Si di*

<proof>

end

1.3.8 Product Metric Spaces

theory *Set-Based-Metric-Product*

imports *Set-Based-Metric-Space*

begin

lemma *nsum-of-r'*:

fixes $r :: \text{real}$

assumes $r:0 < r < 1$

shows $(\sum n. r^{n+k} * K) = r^k / (1 - r) * K$

(is ?lhs = -)
 <proof>

lemma *nsum-of-r-leq*:

fixes $r :: \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{real}$
assumes $r:0 < r$ $r < 1$
and $a:\bigwedge n. 0 \leq a\ n \wedge n. a\ n \leq K$
shows $0 \leq (\sum n. r^\wedge(n+k) * a\ (n+l)) (\sum n. r^\wedge(n+k) * a\ (n+l)) \leq r^\wedge k$
 $/ (1 - r) * K$
 <proof>

lemma *nsum-of-r-le*:

fixes $r :: \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{real}$
assumes $r:0 < r$ $r < 1$
and $a:\bigwedge n. 0 \leq a\ n \wedge n. a\ n \leq K \exists n' \geq l. a\ n' < K$
shows $(\sum n. r^\wedge(n+k) * a\ (n+l)) < r^\wedge k / (1 - r) * K$
 <proof>

definition *product-dist'* :: $[\text{real}, 'i \text{ set}, \text{nat} \Rightarrow 'i, 'i \Rightarrow 'a \text{ set}, 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{real}]$
 $\Rightarrow ('i \Rightarrow 'a) \Rightarrow ('i \Rightarrow 'a) \Rightarrow \text{real}$ **where**

product-dist-def: $\text{product-dist}'\ r\ I\ g\ Mi\ di \equiv (\lambda x\ y. \text{if } x \in (\prod_E i \in I. Mi\ i) \wedge y \in (\prod_E i \in I. Mi\ i) \text{ then } (\sum n. \text{if } g\ n \in I \text{ then } r^\wedge n * di\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n)) \text{ else } 0) \text{ else } 0)$

$$d(x, y) = \sum_{n \in \mathbb{N}} r^n * d_{g_I(i)}(x_{g_I(i)}, y_{g_I(i)}).$$

locale *Product-metric* =

fixes $r :: \text{real}$
and $I :: 'i \text{ set}$
and $f :: 'i \Rightarrow \text{nat}$
and $g :: \text{nat} \Rightarrow 'i$
and $Mi :: 'i \Rightarrow 'a \text{ set}$
and $di :: 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{real}$
and $K :: \text{real}$
assumes $r: 0 < r$ $r < 1$
and I : countable I
and *gf-comp-id* : $\bigwedge i. i \in I \implies g\ (f\ i) = i$
and *gf-if-finite*: $\text{finite } I \implies \text{bij-betw } f\ I\ \{.. < \text{card } I\}$
 $\text{finite } I \implies \text{bij-betw } g\ \{.. < \text{card } I\}\ I$
and *gf-if-infinite*: $\text{infinite } I \implies \text{bij-betw } f\ I\ \text{UNIV}$
 $\text{infinite } I \implies \text{bij-betw } g\ \text{UNIV } I$
 $\bigwedge n. \text{infinite } I \implies f\ (g\ n) = n$
and *Md-metric*: $\bigwedge i. i \in I \implies \text{Metric-space } (Mi\ i)\ (di\ i)$
and *di-nonneg*: $\bigwedge i\ x\ y. 0 \leq di\ i\ x\ y$
and *di-bounded*: $\bigwedge i\ x\ y. di\ i\ x\ y \leq K$
and *K-pos* : $0 < K$

lemma *from-nat-into-to-nat-on-product-metric-pair*:

assumes countable I
shows $\bigwedge i. i \in I \implies \text{from-nat-into } I\ (\text{to-nat-on } I\ i) = i$

and $\text{finite } I \implies \text{bij-betw } (\text{to-nat-on } I) I \{..< \text{card } I\}$
and $\text{finite } I \implies \text{bij-betw } (\text{from-nat-into } I) \{..< \text{card } I\} I$
and $\text{infinite } I \implies \text{bij-betw } (\text{to-nat-on } I) I \text{ UNIV}$
and $\text{infinite } I \implies \text{bij-betw } (\text{from-nat-into } I) \text{ UNIV } I$
and $\bigwedge n. \text{infinite } I \implies \text{to-nat-on } I (\text{from-nat-into } I n) = n$
 $\langle \text{proof} \rangle$

lemma *product-metric-pair-finite-nat*:

$\text{bij-betw } \text{id } \{..n\} \{..< \text{card } \{..n\}\} \text{bij-betw } \text{id } \{..< \text{card } \{..n\}\} \{..n\}$
 $\langle \text{proof} \rangle$

lemma *product-metric-pair-finite-nat'*:

$\text{bij-betw } \text{id } \{..<n\} \{..< \text{card } \{..<n\}\} \text{bij-betw } \text{id } \{..< \text{card } \{..<n\}\} \{..<n\}$
 $\langle \text{proof} \rangle$

context *Product-metric*

begin

abbreviation $\text{product-dist} \equiv \text{product-dist}' r I g M i di$

lemma *nsum-of-rK*: $(\sum n. r \hat{\wedge} (n + k) * K) = r \hat{\wedge} k / (1 - r) * K$
 $\langle \text{proof} \rangle$

lemma *i-min*:

assumes $i \in I \ g \ n = i$
shows $f \ i \leq n$
 $\langle \text{proof} \rangle$

lemma *g-surj*:

assumes $i \in I$
shows $\exists n. g \ n = i$
 $\langle \text{proof} \rangle$

lemma *product-dist-summable'[simp]*:

$\text{summable } (\lambda n. r \hat{\wedge} n * di (g \ n) (x (g \ n)) (y (g \ n)))$
 $\langle \text{proof} \rangle$

lemma *product-dist-summable[simp]*:

$\text{summable } (\lambda n. \text{if } g \ n \in I \text{ then } r \hat{\wedge} n * di (g \ n) (x (g \ n)) (y (g \ n)) \text{ else } 0)$
 $\langle \text{proof} \rangle$

lemma *summable-rK[simp]*: $\text{summable } (\lambda n. r \hat{\wedge} n * K)$

$\langle \text{proof} \rangle$

lemma *Product-metric: Metric-space* $(\Pi_E \ i \in I. M i \ i) \text{ product-dist}$

$\langle \text{proof} \rangle$

sublocale *Product-metric: Metric-space* $\Pi_E \ i \in I. M i \ i \text{ product-dist}$

$\langle \text{proof} \rangle$

lemma *product-dist-leqr*: $\text{product-dist } x \ y \leq 1 / (1 - r) * K$
 ⟨proof⟩

lemma *product-dist-geq*:
 assumes $i \in I$ and $g \ n = i \ x \in (\prod_{E \ i \in I. \ Mi \ i}) \ y \in (\prod_{E \ i \in I. \ Mi \ i})$
 shows $di \ i \ (x \ i) \ (y \ i) \leq (1/r)^{\wedge n} * \text{product-dist } x \ y$
 (is ?lhs \leq ?rhs)
 ⟨proof⟩

lemma *limitin-M-iff-limitin-Mi*:
 shows $\text{limitin } \text{Product-metric.mtopology } \text{xn } x \ \text{sequentially} \longleftrightarrow (\exists N. \forall n \geq N. (\forall i \in I. \ \text{xn } n \ i \in \text{Mi } i) \wedge (\forall i. \ i \notin I \longrightarrow \text{xn } n \ i = \text{undefined})) \wedge (\forall i \in I. \ \text{limitin } (\text{Metric-space.mtopology } (\text{Mi } i) \ (di \ i)) \ (\lambda n. \ \text{xn } n \ i) \ (x \ i) \ \text{sequentially}) \wedge x \in (\prod_{E \ i \in I. \ \text{Mi } i})$
 ⟨proof⟩

lemma *Product-metric-mtopology-eq*: $\text{product-topology } (\lambda i. \ \text{Metric-space.mtopology } (\text{Mi } i) \ (di \ i)) \ I = \text{Product-metric.mtopology}$
 ⟨proof⟩

corollary *separable-Mi-separable-M*:
 assumes $\bigwedge i. \ i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (\text{Mi } i) \ (di \ i))$
 shows $\text{separable-space } \text{Product-metric.mtopology}$
 ⟨proof⟩

lemma *mcomplete-Mi-mcomplete-M*:
 assumes $\bigwedge i. \ i \in I \implies \text{Metric-space.mcomplete } (\text{Mi } i) \ (di \ i)$
 shows $\text{Product-metric.mcomplete}$
 ⟨proof⟩

end

lemma *product-metricI*:
 assumes $0 < r \ r < 1$ countable $I \ \bigwedge i. \ i \in I \implies \text{Metric-space } (\text{Mi } i) \ (di \ i)$
 and $\bigwedge i \ x \ y. \ 0 \leq di \ i \ x \ y \ \bigwedge i \ x \ y. \ di \ i \ x \ y \leq K \ 0 < K$
 shows $\text{Product-metric } r \ I \ (\text{to-nat-on } I) \ (\text{from-nat-into } I) \ \text{Mi } di \ K$
 ⟨proof⟩

lemma *product-metric-natI*:
 assumes $0 < r \ r < 1 \ \bigwedge n. \ \text{Metric-space } (\text{Mi } n) \ (di \ n)$
 and $\bigwedge i \ x \ y. \ 0 \leq di \ i \ x \ y \ \bigwedge i \ x \ y. \ di \ i \ x \ y \leq K \ 0 < K$
 shows $\text{Product-metric } r \ \text{UNIV } id \ id \ \text{Mi } di \ K$
 ⟨proof⟩

end

2 Abstract Polish Spaces

theory *Abstract-Metrizable-Topology*
imports *Set-Based-Metric-Product*
begin

2.1 Polish Spaces

definition *Polish-space* $X \equiv \text{completely-metrizable-space } X \wedge \text{separable-space } X$

lemma(in *Metric-space*) *Polish-space-mtopology*:
assumes *mcomplete separable-space mtopology*
shows *Polish-space mtopology*
<proof>

lemma
assumes *Polish-space X*
shows *Polish-space-imp-completely-metrizable-space: completely-metrizable-space X*
and *Polish-space-imp-metrizable-space: metrizable-space X*
and *Polish-space-imp-second-countable: second-countable X*
and *Polish-space-imp-separable-space: separable-space X*
<proof>

lemma *Polish-space-closedin*:
assumes *Polish-space X closedin X A*
shows *Polish-space (subtopology X A)*
<proof>

lemma *Polish-space-gdelta-in*:
assumes *Polish-space X gdelta-in X A*
shows *Polish-space (subtopology X A)*
<proof>

corollary *Polish-space-openin*:
assumes *Polish-space X openin X A*
shows *Polish-space (subtopology X A)*
<proof>

lemma *homeomorphic-Polish-space-aux*:
assumes *Polish-space X X homeomorphic-space Y*
shows *Polish-space Y*
<proof>

corollary *homeomorphic-Polish-space*:
assumes *X homeomorphic-space Y*
shows *Polish-space X \longleftrightarrow Polish-space Y*
<proof>

lemma *Polish-space-euclidean[simp]*: *Polish-space (euclidean :: ('a :: polish-space))*

topology)
(*proof*)

lemma *Polish-space-countable[simp]*:

Polish-space (euclidean :: 'a :: {countable, discrete-topology} topology)
(*proof*)

lemma *Polish-space-discrete-topology: Polish-space (discrete-topology I) \longleftrightarrow countable I*

(*proof*)

lemma *Polish-space-prod*:

assumes *Polish-space X and Polish-space Y*

shows *Polish-space (prod-topology X Y)*

(*proof*)

lemma *Polish-space-product*:

assumes *countable I and $\bigwedge i. i \in I \implies \text{Polish-space } (S\ i)$*

shows *Polish-space (product-topology S I)*

(*proof*)

lemma(**in** *Product-metric*) *Polish-spaceI*:

assumes $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (M\ i)\ (d\ i))$

and $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (M\ i)\ (d\ i)$

shows *Polish-space Product-metric.mtopology*

(*proof*)

lemma(**in** *Sum-metric*) *Polish-spaceI*:

assumes *countable I*

and $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (M\ i)\ (d\ i))$

and $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (M\ i)\ (d\ i)$

shows *Polish-space Sum-metric.mtopology*

(*proof*)

lemma *compact-metrizable-imp-Polish-space*:

assumes *metrizable-space X compact-space X*

shows *Polish-space X*

(*proof*)

2.2 Extended Reals and Non-Negative Extended Reals

lemma *Polish-space-ereal:Polish-space (euclidean :: ereal topology)*

(*proof*)

corollary *Polish-space-ennreal:Polish-space (euclidean :: ennreal topology)*

(*proof*)

2.3 Continuous Embddings

abbreviation *Hilbert-cube-topology :: (nat \Rightarrow real) topology where*

Hilbert-cube-topology \equiv (*product-topology* ($\lambda n.$ *top-of-set* $\{0..1\}$) *UNIV*)

lemma *topspace-Hilbert-cube: topspace Hilbert-cube-topology* = (Π_E $x \in UNIV.$ $\{0..1\}$)
(*proof*)

lemma *Polish-space-Hilbert-cube: Polish-space Hilbert-cube-topology*
(*proof*)

abbreviation *Cantor-space-topology* :: (*nat* \Rightarrow *real*) *topology* **where**
Cantor-space-topology \equiv (*product-topology* ($\lambda n.$ *top-of-set* $\{0,1\}$) *UNIV*)

lemma *topspace-Cantor-space:*
topspace Cantor-space-topology = (Π_E $x \in UNIV.$ $\{0,1\}$)
(*proof*)

lemma *Polish-space-Cantor-space: Polish-space Cantor-space-topology*
(*proof*)

corollary *completely-metrizable-space-homeo-image-gdelta-in:*
assumes *completely-metrizable-space* X *completely-metrizable-space* Y $B \subseteq$ *topspace*
 Y X *homeomorphic-space* *subtopology* Y B
shows *gdelta-in* Y B
(*proof*)

2.3.1 Embedding into Hilbert Cube

lemma *embedding-into-Hilbert-cube:*
assumes *metrizable-space* X *separable-space* X
shows $\exists A \subseteq$ *topspace Hilbert-cube-topology.* X *homeomorphic-space* (*subtopology*
Hilbert-cube-topology A)
(*proof*)

corollary *embedding-into-Hilbert-cube-gdelta-in:*
assumes *Polish-space* X
shows $\exists A.$ *gdelta-in Hilbert-cube-topology* $A \wedge X$ *homeomorphic-space* (*subtopology*
Hilbert-cube-topology A)
(*proof*)

2.3.2 Embedding from Cantor Space

lemma *embedding-from-Cantor-space:*
assumes *Polish-space* X *uncountable* (*topspace* X)
shows $\exists A.$ *gdelta-in* X $A \wedge$ *Cantor-space-topology* *homeomorphic-space* (*subtopology*
 X A)
(*proof*)

2.4 Borel Spaces generated from Polish Spaces

lemma *closedin-clopen-topology:*
assumes *Polish-space* X *closedin* X a

shows $\exists X'. \text{Polish-space } X' \wedge (\forall u. \text{openin } X u \longrightarrow \text{openin } X' u) \wedge \text{topspace } X = \text{topspace } X' \wedge \text{sets (borel-of } X) = \text{sets (borel-of } X') \wedge \text{openin } X' a \wedge \text{closedin } X' a$
 ⟨proof⟩

lemma *Polish-space-union-Polish:*

fixes $X :: \text{nat} \Rightarrow 'a \text{ topology}$
assumes $\bigwedge n. \text{Polish-space } (X n) \wedge n. \text{topspace } (X n) = Xn \wedge x y. x \in Xn \implies y \in Xn \implies x \neq y \implies \exists Ox Oy. (\forall n. \text{openin } (X n) Ox) \wedge (\forall n. \text{openin } (X n) Oy) \wedge x \in Ox \wedge y \in Oy \wedge \text{disjnt } Ox Oy$
defines $Xun \equiv \text{topology-generated-by } (\bigcup n. \{u. \text{openin } (X n) u\})$
shows *Polish-space* Xun
 ⟨proof⟩

lemma *sets-clopen-topology:*

assumes *Polish-space* $X a \in \text{sets (borel-of } X)$
shows $\exists X'. \text{Polish-space } X' \wedge (\forall u. \text{openin } X u \longrightarrow \text{openin } X' u) \wedge \text{topspace } X = \text{topspace } X' \wedge \text{sets (borel-of } X) = \text{sets (borel-of } X') \wedge \text{openin } X' a \wedge \text{closedin } X' a$
 ⟨proof⟩

end

3 Standard Borel Spaces

3.1 Standard Borel Spaces

theory *StandardBorel*

imports *Abstract-Metrizable-Topology*
begin

locale *standard-borel* =

fixes $M :: 'a \text{ measure}$
assumes *Polish-space:* $\exists S. \text{Polish-space } S \wedge \text{sets } M = \text{sets (borel-of } S)$
begin

lemma *singleton-sets:*

assumes $x \in \text{space } M$
shows $\{x\} \in \text{sets } M$
 ⟨proof⟩

corollary *countable-sets:*

assumes $A \subseteq \text{space } M$ *countable* A
shows $A \in \text{sets } M$
 ⟨proof⟩

lemma *standard-borel-restrict-space:*

assumes $A \in \text{sets } M$
shows *standard-borel (restrict-space* $M A)$

<proof>

end

locale *standard-borel-ne* = *standard-borel* +
 assumes *space-ne*: *space* $M \neq \{\}$
begin

lemma *standard-borel-ne-restrict-space*:
 assumes $A \in \text{sets } M$ $A \neq \{\}$
 shows *standard-borel-ne* (*restrict-space* M A)
 <proof>

lemma *standard-borel*: *standard-borel* M
 <proof>

end

lemma *standard-borel-sets*:
 assumes *standard-borel* M **and** *sets* $M = \text{sets } N$
 shows *standard-borel* N
 <proof>

lemma *standard-borel-ne-sets*:
 assumes *standard-borel-ne* M **and** *sets* $M = \text{sets } N$
 shows *standard-borel-ne* N
 <proof>

lemma *pair-standard-borel*:
 assumes *standard-borel* M *standard-borel* N
 shows *standard-borel* $(M \otimes_M N)$
 <proof>

lemma *pair-standard-borel-ne*:
 assumes *standard-borel-ne* M *standard-borel-ne* N
 shows *standard-borel-ne* $(M \otimes_M N)$
 <proof>

lemma *product-standard-borel*:
 assumes *countable* I
 and $\bigwedge i. i \in I \implies \text{standard-borel } (M i)$
 shows *standard-borel* $(\prod_M i \in I. M i)$
 <proof>

lemma *product-standard-borel-ne*:
 assumes *countable* I
 and $\bigwedge i. i \in I \implies \text{standard-borel-ne } (M i)$
 shows *standard-borel-ne* $(\prod_M i \in I. M i)$
 <proof>

lemma *closed-set-standard-borel*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a$ topology) *closed* U
shows *standard-borel* (*restrict-space borel* U)
<proof>

lemma *closed-set-standard-borel-ne*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a$ topology) *closed* U $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel* U)
<proof>

lemma *open-set-standard-borel*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a$ topology) *open* U
shows *standard-borel* (*restrict-space borel* U)
<proof>

lemma *open-set-standard-borel-ne*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a$ topology) *open* U $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel* U)
<proof>

lemma *standard-borel-ne-borel*[simp]: *standard-borel-ne* (*borel* :: ($'a :: \text{polish-space}$)
measure)
and *standard-borel-ne-lborel*[simp]: *standard-borel-ne lborel*
<proof>

lemma *count-space-standard'*[simp]:
assumes *countable* I
shows *standard-borel* (*count-space* I)
<proof>

lemma *count-space-standard-ne*[simp]: *standard-borel-ne* (*count-space* ($UNIV :: (-$
 $:: \text{countable}) \text{ set}$))
<proof>

corollary *measure-pmf-standard-borel-ne*[simp]: *standard-borel-ne* (*measure-pmf* (p
 $:: (- :: \text{countable}) \text{ pmf}$))
<proof>

corollary *measure-spmf-standard-borel-ne*[simp]: *standard-borel-ne* (*measure-spmf*
($p :: (- :: \text{countable}) \text{ spmf}$))
<proof>

corollary *countable-standard-ne*[simp]:
standard-borel-ne (*borel* :: $'a :: \{\text{countable}, t2\text{-space}\}$ *measure*)

<proof>

lemma(in *standard-borel*) *countable-discrete-space*:

assumes *countable (space M)*

shows *sets M = Pow (space M)*

<proof>

lemma(in *standard-borel*) *measurable-isomorphic-standard*:

assumes *M measurable-isomorphic N*

shows *standard-borel N*

<proof>

lemma(in *standard-borel-ne*) *measurable-isomorphic-standard-ne*:

assumes *M measurable-isomorphic N*

shows *standard-borel-ne N*

<proof>

lemma(in *standard-borel*) *standard-borel-embed-measure*:

assumes *inj-on f (space M)*

shows *standard-borel (embed-measure M f)*

<proof>

corollary(in *standard-borel-ne*) *standard-borel-ne-embed-measure*:

assumes *inj-on f (space M)*

shows *standard-borel-ne (embed-measure M f)*

<proof>

lemma

shows *standard-ne-ereal: standard-borel-ne (borel :: ereal measure)*

and *standard-ne-ennreal: standard-borel-ne (borel :: ennreal measure)*

<proof>

Cantor space \mathcal{C}

definition *Cantor-space :: (nat \Rightarrow real) measure where*

Cantor-space \equiv ($\Pi_M i \in UNIV$. restrict-space borel $\{0,1\}$)

lemma *Cantor-space-standard-ne: standard-borel-ne Cantor-space*

<proof>

lemma *Cantor-space-borel:*

sets (borel-of Cantor-space-topology) = sets Cantor-space

(is ?lhs = -)

<proof>

Hilbert cube \mathcal{H}

definition *Hilbert-cube :: (nat \Rightarrow real) measure where*

Hilbert-cube \equiv ($\Pi_M i \in UNIV$. restrict-space borel $\{0..1\}$)

lemma *Hilbert-cube-standard-ne: standard-borel-ne Hilbert-cube*

<proof>

lemma *Hilbert-cube-borel:*

sets (borel-of Hilbert-cube-topology) = sets Hilbert-cube (is ?lhs = -)
<proof>

3.2 Isomorphism between \mathcal{C} and \mathcal{H}

lemma *Cantor-space-isomorphic-to-Hilbert-cube:*

Cantor-space measurable-isomorphic Hilbert-cube
<proof>

3.3 Final Results

lemma(*in standard-borel*) *embedding-into-Hilbert-cube:*

$\exists A \in \text{sets Hilbert-cube. } M \text{ measurable-isomorphic (restrict-space Hilbert-cube } A)$
<proof>

lemma(*in standard-borel*) *embedding-from-Cantor-space:*

assumes *uncountable (space M)*
shows *$\exists A \in \text{sets } M. \text{ Cantor-space measurable-isomorphic (restrict-space } M A)$*
<proof>

corollary(*in standard-borel*) *uncountable-isomorphic-to-Hilbert-cube:*

assumes *uncountable (space M)*
shows *Hilbert-cube measurable-isomorphic M*
<proof>

corollary(*in standard-borel*) *uncountable-isomorphic-to-real:*

assumes *uncountable (space M)*
shows *M measurable-isomorphic (borel :: real measure)*
<proof>

lemma(*in standard-borel*) *isomorphic-subset-real:*

assumes *$A \in \text{sets (borel :: real measure) uncountable } A$*
obtains *B where $B \in \text{sets borel } B \subseteq A \text{ } M \text{ measurable-isomorphic restrict-space borel } B$*
<proof>

lemma(*in standard-borel*) *countable-isomorphic-to-subset-real:*

assumes *countable (space M)*
obtains *$A :: \text{real set}$*
where *countable A $A \in \text{sets borel } M \text{ measurable-isomorphic restrict-space borel } A$*
<proof>

theorem *Borel-isomorphism-theorem:*

assumes *standard-borel M standard-borel N*
shows *$\text{space } M \approx \text{space } N \iff M \text{ measurable-isomorphic } N$*
<proof>

definition *to-real-on* :: 'a measure \Rightarrow 'a \Rightarrow real **where**
to-real-on $M \equiv$ (if uncountable (space M) then (SOME f . measurable-isomorphic-map
 M (borel :: real measure) f) else (real \circ to-nat-on (space M)))

definition *from-real-into* :: 'a measure \Rightarrow real \Rightarrow 'a **where**
from-real-into $M \equiv$ (if uncountable (space M) then the-inv-into (space M) (to-real-on
 M) else (λr . from-nat-into (space M) (nat $\lfloor r \rfloor$)))

context *standard-borel*
begin

abbreviation *to-real* \equiv *to-real-on* M

abbreviation *from-real* \equiv *from-real-into* M

lemma *to-real-def-countable*:
assumes countable (space M)
shows *to-real* = (λr . real (to-nat-on (space M) r))
 \langle proof \rangle

lemma *from-real-def-countable*:
assumes countable (space M)
shows *from-real* = (λr . from-nat-into (space M) (nat $\lfloor r \rfloor$))
 \langle proof \rangle

lemma *from-real-to-real[simp]*:
assumes $x \in$ space M
shows *from-real* (to-real x) = x
 \langle proof \rangle

lemma *to-real-measurable[measurable]*:
to-real \in $M \rightarrow_M$ borel
 \langle proof \rangle

lemma *from-real-measurable'*:
assumes space $M \neq \{\}$
shows *from-real* \in borel $\rightarrow_M M$
 \langle proof \rangle

lemma *to-real-from-real*:
assumes uncountable (space M)
shows *to-real* (from-real r) = r
 \langle proof \rangle

end

lemma(in *standard-borel-ne*) *from-real-measurable[measurable]*: *from-real* \in borel
 $\rightarrow_M M$
 \langle proof \rangle

end

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