

The Sophomore's Dream

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Abstract

This article provides a brief formalisation of the two equations known as the *Sophomore's Dream*, first discovered by Johann Bernoulli [1] in 1697:

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n} \quad \text{and} \quad \int_0^1 x^x dx = -\sum_{n=1}^{\infty} (-n)^{-n}$$

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1 The Sophomore's Dream

theory *Sophomores-Dream*

imports *HOL-Analysis.Analysis HOL-Real-Asymp.Real-Asymp*

begin

This formalisation mostly follows the very clear proof sketch from Wikipedia [3]. That article also provides an interesting historical perspective. A more detailed exploration of Bernoulli's historical proof can be found in the book by Dunham [2].

The name 'Sophomore's Dream' apparently comes from a book by Borwein et al., in analogy to the 'Freshman's Dream' equation $(x + y)^n = x^n + y^n$ (which is generally *not* true except in rings of characteristic n).

1.1 Continuity and bounds for $x \log x$

lemma *x-log-x-continuous: continuous-on {0..1}* ($\lambda x::real. x * \ln x$)

proof –

have *continuous (at x within {0..1})* ($\lambda x::real. x * \ln x$) **if** $x \in \{0..1\}$ **for** x

```

proof (cases x = 0)
  case True
  have (( $\lambda x::real. x * \ln x$ )  $\longrightarrow$  0) (at-right 0)
  by real-asymp
  thus ?thesis using True
  by (simp add: continuous-def Lim-ident-at at-within-Icc-at-right)
qed (auto intro!: continuous-intros)
thus ?thesis
using continuous-on-eq-continuous-within by blast
qed

lemma x-log-x-within-01-le:
  assumes x  $\in$  {0..(1::real)}
  shows x * ln x  $\in$  {-exp (-1)..0}
proof -
  have x * ln x  $\leq$  0
  using assms by (cases x = 0) (auto simp: mult-nonneg-nonpos)
  let ?f =  $\lambda x::real. x * \ln x$ 
  have diff: (?f has-field-derivative (ln x + 1)) (at x) if x > 0 for x
  using that by (auto intro!: derivative-eq-intros)
  have diff': ?f differentiable at x if x > 0 for x
  using diff[OF that] real-differentiable-def by blast

  consider x = 0 | x = 1 | x = exp (-1) | 0 < x x < exp (-1) | exp (-1) < x x
  < 1
  using assms unfolding atLeastAtMost-iff by linarith
  hence x * ln x  $\geq$  -exp (-1)
  proof cases
  assume x: 0 < x x < exp (-1)
  have  $\exists l z. x < z \wedge z < \exp (-1) \wedge$  (?f has-real-derivative l) (at z)  $\wedge$ 
    ?f (exp (-1)) - ?f x = (exp (-1) - x) * l
  using x by (intro MVT continuous-on-subset [OF x-log-x-continuous] diff')
  auto
  then obtain l z where lz:
    x < z z < exp (-1) (?f has-real-derivative l) (at z)
    ?f x = -exp (-1) - (exp (-1) - x) * l
  by (auto simp: algebra-simps)
  have [simp]: l = ln z + 1
  using DERIV-unique[OF diff[of z] lz(3)] lz(1) x by auto
  have ln z  $\leq$  ln (exp (-1))
  using lz x by (subst ln-le-cancel-iff) auto
  hence (exp (-1) - x) * l  $\leq$  0
  using x lz by (intro mult-nonneg-nonpos) auto
  with lz show ?thesis
  by linarith
next
  assume x: exp (-1) < x x < 1
  have  $\exists l z. \exp (-1) < z \wedge z < x \wedge$  (?f has-real-derivative l) (at z)  $\wedge$ 
    ?f x - ?f (exp (-1)) = (x - exp (-1)) * l

```

```

proof (intro MVT continuous-on-subset [OF x-log-x-continuous] diff')
  fix t :: real assume t: exp (-1) < t
  show t > 0
    by (rule less-trans [OF - t]) auto
qed (use x in auto)
then obtain l z where lz:
  exp (-1) < z z < x (?f has-real-derivative l) (at z)
  ?f x = -exp (-1) - (exp (-1) - x) * l
  by (auto simp: algebra-simps)
have z > 0
  by (rule less-trans [OF - lz(1)]) auto
have [simp]: l = ln z + 1
  using DERIV-unique[OF diff[of z] lz(3)] ⟨z > 0⟩ by auto
have ln z ≥ ln (exp (-1))
  using lz ⟨z > 0⟩ by (subst ln-le-cancel-iff) auto
hence (exp (-1) - x) * l ≤ 0
  using x lz by (intro mult-nonpos-nonneg) auto
with lz show ?thesis
  by linarith
qed auto

with ⟨x * ln x ≤ 0⟩ show ?thesis
  by auto
qed

```

1.2 Convergence, Summability, Integrability

As a first result we can show that the two sums that occur in the two different versions of the Sophomore's Dream are absolutely summable. This is achieved by a simple comparison test with the series $\sum_{k=1}^{\infty} k^{-2}$, as $k^{-k} \in O(k^{-2})$.

```

theorem abs-summable-sophomores-dream: summable (λk. 1 / real (k ^ k))
proof (rule summable-comparison-test-bigo)
  show (λk. 1 / real (k ^ k)) ∈ O(λk. 1 / real k ^ 2)
    by real-asymp
  show summable (λn. norm (1 / real n ^ 2))
    using inverse-power-summable[of 2, where ?'a = real] by (simp add: field-simps)
qed

```

The existence of the integral is also fairly easy to show since the integrand is continuous and the integration domain is compact. There is, however, one hiccup: The integrand is not actually continuous.

We have $\lim_{x \rightarrow 0} x^x = 1$, but in Isabelle 0^0 is defined as 0 (for real numbers). Thus, there is a discontinuity at $x = 0$

However, this is a removable discontinuity since for any $x > 0$ we have $x^x = e^{x \log x}$, and as we have just shown, $e^{x \log x}$ is continuous on $[0, 1]$. Since the two integrands differ only for $x = 0$ (which is negligible), the integral

still exists.

theorem *integrable-sophomores-dream*: $(\lambda x::real. x \text{ powr } x) \text{ integrable-on } \{0..1\}$

proof –

have $(\lambda x::real. \exp (x * \ln x)) \text{ integrable-on } \{0..1\}$

by (*intro integrable-continuous-real continuous-on-exp x-log-x-continuous*)

also have $?this \longleftrightarrow (\lambda x::real. \exp (x * \ln x)) \text{ integrable-on } \{0 < .. < 1\}$

by (*simp add: integrable-on-Icc-iff-Ioo*)

also have $\dots \longleftrightarrow (\lambda x::real. x \text{ powr } x) \text{ integrable-on } \{0 < .. < 1\}$

by (*intro integrable-cong*) (*auto simp: powr-def*)

also have $\dots \longleftrightarrow ?thesis$

by (*simp add: integrable-on-Icc-iff-Ioo*)

finally show *?thesis* .

qed

Next, we have to show the absolute convergence of the two auxiliary sums that will occur in our proofs so that we can exchange the order of integration and summation. This is done with a straightforward application of the Weierstraß M test.

lemma *uniform-limit-sophomores-dream1*:

uniform-limit $\{0..(1::real)\}$

$(\lambda n x. \sum k < n. (x * \ln x) ^ k / \text{fact } k)$

$(\lambda x. \sum k. (x * \ln x) ^ k / \text{fact } k)$

sequentially

proof (*rule Weierstrass-m-test*)

show *summable* $(\lambda k. \exp (-1) ^ k / \text{fact } k :: real)$

using *summable-exp*[*of exp (-1)*] **by** (*simp add: field-simps*)

next

fix $k :: nat$ **and** $x :: real$

assume $x: x \in \{0..1\}$

have *norm* $((x * \ln x) ^ k / \text{fact } k) = \text{norm } (x * \ln x) ^ k / \text{fact } k$

by (*simp add: power-abs*)

also have $\dots \leq \exp (-1) ^ k / \text{fact } k$

by (*intro divide-right-mono power-mono*) (*use x-log-x-within-01-le* [*of x*] **in auto**)

finally show *norm* $((x * \ln x) ^ k / \text{fact } k) \leq \exp (-1) ^ k / \text{fact } k$.

qed

lemma *uniform-limit-sophomores-dream2*:

uniform-limit $\{0..(1::real)\}$

$(\lambda n x. \sum k < n. -(x * \ln x) ^ k / \text{fact } k)$

$(\lambda x. \sum k. -(x * \ln x) ^ k / \text{fact } k)$

sequentially

proof (*rule Weierstrass-m-test*)

show *summable* $(\lambda k. \exp (-1) ^ k / \text{fact } k :: real)$

using *summable-exp*[*of exp (-1)*] **by** (*simp add: field-simps*)

next

fix $k :: nat$ **and** $x :: real$

assume $x: x \in \{0..1\}$

have *norm* $((-x * \ln x) ^ k / \text{fact } k) = \text{norm } (x * \ln x) ^ k / \text{fact } k$

by (*simp add: power-abs*)
 also have $\dots \leq \exp (-1) \wedge k / \text{fact } k$
 by (*intro divide-right-mono power-mono*) (*use x-log-x-within-01-le [of x] x in auto*)
 finally show $\text{norm } ((-x * \ln x) \wedge k / \text{fact } k) \leq \exp (-1) \wedge k / \text{fact } k$ by *simp*
 qed

1.3 An auxiliary integral

Next we compute the integral

$$\int_0^1 (x \log x)^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}},$$

which is a key ingredient in our proof.

lemma *sophomores-dream-aux-integral*:

$((\lambda x. (x * \ln x) \wedge n) \text{ has-integral } (-1) \wedge n * \text{fact } n / \text{real } ((n+1) \wedge (n+1)))$
 $\{0 < .. < 1\}$

proof –

have $((\lambda t. t \text{ powr real } n / \exp t) \text{ has-integral fact } n) \{0..\}$

using *Gamma-integral-real[of n + 1]* by (*auto simp: Gamma-fact powr-realpow*)

also have $?this \longleftrightarrow ((\lambda t. t \text{ powr real } n / \exp t) \text{ has-integral fact } n) \{0 < ..\}$

proof (*rule has-integral-spike-set-eq*)

have $\text{eq: } \{x \in \{0 < ..\} - \{0..\} . x \text{ powr real } n / \exp x \neq 0\} = \{\}$

by *auto*

thus *negligible* $\{x \in \{0 < ..\} - \{0..\} . x \text{ powr real } n / \exp x \neq 0\}$

by (*subst eq*) *auto*

have $\{x \in \{0..\} - \{0 < ..\} . x \text{ powr real } n / \exp x \neq 0\} \subseteq \{0\}$

by *auto*

moreover have *negligible* $\{0::\text{real}\}$

by *simp*

ultimately show *negligible* $\{x \in \{0..\} - \{0 < ..\} . x \text{ powr real } n / \exp x \neq 0\}$

by (*meson negligible-subset*)

qed

also have $\dots \longleftrightarrow ((\lambda t::\text{real} . t \wedge n / \exp t) \text{ has-integral fact } n) \{0 < ..\}$

by (*intro has-integral-spike-eq*) (*auto simp: powr-realpow*)

finally have 1: $((\lambda t::\text{real} . t \wedge n / \exp t) \text{ has-integral fact } n) \{0 < ..\} .$

have $(\lambda x::\text{real} . |x| \wedge n / \exp x) \text{ integrable-on } \{0 < ..\} \longleftrightarrow$

$(\lambda x::\text{real} . x \wedge n / \exp x) \text{ integrable-on } \{0 < ..\}$

by (*intro integrable-cong*) *auto*

hence 2: $(\lambda t::\text{real} . t \wedge n / \exp t) \text{ absolutely-integrable-on } \{0 < ..\}$

using 1 by (*simp add: absolutely-integrable-on-def power-abs has-integral-iff*)

define $g :: \text{real} \Rightarrow \text{real}$ **where** $g = (\lambda x. -\ln x * (n + 1))$

define $g' :: \text{real} \Rightarrow \text{real}$ **where** $g' = (\lambda x. -(n + 1) / x)$

define $h :: \text{real} \Rightarrow \text{real}$ **where** $h = (\lambda u. \exp (-u / (n + 1)))$

have *bij*: *bij-betw* $g \{0 < .. < 1\} \{0 < ..\}$

by (*rule bij-betwI*[*of - - h*]) (*auto simp: g-def h-def mult-neg-pos*)
have *deriv: (g has-real-derivative g' x) (at x within {0<..
if $x \in \{0 < .. < 1\}$ **for** x
unfolding *g-def g'-def* **using** *that by (auto intro!: derivative-eq-intros simp: field-simps)*

have ($\lambda t :: \text{real}. t \wedge n / \exp t$) *absolutely-integrable-on* $g \text{ ' } \{0 < .. < 1\} \wedge$
integral ($g \text{ ' } \{0 < .. < 1\}$) ($\lambda t :: \text{real}. t \wedge n / \exp t$) = *fact n*
using *1 2 bij by (simp add: bij-betw-def has-integral-iff)*
also have *?this $\longleftrightarrow ((\lambda x. |g' x| *_{\mathbb{R}} (g x \wedge n / \exp (g x))) \text{ absolutely-integrable-on } \{0 < .. < 1\} \wedge$*
integral $\{0 < .. < 1\} (\lambda x. |g' x| *_{\mathbb{R}} (g x \wedge n / \exp (g x))) = \text{fact } n$
by (*intro has-absolute-integral-change-of-variables-1' [symmetric] deriv*)
(auto simp: inj-on-def g-def)
finally have ($(\lambda x. |g' x| *_{\mathbb{R}} (g x \wedge n / \exp (g x))) \text{ has-integral fact } n$) $\{0 < .. < 1\}$
using *eq-integralD set-lebesgue-integral-eq-integral(1) by blast*
also have *?this \longleftrightarrow*
*((\lambda x :: real. ((-1) \wedge n * (n+1) \wedge (n+1)) *_{\mathbb{R}} (\ln x \wedge n * x \wedge n)) \text{ has-integral fact } n)*
 $\{0 < .. < 1\}$
proof (*rule has-integral-cong*)
fix $x :: \text{real}$ **assume** $x \in \{0 < .. < 1\}$
have $|g' x| *_{\mathbb{R}} (g x \wedge n / \exp (g x)) =$
 $(-1) \wedge n * (\text{real } n + 1) \wedge (n + 1) * \ln x \wedge n * (\exp (\ln x * (n + 1)) /$
 $x)$
using x **by** (*simp add: g-def g'-def exp-minus power-minus' divide-simps add-ac*)
also have $\exp (\ln x * (n + 1)) = x \text{ powr } \text{real } (n + 1)$
using x **by** (*simp add: powr-def*)
also have $\dots / x = x \wedge n$
using x **by** (*subst powr-realpow*) *auto*
finally show $|g' x| *_{\mathbb{R}} (g x \wedge n / \exp (g x)) =$
 $((-1) \wedge n * (n+1) \wedge (n+1)) *_{\mathbb{R}} (\ln x \wedge n * x \wedge n)$
by (*simp add: algebra-simps*)
qed
also have $\dots \longleftrightarrow ((\lambda x :: \text{real}. \ln x \wedge n * x \wedge n) \text{ has-integral}$
 $\text{fact } n /_{\mathbb{R}} \text{real-of-int } ((-1) \wedge n * \text{int } ((n + 1) \wedge (n + 1))))$
 $\{0 < .. < 1\}$
by (*intro has-integral-cmul-iff'*) (*auto simp del: power-Suc*)
also have $\text{fact } n /_{\mathbb{R}} \text{real-of-int } ((-1) \wedge n * \text{int } ((n + 1) \wedge (n + 1))) =$
 $(-1) \wedge n * \text{fact } n / (n+1) \wedge (n+1)$
by (*auto simp: divide-simps*)
finally show *?thesis*
by (*simp add: power-mult-distrib mult-ac*)
qed*

1.4 Main proofs

We can now show the first formula: $\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$

lemma *sophomores-dream-aux1*:

$\text{summable } (\lambda k. 1 / \text{real } ((k+1) \wedge (k+1)))$
 $\text{integral } \{0..1\} (\lambda x. x \text{ powr } (-x)) = (\sum n. 1 / (n+1) \wedge (n+1))$

proof –

define S **where** $S = (\lambda x::\text{real}. \sum k. (-(x * \ln x)) \wedge k / \text{fact } k)$

have $S\text{-eq}$: $S x = x \text{ powr } (-x)$ **if** $x > 0$ **for** x

proof –

have $S x = \text{exp } (-x * \ln x)$

by (*simp add: S-def exp-def field-simps*)

also have $\dots = x \text{ powr } (-x)$

using $\langle x > 0 \rangle$ **by** (*simp add: powr-def*)

finally show *?thesis* .

qed

have cont : $\text{continuous-on } \{0..1\} (\lambda x::\text{real}. \sum k < n. (-(x * \ln x)) \wedge k / \text{fact } k)$

for n

by (*intro continuous-on-sum continuous-on-divide x-log-x-continuous continuous-on-power*
continuous-on-const continuous-on-minus) *auto*

obtain $I J$ **where** IJ : $\bigwedge n. ((\lambda x. \sum k < n. (-(x * \ln x)) \wedge k / \text{fact } k) \text{ has-integral } I n) \{0..1\}$

$(S \text{ has-integral } J) \{0..1\} I \longrightarrow J$

using *uniform-limit-integral [OF uniform-limit-sophomores-dream2 cont]* **by**
(auto simp: S-def)

note $\langle (S \text{ has-integral } J) \{0..1\} \rangle$

also have $(S \text{ has-integral } J) \{0..1\} \longleftrightarrow (S \text{ has-integral } J) \{0 < .. < 1\}$

by (*simp add: has-integral-Icc-iff-Ioo*)

also have $\dots \longleftrightarrow ((\lambda x. x \text{ powr } (-x)) \text{ has-integral } J) \{0 < .. < 1\}$

by (*intro has-integral-cong*) (*use S-eq in auto*)

also have $\dots \longleftrightarrow ((\lambda x. x \text{ powr } (-x)) \text{ has-integral } J) \{0..1\}$

by (*simp add: has-integral-Icc-iff-Ioo*)

finally have $\text{integral}: ((\lambda x. x \text{ powr } (-x)) \text{ has-integral } J) \{0..1\}$.

have $I\text{-eq}$: $I = (\lambda n. \sum k < n. 1 / \text{real } ((k+1) \wedge (k+1)))$

proof

fix $n :: \text{nat}$

have $((\lambda x::\text{real}. \sum k < n. (-1) \wedge k * ((x * \ln x) \wedge k / \text{fact } k)) \text{ has-integral } (\sum k < n. (-1) \wedge k * ((-1) \wedge k * \text{fact } k / \text{real } ((k+1) \wedge (k+1)) / \text{fact } k))) \{0 < .. < 1\}$

by (*intro has-integral-sum [OF - has-integral-mult-right] has-integral-divide sophomores-dream-aux-integral*) *auto*

also have $(\lambda x::\text{real}. \sum k < n. (-1) \wedge k * ((x * \ln x) \wedge k / \text{fact } k)) = (\lambda x::\text{real}. \sum k < n. (-(x * \ln x)) \wedge k / \text{fact } k)$

by (*simp add: power-minus'*)

also have $(\sum k < n. (-1) \wedge k * ((-1) \wedge k * \text{fact } k / \text{real } ((k+1) \wedge (k+1)) / \text{fact } k)) = (\sum k < n. 1 / \text{real } ((k+1) \wedge (k+1)))$

by *simp*

also note *has-integral-Icc-iff-Ioo* [*symmetric*]
finally show $I n = (\sum_{k < n}. 1 / \text{real } ((k+1) \wedge (k+1)))$
by (*rule has-integral-unique* [*OF IJ(1)*][*of n*])
qed
hence *sums*: $(\lambda k. 1 / \text{real } ((k+1) \wedge (k+1))) \text{ sums } J$
using *IJ(3)* *I-eq* **by** (*simp add: sums-def*)

from *sums show summable* $(\lambda k. 1 / \text{real } ((k+1) \wedge (k+1)))$
by (*simp add: sums-iff*)
from *integral sums show integral* $\{0..1\} (\lambda x. x \text{ powr } (-x)) = (\sum n. 1 / (n+1) \wedge (n+1))$
by (*simp add: sums-iff has-integral-iff*)
qed

theorem *sophomores-dream1*:
 $(\lambda k::\text{nat}. \text{norm } (k \text{ powi } (-k))) \text{ summable-on } \{1..\}$
 $\text{integral } \{0..1\} (\lambda x. x \text{ powr } (-x)) = (\sum_{\infty} k \in \{(1::\text{nat})..\}. k \text{ powi } (-k))$
proof –
let $?I = \text{integral } \{0..1\} (\lambda x. x \text{ powr } (-x))$
have $(\lambda k::\text{nat}. \text{norm } (k \text{ powi } (-k))) \text{ summable-on } \text{UNIV}$
using *abs-summable-sophomores-dream*
by (*intro norm-summable-imp-summable-on*) (*auto simp: power-int-minus field-simps*)
thus $(\lambda k::\text{nat}. \text{norm } (k \text{ powi } (-k))) \text{ summable-on } \{1..\}$
by (*rule summable-on-subset-banach*) *auto*

have $(\lambda n. 1 / (n+1) \wedge (n+1)) \text{ sums } ?I$
using *sophomores-dream-aux1* **by** (*simp add: sums-iff*)
moreover have *summable* $(\lambda n. \text{norm } (1 / \text{real } (\text{Suc } n \wedge \text{Suc } n)))$
by (*subst summable-Suc-iff*) (*use abs-summable-sophomores-dream in <auto simp: field-simps>*)
ultimately have $((\lambda n::\text{nat}. 1 / (n+1) \wedge (n+1)) \text{ has-sum } ?I) \text{ UNIV}$
by (*intro norm-summable-imp-has-sum*) *auto*
also have $?this \longleftrightarrow ((\lambda n::\text{nat}. 1 / n \wedge n) \circ \text{Suc}) \text{ has-sum } ?I) \text{ UNIV}$
by (*simp add: o-def field-simps*)
also have $\dots \longleftrightarrow ((\lambda n::\text{nat}. 1 / n \wedge n) \text{ has-sum } ?I) (\text{Suc } ' \text{UNIV})$
by (*intro has-sum-reindex* [*symmetric*]) *auto*
also have $\text{Suc } ' \text{UNIV} = \{1..\}$
using *greaterThan-0* **by** *auto*
also have $((\lambda n::\text{nat}. (1 / \text{real } (n \wedge n))) \text{ has-sum } ?I) \{1..\} \longleftrightarrow$
 $((\lambda n::\text{nat}. n \text{ powi } (-n)) \text{ has-sum } ?I) \{1..\}$
by (*intro has-sum-cong*) (*auto simp: power-int-minus field-simps power-minus'*)
finally show $\text{integral } \{0..1\} (\lambda x. x \text{ powr } (-x)) = (\sum_{\infty} k \in \{(1::\text{nat})..\}. k \text{ powi } (-k))$
by (*auto dest!: infsumI simp: algebra-simps*)
qed

Next, we show the second formula: $\int_0^1 x^x dx = - \sum_{n=1}^{\infty} (-n)^{-n}$

lemma *sophomores-dream-aux2*:
 $\text{summable } (\lambda k. (-1) \wedge k / \text{real } ((k+1) \wedge (k+1)))$
 $\text{integral } \{0..1\} (\lambda x. x \text{ powr } x) = (\sum n. (-1) \wedge n / (n+1) \wedge (n+1))$

proof –

define S **where** $S = (\lambda x :: \text{real}. \sum k. (x * \ln x) ^ k / \text{fact } k)$

have $S\text{-eq}$: $S x = x \text{ powr } x$ **if** $x > 0$ **for** x

proof –

have $S x = \text{exp } (x * \ln x)$

by (*simp add: S-def exp-def field-simps*)

also have $\dots = x \text{ powr } x$

using $\langle x > 0 \rangle$ **by** (*simp add: powr-def*)

finally show *?thesis* .

qed

have cont : *continuous-on* $\{0..1\}$ $(\lambda x :: \text{real}. \sum k < n. (x * \ln x) ^ k / \text{fact } k)$ **for** n

by (*intro continuous-on-sum continuous-on-divide x-log-x-continuous continuous-on-power*

continuous-on-const) *auto*

obtain $I J$ **where** IJ : $\bigwedge n. ((\lambda x. \sum k < n. (x * \ln x) ^ k / \text{fact } k)$ *has-integral* I $n)$ $\{0..1\}$

$(S \text{ has-integral } J) \{0..1\} I \longrightarrow J$

using *uniform-limit-integral [OF uniform-limit-sophomores-dream1 cont]* **by** (*auto simp: S-def*)

note $\langle (S \text{ has-integral } J) \{0..1\} \rangle$

also have $(S \text{ has-integral } J) \{0..1\} \longleftrightarrow (S \text{ has-integral } J) \{0 < .. < 1\}$

by (*simp add: has-integral-Icc-iff-Ioo*)

also have $\dots \longleftrightarrow ((\lambda x. x \text{ powr } x) \text{ has-integral } J) \{0 < .. < 1\}$

by (*intro has-integral-cong*) (*use S-eq in auto*)

also have $\dots \longleftrightarrow ((\lambda x. x \text{ powr } x) \text{ has-integral } J) \{0..1\}$

by (*simp add: has-integral-Icc-iff-Ioo*)

finally have *integral*: $((\lambda x. x \text{ powr } x) \text{ has-integral } J) \{0..1\}$.

have $I\text{-eq}$: $I = (\lambda n. \sum k < n. (-1) ^ k / \text{real } ((k+1) ^ (k+1)))$

proof

fix $n :: \text{nat}$

have $((\lambda x :: \text{real}. \sum k < n. (x * \ln x) ^ k / \text{fact } k) \text{ has-integral}$

$(\sum k < n. (-1) ^ k * \text{fact } k / \text{real } ((k+1) ^ (k+1)) / \text{fact } k)) \{0 < .. < 1\}$

by (*intro has-integral-sum has-integral-divide sophomores-dream-aux-integral*)

auto

also have $(\sum k < n. (-1) ^ k * \text{fact } k / \text{real } ((k+1) ^ (k+1)) / \text{fact } k) =$
 $(\sum k < n. (-1) ^ k / \text{real } ((k+1) ^ (k+1)))$

by *simp*

also note *has-integral-Icc-iff-Ioo* [*symmetric*]

finally show $I n = (\sum k < n. (-1) ^ k / \text{real } ((k+1) ^ (k+1)))$

by (*rule has-integral-unique [OF IJ(1)[of n]]*)

qed

hence sums : $(\lambda k. (-1) ^ k / \text{real } ((k+1) ^ (k+1))) \text{ sums } J$

using $IJ(3)$ $I\text{-eq}$ **by** (*simp add: sums-def*)

from sums **show** *summable* $(\lambda k. (-1) ^ k / \text{real } ((k+1) ^ (k+1)))$

by (simp add: sums-iff)
 from integral sums show integral $\{0..1\} (\lambda x. x \text{ powr } x) = (\sum n. (-1)^{\wedge n} / (n+1)^{\wedge(n+1)})$
 by (simp add: sums-iff has-integral-iff)
 qed

theorem *sophomores-dream2*:

$(\lambda k::nat. norm ((-k) \text{ powi } (-k))) \text{ summable-on } \{1..\}$
 $\text{integral } \{0..1\} (\lambda x. x \text{ powr } x) = -(\sum_{\infty} k \in \{(1::nat)..\}. (-k) \text{ powi } (-k))$

proof –

let ?I = integral $\{0..1\} (\lambda x. x \text{ powr } x)$
 have $(\lambda k::nat. norm ((-k) \text{ powi } (-k))) \text{ summable-on } UNIV$
 using *abs-summable-sophomores-dream*
 by (intro norm-summable-imp-summable-on) (auto simp: power-int-minus field-simps)
 thus $(\lambda k::nat. norm ((-k) \text{ powi } (-k))) \text{ summable-on } \{1..\}$
 by (rule summable-on-subset-banach) auto

have $(\lambda n. (-1)^{\wedge n} / (n+1)^{\wedge(n+1)}) \text{ sums } ?I$

using *sophomores-dream-aux2* by (simp add: sums-iff)

moreover have summable $(\lambda n. 1 / \text{real } (Suc\ n \wedge Suc\ n))$

by (subst summable-Suc-iff) (use *abs-summable-sophomores-dream* in <auto simp: field-simps>)

hence summable $(\lambda n. norm ((-1)^{\wedge n} / \text{real } (Suc\ n \wedge Suc\ n)))$

by *simp*

ultimately have $((\lambda n::nat. (-1)^{\wedge n} / (n+1)^{\wedge(n+1)}) \text{ has-sum } ?I) UNIV$

by (intro norm-summable-imp-has-sum) auto

also have ?this $\longleftrightarrow ((\lambda n::nat. -((-1)^{\wedge n} / n^{\wedge n})) \circ Suc) \text{ has-sum } ?I) UNIV$

by (simp add: o-def field-simps)

also have ... $\longleftrightarrow ((\lambda n::nat. -((-1)^{\wedge n} / n^{\wedge n})) \text{ has-sum } ?I) (Suc \text{ ' } UNIV)$

by (intro has-sum-reindex [symmetric]) auto

also have $Suc \text{ ' } UNIV = \{1..\}$

using *greaterThan-0* by auto

also have $((\lambda n::nat. -((-1)^{\wedge n} / \text{real } (n^{\wedge n}))) \text{ has-sum } ?I) \{1..\} \longleftrightarrow$
 $((\lambda n::nat. -((-n) \text{ powi } (-n))) \text{ has-sum } ?I) \{1..\}$

by (intro has-sum-cong) (auto simp: power-int-minus field-simps power-minus')

also have ... $\longleftrightarrow ((\lambda n::nat. (-n) \text{ powi } (-n)) \text{ has-sum } (-?I)) \{1..\}$

by (simp add: has-sum-uminus)

finally show integral $\{0..1\} (\lambda x. x \text{ powr } x) = -(\sum_{\infty} k \in \{(1::nat)..\}. (-k) \text{ powi } (-k))$

by (auto dest!: infsumI simp: algebra-simps)

qed

end

References

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