The Classical Seifert–van Kampen Theorem

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Abstract

This entry formalizes a quotient-oriented proof of the classical Seifert–van Kampen theorem in Isabelle/HOL. For open subsets U and V of a topological space with $x_0 \in U \cap V$ and path-connected intersection, it proves that the fundamental group of $U \cup V$ at x_0 is in bijection with the carrier-based amalgamated free product of the fundamental groups of U and V over the fundamental group of $U \cap V$. The development also provides reusable infrastructure for explicit path homotopies, pushout-style quotients, free-product words, carrier-based amalgamated free products, and the abstract encode/decode interface used to organize the final argument. AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

1 Overview

The main unconditional result of the entry is the theorem `classical_seifert_van_kampen_bij_betw` in `Classical_Seifert_Van_Kampen.thy`. It establishes the classical open-union form of Seifert–van Kampen through a quotient-level bijection between the fundamental group of $U \cup V$ and the carrier-based amalgamated free product assembled from the fundamental groups of U, V, and $U \cap V$.

Supporting theories package the quotient-level encode/decode interface and the topological pushout constructions used by the classical proof. These auxiliary layers are internal to the entry; the completed headline result remains the classical open-union theorem above.

2 Development structure

The theories are organized in three layers:

- *Quotients and pushouts:* `Equivalence_Quotients.thy`, `Pushout_Scaffold.thy`, `Binary_Sum_Topology.thy`, and `Topological_`

`Pushout_Scaffold.thy` isolate the generic equivalence-class and topological-pushout constructions.

- *The algebraic target:* `Free_Product_Words.thy`, `Amalgamated_Free_Product.thy`, `Carrier_Group_Scaffold.thy`, `Carrier_Amalgamated_Free_Product.thy`, and `Carrier_Amalgamated_Free_Product_Eval.thy` provide the word calculus, carrier-group locales, and evaluation map for the carrier-based amalgamated free product.
- *The topological argument:* `Explicit_Path_Homotopy_Scaffold.thy`, `Explicit_Fundamental_Group_Scaffold.thy`, and `Fundamental_Group_Scaffold.thy` develop the quotient view of the fundamental group, while `Seifert_Van_Kampen_Scaffold.thy`, `Topological_Seifert_Van_Kampen.thy`, and `Classical_Seifert_Van_Kampen.thy` package the abstract interface and its classical open-union instantiation.

3 Sources

The informal mathematics follows the classical theorem due to Seifert and van Kampen [3, 4] and standard textbook presentations in algebraic topology, especially Hatcher [2] and Brown [1].

Contents

theory *Equivalence-Quotients*
imports *Main*
begin

4 Equivalence classes as quotient infrastructure

The later pushout and fundamental-group constructions are phrased in terms of explicit equivalence classes and quotient carriers. This small theory keeps that quotient infrastructure elementary and reusable, so subsequent theories can concentrate on the specific relations that matter for Seifert–van Kampen.

definition *equiv-class* :: (*'a* => *'a* => *bool*) => *'a* => *'a set* **where**
equiv-class R x = {*y*. *R y x*}

definition *quotient-space* :: (*'a* => *'a* => *bool*) => *'a set set* **where**
quotient-space R = *equiv-class R* ‘ *UNIV*

lemma *quotient-spaceI*:
equiv-class R x ∈ *quotient-space R*
unfolding *quotient-space-def* **by** *blast*

lemma *equiv-class-iff* [*simp*]:
y ∈ *equiv-class R x* ⟷ *R y x*
unfolding *equiv-class-def* **by** *simp*

locale *equivalence-relation* =
fixes *R* :: *'a* => *'a* => *bool*
assumes *refl* [*intro*!, *simp*]: *R x x*

and *sym*: $R\ x\ y \Longrightarrow R\ y\ x$
and *transitive*: $R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
begin

lemma *equiv-class-eqI*:
assumes $R\ x\ y$
shows *equiv-class* $R\ x$ = *equiv-class* $R\ y$
using *assms sym transitive*
unfolding *equiv-class-def*
by *blast*

lemma *equiv-class-eq-iff*:
equiv-class $R\ x$ = *equiv-class* $R\ y \longleftrightarrow R\ x\ y$
proof
assume h : *equiv-class* $R\ x$ = *equiv-class* $R\ y$
have $x \in$ *equiv-class* $R\ x$
unfolding *equiv-class-def* **by** *simp*
with h **show** $R\ x\ y$
unfolding *equiv-class-def* **by** *simp*
next
assume $R\ x\ y$
then show *equiv-class* $R\ x$ = *equiv-class* $R\ y$
by (*rule equiv-class-eqI*)
qed

lemma *quotient-spaceE*:
assumes $Q \in$ *quotient-space* R
obtains x **where** Q = *equiv-class* $R\ x$
using *assms* **unfolding** *quotient-space-def* **by** *blast*

end

end
theory *Free-Product-Words*
imports *Main*
begin

5 Free-product words

Encodings of loops in the Seifert–van Kampen argument are expressed as words that alternate between generators from the left and right factors. This theory provides the basic word combinators, reversal operations, and reduction-oriented bookkeeping used by the later amalgamation quotients.

datatype ('*a*, '*b*) *free-product-word* =
WordNil
| *WordLeft* '*a* ('*a*, '*b*) *free-product-word*
| *WordRight* '*b* ('*a*, '*b*) *free-product-word*

primrec *fpw-concat* ::
  ('a, 'b) *free-product-word* => ('a, 'b) *free-product-word* => ('a, 'b) *free-product-word*
where
  fpw-concat WordNil w2 = w2
| *fpw-concat* (*WordLeft a rest*) *w2* = *WordLeft a* (*fpw-concat rest w2*)
| *fpw-concat* (*WordRight b rest*) *w2* = *WordRight b* (*fpw-concat rest w2*)

primrec *fpw-length* :: ('a, 'b) *free-product-word* => *nat* **where**
  fpw-length WordNil = 0
| *fpw-length* (*WordLeft a rest*) = *Suc* (*fpw-length rest*)
| *fpw-length* (*WordRight b rest*) = *Suc* (*fpw-length rest*)

primrec *fpw-reverse* :: ('a, 'b) *free-product-word* => ('a, 'b) *free-product-word*
where
  fpw-reverse WordNil = WordNil
| *fpw-reverse* (*WordLeft a rest*) = *fpw-concat* (*fpw-reverse rest*) (*WordLeft a WordNil*)
| *fpw-reverse* (*WordRight b rest*) = *fpw-concat* (*fpw-reverse rest*) (*WordRight b WordNil*)

primrec *fpw-map* ::
  ('a => 'c) => ('b => 'd) => ('a, 'b) *free-product-word* => ('c, 'd) *free-product-word*
where
  fpw-map fl fr WordNil = WordNil
| *fpw-map fl fr* (*WordLeft a rest*) = *WordLeft* (*fl a*) (*fpw-map fl fr rest*)
| *fpw-map fl fr* (*WordRight b rest*) = *WordRight* (*fr b*) (*fpw-map fl fr rest*)

lemma *fpw-concat-nil-right* [*simp*]:
  fpw-concat w WordNil = w
  by (*induction w*) *simp-all*

lemma *fpw-concat-assoc* [*simp*]:
  fpw-concat (*fpw-concat u v*) *w* = *fpw-concat u* (*fpw-concat v w*)
  by (*induction u*) *simp-all*

lemma *fpw-reverse-concat* [*simp*]:
  fpw-reverse (*fpw-concat u v*) = *fpw-concat* (*fpw-reverse v*) (*fpw-reverse u*)
  by (*induction u*) *simp-all*

lemma *fpw-reverse-reverse* [*simp*]:
  fpw-reverse (*fpw-reverse w*) = *w*
  by (*induction w*) *simp-all*

fun *fpw-reduced* :: ('a, 'b) *free-product-word* => *bool* **where**
  fpw-reduced WordNil = True
| *fpw-reduced* (*WordLeft a WordNil*) = *True*
| *fpw-reduced* (*WordLeft a* (*WordLeft b rest*)) = *False*

```
| fpw-reduced (WordLeft a (WordRight b rest)) = fpw-reduced (WordRight b rest)
| fpw-reduced (WordRight b WordNil) = True
| fpw-reduced (WordRight b (WordRight c rest)) = False
| fpw-reduced (WordRight b (WordLeft a rest)) = fpw-reduced (WordLeft a rest)

fun fpw-inverse ::
  ('a::group-add, 'b::group-add) free-product-word =>
    ('a, 'b) free-product-word
where
  fpw-inverse WordNil = WordNil
| fpw-inverse (WordLeft a rest) =
     fpw-concat (fpw-inverse rest) (WordLeft (- a) WordNil)
| fpw-inverse (WordRight b rest) =
     fpw-concat (fpw-inverse rest) (WordRight (- b) WordNil)

lemma fpw-inverse-concat:
  fpw-inverse (fpw-concat u v) = fpw-concat (fpw-inverse v) (fpw-inverse u)
  by (induction u) simp-all

lemma fpw-inverse-inverse [simp]:
  fpw-inverse (fpw-inverse w) = w
  by (induction w) (simp-all add: fpw-inverse-concat)

inductive fpw-reduction-step ::
  ('a::group-add, 'b::group-add) free-product-word =>
    ('a, 'b) free-product-word => bool
where
  combine-left:
    fpw-reduction-step
      (WordLeft a (WordLeft b rest))
      (WordLeft (a + b) rest)
| combine-right:
    fpw-reduction-step
      (WordRight a (WordRight b rest))
      (WordRight (a + b) rest)
| remove-left-zero:
    fpw-reduction-step (WordLeft 0 rest) rest
| remove-right-zero:
    fpw-reduction-step (WordRight 0 rest) rest
| context-left:
    fpw-reduction-step u v ==>
      fpw-reduction-step (WordLeft a u) (WordLeft a v)
| context-right:
    fpw-reduction-step u v ==>
      fpw-reduction-step (WordRight b u) (WordRight b v)

inductive fpw-reduction ::
  ('a::group-add, 'b::group-add) free-product-word =>
    ('a, 'b) free-product-word => bool
```

where
  refl [*intro*!, *simp*]: *fpw-reduction w w*
| *sym*: *fpw-reduction u v* ==> *fpw-reduction v u*
| *trans*: *fpw-reduction u v* ==> *fpw-reduction v w* ==> *fpw-reduction u w*
| *step*: *fpw-reduction-step u v* ==> *fpw-reduction u v*

lemma *fpw-reduction-left-context*:
  assumes *fpw-reduction u v*
  shows *fpw-reduction* (*WordLeft a u*) (*WordLeft a v*)
  using *assms*
proof (*induction rule*: *fpw-reduction.induct*)
  case (*refl w*)
  then show *?case* **by** *simp*
next
  case (*sym u v*)
  then show *?case* **by** (*meson fpw-reduction.sym*)
next
  case (*trans u v w*)
  then show *?case* **by** (*meson fpw-reduction.trans*)
next
  case (*step u v*)
  then show *?case*
    by (*meson fpw-reduction.step fpw-reduction-step.context-left*)
qed

lemma *fpw-reduction-right-context*:
  assumes *fpw-reduction u v*
  shows *fpw-reduction* (*WordRight b u*) (*WordRight b v*)
  using *assms*
proof (*induction rule*: *fpw-reduction.induct*)
  case (*refl w*)
  then show *?case* **by** *simp*
next
  case (*sym u v*)
  then show *?case* **by** (*meson fpw-reduction.sym*)
next
  case (*trans u v w*)
  then show *?case* **by** (*meson fpw-reduction.trans*)
next
  case (*step u v*)
  then show *?case*
    by (*meson fpw-reduction.step fpw-reduction-step.context-right*)
qed

end
theory *Amalgamated-Free-Product*
  imports *Equivalence-Quotients Free-Product-Words*
begin

6 Amalgamation quotients of free-product words

This is the purely algebraic target of the later topological theorem. Starting from free-product words, the theory quotients by the relations induced by the common interface and packages the resulting equivalence classes as the abstract amalgamated free product.

inductive *amalgam-step* ::
  ('*h* => '*a*) => ('*h* => '*b*) =>
    ('*a*, '*b*) *free-product-word* => ('*a*, '*b*) *free-product-word* => *bool*
  for *i1* :: '*h* => '*a* **and** *i2* :: '*h* => '*b*
where
  identify:
    amalgam-step i1 i2 (*WordLeft* (*i1 h*) *rest*) (*WordRight* (*i2 h*) *rest*)

inductive *amalgam-equiv* ::
  ('*h* => '*a*) => ('*h* => '*b*) =>
    ('*a*, '*b*) *free-product-word* => ('*a*, '*b*) *free-product-word* => *bool*
  for *i1* :: '*h* => '*a* **and** *i2* :: '*h* => '*b*
where
  refl [*intro*!, *simp*]: *amalgam-equiv i1 i2 w w*
| *sym*: *amalgam-equiv i1 i2 u v* ==> *amalgam-equiv i1 i2 v u*
| *trans*: *amalgam-equiv i1 i2 u v* ==> *amalgam-equiv i1 i2 v w* ==> *amalgam-equiv i1 i2 u w*
| *step*: *amalgam-step i1 i2 u v* ==> *amalgam-equiv i1 i2 u v*
| *left-context*:
    amalgam-equiv i1 i2 u v ==> *amalgam-equiv i1 i2* (*WordLeft a u*) (*WordLeft a v*)
| *right-context*:
    amalgam-equiv i1 i2 u v ==> *amalgam-equiv i1 i2* (*WordRight b u*) (*WordRight b v*)

interpretation *amalgam-equiv-equiv*: *equivalence-relation amalgam-equiv i1 i2*
proof
  show *amalgam-equiv i1 i2 x x* **for** *x*
    by (*rule amalgam-equiv.refl*)
next
  show *amalgam-equiv i1 i2 x y* ==> *amalgam-equiv i1 i2 y x* **for** *x y*
    by (*rule amalgam-equiv.sym*)
next
  show *amalgam-equiv i1 i2 x y* ==> *amalgam-equiv i1 i2 y z* ==> *amalgam-equiv i1 i2 x z* **for** *x y z*
    by (*rule amalgam-equiv.trans*)
qed

definition *amalgam-class* ::
  ('*h* => '*a*) => ('*h* => '*b*) =>
    ('*a*, '*b*) *free-product-word* => (('*a*, '*b*) *free-product-word*) *set*
where
  amalgam-class i1 i2 w = *equiv-class* (*amalgam-equiv i1 i2*) *w*

definition *amalgamated-free-product-space* ::
(*'h* => *'a*) => (*'h* => *'b*) =>
((*'a*, *'b*) *free-product-word*) *set set*
where
amalgamated-free-product-space i1 i2 = *quotient-space* (*amalgam-equiv i1 i2*)

lemma *amalgam-class-eq-iff*:
amalgam-class i1 i2 u = *amalgam-class i1 i2 v* ⟷ *amalgam-equiv i1 i2 u v*
unfolding *amalgam-class-def*
by (*rule amalgam-equiv-equiv.equiv-class-eq-iff*)

inductive *full-amalg-equiv* ::
(*'h* => *'a*::*group-add*) => (*'h* => *'b*::*group-add*) =>
(*'a*, *'b*) *free-product-word* => (*'a*, *'b*) *free-product-word* => *bool*
for *i1* :: *'h* => *'a* **and** *i2* :: *'h* => *'b*
where
refl [*intro*!, *simp*]: *full-amalg-equiv i1 i2 w w*
| *sym*: *full-amalg-equiv i1 i2 u v* ==> *full-amalg-equiv i1 i2 v u*
| *trans*:
full-amalg-equiv i1 i2 u v ==> *full-amalg-equiv i1 i2 v w* ==> *full-amalg-equiv i1 i2 u w*
| *of-amalg*:
amalgam-equiv i1 i2 u v ==> *full-amalg-equiv i1 i2 u v*
| *of-reduction*:
fpw-reduction u v ==> *full-amalg-equiv i1 i2 u v*

interpretation *full-amalg-equiv-equiv*: *equivalence-relation full-amalg-equiv i1 i2*
proof
show *full-amalg-equiv i1 i2 x x* **for** *x*
by (*rule full-amalg-equiv.refl*)
next
show *full-amalg-equiv i1 i2 x y* ==> *full-amalg-equiv i1 i2 y x* **for** *x y*
by (*rule full-amalg-equiv.sym*)
next
show
full-amalg-equiv i1 i2 x y ==> *full-amalg-equiv i1 i2 y z* ==> *full-amalg-equiv i1 i2 x z*
for *x y z*
by (*rule full-amalg-equiv.trans*)
qed

definition *full-amalg-class* ::
(*'h* => *'a*::*group-add*) => (*'h* => *'b*::*group-add*) =>
(*'a*, *'b*) *free-product-word* => ((*'a*, *'b*) *free-product-word*) *set*
where
full-amalg-class i1 i2 w = *equiv-class* (*full-amalg-equiv i1 i2*) *w*

definition *full-amalgamated-free-product-space* ::

('h => 'a::group-add) => ('h => 'b::group-add) =>
  (('a, 'b) free-product-word) set set
where
  full-amalgamated-free-product-space i1 i2 =
    quotient-space (full-amalg-equiv i1 i2)

lemma full-amalg-class-eq-iff:
  full-amalg-class i1 i2 u = full-amalg-class i1 i2 v ⟷ full-amalg-equiv i1 i2 u v
  unfolding full-amalg-class-def
  by (rule full-amalg-equiv-equiv.equiv-class-eq-iff)

lemma full-amalg-class-in-space [intro]:
  full-amalg-class i1 i2 w ∈ full-amalgamated-free-product-space i1 i2
    unfolding full-amalg-class-def full-amalgamated-free-product-space-def quotient-space-def
  by blast

definition full-amalg-one ::
  ('h => 'a::group-add) => ('h => 'b::group-add) =>
    (('a, 'b) free-product-word) set
where
  full-amalg-one i1 i2 = full-amalg-class i1 i2 WordNil

end
theory Carrier-Amalgamated-Free-Product
  imports Amalgamated-Free-Product
begin

7 Carrier-based amalgamated free products

The abstract amalgamated free product is refined here to words whose letters lie in fixed carrier sets. That carrier-level control matches the way loop representatives are produced on the topological side and is what eventually lets the Seifert–van Kampen theorem talk about concrete fundamental-group carriers.

fun fpw-in-space :: 'a set => 'b set => ('a, 'b) free-product-word => bool where
  fpw-in-space G H WordNil = True
| fpw-in-space G H (WordLeft a rest) = (a ∈ G ∧ fpw-in-space G H rest)
| fpw-in-space G H (WordRight b rest) = (b ∈ H ∧ fpw-in-space G H rest)

definition carrier-fpw-space :: 'a set => 'b set => (('a, 'b) free-product-word) set
where
  carrier-fpw-space G H = {w. fpw-in-space G H w}

lemma carrier-fpw-space-iff [simp]:
  w ∈ carrier-fpw-space G H ⟷ fpw-in-space G H w
  unfolding carrier-fpw-space-def by simp

```
inductive carrier-amalgam-step ::
  'h set => ('h => 'a) => ('h => 'b) =>
    ('a, 'b) free-product-word => ('a, 'b) free-product-word => bool
  for K :: 'h set and i1 :: 'h => 'a and i2 :: 'h => 'b
where
  identify:
    h ∈ K ⟹
      carrier-amalgam-step K i1 i2
        (WordLeft (i1 h) rest)
        (WordRight (i2 h) rest)

inductive carrier-amalgam-equiv ::
  'h set => ('h => 'a) => ('h => 'b) =>
    ('a, 'b) free-product-word => ('a, 'b) free-product-word => bool
  for K :: 'h set and i1 :: 'h => 'a and i2 :: 'h => 'b
where
  refl [intro!, simp]: carrier-amalgam-equiv K i1 i2 w w
| sym:
    carrier-amalgam-equiv K i1 i2 u v ⟹
      carrier-amalgam-equiv K i1 i2 v u
| trans:
    carrier-amalgam-equiv K i1 i2 u v ⟹
      carrier-amalgam-equiv K i1 i2 v w ⟹
      carrier-amalgam-equiv K i1 i2 u w
| step:
    carrier-amalgam-step K i1 i2 u v ⟹
      carrier-amalgam-equiv K i1 i2 u v
| left-context:
    carrier-amalgam-equiv K i1 i2 u v ⟹
      carrier-amalgam-equiv K i1 i2 (WordLeft a u) (WordLeft a v)
| right-context:
    carrier-amalgam-equiv K i1 i2 u v ⟹
      carrier-amalgam-equiv K i1 i2 (WordRight b u) (WordRight b v)

interpretation carrier-amalgam-equiv-equiv:
  equivalence-relation carrier-amalgam-equiv K i1 i2
proof
  show carrier-amalgam-equiv K i1 i2 x x for x
    by (rule carrier-amalgam-equiv.refl)
next
  show carrier-amalgam-equiv K i1 i2 x y ⟹ carrier-amalgam-equiv K i1 i2 y x
for x y
    by (rule carrier-amalgam-equiv.sym)
next
  show
    carrier-amalgam-equiv K i1 i2 x y ⟹
      carrier-amalgam-equiv K i1 i2 y z ⟹
      carrier-amalgam-equiv K i1 i2 x z for x y z
    by (rule carrier-amalgam-equiv.trans)
```

qed

definition *carrier-amalgam-class* ::
  'h set => ('h => 'a) => ('h => 'b) =>
    ('a, 'b) free-product-word => (('a, 'b) free-product-word) set
where
  carrier-amalgam-class K i1 i2 w =
    equiv-class (carrier-amalgam-equiv K i1 i2) w

lemma *carrier-amalgam-class-eq-iff*:
  carrier-amalgam-class K i1 i2 u = carrier-amalgam-class K i1 i2 v
    ⟷ carrier-amalgam-equiv K i1 i2 u v
  unfolding *carrier-amalgam-class-def*
  by (*rule carrier-amalgam-equiv-equiv.equiv-class-eq-iff*)

inductive *carrier-fpw-reduction-step* ::
  'a set => 'b set =>
    ('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
    ('a, 'b) free-product-word => ('a, 'b) free-product-word => bool
  for *G1* :: *'a set* **and** *G2* :: *'b set*
    and *mult1* :: *'a => 'a => 'a* **and** *one1* :: *'a*
    and *mult2* :: *'b => 'b => 'b* **and** *one2* :: *'b*
where
  combine-left:
    ⟦a ∈ G1; b ∈ G1; mult1 a b ∈ G1; fpw-in-space G1 G2 rest⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordLeft a (WordLeft b rest))
      (WordLeft (mult1 a b) rest)
| *combine-right*:
    ⟦a ∈ G2; b ∈ G2; mult2 a b ∈ G2; fpw-in-space G1 G2 rest⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordRight a (WordRight b rest))
      (WordRight (mult2 a b) rest)
| *remove-left-one*:
    ⟦one1 ∈ G1; fpw-in-space G1 G2 rest⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordLeft one1 rest) rest
| *remove-right-one*:
    ⟦one2 ∈ G2; fpw-in-space G1 G2 rest⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordRight one2 rest) rest
| *context-left*:
    ⟦a ∈ G1; carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
        (WordLeft a u) (WordLeft a v)
| *context-right*:
    ⟦b ∈ G2; carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v⟧ ⟹
      carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
        (WordRight b u) (WordRight b v)

```
inductive carrier-fpw-reduction ::
  'a set => 'b set =>
    ('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
    ('a, 'b) free-product-word => ('a, 'b) free-product-word => bool
  for G1 :: 'a set and G2 :: 'b set
    and mult1 :: 'a => 'a => 'a and one1 :: 'a
    and mult2 :: 'b => 'b => 'b and one2 :: 'b
where
  refl [intro!, simp]: carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 w w
| sym:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v ⟹
      carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 v u
| trans:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v ⟹
      carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 v w ⟹
      carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u w
| step:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v ⟹
      carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v

lemma carrier-fpw-reduction-left-context:
  assumes carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v
    and a ∈ G1
  shows carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 (WordLeft a u)
(WordLeft a v)
  using assms
proof (induction rule: carrier-fpw-reduction.induct)
  case (refl w)
  then show ?case by simp
next
  case (sym u v)
  then show ?case by (meson carrier-fpw-reduction.sym)
next
  case (trans u v w)
  then show ?case by (meson carrier-fpw-reduction.trans)
next
  case (step u v)
  then show ?case
    by (meson carrier-fpw-reduction.step carrier-fpw-reduction-step.context-left)
qed

lemma carrier-fpw-reduction-right-context:
  assumes carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v
    and b ∈ G2
  shows carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 (WordRight b u)
(WordRight b v)
  using assms
proof (induction rule: carrier-fpw-reduction.induct)
```

```
  case (refl w)
  then show ?case by simp
next
  case (sym u v)
  then show ?case by (meson carrier-fpw-reduction.sym)
next
  case (trans u v w)
  then show ?case by (meson carrier-fpw-reduction.trans)
next
  case (step u v)
  then show ?case
    by (meson carrier-fpw-reduction.step carrier-fpw-reduction-step.context-right)
qed

lemma carrier-fpw-reduction-step-preserves-space:
  assumes step: carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v
  shows fpw-in-space G1 G2 u and fpw-in-space G1 G2 v
  using step
proof (induction rule: carrier-fpw-reduction-step.induct)
  case (combine-left a b rest)
  then show fpw-in-space G1 G2 (WordLeft a (WordLeft b rest))
    by simp
  show fpw-in-space G1 G2 (WordLeft (mult1 a b) rest)
    using combine-left by simp
next
  case (combine-right a b rest)
  then show fpw-in-space G1 G2 (WordRight a (WordRight b rest))
    by simp
  show fpw-in-space G1 G2 (WordRight (mult2 a b) rest)
    using combine-right by simp
next
  case (remove-left-one rest)
  then show fpw-in-space G1 G2 (WordLeft one1 rest)
    by simp
  show fpw-in-space G1 G2 rest
    using remove-left-one by simp
next
  case (remove-right-one rest)
  then show fpw-in-space G1 G2 (WordRight one2 rest)
    by simp
  show fpw-in-space G1 G2 rest
    using remove-right-one by simp
next
  case (context-left a u v)
  then show fpw-in-space G1 G2 (WordLeft a u)
    by simp
  show fpw-in-space G1 G2 (WordLeft a v)
    using context-left by simp
next
```

```
  case (context-right b u v)
  then show fpw-in-space G1 G2 (WordRight b u)
    by simp
  show fpw-in-space G1 G2 (WordRight b v)
    using context-right by simp
qed

lemma carrier-fpw-reduction-step-preserves-space-iff:
  assumes step: carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
  using carrier-fpw-reduction-step-preserves-space[OF step] by blast

lemma carrier-fpw-reduction-preserves-space-iff:
  assumes rel: carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
  using rel
proof (induction rule: carrier-fpw-reduction.induct)
  case (refl w)
  then show ?case by simp
next
  case (sym u v)
  then show ?case by simp
next
  case (trans u v w)
  then show ?case by blast
next
  case (step u v)
  then show ?case
    by (rule carrier-fpw-reduction-step-preserves-space-iff)
qed

inductive carrier-full-amalg-equiv ::
  'a set => 'b set => 'h set => ('h => 'a) => ('h => 'b) =>
    ('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
    ('a, 'b) free-product-word => ('a, 'b) free-product-word => bool
  for G1 :: 'a set and G2 :: 'b set
    and K :: 'h set and i1 :: 'h => 'a and i2 :: 'h => 'b
    and mult1 :: 'a => 'a => 'a and one1 :: 'a
    and mult2 :: 'b => 'b => 'b and one2 :: 'b
where
  refl [intro!, simp]:
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 w w
| sym:
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u v ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 v u
| trans:
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u v ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 v w ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u w
```

| *of-amalg*:
  carrier-amalgam-equiv K i1 i2 u v ⟹
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u v
| *of-reduction*:
  carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v ⟹
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u v

interpretation *carrier-full-amalg-equiv-equiv*:
  equivalence-relation
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2
proof
  show *carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 x x* **for** *x*
    by (*rule carrier-full-amalg-equiv.refl*)
next
  show
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 x y ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 y x **for** *x y*
    by (*rule carrier-full-amalg-equiv.sym*)
next
  show
    carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 x y ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 y z ⟹
      carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 x z **for** *x y z*
    by (*rule carrier-full-amalg-equiv.trans*)
qed

definition *carrier-full-amalg-class* ::
  'a set => 'b set => 'h set => ('h => 'a) => ('h => 'b) =>
    ('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
    ('a, 'b) free-product-word => (('a, 'b) free-product-word) set
where
  carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 w =
    equiv-class (*carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2*) *w*

definition *carrier-full-amalgamated-free-product-space* ::
  'a set => 'b set => 'h set => ('h => 'a) => ('h => 'b) =>
    ('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
    (('a, 'b) free-product-word) set set
where
  carrier-full-amalgamated-free-product-space G1 G2 K i1 i2 mult1 one1 mult2 one2
=
   carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 ‘ carrier-fpw-space G1 G2

lemma *carrier-full-amalg-class-eq-iff*:
  carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 u =
      carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 v
    ⟷ *carrier-full-amalg-equiv G1 G2 K i1 i2 mult1 one1 mult2 one2 u v*
  unfolding *carrier-full-amalg-class-def*

by (*rule carrier-full-amalg-equiv-equiv.equiv-class-eq-iff*)

lemma *carrier-full-amalg-class-in-space* [*intro*]:
assumes *fpw-in-space G1 G2 w*
shows
carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 w ∈
carrier-full-amalgamated-free-product-space G1 G2 K i1 i2 mult1 one1 mult2 one2
using *assms*
unfolding *carrier-full-amalgamated-free-product-space-def*
by *simp*

definition *carrier-full-amalg-one* ::
'a set => 'b set => 'h set => ('h => 'a) => ('h => 'b) =>
('a => 'a => 'a) => 'a => ('b => 'b => 'b) => 'b =>
('a, 'b) free-product-word set
where
carrier-full-amalg-one G1 G2 K i1 i2 mult1 one1 mult2 one2 =
carrier-full-amalg-class G1 G2 K i1 i2 mult1 one1 mult2 one2 WordNil

end
theory *Carrier-Group-Scaffold*
imports *Main*
begin

8 Groups on carriers

The main Seifert–van Kampen statement is formulated on concrete carrier sets rather than on type-class groups. This locale package isolates the carrier algebra laws and homomorphism notions that the amalgamation and fundamental-group constructions need later on.

locale *carrier-group* =
fixes *G* :: *'a set*
and *mult* :: *'a => 'a => 'a*
and *one* :: *'a*
and *inv* :: *'a => 'a*
assumes *one-closed* [*intro*, *simp*]: *one ∈ G*
and *mult-closed* [*intro*]: ⟦*x ∈ G*; *y ∈ G*⟧ ⟹ *mult x y ∈ G*
and *inv-closed* [*intro*]: *x ∈ G ⟹ inv x ∈ G*
and *mult-assoc*: ⟦*x ∈ G*; *y ∈ G*; *z ∈ G*⟧ ⟹ *mult (mult x y) z = mult x (mult y z)*
and *mult-one-left*: *x ∈ G ⟹ mult one x = x*
and *mult-one-right*: *x ∈ G ⟹ mult x one = x*
and *mult-inv-left*: *x ∈ G ⟹ mult (inv x) x = one*
and *mult-inv-right*: *x ∈ G ⟹ mult x (inv x) = one*
begin

lemma *left-cancel*:

```
  assumes x: x ∈ G
    and y: y ∈ G
    and z: z ∈ G
    and eq: mult x y = mult x z
  shows y = z
proof −
  have mult (inv x) (mult x y) = mult (inv x) (mult x z)
    using eq by simp
  moreover have mult (inv x) (mult x y) = y
  proof −
    have mult (inv x) (mult x y) = mult (mult (inv x) x) y
      using x y inv-closed[OF x] by (simp add: mult-assoc[symmetric])
    also have ... = y
      using x y by (simp add: mult-inv-left mult-one-left)
    finally show ?thesis .
  qed
  moreover have mult (inv x) (mult x z) = z
  proof −
    have mult (inv x) (mult x z) = mult (mult (inv x) x) z
      using x z inv-closed[OF x] by (simp add: mult-assoc[symmetric])
    also have ... = z
      using x z by (simp add: mult-inv-left mult-one-left)
    finally show ?thesis .
  qed
  ultimately show ?thesis
    by simp
qed

lemma right-cancel:
  assumes x: x ∈ G
    and y: y ∈ G
    and z: z ∈ G
    and eq: mult y x = mult z x
  shows y = z
proof −
  have mult (mult y x) (inv x) = mult (mult z x) (inv x)
    using eq by simp
  moreover have mult (mult y x) (inv x) = y
  proof −
    have mult (mult y x) (inv x) = mult y (mult x (inv x))
      using x y inv-closed[OF x] by (simp add: mult-assoc)
    also have ... = y
      using x y by (simp add: mult-inv-right mult-one-right)
    finally show ?thesis .
  qed
  moreover have mult (mult z x) (inv x) = z
  proof −
    have mult (mult z x) (inv x) = mult z (mult x (inv x))
      using x z inv-closed[OF x] by (simp add: mult-assoc)
```

```
    also have ... = z
      using x z by (simp add: mult-inv-right mult-one-right)
    finally show ?thesis .
  qed
  ultimately show ?thesis
    by simp
qed

lemma left-inverse-unique:
  assumes x: x ∈ G
    and y: y ∈ G
    and eq: mult y x = one
  shows y = inv x
proof (rule right-cancel[OF x y inv-closed[OF x]])
  show mult y x = mult (inv x) x
    using eq x by (simp add: mult-inv-left)
qed

lemma right-inverse-unique:
  assumes x: x ∈ G
    and y: y ∈ G
    and eq: mult x y = one
  shows y = inv x
proof (rule left-cancel[OF x y inv-closed[OF x]])
  show mult x y = mult x (inv x)
    using eq x by (simp add: mult-inv-right)
qed

end

locale carrier-group-hom =
  G: carrier-group G mult one inv +
  H: carrier-group H mult' one' inv'
  for G :: 'a set and mult :: 'a => 'a => 'a and one :: 'a and inv :: 'a => 'a
    and H :: 'b set and mult' :: 'b => 'b => 'b and one' :: 'b and inv' :: 'b => 'b
    and h :: 'a => 'b +
  assumes map-closed: x ∈ G ⟹ h x ∈ H
    and map-mult: ⟦x ∈ G; y ∈ G⟧ ⟹ h (mult x y) = mult' (h x) (h y)
begin

lemma map-one:
  h one = one'
proof −
  have h-one-closed: h one ∈ H
    using G.one-closed by (rule map-closed)
  have h-one-idem: h one = mult' (h one) (h one)
  proof −
    have h one = h (mult one one)
      using G.one-closed by (simp add: G.mult-one-left)
```

```
    also have ... = mult' (h one) (h one)
      using G.one-closed G.one-closed by (rule map-mult)
    finally show ?thesis .
  qed
  have mult' one' (h one) = mult' (h one) (h one)
    using h-one-closed h-one-idem by (simp add: H.mult-one-left)
  then have one' = h one
    by (rule H.right-cancel[OF h-one-closed H.one-closed h-one-closed])
  then show ?thesis
    by simp
qed

lemma map-inv:
  assumes x: x ∈ G
  shows h (inv x) = inv' (h x)
proof −
  have hx: h x ∈ H
    using x by (rule map-closed)
  have hinvx: h (inv x) ∈ H
    using x G.inv-closed[OF x] by (auto intro: map-closed)
  have eq-left: mult' (h x) (h (inv x)) = one'
  proof −
    have h (mult x (inv x)) = mult' (h x) (h (inv x))
      using x G.inv-closed[OF x] by (rule map-mult)
    then have mult' (h x) (h (inv x)) = h (mult x (inv x))
      by simp
    also have ... = h one
      using x by (simp add: G.mult-inv-right)
    also have ... = one'
      by (rule map-one)
    finally show ?thesis .
  qed
  have eq: mult' (h x) (h (inv x)) = mult' (h x) (inv' (h x))
    using eq-left hx by (simp add: H.mult-inv-right)
  show ?thesis
    by (rule H.left-cancel[OF hx hinvx H.inv-closed[OF hx] eq])
qed

end

end
theory Carrier-Amalgamated-Free-Product-Eval
  imports Carrier-Amalgamated-Free-Product Carrier-Group-Scaffold
begin
```

9 Evaluating carrier amalgamated free-product words

Once the carrier-side amalgamated words have been defined, one still needs a way to evaluate them in a target carrier group. The evaluation locale proved here is the algebraic engine behind the later decoding maps in both the topological and classical Seifert–van Kampen statements.

locale *carrier-full-amalg-word-eval* =
  Grp1: *carrier-group G1 mult1 one1 inv1* +
  Grp2: *carrier-group G2 mult2 one2 inv2* +
  Cod: *carrier-group K multK oneK invK* +
  J1: *carrier-group-hom G1 mult1 one1 inv1 K multK oneK invK j1* +
  J2: *carrier-group-hom G2 mult2 one2 inv2 K multK oneK invK j2*
  for *G1* :: *'a set*
    and *mult1* :: *'a => 'a => 'a*
    and *one1* :: *'a*
    and *inv1* :: *'a => 'a*
    and *G2* :: *'b set*
    and *mult2* :: *'b => 'b => 'b*
    and *one2* :: *'b*
    and *inv2* :: *'b => 'b*
    and *H* :: *'h set*
    and *i1* :: *'h => 'a*
    and *i2* :: *'h => 'b*
    and *K* :: *'k set*
    and *multK* :: *'k => 'k => 'k*
    and *oneK* :: *'k*
    and *invK* :: *'k => 'k*
    and *j1* :: *'a => 'k*
    and *j2* :: *'b => 'k* +
  assumes *i1-closed*: *h ∈ H ⟹ i1 h ∈ G1*
    and *i2-closed*: *h ∈ H ⟹ i2 h ∈ G2*
    and *agree*: *h ∈ H ⟹ j1 (i1 h) = j2 (i2 h)*
begin

fun *carrier-full-amalg-eval* :: *('a, 'b) free-product-word => 'k* **where**
  carrier-full-amalg-eval WordNil = oneK
| *carrier-full-amalg-eval (WordLeft a rest) =*
    multK (j1 a) (carrier-full-amalg-eval rest)
| *carrier-full-amalg-eval (WordRight b rest) =*
    multK (j2 b) (carrier-full-amalg-eval rest)

lemma *carrier-full-amalg-eval-in-carrier*:
  assumes *fpw-in-space G1 G2 w*
  shows *carrier-full-amalg-eval w ∈ K*
  using *assms*
proof (*induction w*)
  case *WordNil*

```
  then show ?case
    by simp
next
  case (WordLeft a rest)
  then have a-in: a ∈ G1 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule WordLeft.IH[OF rest-in])
  then show ?case
    using a-in eval-rest-in by (simp add: Cod.mult-closed J1.map-closed)
next
  case (WordRight b rest)
  then have b-in: b ∈ G2 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule WordRight.IH[OF rest-in])
  then show ?case
    using b-in eval-rest-in by (simp add: Cod.mult-closed J2.map-closed)
qed

lemma carrier-amalgam-step-preserves-space-iff:
  assumes step: carrier-amalgam-step H i1 i2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
  using step
proof cases
  case (identify h rest)
  then show ?thesis
    using i1-closed[OF ‹h ∈ H›] i2-closed[OF ‹h ∈ H›] by simp
qed

lemma carrier-amalgam-equiv-preserves-space-iff:
  assumes rel: carrier-amalgam-equiv H i1 i2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
  using rel
proof (induction rule: carrier-amalgam-equiv.induct)
  case (refl w)
  then show ?case
    by simp
next
  case (sym u v)
  then show ?case
    by simp
next
  case (trans u v w)
  then show ?case
    by blast
next
  case (step u v)
  then show ?case
```

```
    by (rule carrier-amalgam-step-preserves-space-iff)
next
  case (left-context u v a)
  then show ?case
    by simp
next
  case (right-context u v b)
  then show ?case
    by simp
qed

lemma carrier-amalgam-step-preserves-eval:
  assumes step: carrier-amalgam-step H i1 i2 u v
  shows carrier-full-amalg-eval u = carrier-full-amalg-eval v
  using step
proof cases
  case (identify h rest)
  then show ?thesis
    using agree[OF ‹h ∈ H›]
    by (simp add: J1.map-closed J2.map-closed)
qed

lemma carrier-amalgam-equiv-preserves-eval:
  assumes rel: carrier-amalgam-equiv H i1 i2 u v
  shows carrier-full-amalg-eval u = carrier-full-amalg-eval v
  using rel
proof (induction rule: carrier-amalgam-equiv.induct)
  case (refl w)
  then show ?case
    by simp
next
  case (sym u v)
  then show ?case
    by simp
next
  case (trans u v w)
  then show ?case
    by simp
next
  case (step u v)
  then show ?case
    by (rule carrier-amalgam-step-preserves-eval)
next
  case (left-context u v a)
  then show ?case
    by auto
next
  case (right-context u v b)
  then show ?case
```

```
    by auto
qed

lemma carrier-fpw-reduction-step-preserves-eval:
  assumes step: carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2 u v
  shows carrier-full-amalg-eval u = carrier-full-amalg-eval v
  using step
proof (induction rule: carrier-fpw-reduction-step.induct)
  case (combine-left a b rest)
  have a-in: a ∈ G1 and b-in: b ∈ G1 and rest-in: fpw-in-space G1 G2 rest
    using combine-left by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule carrier-full-amalg-eval-in-carrier[OF rest-in])
  have ja-in: j1 a ∈ K
    by (rule J1.map-closed[OF a-in])
  have jb-in: j1 b ∈ K
    by (rule J1.map-closed[OF b-in])
  show ?case
    using a-in b-in eval-rest-in ja-in jb-in
    by (simp add: J1.map-mult Cod.mult-assoc[symmetric])
next
  case (combine-right a b rest)
  have a-in: a ∈ G2 and b-in: b ∈ G2 and rest-in: fpw-in-space G1 G2 rest
    using combine-right by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule carrier-full-amalg-eval-in-carrier[OF rest-in])
  have ja-in: j2 a ∈ K
    by (rule J2.map-closed[OF a-in])
  have jb-in: j2 b ∈ K
    by (rule J2.map-closed[OF b-in])
  show ?case
    using a-in b-in eval-rest-in ja-in jb-in
    by (simp add: J2.map-mult Cod.mult-assoc[symmetric])
next
  case (remove-left-one rest)
  have rest-in: fpw-in-space G1 G2 rest
    using remove-left-one by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule carrier-full-amalg-eval-in-carrier[OF rest-in])
  show ?case
    using eval-rest-in
    by (simp add: J1.map-one Cod.mult-one-left)
next
  case (remove-right-one rest)
  have rest-in: fpw-in-space G1 G2 rest
    using remove-right-one by auto
  have eval-rest-in: carrier-full-amalg-eval rest ∈ K
    by (rule carrier-full-amalg-eval-in-carrier[OF rest-in])
  show ?case
```

```
    using eval-rest-in
    by (simp add: J2.map-one Cod.mult-one-left)
next
  case (context-left a u v)
  then show ?case
    by simp
next
  case (context-right b u v)
  then show ?case
    by simp
qed

lemmas carrier-fpw-reduction-space-iff =
  Carrier-Amalgamated-Free-Product.carrier-fpw-reduction-preserves-space-iff

lemma carrier-fpw-reduction-preserves-space-iff:
  assumes rel: carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
proof −
  show ?thesis
    by (rule carrier-fpw-reduction-space-iff[OF rel])
qed

lemma carrier-fpw-reduction-preserves-eval:
  assumes rel: carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2 u v
  shows carrier-full-amalg-eval u = carrier-full-amalg-eval v
  using rel
proof (induction rule: carrier-fpw-reduction.induct)
  case (refl w)
  then show ?case
    by simp
next
  case (sym u v)
  then show ?case
    by simp
next
  case (trans u v w)
  then show ?case
    by simp
next
  case (step u v)
  then show ?case
    by (rule carrier-fpw-reduction-step-preserves-eval)
qed

lemma carrier-full-amalg-equiv-preserves-space-iff:
  assumes rel: carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v
  shows fpw-in-space G1 G2 u ⟷ fpw-in-space G1 G2 v
  using rel
```

proof (*induction rule*: *carrier-full-amalg-equiv.induct*)
  case (*refl w*)
  then show *?case*
    by *simp*
next
  case (*sym u v*)
  then show *?case*
    by *simp*
next
  case (*trans u v w*)
  then show *?case*
    by *blast*
next
  case (*of-amalg u v*)
  then show *?case*
    by (*rule carrier-amalgam-equiv-preserves-space-iff*)
next
  case (*of-reduction u v*)
  then show *?case*
    by (*rule carrier-fpw-reduction-preserves-space-iff*)
qed

lemma *carrier-full-amalg-equiv-preserves-eval*:
  assumes *rel*: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
  shows *carrier-full-amalg-eval u = carrier-full-amalg-eval v*
  using *rel*
proof (*induction rule*: *carrier-full-amalg-equiv.induct*)
  case (*refl w*)
  then show *?case*
    by *simp*
next
  case (*sym u v*)
  then show *?case*
    by *simp*
next
  case (*trans u v w*)
  then show *?case*
    by *simp*
next
  case (*of-amalg u v*)
  then show *?case*
    by (*rule carrier-amalgam-equiv-preserves-eval*)
next
  case (*of-reduction u v*)
  then show *?case*
    by (*rule carrier-fpw-reduction-preserves-eval*)
qed

definition *carrier-full-amalg-has-good-rep* :: (*'a*, *'b*) *free-product-word* => *bool*

where
carrier-full-amalg-has-good-rep w $\longleftrightarrow$
(∃ *v. carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 w v*
∧ *fpw-in-space G1 G2 v*)

definition *carrier-full-amalg-some-good-rep* ::
('*a*, '*b*) *free-product-word* => ('*a*, '*b*) *free-product-word*
where
carrier-full-amalg-some-good-rep w =
(*SOME v. carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 w v*
∧ *fpw-in-space G1 G2 v*)

definition *carrier-full-amalg-decode* :: ('*a*, '*b*) *free-product-word* => '*k* **where**
carrier-full-amalg-decode w =
(*if carrier-full-amalg-has-good-rep w*
then carrier-full-amalg-eval (*carrier-full-amalg-some-good-rep w*)
else oneK)

lemma *carrier-full-amalg-has-good-repI*:
assumes *fpw-in-space G1 G2 w*
shows *carrier-full-amalg-has-good-rep w*
using *assms* **unfolding** *carrier-full-amalg-has-good-rep-def* **by** *auto*

lemma *carrier-full-amalg-some-good-rep*:
assumes *carrier-full-amalg-has-good-rep w*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
w (*carrier-full-amalg-some-good-rep w*)
and *fpw-in-space G1 G2* (*carrier-full-amalg-some-good-rep w*)
proof −
from *assms* **obtain** *v* **where**
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 w v
and *fpw-in-space G1 G2 v*
unfolding *carrier-full-amalg-has-good-rep-def* **by** *blast*
then have *ex*:
∃ *v. carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 w v*
∧ *fpw-in-space G1 G2 v*
by *blast*
then have
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
w (*carrier-full-amalg-some-good-rep w*)
∧ *fpw-in-space G1 G2* (*carrier-full-amalg-some-good-rep w*)
unfolding *carrier-full-amalg-some-good-rep-def*
by (*rule someI-ex*)
then show
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
w (*carrier-full-amalg-some-good-rep w*)
and *fpw-in-space G1 G2* (*carrier-full-amalg-some-good-rep w*)
by *auto*
qed

lemma *carrier-full-amalg-has-good-rep-respects*:
assumes *uv*: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
shows *carrier-full-amalg-has-good-rep u* $\longleftrightarrow$ *carrier-full-amalg-has-good-rep v*
proof
assume *carrier-full-amalg-has-good-rep u*
then obtain *w* **where**
uw: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u w*
and *w-in*: *fpw-in-space G1 G2 w*
unfolding *carrier-full-amalg-has-good-rep-def* **by** *blast*
have *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 v w*
using *uv uw*
by (*meson carrier-full-amalg-equiv.sym carrier-full-amalg-equiv.trans*)
then show *carrier-full-amalg-has-good-rep v*
using *w-in* **unfolding** *carrier-full-amalg-has-good-rep-def* **by** *blast*
next
assume *carrier-full-amalg-has-good-rep v*
then obtain *w* **where**
vw: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 v w*
and *w-in*: *fpw-in-space G1 G2 w*
unfolding *carrier-full-amalg-has-good-rep-def* **by** *blast*
have *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u w*
using *uv vw*
by (*meson carrier-full-amalg-equiv.trans*)
then show *carrier-full-amalg-has-good-rep u*
using *w-in* **unfolding** *carrier-full-amalg-has-good-rep-def* **by** *blast*
qed

lemma *carrier-full-amalg-decode-in-carrier*:
carrier-full-amalg-decode w $\in$ *K*
proof (*cases carrier-full-amalg-has-good-rep w*)
case *True*
then have *good-rep*:
fpw-in-space G1 G2 (*carrier-full-amalg-some-good-rep w*)
by (*rule carrier-full-amalg-some-good-rep*)
show *?thesis*
unfolding *carrier-full-amalg-decode-def*
using *True carrier-full-amalg-eval-in-carrier*[*OF good-rep*] **by** *simp*
next
case *False*
then show *?thesis*
unfolding *carrier-full-amalg-decode-def* **by** *simp*
qed

lemma *carrier-full-amalg-decode-respects*:
assumes *uv*: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
shows *carrier-full-amalg-decode u* = *carrier-full-amalg-decode v*
proof (*cases carrier-full-amalg-has-good-rep u*)
case *False*

```
  then have ¬ carrier-full-amalg-has-good-rep v
    using carrier-full-amalg-has-good-rep-respects[OF uv] by blast
  then show ?thesis
    unfolding carrier-full-amalg-decode-def using False by simp
next
  case True
  then have v-has: carrier-full-amalg-has-good-rep v
    using carrier-full-amalg-has-good-rep-respects[OF uv] by blast
  have u-rep-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        u (carrier-full-amalg-some-good-rep u)
    using True by (rule carrier-full-amalg-some-good-rep)+
  have v-rep-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        v (carrier-full-amalg-some-good-rep v)
    using v-has by (rule carrier-full-amalg-some-good-rep)+
  have rep-eq:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (carrier-full-amalg-some-good-rep u)
      (carrier-full-amalg-some-good-rep v)
    using u-rep-rel v-rep-rel uv
    by (meson carrier-full-amalg-equiv.sym carrier-full-amalg-equiv.trans)
  have eval-eq:
    carrier-full-amalg-eval (carrier-full-amalg-some-good-rep u) =
      carrier-full-amalg-eval (carrier-full-amalg-some-good-rep v)
    by (rule carrier-full-amalg-equiv-preserves-eval[OF rep-eq])
  show ?thesis
    unfolding carrier-full-amalg-decode-def
    using True v-has eval-eq by simp
qed

lemma carrier-full-amalg-decode-eq-eval:
  assumes w-in: fpw-in-space G1 G2 w
  shows carrier-full-amalg-decode w = carrier-full-amalg-eval w
proof −
  have has: carrier-full-amalg-has-good-rep w
    by (rule carrier-full-amalg-has-good-repI[OF w-in])
  have rep-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        w (carrier-full-amalg-some-good-rep w)
    using has by (rule carrier-full-amalg-some-good-rep)+
  have eval-eq:
    carrier-full-amalg-eval w =
      carrier-full-amalg-eval (carrier-full-amalg-some-good-rep w)
    by (rule carrier-full-amalg-equiv-preserves-eval[OF rep-rel])
  show ?thesis
    unfolding carrier-full-amalg-decode-def
    using has eval-eq by simp
qed
```

end

end
theory *Explicit-Path-Homotopy-Scaffold*
imports *HOL−Analysis.Homotopy*
begin

10 Explicit-topology paths and homotopies

HOL-Analysis already provides paths and homotopies for subsets of a type. For the pushout-oriented development, however, it is convenient to work directly with arbitrary topologies. This theory therefore rephrases the basic path and homotopy notions in explicit-topology form.

definition *loopin-space* ::
'a topology ⇒ *'a* ⇒ (*real* ⇒ *'a*) *set*
where
loopin-space X x = {*p*. *pathin X p* ∧ *p 0* = *x* ∧ *p 1* = *x*}

definition *homotopic-pathsin* ::
'a topology ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool*
where
homotopic-pathsin X p q ≡
homotopic-with (λ*r*. *r 0* = *p 0* ∧ *r 1* = *p 1*) (*top-of-set* {*0..1*}) *X p q*

definition *homotopic-loopsin* ::
'a topology ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool*
where
homotopic-loopsin X p q ≡
homotopic-with (λ*r*. *r 1* = *r 0*) (*top-of-set* {*0..1*}) *X p q*

definition *reversepathin* :: (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*
where
reversepathin p = (λ*t*. *p* (*1* − *t*))

definition *joinpathin* :: (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*
where
joinpathin p q = (λ*t*. *if t* ≤ *1* / *2 then p* (*2* ∗ *t*) *else q* (*2* ∗ *t* − *1*))

definition *subpathin* :: *real* ⇒ *real* ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*
where
subpathin u v p = (λ*t*. *p* ((*v* − *u*) ∗ *t* + *u*))

lemma *reversepathin-eq-reversepath* [*simp*]:
reversepathin p = *reversepath p*
unfolding *reversepathin-def reversepath-def* **by** *simp*

lemma *reversepathin-reversepathin* [*simp*]:

reversepathin (*reversepathin p*) = *p*
unfolding *reversepathin-def* **by** (*rule ext*) *simp*

lemma *joinpathin-eq-joinpaths* [*simp*]:
joinpathin p q = *p* +++ *q*
unfolding *joinpathin-def joinpaths-def* **by** *simp*

lemma *subpathin-image*:
subpathin u v p ‘ {*0..1*} = *p* ‘ *closed-segment u v*
by (*simp add*: *subpathin-def image-image closed-segment-real-eq algebra-simps*)

lemma *subpathin-image-eq*:
assumes *u* ≤ *v*
shows *subpathin u v p* ‘ {*0..1*} = *p* ‘ {*u..v*}
using *assms* **by** (*simp add*: *subpathin-image closed-segment-eq-real-ivl*)

lemma *subpathin-0-1* [*simp*]:
subpathin 0 1 p = *p*
unfolding *subpathin-def* **by** (*rule ext*) *simp*

lemma *joinpathin-subpathin-middle* [*simp*]:
joinpathin (*subpathin 0* (*1 / 2*) *p*) (*subpathin* (*1 / 2*) *1 p*) = *p*
unfolding *joinpathin-def subpathin-def*
by (*rule ext*) (*auto simp*: *field-simps*)

lemma *continuous-map-top01-id*:
continuous-map (*top-of-set* {*0..1*}) *euclideanreal id*
by (*rule continuous-map-into-fulltopology*[*OF continuous-map-id*])

lemma *loopin-spaceI*:
assumes *pathin X p*
and *p 0* = *x*
and *p 1* = *x*
shows *p* ∈ *loopin-space X x*
using *assms* **unfolding** *loopin-space-def* **by** *blast*

lemma *loopin-spaceE*:
assumes *p* ∈ *loopin-space X x*
obtains *pathin X p p 0* = *x p 1* = *x*
using *assms* **unfolding** *loopin-space-def* **by** *blast*

lemma *constant-loopin-in-space*:
assumes *x* ∈ *topspace X*
shows (λ-. *x*) ∈ *loopin-space X x*
using *assms* **unfolding** *loopin-space-def* **by** *auto*

lemma *pathin-image-subset-topspace*:
assumes *pathin X p*
shows *p* ‘ {*0..1*} ⊆ *topspace X*

proof −
have $p \in \{0..1\} \to$ *topspace* X
using *assms* **by** (*rule path-image-subset-topspace*)
then show *?thesis*
by *auto*
qed

lemma *pathin-reversepathin*:
assumes *pathin* X p
shows *pathin* X (*reversepathin* p)
proof −
have *rev-cont*: *continuous-map* (*top-of-set* $\{0..1\}$) (*top-of-set* $\{0..1\}$) (λt::*real*. $1 - t$)
by (*intro continuous-map-into-subtopology continuous-intros*) *auto*
have *continuous-map* (*top-of-set* $\{0..1\}$) X ($p \circ (\lambda t$::*real*. $1 - t)$)
using *rev-cont assms* **unfolding** *pathin-def* **by** (*rule continuous-map-compose*)
then show *?thesis*
unfolding *pathin-def reversepathin-def* **by** (*simp add*: *o-def*)
qed

lemma *pathin-joinpathin*:
assumes *p*: *pathin* X p
and *q*: *pathin* X q
and *pq*: $p\ 1 = q\ 0$
shows *pathin* X (*joinpathin* p q)
proof −
let *?T01* = *subtopology euclideanreal* $\{0..1\}$
have *p-cont*: *continuous-map ?T01 X p*
using *p* **unfolding** *pathin-def* **.**
have *q-cont*: *continuous-map ?T01 X q*
using *q* **unfolding** *pathin-def* **.**
have *left-scale-eu*:
continuous-map (*subtopology ?T01* $\{x \in$ *topspace ?T01*. $2 * x \leq 1\}$) *euclideanreal* (λx::*real*. $2 * x$)
proof −
have *continuous-map ?T01 euclideanreal* (λx::*real*. $2 * x$)
by (*simp add*: *continuous-map-iff-continuous continuous-intros*)
then show *?thesis*
by (*rule continuous-map-from-subtopology*)
qed
have *left-scale-range*:
(λx::*real*. $2 * x$) $\in$ *topspace* (*subtopology ?T01* $\{x \in$ *topspace ?T01*. $2 * x \leq 1\}$) $\to \{0..1\}$
proof
fix x :: *real*
assume *x-in*: $x \in$ *topspace* (*subtopology ?T01* $\{x \in$ *topspace ?T01*. $2 * x \leq 1\}$)
then have *x01*: $x \in \{0..1\}$ **and** *x-half*: $2 * x \leq 1$
by *auto*

```
    from x01 have x-ge0: 0 ≤ x
      by auto
    have lower: 0 ≤ 2 * x
      using x-ge0 by linarith
    show 2 * x ∈ {0..1}
      using lower x-half by simp
  qed
  have left-scale-cont:
     continuous-map (subtopology ?T01 {x ∈ topspace ?T01. 2 * x ≤ 1}) ?T01
(λx::real. 2 * x)
    by (rule continuous-map-into-subtopology[OF left-scale-eu left-scale-range])
  have left-cont-comp:
    continuous-map (subtopology ?T01 {x ∈ topspace ?T01. 2 * x ≤ 1}) X
      (p ∘ (λx::real. 2 * x))
    by (rule continuous-map-compose[OF left-scale-cont p-cont])
  have left-eq:
    (p ∘ (λx::real. 2 * x)) = (λx::real. p (2 * x))
    by (auto simp: fun-eq-iff o-def)
  from left-cont-comp have left-cont:
   continuous-map (subtopology ?T01 {x ∈ topspace ?T01. 2 * x ≤ 1}) X (λx::real.
p (2 * x))
    by (simp only: left-eq)
  have right-shift-eu:
    continuous-map (subtopology ?T01 {x ∈ topspace ?T01. 1 ≤ 2 * x}) euclidean-
real (λx::real. 2 * x - 1)
  proof -
    have continuous-map ?T01 euclideanreal (λx::real. 2 * x - 1)
      by (simp add: continuous-map-iff-continuous continuous-intros)
    then show ?thesis
      by (rule continuous-map-from-subtopology)
  qed
  have right-shift-range:
    (λx::real. 2 * x - 1) ∈ topspace (subtopology ?T01 {x ∈ topspace ?T01. 1 ≤
2 * x}) → {0..1}
  proof
    fix x :: real
    assume x-in: x ∈ topspace (subtopology ?T01 {x ∈ topspace ?T01. 1 ≤ 2 *
x})
    then have x01: x ∈ {0..1} and x-half: 1 ≤ 2 * x
      by auto
    from x01 have x-le1: x ≤ 1
      by auto
    have lower: 0 ≤ 2 * x - 1
      using x-half by linarith
    have upper: 2 * x - 1 ≤ 1
      using x-le1 by linarith
    show 2 * x - 1 ∈ {0..1}
      using lower upper by simp
  qed
```

have *right-shift-cont*:
continuous-map (*subtopology ?T01* {$x \in$ *topspace ?T01*. $1 \leq 2 * x$}) *?T01* (λx::*real*. $2 * x - 1$)
by (*rule continuous-map-into-subtopology*[*OF right-shift-eu right-shift-range*])
have *right-cont-comp*:
continuous-map (*subtopology ?T01* {$x \in$ *topspace ?T01*. $1 \leq 2 * x$}) *X*
($q \circ (\lambda x$::*real*. $2 * x - 1$))
by (*rule continuous-map-compose*[*OF right-shift-cont q-cont*])
have *right-eq*:
($q \circ (\lambda x$::*real*. $2 * x - 1$)) = (λx::*real*. q ($2 * x - 1$))
by (*auto simp*: *fun-eq-iff o-def*)
from *right-cont-comp* **have** *right-cont*:
continuous-map (*subtopology ?T01* {$x \in$ *topspace ?T01*. $1 \leq 2 * x$}) *X* (λx::*real*. q ($2 * x - 1$))
by (*simp only*: *right-eq*)
have *cont-if*:
continuous-map ?T01 X (λx::*real*. *if* $2 * x \leq 1$ *then* p ($2 * x$) *else* q ($2 * x - 1$))
proof (*rule continuous-map-cases-le*[**where** $p = \lambda x$::*real*. $2 * x$ **and** $q = \lambda$-. 1])
show *continuous-map ?T01 euclideanreal* (λx::*real*. $2 * x$)
by (*simp add*: *continuous-map-iff-continuous continuous-intros*)
show *continuous-map ?T01 euclideanreal* (λ-. 1)
by *simp*
show *continuous-map* (*subtopology ?T01* {$x \in$ *topspace ?T01*. $2 * x \leq 1$}) *X* (λx::*real*. p ($2 * x$))
by (*rule left-cont*)
show *continuous-map* (*subtopology ?T01* {$x \in$ *topspace ?T01*. $1 \leq 2 * x$}) *X* (λx::*real*. q ($2 * x - 1$))
by (*rule right-cont*)
show p ($2 * x$) = q ($2 * x - 1$)
if $x \in$ *topspace ?T01* (λx::*real*. $2 * x$) x = (λ-. 1) x **for** x
using *pq that* **by** *simp*
qed
have *cont-join*: *continuous-map ?T01 X* (*joinpathin p q*)
proof (*rule continuous-map-eq*[*OF cont-if*])
fix x :: *real*
assume *x-in*: $x \in$ *topspace ?T01*
show (λx::*real*. *if* $2 * x \leq 1$ *then* p ($2 * x$) *else* q ($2 * x - 1$)) x = *joinpathin p q x*
using *x-in* **by** (*simp add*: *joinpathin-def field-simps*)
qed
show *?thesis*
using *cont-join* **unfolding** *pathin-def* **.**
qed

lemma *pathin-subpathin*:
assumes *p*: *pathin X p*
and *u*: $u \in \{0..1\}$
and *v*: $v \in \{0..1\}$

shows *pathin X* (*subpathin u v p*)
proof −
have *aff-cont*:
continuous-map (*top-of-set* {*0..1*}) (*top-of-set* {*0..1*}) (λ*t*::*real*. (*v* − *u*) ∗ *t* + *u*)
proof −
have *aff-cont-eu*:
continuous-map (*top-of-set* {*0..1*}) *euclideanreal* (λ*t*::*real*. (*v* − *u*) ∗ *t* + *u*)
by (*simp add*: *continuous-map-iff-continuous continuous-intros*)
have *aff-range*:
(λ*t*::*real*. (*v* − *u*) ∗ *t* + *u*) ∈ *topspace* (*top-of-set* {*0..1*}) → {*0..1*}
proof
fix *x* :: *real*
assume *x-in*: *x* ∈ *topspace* (*top-of-set* {*0..1*})
have *x01*: *x* ∈ {*0..1*}
using *x-in* **by** *simp*
have *nonneg*: *0* ≤ *x* ∗ *v* + (*1* − *x*) ∗ *u*
using *u v x01* **by** (*intro add-nonneg-nonneg mult-nonneg-nonneg*) *auto*
moreover have *le1*: *x* ∗ *v* + (*1* − *x*) ∗ *u* ≤ *1*
using *u v x01* **by** (*intro convex-bound-le*) *auto*
ultimately show ((*v* − *u*) ∗ *x* + *u*) ∈ {*0..1*}
by (*simp add*: *algebra-simps*)
qed
show *?thesis*
using *aff-cont-eu aff-range* **by** (*auto simp*: *continuous-map-in-subtopology*)
qed
have *continuous-map* (*top-of-set* {*0..1*}) *X* (*p* ∘ (λ*t*::*real*. (*v* − *u*) ∗ *t* + *u*))
using *aff-cont p* **unfolding** *pathin-def* **by** (*rule continuous-map-compose*)
then show *?thesis*
unfolding *pathin-def subpathin-def* **by** (*simp add*: *o-def*)
qed

lemma *pathin-subdivision-open-cover*:
assumes *p*: *pathin X p*
and *cover*: *p* ‘ {*0..1*} ⊆ ⋃ *U*
and *openU*: ⋀ *U*. *U* ∈ *U* ⟹ *openin X U*
shows ∃ *n*::*nat*. *0* < *n* ∧
(∀ *i*<*n*. ∃ *U*∈*U*.
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ {*0..1*} ⊆ *U*)
proof −
have *U-nonempty*: *U* ≠ {}
proof
assume *U* = {}
moreover have *p 0* ∈ *p* ‘ {*0..1*}
by *auto*
ultimately show *False*
using *cover* **by** *auto*
qed
define *W* **where**

$W\ U = (SOME\ T.\ open\ T \wedge \{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap T)$ **for** U
have *pre-open*: *openin* (*top-of-set* $\{0..1\}$) $\{t \in \{0..1\}.\ p\ t \in U\}$ **if** *U-in*: $U \in \mathcal{U}$ **for** U
using p *openU*[*OF U-in*]
unfolding *pathin-def continuous-map-def*
by *auto*
have *W-spec*:
open $(W\ U)$
$\{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap W\ U$
if *U-in*: $U \in \mathcal{U}$ **for** U
proof −
from *pre-open*[*OF U-in*] **obtain** T **where**
T-open: *open* T
and *T-eq*: $\{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap T$
by (*auto simp*: *openin-open*)
have *open* $(W\ U) \wedge \{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap W\ U$
unfolding *W-def*
by (*rule someI-ex*) (*use T-open T-eq* **in** *blast*)
then show *open* $(W\ U)$ $\{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap W\ U$
by *auto*
qed
have *interval-cover*: $\{0..1\} \subseteq \bigcup (W\ `\ \mathcal{U})$
proof
fix t :: *real*
assume *t-in*: $t \in \{0..1\}$
have $p\ t \in p\ `\ \{0..1\}$
using *t-in* **by** *blast*
then obtain U **where** *U-in*: $U \in \mathcal{U}$ **and** pU: $p\ t \in U$
using *cover* **by** *blast*
have $\{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap W\ U$
by (*rule W-spec*(*2*)[*OF U-in*])
then have $t \in W\ U$
using *t-in pU* **by** *auto*
then show $t \in \bigcup (W\ `\ \mathcal{U})$
using *U-in* **by** *blast*
qed
have *open-W-image*: *open* B **if** $B \in W\ `\ \mathcal{U}$ **for** B
proof −
from *that* **obtain** U **where** *U-in*: $U \in \mathcal{U}$ **and** *B-eq*: $B = W\ U$
by *blast*
show *open* B
unfolding *B-eq* **by** (*rule W-spec*(*1*)[*OF U-in*])
qed
have *W-nonempty*: $W\ `\ \mathcal{U} \neq \{\}$
using *U-nonempty* **by** *auto*
have *compact01*: *compact* $(\{0..1::real\})$
by (*rule compact-Icc*)
from *Lebesgue-number-lemma*[*OF compact01 W-nonempty interval-cover open-W-image*]

obtain δ **where** δ-*pos*: $0 < \delta$
  and δ-*cover*:
    $\bigwedge T.\ T \subseteq \{0..1\} \Longrightarrow \mathit{diameter}\ T < \delta \Longrightarrow \exists\, B{\in}W\ `\ \mathcal{U}.\ T \subseteq B$
  by *blast*
obtain m **where** m: *inverse* (*of-nat* (*Suc* m)) $< \delta$
  using *reals-Archimedean*[*OF* δ-*pos*] **by** *blast*
let *?n* = *Suc m*
have *segment-cover*:
  $\exists\, U{\in}\mathcal{U}.\ \mathit{subpathin}\ (\mathit{real}\ i\ /\ \mathit{real}\ ?n)\ (\mathit{real}\ (\mathit{Suc}\ i)\ /\ \mathit{real}\ ?n)\ p\ `\ \{0..1\} \subseteq U$
  if *i-lt*: $i < ?n$ **for** i
proof −
  let *?a* = *real i* / *real ?n*
  let *?b* = *real* (*Suc i*) / *real ?n*
  let *?T* = {*?a*..*?b*}
  have *a01*: $?a \in \{0..1\}$
    using *i-lt* **by** *auto*
  have *b01*: $?b \in \{0..1\}$
    using *i-lt* **by** *auto*
  have *ab*: $?a \leq ?b$
    using *i-lt* **by** (*simp add*: *field-simps*)
  have *T-subset*: $?T \subseteq \{0..1\}$
    using *a01 b01* **by** *auto*
  have *diameter ?T* = *?b* − *?a*
    using *ab* **by** *simp*
  also have ... = *1* / *real ?n*
    by (*simp add*: *field-simps*)
  also have ... $< \delta$
    using *m* **by** (*simp add*: *divide-inverse*)
  finally have *T-small*: *diameter ?T* $< \delta$ **.**
  from δ-*cover*[*OF T-subset T-small*] **obtain** B **where**
      B-in: $B \in W\ `\ \mathcal{U}$
      and *T-B*: $?T \subseteq B$
    by *blast*
  obtain U **where** *U-in*: $U \in \mathcal{U}$ **and** *B-eq*: $B = W\ U$
    using *B-in* **by** *blast*
  have *pre-eq*: $\{t \in \{0..1\}.\ p\ t \in U\} = \{0..1\} \cap W\ U$
    by (*rule W-spec*(*2*)[*OF U-in*])
  have *subpath-subset*: *subpathin ?a ?b p* ` {*0..1*} ⊆ *U*
  proof
    fix x
    assume *x-in*: $x \in \mathit{subpathin}\ ?a\ ?b\ p\ `\ \{0..1\}$
    then obtain t **where** *t-in*: $t \in \{0..1\}$ **and** *x-eq*: $x = \mathit{subpathin}\ ?a\ ?b\ p\ t$
      by *blast*
    have *s-in-T*: $((?b - ?a) * t + ?a) \in ?T$
    proof −
      from *t-in* **have** *t-ge0*: $0 \leq t$ **and** *t-le1*: $t \leq 1$
        by *auto*
      have *diff-nonneg*: $0 \leq ?b - ?a$
        using *ab* **by** *linarith*

```
      have term-nonneg: 0 ≤ (?b − ?a) ∗ t
        by (rule mult-nonneg-nonneg[OF diff-nonneg t-ge0])
      have term-le: (?b − ?a) ∗ t ≤ (?b − ?a) ∗ 1
        using t-le1 diff-nonneg by (rule mult-left-mono)
      have lower: ?a ≤ (?b − ?a) ∗ t + ?a
        using term-nonneg by linarith
      have upper: (?b − ?a) ∗ t + ?a ≤ ?b
        using term-le by simp
      show ?thesis
        using lower upper by simp
    qed
    have s-in-unit: ((?b − ?a) ∗ t + ?a) ∈ {0..1}
      using s-in-T T-subset by blast
    have s-in-W: ((?b − ?a) ∗ t + ?a) ∈ W U
      using s-in-T T-B B-eq by auto
    have ((?b − ?a) ∗ t + ?a) ∈ {t ∈ {0..1}. p t ∈ U}
      using s-in-unit s-in-W pre-eq by auto
    then show x ∈ U
      using x-eq by (simp add: subpathin-def)
  qed
  then show ?thesis
    using U-in by blast
 qed
 show ?thesis
  by (rule exI[of - ?n]) (use segment-cover in auto)
qed

lemma loopin-space-joinpathin:
 assumes p: p ∈ loopin-space X x
  and q: q ∈ loopin-space X x
 shows joinpathin p q ∈ loopin-space X x
proof −
 from p obtain p-in where p-in: pathin X p p 0 = x p 1 = x
  by (rule loopin-spaceE)
 from q obtain q-in where q-in: pathin X q q 0 = x q 1 = x
  by (rule loopin-spaceE)
 have pathin X (joinpathin p q)
  by (rule pathin-joinpathin[OF p-in(1) q-in(1)]) (simp add: p-in q-in)
 moreover have joinpathin p q 0 = x joinpathin p q 1 = x
  by (simp-all add: joinpathin-def p-in q-in)
 ultimately show ?thesis
  by (rule loopin-spaceI)
qed

lemma homotopic-pathsin-top-of-set [simp]:
 homotopic-pathsin (top-of-set S) p q ⟷ homotopic-paths S p q
 unfolding homotopic-pathsin-def homotopic-paths-def
 by (simp add: pathstart-def pathfinish-def)
```

lemma *homotopic-loopsin-top-of-set* [*simp*]:
homotopic-loopsin (*top-of-set S*) *p q* ⟷ *homotopic-loops S p q*
unfolding *homotopic-loopsin-def homotopic-loops-def*
by (*simp add*: *pathstart-def pathfinish-def*)

lemma *homotopic-pathsin-imp-pathin*:
assumes *homotopic-pathsin X p q*
shows *pathin X p pathin X q*
using *assms* **unfolding** *homotopic-pathsin-def pathin-def*
by (*auto dest*: *homotopic-with-imp-continuous-maps*)

lemma *homotopic-pathsin-imp-endpoints*:
assumes *homotopic-pathsin X p q*
shows *q 0 = p 0 q 1 = p 1*
using *assms* **unfolding** *homotopic-pathsin-def*
by (*auto dest*: *homotopic-with-imp-property*)

lemma *homotopic-pathsin-refl* [*simp*]:
homotopic-pathsin X p p ⟷ *pathin X p*
unfolding *homotopic-pathsin-def pathin-def*
by (*simp add*: *homotopic-with-refl*)

lemma *homotopic-pathsin-sym*:
assumes *homotopic-pathsin X p q*
shows *homotopic-pathsin X q p*
proof −
have *q 0 = p 0 q 1 = p 1*
using *assms* **by** (*rule homotopic-pathsin-imp-endpoints*)+
with *assms* **show** *?thesis*
unfolding *homotopic-pathsin-def*
by (*simp add*: *homotopic-with-sym*)
qed

lemma *homotopic-pathsin-trans*:
assumes *homotopic-pathsin X p q*
and *homotopic-pathsin X q r*
shows *homotopic-pathsin X p r*
proof −
have *q*: *q 0 = p 0 q 1 = p 1*
using *assms*(*1*) **by** (*rule homotopic-pathsin-imp-endpoints*)+
have *pq*: *homotopic-with* (λ*s*. *s 0 = p 0* ∧ *s 1 = p 1*) (*top-of-set* {*0..1*}) *X p q*
using *assms*(*1*) **unfolding** *homotopic-pathsin-def* **.**
have *qr*: *homotopic-with* (λ*s*. *s 0 = p 0* ∧ *s 1 = p 1*) (*top-of-set* {*0..1*}) *X q r*
using *assms*(*2*) *q* **unfolding** *homotopic-pathsin-def* **by** *simp*
from *pq qr* **show** *?thesis*
unfolding *homotopic-pathsin-def* **by** (*rule homotopic-with-trans*)
qed

lemma *homotopic-pathsin-eq*:

assumes *pathin X p*
and $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t = q\ t$
shows *homotopic-pathsin X p q*
proof −
have *continuous-map* (*top-of-set* $\{0..1\}$) *X p*
using *assms*(*1*) **unfolding** *pathin-def* **by** *simp*
moreover have $\forall\, x{\in}$*topspace* (*top-of-set* $\{0..1\}$). *p x* = *q x*
using *assms*(*2*) **by** *simp*
ultimately show *?thesis*
unfolding *homotopic-pathsin-def*
by (*intro homotopic-with-equal*) *auto*
qed

lemma *continuous-map-homotopic-joinpathin-lemma*:
fixes *p q* :: *real* ⇒ *real* ⇒ *'a*
assumes *p*:
continuous-map (*prod-topology* (*top-of-set* $\{0..1\}$) (*top-of-set* $\{0..1\}$)) *X*
(λ*y*. *p* (*fst y*) (*snd y*))
and *q*:
continuous-map (*prod-topology* (*top-of-set* $\{0..1\}$) (*top-of-set* $\{0..1\}$)) *X*
(λ*y*. *q* (*fst y*) (*snd y*))
and *pq*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t\ 1 = q\ t\ 0$
shows
continuous-map (*prod-topology* (*top-of-set* $\{0..1\}$) (*top-of-set* $\{0..1\}$)) *X*
(λ*y*. *joinpathin* (*p* (*fst y*)) (*q* (*fst y*)) (*snd y*))
proof −
let *?T01* = *subtopology euclideanreal* $\{0..1\}$
let *?A* = *prod-topology ?T01 ?T01*
have *sect*:
(λ*t*. *p* (*fst t*) (*2* ∗ *snd t*)) = (λ*y*. *p* (*fst y*) (*snd y*)) ∘ (λ*y*. (*fst y*, *2* ∗ *snd y*))
(λ*t*. *q* (*fst t*) (*2* ∗ *snd t* − *1*)) = (λ*y*. *q* (*fst y*) (*snd y*)) ∘ (λ*y*. (*fst y*, *2* ∗ *snd y*
− *1*))
by *force*+
show *?thesis*
unfolding *joinpathin-def*
proof (*rule continuous-map-cases-le*)
show *continuous-map ?A euclideanreal snd*
by (*rule continuous-map-into-fulltopology*[*OF continuous-map-snd*])
show *continuous-map ?A euclideanreal* (λ*-*. *1* / *2*)
by *simp*
have *left-pair-cont*:
continuous-map (*subtopology ?A* $\{y \in$ *topspace ?A*. *snd y* ≤ *1* / *2*$\}$) *?A*
(λ*t*. (*fst t*, *2* ∗ *snd t*))
proof (*rule continuous-map-pairedI*)
show *continuous-map* (*subtopology ?A* $\{y \in$ *topspace ?A*. *snd y* ≤ *1* / *2*$\}$)
?T01 fst
by (*rule continuous-map-from-subtopology*[*OF continuous-map-fst*])
show *continuous-map* (*subtopology ?A* $\{y \in$ *topspace ?A*. *snd y* ≤ *1* / *2*$\}$)
?T01 (λ*t*. *2* ∗ *snd t*)

```
    proof −
      have snd-cont:
          continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2})
euclideanreal snd
                    by (rule continuous-map-into-fulltopology[OF continu-
ous-map-subtopology-snd])
      have scale-cont:
          continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2})
euclideanreal
        (λt. 2 ∗ snd t)
        using snd-cont by (intro continuous-intros)
      have scale-range:
        (λt. 2 ∗ snd t) ∈
          topspace (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) → {0..1}
      proof
        fix t :: real × real
        assume t-in: t ∈ topspace (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1
/ 2})
        then have tA: t ∈ topspace ?A and t-half: snd t ≤ 1 / 2
          by auto
        have snd01: snd t ∈ {0..1}
          using tA by (auto simp: topspace-prod-topology)
        from snd01 have snd-ge0: 0 ≤ snd t and snd-le1: snd t ≤ 1
          by auto
        have lower: 0 ≤ 2 ∗ (snd t :: real)
          using snd-ge0 by linarith
        have upper: 2 ∗ (snd t :: real) ≤ 1
          using t-half by linarith
        have bounds: 0 ≤ 2 ∗ (snd t :: real) ∧ 2 ∗ (snd t :: real) ≤ 1
          using lower upper by blast
        show 2 ∗ (snd t :: real) ∈ {0..1}
        proof −
          from bounds have lower′: 0 ≤ 2 ∗ (snd t :: real) and upper′: 2 ∗ (snd
t :: real) ≤ 1
            by auto
          show ?thesis
            using lower′ upper′ by (auto simp: atLeastAtMost-iff)
        qed
      qed
      show ?thesis
        by (rule continuous-map-into-subtopology[OF scale-cont scale-range])
    qed
  qed
  have left-comp:
    continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) X
      (((λy. p (fst y) (snd y)) ∘ (λy. (fst y, 2 ∗ snd y))))
    by (rule continuous-map-compose[OF left-pair-cont p])
  show continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) X
        (λt. p (fst t) (2 ∗ snd t))
```

```
    using left-comp by (simp add: o-def)
  have right-pair-cont:
    continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) ?A
     (λt. (fst t, 2 * snd t - 1))
 proof (rule continuous-map-pairedI)
    show continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y})
?T01 fst
     by (rule continuous-map-from-subtopology[OF continuous-map-fst])
    show continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y})
?T01 (λt. 2 * (snd t :: real) - 1)
    proof -
     have snd-cont:
         continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y})
euclideanreal snd
                    by (rule continuous-map-into-fulltopology[OF continu-
ous-map-subtopology-snd])
     have affine-cont:
         continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y})
euclideanreal
        (λt. 2 * (snd t :: real) - 1)
       using snd-cont by (intro continuous-intros)
     have affine-range:
      (λt. 2 * (snd t :: real) - 1) ∈
       topspace (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) → {0..1}
     proof
      fix t :: real × real
      assume t-in: t ∈ topspace (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd
y})
       then have tA: t ∈ topspace ?A and t-half: 1 / 2 ≤ snd t
        by auto
       have snd01: snd t ∈ {0..1}
        using tA by (auto simp: topspace-prod-topology)
       from snd01 have snd-ge0: 0 ≤ snd t and snd-le1: snd t ≤ 1
        by auto
       have lower: 0 ≤ 2 * (snd t :: real) - 1
        using t-half by linarith
       have upper: 2 * (snd t :: real) - 1 ≤ 1
        using snd-le1 by linarith
       show 2 * (snd t :: real) - 1 ∈ {0..1}
        using lower upper by (auto simp: atLeastAtMost-iff)
     qed
     show ?thesis
      by (rule continuous-map-into-subtopology[OF affine-cont affine-range])
    qed
  qed
  have right-comp:
    continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) X
     (((λy. q (fst y) (snd y)) ∘ (λy. (fst y, 2 * snd y - 1))))
    by (rule continuous-map-compose[OF right-pair-cont q])
```

```
    show continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) X
          (λt. q (fst t) (2 * snd t - 1))
      using right-comp by (simp add: o-def)
    show p (fst y) (2 * snd y) = q (fst y) (2 * snd y - 1)
      if y ∈ topspace ?A snd y = 1 / 2 for y
    proof -
      have fst01: fst y ∈ {0..1}
        using that(1) by (auto simp: topspace-prod-topology)
      have mid1: 2 * snd y = 1
        using that(2) by simp
      have mid0: 2 * snd y - 1 = 0
        using that(2) by simp
      have p (fst y) 1 = q (fst y) 0
        by (rule pq[OF fst01])
      then show ?thesis
        using mid1 mid0 by simp
    qed
  qed
qed

lemma homotopic-pathsin-continuous-image:
  assumes pq: homotopic-pathsin X p q
    and h: continuous-map X Y f
  shows homotopic-pathsin Y (f ∘ p) (f ∘ q)
proof -
  have pq': homotopic-with (λr. r 0 = p 0 ∧ r 1 = p 1) (top-of-set {0..1}) X p q
    using pq unfolding homotopic-pathsin-def .
  show ?thesis
    unfolding homotopic-pathsin-def
     by (rule homotopic-with-compose-continuous-map-left[OF pq' h]) simp
qed

lemma homotopic-pathsin-reversepathin-D:
  assumes homotopic-pathsin X p q
  shows homotopic-pathsin X (reversepathin p) (reversepathin q)
proof -
  have rev-cont: continuous-map (top-of-set {0..1}) (top-of-set {0..1}) (λt::real.
1 - t)
    by (intro continuous-map-into-subtopology continuous-intros) auto
  have pq':
    homotopic-with (λr. r 0 = p 0 ∧ r 1 = p 1) (top-of-set {0..1}) X p q
    using assms unfolding homotopic-pathsin-def .
  have homotopic-with (λr. r 0 = reversepathin p 0 ∧ r 1 = reversepathin p 1)
      (top-of-set {0..1}) X (p ∘ (λt::real. 1 - t)) (q ∘ (λt::real. 1 - t))
  proof (rule homotopic-with-compose-continuous-map-right[OF pq' rev-cont])
    fix j :: real ⇒ 'a
    assume j-end: j 0 = p 0 ∧ j 1 = p 1
    show (λr. r 0 = reversepathin p 0 ∧ r 1 = reversepathin p 1)
       (j ∘ (λt::real. 1 - t))
```

```
    using j-end by (simp add: reversepathin-def)
  qed
  then show ?thesis
    unfolding homotopic-pathsin-def reversepathin-def o-def .
qed

lemma homotopic-pathsin-reversepathin:
  homotopic-pathsin X (reversepathin p) (reversepathin q) ⟷ homotopic-pathsin
X p q
  using homotopic-pathsin-reversepathin-D by force

lemma homotopic-pathsin-joinpathin:
  assumes pp': homotopic-pathsin X p p'
    and qq': homotopic-pathsin X q q'
    and pq: p 1 = q 0
  shows homotopic-pathsin X (joinpathin p q) (joinpathin p' q')
proof −
  let ?T01 = subtopology euclideanreal {0..1}
  let ?A = prod-topology ?T01 ?T01
  obtain h1 :: real × real ⇒ 'a where
    h1-cont: continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
X h1
    and h1-0: ∀ x. h1 (0, x) = p x
    and h1-1: ∀ x. h1 (1, x) = p' x
    and h1-prop: ∀ t∈{0..1}. h1 (t, 0) = p 0 ∧ h1 (t, 1) = p 1
    using pp' unfolding homotopic-pathsin-def homotopic-with-def by auto
  obtain h2 :: real × real ⇒ 'a where
    h2-cont: continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
X h2
    and h2-0: ∀ x. h2 (0, x) = q x
    and h2-1: ∀ x. h2 (1, x) = q' x
    and h2-prop: ∀ t∈{0..1}. h2 (t, 0) = q 0 ∧ h2 (t, 1) = q 1
    using qq' unfolding homotopic-pathsin-def homotopic-with-def by auto
  define k1 where k1 t x = h1 (t, x) for t x
  define k2 where k2 t x = h2 (t, x) for t x
  have k1-cont: continuous-map ?A X (λy. k1 (fst y) (snd y))
  proof −
    have (λy. k1 (fst y) (snd y)) = h1
      unfolding k1-def by (auto simp: fun-eq-iff)
    then show ?thesis
      using h1-cont by simp
  qed
  have k2-cont: continuous-map ?A X (λy. k2 (fst y) (snd y))
  proof −
    have (λy. k2 (fst y) (snd y)) = h2
      unfolding k2-def by (auto simp: fun-eq-iff)
    then show ?thesis
      using h2-cont by simp
  qed
```

```
  have k1-0: ∀ x. k1 0 x = p x
    unfolding k1-def using h1-0 by simp
  have k1-1: ∀ x. k1 1 x = p' x
    unfolding k1-def using h1-1 by simp
  have k2-0: ∀ x. k2 0 x = q x
    unfolding k2-def using h2-0 by simp
  have k2-1: ∀ x. k2 1 x = q' x
    unfolding k2-def using h2-1 by simp
  have k1-prop: ∀ t∈{0..1}. k1 t 0 = p 0 ∧ k1 t 1 = p 1
    unfolding k1-def using h1-prop by simp
  have k2-prop: ∀ t∈{0..1}. k2 t 0 = q 0 ∧ k2 t 1 = q 1
    unfolding k2-def using h2-prop by simp
  have mid-eq: k1 t 1 = k2 t 0 if t ∈ {0..1} for t
    using k1-prop[rule-format, OF that] k2-prop[rule-format, OF that] pq by auto
  show ?thesis
    unfolding homotopic-pathsin-def homotopic-with-def
  proof (rule exI[where x = λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)], intro
conjI)
    show continuous-map ?A X (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y))
      by (rule continuous-map-homotopic-joinpathin-lemma[OF k1-cont k2-cont])
(rule mid-eq)
     show ∀ x::real. (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (0, x) =
joinpathin p q x
    proof
      fix x :: real
      show (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (0, x) = joinpathin p
q x
        using k1-0 k2-0 by (cases x ≤ 1 / 2) (auto simp: joinpathin-def)
    qed
     show ∀ x::real. (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (1, x) =
joinpathin p' q' x
    proof
      fix x :: real
      show (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (1, x) = joinpathin p'
q' x
        using k1-1 k2-1 by (cases x ≤ 1 / 2) (auto simp: joinpathin-def)
    qed
    show ∀ t∈{0..1}. (λr. r 0 = joinpathin p q 0 ∧ r 1 = joinpathin p q 1)
         (λx. (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (t, x))
    proof
      fix t :: real
      assume t: t ∈ {0..1}
      have k1t0: k1 t 0 = p 0
        using k1-prop[rule-format, OF t] by auto
      have k2t1: k2 t 1 = q 1
        using k2-prop[rule-format, OF t] by auto
      show (λr. r 0 = joinpathin p q 0 ∧ r 1 = joinpathin p q 1)
           (λx. (λy. joinpathin (k1 (fst y)) (k2 (fst y)) (snd y)) (t, x))
        using k1t0 k2t1 by (simp add: joinpathin-def)
```

```
    qed
  qed
qed

lemma homotopic-pathsin-reparametrize:
  assumes p: pathin X p
    and contf: continuous-map (top-of-set {0..1}) (top-of-set {0..1}) f
    and f01: f ∈ {0..1} → {0..1}
    and [simp]: f 0 = 0 f 1 = 1
    and q: ⋀t. t ∈ {0..1} ⟹ q t = p (f t)
  shows homotopic-pathsin X p q
proof −
  have pf: pathin X (p ∘ f)
    using contf p unfolding pathin-def by (rule continuous-map-compose)
  have qf': homotopic-pathsin X (p ∘ f) q
    using pf q by (intro homotopic-pathsin-eq) auto
  have qf: homotopic-pathsin X q (p ∘ f)
    by (rule homotopic-pathsin-sym[OF qf'])
  have fp: homotopic-pathsin X (p ∘ f) p
    unfolding homotopic-pathsin-def homotopic-with-def
  proof (rule exI[where x = p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd
y)], intro conjI)
    have snd-cont:
      continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
        (top-of-set {0..1}) snd
      by (rule continuous-map-snd)
    have fst-cont:
      continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
        (top-of-set {0..1}) fst
      by (rule continuous-map-fst)
    have fsnd-cont:
      continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
        (top-of-set {0..1}) (f ∘ snd)
      by (rule continuous-map-compose[OF snd-cont contf])
    have param-cont-eu:
      continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1})) eu-
clideanreal
        (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)
      using fst-cont snd-cont fsnd-cont
      by (intro continuous-intros
          continuous-map-into-fulltopology[OF fst-cont]
          continuous-map-into-fulltopology[OF snd-cont]
          continuous-map-into-fulltopology[OF fsnd-cont])
    have param-range:
      (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y) ∈
        topspace (prod-topology (top-of-set {0..1}) (top-of-set {0..1})) → {0..1}
    proof
      fix y :: real × real
        assume y-in: y ∈ topspace (prod-topology (top-of-set {0..1}) (top-of-set
```

```
{0..1}))
      have fst01: fst y ∈ {0..1} and snd01: snd y ∈ {0..1}
        using y-in by (auto simp: topspace-prod-topology)
      have fsnd01: f (snd y) ∈ {0..1}
        using f01 snd01 by auto
      have nonneg: 0 ≤ (1 − fst y) ∗ (f (snd y)) + fst y ∗ snd y
       using fst01 snd01 fsnd01 by (intro add-nonneg-nonneg mult-nonneg-nonneg)
auto
      moreover have le1: (1 − fst y) ∗ (f (snd y)) + fst y ∗ snd y ≤ 1
        using fst01 snd01 fsnd01 by (intro convex-bound-le) auto
      ultimately show ((1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y) ∈ {0..1}
        by simp
    qed
    have param-cont:
      continuous-map (prod-topology (top-of-set {0..1}) (top-of-set {0..1}))
        (top-of-set {0..1}) (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)
              using param-cont-eu param-range by (simp add: continu-
ous-map-in-subtopology)
    show continuous-map
           (prod-topology (top-of-set {0..1}) (top-of-set {0..1})) X
           (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y))
         using p unfolding pathin-def by (rule continuous-map-compose[OF
param-cont])
    show ∀x::real. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (0, x) =
(p ∘ f) x
    proof
      fix x :: real
      show (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (0, x) = (p ∘ f)
x
        by (simp add: o-def algebra-simps)
    qed
    show ∀x::real. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (1, x) =
p x
    proof
      fix x :: real
      show (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (1, x) = p x
        by (simp add: o-def algebra-simps)
    qed
    show ∀t∈{0..1}. (λr. r 0 = (p ∘ f) 0 ∧ r 1 = (p ∘ f) 1)
          (λx. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (t, x))
    proof
      fix t :: real
      assume t: t ∈ {0..1}
      show (λr. r 0 = (p ∘ f) 0 ∧ r 1 = (p ∘ f) 1)
          (λx::real. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y)) (t, x))
      proof −
        have end0: (λx::real. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y))
(t, x)) 0 = (p ∘ f) 0
          by (simp add: o-def algebra-simps)
```

```
      have end1: (λx::real. (p ∘ (λy. (1 − fst y) ∗ ((f ∘ snd) y) + fst y ∗ snd y))
(t, x)) 1 = (p ∘ f) 1
        by (simp add: o-def algebra-simps)
       from end0 end1 show ?thesis
        by (simp add: o-def)
     qed
   qed
  qed
  have homotopic-pathsin X q p
   by (rule homotopic-pathsin-trans[OF qf fp])
  then show ?thesis
   by (rule homotopic-pathsin-sym)
qed

lemma homotopic-pathsin-rid-const:
  assumes p: pathin X p
  shows homotopic-pathsin X (joinpathin p (λ-. p 1)) p
proof −
  have contf: continuous-map (top-of-set {0..1}) (top-of-set {0..1})
              (λt::real. if t ≤ 1 / 2 then 2 ∗ t else 1)
  proof (rule continuous-map-into-subtopology)
   show continuous-map (top-of-set {0..1}) euclideanreal
         (λt::real. if t ≤ 1 / 2 then 2 ∗ t else 1)
   proof (rule continuous-map-cases-le[where p = λt::real. t and q = λ-. 1 / 2])
     show continuous-map (top-of-set {0..1}) euclideanreal (λt::real. t)
      by (simp add: continuous-map-iff-continuous)
     show continuous-map (top-of-set {0..1}) euclideanreal (λ-. 1 / 2)
      by simp
    show continuous-map (subtopology (top-of-set {0..1}) {x ∈ topspace (top-of-set
{0..1}). x ≤ 1 / 2})
           euclideanreal (λt::real. 2 ∗ t)
     proof −
      have continuous-map (top-of-set {0..1}) euclideanreal (λt::real. 2 ∗ t)
       by (simp add: continuous-map-iff-continuous continuous-intros)
      then show ?thesis
       by (rule continuous-map-from-subtopology)
     qed
    show continuous-map (subtopology (top-of-set {0..1}) {x ∈ topspace (top-of-set
{0..1}). 1 / 2 ≤ x})
           euclideanreal (λ-::real. 1)
      by simp
     show 2 ∗ x = (1::real) if x ∈ topspace (top-of-set {0..1}) x = 1 / 2 for x
      using that by simp
   qed
   show (λt::real. if t ≤ 1 / 2 then 2 ∗ t else 1) ∈ topspace (top-of-set {0..1})
→ {0..1}
    by auto
  qed
  show ?thesis
```

apply (*rule homotopic-pathsin-sym*)
apply (*rule homotopic-pathsin-reparametrize*[**where** $f = \lambda t{::}real.$ *if* $t \leq 1 / 2$ *then* $2 * t$ *else* 1])
using *p contf*
apply (*auto simp*: *joinpathin-def*)
done
qed

lemma *homotopic-pathsin-lid-const*:
assumes *p*: *pathin X p*
shows *homotopic-pathsin X* (*joinpathin* (λ-. *p 0*) *p*) *p*
proof −
have *contf*: *continuous-map* (*top-of-set* {*0..1*}) (*top-of-set* {*0..1*})
(λ*t*::*real*. *if* $t \leq 1 / 2$ *then 0 else* $2 * t - 1$)
proof (*rule continuous-map-into-subtopology*)
show *continuous-map* (*top-of-set* {*0..1*}) *euclideanreal*
(λ*t*::*real*. *if* $t \leq 1 / 2$ *then 0 else* $2 * t - 1$)
proof (*rule continuous-map-cases-le*[**where** $p = \lambda t{::}real.$ *t* **and** $q = \lambda$-. $1 / 2$])
show *continuous-map* (*top-of-set* {*0..1*}) *euclideanreal* (λ*t*::*real*. *t*)
by (*simp add*: *continuous-map-iff-continuous*)
show *continuous-map* (*top-of-set* {*0..1*}) *euclideanreal* (λ-. $1 / 2$)
by *simp*
show *continuous-map* (*subtopology* (*top-of-set* {*0..1*}) {*x* ∈ *topspace* (*top-of-set* {*0..1*}). $x \leq 1 / 2$})
euclideanreal (λ-::*real*. *0*)
by *simp*
show *continuous-map* (*subtopology* (*top-of-set* {*0..1*}) {*x* ∈ *topspace* (*top-of-set* {*0..1*}). $1 / 2 \leq x$})
euclideanreal (λ*t*::*real*. $2 * t - 1$)
proof −
have *continuous-map* (*top-of-set* {*0..1*}) *euclideanreal* (λ*t*::*real*. $2 * t - 1$)
by (*simp add*: *continuous-map-iff-continuous continuous-intros*)
then show *?thesis*
by (*rule continuous-map-from-subtopology*)
qed
show (*0*::*real*) = $2 * x - 1$ **if** *x* ∈ *topspace* (*top-of-set* {*0..1*}) $x = 1 / 2$ **for** *x*
using *that* **by** *simp*
qed
show (λ*t*::*real*. *if* $t \leq 1 / 2$ *then 0 else* $2 * t - 1$) ∈ *topspace* (*top-of-set* {*0..1*}) → {*0..1*}
by *auto*
qed
show *?thesis*
apply (*rule homotopic-pathsin-sym*)
apply (*rule homotopic-pathsin-reparametrize*[**where** $f = \lambda t{::}real.$ *if* $t \leq 1 / 2$ *then 0 else* $2 * t - 1$])
using *p contf*
apply (*auto simp*: *joinpathin-def*)

```
    done
qed

lemma homotopic-pathsin-assoc:
  assumes p: pathin X p
    and q: pathin X q
    and r: pathin X r
    and pq: p 1 = q 0
    and qr: q 1 = r 0
  shows homotopic-pathsin X (joinpathin p (joinpathin q r)) (joinpathin (joinpathin
p q) r)
proof −
  let ?f = λt::real. if t ≤ 1 / 2 then inverse 2 ∗ t else if t ≤ 3 / 4 then t − 1 / 4
else 2 ∗ t − 1
  have contf-on: continuous-on {0..1} ?f
  proof (rule continuous-on-cases-1)
    show continuous-on {t ∈ {0..1}. t ≤ 1 / 2} (λt::real. inverse 2 ∗ t)
      by (intro continuous-intros)
    show continuous-on {t ∈ {0..1}. 1 / 2 ≤ t}
          (λt::real. if t ≤ 3 / 4 then t − 1 / 4 else 2 ∗ t − 1)
    proof (rule continuous-on-cases-1)
      show continuous-on {t ∈ {t ∈ {0..1}. 1 / 2 ≤ t}. t ≤ 3 / 4} (λt::real. t −
1 / 4)
        by (intro continuous-intros)
      show continuous-on {t ∈ {t ∈ {0..1}. 1 / 2 ≤ t}. 3 / 4 ≤ t} (λt::real. 2 ∗
t − 1)
        by (intro continuous-intros)
      show (3 / 4::real) ∈ {t ∈ {0..1}. (1 / 2::real) ≤ t} ⟹
        (3 / 4::real) − 1 / 4 = 2 ∗ (3 / 4) − 1
        by simp
    qed
    show (1 / 2::real) ∈ {0..1} ⟹
      inverse 2 ∗ (1 / 2::real) =
        (if (1 / 2::real) ≤ 3 / 4 then 1 / 2 − 1 / 4 else 2 ∗ (1 / 2) − 1)
      by simp
  qed
  have contf-eu: continuous-map (top-of-set {0..1}) euclideanreal ?f
    using contf-on by simp
  have contf:
    continuous-map (top-of-set {0..1}) (top-of-set {0..1}) ?f
    using contf-eu by (rule continuous-map-into-subtopology) auto
  have join-pq: pathin X (joinpathin p q)
    by (rule pathin-joinpathin[OF p q pq])
  have join-qr: pathin X (joinpathin q r)
    by (rule pathin-joinpathin[OF q r qr])
  have join-pq-end: joinpathin p q 1 = r 0
    using qr by (simp add: joinpathin-def)
  have join-pq-r: pathin X (joinpathin (joinpathin p q) r)
    by (rule pathin-joinpathin[OF join-pq r join-pq-end])
```

```
  have join-pq-r-expanded:
    pathin X (λt. if t * 2 ≤ 1
        then if 2 * t * 2 ≤ 1 then p (2 * (2 * t)) else q (2 * (2 * t) - 1)
        else r (2 * t - 1))
    using join-pq-r by (simp add: joinpathin-def)
  show ?thesis
    apply (rule homotopic-pathsin-sym)
    apply (rule homotopic-pathsin-reparametrize[
        where f = ?f])
    using contf join-pq-r-expanded join-pq-r join-pq r qr join-qr p q pq
    apply (auto simp: joinpathin-def field-simps algebra-simps)
    apply (rule arg-cong[where f = q])
    apply (simp add: field-simps)
    done
qed

lemma homotopic-pathsin-rinv-const:
  assumes p: pathin X p
  shows homotopic-pathsin X (joinpathin p (reversepathin p)) (λ-. p 0)
proof -
  let ?T01 = subtopology euclideanreal {0..1}
  let ?A = prod-topology ?T01 ?T01
  let ?h =
    λy. if snd y ≤ 1 / 2
      then p (fst y * (2 * snd y))
      else p (fst y * (1 - (2 * snd y - 1)))
  have p-cont: continuous-map ?T01 X p
    using p unfolding pathin-def .
  have h-cont: continuous-map ?A X ?h
  proof (rule continuous-map-cases-le)
    show continuous-map ?A euclideanreal snd
      by (rule continuous-map-into-fulltopology[OF continuous-map-snd])
    show continuous-map ?A euclideanreal (λ-. 1 / 2)
      by simp
    have left-param-cont-eu:
     continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) euclidean-
real
       (λt. fst t * (2 * snd t))
      by (intro continuous-intros
          continuous-map-into-fulltopology[OF continuous-map-subtopology-fst]
          continuous-map-into-fulltopology[OF continuous-map-subtopology-snd])
    have left-param-cont:
      continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) ?T01
       (λt. fst t * (2 * snd t))
    proof -
      have left-param-range:
       (λt. fst t * (2 * snd t)) ∈
         topspace (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) → {0..1}
      proof
```

```
      fix t :: real × real
      assume t-in: t ∈ topspace (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 /
2})
      then have tA: t ∈ topspace ?A and t-half: snd t ≤ 1 / 2
        by auto
      have fst01: fst t ∈ {0..1} and snd01: snd t ∈ {0..1}
        using tA by (auto simp: topspace-prod-topology)
      from snd01 have snd-ge0: 0 ≤ snd t and snd-le1: snd t ≤ 1
        by auto
      have scale01: 2 * (snd t :: real) ∈ {0..1}
      proof −
        have lower: 0 ≤ 2 * (snd t :: real)
          using snd-ge0 by linarith
        have upper: 2 * (snd t :: real) ≤ 1
          using t-half by linarith
        show 2 * (snd t :: real) ∈ {0..1}
          using lower upper by auto
      qed
      show fst t * (2 * (snd t :: real)) ∈ {0..1}
      proof −
        from fst01 have fst-ge0: 0 ≤ fst t and fst-le1: fst t ≤ 1
          by auto
         from scale01 have scale-ge0: 0 ≤ 2 * (snd t :: real) and scale-le1: 2 *
(snd t :: real) ≤ 1
          by auto
        have lower: 0 ≤ fst t * (2 * (snd t :: real))
          by (rule mult-nonneg-nonneg[OF fst-ge0 scale-ge0])
        have upper: fst t * (2 * (snd t :: real)) ≤ 1
          by (rule mult-le-one[OF fst-le1 scale-ge0 scale-le1])
        have bounds:
          0 ≤ fst t * (2 * (snd t :: real)) ∧ fst t * (2 * (snd t :: real)) ≤ 1
          using lower upper by blast
        show fst t * (2 * (snd t :: real)) ∈ {0..1}
        proof −
          from bounds have lower': 0 ≤ fst t * (2 * (snd t :: real))
            and upper': fst t * (2 * (snd t :: real)) ≤ 1
            by auto
          show ?thesis
            using lower' upper' by (auto simp: atLeastAtMost-iff)
        qed
      qed
    qed
    show ?thesis
            by (rule continuous-map-into-subtopology[OF left-param-cont-eu
left-param-range])
  qed
  show continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) X
       (λt. p (fst t * (2 * snd t)))
  proof −
```

```
    have left-comp:
      continuous-map (subtopology ?A {y ∈ topspace ?A. snd y ≤ 1 / 2}) X
        (p ∘ (λt. fst t * (2 * snd t)))
      by (rule continuous-map-compose[OF left-param-cont p-cont])
    have left-eq: (λt. p (fst t * (2 * snd t))) = p ∘ (λt. fst t * (2 * snd t))
      by (auto simp: fun-eq-iff)
    show ?thesis
      using left-comp left-eq by simp
  qed
  have right-param-cont-eu:
   continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) euclidean-
real
    (λt. fst t * (1 − (2 * snd t − 1)))
    by (intro continuous-intros
        continuous-map-into-fulltopology[OF continuous-map-subtopology-fst]
        continuous-map-into-fulltopology[OF continuous-map-subtopology-snd])
  have right-param-cont:
    continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) ?T01
      (λt. fst t * (1 − (2 * snd t − 1)))
  proof −
    have right-param-range:
      (λt. fst t * (1 − (2 * snd t − 1))) ∈
        topspace (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) → {0..1}
    proof
      fix t :: real × real
      assume t-in: t ∈ topspace (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd
y})
      then have tA: t ∈ topspace ?A and t-half: 1 / 2 ≤ snd t
        by auto
      have fst01: fst t ∈ {0..1} and snd01: snd t ∈ {0..1}
        using tA by (auto simp: topspace-prod-topology)
      from snd01 have snd-ge0: 0 ≤ snd t and snd-le1: snd t ≤ 1
        by auto
      have scale01: 1 − (2 * (snd t :: real) − 1) ∈ {0..1}
      proof −
        have lower: 0 ≤ 1 − (2 * (snd t :: real) − 1)
          using snd-le1 by linarith
        have upper: 1 − (2 * (snd t :: real) − 1) ≤ 1
          using t-half by linarith
        show 1 − (2 * (snd t :: real) − 1) ∈ {0..1}
          using lower upper by force
      qed
      show fst t * (1 − (2 * (snd t :: real) − 1)) ∈ {0..1}
      proof −
        from fst01 have fst-ge0: 0 ≤ fst t and fst-le1: fst t ≤ 1
          by auto
        from scale01 have scale-ge0: 0 ≤ 1 − (2 * (snd t :: real) − 1)
          and scale-le1: 1 − (2 * (snd t :: real) − 1) ≤ 1
          by auto
```

```
      have lower: 0 ≤ fst t * (1 - (2 * (snd t :: real) - 1))
        by (rule mult-nonneg-nonneg[OF fst-ge0 scale-ge0])
      have upper: fst t * (1 - (2 * (snd t :: real) - 1)) ≤ 1
        by (rule mult-le-one[OF fst-le1 scale-ge0 scale-le1])
      show fst t * (1 - (2 * (snd t :: real) - 1)) ∈ {0..1}
        using lower upper by force
    qed
  qed
  show ?thesis
      using right-param-cont-eu right-param-range by (simp add: continuous-map-in-subtopology)
 qed
 show continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) X
      (λt. p (fst t * (1 - (2 * snd t - 1))))
 proof -
  have right-comp:
    continuous-map (subtopology ?A {y ∈ topspace ?A. 1 / 2 ≤ snd y}) X
     (p ∘ (λt. fst t * (1 - (2 * snd t - 1))))
    by (rule continuous-map-compose[OF right-param-cont p-cont])
  have right-eq:
    (λt. p (fst t * (1 - (2 * snd t - 1)))) =
      p ∘ (λt. fst t * (1 - (2 * snd t - 1)))
    by (auto simp: fun-eq-iff)
  show ?thesis
    using right-comp right-eq by simp
 qed
 show p (fst y * (2 * snd y)) = p (fst y * (1 - (2 * snd y - 1)))
  if y ∈ topspace ?A snd y = 1 / 2 for y
 proof -
  have mid1: 2 * snd y = (1::real)
    using that(2) by simp
  have mid0: 2 * snd y - 1 = (0::real)
    using that(2) by simp
  show ?thesis
    using mid1 mid0 by simp
 qed
qed
have hom-const-join: homotopic-pathsin X (λ-. p 0) (joinpathin p (reversepathin p))
 unfolding homotopic-pathsin-def homotopic-with-def
proof (rule exI[where x = ?h], intro conjI)
 show continuous-map ?A X ?h
  by (rule h-cont)
 show ∀x::real. ?h (0, x) = (λ-. p 0) x
  by simp
 show ∀x::real. ?h (1, x) = joinpathin p (reversepathin p) x
  by (auto simp: joinpathin-def reversepathin-def)
 show ∀t∈{0..1}. (λr. r 0 = p 0 ∧ r 1 = p 0) (λx. ?h (t, x))
  by simp
```

qed
show *?thesis*
by (*rule homotopic-pathsin-sym*[*OF hom-const-join*])
qed

lemma *homotopic-pathsin-linv-const*:
assumes *p*: *pathin X p*
shows *homotopic-pathsin X* (*joinpath* (*reversepathin p*) *p*) (λ-. *p 1*)
proof −
have *homotopic-pathsin X* (*joinpath* (*reversepathin p*) (*reversepathin* (*reversepathin p*)))
(λ-. *reversepathin p 0*)
using *pathin-reversepathin*[*OF p*] **by** (*rule homotopic-pathsin-rinv-const*)
then show *?thesis*
by (*simp add*: *reversepathin-def*)
qed

lemma *homotopic-loopsin-imp-pathin*:
assumes *homotopic-loopsin X p q*
shows *pathin X p pathin X q*
using *assms* **unfolding** *homotopic-loopsin-def pathin-def*
by (*auto dest*: *homotopic-with-imp-continuous-maps*)

lemma *homotopic-loopsin-imp-loop*:
assumes *homotopic-loopsin X p q*
shows *p 1 = p 0 q 1 = q 0*
using *assms* **unfolding** *homotopic-loopsin-def*
by (*auto dest*: *homotopic-with-imp-property*)

lemma *homotopic-loopsin-refl* [*simp*]:
homotopic-loopsin X p p ⟷ *pathin X p* ∧ *p 1 = p 0*
unfolding *homotopic-loopsin-def pathin-def*
by (*simp add*: *homotopic-with-refl*)

lemma *homotopic-loopsin-sym*:
assumes *homotopic-loopsin X p q*
shows *homotopic-loopsin X q p*
using *assms* **unfolding** *homotopic-loopsin-def*
by (*simp add*: *homotopic-with-sym*)

lemma *homotopic-loopsin-trans*:
assumes *homotopic-loopsin X p q*
and *homotopic-loopsin X q r*
shows *homotopic-loopsin X p r*
using *assms* **unfolding** *homotopic-loopsin-def*
by (*blast intro*: *homotopic-with-trans*)

lemma *loopin-space-reversepathin*:
assumes *p* ∈ *loopin-space X x*

shows *reversepathin p ∈ loopin-space X x*
proof −
from *assms* **obtain** *p-in* **where** *p-in*: *pathin X p p 0 = x p 1 = x*
by (*rule loopin-spaceE*)
have *path-rev*: *pathin X* (*reversepathin p*)
using *p-in*(*1*) **by** (*rule pathin-reversepathin*)
have *start-rev*: *reversepathin p 0 = x*
using *p-in*(*3*) **by** (*simp add*: *reversepathin-def*)
have *finish-rev*: *reversepathin p 1 = x*
using *p-in*(*2*) **by** (*simp add*: *reversepathin-def*)
show *?thesis*
unfolding *loopin-space-def* **using** *path-rev start-rev finish-rev* **by** *blast*
qed

end
theory *Fundamental-Group-Scaffold*
imports *HOL−Analysis.Homotopy Carrier-Group-Scaffold*
begin

11 Fundamental-group quotients

This is the set-based counterpart to the explicit-topology quotient development. It works directly with subsets of a topological space and the standard HOL-Analysis path notions, which is exactly the level needed for the final classical theorem about open sets U and V.

definition *loop-space* ::
'a::*topological-space set* => *'a* => (*real* => *'a*) *set*
where
loop-space S x =
{*p*. *path p* ∧ *path-image p* ⊆ *S* ∧ *pathstart p* = *x* ∧ *pathfinish p* = *x*}

definition *loop-class* ::
'a::*topological-space set* => *'a* => (*real* => *'a*) => (*real* => *'a*) *set*
where
loop-class S x p = {*q*. *q* ∈ *loop-space S x* ∧ *homotopic-paths S q p*}

definition *fundamental-group-space* ::
'a::*topological-space set* => *'a* => ((*real* => *'a*) *set*) *set*
where
fundamental-group-space S x = *loop-class S x* ‘ *loop-space S x*

definition *some-loop* ::
'a::*topological-space set* => *'a* => (*real* => *'a*) *set* => (*real* => *'a*)
where
some-loop S x Q = (*SOME p*. *p* ∈ *loop-space S x* ∧ *Q* = *loop-class S x p*)

definition *fundamental-group-one* ::
'a::*topological-space set* => *'a* => (*real* => *'a*) *set*

where
fundamental-group-one S x = loop-class S x (λ-. x)

definition *fundamental-group-mult* ::
'a::topological-space set => 'a =>
(real => 'a) set => (real => 'a) set => (real => 'a) set
where
fundamental-group-mult S x A B =
loop-class S x (some-loop S x A +++ some-loop S x B)

definition *fundamental-group-inv* ::
'a::topological-space set => 'a =>
(real => 'a) set => (real => 'a) set
where
fundamental-group-inv S x A = loop-class S x (reversepath (some-loop S x A))

definition *loop-image* ::
('a::topological-space => 'b::topological-space) =>
(real => 'a) => (real => 'b)
where
loop-image h p = h ∘ p

definition *fundamental-group-map* ::
'a::topological-space set => 'a =>
'b::topological-space set => 'b =>
('a => 'b) => (real => 'a) set => (real => 'b) set
where
fundamental-group-map S x T y h A =
loop-class T y (loop-image h (some-loop S x A))

lemma *loop-spaceI*:
assumes *path p*
and *path-image p ⊆ S*
and *pathstart p = x*
and *pathfinish p = x*
shows *p ∈ loop-space S x*
using *assms* **unfolding** *loop-space-def* **by** *blast*

lemma *constant-loop-in-space*:
assumes *x ∈ S*
shows *(λ-. x) ∈ loop-space S x*
proof *(rule loop-spaceI)*
show *path (λ-. x)*
by *(simp add: path-def)*
show *path-image (λ-. x) ⊆ S*
using *assms* **by** *(auto simp: path-image-def)*
show *pathstart (λ-. x) = x*
by *(simp add: pathstart-def)*
show *pathfinish (λ-. x) = x*

by (*simp add*: *pathfinish-def*)
qed

lemma *loop-class-in-space*:
assumes $p \in$ *loop-space* $S\ x$
shows *loop-class* $S\ x\ p \in$ *fundamental-group-space* $S\ x$
using *assms* **unfolding** *fundamental-group-space-def* **by** *blast*

lemma *fundamental-group-spaceE*:
assumes $Q \in$ *fundamental-group-space* $S\ x$
obtains p **where** $p \in$ *loop-space* $S\ x\ Q =$ *loop-class* $S\ x\ p$
using *assms* **unfolding** *fundamental-group-space-def* **by** *blast*

lemma *some-loop-spec*:
assumes $Q \in$ *fundamental-group-space* $S\ x$
shows *some-loop* $S\ x\ Q \in$ *loop-space* $S\ x$
and $Q =$ *loop-class* $S\ x$ (*some-loop* $S\ x\ Q$)
proof −
from *assms* **obtain** p **where** p: $p \in$ *loop-space* $S\ x\ Q =$ *loop-class* $S\ x\ p$
by (*rule fundamental-group-spaceE*)
have *ex*: $\exists p.\ p \in$ *loop-space* $S\ x \wedge Q =$ *loop-class* $S\ x\ p$
using p **by** *blast*
have *some-loop* $S\ x\ Q \in$ *loop-space* $S\ x \wedge Q =$ *loop-class* $S\ x$ (*some-loop* $S\ x\ Q$)
unfolding *some-loop-def* **by** (*rule someI-ex*[*OF ex*])
then show *some-loop* $S\ x\ Q \in$ *loop-space* $S\ x$
and $Q =$ *loop-class* $S\ x$ (*some-loop* $S\ x\ Q$)
by *auto*
qed

lemma *loop-class-eqI*:
assumes p: $p \in$ *loop-space* $S\ x$
and q: $q \in$ *loop-space* $S\ x$
and *pq*: *homotopic-paths* $S\ p\ q$
shows *loop-class* $S\ x\ p =$ *loop-class* $S\ x\ q$
proof (*auto simp*: *loop-class-def*)
fix r
assume *homotopic-paths* $S\ r\ p$
then show *homotopic-paths* $S\ r\ q$
using *pq* **by** (*rule homotopic-paths-trans*)
next
fix r
assume *homotopic-paths* $S\ r\ q$
moreover have *homotopic-paths* $S\ q\ p$
using *pq* **by** (*rule homotopic-paths-sym*)
ultimately show *homotopic-paths* $S\ r\ p$
by (*rule homotopic-paths-trans*)
qed

lemma *loop-class-eq-iff*:

```
  assumes p: p ∈ loop-space S x
    and q: q ∈ loop-space S x
  shows loop-class S x p = loop-class S x q ⟷ homotopic-paths S p q
proof
  assume h: loop-class S x p = loop-class S x q
  have p ∈ loop-class S x p
  proof −
    from p have p-props: path p path-image p ⊆ S
      unfolding loop-space-def by auto
    have homotopic-paths S p p
      using p-props by simp
    with p show ?thesis
      unfolding loop-class-def by auto
  qed
  with h show homotopic-paths S p q
    unfolding loop-class-def by blast
next
  assume homotopic-paths S p q
  then show loop-class S x p = loop-class S x q
    by (rule loop-class-eqI[OF p q])
qed

lemma loop-space-join:
  assumes p: p ∈ loop-space S x
    and q: q ∈ loop-space S x
  shows p +++ q ∈ loop-space S x
proof −
  from p have p-props:
    path p path-image p ⊆ S pathstart p = x pathfinish p = x
    unfolding loop-space-def by auto
  from q have q-props:
    path q path-image q ⊆ S pathstart q = x pathfinish q = x
    unfolding loop-space-def by auto
  show ?thesis
  proof (rule loop-spaceI)
    show path (p +++ q)
      using p-props q-props by simp
    show path-image (p +++ q) ⊆ S
      using p-props(2) q-props(2) by (rule subset-path-image-join)
    show pathstart (p +++ q) = x
      using p-props by simp
    show pathfinish (p +++ q) = x
      using q-props by simp
  qed
qed

lemma loop-space-reversepath:
  assumes p ∈ loop-space S x
  shows reversepath p ∈ loop-space S x
```

proof −
from *assms* **have** *p-props*:
path p path-image p ⊆ *S pathstart p* = *x pathfinish p* = *x*
unfolding *loop-space-def* **by** *auto*
show *?thesis*
proof (*rule loop-spaceI*)
show *path* (*reversepath p*)
using *p-props* **by** *simp*
show *path-image* (*reversepath p*) ⊆ *S*
using *p-props* **by** *simp*
show *pathstart* (*reversepath p*) = *x*
using *p-props* **by** *simp*
show *pathfinish* (*reversepath p*) = *x*
using *p-props* **by** *simp*
qed
qed

lemma *loop-space-continuous-image*:
assumes *p*: *p* ∈ *loop-space S x*
and *cont-h*: *continuous-on S h*
and *map-h*: *h* ∈ *S* → *T*
and *hx*: *h x* = *y*
shows *loop-image h p* ∈ *loop-space T y*
proof −
from *p* **have** *p-props*:
path p path-image p ⊆ *S pathstart p* = *x pathfinish p* = *x*
unfolding *loop-space-def* **by** *auto*
show *?thesis*
unfolding *loop-image-def*
proof (*rule loop-spaceI*)
show *path* (*h* ∘ *p*)
using *p-props*(*1*) *continuous-on-subset*[*OF cont-h p-props*(*2*)]
by (*rule Path-Connected.path-continuous-image*)
show *path-image* (*h* ∘ *p*) ⊆ *T*
using *p-props*(*2*) *map-h*
by (*auto simp*: *path-image-def*)
show *pathstart* (*h* ∘ *p*) = *y*
using *p-props*(*3*) *hx* **by** (*simp add*: *pathstart-compose*)
show *pathfinish* (*h* ∘ *p*) = *y*
using *p-props*(*4*) *hx* **by** (*simp add*: *pathfinish-compose*)
qed
qed

lemma *loop-image-join*:
loop-image h (*p* +++ *q*) = *loop-image h p* +++ *loop-image h q*
unfolding *loop-image-def*
by (*rule ext*) (*simp add*: *joinpaths-def*)

lemma *loop-image-reversepath*:

loop-image h (reversepath p) = reversepath (loop-image h p)
unfolding *loop-image-def reversepath-def*
by (*rule ext*) *simp*

lemma *loop-class-join-eqI*:
assumes *p*: $p \in$ *loop-space S x*
and *p'*: $p' \in$ *loop-space S x*
and *q*: $q \in$ *loop-space S x*
and *q'*: $q' \in$ *loop-space S x*
and *pp'*: *homotopic-paths S p p'*
and *qq'*: *homotopic-paths S q q'*
shows *loop-class S x (p +++ q) = loop-class S x (p' +++ q')*
proof (*rule loop-class-eqI*)
show $p +++ q \in$ *loop-space S x*
using *p q* **by** (*rule loop-space-join*)
show $p' +++ q' \in$ *loop-space S x*
using *p' q'* **by** (*rule loop-space-join*)
from *p* **have** *px*: *pathfinish p = x*
unfolding *loop-space-def* **by** *auto*
from *q* **have** *qx*: *pathstart q = x*
unfolding *loop-space-def* **by** *auto*
show *homotopic-paths S (p +++ q) (p' +++ q')*
proof (*rule homotopic-paths-join*)
show *homotopic-paths S p p'*
by (*rule pp'*)
show *homotopic-paths S q q'*
by (*rule qq'*)
show *pathfinish p = pathstart q*
using *px qx* **by** *simp*
qed
qed

lemma *loop-class-reverse-eqI*:
assumes *p*: $p \in$ *loop-space S x*
and *q*: $q \in$ *loop-space S x*
and *pq*: *homotopic-paths S p q*
shows *loop-class S x (reversepath p) = loop-class S x (reversepath q)*
proof (*rule loop-class-eqI*)
show *reversepath p* $\in$ *loop-space S x*
using *p* **by** (*rule loop-space-reversepath*)
show *reversepath q* $\in$ *loop-space S x*
using *q* **by** (*rule loop-space-reversepath*)
from *pq* **show** *homotopic-paths S (reversepath p) (reversepath q)*
by (*rule homotopic-paths-reversepath-D*)
qed

lemma *homotopic-paths-rid-const*:
assumes *path p*
and *path-image p* $\subseteq$ *S*

```
  shows homotopic-paths S (p +++ (λ-. pathfinish p)) p
proof −
  let ?f = λt::real. if t ≤ 1/2 then 2 *R t else 1
  have contf: continuous-on {0..1} ?f
    unfolding split-01
    by (rule continuous-on-cases continuous-intros | auto)+
  have f01: ?f ∈ {0..1} → {0..1}
    by auto
  have homotopic-paths S p (p +++ (λ-. pathfinish p))
  proof (rule homotopic-paths-reparametrize[where f = ?f])
    show path p
      by (rule assms(1))
    show path-image p ⊆ S
      by (rule assms(2))
    show continuous-on {0..1} ?f
      by (rule contf)
    show ?f ∈ {0..1} → {0..1}
      by (rule f01)
    show ?f 0 = 0 ?f 1 = 1
      by simp-all
    show (p +++ (λ-. pathfinish p)) t = p (?f t) if t ∈ {0..1} for t
      using that by (auto simp: joinpaths-def pathfinish-def)
  qed
  then show ?thesis
    by (rule homotopic-paths-sym)
qed

lemma homotopic-paths-lid-const:
  assumes path p
    and path-image p ⊆ S
  shows homotopic-paths S ((λ-. pathstart p) +++ p) p
proof −
  let ?f = λt::real. if t ≤ 1/2 then 0 else 2 *R t − 1
  have contf: continuous-on {0..1} ?f
    unfolding split-01
    by (rule continuous-on-cases continuous-intros | auto)+
  have f01: ?f ∈ {0..1} → {0..1}
    by auto
  have homotopic-paths S p ((λ-. pathstart p) +++ p)
  proof (rule homotopic-paths-reparametrize[where f = ?f])
    show path p
      by (rule assms(1))
    show path-image p ⊆ S
      by (rule assms(2))
    show continuous-on {0..1} ?f
      by (rule contf)
    show ?f ∈ {0..1} → {0..1}
      by (rule f01)
    show ?f 0 = 0 ?f 1 = 1
```

```
    by simp-all
  show ((λ-. pathstart p) +++ p) t = p (?f t) if t ∈ {0..1} for t
    using that by (auto simp: joinpaths-def pathstart-def)
 qed
 then show ?thesis
  by (rule homotopic-paths-sym)
qed

lemma homotopic-paths-rinv-const:
 assumes path p
  and path-image p ⊆ S
 shows homotopic-paths S (p +++ reversepath p) (λ-. pathstart p)
proof −
 let ?A = {0..1} × {0..1}
 let ?h =
  λy. if snd y ≤ 1/2
   then p (fst y ∗ (2 ∗ snd y))
   else p (fst y ∗ (1 − (2 ∗ snd y − 1)))
 define H where H = ?h
 have p-cont: continuous-on {0..1} p
  using assms by (auto simp: path-def)
 have h-cont: continuous-on ?A H
  unfolding H-def
 proof (rule continuous-on-cases-le)
  show continuous-on {x ∈ ?A. snd x ≤ 1/2} (λt. p (fst t ∗ (2 ∗ snd t)))
       continuous-on {x ∈ ?A. 1/2 ≤ snd x} (λt. p (fst t ∗ (1 − (2 ∗ snd t −
1))))
   continuous-on ?A snd
   by (intro continuous-on-compose2 [OF p-cont] continuous-intros; auto simp:
mult-le-one)+
 qed (auto simp: algebra-simps)
 have h-subset: H ∈ ?A → S
 proof
  fix y :: real × real
  assume y: y ∈ ?A
  with assms show H y ∈ S
   unfolding H-def
   by (force simp: path-image-def mult-le-one)
 qed
 have h0: ∀x∈{0..1}. H (0, x) = (λ-. pathstart p) x
  unfolding H-def by (simp add: pathstart-def)
 have h1: ∀x∈{0..1}. H (1, x) = (p +++ reversepath p) x
  unfolding H-def by (auto simp: joinpaths-def reversepath-def)
 have hends:
  ∀t∈{0..1}. pathstart (H ∘ Pair t) = pathstart (λ-. pathstart p) ∧
   pathfinish (H ∘ Pair t) = pathfinish (λ-. pathstart p)
 proof
  fix t :: real
  assume t ∈ {0..1}
```

```
    show pathstart (H ∘ Pair t) = pathstart (λ-. pathstart p) ∧
      pathfinish (H ∘ Pair t) = pathfinish (λ-. pathstart p)
    proof
      show pathstart (H ∘ Pair t) = pathstart (λ-. pathstart p)
        unfolding H-def by (simp add: pathstart-def)
      show pathfinish (H ∘ Pair t) = pathfinish (λ-. pathstart p)
        unfolding H-def by (simp add: pathstart-def pathfinish-def)
    qed
  qed
  have hom-const:
    homotopic-paths S (λ-. pathstart p) (p +++ reversepath p)
    apply (rule iffD2[OF homotopic-paths])
    apply (rule-tac
        x = (λy. if snd y ≤ 1/2
          then p (fst y * (2 * snd y))
          else p (fst y * (1 - (2 * snd y - 1))))
        in exI)
    using h-cont h-subset h0 h1 hends
    unfolding H-def
    apply blast
    done
  show ?thesis
    using hom-const by (rule homotopic-paths-sym)
qed

lemma homotopic-paths-linv-const:
  assumes path p
    and path-image p ⊆ S
  shows homotopic-paths S (reversepath p +++ p) (λ-. pathfinish p)
  using homotopic-paths-rinv-const [of reversepath p S] assms
  by simp

lemma fundamental-group-one-in-space:
  assumes x ∈ S
  shows fundamental-group-one S x ∈ fundamental-group-space S x
  unfolding fundamental-group-one-def
  using constant-loop-in-space[OF assms] by (rule loop-class-in-space)

lemma fundamental-group-mult-eqI:
  assumes A-in: A ∈ fundamental-group-space S x
    and B-in: B ∈ fundamental-group-space S x
    and p: p ∈ loop-space S x
    and q: q ∈ loop-space S x
    and A: A = loop-class S x p
    and B: B = loop-class S x q
  shows fundamental-group-mult S x A B = loop-class S x (p +++ q)
proof −
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
```

```
    using some-loop-spec[OF A-in] by auto
  have repB: some-loop S x B ∈ loop-space S x B = loop-class S x (some-loop S x
B)
    using some-loop-spec[OF B-in] by auto
  have homoA': homotopic-paths S p (some-loop S x A)
    using repA p A by (simp add: loop-class-eq-iff)
  have homoA: homotopic-paths S (some-loop S x A) p
    using homoA' by (rule homotopic-paths-sym)
  have homoB': homotopic-paths S q (some-loop S x B)
    using repB q B by (simp add: loop-class-eq-iff)
  have homoB: homotopic-paths S (some-loop S x B) q
    using homoB' by (rule homotopic-paths-sym)
  have loop-class S x (some-loop S x A +++ some-loop S x B) = loop-class S x (p
+++ q)
    by (rule loop-class-join-eqI[OF repA(1) p repB(1) q homoA homoB])
  then show ?thesis
    unfolding fundamental-group-mult-def .
qed

lemma fundamental-group-mult-in-space:
  assumes A ∈ fundamental-group-space S x
    and B ∈ fundamental-group-space S x
  shows fundamental-group-mult S x A B ∈ fundamental-group-space S x
proof −
  have repA: some-loop S x A ∈ loop-space S x
    using some-loop-spec[OF assms(1)] by auto
  have repB: some-loop S x B ∈ loop-space S x
    using some-loop-spec[OF assms(2)] by auto
  show ?thesis
    unfolding fundamental-group-mult-def
    using loop-space-join[OF repA repB] by (rule loop-class-in-space)
qed

lemma fundamental-group-inv-eqI:
  assumes A-in: A ∈ fundamental-group-space S x
    and p: p ∈ loop-space S x
    and A: A = loop-class S x p
  shows fundamental-group-inv S x A = loop-class S x (reversepath p)
proof −
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
    using some-loop-spec[OF A-in] by auto
  have homoA': homotopic-paths S p (some-loop S x A)
    using repA p A by (simp add: loop-class-eq-iff)
  have homoA: homotopic-paths S (some-loop S x A) p
    using homoA' by (rule homotopic-paths-sym)
  have loop-class S x (reversepath (some-loop S x A)) = loop-class S x (reversepath
p)
    by (rule loop-class-reverse-eqI[OF repA(1) p homoA])
```

```
  then show ?thesis
    unfolding fundamental-group-inv-def .
qed

lemma fundamental-group-inv-in-space:
  assumes A ∈ fundamental-group-space S x
  shows fundamental-group-inv S x A ∈ fundamental-group-space S x
proof −
  have repA: some-loop S x A ∈ loop-space S x
    using some-loop-spec[OF assms] by auto
  show ?thesis
    unfolding fundamental-group-inv-def
    using loop-space-reversepath[OF repA] by (rule loop-class-in-space)
qed

lemma fundamental-group-map-eqI:
  assumes A-in: A ∈ fundamental-group-space S x
    and p: p ∈ loop-space S x
    and A: A = loop-class S x p
    and cont-h: continuous-on S h
    and map-h: h ∈ S → T
    and hx: h x = y
  shows fundamental-group-map S x T y h A = loop-class T y (loop-image h p)
proof −
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
    using some-loop-spec[OF A-in] by auto
  have homo′:
    homotopic-paths S p (some-loop S x A)
    using repA(1) p repA(2) A
    by (simp add: loop-class-eq-iff)
  have homo:
    homotopic-paths S (some-loop S x A) p
    using homo′ by (rule homotopic-paths-sym)
  have map-rep: loop-image h (some-loop S x A) ∈ loop-space T y
    by (rule loop-space-continuous-image[OF repA(1) cont-h map-h hx])
  have map-p: loop-image h p ∈ loop-space T y
    by (rule loop-space-continuous-image[OF p cont-h map-h hx])
  have map-homo:
    homotopic-paths T (loop-image h (some-loop S x A)) (loop-image h p)
    unfolding loop-image-def
    by (rule homotopic-paths-continuous-image[OF homo cont-h map-h])
  have
    loop-class T y (loop-image h (some-loop S x A)) =
      loop-class T y (loop-image h p)
    by (rule loop-class-eqI[OF map-rep map-p map-homo])
  then show ?thesis
    unfolding fundamental-group-map-def .
qed
```

lemma *fundamental-group-map-in-space*:
assumes *A-in*: $A \in$ *fundamental-group-space* $S\ x$
and *cont-h*: *continuous-on* $S\ h$
and *map-h*: $h \in S \to T$
and *hx*: $h\ x = y$
shows *fundamental-group-map* $S\ x\ T\ y\ h\ A \in$ *fundamental-group-space* $T\ y$
proof −
have *repA*: *some-loop* $S\ x\ A \in$ *loop-space* $S\ x$
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *loop-image* h (*some-loop* $S\ x\ A$) $\in$ *loop-space* $T\ y$
by (*rule loop-space-continuous-image*[*OF repA cont-h map-h hx*])
then show *?thesis*
unfolding *fundamental-group-map-def* **by** (*rule loop-class-in-space*)
qed

lemma *fundamental-group-map-mult*:
assumes *A-in*: $A \in$ *fundamental-group-space* $S\ x$
and *B-in*: $B \in$ *fundamental-group-space* $S\ x$
and *cont-h*: *continuous-on* $S\ h$
and *map-h*: $h \in S \to T$
and *hx*: $h\ x = y$
shows
fundamental-group-map $S\ x\ T\ y\ h$ (*fundamental-group-mult* $S\ x\ A\ B$) =
fundamental-group-mult $T\ y$
(*fundamental-group-map* $S\ x\ T\ y\ h\ A$)
(*fundamental-group-map* $S\ x\ T\ y\ h\ B$)
proof −
have *repA*: *some-loop* $S\ x\ A \in$ *loop-space* $S\ x\ A =$ *loop-class* $S\ x$ (*some-loop* $S\ x$ A)
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *repB*: *some-loop* $S\ x\ B \in$ *loop-space* $S\ x\ B =$ *loop-class* $S\ x$ (*some-loop* $S\ x$ B)
using *some-loop-spec*[*OF B-in*] **by** *auto*
have *AB-in*: *fundamental-group-mult* $S\ x\ A\ B \in$ *fundamental-group-space* $S\ x$
by (*rule fundamental-group-mult-in-space*[*OF A-in B-in*])
have *AB-eq*:
fundamental-group-mult $S\ x\ A\ B =$
loop-class $S\ x$ (*some-loop* $S\ x\ A$ +++ *some-loop* $S\ x\ B$)
by (*rule fundamental-group-mult-eqI*[*OF A-in B-in repA*(*1*) *repB*(*1*) *repA*(*2*) *repB*(*2*)])
have *map-A-in*:
fundamental-group-map $S\ x\ T\ y\ h\ A \in$ *fundamental-group-space* $T\ y$
by (*rule fundamental-group-map-in-space*[*OF A-in cont-h map-h hx*])
have *map-B-in*:
fundamental-group-map $S\ x\ T\ y\ h\ B \in$ *fundamental-group-space* $T\ y$
by (*rule fundamental-group-map-in-space*[*OF B-in cont-h map-h hx*])
have *map-A-eq*:
fundamental-group-map $S\ x\ T\ y\ h\ A =$

```
    loop-class T y (loop-image h (some-loop S x A))
   by (rule fundamental-group-map-eqI[OF A-in repA(1) repA(2) cont-h map-h
hx])
  have map-B-eq:
   fundamental-group-map S x T y h B =
    loop-class T y (loop-image h (some-loop S x B))
   by (rule fundamental-group-map-eqI[OF B-in repB(1) repB(2) cont-h map-h
hx])
  have left-eq:
   fundamental-group-map S x T y h (fundamental-group-mult S x A B) =
    loop-class T y (loop-image h (some-loop S x A +++ some-loop S x B))
    by (rule fundamental-group-map-eqI[OF AB-in loop-space-join[OF repA(1)
repB(1)] AB-eq cont-h map-h hx])
  have map-loop-A: loop-image h (some-loop S x A) ∈ loop-space T y
   by (rule loop-space-continuous-image[OF repA(1) cont-h map-h hx])
  have map-loop-B: loop-image h (some-loop S x B) ∈ loop-space T y
   by (rule loop-space-continuous-image[OF repB(1) cont-h map-h hx])
  have right-eq:
   fundamental-group-mult T y
      (fundamental-group-map S x T y h A)
      (fundamental-group-map S x T y h B) =
    loop-class T y
      (loop-image h (some-loop S x A) +++ loop-image h (some-loop S x B))
     by (rule fundamental-group-mult-eqI[OF map-A-in map-B-in map-loop-A
map-loop-B map-A-eq map-B-eq])
  have
   loop-class T y (loop-image h (some-loop S x A +++ some-loop S x B)) =
    loop-class T y (loop-image h (some-loop S x A) +++ loop-image h (some-loop
S x B))
   by (rule arg-cong[where f = loop-class T y], simp add: loop-image-join)
  then show ?thesis
   using left-eq right-eq by simp
qed

lemma fundamental-group-map-id:
  assumes A-in: A ∈ fundamental-group-space S x
  shows fundamental-group-map S x S x id A = A
proof −
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
   using some-loop-spec[OF A-in] by auto
  have fundamental-group-map S x S x id A =
    loop-class S x (loop-image id (some-loop S x A))
   by (rule fundamental-group-map-eqI[OF A-in repA(1) repA(2)]) simp-all
  then show ?thesis
   using repA(2) by (simp add: loop-image-def)
qed

lemma fundamental-group-map-one:
```

assumes *x-in*: $x \in S$
and *cont-h*: *continuous-on S h*
and *map-h*: $h \in S \to T$
and *hx*: $h\ x = y$
shows
fundamental-group-map S x T y h (*fundamental-group-one S x*) =
fundamental-group-one T y
proof −
have *one-in*: *fundamental-group-one S x* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-one-in-space*[*OF x-in*])
have *const-in*: (λ-. *x*) ∈ *loop-space S x*
by (*rule constant-loop-in-space*[*OF x-in*])
have *fundamental-group-map S x T y h* (*fundamental-group-one S x*) =
loop-class T y (*loop-image h* (λ-. *x*))
by (*rule fundamental-group-map-eqI*[*OF one-in const-in*]) (*simp-all add*: *fundamental-group-one-def cont-h map-h hx*)
then show *?thesis*
by (*simp add*: *fundamental-group-one-def loop-image-def hx*)
qed

lemma *fundamental-group-map-compose*:
assumes *A-in*: *A* ∈ *fundamental-group-space S x*
and *cont-h*: *continuous-on S h*
and *map-h*: $h \in S \to T$
and *hx*: $h\ x = y$
and *cont-k*: *continuous-on T k*
and *map-k*: $k \in T \to U$
and *ky*: $k\ y = z$
shows
fundamental-group-map T y U z k (*fundamental-group-map S x T y h A*) =
fundamental-group-map S x U z (*k* ∘ *h*) *A*
proof −
have *repA*: *some-loop S x A* ∈ *loop-space S x A* = *loop-class S x* (*some-loop S x A*)
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *map-A-in*:
fundamental-group-map S x T y h A ∈ *fundamental-group-space T y*
by (*rule fundamental-group-map-in-space*[*OF A-in cont-h map-h hx*])
have *map-A-eq*:
fundamental-group-map S x T y h A =
loop-class T y (*loop-image h* (*some-loop S x A*))
by (*rule fundamental-group-map-eqI*[*OF A-in repA*(*1*) *repA*(*2*) *cont-h map-h hx*])
have *left-eq*:
fundamental-group-map T y U z k (*fundamental-group-map S x T y h A*) =
loop-class U z (*loop-image k* (*loop-image h* (*some-loop S x A*)))
by (*rule fundamental-group-map-eqI*[*OF map-A-in loop-space-continuous-image*[*OF repA*(*1*) *cont-h map-h hx*] *map-A-eq cont-k map-k ky*])

```
  have comp-cont: continuous-on S (k ∘ h)
  proof −
    have hs: h ‘ S ⊆ T
      using map-h by auto
    have hk: continuous-on (h ‘ S) k
      using cont-k hs by (rule continuous-on-subset)
    from cont-h hk show ?thesis
      by (rule continuous-on-compose)
  qed
  have comp-map: (k ∘ h) ∈ S → U
    using map-h map-k by auto
  have right-eq:
    fundamental-group-map S x U z (k ∘ h) A =
      loop-class U z (loop-image (k ∘ h) (some-loop S x A))
      by (rule fundamental-group-map-eqI[OF A-in repA(1) repA(2) comp-cont
comp-map])
       (simp add: hx ky)
  have
    loop-class U z (loop-image k (loop-image h (some-loop S x A))) =
      loop-class U z (loop-image (k ∘ h) (some-loop S x A))
   by (rule arg-cong[where f = loop-class U z]) (simp add: loop-image-def o-assoc)
  then show ?thesis
    using left-eq right-eq by simp
qed

lemma fundamental-group-mult-one-left:
  assumes x ∈ S
    and A ∈ fundamental-group-space S x
  shows fundamental-group-mult S x (fundamental-group-one S x) A = A
proof −
  have const: (λ-. x) ∈ loop-space S x
    using constant-loop-in-space[OF assms(1)] .
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
    using some-loop-spec[OF assms(2)] by auto
  have loopA:
    path (some-loop S x A)
    path-image (some-loop S x A) ⊆ S
    pathstart (some-loop S x A) = x
    pathfinish (some-loop S x A) = x
    using repA(1) unfolding loop-space-def by auto
  have mult-eq:
    fundamental-group-mult S x (fundamental-group-one S x) A =
      loop-class S x ((λ-. x) +++ some-loop S x A)
  proof (rule fundamental-group-mult-eqI)
    show fundamental-group-one S x ∈ fundamental-group-space S x
      by (rule fundamental-group-one-in-space[OF assms(1)])
    show A ∈ fundamental-group-space S x
      by (rule assms(2))
```

```
    show (λ-. x) ∈ loop-space S x
      by (rule const)
    show some-loop S x A ∈ loop-space S x
      by (rule repA(1))
    show fundamental-group-one S x = loop-class S x (λ-. x)
      by (simp add: fundamental-group-one-def)
    show A = loop-class S x (some-loop S x A)
      by (rule repA(2))
  qed
  have join-loop: (λ-. x) +++ some-loop S x A ∈ loop-space S x
    using const repA(1) by (rule loop-space-join)
  have hom: homotopic-paths S ((λ-. x) +++ some-loop S x A) (some-loop S x A)
    using homotopic-paths-lid-const[OF loopA(1) loopA(2)] loopA(3) by simp
  have class-eq:
    loop-class S x ((λ-. x) +++ some-loop S x A) = loop-class S x (some-loop S x
A)
    by (rule loop-class-eqI[OF join-loop repA(1) hom])
  show ?thesis
    using mult-eq class-eq repA(2) by simp
qed

lemma fundamental-group-mult-one-right:
  assumes x ∈ S
    and A ∈ fundamental-group-space S x
  shows fundamental-group-mult S x A (fundamental-group-one S x) = A
proof −
  have const: (λ-. x) ∈ loop-space S x
    using constant-loop-in-space[OF assms(1)] .
  have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
    using some-loop-spec[OF assms(2)] by auto
  have loopA:
    path (some-loop S x A)
    path-image (some-loop S x A) ⊆ S
    pathstart (some-loop S x A) = x
    pathfinish (some-loop S x A) = x
    using repA(1) unfolding loop-space-def by auto
  have mult-eq:
    fundamental-group-mult S x A (fundamental-group-one S x) =
      loop-class S x (some-loop S x A +++ (λ-. x))
  proof (rule fundamental-group-mult-eqI)
    show A ∈ fundamental-group-space S x
      by (rule assms(2))
    show fundamental-group-one S x ∈ fundamental-group-space S x
      by (rule fundamental-group-one-in-space[OF assms(1)])
    show some-loop S x A ∈ loop-space S x
      by (rule repA(1))
    show (λ-. x) ∈ loop-space S x
      by (rule const)
```

show $A = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ A)$
by (*rule repA*(*2*))
show $\textit{fundamental-group-one}\ S\ x = \textit{loop-class}\ S\ x\ (\lambda\text{-}.\ x)$
by (*simp add*: *fundamental-group-one-def*)
qed
have *join-loop*: $\textit{some-loop}\ S\ x\ A +\!\!+\!\!+ (\lambda\text{-}.\ x) \in \textit{loop-space}\ S\ x$
using *repA*(*1*) *const* **by** (*rule loop-space-join*)
have *hom*: $\textit{homotopic-paths}\ S\ (\textit{some-loop}\ S\ x\ A +\!\!+\!\!+ (\lambda\text{-}.\ x))\ (\textit{some-loop}\ S\ x\ A)$
using *homotopic-paths-rid-const*[*OF loopA*(*1*) *loopA*(*2*)] *loopA*(*4*) **by** *simp*
have *class-eq*:
$\textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ A +\!\!+\!\!+ (\lambda\text{-}.\ x)) = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ A)$
by (*rule loop-class-eqI*[*OF join-loop repA*(*1*) *hom*])
show *?thesis*
using *mult-eq class-eq repA*(*2*) **by** *simp*
qed

lemma *fundamental-group-mult-assoc*:
assumes *A-in*: $A \in \textit{fundamental-group-space}\ S\ x$
and *B-in*: $B \in \textit{fundamental-group-space}\ S\ x$
and *C-in*: $C \in \textit{fundamental-group-space}\ S\ x$
shows $\textit{fundamental-group-mult}\ S\ x\ A\ (\textit{fundamental-group-mult}\ S\ x\ B\ C) =$
$\textit{fundamental-group-mult}\ S\ x\ (\textit{fundamental-group-mult}\ S\ x\ A\ B)\ C$
proof −
have *repA*: $\textit{some-loop}\ S\ x\ A \in \textit{loop-space}\ S\ x\ A = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ A)$
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *repB*: $\textit{some-loop}\ S\ x\ B \in \textit{loop-space}\ S\ x\ B = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ B)$
using *some-loop-spec*[*OF B-in*] **by** *auto*
have *repC*: $\textit{some-loop}\ S\ x\ C \in \textit{loop-space}\ S\ x\ C = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ C)$
using *some-loop-spec*[*OF C-in*] **by** *auto*
have *BC-eq*:
$\textit{fundamental-group-mult}\ S\ x\ B\ C =$
$\textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ B +\!\!+\!\!+ \textit{some-loop}\ S\ x\ C)$
proof (*rule fundamental-group-mult-eqI*)
show $B \in \textit{fundamental-group-space}\ S\ x$
by (*rule B-in*)
show $C \in \textit{fundamental-group-space}\ S\ x$
by (*rule C-in*)
show $\textit{some-loop}\ S\ x\ B \in \textit{loop-space}\ S\ x$
by (*rule repB*(*1*))
show $\textit{some-loop}\ S\ x\ C \in \textit{loop-space}\ S\ x$
by (*rule repC*(*1*))
show $B = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ B)$
by (*rule repB*(*2*))
show $C = \textit{loop-class}\ S\ x\ (\textit{some-loop}\ S\ x\ C)$
by (*rule repC*(*2*))

qed
have *AB-eq*:
fundamental-group-mult S x A B =
loop-class S x (*some-loop S x A* +++ *some-loop S x B*)
proof (*rule fundamental-group-mult-eqI*)
show *A* ∈ *fundamental-group-space S x*
by (*rule A-in*)
show *B* ∈ *fundamental-group-space S x*
by (*rule B-in*)
show *some-loop S x A* ∈ *loop-space S x*
by (*rule repA*(*1*))
show *some-loop S x B* ∈ *loop-space S x*
by (*rule repB*(*1*))
show *A* = *loop-class S x* (*some-loop S x A*)
by (*rule repA*(*2*))
show *B* = *loop-class S x* (*some-loop S x B*)
by (*rule repB*(*2*))
qed
have *join-BC*: *some-loop S x B* +++ *some-loop S x C* ∈ *loop-space S x*
using *repB*(*1*) *repC*(*1*) **by** (*rule loop-space-join*)
have *join-AB*: *some-loop S x A* +++ *some-loop S x B* ∈ *loop-space S x*
using *repA*(*1*) *repB*(*1*) **by** (*rule loop-space-join*)
have *left-eq*:
fundamental-group-mult S x A (*fundamental-group-mult S x B C*) =
loop-class S x (*some-loop S x A* +++ (*some-loop S x B* +++ *some-loop S x C*))
proof (*rule fundamental-group-mult-eqI*)
show *A* ∈ *fundamental-group-space S x*
by (*rule A-in*)
show *fundamental-group-mult S x B C* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-mult-in-space*[*OF B-in C-in*])
show *some-loop S x A* ∈ *loop-space S x*
by (*rule repA*(*1*))
show *some-loop S x B* +++ *some-loop S x C* ∈ *loop-space S x*
by (*rule join-BC*)
show *A* = *loop-class S x* (*some-loop S x A*)
by (*rule repA*(*2*))
show *fundamental-group-mult S x B C* =
loop-class S x (*some-loop S x B* +++ *some-loop S x C*)
by (*rule BC-eq*)
qed
have *right-eq*:
fundamental-group-mult S x (*fundamental-group-mult S x A B*) *C* =
loop-class S x ((*some-loop S x A* +++ *some-loop S x B*) +++ *some-loop S x C*)
proof (*rule fundamental-group-mult-eqI*)
show *fundamental-group-mult S x A B* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-mult-in-space*[*OF A-in B-in*])
show *C* ∈ *fundamental-group-space S x*

```
    by (rule C-in)
  show some-loop S x A +++ some-loop S x B ∈ loop-space S x
    by (rule join-AB)
  show some-loop S x C ∈ loop-space S x
    by (rule repC(1))
  show fundamental-group-mult S x A B =
    loop-class S x (some-loop S x A +++ some-loop S x B)
    by (rule AB-eq)
  show C = loop-class S x (some-loop S x C)
    by (rule repC(2))
 qed
 have left-loop: some-loop S x A +++ (some-loop S x B +++ some-loop S x C)
∈ loop-space S x
   using repA(1) join-BC by (rule loop-space-join)
 have right-loop: (some-loop S x A +++ some-loop S x B) +++ some-loop S x C
∈ loop-space S x
   using join-AB repC(1) by (rule loop-space-join)
 have hom-assoc:
   homotopic-paths S (some-loop S x A +++ (some-loop S x B +++ some-loop S
x C))
     ((some-loop S x A +++ some-loop S x B) +++ some-loop S x C)
   using repA(1) repB(1) repC(1)
   unfolding loop-space-def
   by (auto intro: homotopic-paths-assoc)
 have class-eq:
    loop-class S x (some-loop S x A +++ (some-loop S x B +++ some-loop S x
C)) =
     loop-class S x ((some-loop S x A +++ some-loop S x B) +++ some-loop S x
C)
   by (rule loop-class-eqI[OF left-loop right-loop hom-assoc])
 show ?thesis
   using left-eq right-eq class-eq by simp
qed

lemma fundamental-group-mult-inv-right:
 assumes x ∈ S
   and A ∈ fundamental-group-space S x
 shows fundamental-group-mult S x A (fundamental-group-inv S x A) = funda-
mental-group-one S x
proof −
 have const: (λ-. x) ∈ loop-space S x
   using constant-loop-in-space[OF assms(1)] .
 have repA: some-loop S x A ∈ loop-space S x A = loop-class S x (some-loop S x
A)
   using some-loop-spec[OF assms(2)] by auto
 have loopA:
   path (some-loop S x A)
   path-image (some-loop S x A) ⊆ S
   pathstart (some-loop S x A) = x
```

pathfinish (*some-loop S x A*) = *x*
using *repA*(*1*) **unfolding** *loop-space-def* **by** *auto*
have *inv-eq*:
fundamental-group-inv S x A = *loop-class S x* (*reversepath* (*some-loop S x A*))
by (*rule fundamental-group-inv-eqI*[*OF assms*(*2*) *repA*(*1*) *repA*(*2*)])
have *mult-eq*:
fundamental-group-mult S x A (*fundamental-group-inv S x A*) =
loop-class S x (*some-loop S x A* +++ *reversepath* (*some-loop S x A*))
proof (*rule fundamental-group-mult-eqI*)
show *A* ∈ *fundamental-group-space S x*
by (*rule assms*(*2*))
show *fundamental-group-inv S x A* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-inv-in-space*[*OF assms*(*2*)])
show *some-loop S x A* ∈ *loop-space S x*
by (*rule repA*(*1*))
show *reversepath* (*some-loop S x A*) ∈ *loop-space S x*
by (*rule loop-space-reversepath*[*OF repA*(*1*)])
show *A* = *loop-class S x* (*some-loop S x A*)
by (*rule repA*(*2*))
show *fundamental-group-inv S x A* = *loop-class S x* (*reversepath* (*some-loop S x A*))
by (*rule inv-eq*)
qed
have *join-loop*: *some-loop S x A* +++ *reversepath* (*some-loop S x A*) ∈ *loop-space S x*
using *repA*(*1*) *loop-space-reversepath*[*OF repA*(*1*)] **by** (*rule loop-space-join*)
have *hom*: *homotopic-paths S* (*some-loop S x A* +++ *reversepath* (*some-loop S x A*)) (λ-. *x*)
using *homotopic-paths-rinv-const*[*OF loopA*(*1*) *loopA*(*2*)] *loopA*(*3*) **by** *simp*
have *class-eq*:
loop-class S x (*some-loop S x A* +++ *reversepath* (*some-loop S x A*)) = *loop-class S x* (λ-. *x*)
by (*rule loop-class-eqI*[*OF join-loop const hom*])
show *?thesis*
using *mult-eq class-eq* **unfolding** *fundamental-group-one-def* **by** *simp*
qed

lemma *fundamental-group-mult-inv-left*:
assumes *x* ∈ *S*
and *A* ∈ *fundamental-group-space S x*
shows *fundamental-group-mult S x* (*fundamental-group-inv S x A*) *A* = *fundamental-group-one S x*
proof −
have *const*: (λ-. *x*) ∈ *loop-space S x*
using *constant-loop-in-space*[*OF assms*(*1*)] **.**
have *repA*: *some-loop S x A* ∈ *loop-space S x A* = *loop-class S x* (*some-loop S x A*)
using *some-loop-spec*[*OF assms*(*2*)] **by** *auto*
have *loopA*:

path (*some-loop S x A*)
path-image (*some-loop S x A*) ⊆ *S*
pathstart (*some-loop S x A*) = *x*
pathfinish (*some-loop S x A*) = *x*
using *repA*(*1*) **unfolding** *loop-space-def* **by** *auto*
have *inv-eq*:
fundamental-group-inv S x A = *loop-class S x* (*reversepath* (*some-loop S x A*))
by (*rule fundamental-group-inv-eqI*[*OF assms*(*2*) *repA*(*1*) *repA*(*2*)])
have *mult-eq*:
fundamental-group-mult S x (*fundamental-group-inv S x A*) *A* =
loop-class S x (*reversepath* (*some-loop S x A*) +++ *some-loop S x A*)
proof (*rule fundamental-group-mult-eqI*)
show *fundamental-group-inv S x A* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-inv-in-space*[*OF assms*(*2*)])
show *A* ∈ *fundamental-group-space S x*
by (*rule assms*(*2*))
show *reversepath* (*some-loop S x A*) ∈ *loop-space S x*
by (*rule loop-space-reversepath*[*OF repA*(*1*)])
show *some-loop S x A* ∈ *loop-space S x*
by (*rule repA*(*1*))
show *fundamental-group-inv S x A* = *loop-class S x* (*reversepath* (*some-loop S x A*))
by (*rule inv-eq*)
show *A* = *loop-class S x* (*some-loop S x A*)
by (*rule repA*(*2*))
qed
have *join-loop*: *reversepath* (*some-loop S x A*) +++ *some-loop S x A* ∈ *loop-space S x*
using *loop-space-reversepath*[*OF repA*(*1*)] *repA*(*1*) **by** (*rule loop-space-join*)
have *hom*: *homotopic-paths S* (*reversepath* (*some-loop S x A*) +++ *some-loop S x A*) (λ-. *x*)
using *homotopic-paths-linv-const*[*OF loopA*(*1*) *loopA*(*2*)] *loopA*(*4*) **by** *simp*
have *class-eq*:
loop-class S x (*reversepath* (*some-loop S x A*) +++ *some-loop S x A*) = *loop-class S x* (λ-. *x*)
by (*rule loop-class-eqI*[*OF join-loop const hom*])
show *?thesis*
using *mult-eq class-eq* **unfolding** *fundamental-group-one-def* **by** *simp*
qed

lemma *trivial-loop-class-in-space*:
assumes *x* ∈ *S*
shows *loop-class S x* (λ-. *x*) ∈ *fundamental-group-space S x*
using *constant-loop-in-space*[*OF assms*] **by** (*rule loop-class-in-space*)

lemma *fundamental-group-carrier-group*:
assumes *x-in*: *x* ∈ *S*
shows *carrier-group*
(*fundamental-group-space S x*)

(*fundamental-group-mult S x*)
(*fundamental-group-one S x*)
(*fundamental-group-inv S x*)
proof
show *fundamental-group-one S x* ∈ *fundamental-group-space S x*
using *x-in* **by** (*rule fundamental-group-one-in-space*)
next
fix *A B*
assume *A* ∈ *fundamental-group-space S x B* ∈ *fundamental-group-space S x*
then show *fundamental-group-mult S x A B* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-mult-in-space*)
next
fix *A*
assume *A* ∈ *fundamental-group-space S x*
then show *fundamental-group-inv S x A* ∈ *fundamental-group-space S x*
by (*rule fundamental-group-inv-in-space*)
next
fix *A B C*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
and *B-in*: *B* ∈ *fundamental-group-space S x*
and *C-in*: *C* ∈ *fundamental-group-space S x*
show
fundamental-group-mult S x (*fundamental-group-mult S x A B*) *C* =
fundamental-group-mult S x A (*fundamental-group-mult S x B C*)
by (*rule sym*, *rule fundamental-group-mult-assoc*[*OF A-in B-in C-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x* (*fundamental-group-one S x*) *A* = *A*
by (*rule fundamental-group-mult-one-left*[*OF x-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x A* (*fundamental-group-one S x*) = *A*
by (*rule fundamental-group-mult-one-right*[*OF x-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x* (*fundamental-group-inv S x A*) *A* = *fundamental-group-one S x*
by (*rule fundamental-group-mult-inv-left*[*OF x-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x A* (*fundamental-group-inv S x A*) = *fundamental-group-one S x*
by (*rule fundamental-group-mult-inv-right*[*OF x-in A-in*])
qed

lemma *fundamental-group-map-carrier-group-hom*:
assumes *x-in*: $x \in S$
and *cont-h*: *continuous-on* S h
and *map-h*: $h \in S \to T$
and *hx*: $h\ x = y$
shows *carrier-group-hom*
(*fundamental-group-space* S x)
(*fundamental-group-mult* S x)
(*fundamental-group-one* S x)
(*fundamental-group-inv* S x)
(*fundamental-group-space* T y)
(*fundamental-group-mult* T y)
(*fundamental-group-one* T y)
(*fundamental-group-inv* T y)
(*fundamental-group-map* S x T y h)
proof −
have *y-in*: $y \in T$
using *x-in map-h hx* **by** *auto*
show *?thesis*
proof *unfold-locales*
show *fundamental-group-one* S $x \in$ *fundamental-group-space* S x
by (*rule fundamental-group-one-in-space*[*OF x-in*])
next
fix A B
assume *A-in*: $A \in$ *fundamental-group-space* S x
and *B-in*: $B \in$ *fundamental-group-space* S x
show *fundamental-group-mult* S x A $B \in$ *fundamental-group-space* S x
by (*rule fundamental-group-mult-in-space*[*OF A-in B-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-group-space* S x
show *fundamental-group-inv* S x $A \in$ *fundamental-group-space* S x
by (*rule fundamental-group-inv-in-space*[*OF A-in*])
next
fix A B C
assume *A-in*: $A \in$ *fundamental-group-space* S x
and *B-in*: $B \in$ *fundamental-group-space* S x
and *C-in*: $C \in$ *fundamental-group-space* S x
show *fundamental-group-mult* S x (*fundamental-group-mult* S x A B) $C =$
fundamental-group-mult S x A (*fundamental-group-mult* S x B C)
by (*rule sym*, *rule fundamental-group-mult-assoc*[*OF A-in B-in C-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-group-space* S x
show *fundamental-group-mult* S x (*fundamental-group-one* S x) $A = A$
by (*rule fundamental-group-mult-one-left*[*OF x-in A-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-group-space* S x

show *fundamental-group-mult S x A* (*fundamental-group-one S x*) = *A*
by (*rule fundamental-group-mult-one-right*[*OF x-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x* (*fundamental-group-inv S x A*) *A* =
fundamental-group-one S x
by (*rule fundamental-group-mult-inv-left*[*OF x-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space S x*
show *fundamental-group-mult S x A* (*fundamental-group-inv S x A*) =
fundamental-group-one S x
by (*rule fundamental-group-mult-inv-right*[*OF x-in A-in*])
next
show *fundamental-group-one T y* ∈ *fundamental-group-space T y*
by (*rule fundamental-group-one-in-space*[*OF y-in*])
next
fix *A B*
assume *A-in*: *A* ∈ *fundamental-group-space T y*
and *B-in*: *B* ∈ *fundamental-group-space T y*
show *fundamental-group-mult T y A B* ∈ *fundamental-group-space T y*
by (*rule fundamental-group-mult-in-space*[*OF A-in B-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space T y*
show *fundamental-group-inv T y A* ∈ *fundamental-group-space T y*
by (*rule fundamental-group-inv-in-space*[*OF A-in*])
next
fix *A B C*
assume *A-in*: *A* ∈ *fundamental-group-space T y*
and *B-in*: *B* ∈ *fundamental-group-space T y*
and *C-in*: *C* ∈ *fundamental-group-space T y*
show *fundamental-group-mult T y* (*fundamental-group-mult T y A B*) *C* =
fundamental-group-mult T y A (*fundamental-group-mult T y B C*)
by (*rule sym*, *rule fundamental-group-mult-assoc*[*OF A-in B-in C-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space T y*
show *fundamental-group-mult T y* (*fundamental-group-one T y*) *A* = *A*
by (*rule fundamental-group-mult-one-left*[*OF y-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space T y*
show *fundamental-group-mult T y A* (*fundamental-group-one T y*) = *A*
by (*rule fundamental-group-mult-one-right*[*OF y-in A-in*])
next
fix *A*
assume *A-in*: *A* ∈ *fundamental-group-space T y*

```
    show fundamental-group-mult T y (fundamental-group-inv T y A) A =
      fundamental-group-one T y
      by (rule fundamental-group-mult-inv-left[OF y-in A-in])
  next
    fix A
    assume A-in: A ∈ fundamental-group-space T y
    show fundamental-group-mult T y A (fundamental-group-inv T y A) =
      fundamental-group-one T y
      by (rule fundamental-group-mult-inv-right[OF y-in A-in])
  next
    fix A
    assume A-in: A ∈ fundamental-group-space S x
    show fundamental-group-map S x T y h A ∈ fundamental-group-space T y
      by (rule fundamental-group-map-in-space[OF A-in cont-h map-h hx])
  next
    fix A B
    assume A-in: A ∈ fundamental-group-space S x
      and B-in: B ∈ fundamental-group-space S x
    show fundamental-group-map S x T y h (fundamental-group-mult S x A B) =
      fundamental-group-mult T y
        (fundamental-group-map S x T y h A)
        (fundamental-group-map S x T y h B)
      by (rule fundamental-group-map-mult[OF A-in B-in cont-h map-h hx])
  qed
qed

end
theory Explicit-Fundamental-Group-Scaffold
  imports Explicit-Path-Homotopy-Scaffold Fundamental-Group-Scaffold
begin
```

12 Explicit-topology fundamental-group quotients

Building on the explicit-topology path layer, this theory forms loop classes, quotient carriers, and induced maps for arbitrary topological spaces. It is the main bridge from point-set topology to the carrier-group algebra used in the Seifert–van Kampen interface.

```
definition loopin-class ::
  'a topology ⇒ 'a ⇒ (real ⇒ 'a) ⇒ (real ⇒ 'a) set
where
  loopin-class X x p = {q. q ∈ loopin-space X x ∧ homotopic-pathsin X q p}

definition fundamental-groupin-space ::
  'a topology ⇒ 'a ⇒ ((real ⇒ 'a) set) set
where
  fundamental-groupin-space X x = loopin-class X x ‘ loopin-space X x
```

definition *some-loopin* ::
  'a topology ⇒ *'a* ⇒ (*real* ⇒ *'a*) *set* ⇒ (*real* ⇒ *'a*)
where
  some-loopin X x Q = (*SOME p. p* ∈ *loopin-space X x* ∧ *Q* = *loopin-class X x p*)

definition *fundamental-groupin-one* ::
  'a topology ⇒ *'a* ⇒ (*real* ⇒ *'a*) *set*
where
  fundamental-groupin-one X x = *loopin-class X x* (λ-. *x*)

definition *fundamental-groupin-mult* ::
  'a topology ⇒ *'a* ⇒
    (*real* ⇒ *'a*) *set* ⇒ (*real* ⇒ *'a*) *set* ⇒ (*real* ⇒ *'a*) *set*
where
  fundamental-groupin-mult X x A B =
     loopin-class X x (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*))

definition *fundamental-groupin-inv* ::
  'a topology ⇒ *'a* ⇒ (*real* ⇒ *'a*) *set* ⇒ (*real* ⇒ *'a*) *set*
where
  fundamental-groupin-inv X x A = *loopin-class X x* (*reversepathin* (*some-loopin X x A*))

definition *loopin-image* ::
  (*'a* ⇒ *'b*) ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'b*)
where
  loopin-image h p = *h* ∘ *p*

definition *fundamental-groupin-map* ::
  'a topology ⇒ *'a* ⇒ *'b topology* ⇒ *'b* ⇒
    (*'a* ⇒ *'b*) ⇒ (*real* ⇒ *'a*) *set* ⇒ (*real* ⇒ *'b*) *set*
where
  fundamental-groupin-map X x Y y h A =
     loopin-class Y y (*loopin-image h* (*some-loopin X x A*))

lemma *loopin-class-in-space*:
  assumes *p* ∈ *loopin-space X x*
  shows *loopin-class X x p* ∈ *fundamental-groupin-space X x*
  using *assms* **unfolding** *fundamental-groupin-space-def* **by** *blast*

lemma *fundamental-groupin-spaceE*:
  assumes *Q* ∈ *fundamental-groupin-space X x*
  obtains *p* **where** *p* ∈ *loopin-space X x Q* = *loopin-class X x p*
  using *assms* **unfolding** *fundamental-groupin-space-def* **by** *blast*

lemma *some-loopin-spec*:
  assumes *Q* ∈ *fundamental-groupin-space X x*
  shows *some-loopin X x Q* ∈ *loopin-space X x*

```
    and Q = loopin-class X x (some-loopin X x Q)
proof −
  from assms obtain p where p: p ∈ loopin-space X x Q = loopin-class X x p
    by (rule fundamental-groupin-spaceE)
  have ex: ∃ p. p ∈ loopin-space X x ∧ Q = loopin-class X x p
    using p by blast
  have some-loopin X x Q ∈ loopin-space X x ∧ Q = loopin-class X x (some-loopin
X x Q)
    unfolding some-loopin-def by (rule someI-ex[OF ex])
  then show some-loopin X x Q ∈ loopin-space X x
    and Q = loopin-class X x (some-loopin X x Q)
    by auto
qed

lemma loopin-class-eqI:
  assumes p: p ∈ loopin-space X x
    and q: q ∈ loopin-space X x
    and pq: homotopic-pathsin X p q
  shows loopin-class X x p = loopin-class X x q
proof (auto simp: loopin-class-def)
  fix r
  assume homotopic-pathsin X r p
  then show homotopic-pathsin X r q
    using pq by (rule homotopic-pathsin-trans)
next
  fix r
  assume homotopic-pathsin X r q
  moreover have homotopic-pathsin X q p
    using pq by (rule homotopic-pathsin-sym)
  ultimately show homotopic-pathsin X r p
    by (rule homotopic-pathsin-trans)
qed

lemma loopin-class-eq-iff:
  assumes p: p ∈ loopin-space X x
    and q: q ∈ loopin-space X x
  shows loopin-class X x p = loopin-class X x q ⟷ homotopic-pathsin X p q
proof
  assume h: loopin-class X x p = loopin-class X x q
  have p ∈ loopin-class X x p
    using p by (auto simp: loopin-class-def elim: loopin-spaceE)
  then have p ∈ loopin-class X x q
    using h by simp
  then show homotopic-pathsin X p q
    unfolding loopin-class-def using p by auto
next
  assume pq: homotopic-pathsin X p q
  show loopin-class X x p = loopin-class X x q
    by (rule loopin-class-eqI[OF p q pq])
```

qed

lemma *fundamental-groupin-one-in-space*:
assumes $x \in$ *topspace* X
shows *fundamental-groupin-one* $X\ x \in$ *fundamental-groupin-space* $X\ x$
unfolding *fundamental-groupin-one-def*
by (*rule loopin-class-in-space*, *rule constant-loopin-in-space*, *fact assms*)

lemma *fundamental-groupin-mult-eqI*:
assumes *A-in*: $A \in$ *fundamental-groupin-space* $X\ x$
and *B-in*: $B \in$ *fundamental-groupin-space* $X\ x$
and *p*: $p \in$ *loopin-space* $X\ x$
and *q*: $q \in$ *loopin-space* $X\ x$
and *A*: $A =$ *loopin-class* $X\ x\ p$
and *B*: $B =$ *loopin-class* $X\ x\ q$
shows *fundamental-groupin-mult* $X\ x\ A\ B =$ *loopin-class* $X\ x$ (*joinpathin* $p\ q$)
proof −
have *repA*: *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x\ A =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ A$)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *repB*: *some-loopin* $X\ x\ B \in$ *loopin-space* $X\ x\ B =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ B$)
using *some-loopin-spec*[*OF B-in*] **by** *auto*
have *homoA'*: *homotopic-pathsin* $X\ p$ (*some-loopin* $X\ x\ A$)
using *repA* $p\ A$ **by** (*simp add*: *loopin-class-eq-iff*)
have *homoA*: *homotopic-pathsin* X (*some-loopin* $X\ x\ A$) p
using *homoA'* **by** (*rule homotopic-pathsin-sym*)
have *homoB'*: *homotopic-pathsin* $X\ q$ (*some-loopin* $X\ x\ B$)
using *repB* $q\ B$ **by** (*simp add*: *loopin-class-eq-iff*)
have *homoB*: *homotopic-pathsin* X (*some-loopin* $X\ x\ B$) q
using *homoB'* **by** (*rule homotopic-pathsin-sym*)
have *join-hom*:
homotopic-pathsin X
(*joinpathin* (*some-loopin* $X\ x\ A$) (*some-loopin* $X\ x\ B$))
(*joinpathin* $p\ q$)
proof −
have *repA-end*: *some-loopin* $X\ x\ A\ 1 = x$
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *repB-start*: *some-loopin* $X\ x\ B\ 0 = x$
using *repB*(*1*) **by** (*auto elim*: *loopin-spaceE*)
show *?thesis*
by (*rule homotopic-pathsin-joinpathin*[*OF homoA homoB*]) (*simp add*: *repA-end repB-start*)
qed
have *loopin-class* $X\ x$ (*joinpathin* (*some-loopin* $X\ x\ A$) (*some-loopin* $X\ x\ B$)) = *loopin-class* $X\ x$ (*joinpathin* $p\ q$)
by (*rule loopin-class-eqI*[*OF loopin-space-joinpathin*[*OF repA*(*1*) *repB*(*1*)] *loopin-space-joinpathin*[*OF p q*] *join-hom*])
then show *?thesis*

unfolding *fundamental-groupin-mult-def* **.**
qed

lemma *fundamental-groupin-mult-in-space*:
assumes $A \in$ *fundamental-groupin-space* $X\ x$
and $B \in$ *fundamental-groupin-space* $X\ x$
shows *fundamental-groupin-mult* $X\ x\ A\ B \in$ *fundamental-groupin-space* $X\ x$
proof −
have *repA*: *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x$
using *some-loopin-spec*[*OF assms*(*1*)] **by** *auto*
have *repB*: *some-loopin* $X\ x\ B \in$ *loopin-space* $X\ x$
using *some-loopin-spec*[*OF assms*(*2*)] **by** *auto*
show *?thesis*
unfolding *fundamental-groupin-mult-def*
using *loopin-space-joinpathin*[*OF repA repB*] **by** (*rule loopin-class-in-space*)
qed

lemma *fundamental-groupin-inv-eqI*:
assumes *A-in*: $A \in$ *fundamental-groupin-space* $X\ x$
and *p*: $p \in$ *loopin-space* $X\ x$
and *A*: $A =$ *loopin-class* $X\ x\ p$
shows *fundamental-groupin-inv* $X\ x\ A =$ *loopin-class* $X\ x$ (*reversepathin* p)
proof −
have *repA*: *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x\ A =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ A$)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *homoA'*: *homotopic-pathsin* $X\ p$ (*some-loopin* $X\ x\ A$)
using *repA* $p\ A$ **by** (*simp add*: *loopin-class-eq-iff*)
have *homoA*: *homotopic-pathsin* X (*some-loopin* $X\ x\ A$) p
using *homoA'* **by** (*rule homotopic-pathsin-sym*)
have *rev-hom*:
homotopic-pathsin X (*reversepathin* (*some-loopin* $X\ x\ A$)) (*reversepathin* p)
using *homoA* **by** (*rule homotopic-pathsin-reversepathin-D*)
have *loopin-class* $X\ x$ (*reversepathin* (*some-loopin* $X\ x\ A$)) = *loopin-class* $X\ x$ (*reversepathin* p)
by (*rule loopin-class-eqI*[*OF loopin-space-reversepathin*[*OF repA*(*1*)]
loopin-space-reversepathin[*OF p*] *rev-hom*])
then show *?thesis*
unfolding *fundamental-groupin-inv-def* **.**
qed

lemma *fundamental-groupin-inv-in-space*:
assumes $A \in$ *fundamental-groupin-space* $X\ x$
shows *fundamental-groupin-inv* $X\ x\ A \in$ *fundamental-groupin-space* $X\ x$
proof −
have *repA*: *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x$
using *some-loopin-spec*[*OF assms*] **by** *auto*
show *?thesis*
unfolding *fundamental-groupin-inv-def*

using *loopin-space-reversepathin*[*OF repA*] **by** (*rule loopin-class-in-space*)
qed

lemma *loopin-image-in-space*:
assumes *p*: $p \in$ *loopin-space X x*
and *h*: *continuous-map X Y f*
and *fx*: $f\ x = y$
shows *loopin-image f p* $\in$ *loopin-space Y y*
proof −
from *p* **obtain** *p-in* **where** *p-in*: *pathin X p p 0* $= x$ *p 1* $= x$
by (*rule loopin-spaceE*)
have *pathin Y* (*loopin-image f p*)
using *p-in*(*1*) *h* **unfolding** *loopin-image-def* **by** (*rule pathin-compose*)
moreover have *loopin-image f p 0* $= y$ *loopin-image f p 1* $= y$
using *p-in fx* **unfolding** *loopin-image-def* **by** *simp-all*
ultimately show *?thesis*
unfolding *loopin-space-def* **by** *blast*
qed

lemma *fundamental-groupin-map-rep*:
assumes *A*: $A \in$ *fundamental-groupin-space X x*
and *p*: $p \in$ *loopin-space X x*
and *rep*: $A =$ *loopin-class X x p*
and *h*: *continuous-map X Y f*
and *fx*: $f\ x = y$
shows *fundamental-groupin-map X x Y y f A* $=$ *loopin-class Y y* (*loopin-image f p*)
proof −
have *sp*: *some-loopin X x A* $\in$ *loopin-space X x*
using *A* **by** (*rule some-loopin-spec*)
have *A'*: $A =$ *loopin-class X x* (*some-loopin X x A*)
using *A* **by** (*rule some-loopin-spec*)
have *class-eq*: *loopin-class X x* (*some-loopin X x A*) $=$ *loopin-class X x p*
using *A' rep* **by** *simp*
have *hom*: *homotopic-pathsin X* (*some-loopin X x A*) *p*
using *sp p class-eq* **by** (*simp add*: *loopin-class-eq-iff*)
have *img-some*: *loopin-image f* (*some-loopin X x A*) $\in$ *loopin-space Y y*
by (*rule loopin-image-in-space*[*OF sp h fx*])
have *img-p*: *loopin-image f p* $\in$ *loopin-space Y y*
by (*rule loopin-image-in-space*[*OF p h fx*])
have *img-hom*: *homotopic-pathsin Y* (*loopin-image f* (*some-loopin X x A*)) (*loopin-image f p*)
using *hom h* **unfolding** *loopin-image-def* **by** (*rule homotopic-pathsin-continuous-image*)
have *loopin-class Y y* (*loopin-image f* (*some-loopin X x A*)) $=$ *loopin-class Y y* (*loopin-image f p*)
by (*rule loopin-class-eqI*[*OF img-some img-p img-hom*])
then show *?thesis*
unfolding *fundamental-groupin-map-def* **by** *simp*

qed

lemma *fundamental-groupin-map-in-space*:
assumes *A*: *A* ∈ *fundamental-groupin-space X x*
and *h*: *continuous-map X Y f*
and *fx*: *f x* = *y*
shows *fundamental-groupin-map X x Y y f A* ∈ *fundamental-groupin-space Y y*
proof −
have *sp*: *some-loopin X x A* ∈ *loopin-space X x*
using *A* **by** (*rule some-loopin-spec*)
have *img*: *loopin-image f* (*some-loopin X x A*) ∈ *loopin-space Y y*
by (*rule loopin-image-in-space*[*OF sp h fx*])
show *?thesis*
unfolding *fundamental-groupin-map-def* **by** (*rule loopin-class-in-space*[*OF img*])
qed

lemma *loopin-image-joinpathin* [*simp*]:
loopin-image h (*joinpathin p q*) = *joinpathin* (*loopin-image h p*) (*loopin-image h q*)
unfolding *loopin-image-def joinpathin-def* **by** (*rule ext*) *simp*

lemma *loopin-image-reversepathin* [*simp*]:
loopin-image h (*reversepathin p*) = *reversepathin* (*loopin-image h p*)
unfolding *loopin-image-def reversepathin-def* **by** (*rule ext*) *simp*

lemma *fundamental-groupin-map-mult*:
assumes *A-in*: *A* ∈ *fundamental-groupin-space X x*
and *B-in*: *B* ∈ *fundamental-groupin-space X x*
and *h*: *continuous-map X Y f*
and *fx*: *f x* = *y*
shows *fundamental-groupin-map X x Y y f* (*fundamental-groupin-mult X x A B*) =
fundamental-groupin-mult Y y
(*fundamental-groupin-map X x Y y f A*)
(*fundamental-groupin-map X x Y y f B*)
proof −
have *repA*: *some-loopin X x A* ∈ *loopin-space X x A* = *loopin-class X x* (*some-loopin X x A*)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *repB*: *some-loopin X x B* ∈ *loopin-space X x B* = *loopin-class X x* (*some-loopin X x B*)
using *some-loopin-spec*[*OF B-in*] **by** *auto*
have *AB-in*: *fundamental-groupin-mult X x A B* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-mult-in-space*[*OF A-in B-in*])
have *AB-eq*:
fundamental-groupin-mult X x A B =
loopin-class X x (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*))

by (*rule fundamental-groupin-mult-eqI*[*OF A-in B-in repA*(*1*) *repB*(*1*) *repA*(*2*) *repB*(*2*)])
have *map-A-in*:
fundamental-groupin-map X x Y y f A ∈ *fundamental-groupin-space Y y*
by (*rule fundamental-groupin-map-in-space*[*OF A-in h fx*])
have *map-B-in*:
fundamental-groupin-map X x Y y f B ∈ *fundamental-groupin-space Y y*
by (*rule fundamental-groupin-map-in-space*[*OF B-in h fx*])
have *map-A-eq*:
fundamental-groupin-map X x Y y f A =
loopin-class Y y (*loopin-image f* (*some-loopin X x A*))
by (*rule fundamental-groupin-map-rep*[*OF A-in repA*(*1*) *repA*(*2*) *h fx*])
have *map-B-eq*:
fundamental-groupin-map X x Y y f B =
loopin-class Y y (*loopin-image f* (*some-loopin X x B*))
by (*rule fundamental-groupin-map-rep*[*OF B-in repB*(*1*) *repB*(*2*) *h fx*])
have *left-eq*:
fundamental-groupin-map X x Y y f (*fundamental-groupin-mult X x A B*) =
loopin-class Y y (*loopin-image f* (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*)))
by (*rule fundamental-groupin-map-rep*[*OF AB-in loopin-space-joinpathin*[*OF repA*(*1*) *repB*(*1*)] *AB-eq h fx*])
have *map-loop-A*: *loopin-image f* (*some-loopin X x A*) ∈ *loopin-space Y y*
by (*rule loopin-image-in-space*[*OF repA*(*1*) *h fx*])
have *map-loop-B*: *loopin-image f* (*some-loopin X x B*) ∈ *loopin-space Y y*
by (*rule loopin-image-in-space*[*OF repB*(*1*) *h fx*])
have *right-eq*:
fundamental-groupin-mult Y y
(*fundamental-groupin-map X x Y y f A*)
(*fundamental-groupin-map X x Y y f B*) =
loopin-class Y y
(*joinpathin* (*loopin-image f* (*some-loopin X x A*))
(*loopin-image f* (*some-loopin X x B*)))
by (*rule fundamental-groupin-mult-eqI*[*OF map-A-in map-B-in map-loop-A map-loop-B map-A-eq map-B-eq*])
then show *?thesis*
using *left-eq* **by** *simp*
qed

lemma *fundamental-groupin-mult-one-left*:
assumes *x-in*: *x* ∈ *topspace X*
and *A-in*: *A* ∈ *fundamental-groupin-space X x*
shows *fundamental-groupin-mult X x* (*fundamental-groupin-one X x*) *A* = *A*
proof −
have *const*: (λ-. *x*) ∈ *loopin-space X x*
by (*rule constant-loopin-in-space*[*OF x-in*])
have *repA*: *some-loopin X x A* ∈ *loopin-space X x A* = *loopin-class X x* (*some-loopin X x A*)
using *some-loopin-spec*[*OF A-in*] **by** *auto*

have *loopA*: *pathin X* (*some-loopin X x A*) *some-loopin X x A 0 = x some-loopin X x A 1 = x*
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *mult-eq*:
fundamental-groupin-mult X x (*fundamental-groupin-one X x*) *A* =
loopin-class X x (*joinpathin* (λ-. *x*) (*some-loopin X x A*))
proof (*rule fundamental-groupin-mult-eqI*)
show *fundamental-groupin-one X x* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-one-in-space*[*OF x-in*])
show *A* ∈ *fundamental-groupin-space X x*
by (*rule A-in*)
show (λ-. *x*) ∈ *loopin-space X x*
by (*rule const*)
show *some-loopin X x A* ∈ *loopin-space X x*
by (*rule repA*(*1*))
show *fundamental-groupin-one X x* = *loopin-class X x* (λ-. *x*)
by (*simp add*: *fundamental-groupin-one-def*)
show *A* = *loopin-class X x* (*some-loopin X x A*)
by (*rule repA*(*2*))
qed
have *join-loop*: *joinpathin* (λ-. *x*) (*some-loopin X x A*) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF const repA*(*1*)])
have *hom*: *homotopic-pathsin X* (*joinpathin* (λ-. *x*) (*some-loopin X x A*)) (*some-loopin X x A*)
using *homotopic-pathsin-lid-const*[*OF loopA*(*1*)] *loopA*(*2*) **by** *simp*
have *class-eq*:
loopin-class X x (*joinpathin* (λ-. *x*) (*some-loopin X x A*)) = *loopin-class X x* (*some-loopin X x A*)
by (*rule loopin-class-eqI*[*OF join-loop repA*(*1*) *hom*])
show *?thesis*
using *mult-eq class-eq repA*(*2*) **by** *simp*
qed

lemma *fundamental-groupin-mult-one-right*:
assumes *x-in*: *x* ∈ *topspace X*
and *A-in*: *A* ∈ *fundamental-groupin-space X x*
shows *fundamental-groupin-mult X x A* (*fundamental-groupin-one X x*) = *A*
proof −
have *const*: (λ-. *x*) ∈ *loopin-space X x*
by (*rule constant-loopin-in-space*[*OF x-in*])
have *repA*: *some-loopin X x A* ∈ *loopin-space X x A* = *loopin-class X x* (*some-loopin X x A*)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *loopA*: *pathin X* (*some-loopin X x A*) *some-loopin X x A 0 = x some-loopin X x A 1 = x*
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *mult-eq*:
fundamental-groupin-mult X x A (*fundamental-groupin-one X x*) =
loopin-class X x (*joinpathin* (*some-loopin X x A*) (λ-. *x*))

proof (*rule fundamental-groupin-mult-eqI*)
show $A \in$ *fundamental-groupin-space* $X\ x$
by (*rule A-in*)
show *fundamental-groupin-one* $X\ x \in$ *fundamental-groupin-space* $X\ x$
by (*rule fundamental-groupin-one-in-space*[*OF x-in*])
show *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x$
by (*rule repA*(*1*))
show $(\lambda\text{-}.\ x) \in$ *loopin-space* $X\ x$
by (*rule const*)
show $A =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ A$)
by (*rule repA*(*2*))
show *fundamental-groupin-one* $X\ x =$ *loopin-class* $X\ x\ (\lambda\text{-}.\ x)$
by (*simp add*: *fundamental-groupin-one-def*)
qed
have *join-loop*: *joinpathin* (*some-loopin* $X\ x\ A$) $(\lambda\text{-}.\ x) \in$ *loopin-space* $X\ x$
by (*rule loopin-space-joinpathin*[*OF repA*(*1*) *const*])
have *hom*: *homotopic-pathsin* X (*joinpathin* (*some-loopin* $X\ x\ A$) $(\lambda\text{-}.\ x)$) (*some-loopin* $X\ x\ A$)
using *homotopic-pathsin-rid-const*[*OF loopA*(*1*)] *loopA*(*3*) **by** *simp*
have *class-eq*:
loopin-class $X\ x$ (*joinpathin* (*some-loopin* $X\ x\ A$) $(\lambda\text{-}.\ x)$) = *loopin-class* $X\ x$ (*some-loopin* $X\ x\ A$)
by (*rule loopin-class-eqI*[*OF join-loop repA*(*1*) *hom*])
show *?thesis*
using *mult-eq class-eq repA*(*2*) **by** *simp*
qed

lemma *fundamental-groupin-mult-assoc*:
assumes *A-in*: $A \in$ *fundamental-groupin-space* $X\ x$
and *B-in*: $B \in$ *fundamental-groupin-space* $X\ x$
and *C-in*: $C \in$ *fundamental-groupin-space* $X\ x$
shows *fundamental-groupin-mult* $X\ x\ A$ (*fundamental-groupin-mult* $X\ x\ B\ C$) = *fundamental-groupin-mult* $X\ x$ (*fundamental-groupin-mult* $X\ x\ A\ B$) C
proof −
have *repA*: *some-loopin* $X\ x\ A \in$ *loopin-space* $X\ x\ A =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ A$)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *repB*: *some-loopin* $X\ x\ B \in$ *loopin-space* $X\ x\ B =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ B$)
using *some-loopin-spec*[*OF B-in*] **by** *auto*
have *repC*: *some-loopin* $X\ x\ C \in$ *loopin-space* $X\ x\ C =$ *loopin-class* $X\ x$ (*some-loopin* $X\ x\ C$)
using *some-loopin-spec*[*OF C-in*] **by** *auto*
have *loopA*: *pathin* X (*some-loopin* $X\ x\ A$) *some-loopin* $X\ x\ A\ 0 = x$ *some-loopin* $X\ x\ A\ 1 = x$
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *loopB*: *pathin* X (*some-loopin* $X\ x\ B$) *some-loopin* $X\ x\ B\ 0 = x$ *some-loopin* $X\ x\ B\ 1 = x$
using *repB*(*1*) **by** (*auto elim*: *loopin-spaceE*)

have *loopC*: *pathin X* (*some-loopin X x C*) *some-loopin X x C 0 = x some-loopin X x C 1 = x*
using *repC*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *BC-eq*:
fundamental-groupin-mult X x B C =
loopin-class X x (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*))
by (*rule fundamental-groupin-mult-eqI*[*OF B-in C-in repB*(*1*) *repC*(*1*) *repB*(*2*) *repC*(*2*)])
have *AB-eq*:
fundamental-groupin-mult X x A B =
loopin-class X x (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*))
by (*rule fundamental-groupin-mult-eqI*[*OF A-in B-in repA*(*1*) *repB*(*1*) *repA*(*2*) *repB*(*2*)])
have *join-BC*: *joinpathin* (*some-loopin X x B*) (*some-loopin X x C*) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF repB*(*1*) *repC*(*1*)])
have *join-AB*: *joinpathin* (*some-loopin X x A*) (*some-loopin X x B*) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF repA*(*1*) *repB*(*1*)])
have *left-eq*:
fundamental-groupin-mult X x A (*fundamental-groupin-mult X x B C*) =
loopin-class X x
(*joinpathin* (*some-loopin X x A*) (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*)))
proof (*rule fundamental-groupin-mult-eqI*)
show *A* ∈ *fundamental-groupin-space X x*
by (*rule A-in*)
show *fundamental-groupin-mult X x B C* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-mult-in-space*[*OF B-in C-in*])
show *some-loopin X x A* ∈ *loopin-space X x*
by (*rule repA*(*1*))
show *joinpathin* (*some-loopin X x B*) (*some-loopin X x C*) ∈ *loopin-space X x*
by (*rule join-BC*)
show *A = loopin-class X x* (*some-loopin X x A*)
by (*rule repA*(*2*))
show *fundamental-groupin-mult X x B C =*
loopin-class X x (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*))
by (*rule BC-eq*)
qed
have *right-eq*:
fundamental-groupin-mult X x (*fundamental-groupin-mult X x A B*) *C =*
loopin-class X x
(*joinpathin* (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*)) (*some-loopin X x C*))
proof (*rule fundamental-groupin-mult-eqI*)
show *fundamental-groupin-mult X x A B* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-mult-in-space*[*OF A-in B-in*])
show *C* ∈ *fundamental-groupin-space X x*
by (*rule C-in*)

show *joinpathin* (*some-loopin X x A*) (*some-loopin X x B*) ∈ *loopin-space X x*
by (*rule join-AB*)
show *some-loopin X x C* ∈ *loopin-space X x*
by (*rule repC(1)*)
show *fundamental-groupin-mult X x A B* =
loopin-class X x (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*))
by (*rule AB-eq*)
show *C* = *loopin-class X x* (*some-loopin X x C*)
by (*rule repC(2)*)
qed
have *left-loop*:
joinpathin (*some-loopin X x A*) (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*)) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF repA(1) join-BC*])
have *right-loop*:
joinpathin (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*)) (*some-loopin X x C*) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF join-AB repC(1)*])
have *hom-assoc*:
homotopic-pathsin X
(*joinpathin* (*some-loopin X x A*) (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*)))
(*joinpathin* (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*)) (*some-loopin X x C*))
by (*rule homotopic-pathsin-assoc*[*OF loopA(1) loopB(1) loopC(1)*]) (*simp-all add*: *loopA loopB loopC*)
have *class-eq*:
loopin-class X x
(*joinpathin* (*some-loopin X x A*) (*joinpathin* (*some-loopin X x B*) (*some-loopin X x C*))) =
loopin-class X x
(*joinpathin* (*joinpathin* (*some-loopin X x A*) (*some-loopin X x B*)) (*some-loopin X x C*))
by (*rule loopin-class-eqI*[*OF left-loop right-loop hom-assoc*])
show *?thesis*
using *left-eq right-eq class-eq* **by** *simp*
qed

lemma *fundamental-groupin-mult-inv-right*:
assumes *x-in*: *x* ∈ *topspace X*
and *A-in*: *A* ∈ *fundamental-groupin-space X x*
shows *fundamental-groupin-mult X x A* (*fundamental-groupin-inv X x A*) = *fundamental-groupin-one X x*
proof −
have *const*: (λ-. *x*) ∈ *loopin-space X x*
by (*rule constant-loopin-in-space*[*OF x-in*])
have *repA*: *some-loopin X x A* ∈ *loopin-space X x A* = *loopin-class X x* (*some-loopin X x A*)
using *some-loopin-spec*[*OF A-in*] **by** *auto*

have *loopA*: *pathin X* (*some-loopin X x A*) *some-loopin X x A 0 = x some-loopin X x A 1 = x*
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *inv-eq*:
fundamental-groupin-inv X x A = loopin-class X x (*reversepathin* (*some-loopin X x A*))
by (*rule fundamental-groupin-inv-eqI*[*OF A-in repA*(*1*) *repA*(*2*)])
have *mult-eq*:
fundamental-groupin-mult X x A (*fundamental-groupin-inv X x A*) =
loopin-class X x (*joinpathin* (*some-loopin X x A*) (*reversepathin* (*some-loopin X x A*)))
proof (*rule fundamental-groupin-mult-eqI*)
show *A* ∈ *fundamental-groupin-space X x*
by (*rule A-in*)
show *fundamental-groupin-inv X x A* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-inv-in-space*[*OF A-in*])
show *some-loopin X x A* ∈ *loopin-space X x*
by (*rule repA*(*1*))
show *reversepathin* (*some-loopin X x A*) ∈ *loopin-space X x*
by (*rule loopin-space-reversepathin*[*OF repA*(*1*)])
show *A = loopin-class X x* (*some-loopin X x A*)
by (*rule repA*(*2*))
show *fundamental-groupin-inv X x A = loopin-class X x* (*reversepathin* (*some-loopin X x A*))
by (*rule inv-eq*)
qed
have *join-loop*: *joinpathin* (*some-loopin X x A*) (*reversepathin* (*some-loopin X x A*)) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF repA*(*1*) *loopin-space-reversepathin*[*OF repA*(*1*)]])
have *hom*:
homotopic-pathsin X
(*joinpathin* (*some-loopin X x A*) (*reversepathin* (*some-loopin X x A*)))
(λ-. *x*)
using *homotopic-pathsin-rinv-const*[*OF loopA*(*1*)] *loopA*(*2*) **by** *simp*
have *class-eq*:
loopin-class X x (*joinpathin* (*some-loopin X x A*) (*reversepathin* (*some-loopin X x A*))) =
loopin-class X x (λ-. *x*)
by (*rule loopin-class-eqI*[*OF join-loop const hom*])
show *?thesis*
using *mult-eq class-eq* **unfolding** *fundamental-groupin-one-def* **by** *simp*
qed

lemma *fundamental-groupin-mult-inv-left*:
assumes *x-in*: *x* ∈ *topspace X*
and *A-in*: *A* ∈ *fundamental-groupin-space X x*
shows *fundamental-groupin-mult X x* (*fundamental-groupin-inv X x A*) *A = fundamental-groupin-one X x*

proof −
have *const*: (λ-. *x*) ∈ *loopin-space X x*
by (*rule constant-loopin-in-space*[*OF x-in*])
have *repA*: *some-loopin X x A* ∈ *loopin-space X x A* = *loopin-class X x* (*some-loopin X x A*)
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *loopA*: *pathin X* (*some-loopin X x A*) *some-loopin X x A 0* = *x some-loopin X x A 1* = *x*
using *repA*(*1*) **by** (*auto elim*: *loopin-spaceE*)
have *inv-eq*:
fundamental-groupin-inv X x A = *loopin-class X x* (*reversepathin* (*some-loopin X x A*))
by (*rule fundamental-groupin-inv-eqI*[*OF A-in repA*(*1*) *repA*(*2*)])
have *mult-eq*:
fundamental-groupin-mult X x (*fundamental-groupin-inv X x A*) *A* =
loopin-class X x (*joinpathin* (*reversepathin* (*some-loopin X x A*)) (*some-loopin X x A*))
proof (*rule fundamental-groupin-mult-eqI*)
show *fundamental-groupin-inv X x A* ∈ *fundamental-groupin-space X x*
by (*rule fundamental-groupin-inv-in-space*[*OF A-in*])
show *A* ∈ *fundamental-groupin-space X x*
by (*rule A-in*)
show *reversepathin* (*some-loopin X x A*) ∈ *loopin-space X x*
by (*rule loopin-space-reversepathin*[*OF repA*(*1*)])
show *some-loopin X x A* ∈ *loopin-space X x*
by (*rule repA*(*1*))
show *fundamental-groupin-inv X x A* = *loopin-class X x* (*reversepathin* (*some-loopin X x A*))
by (*rule inv-eq*)
show *A* = *loopin-class X x* (*some-loopin X x A*)
by (*rule repA*(*2*))
qed
have *join-loop*: *joinpathin* (*reversepathin* (*some-loopin X x A*)) (*some-loopin X x A*) ∈ *loopin-space X x*
by (*rule loopin-space-joinpathin*[*OF loopin-space-reversepathin*[*OF repA*(*1*)] *repA*(*1*)])
have *hom*:
homotopic-pathsin X
(*joinpathin* (*reversepathin* (*some-loopin X x A*)) (*some-loopin X x A*))
(λ-. *x*)
using *homotopic-pathsin-linv-const*[*OF loopA*(*1*)] *loopA*(*3*) **by** *simp*
have *class-eq*:
loopin-class X x (*joinpathin* (*reversepathin* (*some-loopin X x A*)) (*some-loopin X x A*)) =
loopin-class X x (λ-. *x*)
by (*rule loopin-class-eqI*[*OF join-loop const hom*])
show *?thesis*
using *mult-eq class-eq* **unfolding** *fundamental-groupin-one-def* **by** *simp*
qed

lemma *fundamental-groupin-carrier-group*:
assumes *x-in*: $x \in$ *topspace* X
shows *carrier-group*
(*fundamental-groupin-space* X x)
(*fundamental-groupin-mult* X x)
(*fundamental-groupin-one* X x)
(*fundamental-groupin-inv* X x)
proof
show *fundamental-groupin-one* X $x \in$ *fundamental-groupin-space* X x
by (*rule fundamental-groupin-one-in-space*[*OF x-in*])
next
fix A B
assume $A \in$ *fundamental-groupin-space* X x $B \in$ *fundamental-groupin-space* X x
then show *fundamental-groupin-mult* X x A $B \in$ *fundamental-groupin-space* X x
by (*rule fundamental-groupin-mult-in-space*)
next
fix A
assume $A \in$ *fundamental-groupin-space* X x
then show *fundamental-groupin-inv* X x $A \in$ *fundamental-groupin-space* X x
by (*rule fundamental-groupin-inv-in-space*)
next
fix A B C
assume *A-in*: $A \in$ *fundamental-groupin-space* X x
and *B-in*: $B \in$ *fundamental-groupin-space* X x
and *C-in*: $C \in$ *fundamental-groupin-space* X x
show
fundamental-groupin-mult X x (*fundamental-groupin-mult* X x A B) C =
fundamental-groupin-mult X x A (*fundamental-groupin-mult* X x B C)
by (*rule sym*, *rule fundamental-groupin-mult-assoc*[*OF A-in B-in C-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-groupin-space* X x
show *fundamental-groupin-mult* X x (*fundamental-groupin-one* X x) $A = A$
by (*rule fundamental-groupin-mult-one-left*[*OF x-in A-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-groupin-space* X x
show *fundamental-groupin-mult* X x A (*fundamental-groupin-one* X x) $= A$
by (*rule fundamental-groupin-mult-one-right*[*OF x-in A-in*])
next
fix A
assume *A-in*: $A \in$ *fundamental-groupin-space* X x
show *fundamental-groupin-mult* X x (*fundamental-groupin-inv* X x A) $A =$ *fundamental-groupin-one* X x
by (*rule fundamental-groupin-mult-inv-left*[*OF x-in A-in*])
next
fix A

assume *A-in*: $A \in$ *fundamental-groupin-space X x*
show *fundamental-groupin-mult X x A* (*fundamental-groupin-inv X x A*) = *fundamental-groupin-one X x*
by (*rule fundamental-groupin-mult-inv-right*[*OF x-in A-in*])
qed

lemma *fundamental-groupin-map-carrier-group-hom*:
assumes *x-in*: $x \in$ *topspace X*
and *h*: *continuous-map X Y f*
and *fx*: $f\ x = y$
shows *carrier-group-hom*
(*fundamental-groupin-space X x*)
(*fundamental-groupin-mult X x*)
(*fundamental-groupin-one X x*)
(*fundamental-groupin-inv X x*)
(*fundamental-groupin-space Y y*)
(*fundamental-groupin-mult Y y*)
(*fundamental-groupin-one Y y*)
(*fundamental-groupin-inv Y y*)
(*fundamental-groupin-map X x Y y f*)
proof −
have *y-in*: $y \in$ *topspace Y*
using *x-in h fx* **unfolding** *continuous-map-def* **by** *blast*
show *?thesis*
proof (*rule carrier-group-hom.intro*)
show *carrier-group*
(*fundamental-groupin-space X x*)
(*fundamental-groupin-mult X x*)
(*fundamental-groupin-one X x*)
(*fundamental-groupin-inv X x*)
by (*rule fundamental-groupin-carrier-group*[*OF x-in*])
show *carrier-group*
(*fundamental-groupin-space Y y*)
(*fundamental-groupin-mult Y y*)
(*fundamental-groupin-one Y y*)
(*fundamental-groupin-inv Y y*)
by (*rule fundamental-groupin-carrier-group*[*OF y-in*])
show *carrier-group-hom-axioms*
(*fundamental-groupin-space X x*)
(*fundamental-groupin-mult X x*)
(*fundamental-groupin-space Y y*)
(*fundamental-groupin-mult Y y*)
(*fundamental-groupin-map X x Y y f*)
proof (*rule carrier-group-hom-axioms.intro*)
fix *A*
assume *A-in*: $A \in$ *fundamental-groupin-space X x*
show *fundamental-groupin-map X x Y y f A* $\in$ *fundamental-groupin-space Y y*
by (*rule fundamental-groupin-map-in-space*[*OF A-in h fx*])

```
    next
      fix A B
      assume A-in: A ∈ fundamental-groupin-space X x
        and B-in: B ∈ fundamental-groupin-space X x
      show fundamental-groupin-map X x Y y f (fundamental-groupin-mult X x A
B) =
          fundamental-groupin-mult Y y
            (fundamental-groupin-map X x Y y f A)
            (fundamental-groupin-map X x Y y f B)
        by (rule fundamental-groupin-map-mult[OF A-in B-in h fx])
    qed
  qed
qed

lemma loopin-space-top-of-set [simp]:
  loopin-space (top-of-set S) x = loop-space S x
  unfolding loopin-space-def loop-space-def
   by (auto simp: pathin-canon-iff pathstart-def pathfinish-def path-image-def im-
age-subset-iff-funcset)

lemma loopin-class-top-of-set [simp]:
  loopin-class (top-of-set S) x p = loop-class S x p
  unfolding loopin-class-def loop-class-def by simp

lemma fundamental-groupin-space-top-of-set [simp]:
  fundamental-groupin-space (top-of-set S) x = fundamental-group-space S x
  unfolding fundamental-groupin-space-def fundamental-group-space-def by simp

lemma fundamental-groupin-one-top-of-set [simp]:
  fundamental-groupin-one (top-of-set S) x = fundamental-group-one S x
  unfolding fundamental-groupin-one-def fundamental-group-one-def by simp

lemma loopin-image-eq-loop-image [simp]:
  loopin-image h p = loop-image h p
  unfolding loopin-image-def loop-image-def by simp

end
theory Pushout-Scaffold
  imports Equivalence-Quotients
begin
```

13 Pushout-style quotients of disjoint sums

The pushout of two maps is first presented here as a quotient of the disjoint sum by the obvious glue relation. This algebraic infrastructure is independent of any topology; the topological quotient is added later, once the point-set infrastructure is in place.

```
inductive pushout-rel ::
```

```
  ('c => 'a) => ('c => 'b) => ('a + 'b) => ('a + 'b) => bool
  for f :: 'c => 'a and g :: 'c => 'b
where
  refl [intro!, simp]: pushout-rel f g x x
| sym: pushout-rel f g x y ==> pushout-rel f g y x
| trans: pushout-rel f g x y ==> pushout-rel f g y z ==> pushout-rel f g x z
| glue [intro]: pushout-rel f g (Inl (f c)) (Inr (g c))

interpretation pushout-equiv: equivalence-relation pushout-rel f g
proof
  show pushout-rel f g x x for x
    by (rule pushout-rel.refl)
next
  show pushout-rel f g x y ==> pushout-rel f g y x for x y
    by (rule pushout-rel.sym)
next
  show pushout-rel f g x y ==> pushout-rel f g y z ==> pushout-rel f g x z for x
y z
    by (rule pushout-rel.trans)
qed

definition pushout-class ::
  ('c => 'a) => ('c => 'b) => ('a + 'b) => ('a + 'b) set
where
  pushout-class f g x = equiv-class (pushout-rel f g) x

definition pushout-space ::
  ('c => 'a) => ('c => 'b) => ('a + 'b) set set
where
  pushout-space f g = quotient-space (pushout-rel f g)

definition pushout-inl ::
  ('c => 'a) => ('c => 'b) => 'a => ('a + 'b) set
where
  pushout-inl f g a = pushout-class f g (Inl a)

definition pushout-inr ::
  ('c => 'a) => ('c => 'b) => 'b => ('a + 'b) set
where
  pushout-inr f g b = pushout-class f g (Inr b)

lemma pushout-class-eq-iff:
  pushout-class f g x = pushout-class f g y ⟷ pushout-rel f g x y
  unfolding pushout-class-def
  by (rule pushout-equiv.equiv-class-eq-iff)

lemma pushout-inl-in-space [intro]:
  pushout-inl f g a ∈ pushout-space f g
  unfolding pushout-inl-def pushout-space-def pushout-class-def quotient-space-def
```

by *blast*

lemma *pushout-inr-in-space* [*intro*]:
pushout-inr f g b ∈ *pushout-space f g*
unfolding *pushout-inr-def pushout-space-def pushout-class-def quotient-space-def*
by *blast*

lemma *pushout-glue*:
pushout-inl f g (*f c*) = *pushout-inr f g* (*g c*)
by (*simp add*: *pushout-class-eq-iff pushout-inl-def pushout-inr-def pushout-rel.glue*)

definition *pushout-case-compatible* ::
('*c* => '*a*) => ('*c* => '*b*) => ('*a* => '*d*) => ('*b* => '*d*) => *bool*
where
pushout-case-compatible f g left right ⟷ (∀ *c*. *left* (*f c*) = *right* (*g c*))

lemma *pushout-rel-case-cong*:
assumes *compat*: *pushout-case-compatible f g left right*
and *rel*: *pushout-rel f g x y*
shows *case-sum left right x* = *case-sum left right y*
using *rel*
proof (*induction rule*: *pushout-rel.induct*)
case (*refl x*)
then show *?case* **by** *simp*
next
case (*sym x y*)
then show *?case* **by** *simp*
next
case (*trans x y z*)
then show *?case* **by** *simp*
next
case (*glue c*)
then show *?case*
using *compat* **unfolding** *pushout-case-compatible-def* **by** *simp*
qed

definition *pushout-rec* ::
('*c* => '*a*) => ('*c* => '*b*) => ('*a* => '*d*) => ('*b* => '*d*) => ('*a* + '*b*) *set* => '*d*
where
pushout-rec f g left right X =
(*THE z*. ∃ *x*. *X* = *pushout-class f g x* ∧ *z* = *case-sum left right x*)

lemma *pushout-rec-well-defined*:
assumes *compat*: *pushout-case-compatible f g left right*
and *X-in*: *X* ∈ *pushout-space f g*
shows ∃! *z*. ∃ *x*. *X* = *pushout-class f g x* ∧ *z* = *case-sum left right x*
proof −

```
  from X-in obtain rep where X-rep: X = pushout-class f g rep
    unfolding pushout-space-def quotient-space-def pushout-class-def by blast
  show ?thesis
  proof (rule ex1I[of - case-sum left right rep])
    show ∃ x. X = pushout-class f g x ∧ case-sum left right rep = case-sum left
right x
      by (rule exI[where x = rep], simp add: X-rep)
  next
    fix z
    assume hz: ∃ x. X = pushout-class f g x ∧ z = case-sum left right x
    from hz obtain x where
      x: X = pushout-class f g x z = case-sum left right x
      by blast
    have pushout-rel f g x rep
      using X-rep x(1) by (simp add: pushout-class-eq-iff)
    then have rel-rep-x: pushout-rel f g rep x
      by (rule pushout-rel.sym)
    from compat rel-rep-x have case-sum left right rep = case-sum left right x
      by (rule pushout-rel-case-cong)
    with x show z = case-sum left right rep
      by simp
  qed
qed

lemma pushout-rec-inl:
  assumes compat: pushout-case-compatible f g left right
  shows pushout-rec f g left right (pushout-inl f g a) = left a
proof −
  have uniq:
    ∃! z. ∃ x. pushout-inl f g a = pushout-class f g x ∧ z = case-sum left right x
    using pushout-rec-well-defined[OF compat pushout-inl-in-space[of f g a]] .
  have witness:
    ∃ x. pushout-inl f g a = pushout-class f g x ∧ left a = case-sum left right x
    unfolding pushout-inl-def by (intro exI[of - Inl a]) simp
  show ?thesis
    unfolding pushout-rec-def
    by (rule the1-equality[OF uniq witness])
qed

lemma pushout-rec-inr:
  assumes compat: pushout-case-compatible f g left right
  shows pushout-rec f g left right (pushout-inr f g b) = right b
proof −
  have uniq:
    ∃! z. ∃ x. pushout-inr f g b = pushout-class f g x ∧ z = case-sum left right x
    using pushout-rec-well-defined[OF compat pushout-inr-in-space[of f g b]] .
  have witness:
    ∃ x. pushout-inr f g b = pushout-class f g x ∧ right b = case-sum left right x
    unfolding pushout-inr-def by (intro exI[of - Inr b]) simp
```

show *?thesis*
unfolding *pushout-rec-def*
by (*rule the1-equality*[*OF uniq witness*])
qed

end
theory *Seifert-Van-Kampen-Scaffold*
imports *Pushout-Scaffold Carrier-Amalgamated-Free-Product*
begin

14 Conditional Seifert–van Kampen interface

This theory packages a general quotient-level Seifert–van Kampen argument behind explicit encode/decode obligations. Once a decoding map factors through the full amalgamation quotient, and once the usual round-trip laws are available, the quotient-level bijection follows abstractly.

locale *svk-encode-decode-interface* =
fixes *i1* :: *'h* => *'a*::*group-add*
and *i2* :: *'h* => *'b*::*group-add*
and *encode* :: *'pi1-pushout* =>
(*'a*, *'b*) *free-product-word*
and *decode* :: (*'a*, *'b*) *free-product-word* => *'pi1-pushout*
assumes *decode-respects*:
full-amalg-equiv i1 i2 u v ==> *decode u* = *decode v*
and *decode-encode*:
decode (*encode x*) = *x*
and *encode-decode*:
full-amalg-equiv i1 i2 (*encode* (*decode w*)) *w*
begin

definition *svk-quotient-map* ::
'pi1-pushout => ((*'a*, *'b*) *free-product-word*) *set*
where
svk-quotient-map x = *full-amalg-class i1 i2* (*encode x*)

lemma *svk-quotient-map-in-space*:
svk-quotient-map x ∈ *full-amalgamated-free-product-space i1 i2*
unfolding *svk-quotient-map-def* **by** (*rule full-amalg-class-in-space*)

lemma *svk-quotient-map-injective*:
assumes *svk-quotient-map x* = *svk-quotient-map y*
shows *x* = *y*
proof −
have *full-amalg-equiv i1 i2* (*encode x*) (*encode y*)
using *assms*
unfolding *svk-quotient-map-def*
by (*simp add*: *full-amalg-class-eq-iff*)
then have *decode* (*encode x*) = *decode* (*encode y*)

```
    by (rule decode-respects)
  then show ?thesis
    by (simp add: decode-encode)
qed

lemma svk-quotient-map-surjective:
  assumes Q ∈ full-amalgamated-free-product-space i1 i2
  shows ∃ x. svk-quotient-map x = Q
proof −
  from assms obtain w where Q-rep: Q = full-amalg-class i1 i2 w
          unfolding  full-amalgamated-free-product-space-def  quotient-space-def
full-amalg-class-def
    by blast
  have svk-quotient-map (decode w) = Q
    unfolding svk-quotient-map-def Q-rep
    by (simp add: full-amalg-class-eq-iff encode-decode)
  then show ?thesis
    by blast
qed

theorem seifert-van-kampen-bij-betw:
  bij-betw svk-quotient-map UNIV (full-amalgamated-free-product-space i1 i2)
  unfolding bij-betw-def
proof
  show inj-on svk-quotient-map UNIV
    unfolding inj-on-def
    using svk-quotient-map-injective by blast
next
  show svk-quotient-map ‘ UNIV = full-amalgamated-free-product-space i1 i2
  proof
    show svk-quotient-map ‘ UNIV ⊆ full-amalgamated-free-product-space i1 i2
      using svk-quotient-map-in-space by blast
  next
    show full-amalgamated-free-product-space i1 i2 ⊆ svk-quotient-map ‘ UNIV
      using svk-quotient-map-surjective by blast
  qed
qed

end

locale carrier-svk-encode-decode-interface =
  fixes G1 :: 'a set
    and G2 :: 'b set
    and H :: 'h set
    and i1 :: 'h => 'a
    and i2 :: 'h => 'b
    and mult1 :: 'a => 'a => 'a
    and one1 :: 'a
    and mult2 :: 'b => 'b => 'b
```

```
    and one2 :: 'b
    and encode :: 'pi1-pushout => ('a, 'b) free-product-word
    and decode :: ('a, 'b) free-product-word => 'pi1-pushout
  assumes encode-in-space:
      fpw-in-space G1 G2 (encode x)
    and decode-respects:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v ==> decode
u = decode v
    and decode-encode:
      decode (encode x) = x
    and encode-decode:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (encode (decode
w)) w
begin

definition carrier-svk-quotient-map ::
  'pi1-pushout => (('a, 'b) free-product-word) set
where
  carrier-svk-quotient-map x =
    carrier-full-amalg-class G1 G2 H i1 i2 mult1 one1 mult2 one2 (encode x)

lemma carrier-svk-quotient-map-in-space:
  carrier-svk-quotient-map x ∈
     carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
  unfolding carrier-svk-quotient-map-def
  by (rule carrier-full-amalg-class-in-space[OF encode-in-space])

lemma carrier-svk-quotient-map-injective:
  assumes carrier-svk-quotient-map x = carrier-svk-quotient-map y
  shows x = y
proof −
  have carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (encode x)
(encode y)
    using assms
    unfolding carrier-svk-quotient-map-def
    by (simp add: carrier-full-amalg-class-eq-iff)
  then have decode (encode x) = decode (encode y)
    by (rule decode-respects)
  then show ?thesis
    by (simp add: decode-encode)
qed

lemma carrier-svk-quotient-map-surjective:
  assumes
     Q ∈ carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1
mult2 one2
  shows ∃ x. carrier-svk-quotient-map x = Q
proof −
```

```
  from assms obtain w where
      w ∈ carrier-fpw-space G1 G2
    and Q-rep: Q = carrier-full-amalg-class G1 G2 H i1 i2 mult1 one1 mult2 one2
w
    unfolding carrier-full-amalgamated-free-product-space-def
    by blast
  have carrier-svk-quotient-map (decode w) = Q
    unfolding carrier-svk-quotient-map-def Q-rep
    by (simp add: carrier-full-amalg-class-eq-iff encode-decode)
  then show ?thesis
    by blast
qed

theorem carrier-seifert-van-kampen-bij-betw:
  bij-betw carrier-svk-quotient-map UNIV
    (carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2)
  unfolding bij-betw-def
proof
  show inj-on carrier-svk-quotient-map UNIV
    unfolding inj-on-def
    using carrier-svk-quotient-map-injective by blast
next
  show carrier-svk-quotient-map ‘ UNIV =
      carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
  proof
    show carrier-svk-quotient-map ‘ UNIV ⊆
       carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
      using carrier-svk-quotient-map-in-space by blast
  next
     show carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1
mult2 one2
        ⊆ carrier-svk-quotient-map ‘ UNIV
      using carrier-svk-quotient-map-surjective by blast
  qed
qed

end

end
theory Classical-Seifert-Van-Kampen
  imports
    Carrier-Amalgamated-Free-Product-Eval
    Explicit-Fundamental-Group-Scaffold
    Seifert-Van-Kampen-Scaffold
begin
```

15 Classical Seifert–van Kampen for open unions

This theory specializes the general encode/decode interface to the classical open-cover hypotheses. Its long middle portion constructs encodings of loops by subdividing them into pieces that lie in U or V, proves invariance under refinement and homotopy, and then packages the resulting quotient as the carrier-based amalgamated free product of the three relevant fundamental groups.

lemma *path-top-of-setI*:
  assumes *path p*
    and *path-image p* $\subseteq$ *S*
  shows *pathin* (*top-of-set S*) *p*
  using *assms*
  by (*auto simp*: *pathin-canon-iff path-image-def image-subset-iff-funcset*)

locale *classical-svk-setup* =
  fixes *U* :: *'a*::*topological-space set*
    and *V* :: *'a set*
    and *x0* :: *'a*
  assumes *U-open*: *open U*
    and *V-open*: *open V*
    and *x0-in*: *x0* $\in$ *U* $\cap$ *V*
    and *UV-path-connected*: *path-connected* (*U* $\cap$ *V*)
begin

abbreviation *W* **where** *W* $\equiv$ *U* $\cup$ *V*

abbreviation *G1* **where** *G1* $\equiv$ *fundamental-group-space U x0*
abbreviation *G2* **where** *G2* $\equiv$ *fundamental-group-space V x0*
abbreviation *H* **where** *H* $\equiv$ *fundamental-group-space* (*U* $\cap$ *V*) *x0*

abbreviation *mult1* **where** *mult1* $\equiv$ *fundamental-group-mult U x0*
abbreviation *one1* **where** *one1* $\equiv$ *fundamental-group-one U x0*
abbreviation *mult2* **where** *mult2* $\equiv$ *fundamental-group-mult V x0*
abbreviation *one2* **where** *one2* $\equiv$ *fundamental-group-one V x0*
abbreviation *multW* **where** *multW* $\equiv$ *fundamental-group-mult W x0*
abbreviation *oneW* **where** *oneW* $\equiv$ *fundamental-group-one W x0*

abbreviation *i1* **where** *i1* $\equiv$ *fundamental-group-map* (*U* $\cap$ *V*) *x0 U x0 id*
abbreviation *i2* **where** *i2* $\equiv$ *fundamental-group-map* (*U* $\cap$ *V*) *x0 V x0 id*
abbreviation *j1* **where** *j1* $\equiv$ *fundamental-group-map U x0 W x0 id*
abbreviation *j2* **where** *j2* $\equiv$ *fundamental-group-map V x0 W x0 id*

15.1 Carrier-side setup

The first part of the theory fixes the inclusion-induced maps between the three fundamental groups and packages them into the carrier-side evaluation locale used by the later decode map. This isolates the algebraic compatibility

conditions that are immediate from the open-union inclusions.

lemma *x0-in-U* [*simp*]: *x0* ∈ *U*
  using *x0-in* **by** *blast*

lemma *x0-in-V* [*simp*]: *x0* ∈ *V*
  using *x0-in* **by** *blast*

lemma *x0-in-W* [*simp*]: *x0* ∈ *W*
  using *x0-in* **by** *blast*

lemma *x0-in-UV* [*simp*]: *x0* ∈ *U* ∩ *V*
  using *x0-in* **by** *blast*

lemma *i1-in-G1*:
  assumes *h* ∈ *H*
  shows *i1 h* ∈ *G1*
  by (*rule fundamental-group-map-in-space*[*OF assms*]) *auto*

lemma *i2-in-G2*:
  assumes *h* ∈ *H*
  shows *i2 h* ∈ *G2*
  by (*rule fundamental-group-map-in-space*[*OF assms*]) *auto*

lemma *union-fundamental-group-maps-agree*:
  assumes *h-in*: *h* ∈ *H*
  shows *j1* (*i1 h*) = *j2* (*i2 h*)
proof −
  have *left-comp*:
    fundamental-group-map U x0 W x0 id
      (*fundamental-group-map* (*U* ∩ *V*) *x0 U x0 id h*) =
      fundamental-group-map (*U* ∩ *V*) *x0 W x0* (*id* ∘ *id*) *h*
    by (*rule fundamental-group-map-compose*[*OF h-in*]) *auto*
  have *right-comp*:
    fundamental-group-map V x0 W x0 id
      (*fundamental-group-map* (*U* ∩ *V*) *x0 V x0 id h*) =
      fundamental-group-map (*U* ∩ *V*) *x0 W x0* (*id* ∘ *id*) *h*
    by (*rule fundamental-group-map-compose*[*OF h-in*]) *auto*
  show *?thesis*
    using *left-comp right-comp* **by** *simp*
qed

lemma *decode-locale*:
  carrier-full-amalg-word-eval
    G1 mult1 one1 (*fundamental-group-inv U x0*)
    G2 mult2 one2 (*fundamental-group-inv V x0*)
    H i1 i2
    (*fundamental-group-space W x0*) *multW oneW* (*fundamental-group-inv W x0*)
    j1 j2
proof (*rule carrier-full-amalg-word-eval.intro*)

show *carrier-group*
(*fundamental-group-space U x0*)
(*fundamental-group-mult U x0*)
(*fundamental-group-one U x0*)
(*fundamental-group-inv U x0*)
by (*rule fundamental-group-carrier-group*[*OF x0-in-U*])
show *carrier-group*
(*fundamental-group-space V x0*)
(*fundamental-group-mult V x0*)
(*fundamental-group-one V x0*)
(*fundamental-group-inv V x0*)
by (*rule fundamental-group-carrier-group*[*OF x0-in-V*])
show *carrier-group*
(*fundamental-group-space W x0*)
(*fundamental-group-mult W x0*)
(*fundamental-group-one W x0*)
(*fundamental-group-inv W x0*)
by (*rule fundamental-group-carrier-group*[*OF x0-in-W*])
show *carrier-group-hom*
(*fundamental-group-space U x0*)
(*fundamental-group-mult U x0*)
(*fundamental-group-one U x0*)
(*fundamental-group-inv U x0*)
(*fundamental-group-space W x0*)
(*fundamental-group-mult W x0*)
(*fundamental-group-one W x0*)
(*fundamental-group-inv W x0*)
(*fundamental-group-map U x0 W x0 id*)
by (*rule fundamental-group-map-carrier-group-hom*[*OF x0-in-U*]) *auto*
show *carrier-group-hom*
(*fundamental-group-space V x0*)
(*fundamental-group-mult V x0*)
(*fundamental-group-one V x0*)
(*fundamental-group-inv V x0*)
(*fundamental-group-space W x0*)
(*fundamental-group-mult W x0*)
(*fundamental-group-one W x0*)
(*fundamental-group-inv W x0*)
(*fundamental-group-map V x0 W x0 id*)
by (*rule fundamental-group-map-carrier-group-hom*[*OF x0-in-V*]) *auto*
show *carrier-full-amalg-word-eval-axioms G1 G2 H i1 i2 j1 j2*
proof (*rule carrier-full-amalg-word-eval-axioms.intro*)
show $h \in H \Longrightarrow i1\ h \in G1$ **for** *h*
by (*rule i1-in-G1*)
show $h \in H \Longrightarrow i2\ h \in G2$ **for** *h*
by (*rule i2-in-G2*)
show $h \in H \Longrightarrow j1\ (i1\ h) = j2\ (i2\ h)$ **for** *h*
by (*rule union-fundamental-group-maps-agree*)
qed

qed

interpretation *decode*:
carrier-full-amalg-word-eval
G1 mult1 one1 fundamental-group-inv U x0
G2 mult2 one2 fundamental-group-inv V x0
H i1 i2
fundamental-group-space W x0 multW oneW fundamental-group-inv W x0
j1 j2
by (*rule decode-locale*)

abbreviation *svk-word-eval* **where** *svk-word-eval* ≡ *decode.carrier-full-amalg-eval*
abbreviation *svk-decode* **where** *svk-decode* ≡ *decode.carrier-full-amalg-decode*

lemma *svk-decode-in-space*:
svk-decode w ∈ *fundamental-group-space W x0*
by (*rule decode.carrier-full-amalg-decode-in-carrier*)

lemma *svk-decode-respects*:
assumes *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
shows *svk-decode u* = *svk-decode v*
using *assms* **by** (*rule decode.carrier-full-amalg-decode-respects*)

lemma *svk-decode-eq-eval*:
assumes *fpw-in-space G1 G2 w*
shows *svk-decode w* = *svk-word-eval w*
using *assms* **by** (*rule decode.carrier-full-amalg-decode-eq-eval*)

lemma *carrier-full-amalg-equiv-left-context*:
assumes *rel*: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
and *a-in*: *a* ∈ *G1*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*WordLeft a u*) (*WordLeft a v*)
using *rel a-in*
proof (*induction rule*: *carrier-full-amalg-equiv.induct*)
case (*refl w*)
then show *?case* **by** *simp*
next
case (*sym u v*)
then show *?case*
by (*meson carrier-full-amalg-equiv.sym*)
next
case (*trans u v w*)
then show *?case*
by (*meson carrier-full-amalg-equiv.trans*)
next
case (*of-amalg u v*)
then show *?case*
by (*meson carrier-amalgam-equiv.left-context carrier-full-amalg-equiv.of-amalg*)

next
case (*of-reduction u v*)
then show *?case*
by (*meson carrier-fpw-reduction-left-context carrier-full-amalg-equiv.of-reduction*)
qed

lemma *carrier-full-amalg-equiv-right-context*:
assumes *rel*: *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
and *b-in*: *b* ∈ *G2*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*WordRight b u*) (*WordRight b v*)
using *rel b-in*
proof (*induction rule*: *carrier-full-amalg-equiv.induct*)
case (*refl w*)
then show *?case* **by** *simp*
next
case (*sym u v*)
then show *?case*
by (*meson carrier-full-amalg-equiv.sym*)
next
case (*trans u v w*)
then show *?case*
by (*meson carrier-full-amalg-equiv.trans*)
next
case (*of-amalg u v*)
then show *?case*
by (*meson carrier-amalgam-equiv.right-context carrier-full-amalg-equiv.of-amalg*)
next
case (*of-reduction u v*)
then show *?case*
by (*meson carrier-fpw-reduction-right-context carrier-full-amalg-equiv.of-reduction*)
qed

lemma *carrier-full-amalg-equiv-left-pair-eq*:
assumes *a-in*: *a* ∈ *G1*
and *b-in*: *b* ∈ *G1*
and *ab-in*: *mult1 a b* ∈ *G1*
and *c-in*: *c* ∈ *G1*
and *d-in*: *d* ∈ *G1*
and *cd-in*: *mult1 c d* ∈ *G1*
and *rest-in*: *fpw-in-space G1 G2 rest*
and *eq*: *mult1 a b* = *mult1 c d*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*WordLeft a* (*WordLeft b rest*))
(*WordLeft c* (*WordLeft d rest*))
proof −

```
  have step-left:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordLeft a (WordLeft b rest))
      (WordLeft (mult1 a b) rest)
  proof (rule carrier-fpw-reduction-step.combine-left)
    show a ∈ G1 by (rule a-in)
    show b ∈ G1 by (rule b-in)
    show mult1 a b ∈ G1 by (rule ab-in)
    show fpw-in-space G1 G2 rest by (rule rest-in)
  qed
  have red-left:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordLeft a (WordLeft b rest))
      (WordLeft (mult1 a b) rest)
    by (rule carrier-fpw-reduction.step[OF step-left])
  have step-right:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordLeft c (WordLeft d rest))
      (WordLeft (mult1 c d) rest)
  proof (rule carrier-fpw-reduction-step.combine-left)
    show c ∈ G1 by (rule c-in)
    show d ∈ G1 by (rule d-in)
    show mult1 c d ∈ G1 by (rule cd-in)
    show fpw-in-space G1 G2 rest by (rule rest-in)
  qed
  have red-right:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordLeft c (WordLeft d rest))
      (WordLeft (mult1 c d) rest)
    by (rule carrier-fpw-reduction.step[OF step-right])
  have rel-left:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft a (WordLeft b rest))
      (WordLeft (mult1 a b) rest)
    by (rule carrier-full-amalg-equiv.of-reduction[OF red-left])
  have rel-right:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft c (WordLeft d rest))
      (WordLeft (mult1 a b) rest)
    using eq by (simp add: carrier-full-amalg-equiv.of-reduction[OF red-right])
  show ?thesis
    by (rule carrier-full-amalg-equiv.trans[OF rel-left])
       (rule carrier-full-amalg-equiv.sym[OF rel-right])
qed

lemma carrier-full-amalg-equiv-right-pair-eq:
  assumes a-in: a ∈ G2
    and b-in: b ∈ G2
    and ab-in: mult2 a b ∈ G2
```

```
    and c-in: c ∈ G2
    and d-in: d ∈ G2
    and cd-in: mult2 c d ∈ G2
    and rest-in: fpw-in-space G1 G2 rest
    and eq: mult2 a b = mult2 c d
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (WordRight a (WordRight b rest))
    (WordRight c (WordRight d rest))
proof −
  have step-left:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordRight a (WordRight b rest))
      (WordRight (mult2 a b) rest)
  proof (rule carrier-fpw-reduction-step.combine-right)
    show a ∈ G2 by (rule a-in)
    show b ∈ G2 by (rule b-in)
    show mult2 a b ∈ G2 by (rule ab-in)
    show fpw-in-space G1 G2 rest by (rule rest-in)
  qed
  have red-left:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordRight a (WordRight b rest))
      (WordRight (mult2 a b) rest)
    by (rule carrier-fpw-reduction.step[OF step-left])
  have step-right:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordRight c (WordRight d rest))
      (WordRight (mult2 c d) rest)
  proof (rule carrier-fpw-reduction-step.combine-right)
    show c ∈ G2 by (rule c-in)
    show d ∈ G2 by (rule d-in)
    show mult2 c d ∈ G2 by (rule cd-in)
    show fpw-in-space G1 G2 rest by (rule rest-in)
  qed
  have red-right:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordRight c (WordRight d rest))
      (WordRight (mult2 c d) rest)
    by (rule carrier-fpw-reduction.step[OF step-right])
  have rel-left:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight a (WordRight b rest))
      (WordRight (mult2 a b) rest)
    by (rule carrier-full-amalg-equiv.of-reduction[OF red-left])
  have rel-right:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight c (WordRight d rest))
      (WordRight (mult2 a b) rest)
    using eq by (simp add: carrier-full-amalg-equiv.of-reduction[OF red-right])
```

show *?thesis*
by (*rule carrier-full-amalg-equiv.trans*[*OF rel-left*])
(*rule carrier-full-amalg-equiv.sym*[*OF rel-right*])
qed

lemma *loop-subdivision-by-cover*:
assumes *p-loop*: $p \in$ *loop-space W x0*
shows $\exists\, n{::}nat.\ 0 < n \wedge$
$(\forall\, i{<}n.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U \vee$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq V$)
proof −
have *p-path*: *path p*
using *p-loop* **unfolding** *loop-space-def* **by** *auto*
have *p-img*: *path-image p* $\subseteq$ *W*
using *p-loop* **unfolding** *loop-space-def* **by** *auto*
have *p-pathin*: *pathin* (*top-of-set W*) *p*
by (*rule path-top-of-setI*[*OF p-path p-img*])
have *cover*: *p* ‘ $\{0..1\} \subseteq \bigcup \{U,\ V\}$
using *p-img* **by** (*auto simp*: *path-image-def*)
have *open-cover*: *openin* (*top-of-set W*) *S* **if** $S \in \{U,\ V\}$ **for** *S*
using *that U-open V-open* **by** (*auto simp*: *openin-open*)
from *pathin-subdivision-open-cover*[*OF p-pathin cover open-cover*]
obtain *n* :: *nat* **where** *n-pos*: $0 < n$
and *n-cover*:
$\forall\, i{<}n.\ \exists\, S{\in}\{U,\ V\}.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq S$
by *blast*
show *?thesis*
using *n-pos n-cover* **by** *auto*
qed

definition *connector* :: *'a* ⇒ *real* ⇒ *'a* **where**
connector a =
(*if a* = *x0 then* (λ*-. x0*)
else (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*))

lemma *connector-x0* [*simp*]:
connector x0 = (λ*-. x0*)
unfolding *connector-def* **by** *simp*

lemma *connector-witness*:
assumes *a-in*: *a* ∈ *U* ∩ *V*
shows ∃ *p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*
proof (*cases a* = *x0*)
case *True*
have *path* (λ*-. x0*) ∧

path-image (λ-. *x0*) ⊆ *U* ∩ *V* ∧
pathstart (λ-. *x0*) = *x0* ∧
pathfinish (λ-. *x0*) = *a*
using *a-in True* **by** (*auto simp*: *path-def path-image-def pathstart-def pathfinish-def*)
then show *?thesis* **by** *blast*
next
case *False*
then show *?thesis*
using *UV-path-connected x0-in-UV a-in* **unfolding** *path-connected-def* **by** *blast*
qed

lemma *connector-path*:
assumes *a-in*: *a* ∈ *U* ∩ *V*
shows *path* (*connector a*)
proof (*cases a* = *x0*)
case *True*
then show *?thesis*
by (*simp add*: *connector-def path-def*)
next
case *False*
have *some*:
path (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ∧
path-image (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ⊆ *U* ∩ *V* ∧
pathstart (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *x0* ∧
pathfinish (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *a*
by (*rule someI-ex*[*OF connector-witness*[*OF a-in*]])
then show *?thesis*
using *False* **by** (*simp add*: *connector-def*)
qed

lemma *connector-image-subset*:
assumes *a-in*: *a* ∈ *U* ∩ *V*
shows *path-image* (*connector a*) ⊆ *U* ∩ *V*
proof (*cases a* = *x0*)
case *True*
then show *?thesis*
using *a-in* **by** (*auto simp*: *connector-def path-image-def*)
next
case *False*
have *some*:
path (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ∧
path-image (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ⊆ *U* ∩ *V* ∧

pathstart (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *x0* ∧
pathfinish (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *a*
by (*rule someI-ex*[*OF connector-witness*[*OF a-in*]])
then show *?thesis*
using *False* **by** (*simp add*: *connector-def*)
qed

lemma *connector-start*:
assumes *a-in*: *a* ∈ *U* ∩ *V*
shows *pathstart* (*connector a*) = *x0*
proof (*cases a* = *x0*)
case *True*
then show *?thesis*
by (*simp add*: *connector-def pathstart-def*)
next
case *False*
have *some*:
path (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ∧
path-image (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ⊆ *U* ∩ *V* ∧
pathstart (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *x0* ∧
pathfinish (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *a*
by (*rule someI-ex*[*OF connector-witness*[*OF a-in*]])
then show *?thesis*
using *False* **by** (*simp add*: *connector-def*)
qed

lemma *connector-finish*:
assumes *a-in*: *a* ∈ *U* ∩ *V*
shows *pathfinish* (*connector a*) = *a*
proof (*cases a* = *x0*)
case *True*
then show *?thesis*
by (*simp add*: *connector-def pathfinish-def*)
next
case *False*
have *some*:
path (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ∧
path-image (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) ⊆ *U* ∩ *V* ∧
pathstart (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧ *pathfinish p* = *a*) = *x0* ∧
pathfinish (*SOME p. path p* ∧ *path-image p* ⊆ *U* ∩ *V* ∧ *pathstart p* = *x0* ∧

pathfinish p = a) = a
by (*rule someI-ex*[*OF connector-witness*[*OF a-in*]])
then show *?thesis*
using *False* **by** (*simp add*: *connector-def*)
qed

lemmas *connector-spec = connector-path connector-image-subset connector-start connector-finish*

definition *segment-loop* :: (*real* ⇒ *'a*) ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *'a* **where**
segment-loop p u v =
(*connector* (*p u*) +++ *subpathin u v p*) +++ *reversepath* (*connector* (*p v*))

lemma *segment-loop-in-set*:
assumes *p-path*: *path p*
and *p-image*: *path-image p* ⊆ *W*
and *uv01*: *u* ∈ {*0..1*} *v* ∈ {*0..1*}
and *puv-in*: *p u* ∈ *U* ∩ *V p v* ∈ *U* ∩ *V*
and *conn-u*: *path-image* (*connector* (*p u*)) ⊆ *S*
and *conn-v*: *path-image* (*connector* (*p v*)) ⊆ *S*
and *seg-in*: *subpathin u v p* ' {*0..1*} ⊆ *S*
and *x0S*: *x0* ∈ *S*
shows *segment-loop p u v* ∈ *loop-space S x0*
proof −
have *cu-path*: *path* (*connector* (*p u*))
using *puv-in connector-spec*(*1*)[*of p u*] **by** *blast*
have *cv-path*: *path* (*connector* (*p v*))
using *puv-in connector-spec*(*1*)[*of p v*] **by** *blast*
have *cu-start*: *pathstart* (*connector* (*p u*)) = *x0*
using *puv-in connector-spec*(*3*)[*of p u*] **by** *blast*
have *cu-finish*: *pathfinish* (*connector* (*p u*)) = *p u*
using *puv-in connector-spec*(*4*)[*of p u*] **by** *blast*
have *cv-start*: *pathstart* (*connector* (*p v*)) = *x0*
using *puv-in connector-spec*(*3*)[*of p v*] **by** *blast*
have *cv-finish*: *pathfinish* (*connector* (*p v*)) = *p v*
using *puv-in connector-spec*(*4*)[*of p v*] **by** *blast*
have *p-pathin*: *pathin* (*top-of-set W*) *p*
by (*rule path-top-of-setI*[*OF p-path p-image*])
have *subpath-pathin*: *pathin* (*top-of-set W*) (*subpathin u v p*)
by (*rule pathin-subpathin*[*OF p-pathin uv01*])
have *subpath-path*: *path* (*subpathin u v p*)
using *subpath-pathin* **by** (*simp add*: *pathin-canon-iff*)
have *subpath-start*: *pathstart* (*subpathin u v p*) = *p u*
by (*simp add*: *pathstart-def subpathin-def*)
have *subpath-finish*: *pathfinish* (*subpathin u v p*) = *p v*
by (*simp add*: *pathfinish-def subpathin-def*)
have *first-join*: *path* (*connector* (*p u*) +++ *subpathin u v p*)
using *cu-path subpath-path cu-finish subpath-start* **by** *simp*
have *first-finish*: *pathfinish* (*connector* (*p u*) +++ *subpathin u v p*) = *p v*

```
    using first-join subpath-finish by simp
  have rev-cv-path: path (reversepath (connector (p v)))
    using cv-path by simp
  have rev-cv-start: pathstart (reversepath (connector (p v))) = p v
    using cv-finish by simp
  have second-join:
    path ((connector (p u) +++ subpathin u v p) +++ reversepath (connector (p
v)))
    using first-join rev-cv-path first-finish rev-cv-start by simp
  show ?thesis
  proof (rule loop-spaceI)
    show path (segment-loop p u v)
      using second-join unfolding segment-loop-def by simp
    show path-image (segment-loop p u v) ⊆ S
    proof −
      have seg-img: path-image (subpathin u v p) ⊆ S
        using seg-in by (simp add: path-image-def)
      have left-img: path-image (connector (p u) +++ subpathin u v p) ⊆ S
        by (rule subset-path-image-join[OF conn-u seg-img])
      have right-img: path-image (reversepath (connector (p v))) ⊆ S
        using conn-v by simp
      show ?thesis
          unfolding segment-loop-def by (rule subset-path-image-join[OF left-img
right-img])
    qed
    show pathstart (segment-loop p u v) = x0
      unfolding segment-loop-def using cu-start by simp
    show pathfinish (segment-loop p u v) = x0
      unfolding segment-loop-def using cv-start by simp
  qed
qed

lemma segment-loop-in-U:
  assumes p-path: path p
    and p-image: path-image p ⊆ W
    and uv01: u ∈ {0..1} v ∈ {0..1}
    and puv-in: p u ∈ U ∩ V p v ∈ U ∩ V
    and seg-in: subpathin u v p ' {0..1} ⊆ U
  shows segment-loop p u v ∈ loop-space U x0
proof (rule segment-loop-in-set[where S = U])
  show path p
    by (rule p-path)
  show path-image p ⊆ W
    by (rule p-image)
  show u ∈ {0..1} v ∈ {0..1}
    by (rule uv01)+
  show p u ∈ U ∩ V p v ∈ U ∩ V
    by (rule puv-in)+
  show path-image (connector (p u)) ⊆ U
```

```
    using connector-spec(2)[OF puv-in(1)] by blast
  show path-image (connector (p v)) ⊆ U
    using connector-spec(2)[OF puv-in(2)] by blast
  show subpathin u v p ‘ {0..1} ⊆ U
    by (rule seg-in)
  show x0 ∈ U
    by simp
qed

lemma segment-loop-in-V:
  assumes p-path: path p
    and p-image: path-image p ⊆ W
    and uv01: u ∈ {0..1} v ∈ {0..1}
    and puv-in: p u ∈ U ∩ V p v ∈ U ∩ V
    and seg-in: subpathin u v p ‘ {0..1} ⊆ V
  shows segment-loop p u v ∈ loop-space V x0
proof (rule segment-loop-in-set[where S = V])
  show path p
    by (rule p-path)
  show path-image p ⊆ W
    by (rule p-image)
  show u ∈ {0..1} v ∈ {0..1}
    by (rule uv01)+
  show p u ∈ U ∩ V p v ∈ U ∩ V
    by (rule puv-in)+
  show path-image (connector (p u)) ⊆ V
    using connector-spec(2)[OF puv-in(1)] by blast
  show path-image (connector (p v)) ⊆ V
    using connector-spec(2)[OF puv-in(2)] by blast
  show subpathin u v p ‘ {0..1} ⊆ V
    by (rule seg-in)
  show x0 ∈ V
    by simp
qed

lemma segment-loop-in-W:
  assumes p-path: path p
    and p-image: path-image p ⊆ W
    and uv01: u ∈ {0..1} v ∈ {0..1}
    and puv-in: p u ∈ U ∩ V p v ∈ U ∩ V
    and seg-in: subpathin u v p ‘ {0..1} ⊆ W
  shows segment-loop p u v ∈ loop-space W x0
proof (rule segment-loop-in-set[where S = W])
  show path p
    by (rule p-path)
  show path-image p ⊆ W
    by (rule p-image)
  show u ∈ {0..1} v ∈ {0..1}
    by (rule uv01)+
```

```
  show p u ∈ U ∩ V p v ∈ U ∩ V
    by (rule puv-in)+
  show path-image (connector (p u)) ⊆ W
    using connector-spec(2)[OF puv-in(1)] by blast
  show path-image (connector (p v)) ⊆ W
    using connector-spec(2)[OF puv-in(2)] by blast
  show subpathin u v p ` {0..1} ⊆ W
    by (rule seg-in)
  show x0 ∈ W
    by simp
qed

lemma path-subpathin:
  assumes path p
    and u ∈ {0..1}
    and v ∈ {0..1}
  shows path (subpathin u v p)
proof −
  have pathin (top-of-set (path-image p)) p
    by (rule path-top-of-setI[OF assms(1)]) auto
  then have pathin (top-of-set (path-image p)) (subpathin u v p)
    by (rule pathin-subpathin[OF - assms(2,3)])
  then show ?thesis
    by (simp add: pathin-canon-iff)
qed

lemma path-image-subpathin-subset:
  assumes u ∈ {0..1}
    and v ∈ {0..1}
  shows path-image (subpathin u v p) ⊆ path-image p
proof −
  have closed-segment u v ⊆ {0..1}
    using assms by (auto simp: closed-segment-eq-real-ivl)
  then show ?thesis
    by (simp add: path-image-def subpathin-image image-mono)
qed

lemma reversepath-subpathin [simp]:
  reversepath (subpathin u v p) = subpathin v u p
  unfolding reversepath-def subpathin-def by (rule ext) (simp add: algebra-simps)

lemma subpathin-refl [simp]:
  subpathin u u p = (λ-. p u)
  unfolding subpathin-def by (rule ext) simp

fun svk-partition :: (real ⇒ 'a) ⇒ real list ⇒ bool list ⇒ bool where
  svk-partition p [] bs = False
| svk-partition p [t] [] = (t = 1 ∧ p t ∈ U ∩ V)
| svk-partition p [t] (b # bs) = False
```

```
| svk-partition p (t # u # ts) [] = False
| svk-partition p (t # u # ts) (b # bs) =
    (t ∈ {0..1} ∧ p t ∈ U ∩ V ∧ u ∈ {0..1} ∧ t < u ∧
      (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V) ∧
      svk-partition p (u # ts) bs)
```

15.2 Partitions and encoded loop words

A loop is encoded by subdividing the unit interval into pieces whose images lie entirely in U or entirely in V. The resulting bitstring records which side each segment uses, and the partition word records the corresponding loop classes in the two factors of the amalgamated free product.

```
definition valid-partition :: (real ⇒ 'a) ⇒ real list ⇒ bool list ⇒ bool where
  valid-partition p ts bs ⟷ ts ≠ [] ∧ hd ts = 0 ∧ svk-partition p ts bs

fun cover-partition :: (real ⇒ 'a) ⇒ real list ⇒ bool list ⇒ bool where
  cover-partition p [t] [] = (t = 1)
| cover-partition p (t # u # ts) (b # bs) =
    (t ∈ {0..1} ∧ u ∈ {0..1} ∧ t < u ∧
      (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V) ∧
      cover-partition p (u # ts) bs)
| cover-partition p ts bs = False

fun rectangle-partition ::
  ((real × real) ⇒ 'a) ⇒ real ⇒ real ⇒ real list ⇒ bool list ⇒ bool
where
  rectangle-partition h c d [t] [] = (t = 1)
| rectangle-partition h c d (t # u # ts) (b # bs) =
    (t ∈ {0..1} ∧ u ∈ {0..1} ∧ t < u ∧
      (if b then h ‘ ({t..u} × {c..d}) ⊆ U else h ‘ ({t..u} × {c..d}) ⊆ V) ∧
      rectangle-partition h c d (u # ts) bs)
| rectangle-partition h c d ts bs = False

fun alternating-bits :: bool list ⇒ bool where
  alternating-bits [] = True
| alternating-bits [b] = True
| alternating-bits (b # c # bs) = (b ≠ c ∧ alternating-bits (c # bs))

fun partition-word ::
  (real ⇒ 'a) ⇒ real list ⇒ bool list ⇒
    ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word
where
  partition-word p (t # u # ts) (b # bs) =
    (if b then WordLeft (loop-class U x0 (segment-loop p t u))
     else WordRight (loop-class V x0 (segment-loop p t u)))
      (partition-word p (u # ts) bs)
| partition-word p ts bs = WordNil

fun partition-word-with-tail ::
```

```
  (real ⇒ 'a) ⇒ real list ⇒ bool list ⇒
    ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word ⇒
    ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word
where
  partition-word-with-tail p (t # u # ts) (b # bs) tail =
    (if b then WordLeft (loop-class U x0 (segment-loop p t u))
     else WordRight (loop-class V x0 (segment-loop p t u)))
      (partition-word-with-tail p (u # ts) bs tail)
| partition-word-with-tail p ts bs tail = tail

fun bridge-word ::
  bool ⇒ (real ⇒ 'a) set ⇒
    ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word ⇒
    ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word
where
  bridge-word True h rest = WordLeft (i1 h) rest
| bridge-word False h rest = WordRight (i2 h) rest

fun partition-loop :: (real ⇒ 'a) ⇒ real list ⇒ real ⇒ 'a where
  partition-loop p (t # u # ts) = segment-loop p t u +++ partition-loop p (u #
ts)
| partition-loop p ts = (λ-. x0)

lemma partition-word-with-tail-nil [simp]:
  partition-word-with-tail p ts bs WordNil = partition-word p ts bs
  by (induction p ts bs
      WordNil :: ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word
      rule: partition-word-with-tail.induct) simp-all

lemma subpathin-joinpaths-left-half [simp]:
  subpathin 0 (1 / 2) (p +++ q) ‘ {0..1} = p ‘ {0..1}
proof
  show subpathin 0 (1 / 2) (p +++ q) ‘ {0..1} ⊆ p ‘ {0..1}
    by (auto simp: subpathin-def joinpaths-def)
next
  show p ‘ {0..1} ⊆ subpathin 0 (1 / 2) (p +++ q) ‘ {0..1}
  proof
    fix x
    assume x ∈ p ‘ {0..1}
    then obtain t where t01: t ∈ {0..1} and x-eq: x = p t
      by blast
    have x-eq': x = subpathin 0 (1 / 2) (p +++ q) t
      using x-eq t01 by (simp add: subpathin-def joinpaths-def)
    show x ∈ subpathin 0 (1 / 2) (p +++ q) ‘ {0..1}
      using t01 x-eq' by blast
  qed
qed

lemma affine-unit-interval:
```

```
  fixes u v t :: real
  assumes u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and t01: t ∈ {0..1}
  shows (v − u) * t + u ∈ {0..1}
proof −
  have t-bounds: 0 ≤ t t ≤ 1
    using t01 by auto
  have u-bounds: 0 ≤ u u ≤ 1
    using u01 by auto
  have v-bounds: 0 ≤ v v ≤ 1
    using v01 by auto
  have one-minus-t-nonneg: 0 ≤ 1 − t
    using t-bounds by linarith
  have lower1: 0 ≤ (1 − t) * u
    using one-minus-t-nonneg u-bounds by (intro mult-nonneg-nonneg) auto
  have lower2: 0 ≤ t * v
    using t-bounds v-bounds by (intro mult-nonneg-nonneg) auto
  have lower: 0 ≤ (1 − t) * u + t * v
    using lower1 lower2 by simp
  have upper1: (1 − t) * u ≤ (1 − t) * 1
    using one-minus-t-nonneg u-bounds by (intro mult-left-mono) auto
  have upper2: t * v ≤ t * 1
    using t-bounds v-bounds by (intro mult-left-mono) auto
  have upper: (1 − t) * u + t * v ≤ 1
    using upper1 upper2 by simp
  have (v − u) * t + u = (1 − t) * u + t * v
    by (simp add: algebra-simps)
  then show ?thesis
    using lower upper by auto
qed

lemma subpathin-joinpaths-tail-scaled-pointwise:
  assumes q0: pathstart q = pathfinish p
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and t01: t ∈ {0..1}
  shows subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) t = subpathin u v q t
proof −
  let ?s = (v − u) * t + u
  have s01: ?s ∈ {0..1}
    by (rule affine-unit-interval[OF u01 v01 t01])
  have param-eq:
    ((1 + v) / 2 − (1 + u) / 2) * t + (1 + u) / 2 = (1 + ?s) / 2
    by (simp add: field-simps algebra-simps)
  show ?thesis
  proof (cases ?s = 0)
    case True
    then show ?thesis
```

```
    using q0 by (simp add: subpathin-def joinpaths-def pathstart-def pathfinish-def
param-eq)
  next
    case False
    then have s-pos: 0 < ?s
      using s01 by auto
    then have param-gt:
      1 / 2 < ((1 + v) / 2 − (1 + u) / 2) ∗ t + (1 + u) / 2
      by (simp add: param-eq)
    have param-not-le:
      ¬ (((1 + v) / 2 − (1 + u) / 2) ∗ t + (1 + u) / 2 ≤ 1 / 2)
      using param-gt by simp
    then show ?thesis
    proof −
      have subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) t =
          q (2 ∗ (((1 + v) / 2 − (1 + u) / 2) ∗ t + (1 + u) / 2) − 1)
        using param-not-le by (simp add: subpathin-def joinpaths-def)
      also have ... = q ?s
      proof −
        have 2 ∗ (((1 + v) / 2 − (1 + u) / 2) ∗ t + (1 + u) / 2) − 1 = ?s
          by (simp add: field-simps algebra-simps)
        then show ?thesis
          by simp
      qed
      also have ... = subpathin u v q t
        by (simp add: subpathin-def)
      finally show ?thesis .
    qed
  qed
qed

lemma subpathin-joinpaths-tail-scaled [simp]:
  assumes q0: pathstart q = pathfinish p
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
  shows subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) ‘ {0..1} = subpathin
u v q ‘ {0..1}
proof
  show subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) ‘ {0..1} ⊆ subpathin u
v q ‘ {0..1}
  proof
    fix x
    assume x-in: x ∈ subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) ‘ {0..1}
    then obtain t where t01: t ∈ {0..1}
      and x-eq: x = subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) t
      by blast
    have x-eq′: x = subpathin u v q t
      using x-eq subpathin-joinpaths-tail-scaled-pointwise[OF q0 u01 v01 t01] by
simp
```

```
    show x ∈ subpathin u v q ` {0..1}
      using t01 x-eq' by blast
  qed
next
  show subpathin u v q ` {0..1} ⊆ subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++
q) ` {0..1}
  proof
    fix x
    assume x-in: x ∈ subpathin u v q ` {0..1}
    then obtain t where t01: t ∈ {0..1} and x-eq: x = subpathin u v q t
      by blast
    have x-eq': x = subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) t
       using x-eq subpathin-joinpaths-tail-scaled-pointwise[OF q0 u01 v01 t01] by
simp
    show x ∈ subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) ` {0..1}
      using t01 x-eq' by blast
  qed
qed

lemma homotopic-paths-join-left:
  assumes qr: homotopic-paths S q r
    and p-path: path p
    and p-img: path-image p ⊆ S
    and pq: pathfinish p = pathstart q
  shows homotopic-paths S (p +++ q) (p +++ r)
proof (rule homotopic-paths-join)
  show homotopic-paths S p p
    using p-path p-img by simp
  show homotopic-paths S q r
    by (rule qr)
  show pathfinish p = pathstart q
    by (rule pq)
qed

lemma homotopic-paths-join-right:
  assumes pq: homotopic-paths S p q
    and r-path: path r
    and r-img: path-image r ⊆ S
    and qr: pathfinish p = pathstart r
  shows homotopic-paths S (p +++ r) (q +++ r)
proof (rule homotopic-paths-join)
  show homotopic-paths S p q
    by (rule pq)
  show homotopic-paths S r r
    using r-path r-img by simp
  show pathfinish p = pathstart r
    by (rule qr)
qed
```

lemma *homotopic-paths-cancel-middle-local*:
assumes *r-path*: *path r*
and *r-img*: *path-image r* $\subseteq$ *S*
and *c-path*: *path c*
and *c-img*: *path-image c* $\subseteq$ *S*
and *s-path*: *path s*
and *s-img*: *path-image s* $\subseteq$ *S*
and *rc*: *pathfinish r* = *pathfinish c*
and *cs*: *pathstart s* = *pathfinish c*
shows *homotopic-paths S* ((((*r* +++ *reversepath c*) +++ *c*) +++ *s*)) (*r* +++ *s*)
proof −
have *revc-path*: *path* (*reversepath c*)
using *c-path* **by** *simp*
have *revc-img*: *path-image* (*reversepath c*) $\subseteq$ *S*
using *c-img* **by** *simp*
have *mid-path*: *path* (*reversepath c* +++ *c*)
using *revc-path c-path* **by** *simp*
have *mid-img*: *path-image* (*reversepath c* +++ *c*) $\subseteq$ *S*
by (*rule subset-path-image-join*[*OF revc-img c-img*])
have *hom-cancel0*: *homotopic-paths S* (*reversepath c* +++ *c*) (λ-. *pathfinish c*)
by (*rule homotopic-paths-linv-const*[*OF c-path c-img*])
have *hom-cancel1*:
homotopic-paths S (((*reversepath c* +++ *c*) +++ *s*)) (((λ-. *pathfinish c*) +++ *s*))
proof (*rule homotopic-paths-join-right*[*OF hom-cancel0 s-path s-img*])
show *pathfinish* (*reversepath c* +++ *c*) = *pathstart s*
using *cs* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *hom-cancel2*: *homotopic-paths S* (((λ-. *pathfinish c*) +++ *s*)) *s*
using *homotopic-paths-lid-const*[*OF s-path s-img*] *cs* **by** (*simp add*: *pathstart-def*)
have *hom-cancel*: *homotopic-paths S* (((*reversepath c* +++ *c*) +++ *s*)) *s*
by (*rule homotopic-paths-trans*[*OF hom-cancel1 hom-cancel2*])
have *hom-left*:
homotopic-paths S (*r* +++ ((*reversepath c* +++ *c*) +++ *s*)) (*r* +++ *s*)
proof (*rule homotopic-paths-join-left*[*OF hom-cancel r-path r-img*])
show *pathfinish r* = *pathstart* ((*reversepath c* +++ *c*) +++ *s*)
using *rc* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *assoc1*:
homotopic-paths S (((*r* +++ *reversepath c*) +++ *c*)) (*r* +++ (*reversepath c* +++ *c*))
proof −
have *homotopic-paths S* (*r* +++ (*reversepath c* +++ *c*)) (((*r* +++ *reversepath c*) +++ *c*))
by (*rule homotopic-paths-assoc*[*OF r-path r-img revc-path revc-img c-path*

c-img]) (*use rc* **in** *simp-all*)
  then show *?thesis*
  by (*rule homotopic-paths-sym*)
qed
have *assoc1-join*:
  homotopic-paths S ((((*r* +++ *reversepath c*) +++ *c*) +++ *s*)) (((*r* +++ (*reversepath c* +++ *c*)) +++ *s*))
proof (*rule homotopic-paths-join-right*[*OF assoc1 s-path s-img*])
  show *pathfinish* ((*r* +++ *reversepath c*) +++ *c*) = *pathstart s*
    using *rc cs* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *assoc2*:
  homotopic-paths S (((*r* +++ (*reversepath c* +++ *c*)) +++ *s*)) (*r* +++ ((*reversepath c* +++ *c*) +++ *s*))
proof −
  have *homotopic-paths S* (*r* +++ ((*reversepath c* +++ *c*) +++ *s*)) (((*r* +++ (*reversepath c* +++ *c*)) +++ *s*))
    by (*rule homotopic-paths-assoc*[*OF r-path r-img mid-path mid-img s-path s-img*]) (*use rc cs* **in** *simp-all*)
  then show *?thesis*
  by (*rule homotopic-paths-sym*)
qed
have *homotopic-paths S* ((((*r* +++ *reversepath c*) +++ *c*) +++ *s*)) (*r* +++ ((*reversepath c* +++ *c*) +++ *s*))
  by (*rule homotopic-paths-trans*[*OF assoc1-join assoc2*])
then show *?thesis*
  by (*rule homotopic-paths-trans*[*OF - hom-left*])
qed

lemma *segment-loop-base-full-in-set*:
assumes *p-loop*: *p* ∈ *loop-space S x0*
shows *homotopic-paths S* (*segment-loop p 0 1*) *p*
proof −
have *p-path*: *path p*
  and *p-img*: *path-image p* ⊆ *S*
  and *p0*: *p 0* = *x0*
  and *p1*: *p 1* = *x0*
  using *p-loop* **unfolding** *loop-space-def pathstart-def pathfinish-def* **by** *auto*
have *x0-in-S*: *x0* ∈ *S*
  using *p-img p0* **by** (*auto simp*: *path-image-def*)
have *const-path*: *path* (λ-. *x0*)
  by (*simp add*: *path-def*)
have *const-img*: *path-image* (λ-. *x0*) ⊆ *S*
  using *x0-in-S* **by** (*auto simp*: *path-image-def*)
have *hom-lid*: *homotopic-paths S* ((λ-. *x0*) +++ *p*) *p*
  using *homotopic-paths-lid-const*[*OF p-path p-img*] *p0* **by** (*simp add*: *pathstart-def*)
have *hom-join*:

```
    homotopic-paths S ((((λ-. x0) +++ p) +++ (λ-. x0))) (p +++ (λ-. x0))
  proof (rule homotopic-paths-join-right[OF hom-lid const-path const-img])
    show pathfinish ((λ-. x0) +++ p) = pathstart (λ-. x0)
      using p0 p1 by (simp add: pathstart-def pathfinish-def joinpaths-def)
  qed
  have hom-rid: homotopic-paths S (p +++ (λ-. x0)) p
    using homotopic-paths-rid-const[OF p-path p-img] p1 by (simp add: pathfin-
ish-def)
  have conn0: connector (p 0) = (λ-. x0)
    using p0 by (simp add: connector-def fun-eq-iff)
  have conn1: reversepath (connector (p 1)) = (λ-. x0)
    using p1 by (simp add: connector-def reversepath-def fun-eq-iff)
  have seg-eq: segment-loop p 0 1 = (((λ-. x0) +++ p) +++ (λ-. x0))
    unfolding segment-loop-def using conn0 conn1 by simp
  have homotopic-paths S (segment-loop p 0 1) (p +++ (λ-. x0))
    unfolding seg-eq by (rule hom-join)
  then show ?thesis
    by (rule homotopic-paths-trans[OF - hom-rid])
qed

lemma segment-loop-joinpaths-head [simp]:
  assumes p-loop: p ∈ loop-space S x0
    and q-loop: q ∈ loop-space W x0
  shows segment-loop (p +++ q) 0 (1 / 2) = segment-loop p 0 1
proof −
  have p0: p 0 = x0 and p1: p 1 = x0
    using p-loop unfolding loop-space-def pathstart-def pathfinish-def by auto
  show ?thesis
  proof
    fix t
    show segment-loop (p +++ q) 0 (1 / 2) t = segment-loop p 0 1 t
    proof (cases t ≤ 1 / 4)
      case True
      then show ?thesis
        unfolding segment-loop-def
        using p0 p1
          by (simp add: connector-def joinpaths-def subpathin-def reversepath-def
field-simps algebra-simps)
    next
      case False
      show ?thesis
      proof (cases t ≤ 1 / 2)
        case True
        have mid-gt: ¬ (2 * t ≤ 1 / 2)
          using False by linarith
        have sub-le: (4 * t − 1) / 2 ≤ 1 / 2
        proof −
          have 4 * t − 1 ≤ 1
            using True by linarith
```

```
      then show ?thesis
        by (simp add: divide-right-mono)
    qed
    have start-eq: (p +++ q) 0 = x0
      using p0 by (simp add: joinpaths-def)
    have mid-eq: (p +++ q) (1 / 2) = x0
      using p1 by (simp add: joinpaths-def)
    have conn-start: connector ((p +++ q) 0) = (λ-. x0)
      using start-eq by (simp add: connector-def fun-eq-iff)
    have conn-mid: reversepath (connector ((p +++ q) (1 / 2))) = (λ-. x0)
      using mid-eq by (simp add: connector-def reversepath-def fun-eq-iff)
    have arg-eq: ((8 * t - 2) / 2 :: real) = 4 * t - 1
    proof -
      have ((8 * t - 2) / 2 :: real) = (8 * t) / 2 - (2 :: real) / 2
        by (simp add: diff-divide-distrib)
      also have ... = 4 * t - 1
        by simp
      finally show ?thesis .
    qed
    have lhs-eq: segment-loop (p +++ q) 0 (1 / 2) t = p (4 * t - 1)
    proof -
      have segment-loop (p +++ q) 0 (1 / 2) t =
          (((λ-. x0) +++ subpathin 0 (1 / 2) (p +++ q)) +++ (λ-. x0)) t
        unfolding segment-loop-def
        using conn-start conn-mid by simp
      also have ... = subpathin 0 (1 / 2) (p +++ q) (4 * t - 1)
        using False True mid-gt
        by (simp add: joinpaths-def field-simps algebra-simps)
      also have ... = (p +++ q) ((4 * t - 1) / 2)
        by (simp add: subpathin-def)
      also have ... = p ((8 * t - 2) / 2)
        using sub-le by (simp add: joinpaths-def field-simps algebra-simps)
      also have ... = p (4 * t - 1)
        by (subst arg-eq) simp
      finally show ?thesis .
    qed
    have rhs-eq: segment-loop p 0 1 t = p (4 * t - 1)
      unfolding segment-loop-def
      using p0 p1 False True mid-gt
        by (simp add: connector-def joinpaths-def subpathin-def reversepath-def
field-simps algebra-simps)
    show ?thesis
      using lhs-eq rhs-eq by simp
  next
    case False
    then show ?thesis
      unfolding segment-loop-def
      using p0 p1
        by (simp add: connector-def joinpaths-def subpathin-def reversepath-def
```

field-simps algebra-simps)
    qed
  qed
 qed
qed

lemma *segment-loop-joinpaths-tail-scaled* [*simp*]:
 assumes *p-loop*: $p \in$ *loop-space W x0*
  and *q-loop*: $q \in$ *loop-space W x0*
  and *u01*: $u \in \{0..1\}$
  and *v01*: $v \in \{0..1\}$
 shows *segment-loop* $(p +\!\!+\!\!+ q)$ $((1 + u) / 2)$ $((1 + v) / 2) =$ *segment-loop q u v*
proof −
 have *p1*: $p\ 1 =$ *x0* **and** *q0*: $q\ 0 =$ *x0*
  using *p-loop q-loop* **unfolding** *loop-space-def pathfinish-def pathstart-def* **by** *auto*
 have *pq*: *pathstart q* = *pathfinish p*
  using *p-loop q-loop* **unfolding** *loop-space-def* **by** *auto*
 have *start-eq*: $(p +\!\!+\!\!+ q)$ $((1 + u) / 2) = q\ u$
 proof (*cases u* = *0*)
  case *True*
  then show *?thesis*
   using *p1 q0* **by** (*simp add*: *joinpaths-def*)
 next
  case *False*
  then have (*1 / 2* :: *real*) < (*1* + *u*) / *2*
   using *u01* **by** *auto*
  then show *?thesis*
   by (*simp add*: *joinpaths-def field-simps algebra-simps*)
 qed
 have *finish-eq*: $(p +\!\!+\!\!+ q)$ $((1 + v) / 2) = q\ v$
 proof (*cases v* = *0*)
  case *True*
  then show *?thesis*
   using *p1 q0* **by** (*simp add*: *joinpaths-def*)
 next
  case *False*
  then have (*1 / 2* :: *real*) < (*1* + *v*) / *2*
   using *v01* **by** *auto*
  then show *?thesis*
   by (*simp add*: *joinpaths-def field-simps algebra-simps*)
 qed
 show *?thesis*
 proof
  fix *t* :: *real*
  show *segment-loop* $(p +\!\!+\!\!+ q)$ $((1 + u) / 2)$ $((1 + v) / 2)$ *t* = *segment-loop q u v t*
   proof (*cases t* ≤ *1 / 4*)

```
      case True
      then show ?thesis
        unfolding segment-loop-def
        using start-eq finish-eq
        by (simp add: joinpaths-def field-simps algebra-simps)
    next
      case False
      show ?thesis
      proof (cases t ≤ 1 / 2)
        case True
        have s01: 4 * t - 1 ∈ {0..1}
          using False True by auto
        have sub-eq:
          subpathin ((1 + u) / 2) ((1 + v) / 2) (p +++ q) (4 * t - 1) =
            subpathin u v q (4 * t - 1)
          by (rule subpathin-joinpaths-tail-scaled-pointwise[OF pq u01 v01 s01])
        show ?thesis
          unfolding segment-loop-def
          using False True start-eq finish-eq sub-eq
          by (simp add: joinpaths-def field-simps algebra-simps)
      next
        case False
        then show ?thesis
          unfolding segment-loop-def
          using finish-eq
          by (simp add: joinpaths-def field-simps algebra-simps)
      qed
    qed
  qed
qed

fun word-loop ::
  ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word ⇒ real ⇒ 'a
where
  word-loop WordNil = (λ-. x0)
| word-loop (WordLeft a rest) =
    (if rest = WordNil then some-loop U x0 a else some-loop U x0 a +++ word-loop
rest)
| word-loop (WordRight b rest) =
    (if rest = WordNil then some-loop V x0 b else some-loop V x0 b +++ word-loop
rest)

fun word-partition-times ::
  ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word ⇒ real list
where
  word-partition-times WordNil = [0, 1]
| word-partition-times (WordLeft a rest) =
    (if rest = WordNil
     then [0, 1]
```

```
    else 0 # map (λt. (1 + t) / 2) (word-partition-times rest))
| word-partition-times (WordRight b rest) =
    (if rest = WordNil
     then [0, 1]
     else 0 # map (λt. (1 + t) / 2) (word-partition-times rest))

fun word-partition-bits ::
  ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word ⇒ bool list
where
  word-partition-bits WordNil = [True]
| word-partition-bits (WordLeft a rest) =
    (if rest = WordNil then [True] else True # word-partition-bits rest)
| word-partition-bits (WordRight b rest) =
    (if rest = WordNil then [False] else False # word-partition-bits rest)

lemma joinpaths-tail-scaled-point [simp]:
  assumes p-loop: p ∈ loop-space W x0
    and q-loop: q ∈ loop-space W x0
    and t01: t ∈ {0..1}
  shows (p +++ q) ((1 + t) / 2) = q t
proof (cases t = 0)
  case True
  then show ?thesis
    using p-loop q-loop
    unfolding loop-space-def pathstart-def pathfinish-def joinpaths-def
    by simp
next
  case False
  have t-pos: 0 < t
    using False t01 by auto
  then have param-gt: (1 + t) / 2 > 1 / 2
    by (simp add: field-simps)
  have param-not-le: ¬ ((1 + t) / 2 ≤ 1 / 2)
    using param-gt by simp
  have arg-eq: 2 * ((1 + t) / 2) - 1 = (t :: real)
    by (simp add: field-simps algebra-simps)
  have (p +++ q) ((1 + t) / 2) = q (2 * ((1 + t) / 2) - 1)
    using param-not-le by (simp add: joinpaths-def algebra-simps)
  also have ... = q t
    by (subst arg-eq) simp
  finally show ?thesis .
qed

lemma word-loop-in-W:
  assumes w-in: fpw-in-space G1 G2 w
  shows word-loop w ∈ loop-space W x0
  using w-in
proof (induction w)
  case WordNil
```

```
  then show ?case
    by (simp add: constant-loop-in-space[OF x0-in-W])
next
  case (WordLeft a rest)
  then have a-in: a ∈ G1 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have a-loopU: some-loop U x0 a ∈ loop-space U x0
    using some-loop-spec[OF a-in] by auto
  have a-loopW: some-loop U x0 a ∈ loop-space W x0
    using a-loopU unfolding loop-space-def by auto
  show ?case
  proof (cases rest = WordNil)
    case True
    then show ?thesis
      using a-loopW by simp
  next
    case False
    have tail-loop: word-loop rest ∈ loop-space W x0
      by (rule WordLeft.IH[OF rest-in])
    have a-path: path (some-loop U x0 a)
      and a-img: path-image (some-loop U x0 a) ⊆ W
      and a-start: pathstart (some-loop U x0 a) = x0
      and a-finish: pathfinish (some-loop U x0 a) = x0
      using a-loopW unfolding loop-space-def by auto
    have tail-path: path (word-loop rest)
      and tail-img: path-image (word-loop rest) ⊆ W
      and tail-start: pathstart (word-loop rest) = x0
      and tail-finish: pathfinish (word-loop rest) = x0
      using tail-loop unfolding loop-space-def by auto
    have join-loop: some-loop U x0 a +++ word-loop rest ∈ loop-space W x0
    proof −
      have join-path: path (some-loop U x0 a +++ word-loop rest)
        using a-path tail-path a-finish tail-start by simp
      have join-img: path-image (some-loop U x0 a +++ word-loop rest) ⊆ W
        by (rule subset-path-image-join[OF a-img tail-img])
      show ?thesis
        unfolding loop-space-def
        using join-path join-img a-start tail-finish by simp
    qed
    then show ?thesis
      using False by simp
  qed
next
  case (WordRight b rest)
  then have b-in: b ∈ G2 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have b-loopV: some-loop V x0 b ∈ loop-space V x0
    using some-loop-spec[OF b-in] by auto
  have b-loopW: some-loop V x0 b ∈ loop-space W x0
```

```
    using b-loopV unfolding loop-space-def by auto
  show ?case
  proof (cases rest = WordNil)
    case True
    then show ?thesis
      using b-loopW by simp
  next
    case False
    have tail-loop: word-loop rest ∈ loop-space W x0
      by (rule WordRight.IH[OF rest-in])
    have b-path: path (some-loop V x0 b)
      and b-img: path-image (some-loop V x0 b) ⊆ W
      and b-start: pathstart (some-loop V x0 b) = x0
      and b-finish: pathfinish (some-loop V x0 b) = x0
      using b-loopW unfolding loop-space-def by auto
    have tail-path: path (word-loop rest)
      and tail-img: path-image (word-loop rest) ⊆ W
      and tail-start: pathstart (word-loop rest) = x0
      and tail-finish: pathfinish (word-loop rest) = x0
      using tail-loop unfolding loop-space-def by auto
    have join-loop: some-loop V x0 b +++ word-loop rest ∈ loop-space W x0
    proof −
      have join-path: path (some-loop V x0 b +++ word-loop rest)
        using b-path tail-path b-finish tail-start by simp
      have join-img: path-image (some-loop V x0 b +++ word-loop rest) ⊆ W
        by (rule subset-path-image-join[OF b-img tail-img])
      show ?thesis
        unfolding loop-space-def
        using join-path join-img b-start tail-finish by simp
    qed
    then show ?thesis
      using False by simp
  qed
qed

lemma svk-partition-joinpaths-tail-scaled:
  assumes p-loop: p ∈ loop-space W x0
    and q-loop: q ∈ loop-space W x0
    and q-part: svk-partition q ts bs
  shows svk-partition (p +++ q) (map (λt. (1 + t) / 2) ts) bs
  using q-part
proof (induction ts arbitrary: bs)
  case Nil
  then show ?case by simp
next
  case (Cons t ts)
  show ?case
  proof (cases ts)
    case Nil
```

```
    have bs-nil: bs = []
      using Cons.prems Nil by (cases bs) simp-all
    have t1: t = 1
      using Cons.prems Nil bs-nil by simp
    have qt1UV: q 1 ∈ U ∩ V
      using Cons.prems Nil bs-nil t1 by simp
    have qtUV: q t ∈ U ∩ V
      using t1 qt1UV by simp
    have t01: t ∈ {0..1}
      using t1 by simp
    have joined-tUV: (p +++ q) ((1 + t) / 2) ∈ U ∩ V
      using qtUV joinpaths-tail-scaled-point[OF p-loop q-loop t01] by simp
    show ?thesis
      using Nil bs-nil t1 joined-tUV by simp
  next
    case (Cons u us)
    then obtain b bs' where bs: bs = b # bs'
      using Cons.prems by (cases bs) auto
    have t01: t ∈ {0..1} and qtUV: q t ∈ U ∩ V
      and u01: u ∈ {0..1} and tu: t < u
       and seg-side: (if b then subpathin t u q ‘ {0..1} ⊆ U else subpathin t u q ‘
{0..1} ⊆ V)
      and q-tail: svk-partition q (u # us) bs'
      using Cons.prems bs Cons by simp-all
    have pq: pathstart q = pathfinish p
      using p-loop q-loop unfolding loop-space-def by auto
    have scaled-seg-side:
      (if b
        then subpathin ((1 + t) / 2) ((1 + u) / 2) (p +++ q) ‘ {0..1} ⊆ U
        else subpathin ((1 + t) / 2) ((1 + u) / 2) (p +++ q) ‘ {0..1} ⊆ V)
      using seg-side by (simp add: subpathin-joinpaths-tail-scaled[OF pq t01 u01])
    have q-tail-ts: svk-partition q ts bs'
      using q-tail Cons by simp
    have tail-scaled-ts:
      svk-partition (p +++ q) (map (λt. (1 + t) / 2) ts) bs'
      by (rule Cons.IH[OF q-tail-ts])
    have tail-scaled:
      svk-partition (p +++ q) (map (λt. (1 + t) / 2) (u # us)) bs'
      using tail-scaled-ts Cons by simp
    have joined-tUV: (p +++ q) ((1 + t) / 2) ∈ U ∩ V
      using qtUV joinpaths-tail-scaled-point[OF p-loop q-loop t01] by simp
    show ?thesis
      using bs Cons t01 joined-tUV u01 tu scaled-seg-side tail-scaled
      by simp
  qed
qed

lemma word-loop-valid-partition:
  assumes w-in: fpw-in-space G1 G2 w
```

shows *valid-partition* (*word-loop w*) (*word-partition-times w*) (*word-partition-bits w*)
using *w-in*
proof (*induction w*)
case *WordNil*
have *const-subU*: (λ-. *x0*) ` {*0..1*} ⊆ *U*
using *x0-in-UV* **by** *auto*
then show *?case*
unfolding *valid-partition-def*
using *x0-in-UV const-subU*
by (*auto simp*: *path-image-def subpathin-def*)
next
case (*WordLeft a rest*)
then have *a-in*: *a* ∈ *G1* **and** *rest-in*: *fpw-in-space G1 G2 rest*
by *auto*
have *a-loopU*: *some-loop U x0 a* ∈ *loop-space U x0*
using *some-loop-spec*[*OF a-in*] **by** *auto*
have *a-loopW*: *some-loop U x0 a* ∈ *loop-space W x0*
using *a-loopU* **unfolding** *loop-space-def* **by** *auto*
show *?case*
proof (*cases rest* = *WordNil*)
case *True*
then show *?thesis*
using *a-loopU x0-in-UV*
unfolding *valid-partition-def loop-space-def*
by (*auto simp*: *pathstart-def pathfinish-def path-image-def*)
next
case *False*
have *rest-valid*:
valid-partition (*word-loop rest*) (*word-partition-times rest*) (*word-partition-bits rest*)
by (*rule WordLeft.IH*[*OF rest-in*])
have *rest-svk*:
svk-partition (*word-loop rest*) (*word-partition-times rest*) (*word-partition-bits rest*)
using *rest-valid* **unfolding** *valid-partition-def* **by** *auto*
have *tail-scaled*:
svk-partition (*some-loop U x0 a* +++ *word-loop rest*)
(*map* (λ*t*. (*1* + *t*) / *2*) (*word-partition-times rest*))
(*word-partition-bits rest*)
by (*rule svk-partition-joinpaths-tail-scaled*[*OF a-loopW word-loop-in-W*[*OF rest-in*] *rest-svk*])
have *headU*:
subpathin 0 (*1* / *2*) (*some-loop U x0 a* +++ *word-loop rest*) ` {*0..1*} ⊆ *U*
using *a-loopU* **by** (*simp add*: *loop-space-def path-image-def*)
have *startUV*: (*some-loop U x0 a* +++ *word-loop rest*) *0* ∈ *U* ∩ *V*
using *a-loopU x0-in-UV* **unfolding** *loop-space-def pathstart-def joinpaths-def* **by** *simp*
have *midUV*: (*some-loop U x0 a* +++ *word-loop rest*) (*1* / *2*) ∈ *U* ∩ *V*

using *a-loopU x0-in-UV* **unfolding** *loop-space-def pathfinish-def joinpaths-def*
by *simp*
have *rest-times-nonempty*: *word-partition-times rest* ≠ []
using *rest-valid* **unfolding** *valid-partition-def* **by** *auto*
have *rest-times-hd*: *hd* (*word-partition-times rest*) = *0*
using *rest-valid* **unfolding** *valid-partition-def* **by** *auto*
have *rest-times*: *word-partition-times rest* = *0* # *tl* (*word-partition-times rest*)
proof (*cases word-partition-times rest*)
case *Nil*
with *rest-times-nonempty* **show** *?thesis*
by *simp*
next
case (*Cons s ss*)
have *s* = *0*
using *rest-times-hd Cons* **by** *simp*
then show *?thesis*
using *Cons* **by** *simp*
qed
have *scaled-times*:
map (λ*t*. (*1* + *t*) / *2*) (*word-partition-times rest*) =
1 / *2* # *map* (λ*t*. (*1* + *t*) / *2*) (*tl* (*word-partition-times rest*))
proof −
have *map* (λ*t*. (*1* + *t*) / *2*) (*word-partition-times rest*) =
map (λ*t*. (*1* + *t*) / *2*) (*0* # *tl* (*word-partition-times rest*))
using *rest-times* **by** *simp*
also have ... = *1* / *2* # *map* (λ*t*. (*1* + *t*) / *2*) (*tl* (*word-partition-times*
rest))
by *simp*
finally show *?thesis* **.**
qed
have *tail-scaled′*:
svk-partition (*some-loop U x0 a* +++ *word-loop rest*)
(*1* / *2* # *map* (λ*t*. (*1* + *t*) / *2*) (*tl* (*word-partition-times rest*)))
(*word-partition-bits rest*)
using *tail-scaled scaled-times* **by** *simp*
have *step-svk*:
svk-partition (*some-loop U x0 a* +++ *word-loop rest*)
(*0* # *1* / *2* # *map* (λ*t*. (*1* + *t*) / *2*) (*tl* (*word-partition-times rest*)))
(*True* # *word-partition-bits rest*)
using *False tail-scaled′ headU startUV midUV*
by *simp*
have *step-svk′*:
svk-partition (*some-loop U x0 a* +++ *word-loop rest*)
(*0* # *map* (λ*t*. (*1* + *t*) / *2*) (*word-partition-times rest*))
(*True* # *word-partition-bits rest*)
using *step-svk scaled-times* **by** *simp*
show *?thesis*
unfolding *valid-partition-def*
using *False step-svk′*

```
    by simp
  qed
next
  case (WordRight b rest)
  then have b-in: b ∈ G2 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have b-loopV: some-loop V x0 b ∈ loop-space V x0
    using some-loop-spec[OF b-in] by auto
  have b-loopW: some-loop V x0 b ∈ loop-space W x0
    using b-loopV unfolding loop-space-def by auto
  show ?case
  proof (cases rest = WordNil)
    case True
    then show ?thesis
      using b-loopV x0-in-UV
      unfolding valid-partition-def loop-space-def
      by (auto simp: pathstart-def pathfinish-def path-image-def)
  next
    case False
    have rest-valid:
     valid-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      by (rule WordRight.IH[OF rest-in])
    have rest-svk:
      svk-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      using rest-valid unfolding valid-partition-def by auto
    have tail-scaled:
      svk-partition (some-loop V x0 b +++ word-loop rest)
        (map (λt. (1 + t) / 2) (word-partition-times rest))
        (word-partition-bits rest)
       by (rule svk-partition-joinpaths-tail-scaled[OF b-loopW word-loop-in-W[OF
rest-in] rest-svk])
    have headV:
      subpathin 0 (1 / 2) (some-loop V x0 b +++ word-loop rest) ` {0..1} ⊆ V
      using b-loopV by (simp add: loop-space-def path-image-def)
    have startUV: (some-loop V x0 b +++ word-loop rest) 0 ∈ U ∩ V
      using b-loopV x0-in-UV unfolding loop-space-def pathstart-def joinpaths-def
by simp
    have midUV: (some-loop V x0 b +++ word-loop rest) (1 / 2) ∈ U ∩ V
     using b-loopV x0-in-UV unfolding loop-space-def pathfinish-def joinpaths-def
by simp
    have rest-times-nonempty: word-partition-times rest ≠ []
      using rest-valid unfolding valid-partition-def by auto
    have rest-times-hd: hd (word-partition-times rest) = 0
      using rest-valid unfolding valid-partition-def by auto
    have rest-times: word-partition-times rest = 0 # tl (word-partition-times rest)
    proof (cases word-partition-times rest)
      case Nil
```

```
    with rest-times-nonempty show ?thesis
      by simp
  next
    case (Cons s ss)
    have s = 0
      using rest-times-hd Cons by simp
    then show ?thesis
      using Cons by simp
  qed
  have scaled-times:
    map (λt. (1 + t) / 2) (word-partition-times rest) =
      1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest))
  proof −
    have map (λt. (1 + t) / 2) (word-partition-times rest) =
        map (λt. (1 + t) / 2) (0 # tl (word-partition-times rest))
      using rest-times by simp
     also have ... = 1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times
rest))
      by simp
    finally show ?thesis .
  qed
  have tail-scaled':
    svk-partition (some-loop V x0 b +++ word-loop rest)
      (1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
      (word-partition-bits rest)
    using tail-scaled scaled-times by simp
  have step-svk:
    svk-partition (some-loop V x0 b +++ word-loop rest)
      (0 # 1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
      (False # word-partition-bits rest)
    using False tail-scaled' headV startUV midUV
    by simp
  have step-svk':
    svk-partition (some-loop V x0 b +++ word-loop rest)
      (0 # map (λt. (1 + t) / 2) (word-partition-times rest))
      (False # word-partition-bits rest)
    using step-svk scaled-times by simp
  show ?thesis
    unfolding valid-partition-def
    using False step-svk'
    by simp
 qed
qed

lemma partition-word-joinpaths-tail-scaled:
 assumes p-loop: p ∈ loop-space W x0
   and q-loop: q ∈ loop-space W x0
   and q-part: svk-partition q ts bs
 shows partition-word (p +++ q) (map (λt. (1 + t) / 2) ts) bs = partition-word
```

```
  q ts bs
  using q-part
proof (induction ts arbitrary: bs)
  case Nil
  then show ?case
    by simp
next
  case (Cons t ts)
  show ?case
  proof (cases ts)
    case Nil
    then show ?thesis
      using Cons.prems by (cases bs) simp-all
  next
    case (Cons u us)
    then obtain b bs' where bs: bs = b # bs'
      using Cons.prems by (cases bs) auto
    have tail: svk-partition q (u # us) bs'
      using Cons.prems Cons bs by simp
    have t01: t ∈ {0..1} and u01: u ∈ {0..1}
      using Cons.prems Cons bs by simp-all
    have tail-ts: svk-partition q ts bs'
      using tail Cons by simp
    have ih0:
      partition-word (p +++ q) (map (λt. (1 + t) / 2) ts) bs' =
        partition-word q ts bs'
      by (rule Cons.IH[of bs', OF tail-ts])
    have ih:
      partition-word (p +++ q) (map (λt. (1 + t) / 2) (u # us)) bs' =
        partition-word q (u # us) bs'
      using ih0 Cons by simp
    have seg-eq:
      segment-loop (p +++ q) ((1 + t) / 2) ((1 + u) / 2) = segment-loop q t u
      by (rule segment-loop-joinpaths-tail-scaled[OF p-loop q-loop t01 u01])
    show ?thesis
      using Cons bs ih seg-eq
      by simp
  qed
qed

lemma segment-loop-some-loop-left-class:
  assumes a-in: a ∈ G1
  shows loop-class U x0 (segment-loop (some-loop U x0 a) 0 1) = a
proof −
  have p-loop: some-loop U x0 a ∈ loop-space U x0
    and a-eq: a = loop-class U x0 (some-loop U x0 a)
    using some-loop-spec[OF a-in] by auto
  have p-path: path (some-loop U x0 a)
    and p-imgU: path-image (some-loop U x0 a) ⊆ U
```

```
    and p0: some-loop U x0 a 0 = x0
    and p1: some-loop U x0 a 1 = x0
    using p-loop unfolding loop-space-def pathstart-def pathfinish-def by auto
  have segU: segment-loop (some-loop U x0 a) 0 1 ∈ loop-space U x0
  proof (rule segment-loop-in-U[OF p-path])
    show path-image (some-loop U x0 a) ⊆ W
      using p-imgU by auto
    show (0::real) ∈ {0..1}
      by simp
    show (1::real) ∈ {0..1}
      by simp
    show some-loop U x0 a 0 ∈ U ∩ V some-loop U x0 a 1 ∈ U ∩ V
      using p0 p1 x0-in-UV by simp-all
    show subpathin 0 1 (some-loop U x0 a) ‘ {0..1} ⊆ U
      using p-imgU by (simp add: path-image-def)
  qed
  have seg-eq:
    loop-class U x0 (segment-loop (some-loop U x0 a) 0 1) =
      loop-class U x0 (some-loop U x0 a)
   by (rule loop-class-eqI[OF segU p-loop segment-loop-base-full-in-set[OF p-loop]])
  show ?thesis
    using seg-eq a-eq by simp
qed

lemma segment-loop-some-loop-right-class:
  assumes b-in: b ∈ G2
  shows loop-class V x0 (segment-loop (some-loop V x0 b) 0 1) = b
proof −
  have p-loop: some-loop V x0 b ∈ loop-space V x0
    and b-eq: b = loop-class V x0 (some-loop V x0 b)
    using some-loop-spec[OF b-in] by auto
  have p-path: path (some-loop V x0 b)
    and p-imgV: path-image (some-loop V x0 b) ⊆ V
    and p0: some-loop V x0 b 0 = x0
    and p1: some-loop V x0 b 1 = x0
    using p-loop unfolding loop-space-def pathstart-def pathfinish-def by auto
  have segV: segment-loop (some-loop V x0 b) 0 1 ∈ loop-space V x0
  proof (rule segment-loop-in-V[OF p-path])
    show path-image (some-loop V x0 b) ⊆ W
      using p-imgV by auto
    show (0::real) ∈ {0..1}
      by simp
    show (1::real) ∈ {0..1}
      by simp
    show some-loop V x0 b 0 ∈ U ∩ V some-loop V x0 b 1 ∈ U ∩ V
      using p0 p1 x0-in-UV by simp-all
    show subpathin 0 1 (some-loop V x0 b) ‘ {0..1} ⊆ V
      using p-imgV by (simp add: path-image-def)
  qed
```

```
  have seg-eq:
    loop-class V x0 (segment-loop (some-loop V x0 b) 0 1) =
      loop-class V x0 (some-loop V x0 b)
   by (rule loop-class-eqI[OF segV p-loop segment-loop-base-full-in-set[OF p-loop]])
  show ?thesis
    using seg-eq b-eq by simp
qed

lemma partition-word-word-loop-equiv:
  assumes w-in: fpw-in-space G1 G2 w
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (partition-word (word-loop w) (word-partition-times w) (word-partition-bits w))
w
  using w-in
proof (induction w)
  case WordNil
  have nil-loopU: word-loop WordNil ∈ loop-space U x0
    by (simp add: constant-loop-in-space[OF x0-in-U])
  have nil-segU: segment-loop (word-loop WordNil) 0 1 ∈ loop-space U x0
  proof (rule segment-loop-in-U)
    show path (word-loop WordNil)
      using nil-loopU unfolding loop-space-def by simp
    show path-image (word-loop WordNil) ⊆ W
      using nil-loopU unfolding loop-space-def by simp
    show (0::real) ∈ {0..1}
      by simp
    show (1::real) ∈ {0..1}
      by simp
    show word-loop WordNil 0 ∈ U ∩ V word-loop WordNil 1 ∈ U ∩ V
      using x0-in-UV by simp-all
    show subpathin 0 1 (word-loop WordNil) ` {0..1} ⊆ U
    proof
      fix y
      assume y ∈ subpathin 0 1 (word-loop WordNil) ` {0..1}
       then obtain t where t ∈ {0..1} and y-eq: y = subpathin 0 1 (word-loop
WordNil) t
        by blast
      have y = x0
        using y-eq by (simp add: word-loop.simps subpathin-def)
      then show y ∈ U
        using x0-in-U by simp
    qed
  qed
  have nil-hom:
   homotopic-paths U (segment-loop (word-loop WordNil) 0 1) (word-loop WordNil)
    by (rule segment-loop-base-full-in-set[OF nil-loopU])
  have nil-class:
    loop-class U x0 (segment-loop (word-loop WordNil) 0 1) = one1
  proof −
```

```
    have loop-class U x0 (segment-loop (word-loop WordNil) 0 1) =
        loop-class U x0 (word-loop WordNil)
      by (rule loop-class-eqI[OF nil-segU nil-loopU nil-hom])
    then show ?thesis
      unfolding word-loop.simps fundamental-group-one-def by simp
  qed
  have one1-in: one1 ∈ G1
    by (rule fundamental-group-one-in-space[OF x0-in-U])
  have red:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordLeft one1 WordNil) WordNil
    by (rule carrier-fpw-reduction.step,
        rule carrier-fpw-reduction-step.remove-left-one[OF one1-in], simp)
  show ?case
    using nil-class red
    by (simp add: carrier-full-amalg-equiv.of-reduction)
next
  case (WordLeft a rest)
  then have a-in: a ∈ G1 and rest-in: fpw-in-space G1 G2 rest
    by auto
  have a-loopU: some-loop U x0 a ∈ loop-space U x0
    using some-loop-spec[OF a-in] by auto
  have a-loopW: some-loop U x0 a ∈ loop-space W x0
    using a-loopU unfolding loop-space-def by auto
  show ?case
  proof (cases rest = WordNil)
    case True
    then show ?thesis
      using segment-loop-some-loop-left-class[OF a-in]
      by simp
  next
    case False
    have rest-loopW: word-loop rest ∈ loop-space W x0
      by (rule word-loop-in-W[OF rest-in])
    have rest-valid:
      valid-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      by (rule word-loop-valid-partition[OF rest-in])
    have rest-svk:
       svk-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      using rest-valid unfolding valid-partition-def by auto
    have rest-times-nonempty: word-partition-times rest ≠ []
      using rest-valid unfolding valid-partition-def by auto
    have rest-times-hd: hd (word-partition-times rest) = 0
      using rest-valid unfolding valid-partition-def by auto
    have rest-times: word-partition-times rest = 0 # tl (word-partition-times rest)
    proof (cases word-partition-times rest)
      case Nil
```

```
    with rest-times-nonempty show ?thesis
      by simp
  next
    case (Cons s ss)
    have s = 0
      using rest-times-hd Cons by simp
    then show ?thesis
      using Cons by simp
  qed
  have scaled-times:
    map (λt. (1 + t) / 2) (word-partition-times rest) =
      1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest))
  proof −
    have map (λt. (1 + t) / 2) (word-partition-times rest) =
        map (λt. (1 + t) / 2) (0 # tl (word-partition-times rest))
      using rest-times by simp
     also have ... = 1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times
rest))
      by simp
    finally show ?thesis .
  qed
  have tail-eq:
    partition-word (some-loop U x0 a +++ word-loop rest)
        (map (λt. (1 + t) / 2) (word-partition-times rest))
        (word-partition-bits rest) =
    partition-word (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
    by (rule partition-word-joinpaths-tail-scaled[OF a-loopW rest-loopW rest-svk])
  have head-eq:
    loop-class U x0
        (segment-loop (some-loop U x0 a +++ word-loop rest) 0 (1 / 2)) = a
    using segment-loop-some-loop-left-class[OF a-in]
    by (simp add: segment-loop-joinpaths-head[OF a-loopU rest-loopW])
  have tail-eq′:
    partition-word (some-loop U x0 a +++ word-loop rest)
        (1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
        (word-partition-bits rest) =
    partition-word (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
    using tail-eq scaled-times by simp
  have step-eq:
    partition-word (word-loop (WordLeft a rest))
        (word-partition-times (WordLeft a rest))
        (word-partition-bits (WordLeft a rest)) =
      WordLeft a
                  (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
  proof −
    have partition-word (word-loop (WordLeft a rest))
```

```
        (word-partition-times (WordLeft a rest))
        (word-partition-bits (WordLeft a rest)) =
       WordLeft (loop-class U x0
         (segment-loop (some-loop U x0 a +++ word-loop rest) 0 (1 / 2)))
        (partition-word (some-loop U x0 a +++ word-loop rest)
         (1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
         (word-partition-bits rest))
       using False scaled-times by simp
     also have ... =
       WordLeft a
                  (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
       using head-eq tail-eq' by simp
     finally show ?thesis .
   qed
   have tail-rel:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
                  (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
       rest
     by (rule WordLeft.IH[OF rest-in])
   have
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (WordLeft a
                  (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest)))
       (WordLeft a rest)
     by (rule carrier-full-amalg-equiv-left-context[OF tail-rel a-in])
   then show ?thesis
     by (subst step-eq) simp
 qed
next
 case (WordRight b rest)
 then have b-in: b ∈ G2 and rest-in: fpw-in-space G1 G2 rest
   by auto
 have b-loopV: some-loop V x0 b ∈ loop-space V x0
   using some-loop-spec[OF b-in] by auto
 have b-loopW: some-loop V x0 b ∈ loop-space W x0
   using b-loopV unfolding loop-space-def by auto
 show ?case
 proof (cases rest = WordNil)
   case True
   then show ?thesis
     using segment-loop-some-loop-right-class[OF b-in]
     by simp
 next
   case False
   have rest-loopW: word-loop rest ∈ loop-space W x0
     by (rule word-loop-in-W[OF rest-in])
```

```
    have rest-valid:
     valid-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      by (rule word-loop-valid-partition[OF rest-in])
    have rest-svk:
      svk-partition (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      using rest-valid unfolding valid-partition-def by auto
    have rest-times-nonempty: word-partition-times rest ≠ []
      using rest-valid unfolding valid-partition-def by auto
    have rest-times-hd: hd (word-partition-times rest) = 0
      using rest-valid unfolding valid-partition-def by auto
    have rest-times: word-partition-times rest = 0 # tl (word-partition-times rest)
    proof (cases word-partition-times rest)
      case Nil
      with rest-times-nonempty show ?thesis
        by simp
    next
      case (Cons s ss)
      have s = 0
        using rest-times-hd Cons by simp
      then show ?thesis
        using Cons by simp
    qed
    have scaled-times:
      map (λt. (1 + t) / 2) (word-partition-times rest) =
        1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest))
    proof −
      have map (λt. (1 + t) / 2) (word-partition-times rest) =
          map (λt. (1 + t) / 2) (0 # tl (word-partition-times rest))
        using rest-times by simp
       also have ... = 1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times
rest))
        by simp
      finally show ?thesis .
    qed
    have tail-eq:
      partition-word (some-loop V x0 b +++ word-loop rest)
          (map (λt. (1 + t) / 2) (word-partition-times rest))
          (word-partition-bits rest) =
       partition-word (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
      by (rule partition-word-joinpaths-tail-scaled[OF b-loopW rest-loopW rest-svk])
    have head-eq:
      loop-class V x0
          (segment-loop (some-loop V x0 b +++ word-loop rest) 0 (1 / 2)) = b
      using segment-loop-some-loop-right-class[OF b-in]
      by (simp add: segment-loop-joinpaths-head[OF b-loopV rest-loopW])
    have tail-eq′:
```

```
    partition-word (some-loop V x0 b +++ word-loop rest)
      (1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
      (word-partition-bits rest) =
    partition-word (word-loop rest) (word-partition-times rest) (word-partition-bits
rest)
    using tail-eq scaled-times by simp
  have step-eq:
    partition-word (word-loop (WordRight b rest))
      (word-partition-times (WordRight b rest))
      (word-partition-bits (WordRight b rest)) =
     WordRight b
                (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
  proof −
    have partition-word (word-loop (WordRight b rest))
      (word-partition-times (WordRight b rest))
      (word-partition-bits (WordRight b rest)) =
     WordRight (loop-class V x0
       (segment-loop (some-loop V x0 b +++ word-loop rest) 0 (1 / 2)))
      (partition-word (some-loop V x0 b +++ word-loop rest)
       (1 / 2 # map (λt. (1 + t) / 2) (tl (word-partition-times rest)))
       (word-partition-bits rest))
     using False scaled-times by simp
    also have ... =
     WordRight b
                (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
     using head-eq tail-eq′ by simp
    finally show ?thesis .
  qed
  have tail-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
              (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest))
     rest
    by (rule WordRight.IH[OF rest-in])
  have
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (WordRight b
                (partition-word (word-loop rest) (word-partition-times rest)
(word-partition-bits rest)))
     (WordRight b rest)
    by (rule carrier-full-amalg-equiv-right-context[OF tail-rel b-in])
  then show ?thesis
    by (subst step-eq) simp
 qed
qed

lemma valid-partition-hd:
```

assumes *valid-partition p ts bs*
shows *ts* $\neq$ [] *hd ts = 0*
using *assms* **unfolding** *valid-partition-def* **by** *auto*

lemma *valid-partition-cases*:
assumes *valid-partition p* (*t # ts*) *bs*
shows *t = 0* **and** *svk-partition p* (*t # ts*) *bs*
using *assms* **unfolding** *valid-partition-def* **by** *auto*

lemma *svk-partition-head-props*:
assumes *svk-partition p* (*t # ts*) *bs*
shows *t* $\in$ {*0..1*} **and** *p t* $\in$ *U* $\cap$ *V*
proof −
show *t* $\in$ {*0..1*}
proof (*cases ts*)
case *Nil*
then show *?thesis*
using *assms* **by** (*cases bs*) *auto*
next
case (*Cons u us*)
then show *?thesis*
using *assms* **by** (*cases bs*) *auto*
qed
next
show *p t* $\in$ *U* $\cap$ *V*
proof (*cases ts*)
case *Nil*
then show *?thesis*
using *assms* **by** (*cases bs*) *auto*
next
case (*Cons u us*)
then show *?thesis*
using *assms* **by** (*cases bs*) *auto*
qed
qed

lemma *svk-partition-tail*:
assumes *svk-partition p* (*t # u # ts*) (*b # bs*)
shows *svk-partition p* (*u # ts*) *bs*
using *assms* **by** *simp*

lemma *svk-partition-step-props*:
assumes *svk-partition p* (*t # u # ts*) (*b # bs*)
shows *t* $\in$ {*0..1*}
and *p t* $\in$ *U* $\cap$ *V*
and *u* $\in$ {*0..1*}
and *t* < *u*
and (*if b then subpathin t u p* ‘ {*0..1*} $\subseteq$ *U else subpathin t u p* ‘ {*0..1*} $\subseteq$ *V*)
using *assms* **by** *simp-all*

lemma *svk-partition-next-in-intersection*:
assumes *svk-partition p* (*t # u # ts*) (*b # bs*)
shows *p u* ∈ *U* ∩ *V*
proof −
have *svk-partition p* (*u # ts*) *bs*
using *assms* **by** *simp*
then show *?thesis*
by (*rule svk-partition-head-props*(*2*))
qed

lemma *svk-partition-nonempty*:
assumes *svk-partition p ts bs*
shows *ts* ≠ []
using *assms* **by** (*cases ts*) *simp-all*

lemma *svk-partition-last-eq-one*:
assumes *part*: *svk-partition p ts bs*
shows *last ts* = *1*
using *part*
proof (*induction ts arbitrary*: *bs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons t ts*)
show *?case*
proof (*cases ts*)
case *Nil*
then show *?thesis*
using *Cons.prems* **by** (*cases bs*) *auto*
next
case (*Cons u us*)
then obtain *b bs′* **where** *bs*: *bs* = *b # bs′*
using *Cons.prems* **by** (*cases bs*) *auto*
have *tail*: *svk-partition p* (*u # us*) *bs′*
using *Cons.prems Cons bs* **by** *simp*
from *Cons.IH*[*of bs′*] *Cons tail* **show** *?thesis*
by *simp*
qed
qed

lemma *svk-partition-last-in-intersection*:
assumes *part*: *svk-partition p ts bs*
shows *p* (*last ts*) ∈ *U* ∩ *V*
using *part*
proof (*induction ts arbitrary*: *bs*)
case *Nil*
then show *?case*

```
    by simp
next
  case (Cons t ts)
  show ?case
  proof (cases ts)
    case Nil
    then show ?thesis
      using Cons.prems by (cases bs) auto
  next
    case (Cons u us)
    then obtain b bs' where bs: bs = b # bs'
      using Cons.prems by (cases bs) auto
    have tail: svk-partition p (u # us) bs'
      using Cons.prems Cons bs by simp
    from Cons.IH[of bs'] Cons tail show ?thesis
      by simp
  qed
qed

lemma svk-partition-last-props:
  assumes part: svk-partition p ts bs
  shows ts ≠ [] and last ts = 1 and p (last ts) ∈ U ∩ V
  using svk-partition-nonempty[OF part]
    svk-partition-last-eq-one[OF part]
    svk-partition-last-in-intersection[OF part]
  by auto

lemma valid-partition-last-props:
  assumes valid-partition p ts bs
  shows ts ≠ [] and last ts = 1 and p (last ts) ∈ U ∩ V
proof −
  have ts-ne: ts ≠ []
    using assms unfolding valid-partition-def by auto
  have part: svk-partition p ts bs
    using assms unfolding valid-partition-def by auto
  show ts ≠ []
    by (rule ts-ne)
  from svk-partition-last-props[OF part] ts-ne
  show last ts = 1 and p (last ts) ∈ U ∩ V
    by auto
qed

lemma subpathin-endpoints-in-set:
  assumes seg-in: subpathin u v p ‘ {0..1} ⊆ S
  shows p u ∈ S and p v ∈ S
proof −
  have u-in: subpathin u v p 0 ∈ S
    using seg-in by auto
  then show p u ∈ S
```

by (*simp add*: *subpathin-def*)
have *v-in*: *subpathin u v p 1* ∈ *S*
using *seg-in* **by** *auto*
then show *p v* ∈ *S*
by (*simp add*: *subpathin-def*)
qed

lemma *subpathin-image-subset-union*:
assumes *tu*: *t* ≤ *u*
and *uv*: *u* ≤ *v*
shows *subpathin t v p* ‘ {*0..1*} ⊆ *subpathin t u p* ‘ {*0..1*} ∪ *subpathin u v p* ‘ {*0..1*}
proof −
have *seg-subset*: *closed-segment t v* ⊆ *closed-segment t u* ∪ *closed-segment u v*
using *tu uv* **by** (*auto simp*: *closed-segment-eq-real-ivl*)
show *?thesis*
using *seg-subset* **by** (*auto simp*: *subpathin-image*)
qed

lemma *subpathin-image-subset-trans*:
assumes *tu*: *t* ≤ *u*
and *uv*: *u* ≤ *v*
and *left*: *subpathin t u p* ‘ {*0..1*} ⊆ *S*
and *right*: *subpathin u v p* ‘ {*0..1*} ⊆ *S*
shows *subpathin t v p* ‘ {*0..1*} ⊆ *S*
using *subpathin-image-subset-union*[*OF tu uv*, *of p*] *left right* **by** *blast*

lemma *cover-partition-step-props*:
assumes *cover-partition p* (*t* # *u* # *ts*) (*b* # *bs*)
shows *t* ∈ {*0..1*}
and *u* ∈ {*0..1*}
and *t* < *u*
and (*if b then subpathin t u p* ‘ {*0..1*} ⊆ *U else subpathin t u p* ‘ {*0..1*} ⊆ *V*)
and *cover-partition p* (*u* # *ts*) *bs*
using *assms* **by** *simp-all*

lemma *cover-partition-consI*:
assumes *t* ∈ {*0..1*}
and *u* ∈ {*0..1*}
and *t* < *u*
and (*if b then subpathin t u p* ‘ {*0..1*} ⊆ *U else subpathin t u p* ‘ {*0..1*} ⊆ *V*)
and *cover-partition p* (*u* # *ts*) *bs*
shows *cover-partition p* (*t* # *u* # *ts*) (*b* # *bs*)
using *assms* **by** *simp*

lemma *cover-partition-switch-point*:
assumes *cp*: *cover-partition p* (*t* # *u* # *v* # *ts*) (*b* # *c* # *bs*)
and *diff*: *b* ≠ *c*
shows *p u* ∈ *U* ∩ *V*

proof (*cases b*)
case *True*
then have *leftU*: *subpathin t u p ‘ {0..1} ⊆ U*
using *cp* **by** *simp*
from *diff True* **have** *c-false*: ¬ *c*
by *simp*
then have *rightV*: *subpathin u v p ‘ {0..1} ⊆ V*
using *cp* **by** *simp*
show *?thesis*
using *subpathin-endpoints-in-set*(*2*)[*OF leftU*] *subpathin-endpoints-in-set*(*1*)[*OF rightV*]
by *blast*
next
case *False*
then have *leftV*: *subpathin t u p ‘ {0..1} ⊆ V*
using *cp* **by** *simp*
from *diff False* **have** *c-true*: *c*
by *simp*
then have *rightU*: *subpathin u v p ‘ {0..1} ⊆ U*
using *cp* **by** *simp*
show *?thesis*
using *subpathin-endpoints-in-set*(*1*)[*OF rightU*] *subpathin-endpoints-in-set*(*2*)[*OF leftV*]
by *blast*
qed

lemma *cover-partition-pair-svk-partition*:
assumes *cp*: *cover-partition p* [*t*, *u*] [*b*]
and *ptUV*: *p t ∈ U ∩ V*
and *puUV*: *p u ∈ U ∩ V*
shows *svk-partition p* [*t*, *u*] [*b*]
proof −
have *t01*: *t ∈ {0..1}*
using *cp* **by** *simp*
have *u01*: *u ∈ {0..1}*
using *cp* **by** *simp*
have *tu*: *t < u*
using *cp* **by** *simp*
have *u1*: *u = 1*
using *cp* **by** *simp*
have *seg*: (*if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V*)
proof (*cases b*)
case *True*
then show *?thesis*
using *cp u1* **by** *simp*
next
case *False*
then show *?thesis*

```
    using cp u1 by simp
  qed
  have tail: svk-partition p [u] []
    using u1 puUV by simp
  show ?thesis
  proof (cases b)
    case True
    then show ?thesis
      using t01 ptUV u01 tu seg u1 puUV by simp
  next
    case False
    then show ?thesis
      using t01 ptUV u01 tu seg u1 puUV by simp
  qed
qed

lemma cover-partition-compress-svk-partition:
  assumes cp: cover-partition p (t # ts) bs
    and ptUV: p t ∈ U ∩ V
    and plastUV: p (last (t # ts)) ∈ U ∩ V
  shows ∃ ts' bs'. svk-partition p (t # ts') bs'
  using assms
proof (induction length bs arbitrary: bs t ts rule: less-induct)
  case less
  show ?case
  proof (cases bs)
    case Nil
    show ?thesis
    proof (cases ts)
      case Nil-ts: Nil
      then have t1: t = 1
        using less.prems(1) Nil by simp
      have svk-partition p [t] []
        using t1 less.prems(2) by simp
      show ?thesis
      proof
        show ∃ bs'. svk-partition p (t # []) bs'
        proof
          show svk-partition p (t # []) []
            using t1 less.prems(2) by simp
        qed
      qed
    next
      case (Cons u us)
      then have False
        using less.prems(1) Nil by simp
      then show ?thesis
        by simp
    qed
```

```
  next
    case (Cons b bs0)
    note bs-cons = Cons
    show ?thesis
    proof (cases bs0)
      case Nil
      then obtain u where ts: ts = [u]
      proof (cases ts)
        case Nil
        then have False
          using less.prems(1) bs-cons Nil by simp
        then show thesis
          by simp
      next
        case (Cons u ts0)
        note ts-cons = Cons
        then show thesis
        proof (cases ts0)
          case Nil
          have ts-single: ts = [u]
            using ts-cons Nil by simp
          from ts-single show thesis
            by (rule that[of u])
        next
          case (Cons v us)
          then have False
            using less.prems(1) bs-cons Nil ts-cons by simp
          then show thesis
            by simp
        qed
      qed
      have puUV: p u ∈ U ∩ V
        using less.prems(3) unfolding ts by simp
      have t01: t ∈ {0..1}
        using less.prems(1) bs-cons Nil ts by simp
      have u01: u ∈ {0..1}
        using less.prems(1) bs-cons Nil ts by simp
      have tu: t < u
        using less.prems(1) bs-cons Nil ts by simp
      have u1: u = 1
        using less.prems(1) bs-cons Nil ts by simp
       have seg-tu: (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘
{0..1} ⊆ V)
      proof (cases b)
        case True
        then show ?thesis
          using less.prems(1) bs-cons Nil ts u1 by simp
      next
        case False
```

```
    then show ?thesis
      using less.prems(1) bs-cons Nil ts u1 by simp
  qed
  have part-tu: svk-partition p [t, u] [b]
  proof (cases b)
    case True
    then show ?thesis
      using t01 less.prems(2) u01 tu seg-tu u1 puUV by simp
  next
    case False
    then show ?thesis
      using t01 less.prems(2) u01 tu seg-tu u1 puUV by simp
  qed
  show ?thesis
  proof
    show ∃ bs'. svk-partition p (t # [u]) bs'
    proof
      show svk-partition p (t # [u]) [b]
        by (rule part-tu)
    qed
  qed
next
  case (Cons c bs')
  note bs0-cons = Cons
  have bs-eq: bs = b # c # bs'
    using bs-cons bs0-cons by simp
  then obtain u v us where ts: ts = u # v # us
  proof (cases ts)
    case Nil
    then have False
      using less.prems(1) bs-cons bs0-cons by simp
    then show thesis
      by simp
  next
    case (Cons u ts0)
    note ts-cons = Cons
    then show thesis
    proof (cases ts0)
      case Nil
      then have False
        using less.prems(1) bs-cons bs0-cons ts-cons by simp
      then show thesis
        by simp
    next
      case (Cons v us)
      have ts-long: ts = u # v # us
        using ts-cons Cons by simp
      from ts-long show thesis
        by (rule that[of u v us])
```

```
    qed
  qed
  have cp-step: cover-partition p (t # u # v # us) (b # c # bs')
    using less.prems(1) bs-cons bs0-cons ts by simp
  have t01: t ∈ {0..1} and u01: u ∈ {0..1} and tu: t < u
     and seg-tu: (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘
{0..1} ⊆ V)
    and cp-tail: cover-partition p (u # v # us) (c # bs')
    using cp-step by (rule cover-partition-step-props)+
  show ?thesis
  proof (cases b = c)
    case True
    note bits-eq = True
    have uv: u < v
      using cp-tail by simp
    have tu-le: t ≤ u
      using tu by simp
    have uv-le: u ≤ v
      using uv by simp
     have seg-tv: (if b then subpathin t v p ‘ {0..1} ⊆ U else subpathin t v p ‘
{0..1} ⊆ V)
    proof (cases b)
      case True
      then have leftU: subpathin t u p ‘ {0..1} ⊆ U
        using seg-tu by simp
      have rightU: subpathin u v p ‘ {0..1} ⊆ U
        using cp-tail bits-eq True by simp
      have seg-U: subpathin t v p ‘ {0..1} ⊆ U
        by (rule subpathin-image-subset-trans[OF tu-le uv-le leftU rightU])
      then show ?thesis
        using True by simp
    next
      case False
      then have leftV: subpathin t u p ‘ {0..1} ⊆ V
        using seg-tu by simp
      have rightV: subpathin u v p ‘ {0..1} ⊆ V
        using cp-tail bits-eq False by simp
      have seg-V: subpathin t v p ‘ {0..1} ⊆ V
        by (rule subpathin-image-subset-trans[OF tu-le uv-le leftV rightV])
      then show ?thesis
        using False by simp
    qed
    have cp-merged: cover-partition p (t # v # us) (b # bs')
      using cp-tail t01 seg-tv tu u01 by simp
    have shorter: length (b # bs') < length bs
      using bs-eq by simp
    have plast-merged: p (last (t # v # us)) ∈ U ∩ V
      using less.prems(3) unfolding ts by simp
    have merged-part: ∃ ts' bs''. svk-partition p (t # ts') bs''
```

```
      by (rule less.hyps[of b # bs' t v # us]) (use shorter cp-merged less.prems(2)
plast-merged in simp-all)
      from merged-part obtain ts' bs'' where svk-partition p (t # ts') bs''
        by blast
      then show ?thesis
        by blast
    next
      case False
      have puUV: p u ∈ U ∩ V
        by (rule cover-partition-switch-point[OF cp-step False])
      have shorter: length (c # bs') < length bs
        using bs-eq by simp
      have plast-tail: p (last (u # v # us)) ∈ U ∩ V
        using less.prems(3) unfolding ts by simp
      have tail-exists: ∃ us' bs''. svk-partition p (u # us') bs''
       by (rule less.hyps[of c # bs' u v # us]) (use shorter cp-tail puUV plast-tail
in simp-all)
      from tail-exists obtain us' bs'' where tail-part: svk-partition p (u # us')
bs''
        by blast
      have svk-partition p (t # u # us') (b # bs'')
        using t01 u01 tu seg-tu less.prems(2) puUV tail-part by simp
      then show ?thesis
        by blast
    qed
  qed
 qed
qed

lemma cover-partition-last-eq-one:
 assumes cp: cover-partition p ts bs
   and ts-ne: ts ≠ []
 shows last ts = 1
 using cp ts-ne
proof (induction ts arbitrary: bs)
 case Nil
 then show ?case
   by simp
next
 case (Cons t ts)
 show ?case
 proof (cases ts)
   case Nil
   then show ?thesis
     using Cons.prems by (cases bs) simp-all
 next
   case (Cons u us)
   then obtain b bs' where bs: bs = b # bs'
     using Cons.prems by (cases bs) auto
```

```
    have tail: cover-partition p (u # us) bs'
      using Cons.prems Cons bs by simp
    have ts-ne-tail: ts ≠ []
      using Cons by simp
    have last-ts: last ts = 1
      by (rule Cons.IH[of bs']) (use tail ts-ne-tail Cons in simp-all)
    have last-tail: last (u # us) = 1
      using last-ts Cons by simp
    then show ?thesis
      using Cons by simp
  qed
qed

lemma nat-real-div-in-unit-interval:
  assumes n-pos: 0 < n
    and i-le: i ≤ n
  shows real i / real n ∈ {0..1}
proof −
  have n-real-pos: 0 < real n
  proof −
    have real 0 < real n
      using n-pos by (rule less-imp-of-nat-less)
    then show ?thesis
      by simp
  qed
  have lower: 0 ≤ real i / real n
    using n-real-pos by (simp add: zero-le-divide-iff)
  have i-real-le: real i ≤ real n
    using i-le by simp
  have upper: real i / real n ≤ 1
  proof −
    have real i / real n ≤ real n / real n
      using i-real-le n-real-pos by (intro divide-right-mono) simp-all
    then show ?thesis
      using n-real-pos by simp
  qed
  show ?thesis
    using lower upper by auto
qed

lemma nat-real-div-strict-mono:
  assumes n-pos: 0 < n
    and i-lt: i < n
  shows real i / real n < real (Suc i) / real n
proof −
  have real i < real (Suc i)
    by simp
  then show ?thesis
    using n-pos by (simp add: divide-strict-right-mono)
```

qed

fun *subdivision-times* :: *nat* ⇒ *nat* ⇒ *real list* **where**
  subdivision-times n 0 = [*1*]
| *subdivision-times n* (*Suc k*) = *real* (*n* − *Suc k*) / *real n* # *subdivision-times n k*

fun *subdivision-bits* :: (*nat* ⇒ *bool*) ⇒ *nat* ⇒ *nat* ⇒ *bool list* **where**
  subdivision-bits side n 0 = []
| *subdivision-bits side n* (*Suc k*) = *side* (*n* − *Suc k*) # *subdivision-bits side n k*

lemma *cover-partition-subdivision-from*:
  assumes *n-pos*: *0* < *n*
    and *k-le*: *k* ≤ *n*
    and *cover*: ⋀*i*. *n* − *k* ≤ *i* ⟹ *i* < *n* ⟹
      (*if side i*
       then subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ {*0..1*} ⊆ *U*
       else subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ {*0..1*} ⊆ *V*)
  shows *cover-partition p* (*subdivision-times n k*) (*subdivision-bits side n k*)
  using *k-le cover*
proof (*induction k*)
  case *0*
  then show *?case*
    by *simp*
next
  case (*Suc k*)
  have *k-le-n*: *k* ≤ *n*
    using *Suc.prems* **by** *simp*
  have *i0-lt-n*: *n* − *Suc k* < *n*
    using *Suc.prems n-pos* **by** *arith*
  have *i0-le-n*: *n* − *Suc k* ≤ *n*
    by *arith*
  have *i1-le-n*: *n* − *k* ≤ *n*
    by *arith*
  have *nk*: *n* − *k* = *Suc* (*n* − *Suc k*)
    using *Suc.prems* **by** *arith*
  have *t01*: *real* (*n* − *Suc k*) / *real n* ∈ {*0..1*}
    by (*rule nat-real-div-in-unit-interval*[*OF n-pos i0-le-n*])
  have *u01*: *real* (*n* − *k*) / *real n* ∈ {*0..1*}
    by (*rule nat-real-div-in-unit-interval*[*OF n-pos i1-le-n*])
  have *tu*: *real* (*n* − *Suc k*) / *real n* < *real* (*n* − *k*) / *real n*
    unfolding *nk* **by** (*rule nat-real-div-strict-mono*[*OF n-pos i0-lt-n*])
  have *seg-side-suc*:
    (*if side* (*n* − *Suc k*)
      then subpathin (*real* (*n* − *Suc k*) / *real n*) (*real* (*Suc* (*n* − *Suc k*)) / *real n*)
p ‘ {*0..1*} ⊆ *U*
      else subpathin (*real* (*n* − *Suc k*) / *real n*) (*real* (*Suc* (*n* − *Suc k*)) / *real n*) *p*
‘ {*0..1*} ⊆ *V*)
    by (*rule Suc.prems*(*2*)[*of n* − *Suc k*]) (*use i0-lt-n* **in** *simp-all*)
  have *seg-side*:

```
    (if side (n − Suc k)
      then subpathin (real (n − Suc k) / real n) (real (n − k) / real n) p ‘ {0..1}
⊆ U
      else subpathin (real (n − Suc k) / real n) (real (n − k) / real n) p ‘ {0..1}
⊆ V)
  proof (cases side (n − Suc k))
    case True
    then show ?thesis
      using seg-side-suc nk by simp
  next
    case False
    then show ?thesis
      using seg-side-suc nk by simp
  qed
  have tail-cover:
    ⋀i. n − k ≤ i ⟹ i < n ⟹
      (if side i
       then subpathin (real i / real n) (real (Suc i) / real n) p ‘ {0..1} ⊆ U
       else subpathin (real i / real n) (real (Suc i) / real n) p ‘ {0..1} ⊆ V)
  proof −
    fix i
    assume i-lb: n − k ≤ i
      and i-lt: i < n
    have i-lb′: n − Suc k ≤ i
      using i-lb by arith
    show (if side i
      then subpathin (real i / real n) (real (Suc i) / real n) p ‘ {0..1} ⊆ U
      else subpathin (real i / real n) (real (Suc i) / real n) p ‘ {0..1} ⊆ V)
      by (rule Suc.prems(2)[OF i-lb′ i-lt])
  qed
  have tail-cp: cover-partition p (subdivision-times n k) (subdivision-bits side n k)
    by (rule Suc.IH[OF k-le-n tail-cover])
  show ?case
  proof (cases k)
    case 0
    have t0-nonneg: 0 ≤ real (n − Suc 0) / real n
      and t0-le1: real (n − Suc 0) / real n ≤ 1
      using t01 0 by auto
    have tu0: real (n − Suc 0) / real n < 1
      using tu 0 n-pos by simp
    have seg0-raw:
      (if side (n − Suc 0)
       then subpathin (real (n − Suc 0) / real n) (real (n − 0) / real n) p ‘ {0..1}
⊆ U
       else subpathin (real (n − Suc 0) / real n) (real (n − 0) / real n) p ‘ {0..1}
⊆ V)
      using seg-side
      unfolding 0
      by simp
```

```
    have end-eq: real (n − 0) / real n = 1
      using n-pos by simp
    have seg0:
      (if side (n − Suc 0)
        then subpathin (real (n − Suc 0) / real n) 1 p ‘ {0..1} ⊆ U
        else subpathin (real (n − Suc 0) / real n) 1 p ‘ {0..1} ⊆ V)
      using seg0-raw end-eq by (cases side (n − Suc 0)) simp-all
    have tail0: cover-partition p [1] []
      using tail-cp 0 by simp
    then show ?thesis
      unfolding 0 subdivision-times.simps subdivision-bits.simps
      using t0-nonneg t0-le1 tu0 seg0 tail0
      by simp
  next
    case (Suc j)
    have t01-suc: real (n − Suc (Suc j)) / real n ∈ {0..1}
      using t01 by (simp only: Suc)
    have u01-suc: real (n − Suc j) / real n ∈ {0..1}
      using u01 by (simp only: Suc)
    have tu-suc: real (n − Suc (Suc j)) / real n < real (n − Suc j) / real n
      using tu by (simp only: Suc)
    have seg-suc:
      (if side (n − Suc (Suc j))
        then subpathin (real (n − Suc (Suc j)) / real n) (real (n − Suc j) / real n)
p ‘ {0..1} ⊆ U
        else subpathin (real (n − Suc (Suc j)) / real n) (real (n − Suc j) / real n)
p ‘ {0..1} ⊆ V)
      using seg-side by (simp only: Suc)
    have tail-suc:
      cover-partition p
        (real (n − Suc j) / real n # subdivision-times n j)
        (side (n − Suc j) # subdivision-bits side n j)
          using tail-cp by (simp only: Suc subdivision-times.simps subdivi-
sion-bits.simps)
    have times-suc:
      subdivision-times n (Suc (Suc j)) =
      real (n − Suc (Suc j)) / real n # real (n − Suc j) / real n # subdivision-times
n j
      by simp
    have bits-suc:
      subdivision-bits side n (Suc (Suc j)) =
        side (n − Suc (Suc j)) # side (n − Suc j) # subdivision-bits side n j
      by simp
    show ?thesis
      unfolding Suc times-suc bits-suc
      by (rule cover-partition-consI[OF t01-suc u01-suc tu-suc seg-suc tail-suc])
  qed
qed
```

lemma *subdivision-times-start*:
assumes *n-pos*: $0 < n$
shows *subdivision-times n n* $=$ *0* $\#$ *subdivision-times n* $(n - 1)$
using *n-pos* **by** (*cases n*) *simp-all*

lemma *loop-has-valid-partition*:
assumes *p-loop*: $p \in$ *loop-space W x0*
shows $\exists$ *ts bs. valid-partition p ts bs*
proof −
obtain *n* :: *nat* **where** *n-pos*: $0 < n$
and *n-cover*:
$\forall i{<}n.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U \vee$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq V$
using *loop-subdivision-by-cover*[*OF p-loop*] **by** *blast*
let *?side* $= \lambda i.$ *subpathin* (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U$
have *raw*: *cover-partition p* (*subdivision-times n n*) (*subdivision-bits ?side n n*)
proof (*rule cover-partition-subdivision-from*[*OF n-pos le-refl*])
fix *i*
assume $n - n \leq i$ **and** *i-lt*: $i < n$
from *n-cover*[*rule-format*, *OF i-lt*]
show (*if ?side i*
then subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U$
else subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq V$)
by *auto*
qed
have *times0*: *subdivision-times n n* $=$ *0* $\#$ *subdivision-times n* $(n - 1)$
by (*rule subdivision-times-start*[*OF n-pos*])
have *times-ne*: *subdivision-times n n* $\neq$ []
using *times0* **by** *simp*
have *last1*: *last* (*subdivision-times n n*) $=$ *1*
by (*rule cover-partition-last-eq-one*[*OF raw times-ne*])
have *p0UV*: *p 0* $\in U \cap V$ **and** *p1UV*: *p 1* $\in U \cap V$
using *p-loop x0-in-UV* **unfolding** *loop-space-def pathstart-def pathfinish-def* **by** *auto*
have *raw0*: *cover-partition p* (*0* $\#$ *subdivision-times n* $(n - 1)$) (*subdivision-bits ?side n n*)
using *raw times0* **by** *simp*
have *plastUV*: *p* (*last* (*0* $\#$ *subdivision-times n* $(n - 1)$)) $\in U \cap V$
using *p1UV last1 times0* **by** *simp*
from *cover-partition-compress-svk-partition*[*OF raw0 p0UV plastUV*]
obtain *ts′ bs′* **where** *part*: *svk-partition p* (*0* $\#$ *ts′*) *bs′*
by *blast*
have *valid-partition p* (*0* $\#$ *ts′*) *bs′*
unfolding *valid-partition-def* **using** *part* **by** *simp*
then show *?thesis*
by *blast*
qed

lemma *svk-partition-partition-word-in-space*:
assumes *p-loop*: $p \in$ *loop-space W x0*
and *part*: *svk-partition p ts bs*
shows *fpw-in-space G1 G2* (*partition-word p ts bs*)
using *part*
proof (*induction ts arbitrary*: *bs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons t ts*)
from *p-loop* **have** *p-path*: *path p* **and** *p-image*: *path-image p* $\subseteq$ *W*
unfolding *loop-space-def* **by** *auto*
show *?case*
proof (*cases ts*)
case *Nil*
then show *?thesis*
using *Cons.prems* **by** (*cases bs*) *simp-all*
next
case (*Cons u us*)
then obtain *b bs'* **where** *bs*: $bs = b \# bs'$
using *Cons.prems* **by** (*cases bs*) *auto*
have *tail*: *svk-partition p* (*u # us*) *bs'*
using *Cons.prems Cons bs* **by** *simp*
have *tail-in*: *fpw-in-space G1 G2* (*partition-word p* (*u # us*) *bs'*)
using *p-loop Cons.IH*[*of bs'*] *Cons tail* **by** *simp*
have *t01*: $t \in \{0..1\}$ **and** *ptUV*: $p\ t \in U \cap V$
using *Cons.prems Cons bs* **by** *simp-all*
have *u01*: $u \in \{0..1\}$ **and** *seg-side*:
(*if b then subpathin t u p* ‘ $\{0..1\} \subseteq U$ *else subpathin t u p* ‘ $\{0..1\} \subseteq V$)
using *Cons.prems Cons bs* **by** *simp-all*
have *puUV*: $p\ u \in U \cap V$
using *svk-partition-next-in-intersection*[*of p t u us b bs'*] *Cons.prems Cons bs*
by *simp*
show *?thesis*
proof (*cases b*)
case *True*
have *segU*: *segment-loop p t u* $\in$ *loop-space U x0*
by (*rule segment-loop-in-U*[*OF p-path p-image t01 u01 ptUV puUV*]) (*use*
seg-side True **in** *simp*)
have *loop-class U x0* (*segment-loop p t u*) $\in$ *G1*
by (*rule loop-class-in-space*[*OF segU*])
then show *?thesis*
using *tail-in True bs Cons* **by** *simp*
next
case *False*
have *segV*: *segment-loop p t u* $\in$ *loop-space V x0*
by (*rule segment-loop-in-V*[*OF p-path p-image t01 u01 ptUV puUV*]) (*use*
seg-side False **in** *simp*)

```
      have loop-class V x0 (segment-loop p t u) ∈ G2
        by (rule loop-class-in-space[OF segV])
      then show ?thesis
        using tail-in False bs Cons by simp
    qed
  qed
qed

lemma valid-partition-partition-word-in-space:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p ts bs
  shows fpw-in-space G1 G2 (partition-word p ts bs)
  using assms unfolding valid-partition-def
  by (auto intro: svk-partition-partition-word-in-space)

lemma svk-partition-partition-loop-in-W:
  assumes p-loop: p ∈ loop-space W x0
    and part: svk-partition p ts bs
  shows partition-loop p ts ∈ loop-space W x0
  using part
proof (induction ts arbitrary: bs)
  case Nil
  then show ?case
    by (simp add: constant-loop-in-space[OF x0-in-W])
next
  case (Cons t ts)
  from p-loop have p-path: path p and p-image: path-image p ⊆ W
    unfolding loop-space-def by auto
  show ?case
  proof (cases ts)
    case Nil
    then show ?thesis
      by (simp add: constant-loop-in-space[OF x0-in-W])
  next
    case (Cons u us)
    then obtain b bs' where bs: bs = b # bs'
      using Cons.prems by (cases bs) auto
    have tail: svk-partition p (u # us) bs'
      using Cons.prems Cons bs by simp
    have tail-loop: partition-loop p (u # us) ∈ loop-space W x0
      using p-loop Cons.IH[of bs'] Cons tail by simp
    have t01: t ∈ {0..1} and ptUV: p t ∈ U ∩ V
      using Cons.prems Cons bs by simp-all
    have u01: u ∈ {0..1} and puUV: p u ∈ U ∩ V
      using Cons.prems Cons bs svk-partition-next-in-intersection[of p t u us b bs']
by simp-all
    have segW:
      segment-loop p t u ∈ loop-space W x0
    proof (rule segment-loop-in-W[OF p-path p-image t01 u01 ptUV puUV])
```

```
    have sub-imgW: path-image (subpathin t u p) ⊆ W
      using p-image path-image-subpathin-subset[OF t01 u01, of p] by blast
    show subpathin t u p ` {0..1} ⊆ W
      using sub-imgW by (simp add: path-image-def)
  qed
  have joined-loop:
    segment-loop p t u +++ partition-loop p (u # us) ∈ loop-space W x0
    by (rule loop-space-join[OF segW tail-loop])
  show ?thesis
    using joined-loop Cons by simp
 qed
qed

lemma valid-partition-partition-loop-in-W:
 assumes p-loop: p ∈ loop-space W x0
   and part: valid-partition p ts bs
 shows partition-loop p ts ∈ loop-space W x0
 using assms unfolding valid-partition-def
 by (auto intro: svk-partition-partition-loop-in-W)

lemma i1-loop-class-eq:
 assumes p-loop: p ∈ loop-space (U ∩ V) x0
 shows i1 (loop-class (U ∩ V) x0 p) = loop-class U x0 p
proof −
 have A-in: loop-class (U ∩ V) x0 p ∈ fundamental-group-space (U ∩ V) x0
   by (rule loop-class-in-space[OF p-loop])
 have i1 (loop-class (U ∩ V) x0 p) = loop-class U x0 (loop-image id p)
 proof (rule fundamental-group-map-eqI)
   show loop-class (U ∩ V) x0 p ∈ fundamental-group-space (U ∩ V) x0
     by (rule A-in)
   show p ∈ loop-space (U ∩ V) x0
     by (rule p-loop)
   show loop-class (U ∩ V) x0 p = loop-class (U ∩ V) x0 p
     by simp
   show continuous-on (U ∩ V) id
     by simp
   show id ∈ (U ∩ V) → U
     by auto
   show id x0 = x0
     by simp
 qed
 then show ?thesis
   by (simp add: loop-image-def)
qed

lemma i2-loop-class-eq:
 assumes p-loop: p ∈ loop-space (U ∩ V) x0
 shows i2 (loop-class (U ∩ V) x0 p) = loop-class V x0 p
proof −
```

```
  have A-in: loop-class (U ∩ V) x0 p ∈ fundamental-group-space (U ∩ V) x0
    by (rule loop-class-in-space[OF p-loop])
  have i2 (loop-class (U ∩ V) x0 p) = loop-class V x0 (loop-image id p)
  proof (rule fundamental-group-map-eqI)
    show loop-class (U ∩ V) x0 p ∈ fundamental-group-space (U ∩ V) x0
      by (rule A-in)
    show p ∈ loop-space (U ∩ V) x0
      by (rule p-loop)
    show loop-class (U ∩ V) x0 p = loop-class (U ∩ V) x0 p
      by simp
    show continuous-on (U ∩ V) id
      by simp
    show id ∈ (U ∩ V) → V
      by auto
    show id x0 = x0
      by simp
  qed
  then show ?thesis
    by (simp add: loop-image-def)
qed

lemma j1-segment-loop-eq:
  assumes segU: segment-loop p t u ∈ loop-space U x0
  shows j1 (loop-class U x0 (segment-loop p t u)) =
    loop-class W x0 (segment-loop p t u)
proof −
  have A-in: loop-class U x0 (segment-loop p t u) ∈ fundamental-group-space U x0
    by (rule loop-class-in-space[OF segU])
  have j1 (loop-class U x0 (segment-loop p t u)) =
      loop-class W x0 (loop-image id (segment-loop p t u))
  proof (rule fundamental-group-map-eqI)
    show loop-class U x0 (segment-loop p t u) ∈ fundamental-group-space U x0
      by (rule A-in)
    show segment-loop p t u ∈ loop-space U x0
      by (rule segU)
    show loop-class U x0 (segment-loop p t u) =
        loop-class U x0 (segment-loop p t u)
      by simp
    show continuous-on U id
      by simp
    show id ∈ U → W
      by auto
    show id x0 = x0
      by simp
  qed
  then show ?thesis
    by (simp add: loop-image-def)
qed
```

```
lemma j2-segment-loop-eq:
  assumes segV: segment-loop p t u ∈ loop-space V x0
  shows j2 (loop-class V x0 (segment-loop p t u)) =
    loop-class W x0 (segment-loop p t u)
proof −
  have A-in: loop-class V x0 (segment-loop p t u) ∈ fundamental-group-space V x0
    by (rule loop-class-in-space[OF segV])
  have j2 (loop-class V x0 (segment-loop p t u)) =
      loop-class W x0 (loop-image id (segment-loop p t u))
  proof (rule fundamental-group-map-eqI)
    show loop-class V x0 (segment-loop p t u) ∈ fundamental-group-space V x0
      by (rule A-in)
    show segment-loop p t u ∈ loop-space V x0
      by (rule segV)
    show loop-class V x0 (segment-loop p t u) =
        loop-class V x0 (segment-loop p t u)
      by simp
    show continuous-on V id
      by simp
    show id ∈ V → W
      by auto
    show id x0 = x0
      by simp
  qed
  then show ?thesis
    by (simp add: loop-image-def)
qed

lemma svk-partition-eval-partition-word:
  assumes p-loop: p ∈ loop-space W x0
    and part: svk-partition p ts bs
  shows svk-word-eval (partition-word p ts bs) =
    loop-class W x0 (partition-loop p ts)
  using part
proof (induction ts arbitrary: bs)
  case Nil
  then show ?case
    by (simp add: fundamental-group-one-def)
next
  case (Cons t ts)
  from p-loop have p-path: path p and p-image: path-image p ⊆ W
    unfolding loop-space-def by auto
  show ?case
  proof (cases ts)
    case ts-nil: Nil
    then show ?thesis
    proof (cases bs)
      case bs-nil: Nil
      have pw: partition-word p (t # ts) bs = WordNil
```

```
      using ts-nil bs-nil by simp
    have pl: partition-loop p (t # ts) = (λ-. x0)
      using ts-nil by simp
    have eval-nil0: svk-word-eval WordNil = oneW
      by (rule decode.carrier-full-amalg-eval.simps(1))
    have eval-nil: svk-word-eval WordNil = loop-class W x0 (λ-. x0)
      using eval-nil0 by (simp add: fundamental-group-one-def)
    have svk-word-eval (partition-word p (t # ts) bs) = svk-word-eval WordNil
      using pw by simp
    also have ... = loop-class W x0 (λ-. x0)
      by (rule eval-nil)
    also have ... = loop-class W x0 (partition-loop p (t # ts))
      using pl by simp
    finally show ?thesis
      .
  next
    case bs-cons: (Cons b bs')
    then show ?thesis
      using Cons.prems ts-nil by simp
  qed
next
  case (Cons u us)
  then obtain b bs' where bs: bs = b # bs'
    using Cons.prems by (cases bs) auto
  have tail: svk-partition p (u # us) bs'
    using Cons.prems Cons bs by simp
  have tail-ts: svk-partition p ts bs'
    using Cons tail by simp
  have tail-eval-ts:
    svk-word-eval (partition-word p ts bs') =
      loop-class W x0 (partition-loop p ts)
    by (rule Cons.IH[OF tail-ts])
  have tail-eval:
    svk-word-eval (partition-word p (u # us) bs') =
      loop-class W x0 (partition-loop p (u # us))
    using Cons tail-eval-ts by simp
  have tail-loop: partition-loop p (u # us) ∈ loop-space W x0
    by (rule svk-partition-partition-loop-in-W[OF p-loop tail])
  have t01: t ∈ {0..1} and ptUV: p t ∈ U ∩ V
    using Cons.prems Cons bs by simp-all
  have u01: u ∈ {0..1} and puUV: p u ∈ U ∩ V
    using Cons.prems Cons bs svk-partition-next-in-intersection[of p t u us b bs']
by simp-all
  have seg-side:
    (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V)
    using Cons.prems Cons bs by simp
  have segW:
    segment-loop p t u ∈ loop-space W x0
  proof (rule segment-loop-in-W[OF p-path p-image t01 u01 ptUV puUV])
```

```
    have sub-imgW: path-image (subpathin t u p) ⊆ W
      using p-image path-image-subpathin-subset[OF t01 u01, of p] by blast
    show subpathin t u p ‘ {0..1} ⊆ W
      using sub-imgW by (simp add: path-image-def)
  qed
  have segW-in: loop-class W x0 (segment-loop p t u) ∈ fundamental-group-space
W x0
    by (rule loop-class-in-space[OF segW])
   have tail-in: loop-class W x0 (partition-loop p (u # us)) ∈ fundamen-
tal-group-space W x0
    by (rule loop-class-in-space[OF tail-loop])
  have mult-eq:
    multW (loop-class W x0 (segment-loop p t u))
       (loop-class W x0 (partition-loop p (u # us))) =
     loop-class W x0 (segment-loop p t u +++ partition-loop p (u # us))
      by (rule fundamental-group-mult-eqI[OF segW-in tail-in segW tail-loop])
simp-all
  show ?thesis
  proof (cases b)
    case True
    have segU: segment-loop p t u ∈ loop-space U x0
      by (rule segment-loop-in-U[OF p-path p-image t01 u01 ptUV puUV]) (use
seg-side True in simp)
    have j1-eq:
      j1 (loop-class U x0 (segment-loop p t u)) =
        loop-class W x0 (segment-loop p t u)
      by (rule j1-segment-loop-eq[OF segU])
    show ?thesis
    proof −
      have svk-word-eval (partition-word p (t # ts) (True # bs′)) =
          multW (j1 (loop-class U x0 (segment-loop p t u)))
            (svk-word-eval (partition-word p (u # us) bs′))
        using Cons by simp
      also have ... =
          multW (loop-class W x0 (segment-loop p t u))
            (loop-class W x0 (partition-loop p (u # us)))
        using j1-eq tail-eval by simp
      also have ... =
          loop-class W x0 (segment-loop p t u +++ partition-loop p (u # us))
        by (rule mult-eq)
      also have ... = loop-class W x0 (partition-loop p (t # ts))
        using Cons by simp
      finally have branch-true:
        svk-word-eval (partition-word p (t # ts) (True # bs′)) =
          loop-class W x0 (partition-loop p (t # ts)) .
      have bs-true: bs = True # bs′
        using bs True by simp
      show ?thesis
        unfolding bs-true using branch-true .
```

```
    qed
  next
    case False
    have segV: segment-loop p t u ∈ loop-space V x0
      by (rule segment-loop-in-V[OF p-path p-image t01 u01 ptUV puUV]) (use
seg-side False in simp)
    have j2-eq:
      j2 (loop-class V x0 (segment-loop p t u)) =
        loop-class W x0 (segment-loop p t u)
      by (rule j2-segment-loop-eq[OF segV])
    show ?thesis
    proof −
      have svk-word-eval (partition-word p (t # ts) (False # bs')) =
          multW (j2 (loop-class V x0 (segment-loop p t u)))
            (svk-word-eval (partition-word p (u # us) bs'))
        using Cons by simp
      also have ... =
          multW (loop-class W x0 (segment-loop p t u))
            (loop-class W x0 (partition-loop p (u # us)))
        using j2-eq tail-eval by simp
      also have ... =
          loop-class W x0 (segment-loop p t u +++ partition-loop p (u # us))
        by (rule mult-eq)
      also have ... = loop-class W x0 (partition-loop p (t # ts))
        using Cons by simp
      finally have branch-false:
        svk-word-eval (partition-word p (t # ts) (False # bs')) =
          loop-class W x0 (partition-loop p (t # ts)) .
      have bs-false: bs = False # bs'
        using bs False by simp
      show ?thesis
        unfolding bs-false using branch-false .
    qed
  qed
qed

lemma valid-partition-eval-partition-word:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p ts bs
  shows svk-word-eval (partition-word p ts bs) =
    loop-class W x0 (partition-loop p ts)
proof −
  have svk-part: svk-partition p ts bs
    using part unfolding valid-partition-def by auto
  show ?thesis
    by (rule svk-partition-eval-partition-word[OF p-loop svk-part])
qed
```

lemma *valid-partition-decode-partition-word*:
assumes *p-loop*: $p \in$ *loop-space W x0*
and *part*: *valid-partition p ts bs*
shows *svk-decode* (*partition-word p ts bs*) =
loop-class W x0 (*partition-loop p ts*)
proof −
have *in-space*: *fpw-in-space G1 G2* (*partition-word p ts bs*)
by (*rule valid-partition-partition-word-in-space*[*OF p-loop part*])
show *?thesis*
using *valid-partition-eval-partition-word*[*OF p-loop part*]
by (*simp add*: *svk-decode-eq-eval*[*OF in-space*])
qed

lemma *pair-interval-member*:
fixes *x y* :: *real* × *real* **and** *x1 x2 y1 y2 u v* :: *real*
assumes *x*: $x = (x1, x2)$
and *y*: $y = (y1, y2)$
and *mix1*: $u *_R x1 + v *_R y1 \in \{0..1\}$
and *mix2*: $u *_R x2 + v *_R y2 \in \{0..1\}$
shows $u *_R x + v *_R y \in \{0..1\} \times \{0..1\}$
proof −
have *pair-form*:
$u *_R x + v *_R y =$
$(u *_R x1 + v *_R y1, u *_R x2 + v *_R y2)$
using *x y* **by** *simp*
have *pair-in-Q*:
$(u *_R x1 + v *_R y1, u *_R x2 + v *_R y2) \in \{0..1\} \times \{0..1\}$
using *mix1 mix2* **by** *auto*
from *pair-in-Q* **show** *?thesis*
by (*subst pair-form*) *simp*
qed

lemma *affine-closed-segment-member*:
fixes *a b u* :: *real*
assumes *u01*: $u \in \{0..1\}$
shows $(b - a) * u + a \in$ *closed-segment a b*
proof −
have $(b - a) * u + a =$ *linepath a b u*
by (*simp add*: *linepath-def algebra-simps*)
moreover have *linepath a b u* ∈ *closed-segment a b*
using *u01* **by** (*rule linepath-in-path*)
ultimately show *?thesis*
by *simp*
qed

lemma *affine-subinterval-member*:
fixes *a b u* :: *real*
assumes *ab*: $a \leq b$
and *u01*: $u \in \{0..1\}$

```
  shows (b − a) ∗ u + a ∈ {a..b}
proof −
  have (b − a) ∗ u + a ∈ closed-segment a b
    by (rule affine-closed-segment-member[OF u01])
  also have closed-segment a b = {a..b}
    by (rule closed-segment-eq-real-ivl1[OF ab])
  finally show ?thesis .
qed

lemma affine-unit-interval-member:
  fixes a b u :: real
  assumes a01: a ∈ {0..1}
    and b01: b ∈ {0..1}
    and ab: a ≤ b
    and u01: u ∈ {0..1}
  shows (b − a) ∗ u + a ∈ {0..1}
proof −
  have (b − a) ∗ u + a ∈ {a..b}
    by (rule affine-subinterval-member[OF ab u01])
  moreover have {a..b} ⊆ {0..1}
    using a01 b01 ab by auto
  ultimately show ?thesis
    by blast
qed

lemma square-edge-homotopic:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set S) h
  shows homotopic-paths S
    ((λt. h (t, 0)) +++ (λt. h (1, t)))
    ((λt. h (0, t)) +++ (λt. h (t, 1)))
proof −
  let ?g = linepath (0::real, 0::real) (1, 0) +++ linepath (1, 0) (1, 1)
  let ?k = linepath (0::real, 0::real) (0, 1) +++ linepath (0, 1) (1, 1)
  have hk:
    homotopic-paths ({0..1} × {0..1}) ?g ?k
  proof (rule homotopic-paths-linear)
    show path ?g path ?k
      by simp-all
    show pathstart ?k = pathstart ?g pathfinish ?k = pathfinish ?g
      by simp-all
    show closed-segment (?g t) (?k t) ⊆ {0..1} × {0..1} if t ∈ {0..1} for t
    proof (rule closed-segment-subset)
      show ?g t ∈ {0..1} × {0..1} ?k t ∈ {0..1} × {0..1}
        using that by (auto simp: joinpaths-def linepath-def)
      show convex (({0::real..1}) × ({0::real..1}))
        by (intro convex-Times) auto
    qed
  qed
```

```
  from h-cont have h-on: continuous-on ({0..1} × {0..1}) h
    and h-into: h ∈ ({0..1} × {0..1}) → S
    by simp-all
  have img:
    homotopic-paths S (h ∘ ?g) (h ∘ ?k)
    by (rule homotopic-paths-continuous-image[OF hk h-on h-into])
  have g-eq: h ∘ ?g = ((λt. h (t, 0)) +++ (λt. h (1, t)))
    by (rule ext) (simp add: joinpaths-def linepath-def)
  have k-eq: h ∘ ?k = ((λt. h (0, t)) +++ (λt. h (t, 1)))
    by (rule ext) (simp add: joinpaths-def linepath-def)
  show ?thesis
    using img unfolding g-eq k-eq .
qed

lemma rectangle-edge-homotopic:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set S) h
    and a01: a ∈ {0..1} and b01: b ∈ {0..1}
    and c01: c ∈ {0..1} and d01: d ∈ {0..1}
    and ab: a ≤ b and cd: c ≤ d
  shows homotopic-paths S
    (subpathin a b (λt. h (t, c)) +++ (λt. h (b, (d − c) ∗ t + c)))
    ((λt. h (a, (d − c) ∗ t + c)) +++ subpathin a b (λt. h (t, d)))
proof −
  let ?Q = {0..1} × {0..1}
  let ?r = λz::real × real. ((b − a) ∗ fst z + a, (d − c) ∗ snd z + c)
  have r-on: continuous-on ?Q ?r
    by (intro continuous-intros)
  have r-into: ?r ∈ ?Q → ?Q
  proof
    fix z :: real × real
    assume z-in: z ∈ ?Q
    then have z1: fst z ∈ {0..1} and z2: snd z ∈ {0..1}
      by auto
    have x-in: (b − a) ∗ fst z + a ∈ {0..1}
      by (rule affine-unit-interval-member[OF a01 b01 ab z1])
    have y-in: (d − c) ∗ snd z + c ∈ {0..1}
      by (rule affine-unit-interval-member[OF c01 d01 cd z2])
    show ?r z ∈ ?Q
      using x-in y-in by simp
  qed
  from h-cont have h-on: continuous-on ?Q h and h-into: h ∈ ?Q → S
    by simp-all
  have hr-on: continuous-on ?Q (h ∘ ?r)
  proof −
    have continuous-on ?Q (λx. h (?r x))
    proof (rule continuous-on-compose2[OF h-on])
      show continuous-on ?Q ?r
        by (rule r-on)
```

```
    show ?r ‘ ?Q ⊆ ?Q
      using r-into by auto
  qed
  then show ?thesis
    by (simp add: comp-def)
qed
have hr-into: (h ∘ ?r) ∈ ?Q → S
proof
  fix z :: real × real
  assume z-in: z ∈ ?Q
  obtain x y where z: z = (x, y)
    by (cases z)
  have x01: x ∈ {0..1} and y01: y ∈ {0..1}
    using z-in z by auto
  have rx-in: (b − a) ∗ x + a ∈ {0..1}
    by (rule affine-unit-interval-member[OF a01 b01 ab x01])
  have ry-in: (d − c) ∗ y + c ∈ {0..1}
    by (rule affine-unit-interval-member[OF c01 d01 cd y01])
  have rz-in: ?r z ∈ ?Q
    using z rx-in ry-in by simp
  have hz-in: h (?r z) ∈ S
    using h-into rz-in by auto
  show (h ∘ ?r) z ∈ S
    using hz-in by (simp add: comp-def)
qed
have hr-cont: continuous-map (top-of-set ?Q) (top-of-set S) (h ∘ ?r)
  using hr-on hr-into by simp
have base:
  homotopic-paths S
    ((λt. (h ∘ ?r) (t, 0)) +++ (λt. (h ∘ ?r) (1, t)))
    ((λt. (h ∘ ?r) (0, t)) +++ (λt. (h ∘ ?r) (t, 1)))
  by (rule square-edge-homotopic[OF hr-cont])
have left-eq:
  ((λt. (h ∘ ?r) (t, 0)) +++ (λt. (h ∘ ?r) (1, t))) =
    (subpathin a b (λt. h (t, c)) +++ (λt. h (b, (d − c) ∗ t + c)))
  by (rule ext) (simp add: subpathin-def joinpaths-def)
have right-eq:
  ((λt. (h ∘ ?r) (0, t)) +++ (λt. (h ∘ ?r) (t, 1))) =
    ((λt. h (a, (d − c) ∗ t + c)) +++ subpathin a b (λt. h (t, d)))
  by (rule ext) (simp add: subpathin-def joinpaths-def)
show ?thesis
proof (subst left-eq[symmetric], subst right-eq[symmetric])
  show homotopic-paths S
      (((λt. (h ∘ ?r) (t, 0)) +++ (λt. (h ∘ ?r) (1, t))))
      (((λt. (h ∘ ?r) (0, t)) +++ (λt. (h ∘ ?r) (t, 1))))
    by (rule base)
qed
qed
```

lemma *rectangle-edge-homotopic-in-set*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *h-cont*: *continuous-map* (*top-of-set* ($\{0..1\} \times \{0..1\}$)) (*top-of-set* W) h
and *a01*: $a \in \{0..1\}$ **and** *b01*: $b \in \{0..1\}$
and *c01*: $c \in \{0..1\}$ **and** *d01*: $d \in \{0..1\}$
and *ab*: $a \leq b$ **and** *cd*: $c \leq d$
and *rectS*: $h \text{ ` } (\{a..b\} \times \{c..d\}) \subseteq S$
shows *homotopic-paths* S
(*subpathin a b* ($\lambda t.\ h\ (t,\ c)$)) +++ ($\lambda t.\ h\ (b,\ (d - c) * t + c)$)))
(($\lambda t.\ h\ (a,\ (d - c) * t + c)$) +++ *subpathin a b* ($\lambda t.\ h\ (t,\ d)$)))
proof −
let *?Q* = $\{0..1\} \times \{0..1\}$
let *?R* = $\{a..b\} \times \{c..d\}$
let *?r* = $\lambda z{::}real \times real.\ ((b - a) * fst\ z + a,\ (d - c) * snd\ z + c)$
have *r-on*: *continuous-on ?Q ?r*
by (*intro continuous-intros*)
have *r-into*: $?r \in ?Q \to ?Q$
proof
fix $z :: real \times real$
assume *z-in*: $z \in ?Q$
then have *z1*: $fst\ z \in \{0..1\}$ **and** *z2*: $snd\ z \in \{0..1\}$
by *auto*
have *x-in*: $(b - a) * fst\ z + a \in \{0..1\}$
by (*rule affine-unit-interval-member*[*OF a01 b01 ab z1*])
have *y-in*: $(d - c) * snd\ z + c \in \{0..1\}$
by (*rule affine-unit-interval-member*[*OF c01 d01 cd z2*])
show $?r\ z \in ?Q$
using *x-in y-in* **by** *simp*
qed
have *r-rect*: $?r \text{ ` } ?Q \subseteq ?R$
proof
fix x
assume $x \in ?r \text{ ` } ?Q$
then obtain z **where** *z-in*: $z \in ?Q$ **and** *x-eq*: $x = ?r\ z$
by *blast*
have *x1*: $(b - a) * fst\ z + a \in \{a..b\}$
by (*rule affine-subinterval-member*[*OF ab*]) (*use z-in* **in** *auto*)
have *x2*: $(d - c) * snd\ z + c \in \{c..d\}$
by (*rule affine-subinterval-member*[*OF cd*]) (*use z-in* **in** *auto*)
show $x \in ?R$
using *x1 x2 x-eq* **by** *simp*
qed
from *h-cont* **have** *h-on*: *continuous-on ?Q h*
and *h-intoW*: $h \in ?Q \to W$
by *simp-all*
have *hr-on*: *continuous-on ?Q* ($h \circ ?r$)
proof −
have *continuous-on ?Q* ($\lambda x.\ h\ (?r\ x)$)

```
    proof (rule continuous-on-compose2[OF h-on])
      show continuous-on ?Q ?r
        by (rule r-on)
      show ?r ‘ ?Q ⊆ ?Q
        using r-into by auto
    qed
    then show ?thesis
      by (simp add: comp-def)
  qed
  have hr-into: (h ∘ ?r) ∈ ?Q → S
  proof
    fix z :: real × real
    assume z-in: z ∈ ?Q
    then have rz-in: ?r z ∈ ?R
    proof −
      have ?r z ∈ ?r ‘ ?Q
        using z-in by blast
      then show ?thesis
        using r-rect by blast
    qed
    show (h ∘ ?r) z ∈ S
      using rectS rz-in by auto
  qed
  have hr-cont: continuous-map (top-of-set ?Q) (top-of-set S) (h ∘ ?r)
    using hr-on hr-into by simp
  have base:
    homotopic-paths S
      ((λt. (h ∘ ?r) (t, 0)) +++ (λt. (h ∘ ?r) (1, t)))
      ((λt. (h ∘ ?r) (0, t)) +++ (λt. (h ∘ ?r) (t, 1)))
    by (rule square-edge-homotopic[OF hr-cont])
  have left-eq:
    ((λt. (h ∘ ?r) (t, 0)) +++ (λt. (h ∘ ?r) (1, t))) =
      (subpathin a b (λt. h (t, c)) +++ (λt. h (b, (d − c) ∗ t + c)))
    by (rule ext) (simp add: subpathin-def joinpaths-def)
  have right-eq:
    ((λt. (h ∘ ?r) (0, t)) +++ (λt. (h ∘ ?r) (t, 1))) =
      ((λt. h (a, (d − c) ∗ t + c)) +++ subpathin a b (λt. h (t, d)))
    by (rule ext) (simp add: subpathin-def joinpaths-def)
  show ?thesis
    using base unfolding left-eq right-eq .
qed

definition vertical-strip-path ::
  ((real × real) ⇒ 'a) ⇒ real ⇒ real ⇒ real ⇒ real ⇒ 'a
where
  vertical-strip-path h t c d = (λu. h (t, (d − c) ∗ u + c))

definition bridge-loop ::
  ((real × real) ⇒ 'a) ⇒ real ⇒ real ⇒ real ⇒ real ⇒ 'a
```

where
bridge-loop h t c d =
(*connector* (*h* (*t*, *c*)) +++ *vertical-strip-path h t c d*) +++
reversepath (*connector* (*h* (*t*, *d*)))

lemma *bridge-loop-eq-segment-loop* [*simp*]:
bridge-loop h t c d = *segment-loop* (*vertical-strip-path h t c d*) *0 1*
unfolding *bridge-loop-def vertical-strip-path-def segment-loop-def subpathin-def*
by (*rule ext*) *simp*

lemma *vertical-strip-path-image-subset*:
assumes *cd*: $c \leq d$
shows *vertical-strip-path h t c d* ‘ $\{0..1\} \subseteq h$ ‘ $(\{t\} \times \{c..d\})$
proof
fix *x*
assume *x-in*: $x \in$ *vertical-strip-path h t c d* ‘ $\{0..1\}$
then obtain *u* **where** *u01*: $u \in \{0..1\}$ **and** *x-eq*: $x =$ *vertical-strip-path h t c d u*
by *blast*
have *yc*: $(d - c) * u + c \in \{c..d\}$
by (*rule affine-subinterval-member*[*OF cd u01*])
show $x \in h$ ‘ $(\{t\} \times \{c..d\})$
using *yc x-eq* **unfolding** *vertical-strip-path-def* **by** *auto*
qed

lemma *rectangle-segment-loop-bridge-homotopic*:
fixes *h* :: (*real* × *real*) ⇒ *'a*
assumes *h-cont*: *continuous-map* (*top-of-set* ($\{0..1\} \times \{0..1\}$)) (*top-of-set W*) *h*
and *a01*: $a \in \{0..1\}$ **and** *b01*: $b \in \{0..1\}$
and *c01*: $c \in \{0..1\}$ **and** *d01*: $d \in \{0..1\}$
and *ab*: $a \leq b$ **and** *cd*: $c \leq d$
and *rectS*: *h* ‘ $(\{a..b\} \times \{c..d\}) \subseteq S$
and *leftUV*: *h* ‘ $(\{a\} \times \{c..d\}) \subseteq U \cap V$
and *rightUV*: *h* ‘ $(\{b\} \times \{c..d\}) \subseteq U \cap V$
and *UVS*: $U \cap V \subseteq S$
shows *homotopic-paths S*
(*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b* +++ *bridge-loop h b c d*)
(*bridge-loop h a c d* +++ *segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*)
proof −
let *?pc* = λ*t*. *h* (*t*, *c*)
let *?pd* = λ*t*. *h* (*t*, *d*)
let *?la* = *vertical-strip-path h a c d*
let *?lb* = *vertical-strip-path h b c d*
let *?ac* = *connector* (*h* (*a*, *c*))
let *?ad* = *connector* (*h* (*a*, *d*))
let *?bc* = *connector* (*h* (*b*, *c*))
let *?bd* = *connector* (*h* (*b*, *d*))
let *?bot* = *subpathin a b ?pc*

```
let ?top = subpathin a b ?pd
have h-on: continuous-on ({0..1} × {0..1}) h
  and h-intoW: h ∈ ({0..1} × {0..1}) → W
  using h-cont by simp-all
have x0S: x0 ∈ S
  using x0-in-UV UVS by blast

have acUV: h (a, c) ∈ U ∩ V
  using leftUV cd by auto
have adUV: h (a, d) ∈ U ∩ V
  using leftUV cd by auto
have bcUV: h (b, c) ∈ U ∩ V
  using rightUV cd by auto
have bdUV: h (b, d) ∈ U ∩ V
  using rightUV cd by auto

have pc-path: path ?pc
proof −
  have continuous-on {0..1} ?pc
  proof (rule continuous-on-compose2[OF h-on])
    show continuous-on {0..1} (λt. (t, c))
      by (intro continuous-intros)
    show (λt. (t, c)) ‘ {0..1} ⊆ {0..1} × {0..1}
      using c01 by auto
  qed
  then show ?thesis
    by (simp add: path-def)
qed
have pd-path: path ?pd
proof −
  have continuous-on {0..1} ?pd
  proof (rule continuous-on-compose2[OF h-on])
    show continuous-on {0..1} (λt. (t, d))
      by (intro continuous-intros)
    show (λt. (t, d)) ‘ {0..1} ⊆ {0..1} × {0..1}
      using d01 by auto
  qed
  then show ?thesis
    by (simp add: path-def)
qed
have la-path: path ?la
proof −
  have continuous-on {0..1} ?la
  proof −
    have continuous-on {0..1} (λu. h (a, (d − c) ∗ u + c))
    proof (rule continuous-on-compose2[OF h-on])
      show continuous-on {0..1} (λu. (a, (d − c) ∗ u + c))
        by (intro continuous-intros)
      show (λu. (a, (d − c) ∗ u + c)) ‘ {0..1} ⊆ {0..1} × {0..1}
```

```
      proof
        fix x
        assume x ∈ (λu. (a, (d − c) ∗ u + c)) ‘ {0..1}
        then obtain u where u01: u ∈ {0..1} and x-eq: x = (a, (d − c) ∗ u +
c)
          by blast
        have y-in: (d − c) ∗ u + c ∈ {0..1}
          by (rule affine-unit-interval-member[OF c01 d01 cd u01])
        show x ∈ {0..1} × {0..1}
          using a01 y-in x-eq by auto
      qed
    qed
    then show ?thesis
      by (simp add: vertical-strip-path-def)
  qed
  then show ?thesis
    by (simp add: vertical-strip-path-def path-def)
qed
have lb-path: path ?lb
proof −
  have continuous-on {0..1} ?lb
  proof −
    have continuous-on {0..1} (λu. h (b, (d − c) ∗ u + c))
    proof (rule continuous-on-compose2[OF h-on])
      show continuous-on {0..1} (λu. (b, (d − c) ∗ u + c))
        by (intro continuous-intros)
      show (λu. (b, (d − c) ∗ u + c)) ‘ {0..1} ⊆ {0..1} × {0..1}
      proof
        fix x
        assume x ∈ (λu. (b, (d − c) ∗ u + c)) ‘ {0..1}
        then obtain u where u01: u ∈ {0..1} and x-eq: x = (b, (d − c) ∗ u +
c)
          by blast
        have y-in: (d − c) ∗ u + c ∈ {0..1}
          by (rule affine-unit-interval-member[OF c01 d01 cd u01])
        show x ∈ {0..1} × {0..1}
          using b01 y-in x-eq by auto
      qed
    qed
    then show ?thesis
      by (simp add: vertical-strip-path-def)
  qed
  then show ?thesis
    by (simp add: vertical-strip-path-def path-def)
qed

have pc-imgW: path-image ?pc ⊆ W
  using h-intoW c01 by (auto simp: path-image-def)
have pd-imgW: path-image ?pd ⊆ W
```

```
  using h-intoW d01 by (auto simp: path-image-def)
 have bot-imgS: ?bot ‘ {0..1} ⊆ S
  using rectS ab cd by (auto simp: subpathin-image-eq)
 have top-imgS: ?top ‘ {0..1} ⊆ S
  using rectS ab cd by (auto simp: subpathin-image-eq)
 have la-imgS: path-image ?la ⊆ S
 proof −
  have vs-subset: vertical-strip-path h a c d ‘ {0..1} ⊆ h ‘ ({a} × {c..d})
   by (rule vertical-strip-path-image-subset[OF cd])
  have path-image ?la ⊆ U ∩ V
   using vs-subset leftUV by (auto simp: path-image-def)
  then show ?thesis
   using UVS by blast
 qed
 have lb-imgS: path-image ?lb ⊆ S
 proof −
  have vs-subset: vertical-strip-path h b c d ‘ {0..1} ⊆ h ‘ ({b} × {c..d})
   by (rule vertical-strip-path-image-subset[OF cd])
  have path-image ?lb ⊆ U ∩ V
   using vs-subset rightUV by (auto simp: path-image-def)
  then show ?thesis
   using UVS by blast
 qed
 have la-imgUV: path-image ?la ⊆ U ∩ V
 proof −
  have vs-subset: vertical-strip-path h a c d ‘ {0..1} ⊆ h ‘ ({a} × {c..d})
   by (rule vertical-strip-path-image-subset[OF cd])
  show ?thesis
   using vs-subset leftUV by (auto simp: path-image-def)
 qed
 have lb-imgUV: path-image ?lb ⊆ U ∩ V
 proof −
  have vs-subset: vertical-strip-path h b c d ‘ {0..1} ⊆ h ‘ ({b} × {c..d})
   by (rule vertical-strip-path-image-subset[OF cd])
  show ?thesis
   using vs-subset rightUV by (auto simp: path-image-def)
 qed

 have ac-path: path ?ac and ac-imgS: path-image ?ac ⊆ S
   using connector-path[OF acUV] connector-image-subset[OF acUV] UVS by
blast+
 have ad-path: path ?ad and ad-imgS: path-image ?ad ⊆ S
   using connector-path[OF adUV] connector-image-subset[OF adUV] UVS by
blast+
 have bc-path: path ?bc and bc-imgS: path-image ?bc ⊆ S
   using connector-path[OF bcUV] connector-image-subset[OF bcUV] UVS by
blast+
 have bd-path: path ?bd and bd-imgS: path-image ?bd ⊆ S
   using connector-path[OF bdUV] connector-image-subset[OF bdUV] UVS by
```

blast+

have *segc-loop*: *segment-loop ?pc a b ∈ loop-space S x0*
proof (*rule segment-loop-in-set*[**where** *S = S*])
show *path ?pc*
by (*rule pc-path*)
show *path-image ?pc ⊆ W*
by (*rule pc-imgW*)
show *a ∈ {0..1} b ∈ {0..1}*
by (*rule a01*, *rule b01*)
show *?pc a ∈ U ∩ V ?pc b ∈ U ∩ V*
using *acUV bcUV* **by** *simp-all*
show *path-image* (*connector* (*?pc a*)) *⊆ S*
using *connector-image-subset*[*OF acUV*] *UVS* **by** *blast*
show *path-image* (*connector* (*?pc b*)) *⊆ S*
using *connector-image-subset*[*OF bcUV*] *UVS* **by** *blast*
show *?bot ‘ {0..1} ⊆ S*
by (*rule bot-imgS*)
show *x0 ∈ S*
by (*rule x0S*)
qed
have *segd-loop*: *segment-loop ?pd a b ∈ loop-space S x0*
proof (*rule segment-loop-in-set*[**where** *S = S*])
show *path ?pd*
by (*rule pd-path*)
show *path-image ?pd ⊆ W*
by (*rule pd-imgW*)
show *a ∈ {0..1} b ∈ {0..1}*
by (*rule a01*, *rule b01*)
show *?pd a ∈ U ∩ V ?pd b ∈ U ∩ V*
using *adUV bdUV* **by** *simp-all*
show *path-image* (*connector* (*?pd a*)) *⊆ S*
using *connector-image-subset*[*OF adUV*] *UVS* **by** *blast*
show *path-image* (*connector* (*?pd b*)) *⊆ S*
using *connector-image-subset*[*OF bdUV*] *UVS* **by** *blast*
show *?top ‘ {0..1} ⊆ S*
by (*rule top-imgS*)
show *x0 ∈ S*
by (*rule x0S*)
qed
have *bridge-a-loop*: *bridge-loop h a c d ∈ loop-space S x0*
unfolding *bridge-loop-eq-segment-loop*
proof (*rule segment-loop-in-set*[**where** *S = S*])
show *path ?la*
by (*rule la-path*)
show *path-image ?la ⊆ W*
using *la-imgUV* **by** *blast*
show (*0*::*real*) *∈ {0..1}*
by *simp*

show (*1*::*real*) ∈ {*0*..*1*}
by *simp*
show *?la 0* ∈ *U* ∩ *V ?la 1* ∈ *U* ∩ *V*
using *acUV adUV* **by** (*simp-all add*: *vertical-strip-path-def*)
show *path-image* (*connector* (*?la 0*)) ⊆ *S*
using *connector-image-subset*[*OF acUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
show *path-image* (*connector* (*?la 1*)) ⊆ *S*
using *connector-image-subset*[*OF adUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
have *edge-S*: *h* ‘ ({*a*} × {*c*..*d*}) ⊆ *S*
by (*rule order-trans*[*OF leftUV UVS*])
have *la-imgS'*: *vertical-strip-path h a c d* ‘ {*0*..*1*} ⊆ *S*
by (*rule order-trans*[*OF vertical-strip-path-image-subset*[*OF cd*] *edge-S*])
have *la-eq*: *?la* = *vertical-strip-path h a c d*
by *simp*
show *subpathin 0 1 ?la* ‘ {*0*..*1*} ⊆ *S*
using *la-imgS'* **by** *simp*
show *x0* ∈ *S*
by (*rule x0S*)
qed
have *bridge-b-loop*: *bridge-loop h b c d* ∈ *loop-space S x0*
unfolding *bridge-loop-eq-segment-loop*
proof (*rule segment-loop-in-set*[**where** *S* = *S*])
show *path ?lb*
by (*rule lb-path*)
show *path-image ?lb* ⊆ *W*
using *lb-imgUV* **by** *blast*
show (*0*::*real*) ∈ {*0*..*1*}
by *simp*
show (*1*::*real*) ∈ {*0*..*1*}
by *simp*
show *?lb 0* ∈ *U* ∩ *V ?lb 1* ∈ *U* ∩ *V*
using *bcUV bdUV* **by** (*simp-all add*: *vertical-strip-path-def*)
show *path-image* (*connector* (*?lb 0*)) ⊆ *S*
using *connector-image-subset*[*OF bcUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
show *path-image* (*connector* (*?lb 1*)) ⊆ *S*
using *connector-image-subset*[*OF bdUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
have *edge-S*: *h* ‘ ({*b*} × {*c*..*d*}) ⊆ *S*
by (*rule order-trans*[*OF rightUV UVS*])
have *lb-imgS'*: *vertical-strip-path h b c d* ‘ {*0*..*1*} ⊆ *S*
by (*rule order-trans*[*OF vertical-strip-path-image-subset*[*OF cd*] *edge-S*])
have *lb-eq*: *?lb* = *vertical-strip-path h b c d*
by *simp*
show *subpathin 0 1 ?lb* ‘ {*0*..*1*} ⊆ *S*
using *lb-imgS'* **by** *simp*
show *x0* ∈ *S*

by (*rule x0S*)
qed

have *segc-path*: *path* (*segment-loop ?pc a b*) **and** *segc-imgS*: *path-image* (*segment-loop ?pc a b*) ⊆ *S*
using *segc-loop* **unfolding** *loop-space-def* **by** *auto*
have *segd-path*: *path* (*segment-loop ?pd a b*) **and** *segd-imgS*: *path-image* (*segment-loop ?pd a b*) ⊆ *S*
using *segd-loop* **unfolding** *loop-space-def* **by** *auto*
have *bridge-a-path*: *path* (*bridge-loop h a c d*) **and** *bridge-a-imgS*: *path-image* (*bridge-loop h a c d*) ⊆ *S*
using *bridge-a-loop* **unfolding** *loop-space-def* **by** *auto*
have *bridge-b-path*: *path* (*bridge-loop h b c d*) **and** *bridge-b-imgS*: *path-image* (*bridge-loop h b c d*) ⊆ *S*
using *bridge-b-loop* **unfolding** *loop-space-def* **by** *auto*
have *segc-finish*: *pathfinish* (*segment-loop ?pc a b*) = *x0*
using *segc-loop* **unfolding** *loop-space-def* **by** *auto*
have *bridge-b-start*: *pathstart* (*bridge-loop h b c d*) = *x0*
using *bridge-b-loop* **unfolding** *loop-space-def* **by** *auto*
have *bc-start*: *pathstart ?bc* = *x0*
using *connector-start*[*OF bcUV*] **by** *simp*
have *bc-finish*: *pathfinish ?bc* = *h* (*b*, *c*)
using *connector-finish*[*OF bcUV*] **by** *simp*

have *edge-hom*:
homotopic-paths S (*?bot* +++ *?lb*) (*?la* +++ *?top*)
unfolding *vertical-strip-path-def*
by (*rule rectangle-edge-homotopic-in-set*[*OF h-cont a01 b01 c01 d01 ab cd rectS*])

have *lb-finish*: *pathfinish ?lb* = *h* (*b*, *d*)
using *cd* **by** (*simp add*: *vertical-strip-path-def pathfinish-def*)
have *lb-start*: *pathstart ?lb* = *h* (*b*, *c*)
by (*simp add*: *vertical-strip-path-def pathstart-def*)
have *rev-bd-path*: *path* (*reversepath ?bd*)
using *bd-path* **by** *simp*
have *rev-bd-imgS*: *path-image* (*reversepath ?bd*) ⊆ *S*
using *bd-imgS* **by** *simp*
have *rev-bd-start*: *pathstart* (*reversepath ?bd*) = *h* (*b*, *d*)
using *connector-finish*[*OF bdUV*] **by** *simp*
have *s-b-path*: *path* (*?lb* +++ *reversepath ?bd*)
using *lb-path bd-path lb-finish rev-bd-start* **by** *simp*
have *s-b-imgS*: *path-image* (*?lb* +++ *reversepath ?bd*) ⊆ *S*
by (*rule subset-path-image-join*[*OF lb-imgS*]) (*use bd-imgS* **in** *simp*)
have *ac-finish*: *pathfinish ?ac* = *h* (*a*, *c*)
using *connector-finish*[*OF acUV*] **by** *simp*
have *bot-start*: *pathstart ?bot* = *h* (*a*, *c*)
by (*simp add*: *pathstart-def subpathin-def*)
have *pc-pathin*: *pathin* (*top-of-set W*) *?pc*
by (*rule path-top-of-setI*[*OF pc-path pc-imgW*])

```
  have bot-path: path ?bot
  proof −
    have pathin (top-of-set W) ?bot
      by (rule pathin-subpathin[OF pc-pathin]) (use a01 b01 in auto)
    then show ?thesis
      by (simp add: pathin-canon-iff)
  qed
  have r-b-path: path (?ac +++ ?bot)
    using ac-path bot-path ac-finish bot-start by simp
  have bot-finish: pathfinish ?bot = h (b, c)
    by (simp add: pathfinish-def subpathin-def)
  have r-b-finish: pathfinish (?ac +++ ?bot) = h (b, c)
    using r-b-path bot-finish by simp
  have bot-path-imgS: path-image ?bot ⊆ S
    using bot-imgS by (simp add: path-image-def)
  have pd-pathin: pathin (top-of-set W) ?pd
    by (rule path-top-of-setI[OF pd-path pd-imgW])
  have r-b-imgS: path-image (?ac +++ ?bot) ⊆ S
  proof (rule subset-path-image-join[OF ac-imgS])
    show path-image ?bot ⊆ S
      using bot-imgS by (simp add: path-image-def)
  qed
  have assoc-bridge-b:
    homotopic-paths S (bridge-loop h b c d) (?bc +++ (?lb +++ reversepath ?bd))
  proof −
    have bc-lb: pathfinish ?bc = pathstart ?lb
      using bc-finish lb-start by simp
    have lb-bd: pathfinish ?lb = pathstart (reversepath ?bd)
      using lb-finish rev-bd-start by simp
    have rev-bd-path: path (reversepath ?bd)
      using bd-path by simp
    have rev-bd-imgS: path-image (reversepath ?bd) ⊆ S
      using bd-imgS by simp
     have homotopic-paths S (?bc +++ (?lb +++ reversepath ?bd)) (((?bc +++
?lb) +++ reversepath ?bd))
      by (rule homotopic-paths-assoc[OF bc-path bc-imgS lb-path lb-imgS rev-bd-path
rev-bd-imgS bc-lb lb-bd])
    then show ?thesis
      unfolding bridge-loop-def by (rule homotopic-paths-sym)
  qed
  have s-b-start: pathstart (?lb +++ reversepath ?bd) = h (b, c)
    using lb-start by simp
  have lhs-step1:
    homotopic-paths S
      (segment-loop ?pc a b +++ bridge-loop h b c d)
      (segment-loop ?pc a b +++ (?bc +++ (?lb +++ reversepath ?bd)))
    by (rule homotopic-paths-join-left[OF assoc-bridge-b segc-path segc-imgS]) (use
segc-finish bridge-b-start in simp)
  have lhs-step2:
```

homotopic-paths S
(*segment-loop ?pc a b +++ (?bc +++ (?lb +++ reversepath ?bd)))*
(((*segment-loop ?pc a b +++ ?bc*) *+++ (?lb +++ reversepath ?bd)))*
by (*rule homotopic-paths-assoc*[*OF segc-path segc-imgS bc-path bc-imgS s-b-path s-b-imgS*])
(*use segc-finish bc-start bc-finish s-b-start* **in** *simp-all*)
have *segc-bc-path*: *path* (*segment-loop ?pc a b +++ ?bc*)
using *segc-path bc-path segc-finish bc-start* **by** *simp*
have *segc-bc-imgS*: *path-image* (*segment-loop ?pc a b +++ ?bc*) ⊆ *S*
by (*rule subset-path-image-join*[*OF segc-imgS*]) (*use bc-imgS* **in** *simp*)
have *lhs-step3*:
homotopic-paths S
(((*segment-loop ?pc a b +++ ?bc*) *+++ (?lb +++ reversepath ?bd)))*
((((*?ac +++ ?bot*) *+++ reversepath ?bc*) *+++ ?bc*) *+++ (?lb +++ reversepath ?bd*))
proof (*rule homotopic-paths-eq*[*OF - -*])
show *path* (((*segment-loop ?pc a b +++ ?bc*) *+++ (?lb +++ reversepath ?bd)))*
using *segc-bc-path s-b-path bc-finish s-b-start* **by** *simp*
show *path-image* (((*segment-loop ?pc a b +++ ?bc*) *+++ (?lb +++ reversepath ?bd*))) ⊆ *S*
by (*rule subset-path-image-join*[*OF segc-bc-imgS*]) (*use s-b-imgS* **in** *simp*)
show ((*segment-loop ?pc a b +++ ?bc*) *+++ (?lb +++ reversepath ?bd*)) *t* =
((((*?ac +++ ?bot*) *+++ reversepath ?bc*) *+++ ?bc*) *+++ (?lb +++ reversepath ?bd*)) *t*
if *t* ∈ {*0..1*} **for** *t*
using *that* **by** (*simp add*: *segment-loop-def*)
qed
have *lhs-step4*:
homotopic-paths S
((((*?ac +++ ?bot*) *+++ reversepath ?bc*) *+++ ?bc*) *+++ (?lb +++ reversepath ?bd*))
((*?ac +++ ?bot*) *+++ (?lb +++ reversepath ?bd*))
by (*rule homotopic-paths-cancel-middle-local*[*OF r-b-path r-b-imgS bc-path bc-imgS s-b-path s-b-imgS*])
(*use r-b-finish bc-finish s-b-start* **in** *simp-all*)
have *assoc-ac-bot-lb*:
homotopic-paths S (((*?ac +++ ?bot*) *+++ ?lb*)) (*?ac +++ (?bot +++ ?lb*))
proof −
have *homotopic-paths S* (*?ac +++ (?bot +++ ?lb*)) (((*?ac +++ ?bot*) *+++ ?lb*))
by (*rule homotopic-paths-assoc*[*OF ac-path ac-imgS bot-path bot-path-imgS lb-path lb-imgS*])
(*use ac-finish bot-start bot-finish lb-start* **in** *simp-all*)
then show *?thesis*
by (*rule homotopic-paths-sym*)
qed
have *assoc-r-b-lb*:
homotopic-paths S
((*?ac +++ ?bot*) *+++ (?lb +++ reversepath ?bd*))

```
      ((((?ac +++ ?bot) +++ ?lb) +++ reversepath ?bd))
   by (rule homotopic-paths-assoc[OF r-b-path r-b-imgS lb-path lb-imgS rev-bd-path
rev-bd-imgS])
      (use r-b-finish lb-start lb-finish rev-bd-start in simp-all)
  have assoc-ac-bot-lb-join:
   homotopic-paths S
     ((((?ac +++ ?bot) +++ ?lb) +++ reversepath ?bd))
     ((?ac +++ (?bot +++ ?lb)) +++ reversepath ?bd)
   by (rule homotopic-paths-join-right[OF assoc-ac-bot-lb rev-bd-path rev-bd-imgS])
      (use lb-finish rev-bd-start in simp-all)
  have lhs-step5:
   homotopic-paths S
     ((?ac +++ ?bot) +++ (?lb +++ reversepath ?bd))
     ((?ac +++ (?bot +++ ?lb)) +++ reversepath ?bd)
   by (rule homotopic-paths-trans[OF assoc-r-b-lb assoc-ac-bot-lb-join])
  have lhs-to-boundary:
   homotopic-paths S
     (segment-loop ?pc a b +++ bridge-loop h b c d)
     ((?ac +++ (?bot +++ ?lb)) +++ reversepath ?bd)
   by (rule homotopic-paths-trans[OF lhs-step1])
      (rule homotopic-paths-trans[OF lhs-step2],
       rule homotopic-paths-trans[OF lhs-step3],
       rule homotopic-paths-trans[OF lhs-step4 lhs-step5])

  have top-path: path ?top
  proof −
   have pathin (top-of-set W) ?top
    by (rule pathin-subpathin[OF pd-pathin]) (use a01 b01 in auto)
   then show ?thesis
    by (simp add: pathin-canon-iff)
  qed
  have top-start: pathstart ?top = h (a, d)
   by (simp add: pathstart-def subpathin-def)
  have top-finish: pathfinish ?top = h (b, d)
   by (simp add: pathfinish-def subpathin-def)
  have top-path-imgS: path-image ?top ⊆ S
   using top-imgS by (simp add: path-image-def)
  have s-a-path: path (?top +++ reversepath ?bd)
   using top-path bd-path top-finish rev-bd-start by simp
  have s-a-imgS: path-image (?top +++ reversepath ?bd) ⊆ S
   by (rule subset-path-image-join[OF top-path-imgS rev-bd-imgS])
  have la-start: pathstart ?la = h (a, c)
   by (simp add: vertical-strip-path-def pathstart-def)
  have la-finish: pathfinish ?la = h (a, d)
   by (simp add: vertical-strip-path-def pathfinish-def)
  have ad-start: pathstart ?ad = x0
   using connector-start[OF adUV] by simp
  have ad-finish: pathfinish ?ad = h (a, d)
   using connector-finish[OF adUV] by simp
```

have *rev-ad-path*: *path* (*reversepath ?ad*)
using *ad-path* **by** *simp*
have *rev-ad-imgS*: *path-image* (*reversepath ?ad*) ⊆ *S*
using *ad-imgS* **by** *simp*
have *rev-ad-start*: *pathstart* (*reversepath ?ad*) = *h* (*a*, *d*)
using *connector-finish*[*OF adUV*] **by** *simp*
have *rev-ad-finish*: *pathfinish* (*reversepath ?ad*) = *x0*
using *ad-start* **by** *simp*
have *r-a-path*: *path* (*?ac* +++ *?la*)
using *ac-path la-path ac-finish la-start* **by** *simp*
have *r-a-imgS*: *path-image* (*?ac* +++ *?la*) ⊆ *S*
by (*rule subset-path-image-join*[*OF ac-imgS la-imgS*])
have *r-a-rev-path*: *path* ((*?ac* +++ *?la*) +++ *reversepath ?ad*)
using *r-a-path rev-ad-path la-finish rev-ad-start* **by** *simp*
have *r-a-rev-imgS*: *path-image* ((*?ac* +++ *?la*) +++ *reversepath ?ad*) ⊆ *S*
by (*rule subset-path-image-join*[*OF r-a-imgS rev-ad-imgS*])
have *assoc-bridge-a*:
homotopic-paths S (*bridge-loop h a c d*) (*?ac* +++ (*?la* +++ *reversepath ?ad*))
proof −
have *homotopic-paths S* (*?ac* +++ (*?la* +++ *reversepath ?ad*)) (((*?ac* +++ *?la*) +++ *reversepath ?ad*))
by (*rule homotopic-paths-assoc*[*OF ac-path ac-imgS la-path la-imgS rev-ad-path rev-ad-imgS*])
(*use ac-finish la-start la-finish rev-ad-start* **in** *simp-all*)
then show *?thesis*
unfolding *bridge-loop-def* **by** (*rule homotopic-paths-sym*)
qed
have *bridge-a-finish*: *pathfinish* (*bridge-loop h a c d*) = *x0*
using *bridge-a-loop* **unfolding** *loop-space-def* **by** *auto*
have *segd-start*: *pathstart* (*segment-loop ?pd a b*) = *x0*
using *segd-loop* **unfolding** *loop-space-def* **by** *auto*
have *rhs-step1*:
homotopic-paths S
(*bridge-loop h a c d* +++ *segment-loop ?pd a b*)
((*?ac* +++ (*?la* +++ *reversepath ?ad*)) +++ *segment-loop ?pd a b*)
by (*rule homotopic-paths-join-right*[*OF assoc-bridge-a segd-path segd-imgS*])
(*use bridge-a-finish segd-start* **in** *simp-all*)
have *rhs-step2*:
homotopic-paths S
((*?ac* +++ (*?la* +++ *reversepath ?ad*)) +++ *segment-loop ?pd a b*)
(((*?ac* +++ *?la*) +++ *reversepath ?ad*) +++ *segment-loop ?pd a b*)
by (*rule homotopic-paths-join-right*[
OF homotopic-paths-assoc[*OF ac-path ac-imgS la-path la-imgS rev-ad-path rev-ad-imgS*] *segd-path segd-imgS*])
(*use ac-finish la-start la-finish rev-ad-start ad-start bridge-a-finish segd-start* **in** *simp-all*)
have *rhs-step3*:
homotopic-paths S
(((*?ac* +++ *?la*) +++ *reversepath ?ad*) +++ *segment-loop ?pd a b*)

```
      ((((?ac +++ ?la) +++ reversepath ?ad) +++ ?ad) +++ (?top +++ re-
versepath ?bd))
  proof −
    have segd-assoc:
      homotopic-paths S (segment-loop ?pd a b) (?ad +++ (?top +++ reversepath
?bd))
    proof −
     have homotopic-paths S (?ad +++ (?top +++ reversepath ?bd)) (segment-loop
?pd a b)
       unfolding segment-loop-def
       by (rule homotopic-paths-assoc[OF ad-path ad-imgS top-path top-path-imgS
rev-bd-path rev-bd-imgS])
         (use ad-finish top-start top-finish rev-bd-start in simp-all)
      then show ?thesis
       by (rule homotopic-paths-sym)
    qed
    have step1:
      homotopic-paths S
       (((?ac +++ ?la) +++ reversepath ?ad) +++ segment-loop ?pd a b)
         (((?ac +++ ?la) +++ reversepath ?ad) +++ (?ad +++ (?top +++
reversepath ?bd)))
      by (rule homotopic-paths-join-left[OF segd-assoc r-a-rev-path r-a-rev-imgS])
        (use segd-start ad-start rev-ad-finish in simp-all)
    have step2:
      homotopic-paths S
          (((?ac +++ ?la) +++ reversepath ?ad) +++ (?ad +++ (?top +++
reversepath ?bd)))
         ((((?ac +++ ?la) +++ reversepath ?ad) +++ ?ad) +++ (?top +++
reversepath ?bd))
     by (rule homotopic-paths-assoc[OF r-a-rev-path r-a-rev-imgS ad-path ad-imgS
s-a-path s-a-imgS])
       (use rev-ad-finish ad-start ad-finish top-start in simp-all)
    show ?thesis
      by (rule homotopic-paths-trans[OF step1 step2])
  qed
  have rhs-step4:
    homotopic-paths S
      ((((?ac +++ ?la) +++ reversepath ?ad) +++ ?ad) +++ (?top +++ re-
versepath ?bd))
      ((?ac +++ ?la) +++ (?top +++ reversepath ?bd))
     by (rule homotopic-paths-cancel-middle-local[OF r-a-path r-a-imgS ad-path
ad-imgS s-a-path s-a-imgS])
      (use la-finish ad-finish top-start in simp-all)
  have rhs-step5:
    homotopic-paths S
      ((?ac +++ ?la) +++ (?top +++ reversepath ?bd))
      ((?ac +++ (?la +++ ?top)) +++ reversepath ?bd)
  proof −
    have step5a:
```

```
    homotopic-paths S
      ((?ac +++ ?la) +++ (?top +++ reversepath ?bd))
      (((?ac +++ ?la) +++ ?top) +++ reversepath ?bd)
    by (rule homotopic-paths-assoc[OF r-a-path r-a-imgS top-path top-path-imgS
rev-bd-path rev-bd-imgS])
       (use la-finish top-start top-finish rev-bd-start in simp-all)
  have inner:
    homotopic-paths S
      (((?ac +++ ?la) +++ ?top))
      (?ac +++ (?la +++ ?top))
  proof −
    have homotopic-paths S
       (?ac +++ (?la +++ ?top))
       (((?ac +++ ?la) +++ ?top))
     by (rule homotopic-paths-assoc[OF ac-path ac-imgS la-path la-imgS top-path
top-path-imgS])
        (use ac-finish la-start la-finish top-start in simp-all)
    then show ?thesis
      by (rule homotopic-paths-sym)
  qed
  have step5b:
    homotopic-paths S
      (((?ac +++ ?la) +++ ?top) +++ reversepath ?bd)
      ((?ac +++ (?la +++ ?top)) +++ reversepath ?bd)
    by (rule homotopic-paths-join-right[OF inner rev-bd-path rev-bd-imgS])
      (use top-finish rev-bd-start in simp-all)
  show ?thesis
    by (rule homotopic-paths-trans[OF step5a step5b])
qed
have boundary-to-rhs:
  homotopic-paths S
    ((?ac +++ (?la +++ ?top)) +++ reversepath ?bd)
    (bridge-loop h a c d +++ segment-loop ?pd a b)
proof −
  have homotopic-paths S (bridge-loop h a c d +++ segment-loop ?pd a b)
    ((?ac +++ (?la +++ ?top)) +++ reversepath ?bd)
    by (rule homotopic-paths-trans[OF rhs-step1])
      (rule homotopic-paths-trans[OF rhs-step2],
       rule homotopic-paths-trans[OF rhs-step3],
       rule homotopic-paths-trans[OF rhs-step4 rhs-step5])
  then show ?thesis
    by (rule homotopic-paths-sym)
qed

have edge-boundary:
  homotopic-paths S
    ((?ac +++ (?bot +++ ?lb)) +++ reversepath ?bd)
    ((?ac +++ (?la +++ ?top)) +++ reversepath ?bd)
proof −
```

```
    have pre:
      homotopic-paths S (?ac +++ (?bot +++ ?lb)) (?ac +++ (?la +++ ?top))
        by (rule homotopic-paths-join-left[OF edge-hom ac-path ac-imgS]) (use
ac-finish bot-start la-start in simp-all)
    show ?thesis
    proof (rule homotopic-paths-join-right[OF pre rev-bd-path rev-bd-imgS])
      show pathfinish (?ac +++ (?bot +++ ?lb)) = pathstart (reversepath ?bd)
        using lb-finish rev-bd-start by simp
    qed
  qed

  show ?thesis
    by (rule homotopic-paths-trans[OF lhs-to-boundary])
       (rule homotopic-paths-trans[OF edge-boundary boundary-to-rhs])
qed

lemma horizontal-rectangle-segment-loop-in-set:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
    and a01: a ∈ {0..1} and b01: b ∈ {0..1} and c01: c ∈ {0..1}
    and ab: a ≤ b
    and segS: h ` ({a..b} × {c}) ⊆ S
    and acUV: h (a, c) ∈ U ∩ V
    and bcUV: h (b, c) ∈ U ∩ V
    and UVS: U ∩ V ⊆ S
  shows segment-loop (λt. h (t, c)) a b ∈ loop-space S x0
proof −
  have h-on: continuous-on ({0..1} × {0..1}) h
    and h-intoW: h ∈ ({0..1} × {0..1}) → W
    using h-cont by simp-all
  have pc-path: path (λt. h (t, c))
  proof −
    have continuous-on {0..1} (λt. h (t, c))
    proof (rule continuous-on-compose2[OF h-on])
      show continuous-on {0..1} (λt. (t, c))
        by (intro continuous-intros)
      show (λt. (t, c)) ` {0..1} ⊆ {0..1} × {0..1}
        using c01 by auto
    qed
    then show ?thesis
      by (simp add: path-def)
  qed
  have pc-imgW: path-image (λt. h (t, c)) ⊆ W
    using h-intoW c01 by (auto simp: path-image-def)
  have seg-imgS: subpathin a b (λt. h (t, c)) ` {0..1} ⊆ S
    using segS ab by (auto simp: subpathin-image-eq)
  have x0S: x0 ∈ S
    using x0-in-UV UVS by blast
```

```
  show ?thesis
  proof (rule segment-loop-in-set[where S = S])
    show path (λt. h (t, c))
      by (rule pc-path)
    show path-image (λt. h (t, c)) ⊆ W
      by (rule pc-imgW)
    show a ∈ {0..1} b ∈ {0..1}
      by (rule a01, rule b01)
    show (λt. h (t, c)) a ∈ U ∩ V (λt. h (t, c)) b ∈ U ∩ V
      using acUV bcUV by simp-all
    show path-image (connector ((λt. h (t, c)) a)) ⊆ S
      using connector-image-subset[OF acUV] UVS by blast
    show path-image (connector ((λt. h (t, c)) b)) ⊆ S
      using connector-image-subset[OF bcUV] UVS by blast
    show subpathin a b (λt. h (t, c)) ‘ {0..1} ⊆ S
      by (rule seg-imgS)
    show x0 ∈ S
      by (rule x0S)
  qed
qed

lemma vertical-bridge-loop-in-set:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
    and t01: t ∈ {0..1} and c01: c ∈ {0..1} and d01: d ∈ {0..1}
    and cd: c ≤ d
    and edgeUV: h ‘ ({t} × {c..d}) ⊆ U ∩ V
    and UVS: U ∩ V ⊆ S
  shows bridge-loop h t c d ∈ loop-space S x0
proof −
  have h-on: continuous-on ({0..1} × {0..1}) h
    and h-intoW: h ∈ ({0..1} × {0..1}) → W
    using h-cont by simp-all
  have vt-path: path (vertical-strip-path h t c d)
  proof −
    have continuous-on {0..1} (vertical-strip-path h t c d)
    proof −
      have continuous-on {0..1} (λu. h (t, (d − c) ∗ u + c))
      proof (rule continuous-on-compose2[OF h-on])
        show continuous-on {0..1} (λu. (t, (d − c) ∗ u + c))
          by (intro continuous-intros)
        show (λu. (t, (d − c) ∗ u + c)) ‘ {0..1} ⊆ {0..1} × {0..1}
        proof
          fix x
          assume x ∈ (λu. (t, (d − c) ∗ u + c)) ‘ {0..1}
          then obtain u where u01: u ∈ {0..1} and x-eq: x = (t, (d − c) ∗ u +
c)
            by blast
```

```
        have y-in: (d − c) ∗ u + c ∈ {0..1}
          by (rule affine-unit-interval-member[OF c01 d01 cd u01])
        show x ∈ {0..1} × {0..1}
          using t01 y-in x-eq by auto
      qed
    qed
    then show ?thesis
      by (simp add: vertical-strip-path-def)
  qed
  then show ?thesis
    by (simp add: vertical-strip-path-def path-def)
qed
have vt-imgW: path-image (vertical-strip-path h t c d) ⊆ W
proof
  fix x
  assume x-in: x ∈ path-image (vertical-strip-path h t c d)
  then obtain u where u01: u ∈ {0..1} and x-eq: x = vertical-strip-path h t c
d u
    unfolding path-image-def by blast
  have yu01: (d − c) ∗ u + c ∈ {0..1}
    by (rule affine-unit-interval-member[OF c01 d01 cd u01])
  have point-in: (t, (d − c) ∗ u + c) ∈ {0..1} × {0..1}
    using t01 yu01 by auto
  show x ∈ W
    using h-intoW point-in x-eq by (auto simp: vertical-strip-path-def)
qed
have vt-imgS: vertical-strip-path h t c d ‘ {0..1} ⊆ S
proof −
  have vs-subset: vertical-strip-path h t c d ‘ {0..1} ⊆ h ‘ ({t} × {c..d})
    by (rule vertical-strip-path-image-subset[OF cd])
  show ?thesis
    using vs-subset edgeUV UVS by blast
qed
have tcUV: h (t, c) ∈ U ∩ V and tdUV: h (t, d) ∈ U ∩ V
  using edgeUV cd by auto
have x0S: x0 ∈ S
  using x0-in-UV UVS by blast
have segment-loop (vertical-strip-path h t c d) 0 1 ∈ loop-space S x0
proof (rule segment-loop-in-set[where S = S])
  show path (vertical-strip-path h t c d)
    by (rule vt-path)
  show path-image (vertical-strip-path h t c d) ⊆ W
    by (rule vt-imgW)
  show 0 ∈ {0::real..1}
    by auto
  show 1 ∈ {0::real..1}
    by auto
  show vertical-strip-path h t c d 0 ∈ U ∩ V
    using tcUV by (simp add: vertical-strip-path-def)
```

show *vertical-strip-path h t c d 1* ∈ *U* ∩ *V*
using *tdUV* **by** (*simp add*: *vertical-strip-path-def*)
show *path-image* (*connector* (*vertical-strip-path h t c d 0*)) ⊆ *S*
using *connector-image-subset*[*OF tcUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
show *path-image* (*connector* (*vertical-strip-path h t c d 1*)) ⊆ *S*
using *connector-image-subset*[*OF tdUV*] *UVS* **by** (*simp add*: *vertical-strip-path-def*; *blast*)
show *subpathin 0 1* (*vertical-strip-path h t c d*) ‘ {*0..1*} ⊆ *S*
using *vt-imgS* **by** (*simp add*: *subpathin-def*)
show *x0* ∈ *S*
by (*rule x0S*)
qed
then show *?thesis*
by *simp*
qed

lemma *rectangle-segment-loop-bridge-class-eq*:
fixes *h* :: (*real* × *real*) ⇒ *'a*
assumes *h-cont*: *continuous-map* (*top-of-set* ({*0..1*} × {*0..1*})) (*top-of-set W*) *h*
and *a01*: *a* ∈ {*0..1*} **and** *b01*: *b* ∈ {*0..1*}
and *c01*: *c* ∈ {*0..1*} **and** *d01*: *d* ∈ {*0..1*}
and *ab*: *a* ≤ *b* **and** *cd*: *c* ≤ *d*
and *rectS*: *h* ‘ ({*a..b*} × {*c..d*}) ⊆ *S*
and *leftUV*: *h* ‘ ({*a*} × {*c..d*}) ⊆ *U* ∩ *V*
and *rightUV*: *h* ‘ ({*b*} × {*c..d*}) ⊆ *U* ∩ *V*
and *UVS*: *U* ∩ *V* ⊆ *S*
shows *fundamental-group-mult S x0*
(*loop-class S x0* (*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b*))
(*loop-class S x0* (*bridge-loop h b c d*)) =
fundamental-group-mult S x0
(*loop-class S x0* (*bridge-loop h a c d*))
(*loop-class S x0* (*segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*))
proof −
have *acUV*: *h* (*a*, *c*) ∈ *U* ∩ *V*
using *leftUV cd* **by** *auto*
have *bcUV*: *h* (*b*, *c*) ∈ *U* ∩ *V*
using *rightUV cd* **by** *auto*
have *adUV*: *h* (*a*, *d*) ∈ *U* ∩ *V*
using *leftUV cd* **by** *auto*
have *bdUV*: *h* (*b*, *d*) ∈ *U* ∩ *V*
using *rightUV cd* **by** *auto*

have *segc-in-S*: *h* ‘ ({*a..b*} × {*c*}) ⊆ *S*
proof
fix *x*
assume *x-in*: *x* ∈ *h* ‘ ({*a..b*} × {*c*})
then obtain *aa* **where** *aa-in*: *aa* ∈ {*a..b*} **and** *x-eq*: *x* = *h* (*aa*, *c*)

by *auto*
have $(aa,\ c) \in \{a..b\} \times \{c..d\}$
using *aa-in cd* **by** *auto*
then show $x \in S$
using *rectS x-eq* **by** *blast*
qed
have *segd-in-S*: $h\ `\ (\{a..b\} \times \{d\}) \subseteq S$
proof
fix x
assume *x-in*: $x \in h\ `\ (\{a..b\} \times \{d\})$
then obtain *aa* **where** *aa-in*: $aa \in \{a..b\}$ **and** *x-eq*: $x = h\ (aa,\ d)$
by *auto*
have $(aa,\ d) \in \{a..b\} \times \{c..d\}$
using *aa-in cd* **by** *auto*
then show $x \in S$
using *rectS x-eq* **by** *blast*
qed
have *segc-loop*: *segment-loop* $(\lambda t.\ h\ (t,\ c))\ a\ b \in$ *loop-space S x0*
by (*rule horizontal-rectangle-segment-loop-in-set*[*OF h-cont a01 b01 c01 ab*]) (*use segc-in-S acUV bcUV UVS* **in** *auto*)
have *segd-loop*: *segment-loop* $(\lambda t.\ h\ (t,\ d))\ a\ b \in$ *loop-space S x0*
by (*rule horizontal-rectangle-segment-loop-in-set*[*OF h-cont a01 b01 d01 ab*]) (*use segd-in-S adUV bdUV UVS* **in** *auto*)
have *bridge-b-loop*: *bridge-loop h b c d* $\in$ *loop-space S x0*
by (*rule vertical-bridge-loop-in-set*[*OF h-cont b01 c01 d01 cd*]) (*use rightUV UVS* **in** *auto*)
have *bridge-a-loop*: *bridge-loop h a c d* $\in$ *loop-space S x0*
by (*rule vertical-bridge-loop-in-set*[*OF h-cont a01 c01 d01 cd*]) (*use leftUV UVS* **in** *auto*)

have *left-join-loop*:
segment-loop $(\lambda t.\ h\ (t,\ c))\ a\ b$ *+++ bridge-loop h b c d* $\in$ *loop-space S x0*
by (*rule loop-space-join*[*OF segc-loop bridge-b-loop*])
have *right-join-loop*:
bridge-loop h a c d +++ segment-loop $(\lambda t.\ h\ (t,\ d))\ a\ b \in$ *loop-space S x0*
by (*rule loop-space-join*[*OF bridge-a-loop segd-loop*])

have *segc-in*: *loop-class S x0* (*segment-loop* $(\lambda t.\ h\ (t,\ c))\ a\ b) \in$ *fundamental-group-space S x0*
by (*rule loop-class-in-space*[*OF segc-loop*])
have *segd-in*: *loop-class S x0* (*segment-loop* $(\lambda t.\ h\ (t,\ d))\ a\ b) \in$ *fundamental-group-space S x0*
by (*rule loop-class-in-space*[*OF segd-loop*])
have *bridge-a-in*: *loop-class S x0* (*bridge-loop h a c d*) $\in$ *fundamental-group-space S x0*
by (*rule loop-class-in-space*[*OF bridge-a-loop*])
have *bridge-b-in*: *loop-class S x0* (*bridge-loop h b c d*) $\in$ *fundamental-group-space S x0*
by (*rule loop-class-in-space*[*OF bridge-b-loop*])

have *left-mult*:
fundamental-group-mult S x0
(*loop-class S x0* (*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b*))
(*loop-class S x0* (*bridge-loop h b c d*)) =
loop-class S x0 (*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b* +++ *bridge-loop h b c d*)
by (*rule fundamental-group-mult-eqI*[*OF segc-in bridge-b-in segc-loop bridge-b-loop*]) *simp-all*
have *right-mult*:
fundamental-group-mult S x0
(*loop-class S x0* (*bridge-loop h a c d*))
(*loop-class S x0* (*segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*)) =
loop-class S x0 (*bridge-loop h a c d* +++ *segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*)
by (*rule fundamental-group-mult-eqI*[*OF bridge-a-in segd-in bridge-a-loop segd-loop*]) *simp-all*

have *joined-hom*:
homotopic-paths S
(*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b* +++ *bridge-loop h b c d*)
(*bridge-loop h a c d* +++ *segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*)
by (*rule rectangle-segment-loop-bridge-homotopic*[*OF h-cont a01 b01 c01 d01 ab cd rectS leftUV rightUV UVS*])
have *joined-eq*:
loop-class S x0 (*segment-loop* (λ*t*. *h* (*t*, *c*)) *a b* +++ *bridge-loop h b c d*) =
loop-class S x0 (*bridge-loop h a c d* +++ *segment-loop* (λ*t*. *h* (*t*, *d*)) *a b*)
by (*rule loop-class-eqI*[*OF left-join-loop right-join-loop joined-hom*])

show *?thesis*
using *left-mult right-mult joined-eq* **by** *simp*
qed

lemma *bridge-word-identify*:
assumes *h-in*: *h* ∈ *H*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*bridge-word True h rest*) (*bridge-word False h rest*)
and *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*bridge-word False h rest*) (*bridge-word True h rest*)
proof −
have *step*:
carrier-amalgam-equiv H i1 i2 (*bridge-word True h rest*) (*bridge-word False h rest*)
using *h-in*
by (*auto simp*: *bridge-word.simps intro*: *carrier-amalgam-equiv.step carrier-amalgam-step.identify*)
show *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*bridge-word True h rest*) (*bridge-word False h rest*)
by (*rule carrier-full-amalg-equiv.of-amalg*[*OF step*])
then show *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*bridge-word False h rest*) (*bridge-word True h rest*)

```
    by (rule carrier-full-amalg-equiv.sym)
qed

lemma rectangle-partition-step-props:
  assumes rectangle-partition h c d (t # u # ts) (b # bs)
  shows t ∈ {0..1}
    and u ∈ {0..1}
    and t < u
    and (if b then h ‘ ({t..u} × {c..d}) ⊆ U else h ‘ ({t..u} × {c..d}) ⊆ V)
    and rectangle-partition h c d (u # ts) bs
  using assms by simp-all

lemma rectangle-partition-switch-edge:
  assumes rp: rectangle-partition h c d (t # u # v # ts) (b # e # bs)
    and diff: b ≠ e
  shows h ‘ ({u} × {c..d}) ⊆ U ∩ V
proof (cases b)
  case True
  have step: rectangle-partition h c d (u # v # ts) (e # bs)
    by (rule rectangle-partition-step-props(5)[OF rp])
  have tu: t < u and uv: u < v
    using rp step by simp-all
  have leftU: h ‘ ({t..u} × {c..d}) ⊆ U
    using rectangle-partition-step-props(4)[OF rp] True by simp
  from diff True have e-false: ¬ e
    by simp
  then have rightV: h ‘ ({u..v} × {c..d}) ⊆ V
    using rectangle-partition-step-props(4)[OF step] by simp
  show ?thesis
  proof
    fix x
    assume x-in: x ∈ h ‘ ({u} × {c..d})
    then obtain y where y-cd: y ∈ {c..d} and x-eq: x = h (u, y)
      by auto
    have u-tu: u ∈ {t..u} and u-uv: u ∈ {u..v}
      using tu uv by auto
    have xU: x ∈ U
      using leftU x-eq u-tu y-cd by auto
    have xV: x ∈ V
      using rightV x-eq u-uv y-cd by auto
    show x ∈ U ∩ V
      using xU xV by blast
  qed
next
  case False
  have step: rectangle-partition h c d (u # v # ts) (e # bs)
    by (rule rectangle-partition-step-props(5)[OF rp])
  have tu: t < u and uv: u < v
    using rp step by simp-all
```

```
  have leftV: h ‘ ({t..u} × {c..d}) ⊆ V
    using rectangle-partition-step-props(4)[OF rp] False by simp
  from diff False have e-true: e
    by simp
  then have rightU: h ‘ ({u..v} × {c..d}) ⊆ U
    using rectangle-partition-step-props(4)[OF step] by simp
  show ?thesis
  proof
    fix x
    assume x-in: x ∈ h ‘ ({u} × {c..d})
    then obtain y where y-cd: y ∈ {c..d} and x-eq: x = h (u, y)
      by auto
    have u-tu: u ∈ {t..u} and u-uv: u ∈ {u..v}
      using tu uv by auto
    have xU: x ∈ U
      using rightU x-eq u-uv y-cd by auto
    have xV: x ∈ V
      using leftV x-eq u-tu y-cd by auto
    show x ∈ U ∩ V
      using xU xV by blast
  qed
qed

lemma carrier-full-amalg-equiv-side-context:
  assumes rel: carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v
    and a-in: (if b then a ∈ G1 else a ∈ G2)
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (if b then WordLeft a u else WordRight a u)
      (if b then WordLeft a v else WordRight a v)
  using rel a-in
      by (cases b) (auto intro: carrier-full-amalg-equiv-left-context car-
rier-full-amalg-equiv-right-context)

lemma bridge-word-switch:
  assumes h-in: h ∈ H
    and bc: b ≠ c
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (bridge-word b h rest) (bridge-word c h rest)
  using h-in bc bridge-word-identify
  by (cases b; cases c) auto

lemma rectangle-segment-bridge-left-equiv:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
    and a01: a ∈ {0..1} and b01: b ∈ {0..1}
    and c01: c ∈ {0..1} and d01: d ∈ {0..1}
    and ab: a ≤ b and cd: c ≤ d
    and rectU: h ‘ ({a..b} × {c..d}) ⊆ U
```

```
    and leftUV: h ‘ ({a} × {c..d}) ⊆ U ∩ V
    and rightUV: h ‘ ({b} × {c..d}) ⊆ U ∩ V
    and rest-in: fpw-in-space G1 G2 rest
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop (λt. h (t, c)) a b))
        (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h b c d)) rest))
      (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h a c d))
        (WordLeft (loop-class U x0 (segment-loop (λt. h (t, d)) a b)) rest))
proof −
  define bottom where
    bottom = loop-class U x0 (segment-loop (λt. h (t, c)) a b)
  define top where
    top = loop-class U x0 (segment-loop (λt. h (t, d)) a b)
  define ga where
    ga = loop-class (U ∩ V) x0 (bridge-loop h a c d)
  define gb where
    gb = loop-class (U ∩ V) x0 (bridge-loop h b c d)

  have acUV: h (a, c) ∈ U ∩ V
    using leftUV cd by auto
  have bcUV: h (b, c) ∈ U ∩ V
    using rightUV cd by auto
  have adUV: h (a, d) ∈ U ∩ V
    using leftUV cd by auto
  have bdUV: h (b, d) ∈ U ∩ V
    using rightUV cd by auto

  have segc-in-U: h ‘ ({a..b} × {c}) ⊆ U
  proof
    fix x
    assume x-in: x ∈ h ‘ ({a..b} × {c})
    then obtain aa where aa-in: aa ∈ {a..b} and x-eq: x = h (aa, c)
      by auto
    have (aa, c) ∈ {a..b} × {c..d}
      using aa-in cd by auto
    then show x ∈ U
      using rectU x-eq by blast
  qed
  have segd-in-U: h ‘ ({a..b} × {d}) ⊆ U
  proof
    fix x
    assume x-in: x ∈ h ‘ ({a..b} × {d})
    then obtain aa where aa-in: aa ∈ {a..b} and x-eq: x = h (aa, d)
      by auto
    have (aa, d) ∈ {a..b} × {c..d}
      using aa-in cd by auto
    then show x ∈ U
      using rectU x-eq by blast
  qed
```

```
  have segc-loop: segment-loop (λt. h (t, c)) a b ∈ loop-space U x0
    by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont a01 b01 c01 ab])
      (use segc-in-U acUV bcUV in auto)
  have segd-loop: segment-loop (λt. h (t, d)) a b ∈ loop-space U x0
    by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont a01 b01 d01 ab])
      (use segd-in-U adUV bdUV in auto)
  have bridge-a-loop: bridge-loop h a c d ∈ loop-space (U ∩ V) x0
    by (rule vertical-bridge-loop-in-set[OF h-cont a01 c01 d01 cd leftUV]) simp
  have bridge-b-loop: bridge-loop h b c d ∈ loop-space (U ∩ V) x0
    by (rule vertical-bridge-loop-in-set[OF h-cont b01 c01 d01 cd rightUV]) simp

  have bottom-in: bottom ∈ G1
    unfolding bottom-def by (rule loop-class-in-space[OF segc-loop])
  have top-in: top ∈ G1
    unfolding top-def by (rule loop-class-in-space[OF segd-loop])
  have ga-in-H: ga ∈ H
    unfolding ga-def by (rule loop-class-in-space[OF bridge-a-loop])
  have gb-in-H: gb ∈ H
    unfolding gb-def by (rule loop-class-in-space[OF bridge-b-loop])
  have ga-in: i1 ga ∈ G1
    by (rule i1-in-G1[OF ga-in-H])
  have gb-in: i1 gb ∈ G1
    by (rule i1-in-G1[OF gb-in-H])
  have ab-in: mult1 bottom (i1 gb) ∈ G1
    by (rule fundamental-group-mult-in-space[OF bottom-in gb-in])
  have cd-in: mult1 (i1 ga) top ∈ G1
    by (rule fundamental-group-mult-in-space[OF ga-in top-in])

  have ga-eq: i1 ga = loop-class U x0 (bridge-loop h a c d)
    unfolding ga-def
    by (subst i1-loop-class-eq[OF bridge-a-loop]) simp
  have gb-eq: i1 gb = loop-class U x0 (bridge-loop h b c d)
    unfolding gb-def
    by (subst i1-loop-class-eq[OF bridge-b-loop]) simp
  have mult-eq: mult1 bottom (i1 gb) = mult1 (i1 ga) top
    unfolding bottom-def top-def ga-eq gb-eq
    by (rule rectangle-segment-loop-bridge-class-eq[OF h-cont a01 b01 c01 d01 ab
cd rectU leftUV rightUV])
      auto
  have pair-eq:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft bottom (WordLeft (i1 gb) rest))
      (WordLeft (i1 ga) (WordLeft top rest))
    by (rule carrier-full-amalg-equiv-left-pair-eq[OF bottom-in gb-in ab-in ga-in
top-in cd-in rest-in mult-eq])

  show ?thesis
    using pair-eq
    by (simp only: bottom-def top-def bridge-word.simps ga-def gb-def)
```

qed

lemma *rectangle-segment-bridge-right-equiv*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *h-cont*: *continuous-map* (*top-of-set* ($\{0..1\} \times \{0..1\}$)) (*top-of-set* W) h
and *a01*: $a \in \{0..1\}$ **and** *b01*: $b \in \{0..1\}$
and *c01*: $c \in \{0..1\}$ **and** *d01*: $d \in \{0..1\}$
and *ab*: $a \leq b$ **and** *cd*: $c \leq d$
and *rectV*: $h \text{ ` } (\{a..b\} \times \{c..d\}) \subseteq V$
and *leftUV*: $h \text{ ` } (\{a\} \times \{c..d\}) \subseteq U \cap V$
and *rightUV*: $h \text{ ` } (\{b\} \times \{c..d\}) \subseteq U \cap V$
and *rest-in*: *fpw-in-space G1 G2 rest*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*WordRight* (*loop-class V x0* (*segment-loop* ($\lambda t.$ *h* (*t*, *c*)) *a b*))
(*bridge-word False* (*loop-class* ($U \cap V$) *x0* (*bridge-loop h b c d*)) *rest*))
(*bridge-word False* (*loop-class* ($U \cap V$) *x0* (*bridge-loop h a c d*))
(*WordRight* (*loop-class V x0* (*segment-loop* ($\lambda t.$ *h* (*t*, *d*)) *a b*)) *rest*))
proof −
define *bottom* **where**
bottom = *loop-class V x0* (*segment-loop* ($\lambda t.$ *h* (*t*, *c*)) *a b*)
define *top* **where**
top = *loop-class V x0* (*segment-loop* ($\lambda t.$ *h* (*t*, *d*)) *a b*)
define *ga* **where**
ga = *loop-class* ($U \cap V$) *x0* (*bridge-loop h a c d*)
define *gb* **where**
gb = *loop-class* ($U \cap V$) *x0* (*bridge-loop h b c d*)

have *acUV*: $h\ (a, c) \in U \cap V$
using *leftUV cd* **by** *auto*
have *bcUV*: $h\ (b, c) \in U \cap V$
using *rightUV cd* **by** *auto*
have *adUV*: $h\ (a, d) \in U \cap V$
using *leftUV cd* **by** *auto*
have *bdUV*: $h\ (b, d) \in U \cap V$
using *rightUV cd* **by** *auto*

have *segc-in-V*: $h \text{ ` } (\{a..b\} \times \{c\}) \subseteq V$
proof
fix x
assume *x-in*: $x \in h \text{ ` } (\{a..b\} \times \{c\})$
then obtain *aa* **where** *aa-in*: $aa \in \{a..b\}$ **and** *x-eq*: $x = h\ (aa, c)$
by *auto*
have $(aa, c) \in \{a..b\} \times \{c..d\}$
using *aa-in cd* **by** *auto*
then show $x \in V$
using *rectV x-eq* **by** *blast*
qed
have *segd-in-V*: $h \text{ ` } (\{a..b\} \times \{d\}) \subseteq V$

```
  proof
    fix x
    assume x-in: x ∈ h ‘ ({a..b} × {d})
    then obtain aa where aa-in: aa ∈ {a..b} and x-eq: x = h (aa, d)
      by auto
    have (aa, d) ∈ {a..b} × {c..d}
      using aa-in cd by auto
    then show x ∈ V
      using rectV x-eq by blast
  qed
  have segc-loop: segment-loop (λt. h (t, c)) a b ∈ loop-space V x0
    by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont a01 b01 c01 ab])
      (use segc-in-V acUV bcUV in auto)
  have segd-loop: segment-loop (λt. h (t, d)) a b ∈ loop-space V x0
  proof −
    have UVV: U ∩ V ⊆ V
      by blast
    show ?thesis
      by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont a01 b01 d01 ab
segd-in-V adUV bdUV UVV])
  qed
  have bridge-a-loop: bridge-loop h a c d ∈ loop-space (U ∩ V) x0
    by (rule vertical-bridge-loop-in-set[OF h-cont a01 c01 d01 cd leftUV]) simp
  have bridge-b-loop: bridge-loop h b c d ∈ loop-space (U ∩ V) x0
    by (rule vertical-bridge-loop-in-set[OF h-cont b01 c01 d01 cd rightUV]) simp

  have bottom-in: bottom ∈ G2
    unfolding bottom-def by (rule loop-class-in-space[OF segc-loop])
  have top-in: top ∈ G2
    unfolding top-def by (rule loop-class-in-space[OF segd-loop])
  have ga-in-H: ga ∈ H
    unfolding ga-def by (rule loop-class-in-space[OF bridge-a-loop])
  have gb-in-H: gb ∈ H
    unfolding gb-def by (rule loop-class-in-space[OF bridge-b-loop])
  have ga-in: i2 ga ∈ G2
    by (rule i2-in-G2[OF ga-in-H])
  have gb-in: i2 gb ∈ G2
    by (rule i2-in-G2[OF gb-in-H])
  have ab-in: mult2 bottom (i2 gb) ∈ G2
    by (rule fundamental-group-mult-in-space[OF bottom-in gb-in])
  have cd-in: mult2 (i2 ga) top ∈ G2
    by (rule fundamental-group-mult-in-space[OF ga-in top-in])

  have ga-eq: i2 ga = loop-class V x0 (bridge-loop h a c d)
    unfolding ga-def
    by (subst i2-loop-class-eq[OF bridge-a-loop]) simp
  have gb-eq: i2 gb = loop-class V x0 (bridge-loop h b c d)
    unfolding gb-def
    by (subst i2-loop-class-eq[OF bridge-b-loop]) simp
```

have *mult-eq*: *mult2 bottom* (*i2 gb*) = *mult2* (*i2 ga*) *top*
unfolding *bottom-def top-def ga-eq gb-eq*
by (*rule rectangle-segment-loop-bridge-class-eq*[*OF h-cont a01 b01 c01 d01 ab cd rectV leftUV rightUV*])
auto
have *pair-eq*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*WordRight bottom* (*WordRight* (*i2 gb*) *rest*))
(*WordRight* (*i2 ga*) (*WordRight top rest*))
by (*rule carrier-full-amalg-equiv-right-pair-eq*[*OF bottom-in gb-in ab-in ga-in top-in cd-in rest-in mult-eq*])

show *?thesis*
using *pair-eq*
by (*simp only*: *bottom-def top-def bridge-word.simps ga-def gb-def*)
qed

lemma *rectangle-partition-partition-word-with-tail-in-space*:
fixes *h* :: (*real* × *real*) ⇒ *'a*
assumes *h-cont*: *continuous-map* (*top-of-set* ({*0..1*} × {*0..1*})) (*top-of-set W*) *h*
and *c01*: *c* ∈ {*0..1*} **and** *d01*: *d* ∈ {*0..1*} **and** *cd*: *c* ≤ *d*
and *y-in*: *y* ∈ {*c*, *d*}
and *part*: *rectangle-partition h c d ts bs*
and *edgeUV*: ⋀*t*. *t* ∈ *set ts* ⟹ *h* ‘ ({*t*} × {*c..d*}) ⊆ *U* ∩ *V*
and *tail-in*: *fpw-in-space G1 G2 rest*
shows *fpw-in-space G1 G2* (*partition-word-with-tail* (λ*t*. *h* (*t*, *y*)) *ts bs rest*)
using *part edgeUV*
proof (*induction ts arbitrary*: *bs*)
case *Nil*
then show *?case*
by (*cases bs*) (*simp-all add*: *tail-in*)
next
case (*Cons t ts*)
show *?case*
proof (*cases ts*)
case *Nil*
then show *?thesis*
using *Cons.prems tail-in* **by** (*cases bs*) *simp-all*
next
case (*Cons u us*)
obtain *b bs′* **where** *bs*: *bs* = *b* # *bs′*
using *Cons.prems Cons* **by** (*cases bs*) *auto*
have *tail-part*: *rectangle-partition h c d* (*u* # *us*) *bs′*
using *Cons.prems Cons bs* **by** *simp*
have *tail-edgeUV*:
⋀*x*. *x* ∈ *set* (*u* # *us*) ⟹ *h* ‘ ({*x*} × {*c..d*}) ⊆ *U* ∩ *V*
using *Cons.prems Cons* **by** *auto*
have *tail-in′*:

fpw-in-space G1 G2 (partition-word-with-tail (λt. h (t, y)) (u # us) bs′ rest)
using *h-cont c01 d01 cd y-in Cons.IH[of bs′] Cons tail-part tail-edgeUV tail-in bs*
by *simp*

have *t01*: $t \in \{0..1\}$ **and** *u01*: $u \in \{0..1\}$ **and** *tu*: $t < u$
and *rect-side*:
(*if b then h ‘ ({t..u} × {c..d}) ⊆ U else h ‘ ({t..u} × {c..d}) ⊆ V*)
using *Cons.prems Cons bs* **by** *simp-all*
have *y01*: $y \in \{0..1\}$
using *y-in c01 d01* **by** *auto*
have *y-cd*: $y \in \{c..d\}$
using *y-in cd* **by** *auto*
have *tu-le*: $t \leq u$
using *tu* **by** *simp*
have *tyUV*: $h\ (t,\ y) \in U \cap V$
proof −
have *t-edge*: h ‘ $(\{t\} \times \{c..d\}) \subseteq U \cap V$
using *Cons.prems(2)[of t] Cons* **by** *simp*
moreover have $(t,\ y) \in \{t\} \times \{c..d\}$
using *y-cd* **by** *auto*
ultimately show *?thesis*
using *t-edge*
by *blast*
qed
have *uyUV*: $h\ (u,\ y) \in U \cap V$
proof −
have *u-edge*: h ‘ $(\{u\} \times \{c..d\}) \subseteq U \cap V$
using *Cons.prems(2)[of u] Cons* **by** *simp*
moreover have $(u,\ y) \in \{u\} \times \{c..d\}$
using *y-cd* **by** *auto*
ultimately show *?thesis*
using *u-edge*
by *blast*
qed

show *?thesis*
proof (*cases b*)
case *True*
have *segyU*: h ‘ $(\{t..u\} \times \{y\}) \subseteq U$
proof
fix *z*
assume *z-in*: $z \in h$ ‘ $(\{t..u\} \times \{y\})$
then obtain *a* **where** *a-in*: $a \in \{t..u\}$ **and** *z-eq*: $z = h\ (a,\ y)$
by *auto*
have $(a,\ y) \in \{t..u\} \times \{c..d\}$
using *a-in y-cd* **by** *auto*
then have *z-strip*: $z \in h$ ‘ $(\{t..u\} \times \{c..d\})$
using *z-eq* **by** *blast*

```
      have stripU: h ‘ ({t..u} × {c..d}) ⊆ U
        using rect-side True by simp
      then show z ∈ U
        using z-strip by blast
    qed
    have segU: segment-loop (λt. h (t, y)) t u ∈ loop-space U x0
       by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont t01 u01 y01
tu-le])
        (use segyU tyUV uyUV in auto)
    have seg-in: loop-class U x0 (segment-loop (λt. h (t, y)) t u) ∈ G1
      by (rule loop-class-in-space[OF segU])
    show ?thesis
      using True seg-in tail-in' bs Cons by simp
  next
    case False
    have segyV: h ‘ ({t..u} × {y}) ⊆ V
    proof
      fix z
      assume z-in: z ∈ h ‘ ({t..u} × {y})
      then obtain a where a-in: a ∈ {t..u} and z-eq: z = h (a, y)
        by auto
      have (a, y) ∈ {t..u} × {c..d}
        using a-in y-cd by auto
      then have z-strip: z ∈ h ‘ ({t..u} × {c..d})
        using z-eq by blast
      have stripV: h ‘ ({t..u} × {c..d}) ⊆ V
        using rect-side False by simp
      then show z ∈ V
        using z-strip by blast
    qed
    have segV: segment-loop (λt. h (t, y)) t u ∈ loop-space V x0
       by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont t01 u01 y01
tu-le])
        (use segyV tyUV uyUV in auto)
    have seg-in: loop-class V x0 (segment-loop (λt. h (t, y)) t u) ∈ G2
      by (rule loop-class-in-space[OF segV])
    show ?thesis
      using False seg-in tail-in' bs Cons by simp
  qed
 qed
qed

lemma rectangle-partition-partition-word-with-tail-equiv:
 fixes h :: (real × real) ⇒ 'a
 assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
   and c01: c ∈ {0..1} and d01: d ∈ {0..1} and cd: c ≤ d
   and part: rectangle-partition h c d (t # u # ts) (b # bs)
   and edgeUV: ⋀x. x ∈ set (t # u # ts) ⟹ h ‘ ({x} × {c..d}) ⊆ U ∩ V
```

```
    and rest-in: fpw-in-space G1 G2 rest
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail (λx. h (x, c)) (t # u # ts) (b # bs)
        (bridge-word (last (b # bs))
          (loop-class (U ∩ V) x0 (bridge-loop h (last (t # u # ts)) c d)) rest))
      (bridge-word b (loop-class (U ∩ V) x0 (bridge-loop h t c d))
        (partition-word-with-tail (λx. h (x, d)) (t # u # ts) (b # bs) rest))
  using part rest-in edgeUV
proof (induction bs arbitrary: t u ts b rest)
  case Nil
  have ts-nil: ts = []
  proof (cases ts)
    case Nil
    then show ?thesis .
  next
    case (Cons v vs)
    with Nil.prems(1) show ?thesis
      by simp
  qed
  have u1: u = 1
    using Nil.prems(1) ts-nil by simp
  have t01: t ∈ {0..1} and u01: u ∈ {0..1} and tu: t < u
    using Nil.prems(1) ts-nil by simp-all
  have tu-le: t ≤ u
    using tu by simp
  have rect-side: (if b then h ‘ ({t..u} × {c..d}) ⊆ U else h ‘ ({t..u} × {c..d}) ⊆
V)
    using Nil.prems(1) ts-nil u1 by simp
  have t-in: t ∈ set (t # u # ts)
    by simp
  have u-in: u ∈ set (t # u # ts)
    by simp
  have leftUV: h ‘ ({t} × {c..d}) ⊆ U ∩ V
    by (rule Nil.prems(3)[OF t-in])
  have rightUV: h ‘ ({u} × {c..d}) ⊆ U ∩ V
    by (rule Nil.prems(3)[OF u-in])
  show ?case
  proof (cases b)
    case True
    have rectU: h ‘ ({t..u} × {c..d}) ⊆ U
      using rect-side True by simp
    have rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (WordLeft (loop-class U x0 (segment-loop (λx. h (x, c)) t u))
          (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h u c d)) rest))
        (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h t c d))
          (WordLeft (loop-class U x0 (segment-loop (λx. h (x, d)) t u)) rest))
      by (rule rectangle-segment-bridge-left-equiv[
            where h = h and a = t and b = u and c = c and d = d and rest =
```

rest,
OF h-cont t01 u01 c01 d01 tu-le cd rectU leftUV rightUV Nil.prems(*2*)])
have *lhs-eq*:
partition-word-with-tail (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *ts*) [*b*]
(*bridge-word* (*last* [*b*])
(*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h* (*last* (*t* # *u* # *ts*)) *c d*)) *rest*) =
WordLeft (*loop-class U x0* (*segment-loop* (λ*x*. *h* (*x*, *c*)) *t u*))
(*bridge-word True* (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h u c d*)) *rest*)
using *True ts-nil* **by** *simp*
have *rhs-eq*:
bridge-word b (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h t c d*))
(*partition-word-with-tail* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *ts*) [*b*] *rest*) =
bridge-word True (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h t c d*))
(*WordLeft* (*loop-class U x0* (*segment-loop* (λ*x*. *h* (*x*, *d*)) *t u*)) *rest*)
using *True ts-nil* **by** *simp*
show *?thesis*
unfolding *lhs-eq rhs-eq* **by** (*rule rel*)
next
case *False*
have *rectV*: *h* ‘ ({*t*..*u*} × {*c*..*d*}) ⊆ *V*
using *rect-side False* **by** *simp*
have *rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*WordRight* (*loop-class V x0* (*segment-loop* (λ*x*. *h* (*x*, *c*)) *t u*))
(*bridge-word False* (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h u c d*)) *rest*))
(*bridge-word False* (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h t c d*))
(*WordRight* (*loop-class V x0* (*segment-loop* (λ*x*. *h* (*x*, *d*)) *t u*)) *rest*))
by (*rule rectangle-segment-bridge-right-equiv*[
where *h* = *h* **and** *a* = *t* **and** *b* = *u* **and** *c* = *c* **and** *d* = *d* **and** *rest* =
rest,
OF h-cont t01 u01 c01 d01 tu-le cd rectV leftUV rightUV Nil.prems(*2*)])
have *lhs-eq*:
partition-word-with-tail (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *ts*) [*b*]
(*bridge-word* (*last* [*b*])
(*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h* (*last* (*t* # *u* # *ts*)) *c d*)) *rest*) =
WordRight (*loop-class V x0* (*segment-loop* (λ*x*. *h* (*x*, *c*)) *t u*))
(*bridge-word False* (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h u c d*)) *rest*)
using *False ts-nil* **by** *simp*
have *rhs-eq*:
bridge-word b (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h t c d*))
(*partition-word-with-tail* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *ts*) [*b*] *rest*) =
bridge-word False (*loop-class* (*U* ∩ *V*) *x0* (*bridge-loop h t c d*))
(*WordRight* (*loop-class V x0* (*segment-loop* (λ*x*. *h* (*x*, *d*)) *t u*)) *rest*)
using *False ts-nil* **by** *simp*
show *?thesis*
unfolding *lhs-eq rhs-eq* **by** (*rule rel*)
qed
next
case (*Cons cflag bs'*)

obtain *v us* **where** *ts*: $ts = v \# us$
using *Cons.prems*(*1*) **by** (*cases ts*) *simp-all*
have *t01*: $t \in \{0..1\}$ **and** *u01*: $u \in \{0..1\}$ **and** *tu*: $t < u$
and *rect-side*: (*if b then h* ‘ $(\{t..u\} \times \{c..d\}) \subseteq U$ *else h* ‘ $(\{t..u\} \times \{c..d\}) \subseteq V$)
and *tail-part*: *rectangle-partition h c d* $(u \# v \# us)$ $(cflag \# bs')$
using *Cons.prems*(*1*) *ts* **by** *simp-all*
have *tu-le*: $t \leq u$
using *tu* **by** *simp*
have *t-in*: $t \in set\ (t \# u \# ts)$
by *simp*
have *u-in*: $u \in set\ (t \# u \# ts)$
by *simp*
have *leftUV*: *h* ‘ $(\{t\} \times \{c..d\}) \subseteq U \cap V$
by (*rule Cons.prems*(*3*)[*OF t-in*])
have *midUV*: *h* ‘ $(\{u\} \times \{c..d\}) \subseteq U \cap V$
by (*rule Cons.prems*(*3*)[*OF u-in*])
have *top-tail-in*:
fpw-in-space G1 G2
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ d))$ $(u \# v \# us)$ $(cflag \# bs')$ *rest*)
by (*rule rectangle-partition-partition-word-with-tail-in-space*[*OF h-cont c01 d01 cd*])
(*use tail-part Cons.prems*(*3*) *Cons.prems*(*2*) *ts* **in** *auto*)
have *gu-loop*: *bridge-loop h u c d* ∈ *loop-space* $(U \cap V)$ *x0*
by (*rule vertical-bridge-loop-in-set*[*OF h-cont u01 c01 d01 cd*]) (*use midUV* **in** *auto*)
have *gu-in-H*: *loop-class* $(U \cap V)$ *x0* (*bridge-loop h u c d*) ∈ *H*
by (*rule loop-class-in-space*[*OF gu-loop*])
have *tail-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ c))$ $(u \# v \# us)$ $(cflag \# bs')$
(*bridge-word* (*last* $(cflag \# bs')$)
(*loop-class* $(U \cap V)$ *x0* (*bridge-loop h* (*last* $(u \# v \# us)$) *c d*)) *rest*))
(*bridge-word cflag* (*loop-class* $(U \cap V)$ *x0* (*bridge-loop h u c d*))
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ d))$ $(u \# v \# us)$ $(cflag \# bs')$ *rest*))
by (*rule Cons.IH*[*OF tail-part*])
(*use h-cont c01 d01 cd Cons.prems*(*3*) *Cons.prems*(*2*) *ts* **in** *auto*)
have *tail-switched*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ c))$ $(u \# v \# us)$ $(cflag \# bs')$
(*bridge-word* (*last* $(cflag \# bs')$)
(*loop-class* $(U \cap V)$ *x0* (*bridge-loop h* (*last* $(u \# v \# us)$) *c d*)) *rest*))
(*bridge-word b* (*loop-class* $(U \cap V)$ *x0* (*bridge-loop h u c d*))
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ d))$ $(u \# v \# us)$ $(cflag \# bs')$ *rest*))
proof (*cases b* = *cflag*)
case *True*
then show *?thesis*
using *tail-rel* **by** *simp*
next

```
    case False
    have switch:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (bridge-word cflag (loop-class (U ∩ V) x0 (bridge-loop h u c d))
         (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest))
       (bridge-word b (loop-class (U ∩ V) x0 (bridge-loop h u c d))
         (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest))
        by (rule carrier-full-amalg-equiv.sym[OF bridge-word-switch[OF gu-in-H
False]])
    show ?thesis
      by (rule carrier-full-amalg-equiv.trans[OF tail-rel switch])
  qed
  show ?case
  proof (cases b)
    case True
    have seg-in: loop-class U x0 (segment-loop (λx. h (x, c)) t u) ∈ G1
    proof −
      have line-subset: {t..u} × {c} ⊆ {t..u} × {c..d}
        using c01 cd by auto
      have rectU: h ‘ ({t..u} × {c..d}) ⊆ U
        using rect-side True by simp
      have segU-line: h ‘ ({t..u} × {c}) ⊆ U
        by (rule order-trans[OF image-mono[OF line-subset] rectU])
      have tcUV: h (t, c) ∈ U ∩ V
        by (rule subsetD[OF leftUV]) (use cd in auto)
      have ucUV: h (u, c) ∈ U ∩ V
        by (rule subsetD[OF midUV]) (use cd in auto)
      have seg-loop: segment-loop (λx. h (x, c)) t u ∈ loop-space U x0
       by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont t01 u01 c01 tu-le
segU-line tcUV ucUV]) auto
      show ?thesis
        by (rule loop-class-in-space[OF seg-loop])
    qed
    have pref-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (WordLeft (loop-class U x0 (segment-loop (λx. h (x, c)) t u))
         (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs')
           (bridge-word (last (cflag # bs'))
             (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest)))
       (WordLeft (loop-class U x0 (segment-loop (λx. h (x, c)) t u))
         (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h u c d))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest)))
    proof −
      have tail-true:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
         (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs')
           (bridge-word (last (cflag # bs'))
             (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest))
         (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h u c d))
```

```
        (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs′) rest))
      using tail-switched True by simp
    show ?thesis
      by (rule carrier-full-amalg-equiv-left-context[OF tail-true seg-in])
  qed
  have cell-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop (λx. h (x, c)) t u))
        (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h u c d))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs′) rest)))
      (bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h t c d))
        (WordLeft (loop-class U x0 (segment-loop (λx. h (x, d)) t u))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs′) rest)))
    by (rule rectangle-segment-bridge-left-equiv[OF h-cont t01 u01 c01 d01 tu-le
cd])
      (use rect-side True leftUV midUV top-tail-in in auto)
  have lhs-eq:
    partition-word-with-tail (λx. h (x, c)) (t # u # ts) (b # cflag # bs′)
      (bridge-word (last (b # cflag # bs′))
        (loop-class (U ∩ V) x0 (bridge-loop h (last (t # u # ts)) c d)) rest) =
     WordLeft (loop-class U x0 (segment-loop (λx. h (x, c)) t u))
      (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs′)
        (bridge-word (last (cflag # bs′))
          (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest))
    using True ts by simp
  have rhs-eq:
    bridge-word b (loop-class (U ∩ V) x0 (bridge-loop h t c d))
      (partition-word-with-tail (λx. h (x, d)) (t # u # ts) (b # cflag # bs′) rest)
=
     bridge-word True (loop-class (U ∩ V) x0 (bridge-loop h t c d))
      (WordLeft (loop-class U x0 (segment-loop (λx. h (x, d)) t u))
        (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs′) rest))
    using True ts by simp
  show ?thesis
    unfolding lhs-eq rhs-eq by (rule carrier-full-amalg-equiv.trans[OF pref-rel
cell-rel])
next
  case False
  have seg-in: loop-class V x0 (segment-loop (λx. h (x, c)) t u) ∈ G2
  proof −
    have line-subset: {t..u} × {c} ⊆ {t..u} × {c..d}
      using c01 cd by auto
    have rectV: h ‘ ({t..u} × {c..d}) ⊆ V
      using rect-side False by simp
    have segV-line: h ‘ ({t..u} × {c}) ⊆ V
      by (rule order-trans[OF image-mono[OF line-subset] rectV])
    have tcUV: h (t, c) ∈ U ∩ V
      by (rule subsetD[OF leftUV]) (use cd in auto)
    have ucUV: h (u, c) ∈ U ∩ V
```

```
      by (rule subsetD[OF midUV]) (use cd in auto)
    have seg-loop: segment-loop (λx. h (x, c)) t u ∈ loop-space V x0
     by (rule horizontal-rectangle-segment-loop-in-set[OF h-cont t01 u01 c01 tu-le
segV-line tcUV ucUV]) auto
    show ?thesis
      by (rule loop-class-in-space[OF seg-loop])
  qed
  have pref-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight (loop-class V x0 (segment-loop (λx. h (x, c)) t u))
        (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs')
          (bridge-word (last (cflag # bs'))
            (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest)))
      (WordRight (loop-class V x0 (segment-loop (λx. h (x, c)) t u))
        (bridge-word False (loop-class (U ∩ V) x0 (bridge-loop h u c d))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest)))
  proof −
    have tail-false:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs')
          (bridge-word (last (cflag # bs'))
            (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest))
        (bridge-word False (loop-class (U ∩ V) x0 (bridge-loop h u c d))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest))
      using tail-switched False by simp
    show ?thesis
      by (rule carrier-full-amalg-equiv-right-context[OF tail-false seg-in])
  qed
  have cell-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight (loop-class V x0 (segment-loop (λx. h (x, c)) t u))
        (bridge-word False (loop-class (U ∩ V) x0 (bridge-loop h u c d))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest)))
      (bridge-word False (loop-class (U ∩ V) x0 (bridge-loop h t c d))
        (WordRight (loop-class V x0 (segment-loop (λx. h (x, d)) t u))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest)))
    by (rule rectangle-segment-bridge-right-equiv[OF h-cont t01 u01 c01 d01 tu-le
cd])
       (use rect-side False leftUV midUV top-tail-in in auto)
  have lhs-eq:
    partition-word-with-tail (λx. h (x, c)) (t # u # ts) (b # cflag # bs')
      (bridge-word (last (b # cflag # bs'))
        (loop-class (U ∩ V) x0 (bridge-loop h (last (t # u # ts)) c d)) rest) =
     WordRight (loop-class V x0 (segment-loop (λx. h (x, c)) t u))
      (partition-word-with-tail (λx. h (x, c)) (u # v # us) (cflag # bs')
        (bridge-word (last (cflag # bs'))
          (loop-class (U ∩ V) x0 (bridge-loop h (last (u # v # us)) c d)) rest))
    using False ts by simp
  have rhs-eq:
```

```
      bridge-word b (loop-class (U ∩ V) x0 (bridge-loop h t c d))
        (partition-word-with-tail (λx. h (x, d)) (t # u # ts) (b # cflag # bs') rest)
=
       bridge-word False (loop-class (U ∩ V) x0 (bridge-loop h t c d))
        (WordRight (loop-class V x0 (segment-loop (λx. h (x, d)) t u))
          (partition-word-with-tail (λx. h (x, d)) (u # v # us) (cflag # bs') rest))
      using False ts by simp
    show ?thesis
      unfolding lhs-eq rhs-eq by (rule carrier-full-amalg-equiv.trans[OF pref-rel
cell-rel])
  qed
qed

lemma homotopic-paths-join-subpathin:
  assumes p-path: path p
    and p-img: path-image p ⊆ S
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and w01: w ∈ {0..1}
    and uv: u ≤ v
    and vw: v ≤ w
  shows homotopic-paths S (subpathin u v p +++ subpathin v w p) (subpathin u
w p)
proof (cases w = u)
  case True
  then have uvw: u = v v = w
    using uv vw by linarith+
  have pu-in: p u ∈ S
    using p-img u01 by (auto simp: path-image-def)
  have const-join: subpathin u v p +++ subpathin v w p = (λ-. p u)
    using uvw by (simp add: fun-eq-iff joinpaths-def subpathin-def)
  have const-subpath: subpathin u w p = (λ-. p u)
    using uvw by (simp add: fun-eq-iff subpathin-def)
  show ?thesis
    unfolding const-join const-subpath
    using pu-in by (simp add: path-def)
next
  case False
  define a where a = (v − u) / (w − u)
  have wu-pos: 0 < w − u
    using False uv vw by linarith
  have a-nonneg: 0 ≤ a
    unfolding a-def using uv wu-pos by (simp add: field-simps)
  have a-le1: a ≤ 1
    unfolding a-def using uv vw wu-pos by (simp add: field-simps)
  have a01: a ∈ {0..1}
    using a-nonneg a-le1 by auto
  let ?f = λt::real. if t ≤ 1 / 2 then 2 * a * t else a + (1 − a) * (2 * t − 1)
  have f-eq: ?f = subpathin 0 a id +++ subpathin a 1 id
```

```
  by (rule ext) (simp add: joinpaths-def subpathin-def)
have contf: continuous-on {0..1} ?f
proof −
  have id-path: path id
    by (simp add: path-def)
  have left-path: path (subpathin 0 a id)
    by (rule path-subpathin[OF id-path]) (use a01 in auto)
  have right-path: path (subpathin a 1 id)
    by (rule path-subpathin[OF id-path]) (use a01 in auto)
  have join-path: path (subpathin 0 a id +++ subpathin a 1 id)
    using left-path right-path a01
    by (simp add: pathstart-def pathfinish-def subpathin-def)
  show ?thesis
    unfolding f-eq
    using join-path by (simp add: path-def)
qed
have f01: ?f ∈ {0..1} → {0..1}
proof
  fix t :: real
  assume t01: t ∈ {0..1}
  show ?f t ∈ {0..1}
  proof (cases t ≤ 1 / 2)
    case True
    have t-nonneg: 0 ≤ t
      using t01 by auto
    have two-t-le1: 2 * t ≤ 1
      using t01 True by linarith
    have lower: 0 ≤ 2 * a * t
      using a-nonneg t-nonneg by (intro mult-nonneg-nonneg) auto
    have upper1: a * (2 * t) ≤ a * 1
      using a-nonneg two-t-le1 by (intro mult-left-mono) auto
    have upper: 2 * a * t ≤ 1
      using upper1 a-le1 by (simp add: algebra-simps)
    show ?thesis
      using True lower upper by auto
  next
    case False
    have two-t-minus1-nonneg: 0 ≤ 2 * t − 1
      using t01 False by linarith
    have two-t-minus1-le1: 2 * t − 1 ≤ 1
    proof −
      have t-le1: t ≤ 1
        using t01 by auto
      then show ?thesis
        by linarith
    qed
    have one-minus-a-nonneg: 0 ≤ 1 − a
      using a-le1 by linarith
    have lower: 0 ≤ a + (1 − a) * (2 * t − 1)
```

```
      using a-nonneg one-minus-a-nonneg two-t-minus1-nonneg
      by (intro add-nonneg-nonneg mult-nonneg-nonneg)
    have upper1: (1 − a) ∗ (2 ∗ t − 1) ≤ (1 − a) ∗ 1
     using one-minus-a-nonneg two-t-minus1-le1 by (intro mult-left-mono) auto
    have upper: a + (1 − a) ∗ (2 ∗ t − 1) ≤ 1
    proof −
      have a + (1 − a) ∗ (2 ∗ t − 1) ≤ a + (1 − a) ∗ 1
        using upper1 by linarith
      also have ... = 1
        by simp
      finally show ?thesis .
    qed
    show ?thesis
      using False lower upper by auto
  qed
 qed
 have sub-uw-path: path (subpathin u w p)
  by (rule path-subpathin[OF p-path u01 w01])
 have sub-uw-img: path-image (subpathin u w p) ⊆ S
  using p-img path-image-subpathin-subset[OF u01 w01, of p] by blast
 show ?thesis
 proof (rule homotopic-paths-sym[OF homotopic-paths-reparametrize[where p =
subpathin u w p and f = ?f]])
  show path (subpathin u w p)
   by (rule sub-uw-path)
  show path-image (subpathin u w p) ⊆ S
   by (rule sub-uw-img)
  show continuous-on {0..1} ?f
   by (rule contf)
  show ?f ∈ {0..1} → {0..1}
   by (rule f01)
  show ?f 0 = 0
   by simp
  show ?f 1 = 1
   using a-nonneg a-le1 by simp
  show (subpathin u v p +++ subpathin v w p) t = subpathin u w p (?f t)
   if t01: t ∈ {0..1} for t
  proof (cases t ≤ 1 / 2)
   case True
   have (subpathin u v p +++ subpathin v w p) t = p (u + (v − u) ∗ (2 ∗ t))
    using True by (simp add: joinpaths-def subpathin-def algebra-simps)
   also have ... = p (u + (w − u) ∗ (2 ∗ a ∗ t))
   proof −
    have wa-eq: (w − u) ∗ a = v − u
     unfolding a-def using wu-pos by (simp add: field-simps algebra-simps)
    have scale-eq: (w − u) ∗ (2 ∗ a ∗ t) = (v − u) ∗ (2 ∗ t)
    proof −
     have (w − u) ∗ (2 ∗ a ∗ t) = ((w − u) ∗ a) ∗ (2 ∗ t)
      by (simp add: algebra-simps)
```

```
      also have ... = (v − u) ∗ (2 ∗ t)
        using wa-eq by simp
      finally show ?thesis .
    qed
    have arg-eq: u + (v − u) ∗ (2 ∗ t) = u + (w − u) ∗ (2 ∗ a ∗ t)
      using scale-eq by linarith
    show ?thesis
      by (subst arg-eq) simp
  qed
  also have ... = subpathin u w p (?f t)
    using True by (simp add: subpathin-def algebra-simps)
  finally show ?thesis .
next
  case False
  have (subpathin u v p +++ subpathin v w p) t = p (v + (w − v) ∗ (2 ∗ t −
1))
    using t01 False by (simp add: joinpaths-def subpathin-def algebra-simps)
  also have ... = p (u + (w − u) ∗ (a + (1 − a) ∗ (2 ∗ t − 1)))
  proof −
    have one-minus-a: 1 − a = (w − v) / (w − u)
      unfolding a-def using wu-pos by (simp add: field-simps)
    have ua-eq: u + (w − u) ∗ a = v
      unfolding a-def using wu-pos by (simp add: field-simps)
    have tail-eq: (w − u) ∗ (1 − a) = w − v
      using one-minus-a wu-pos by (simp add: field-simps)
    have arg-eq: u + (w − u) ∗ (a + (1 − a) ∗ (2 ∗ t − 1)) = v + (w − v) ∗
(2 ∗ t − 1)
    proof −
      have u + (w − u) ∗ (a + (1 − a) ∗ (2 ∗ t − 1)) =
          (u + (w − u) ∗ a) + ((w − u) ∗ (1 − a)) ∗ (2 ∗ t − 1)
        by (simp add: algebra-simps)
      also have ... = v + (w − v) ∗ (2 ∗ t − 1)
        using ua-eq tail-eq by simp
      finally show ?thesis .
    qed
    show ?thesis
      using arg-eq by simp
  qed
  also have ... = subpathin u w p (?f t)
    using False by (simp add: subpathin-def algebra-simps)
  finally show ?thesis .
qed
qed
qed

lemma subpathin-full [simp]:
  subpathin 0 1 p = p
  unfolding subpathin-def by (rule ext) simp
```

lemma *segment-loop-refl*:
assumes *puUV*: *p u* ∈ *U* ∩ *V*
shows *homotopic-paths W* (*segment-loop p u u*) (λ-. *x0*)
proof −
let *?c* = *connector* (*p u*)
have *c-path*: *path ?c*
by (*rule connector-path*[*OF puUV*])
have *c-img*: *path-image ?c* ⊆ *W*
using *connector-image-subset*[*OF puUV*] **by** *blast*
have *rev-c-path*: *path* (*reversepath ?c*)
using *c-path* **by** *simp*
have *rev-c-img*: *path-image* (*reversepath ?c*) ⊆ *W*
using *c-img* **by** *simp*
have *hom-rid*: *homotopic-paths W* (*?c* +++ (λ-. *p u*)) *?c*
using *homotopic-paths-rid-const*[*OF c-path c-img*] *connector-finish*[*OF puUV*]
by (*simp add*: *pathfinish-def*)
have *hom-join*:
homotopic-paths W (((*?c* +++ (λ-. *p u*)) +++ *reversepath ?c*)) (*?c* +++ *reversepath ?c*)
proof (*rule homotopic-paths-join-right*[*OF hom-rid rev-c-path rev-c-img*])
show *pathfinish* (*?c* +++ (λ-. *p u*)) = *pathstart* (*reversepath ?c*)
using *connector-finish*[*OF puUV*]
by (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *hom-inv*: *homotopic-paths W* (*?c* +++ *reversepath ?c*) (λ-. *x0*)
using *homotopic-paths-rinv-const*[*OF c-path c-img*] *connector-start*[*OF puUV*]
by *simp*
have *homotopic-paths W* (*segment-loop p u u*) (*?c* +++ *reversepath ?c*)
unfolding *segment-loop-def* **by** (*simp add*: *hom-join*)
then show *?thesis*
by (*rule homotopic-paths-trans*[*OF - hom-inv*])
qed

lemma *segment-loop-full*:
assumes *p-loop*: *p* ∈ *loop-space W x0*
shows *homotopic-paths W* (*segment-loop p 0 1*) *p*
by (*rule segment-loop-base-full-in-set*[*OF p-loop*])

lemma *homotopic-paths-cancel-middle*:
assumes *r-path*: *path r*
and *r-img*: *path-image r* ⊆ *S*
and *c-path*: *path c*
and *c-img*: *path-image c* ⊆ *S*
and *s-path*: *path s*
and *s-img*: *path-image s* ⊆ *S*
and *rc*: *pathfinish r* = *pathfinish c*
and *cs*: *pathstart s* = *pathfinish c*
shows *homotopic-paths S* ((((*r* +++ *reversepath c*) +++ *c*) +++ *s*)) (*r* +++ *s*)

proof −
have *revc-path*: *path* (*reversepath c*)
using *c-path* **by** *simp*
have *revc-img*: *path-image* (*reversepath c*) ⊆ *S*
using *c-img* **by** *simp*
have *mid-path*: *path* (*reversepath c* +++ *c*)
using *revc-path c-path* **by** *simp*
have *mid-img*: *path-image* (*reversepath c* +++ *c*) ⊆ *S*
by (*rule subset-path-image-join*[*OF revc-img c-img*])
have *hom-cancel0*: *homotopic-paths S* (*reversepath c* +++ *c*) (λ-. *pathfinish c*)
by (*rule homotopic-paths-linv-const*[*OF c-path c-img*])
have *hom-cancel1*:
homotopic-paths S (((*reversepath c* +++ *c*) +++ *s*)) (((λ-. *pathfinish c*) +++ *s*))
proof (*rule homotopic-paths-join-right*[*OF hom-cancel0 s-path s-img*])
show *pathfinish* (*reversepath c* +++ *c*) = *pathstart s*
using *cs* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *hom-cancel2*: *homotopic-paths S* (((λ-. *pathfinish c*) +++ *s*)) *s*
using *homotopic-paths-lid-const*[*OF s-path s-img*] *cs* **by** (*simp add*: *pathstart-def*)
have *hom-cancel*: *homotopic-paths S* (((*reversepath c* +++ *c*) +++ *s*)) *s*
by (*rule homotopic-paths-trans*[*OF hom-cancel1 hom-cancel2*])
have *hom-left*:
homotopic-paths S (*r* +++ ((*reversepath c* +++ *c*) +++ *s*)) (*r* +++ *s*)
proof (*rule homotopic-paths-join-left*[*OF hom-cancel r-path r-img*])
show *pathfinish r* = *pathstart* ((*reversepath c* +++ *c*) +++ *s*)
using *rc* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)
qed
have *assoc1*:
homotopic-paths S (((*r* +++ *reversepath c*) +++ *c*)) (*r* +++ (*reversepath c* +++ *c*))
proof −
have *homotopic-paths S* (*r* +++ (*reversepath c* +++ *c*)) (((*r* +++ *reversepath c*) +++ *c*))
by (*rule homotopic-paths-assoc*[*OF r-path r-img revc-path revc-img c-path c-img*]) (*use rc* **in** *simp-all*)
then show *?thesis*
by (*rule homotopic-paths-sym*)
qed
have *assoc1-join*:
homotopic-paths S ((((*r* +++ *reversepath c*) +++ *c*) +++ *s*)) (((*r* +++ (*reversepath c* +++ *c*)) +++ *s*))
proof (*rule homotopic-paths-join-right*[*OF assoc1 s-path s-img*])
show *pathfinish* ((*r* +++ *reversepath c*) +++ *c*) = *pathstart s*
using *rc cs* **by** (*simp add*: *pathstart-def pathfinish-def joinpaths-def reversepath-def*)

```
  qed
  have assoc2:
     homotopic-paths S (((r +++ (reversepath c +++ c)) +++ s)) (r +++
((reversepath c +++ c) +++ s))
  proof −
    have homotopic-paths S (r +++ ((reversepath c +++ c) +++ s)) (((r +++
(reversepath c +++ c)) +++ s))
       by (rule homotopic-paths-assoc[OF r-path r-img mid-path mid-img s-path
s-img]) (use rc cs in simp-all)
    then show ?thesis
      by (rule homotopic-paths-sym)
  qed
  have homotopic-paths S ((((r +++ reversepath c) +++ c) +++ s)) (r +++
((reversepath c +++ c) +++ s))
    by (rule homotopic-paths-trans[OF assoc1-join assoc2])
  then show ?thesis
    by (rule homotopic-paths-trans[OF - hom-left])
qed

lemma segment-loop-join:
  assumes p-path: path p
    and p-img: path-image p ⊆ W
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and w01: w ∈ {0..1}
    and uv: u ≤ v
    and vw: v ≤ w
    and puUV: p u ∈ U ∩ V
    and pvUV: p v ∈ U ∩ V
    and pwUV: p w ∈ U ∩ V
   shows homotopic-paths W (segment-loop p u v +++ segment-loop p v w)
(segment-loop p u w)
proof −
  let ?cu = connector (p u)
  let ?cv = connector (p v)
  let ?cw = connector (p w)
  let ?suv = subpathin u v p
  let ?svw = subpathin v w p
  let ?suw = subpathin u w p
  let ?head = ?cu +++ ?suv
  let ?tail = ?svw +++ reversepath ?cw

  have cu-path: path ?cu
    by (rule connector-path[OF puUV])
  have cv-path: path ?cv
    by (rule connector-path[OF pvUV])
  have cw-path: path ?cw
    by (rule connector-path[OF pwUV])
  have cu-img: path-image ?cu ⊆ W
```

using *connector-image-subset*[*OF puUV*] **by** *blast*
have *cv-img*: *path-image ?cv* ⊆ *W*
using *connector-image-subset*[*OF pvUV*] **by** *blast*
have *cw-img*: *path-image ?cw* ⊆ *W*
using *connector-image-subset*[*OF pwUV*] **by** *blast*

have *suv-path*: *path ?suv*
by (*rule path-subpathin*[*OF p-path u01 v01*])
have *svw-path*: *path ?svw*
by (*rule path-subpathin*[*OF p-path v01 w01*])
have *suw-path*: *path ?suw*
by (*rule path-subpathin*[*OF p-path u01 w01*])
have *suv-img*: *path-image ?suv* ⊆ *W*
using *p-img path-image-subpathin-subset*[*OF u01 v01*, *of p*] **by** *blast*
have *svw-img*: *path-image ?svw* ⊆ *W*
using *p-img path-image-subpathin-subset*[*OF v01 w01*, *of p*] **by** *blast*
have *suw-img*: *path-image ?suw* ⊆ *W*
using *p-img path-image-subpathin-subset*[*OF u01 w01*, *of p*] **by** *blast*
have *suv-start*: *pathstart ?suv* = *p u*
by (*simp add*: *pathstart-def subpathin-def*)
have *svw-start*: *pathstart ?svw* = *p v*
by (*simp add*: *pathstart-def subpathin-def*)
have *svw-finish*: *pathfinish ?svw* = *p w*
by (*simp add*: *pathfinish-def subpathin-def*)
have *rev-cw-path*: *path* (*reversepath ?cw*)
using *cw-path* **by** *simp*
have *rev-cw-img*: *path-image* (*reversepath ?cw*) ⊆ *W*
using *cw-img* **by** *simp*
have *rev-cw-start*: *pathstart* (*reversepath ?cw*) = *p w*
using *connector-finish*[*OF pwUV*] **by** *simp*

have *head-path*: *path ?head*
using *cu-path suv-path connector-finish*[*OF puUV*] *suv-start* **by** *simp*
have *head-img*: *path-image ?head* ⊆ *W*
by (*rule subset-path-image-join*[*OF cu-img suv-img*])
have *tail-path*: *path ?tail*
using *svw-path rev-cw-path svw-finish rev-cw-start* **by** *simp*
have *tail-img*: *path-image ?tail* ⊆ *W*
by (*rule subset-path-image-join*[*OF svw-img rev-cw-img*])

have *seg-uv-loop*: *segment-loop p u v* ∈ *loop-space W x0*
proof (*rule segment-loop-in-W*[*OF p-path p-img u01 v01 puUV pvUV*])
show *subpathin u v p* ‘ {*0..1*} ⊆ *W*
using *suv-img* **by** (*simp add*: *path-image-def*)
qed
then have *seg-uv-path*: *path* (*segment-loop p u v*) **and** *seg-uv-img*: *path-image* (*segment-loop p u v*) ⊆ *W*
unfolding *loop-space-def* **by** *auto*

have *seg-vw-assoc*:
homotopic-paths W (*segment-loop p v w*) (*?cv +++ ?tail*)
proof −
have *homotopic-paths W* (*?cv +++* (*?svw +++ reversepath ?cw*)) ((*?cv +++ ?svw*) *+++ reversepath ?cw*)
by (*rule homotopic-paths-assoc*[*OF cv-path cv-img svw-path svw-img rev-cw-path rev-cw-img*])
(*use connector-finish*[*OF pvUV*] *svw-start svw-finish rev-cw-start* **in** *simp-all*)
then show *?thesis*
unfolding *segment-loop-def* **by** (*rule homotopic-paths-sym*)
qed

have *seg-join-start*: *pathfinish* (*segment-loop p u v*) = *pathstart* (*segment-loop p v w*)
using *pvUV* **by** (*simp add*: *segment-loop-def connector-start connector-finish*)
have *step1*:
homotopic-paths W (*segment-loop p u v +++ segment-loop p v w*) (*segment-loop p u v +++* (*?cv +++ ?tail*))
by (*rule homotopic-paths-join-left*[*OF seg-vw-assoc seg-uv-path seg-uv-img seg-join-start*])

have *seg-cv-start*: *pathfinish* (*segment-loop p u v*) = *pathstart ?cv*
using *pvUV* **by** (*simp add*: *segment-loop-def connector-start connector-finish*)
have *cv-tail-start*: *pathfinish ?cv* = *pathstart ?tail*
using *connector-finish*[*OF pvUV*] *svw-start* **by** *simp*
have *step2*:
homotopic-paths W (*segment-loop p u v +++* (*?cv +++ ?tail*)) ((*segment-loop p u v +++ ?cv*) *+++ ?tail*)
by (*rule homotopic-paths-assoc*[*OF seg-uv-path seg-uv-img cv-path cv-img tail-path tail-img seg-cv-start cv-tail-start*])

have *subpath-uv-finish*: *pathfinish* (*subpathin u v p*) = *p v*
by (*simp add*: *subpathin-def pathfinish-def*)
have *head-finish-pv*: *pathfinish ?head* = *p v*
using *connector-finish*[*OF puUV*] *subpath-uv-finish*
by (*simp add*: *segment-loop-def connector-start connector-finish*)
have *head-cv-finish*: *pathfinish ?head* = *pathfinish ?cv*
using *head-finish-pv connector-finish*[*OF pvUV*] **by** *simp*
have *tail-cv-start*: *pathstart ?tail* = *pathfinish ?cv*
using *connector-finish*[*OF pvUV*] *svw-start* **by** *simp*
have *step3-raw*:
homotopic-paths W ((((*?head +++ reversepath ?cv*) *+++ ?cv*) *+++ ?tail*)) (*?head +++ ?tail*)
by (*rule homotopic-paths-cancel-middle*[*OF head-path head-img cv-path cv-img tail-path tail-img head-cv-finish tail-cv-start*])
have *step3*:
homotopic-paths W ((*segment-loop p u v +++ ?cv*) *+++ ?tail*) (*?head +++ ?tail*)
unfolding *segment-loop-def* **using** *step3-raw* **by** *simp*

have *suv-finish*: *pathfinish ?suv = p v*
by (*simp add*: *subpathin-def pathfinish-def*)
have *svw-start-eq*: *pathstart ?svw = p v*
by (*simp add*: *subpathin-def pathstart-def*)
have *svw-finish*: *pathfinish ?svw = p w*
by (*simp add*: *subpathin-def pathfinish-def*)
have *head-svw-start*: *pathfinish ?head = pathstart ?svw*
using *connector-finish*[*OF puUV*] *suv-finish svw-start-eq*
by (*simp add*: *connector-start connector-finish*)
have *svw-revcw-start*: *pathfinish ?svw = pathstart* (*reversepath ?cw*)
using *connector-finish*[*OF pwUV*] *svw-finish* **by** *simp*
have *step4a*:
homotopic-paths W (*?head +++ ?tail*) ((((*?cu +++ ?suv*) *+++ ?svw*) *+++ reversepath ?cw*)
by (*rule homotopic-paths-assoc*[*OF head-path head-img svw-path svw-img rev-cw-path rev-cw-img head-svw-start svw-revcw-start*])
have *step4b-inner*:
homotopic-paths W ((((*?cu +++ ?suv*) *+++ ?svw*)) (*?cu +++* (*?suv +++ ?svw*))
proof −
have *cu-suv-start*: *pathfinish ?cu = pathstart ?suv*
using *connector-finish*[*OF puUV*] **by** (*simp add*: *subpathin-def pathstart-def*)
have *suv-svw-start*: *pathfinish ?suv = pathstart ?svw*
by (*simp add*: *subpathin-def pathstart-def pathfinish-def*)
have *homotopic-paths W* (*?cu +++* (*?suv +++ ?svw*)) ((((*?cu +++ ?suv*) *+++ ?svw*))
by (*rule homotopic-paths-assoc*[*OF cu-path cu-img suv-path suv-img svw-path svw-img cu-suv-start suv-svw-start*])
then show *?thesis*
by (*rule homotopic-paths-sym*)
qed
have *step4b*:
homotopic-paths W ((((*?cu +++ ?suv*) *+++ ?svw*) *+++ reversepath ?cw*))
(((*?cu +++* (*?suv +++ ?svw*)) *+++ reversepath ?cw*))
proof −
have *cusuv-svw-finish*: *pathfinish* ((((*?cu +++ ?suv*) *+++ ?svw*)) = *p w*
using *connector-finish*[*OF puUV*] *suv-finish svw-finish*
by (*simp add*: *connector-start connector-finish*)
have *cusuv-svw-revcw-start*: *pathfinish* ((((*?cu +++ ?suv*) *+++ ?svw*)) = *pathstart* (*reversepath ?cw*)
using *cusuv-svw-finish connector-finish*[*OF pwUV*] **by** *simp*
show *?thesis*
by (*rule homotopic-paths-join-right*[*OF step4b-inner rev-cw-path rev-cw-img cusuv-svw-revcw-start*])
qed
have *step4*:
homotopic-paths W (*?head +++ ?tail*) (((*?cu +++* (*?suv +++ ?svw*)) *+++ reversepath ?cw*))

```
    by (rule homotopic-paths-trans[OF step4a step4b])

  have step5-inner0: homotopic-paths W (?suv +++ ?svw) ?suw
    by (rule homotopic-paths-join-subpathin[OF p-path p-img u01 v01 w01 uv vw])
  have step5-inner1:
    homotopic-paths W (?cu +++ (?suv +++ ?svw)) (?cu +++ ?suw)
  proof −
    have cu-suvsvw-start: pathfinish ?cu = pathstart (?suv +++ ?svw)
    proof −
      have cu-finish: pathfinish ?cu = p u
        using connector-finish[OF puUV] by simp
      have suvsvw-start: pathstart (?suv +++ ?svw) = p u
        by (simp add: joinpaths-def subpathin-def pathstart-def)
      show ?thesis
        using cu-finish suvsvw-start by simp
    qed
    show ?thesis
          by (rule homotopic-paths-join-left[OF step5-inner0 cu-path cu-img
cu-suvsvw-start])
  qed
  have step5:
     homotopic-paths W (((?cu +++ (?suv +++ ?svw)) +++ reversepath ?cw))
(segment-loop p u w)
  proof −
    have cu-suvw-finish: pathfinish (?cu +++ (?suv +++ ?svw)) = p w
      using connector-finish[OF puUV] suv-finish svw-finish
      by (simp add: connector-start connector-finish)
   have cu-suvw-revcw-start: pathfinish (?cu +++ (?suv +++ ?svw)) = pathstart
(reversepath ?cw)
      using cu-suvw-finish connector-finish[OF pwUV] by simp
    have step5-raw:
      homotopic-paths W (((?cu +++ (?suv +++ ?svw)) +++ reversepath ?cw))
        (((?cu +++ ?suw) +++ reversepath ?cw))
          proof (rule homotopic-paths-join-right[OF step5-inner1 rev-cw-path
rev-cw-img])
     show pathfinish (?cu +++ (?suv +++ ?svw)) = pathstart (reversepath ?cw)
        by (rule cu-suvw-revcw-start)
    qed
    have step5-assoc:
      homotopic-paths W (((?cu +++ ?suw) +++ reversepath ?cw))
        (?cu +++ (?suw +++ reversepath ?cw))
    proof −
      have cu-suw-start: pathfinish ?cu = pathstart ?suw
       using connector-finish[OF puUV] by (simp add: subpathin-def pathstart-def)
      have suw-revcw-start: pathfinish ?suw = pathstart (reversepath ?cw)
      using connector-finish[OF pwUV] by (simp add: subpathin-def pathfinish-def)
      have homotopic-paths W (?cu +++ (?suw +++ reversepath ?cw))
          (((?cu +++ ?suw) +++ reversepath ?cw))
            by (rule homotopic-paths-assoc[OF cu-path cu-img suw-path suw-img
```

rev-cw-path rev-cw-img cu-suw-start suw-revcw-start])
then show *?thesis*
by (*rule homotopic-paths-sym*)
qed
have *step5-assoc'*:
homotopic-paths W (((*?cu* +++ *?suw*) +++ *reversepath ?cw*))
(*connector* (*p u*) +++ (*subpathin u w p* +++ *reversepath* (*connector* (*p w*))))
using *step5-assoc* **by** *simp*
have *step5-final*:
homotopic-paths W (((*?cu* +++ (*?suv* +++ *?svw*)) +++ *reversepath ?cw*))
(*connector* (*p u*) +++ (*subpathin u w p* +++ *reversepath* (*connector* (*p w*))))
by (*rule homotopic-paths-trans*[*OF step5-raw step5-assoc'*])
show *?thesis*
unfolding *segment-loop-def*
by (*rule step5-raw*)
qed

have *homotopic-paths W* (*segment-loop p u v* +++ *segment-loop p v w*) ((*segment-loop p u v* +++ *?cv*) +++ *?tail*)
by (*rule homotopic-paths-trans*[*OF step1 step2*])
then have *homotopic-paths W* (*segment-loop p u v* +++ *segment-loop p v w*) (*?head* +++ *?tail*)
by (*rule homotopic-paths-trans*[*OF - step3*])
then have *homotopic-paths W* (*segment-loop p u v* +++ *segment-loop p v w*)
(((*?cu* +++ (*?suv* +++ *?svw*)) +++ *reversepath ?cw*))
by (*rule homotopic-paths-trans*[*OF - step4*])
then show *?thesis*
by (*rule homotopic-paths-trans*[*OF - step5*])
qed

lemma *segment-loop-join-in-set*:
assumes *p-path*: *path p*
and *p-imgS*: *path-image p* ⊆ *S*
and *SW*: *S* ⊆ *W*
and *UVS*: *U* ∩ *V* ⊆ *S*
and *u01*: *u* ∈ {*0..1*}
and *v01*: *v* ∈ {*0..1*}
and *w01*: *w* ∈ {*0..1*}
and *uv*: *u* ≤ *v*
and *vw*: *v* ≤ *w*
and *puUV*: *p u* ∈ *U* ∩ *V*
and *pvUV*: *p v* ∈ *U* ∩ *V*
and *pwUV*: *p w* ∈ *U* ∩ *V*
shows *homotopic-paths S* (*segment-loop p u v* +++ *segment-loop p v w*) (*segment-loop p u w*)
proof −
let *?cu* = *connector* (*p u*)

let *?cv* = *connector* (*p v*)
let *?cw* = *connector* (*p w*)
let *?suv* = *subpathin u v p*
let *?svw* = *subpathin v w p*
let *?suw* = *subpathin u w p*
let *?head* = *?cu* +++ *?suv*
let *?tail* = *?svw* +++ *reversepath ?cw*

have *p-imgW*: *path-image p* ⊆ *W*
  using *p-imgS SW* **by** *blast*

have *cu-path*: *path ?cu*
  by (*rule connector-path*[*OF puUV*])
have *cv-path*: *path ?cv*
  by (*rule connector-path*[*OF pvUV*])
have *cw-path*: *path ?cw*
  by (*rule connector-path*[*OF pwUV*])
have *cu-img*: *path-image ?cu* ⊆ *S*
  using *connector-image-subset*[*OF puUV*] *UVS* **by** *blast*
have *cv-img*: *path-image ?cv* ⊆ *S*
  using *connector-image-subset*[*OF pvUV*] *UVS* **by** *blast*
have *cw-img*: *path-image ?cw* ⊆ *S*
  using *connector-image-subset*[*OF pwUV*] *UVS* **by** *blast*

have *suv-path*: *path ?suv*
  by (*rule path-subpathin*[*OF p-path u01 v01*])
have *svw-path*: *path ?svw*
  by (*rule path-subpathin*[*OF p-path v01 w01*])
have *suw-path*: *path ?suw*
  by (*rule path-subpathin*[*OF p-path u01 w01*])
have *suv-img*: *path-image ?suv* ⊆ *S*
  using *p-imgS path-image-subpathin-subset*[*OF u01 v01*, *of p*] **by** *blast*
have *svw-img*: *path-image ?svw* ⊆ *S*
  using *p-imgS path-image-subpathin-subset*[*OF v01 w01*, *of p*] **by** *blast*
have *suw-img*: *path-image ?suw* ⊆ *S*
  using *p-imgS path-image-subpathin-subset*[*OF u01 w01*, *of p*] **by** *blast*
have *suv-start*: *pathstart ?suv* = *p u*
  by (*simp add*: *pathstart-def subpathin-def*)
have *svw-start*: *pathstart ?svw* = *p v*
  by (*simp add*: *pathstart-def subpathin-def*)
have *svw-finish*: *pathfinish ?svw* = *p w*
  by (*simp add*: *pathfinish-def subpathin-def*)
have *rev-cw-path*: *path* (*reversepath ?cw*)
  using *cw-path* **by** *simp*
have *rev-cw-img*: *path-image* (*reversepath ?cw*) ⊆ *S*
  using *cw-img* **by** *simp*
have *rev-cw-start*: *pathstart* (*reversepath ?cw*) = *p w*
  using *connector-finish*[*OF pwUV*] **by** *simp*

have *head-path*: *path ?head*
using *cu-path suv-path connector-finish*[*OF puUV*] *suv-start* **by** *simp*
have *head-img*: *path-image ?head* $\subseteq$ *S*
by (*rule subset-path-image-join*[*OF cu-img suv-img*])
have *tail-path*: *path ?tail*
using *svw-path rev-cw-path svw-finish rev-cw-start* **by** *simp*
have *tail-img*: *path-image ?tail* $\subseteq$ *S*
by (*rule subset-path-image-join*[*OF svw-img rev-cw-img*])

have *seg-uv-loop*: *segment-loop p u v* $\in$ *loop-space S x0*
proof (*rule segment-loop-in-set*[**where** *S* = *S*])
show *path p*
by (*rule p-path*)
show *path-image p* $\subseteq$ *W*
by (*rule p-imgW*)
show *u* $\in$ {*0..1*} *v* $\in$ {*0..1*}
by (*rule u01*, *rule v01*)+
show *p u* $\in$ *U* $\cap$ *V p v* $\in$ *U* $\cap$ *V*
by (*rule puUV*, *rule pvUV*)+
show *path-image* (*connector* (*p u*)) $\subseteq$ *S*
by (*rule cu-img*)
show *path-image* (*connector* (*p v*)) $\subseteq$ *S*
by (*rule cv-img*)
show *subpathin u v p* ‘ {*0..1*} $\subseteq$ *S*
using *suv-img* **by** (*simp add*: *path-image-def*)
show *x0* $\in$ *S*
using *x0-in-UV UVS* **by** *blast*
qed
then have *seg-uv-path*: *path* (*segment-loop p u v*)
and *seg-uv-img*: *path-image* (*segment-loop p u v*) $\subseteq$ *S*
unfolding *loop-space-def* **by** *auto*

have *seg-vw-assoc*:
homotopic-paths S (*segment-loop p v w*) (*?cv* +++ *?tail*)
proof −
have *homotopic-paths S* (*?cv* +++ (*?svw* +++ *reversepath ?cw*)) ((*?cv* +++ *?svw*) +++ *reversepath ?cw*)
by (*rule homotopic-paths-assoc*[*OF cv-path cv-img svw-path svw-img rev-cw-path rev-cw-img*])
(*use connector-finish*[*OF pvUV*] *svw-start svw-finish rev-cw-start* **in** *simp-all*)
then show *?thesis*
unfolding *segment-loop-def* **by** (*rule homotopic-paths-sym*)
qed

have *seg-join-start*: *pathfinish* (*segment-loop p u v*) = *pathstart* (*segment-loop p v w*)
using *pvUV* **by** (*simp add*: *segment-loop-def connector-start connector-finish*)
have *step1*:
homotopic-paths S (*segment-loop p u v* +++ *segment-loop p v w*) (*segment-loop*

p u v +++ (?cv +++ ?tail))
by (*rule homotopic-paths-join-left*[*OF seg-vw-assoc seg-uv-path seg-uv-img seg-join-start*])

have *seg-cv-start*: *pathfinish* (*segment-loop p u v*) = *pathstart ?cv*
using *pvUV* **by** (*simp add*: *segment-loop-def connector-start connector-finish*)
have *cv-tail-start*: *pathfinish ?cv* = *pathstart ?tail*
using *connector-finish*[*OF pvUV*] *svw-start* **by** *simp*
have *step2*:
homotopic-paths S (*segment-loop p u v +++ (?cv +++ ?tail)*) ((*segment-loop p u v +++ ?cv*) *+++ ?tail*)
by (*rule homotopic-paths-assoc*[*OF seg-uv-path seg-uv-img cv-path cv-img tail-path tail-img seg-cv-start cv-tail-start*])

have *subpath-uv-finish*: *pathfinish* (*subpathin u v p*) = *p v*
by (*simp add*: *subpathin-def pathfinish-def*)
have *head-finish-pv*: *pathfinish ?head* = *p v*
using *connector-finish*[*OF puUV*] *subpath-uv-finish*
by (*simp add*: *segment-loop-def connector-start connector-finish*)
have *head-cv-finish*: *pathfinish ?head* = *pathfinish ?cv*
using *head-finish-pv connector-finish*[*OF pvUV*] **by** *simp*
have *tail-cv-start*: *pathstart ?tail* = *pathfinish ?cv*
using *connector-finish*[*OF pvUV*] *svw-start* **by** *simp*
have *step3-raw*:
homotopic-paths S ((((*?head +++ reversepath ?cv*) *+++ ?cv*) *+++ ?tail*)) (*?head +++ ?tail*)
by (*rule homotopic-paths-cancel-middle*[*OF head-path head-img cv-path cv-img tail-path tail-img head-cv-finish tail-cv-start*])
have *step3*:
homotopic-paths S ((*segment-loop p u v +++ ?cv*) *+++ ?tail*) (*?head +++ ?tail*)
unfolding *segment-loop-def* **using** *step3-raw* **by** *simp*

have *suv-finish*: *pathfinish ?suv* = *p v*
by (*simp add*: *subpathin-def pathfinish-def*)
have *svw-start-eq*: *pathstart ?svw* = *p v*
by (*simp add*: *subpathin-def pathstart-def*)
have *svw-finish*: *pathfinish ?svw* = *p w*
by (*simp add*: *subpathin-def pathfinish-def*)
have *head-svw-start*: *pathfinish ?head* = *pathstart ?svw*
using *connector-finish*[*OF puUV*] *suv-finish svw-start-eq*
by (*simp add*: *connector-start connector-finish*)
have *svw-revcw-start*: *pathfinish ?svw* = *pathstart* (*reversepath ?cw*)
using *connector-finish*[*OF pwUV*] *svw-finish* **by** *simp*
have *step4a*:
homotopic-paths S (*?head +++ ?tail*) (((*?cu +++ ?suv*) *+++ ?svw*) *+++ reversepath ?cw*)
by (*rule homotopic-paths-assoc*[*OF head-path head-img svw-path svw-img rev-cw-path rev-cw-img head-svw-start svw-revcw-start*])

```
  have step4b-inner:
     homotopic-paths S (((?cu +++ ?suv) +++ ?svw)) (?cu +++ (?suv +++
?svw))
  proof −
    have cu-suv-start: pathfinish ?cu = pathstart ?suv
      using connector-finish[OF puUV] by (simp add: subpathin-def pathstart-def)
    have suv-svw-start: pathfinish ?suv = pathstart ?svw
      by (simp add: subpathin-def pathstart-def pathfinish-def)
   have homotopic-paths S (?cu +++ (?suv +++ ?svw)) (((?cu +++ ?suv) +++
?svw))
      by (rule homotopic-paths-assoc[OF cu-path cu-img suv-path suv-img svw-path
svw-img cu-suv-start suv-svw-start])
    then show ?thesis
      by (rule homotopic-paths-sym)
  qed
  have step4b:
    homotopic-paths S ((((?cu +++ ?suv) +++ ?svw) +++ reversepath ?cw))
      (((?cu +++ (?suv +++ ?svw)) +++ reversepath ?cw))
  proof −
    have cusuv-svw-finish: pathfinish (((?cu +++ ?suv) +++ ?svw)) = p w
      using connector-finish[OF puUV] suv-finish svw-finish
      by (simp add: connector-start connector-finish)
   have cusuv-svw-revcw-start: pathfinish (((?cu +++ ?suv) +++ ?svw)) = path-
start (reversepath ?cw)
      using cusuv-svw-finish connector-finish[OF pwUV] by simp
    show ?thesis
       by (rule homotopic-paths-join-right[OF step4b-inner rev-cw-path rev-cw-img
cusuv-svw-revcw-start])
  qed
  have step4:
     homotopic-paths S (?head +++ ?tail) (((?cu +++ (?suv +++ ?svw)) +++
reversepath ?cw))
    by (rule homotopic-paths-trans[OF step4a step4b])

  have step5-inner0: homotopic-paths S (?suv +++ ?svw) ?suw
   by (rule homotopic-paths-join-subpathin[OF p-path p-imgS u01 v01 w01 uv vw])
  have step5-inner1:
    homotopic-paths S (?cu +++ (?suv +++ ?svw)) (?cu +++ ?suw)
  proof −
    have cu-suvsvw-start: pathfinish ?cu = pathstart (?suv +++ ?svw)
    proof −
      have cu-finish: pathfinish ?cu = p u
        using connector-finish[OF puUV] by simp
      have suvsvw-start: pathstart (?suv +++ ?svw) = p u
        by (simp add: joinpaths-def subpathin-def pathstart-def)
      show ?thesis
        using cu-finish suvsvw-start by simp
    qed
    show ?thesis
```

by (*rule homotopic-paths-join-left*[*OF step5-inner0 cu-path cu-img cu-suvsvw-start*])
qed
have *step5*:
homotopic-paths S (((*?cu* +++ (*?suv* +++ *?svw*)) +++ *reversepath ?cw*)) (*segment-loop p u w*)
proof −
have *cu-suvw-finish*: *pathfinish* (*?cu* +++ (*?suv* +++ *?svw*)) = *p w*
using *connector-finish*[*OF puUV*] *suv-finish svw-finish*
by (*simp add*: *connector-start connector-finish*)
have *cu-suvw-revcw-start*: *pathfinish* (*?cu* +++ (*?suv* +++ *?svw*)) = *pathstart* (*reversepath ?cw*)
using *cu-suvw-finish connector-finish*[*OF pwUV*] **by** *simp*
have *step5-raw*:
homotopic-paths S (((*?cu* +++ (*?suv* +++ *?svw*)) +++ *reversepath ?cw*)) (((*?cu* +++ *?suw*) +++ *reversepath ?cw*))
proof (*rule homotopic-paths-join-right*[*OF step5-inner1 rev-cw-path rev-cw-img*])
show *pathfinish* (*?cu* +++ (*?suv* +++ *?svw*)) = *pathstart* (*reversepath ?cw*)
by (*rule cu-suvw-revcw-start*)
qed
have *step5-assoc*:
homotopic-paths S (((*?cu* +++ *?suw*) +++ *reversepath ?cw*)) (*?cu* +++ (*?suw* +++ *reversepath ?cw*))
proof −
have *cu-suw-start*: *pathfinish ?cu* = *pathstart ?suw*
using *connector-finish*[*OF puUV*] **by** (*simp add*: *subpathin-def pathstart-def*)
have *suw-revcw-start*: *pathfinish ?suw* = *pathstart* (*reversepath ?cw*)
using *connector-finish*[*OF pwUV*] **by** (*simp add*: *subpathin-def pathfinish-def*)
have *homotopic-paths S* (*?cu* +++ (*?suw* +++ *reversepath ?cw*)) (((*?cu* +++ *?suw*) +++ *reversepath ?cw*))
by (*rule homotopic-paths-assoc*[*OF cu-path cu-img suw-path suw-img rev-cw-path rev-cw-img cu-suw-start suw-revcw-start*])
then show *?thesis*
by (*rule homotopic-paths-sym*)
qed
have *step5-assoc′*:
homotopic-paths S (((*?cu* +++ *?suw*) +++ *reversepath ?cw*)) (*connector* (*p u*) +++ (*subpathin u w p* +++ *reversepath* (*connector* (*p w*))))
using *step5-assoc* **by** *simp*
have *step5-final*:
homotopic-paths S (((*?cu* +++ (*?suv* +++ *?svw*)) +++ *reversepath ?cw*)) (*connector* (*p u*) +++ (*subpathin u w p* +++ *reversepath* (*connector* (*p w*))))
by (*rule homotopic-paths-trans*[*OF step5-raw step5-assoc′*])
show *?thesis*
unfolding *segment-loop-def*
by (*rule step5-raw*)

qed

have *homotopic-paths S* (*segment-loop p u v +++ segment-loop p v w*) ((*segment-loop p u v +++ ?cv*) *+++ ?tail*)
by (*rule homotopic-paths-trans*[*OF step1 step2*])
then have *homotopic-paths S* (*segment-loop p u v +++ segment-loop p v w*) (*?head +++ ?tail*)
by (*rule homotopic-paths-trans*[*OF - step3*])
then have *homotopic-paths S* (*segment-loop p u v +++ segment-loop p v w*)
(((*?cu +++* (*?suv +++ ?svw*)) *+++ reversepath ?cw*))
by (*rule homotopic-paths-trans*[*OF - step4*])
then show *?thesis*
by (*rule homotopic-paths-trans*[*OF - step5*])
qed

lemma *partition-loop-nil* [*simp*]:
partition-loop p [] = (λ-. *x0*)
by *simp*

lemma *partition-loop-singleton* [*simp*]:
partition-loop p [*t*] = (λ-. *x0*)
by *simp*

lemma *svk-partition-partition-loop-homotopic-segment-loop*:
assumes *p-loop*: *p* ∈ *loop-space W x0*
and *part*: *svk-partition p* (*t # ts*) *bs*
shows *homotopic-paths W* (*partition-loop p* (*t # ts*)) (*segment-loop p t 1*)
using *part*
proof (*induction ts arbitrary*: *t bs*)
case *Nil*
have *t1*: *t* = *1*
using *svk-partition-last-eq-one*[*OF Nil.prems*] **by** *simp*
have *ptUV*: *p t* ∈ *U* ∩ *V*
by (*rule svk-partition-head-props*(*2*)[*OF Nil.prems*])
have *homotopic-paths W* (*segment-loop p t 1*) (λ-. *x0*)
using *t1 ptUV* **by** (*simp add*: *segment-loop-refl*)
then show *?case*
by (*simp add*: *homotopic-paths-sym*)
next
case (*Cons u us*)
from *p-loop* **have** *p-path*: *path p* **and** *p-img*: *path-image p* ⊆ *W*
unfolding *loop-space-def* **by** *auto*
obtain *b bs′* **where** *bs*: *bs* = *b # bs′*
using *Cons.prems* **by** (*cases bs*) *auto*
have *tail*: *svk-partition p* (*u # us*) *bs′*
using *Cons.prems bs* **by** *simp*
have *tail-hom*:
homotopic-paths W (*partition-loop p* (*u # us*)) (*segment-loop p u 1*)
by (*rule Cons.IH*[*OF tail*])

have *t01*: $t \in \{0..1\}$ **and** *ptUV*: $p\ t \in U \cap V$
using *Cons.prems bs* **by** *simp-all*
have *u01*: $u \in \{0..1\}$ **and** *tu*: $t < u$
using *Cons.prems bs* **by** *simp-all*
have *puUV*: $p\ u \in U \cap V$
using *Cons.prems bs svk-partition-next-in-intersection*[*of p t u us b bs'*] **by** *simp*
have *seg-in*:
subpathin t u p ‘ $\{0..1\} \subseteq W$
by (*cases b*) (*use Cons.prems bs* **in** *auto*)
have *oneUV*: $p\ 1 \in U \cap V$
using *svk-partition-last-in-intersection*[*OF tail*] *svk-partition-last-eq-one*[*OF tail*] **by** *simp*
have *seg-tu-loop*: *segment-loop p t u* $\in$ *loop-space W x0*
by (*rule segment-loop-in-W*[*OF p-path p-img t01 u01 ptUV puUV seg-in*])
then have *seg-tu-path*: *path* (*segment-loop p t u*)
and *seg-tu-img*: *path-image* (*segment-loop p t u*) $\subseteq W$
unfolding *loop-space-def* **by** *auto*
have *tail-loop*: *partition-loop p* (*u # us*) $\in$ *loop-space W x0*
by (*rule svk-partition-partition-loop-in-W*[*OF p-loop tail*])
have *join-hom*:
homotopic-paths W (*segment-loop p t u +++ partition-loop p* (*u # us*))
(*segment-loop p t u +++ segment-loop p u 1*)
proof (*rule homotopic-paths-join-left*[*OF tail-hom seg-tu-path seg-tu-img*])
show *pathfinish* (*segment-loop p t u*) = *pathstart* (*partition-loop p* (*u # us*))
using *seg-tu-loop tail-loop* **unfolding** *loop-space-def* **by** *simp*
qed
have *step1*:
homotopic-paths W (*partition-loop p* (*t # u # us*)) (*segment-loop p t u +++ segment-loop p u 1*)
using *join-hom* **by** *simp*
have *step2*:
homotopic-paths W (*segment-loop p t u +++ segment-loop p u 1*) (*segment-loop p t 1*)
by (*rule segment-loop-join*[*OF p-path p-img t01 u01*]) (*use u01 tu oneUV ptUV puUV* **in** *auto*)
show *?case*
by (*rule homotopic-paths-trans*[*OF step1 step2*])
qed

lemma *valid-partition-partition-loop-homotopic-segment-loop*:
assumes *p-loop*: $p \in$ *loop-space W x0*
and *part*: *valid-partition p ts bs*
shows *homotopic-paths W* (*partition-loop p ts*) (*segment-loop p 0 1*)
proof −
obtain *t ts'* **where** *ts*: $ts = t \# ts'$
using *valid-partition-hd*(*1*)[*OF part*] **by** (*cases ts*) *auto*
have *valid-ts*: *valid-partition p* (*t # ts'*) *bs*
using *part* **unfolding** *ts* **by** *simp*
have *t0*: $t = 0$

```
    by (rule valid-partition-cases(1)[OF valid-ts])
  have svk: svk-partition p (t # ts') bs
    by (rule valid-partition-cases(2)[OF valid-ts])
  have homotopic-paths W (partition-loop p (t # ts')) (segment-loop p t 1)
    by (rule svk-partition-partition-loop-homotopic-segment-loop[OF p-loop svk])
  then show ?thesis
    using ts t0 by simp
qed

lemma valid-partition-partition-loop-homotopic:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p ts bs
  shows homotopic-paths W (partition-loop p ts) p
proof −
  have step1: homotopic-paths W (partition-loop p ts) (segment-loop p 0 1)
    by (rule valid-partition-partition-loop-homotopic-segment-loop[OF p-loop part])
  have step2: homotopic-paths W (segment-loop p 0 1) p
    by (rule segment-loop-full[OF p-loop])
  show ?thesis
    by (rule homotopic-paths-trans[OF step1 step2])
qed

lemma valid-partition-partition-loop-eq:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p ts bs
  shows loop-class W x0 (partition-loop p ts) = loop-class W x0 p
proof −
  have part-loop: partition-loop p ts ∈ loop-space W x0
    by (rule valid-partition-partition-loop-in-W[OF p-loop part])
  have hom: homotopic-paths W (partition-loop p ts) p
    by (rule valid-partition-partition-loop-homotopic[OF p-loop part])
  show ?thesis
    by (rule loop-class-eqI[OF part-loop p-loop hom])
qed

lemma valid-partition-decode-partition-word-eq-loop-class:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p ts bs
  shows svk-decode (partition-word p ts bs) = loop-class W x0 p
  using valid-partition-decode-partition-word[OF p-loop part]
    valid-partition-partition-loop-eq[OF p-loop part]
  by simp

lemma subpathin-image-subset-left:
  assumes t01: t ∈ {0..1}
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and tu: t ≤ u
    and uv: u ≤ v
```

shows *subpathin t u p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}*
proof −
have *closed-segment t u ⊆ closed-segment t v*
using *t01 u01 v01 tu uv* **by** (*auto simp*: *closed-segment-eq-real-ivl*)
then show *?thesis*
by (*auto simp*: *subpathin-image*)
qed

lemma *subpathin-image-subset-right*:
assumes *t01*: $t \in \{0..1\}$
and *u01*: $u \in \{0..1\}$
and *v01*: $v \in \{0..1\}$
and *tu*: $t \leq u$
and *uv*: $u \leq v$
shows *subpathin u v p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}*
proof −
have *closed-segment u v ⊆ closed-segment t v*
using *t01 u01 v01 tu uv* **by** (*auto simp*: *closed-segment-eq-real-ivl*)
then show *?thesis*
by (*auto simp*: *subpathin-image*)
qed

lemma *subpathin-subpathin*:
subpathin a b (*subpathin u v p*) =
subpathin (((*v* − *u*) ∗ *a*) + *u*) (((*v* − *u*) ∗ *b*) + *u*) *p*
unfolding *subpathin-def* **by** (*rule ext*) (*simp add*: *algebra-simps*)

lemma *segment-loop-subpathin*:
segment-loop (*subpathin u v p*) *a b* =
segment-loop p (((*v* − *u*) ∗ *a*) + *u*) (((*v* − *u*) ∗ *b*) + *u*)
unfolding *segment-loop-def subpathin-subpathin subpathin-def*
by (*rule ext*) (*simp add*: *algebra-simps*)

lemma *segment-loop-mult-eq-left*:
assumes *p-path*: *path p*
and *p-imgW*: *path-image p ⊆ W*
and *t01*: $t \in \{0..1\}$
and *u01*: $u \in \{0..1\}$
and *v01*: $v \in \{0..1\}$
and *tu*: $t < u$
and *uv*: $u < v$
and *ptUV*: *p t* ∈ *U* ∩ *V*
and *puUV*: *p u* ∈ *U* ∩ *V*
and *pvUV*: *p v* ∈ *U* ∩ *V*
and *seg-tvU*: *subpathin t v p ‘ {0..1} ⊆ U*
shows *mult1* (*loop-class U x0* (*segment-loop p t u*))
(*loop-class U x0* (*segment-loop p u v*)) =
loop-class U x0 (*segment-loop p t v*)
proof −

```
  define q where q = subpathin t v p
  define a where a = (u − t) / (v − t)

  have tv: t < v
    using tu uv by linarith
  have a01: a ∈ {0..1}
    unfolding a-def using tu uv tv by (auto simp: field-simps)
  have q-path: path q
    unfolding q-def by (rule path-subpathin[OF p-path t01 v01])
  have q-imgU: path-image q ⊆ U
    unfolding q-def using seg-tvU by (simp add: path-image-def)
  have q0UV: q 0 ∈ U ∩ V
    unfolding q-def using ptUV by (simp add: subpathin-def)
  have qaUV: q a ∈ U ∩ V
  proof −
    have qa-eq: q a = p u
    proof −
      have ta-eq: t + (v − t) * a = u
        unfolding a-def using tv by (simp add: field-simps)
      show ?thesis
        unfolding q-def subpathin-def using ta-eq by (simp add: algebra-simps)
    qed
    then show ?thesis
      using puUV by simp
  qed
  have q1UV: q 1 ∈ U ∩ V
    unfolding q-def using pvUV by (simp add: subpathin-def)
  have join-hom-q:
    homotopic-paths U (segment-loop q 0 a +++ segment-loop q a 1) (segment-loop
q 0 1)
      by (rule segment-loop-join-in-set[OF q-path q-imgU]) (use a01 q0UV qaUV
q1UV in auto)

  have seg-tu-eq: segment-loop q 0 a = segment-loop p t u
  proof −
    have ta-eq: ((v − t) * a) + t = u
      unfolding a-def using tv by (simp add: field-simps algebra-simps)
    show ?thesis
      unfolding q-def using ta-eq by (simp add: segment-loop-subpathin)
  qed
  have seg-uv-eq: segment-loop q a 1 = segment-loop p u v
  proof −
    have ta-eq: ((v − t) * a) + t = u
      unfolding a-def using tv by (simp add: field-simps algebra-simps)
    show ?thesis
      unfolding q-def using ta-eq by (simp add: segment-loop-subpathin)
  qed
  have seg-tv-eq: segment-loop q 0 1 = segment-loop p t v
    unfolding q-def by (simp add: segment-loop-subpathin)
```

```
  have seg-tuU: subpathin t u p ‘ {0..1} ⊆ U
    by (rule order-trans[OF subpathin-image-subset-left[OF t01 u01 v01]]) (use tu
uv seg-tvU in auto)
  have seg-uvU: subpathin u v p ‘ {0..1} ⊆ U
    by (rule order-trans[OF subpathin-image-subset-right[OF t01 u01 v01]]) (use tu
uv seg-tvU in auto)

  have loop-tu: segment-loop p t u ∈ loop-space U x0
    by (rule segment-loop-in-U[OF p-path p-imgW t01 u01 ptUV puUV seg-tuU])
  have loop-uv: segment-loop p u v ∈ loop-space U x0
    by (rule segment-loop-in-U[OF p-path p-imgW u01 v01 puUV pvUV seg-uvU])
  have loop-tv: segment-loop p t v ∈ loop-space U x0
    by (rule segment-loop-in-U[OF p-path p-imgW t01 v01 ptUV pvUV seg-tvU])

  have class-tu-in: loop-class U x0 (segment-loop p t u) ∈ G1
    by (rule loop-class-in-space[OF loop-tu])
  have class-uv-in: loop-class U x0 (segment-loop p u v) ∈ G1
    by (rule loop-class-in-space[OF loop-uv])
  have join-loop: segment-loop p t u +++ segment-loop p u v ∈ loop-space U x0
    by (rule loop-space-join[OF loop-tu loop-uv])

  have mult-eq-join:
    mult1 (loop-class U x0 (segment-loop p t u))
      (loop-class U x0 (segment-loop p u v)) =
      loop-class U x0 (segment-loop p t u +++ segment-loop p u v)
    by (rule fundamental-group-mult-eqI[OF class-tu-in class-uv-in loop-tu loop-uv])
simp-all
  have join-eq:
    loop-class U x0 (segment-loop p t u +++ segment-loop p u v) =
      loop-class U x0 (segment-loop p t v)
  proof −
    have join-eq-q:
      loop-class U x0 (segment-loop q 0 a +++ segment-loop q a 1) =
        loop-class U x0 (segment-loop q 0 1)
    proof (rule loop-class-eqI)
      show segment-loop q 0 a +++ segment-loop q a 1 ∈ loop-space U x0
        unfolding seg-tu-eq seg-uv-eq by (rule join-loop)
      show segment-loop q 0 1 ∈ loop-space U x0
        using seg-tv-eq loop-tv by simp
        show homotopic-paths U (segment-loop q 0 a +++ segment-loop q a 1)
(segment-loop q 0 1)
        by (rule join-hom-q)
    qed
    show ?thesis
      using join-eq-q by (simp only: seg-tu-eq seg-uv-eq seg-tv-eq)
  qed
  show ?thesis
    using mult-eq-join join-eq by simp
```

qed

lemma *segment-loop-mult-eq-right*:
  assumes *p-path*: *path p*
    and *p-imgW*: *path-image p* $\subseteq$ *W*
    and *t01*: *t* $\in$ {*0..1*}
    and *u01*: *u* $\in$ {*0..1*}
    and *v01*: *v* $\in$ {*0..1*}
    and *tu*: *t* $<$ *u*
    and *uv*: *u* $<$ *v*
    and *ptUV*: *p t* $\in$ *U* $\cap$ *V*
    and *puUV*: *p u* $\in$ *U* $\cap$ *V*
    and *pvUV*: *p v* $\in$ *U* $\cap$ *V*
    and *seg-tvV*: *subpathin t v p* ' {*0..1*} $\subseteq$ *V*
  shows *mult2* (*loop-class V x0* (*segment-loop p t u*))
      (*loop-class V x0* (*segment-loop p u v*)) =
    loop-class V x0 (*segment-loop p t v*)
proof −
  define *q* **where** *q* = *subpathin t v p*
  define *a* **where** *a* = (*u* − *t*) / (*v* − *t*)

  have *tv*: *t* $<$ *v*
    using *tu uv* **by** *linarith*
  have *a01*: *a* $\in$ {*0..1*}
    unfolding *a-def* **using** *tu uv tv* **by** (*auto simp*: *field-simps*)
  have *q-path*: *path q*
    unfolding *q-def* **by** (*rule path-subpathin*[*OF p-path t01 v01*])
  have *q-imgV*: *path-image q* $\subseteq$ *V*
    unfolding *q-def* **using** *seg-tvV* **by** (*simp add*: *path-image-def*)
  have *q0UV*: *q 0* $\in$ *U* $\cap$ *V*
    unfolding *q-def* **using** *ptUV* **by** (*simp add*: *subpathin-def*)
  have *qaUV*: *q a* $\in$ *U* $\cap$ *V*
  proof −
    have *qa-eq*: *q a* = *p u*
    proof −
      have *ta-eq*: *t* + (*v* − *t*) ∗ *a* = *u*
        unfolding *a-def* **using** *tv* **by** (*simp add*: *field-simps*)
      show *?thesis*
        unfolding *q-def subpathin-def* **using** *ta-eq* **by** (*simp add*: *algebra-simps*)
    qed
    then show *?thesis*
      using *puUV* **by** *simp*
  qed
  have *q1UV*: *q 1* $\in$ *U* $\cap$ *V*
    unfolding *q-def* **using** *pvUV* **by** (*simp add*: *subpathin-def*)
  have *join-hom-q*:
    homotopic-paths V (*segment-loop q 0 a* +++ *segment-loop q a 1*) (*segment-loop q 0 1*)
      by (*rule segment-loop-join-in-set*[*OF q-path q-imgV*]) (*use a01 q0UV qaUV*

q1UV **in** *auto*)

have *seg-tu-eq*: *segment-loop q 0 a = segment-loop p t u*
proof −
have *ta-eq*: $((v - t) * a) + t = u$
unfolding *a-def* **using** *tv* **by** (*simp add*: *field-simps algebra-simps*)
show *?thesis*
unfolding *q-def* **using** *ta-eq* **by** (*simp add*: *segment-loop-subpathin*)
qed
have *seg-uv-eq*: *segment-loop q a 1 = segment-loop p u v*
proof −
have *ta-eq*: $((v - t) * a) + t = u$
unfolding *a-def* **using** *tv* **by** (*simp add*: *field-simps algebra-simps*)
show *?thesis*
unfolding *q-def* **using** *ta-eq* **by** (*simp add*: *segment-loop-subpathin*)
qed
have *seg-tv-eq*: *segment-loop q 0 1 = segment-loop p t v*
unfolding *q-def* **by** (*simp add*: *segment-loop-subpathin*)

have *seg-tuV*: *subpathin t u p ‘ {0..1} ⊆ V*
by (*rule order-trans*[*OF subpathin-image-subset-left*[*OF t01 u01 v01*]]) (*use tu uv seg-tvV* **in** *auto*)
have *seg-uvV*: *subpathin u v p ‘ {0..1} ⊆ V*
by (*rule order-trans*[*OF subpathin-image-subset-right*[*OF t01 u01 v01*]]) (*use tu uv seg-tvV* **in** *auto*)

have *loop-tu*: *segment-loop p t u ∈ loop-space V x0*
by (*rule segment-loop-in-V*[*OF p-path p-imgW t01 u01 ptUV puUV seg-tuV*])
have *loop-uv*: *segment-loop p u v ∈ loop-space V x0*
by (*rule segment-loop-in-V*[*OF p-path p-imgW u01 v01 puUV pvUV seg-uvV*])
have *loop-tv*: *segment-loop p t v ∈ loop-space V x0*
by (*rule segment-loop-in-V*[*OF p-path p-imgW t01 v01 ptUV pvUV seg-tvV*])

have *class-tu-in*: *loop-class V x0* (*segment-loop p t u*) ∈ *G2*
by (*rule loop-class-in-space*[*OF loop-tu*])
have *class-uv-in*: *loop-class V x0* (*segment-loop p u v*) ∈ *G2*
by (*rule loop-class-in-space*[*OF loop-uv*])
have *join-loop*: *segment-loop p t u +++ segment-loop p u v ∈ loop-space V x0*
by (*rule loop-space-join*[*OF loop-tu loop-uv*])

have *mult-eq-join*:
mult2 (*loop-class V x0* (*segment-loop p t u*))
(*loop-class V x0* (*segment-loop p u v*)) =
loop-class V x0 (*segment-loop p t u +++ segment-loop p u v*)
by (*rule fundamental-group-mult-eqI*[*OF class-tu-in class-uv-in loop-tu loop-uv*]) *simp-all*
have *join-eq*:
loop-class V x0 (*segment-loop p t u +++ segment-loop p u v*) =
loop-class V x0 (*segment-loop p t v*)

```
  proof −
    have join-eq-q:
      loop-class V x0 (segment-loop q 0 a +++ segment-loop q a 1) =
        loop-class V x0 (segment-loop q 0 1)
    proof (rule loop-class-eqI)
      show segment-loop q 0 a +++ segment-loop q a 1 ∈ loop-space V x0
        unfolding seg-tu-eq seg-uv-eq by (rule join-loop)
      show segment-loop q 0 1 ∈ loop-space V x0
        using seg-tv-eq loop-tv by simp
        show homotopic-paths V (segment-loop q 0 a +++ segment-loop q a 1)
(segment-loop q 0 1)
        by (rule join-hom-q)
    qed
    show ?thesis
      using join-eq-q by (simp only: seg-tu-eq seg-uv-eq seg-tv-eq)
  qed
  show ?thesis
    using mult-eq-join join-eq by simp
qed

lemma segment-word-split-left-equiv:
  assumes p-loop: p ∈ loop-space W x0
    and t01: t ∈ {0..1}
    and u01: u ∈ {0..1}
    and v01: v ∈ {0..1}
    and tu: t < u
    and uv: u < v
    and ptUV: p t ∈ U ∩ V
    and puUV: p u ∈ U ∩ V
    and pvUV: p v ∈ U ∩ V
    and seg-tvU: subpathin t v p ‘ {0..1} ⊆ U
    and rest-in: fpw-in-space G1 G2 rest
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop p t u))
        (WordLeft (loop-class U x0 (segment-loop p u v)) rest))
      (WordLeft (loop-class U x0 (segment-loop p t v)) rest)
proof −
  from p-loop have p-path: path p and p-imgW: path-image p ⊆ W
    unfolding loop-space-def by auto
  have seg-tuU: subpathin t u p ‘ {0..1} ⊆ U
    by (rule order-trans[OF subpathin-image-subset-left[OF t01 u01 v01]]) (use tu
uv seg-tvU in auto)
  have seg-uvU: subpathin u v p ‘ {0..1} ⊆ U
    by (rule order-trans[OF subpathin-image-subset-right[OF t01 u01 v01]]) (use tu
uv seg-tvU in auto)
  have loop-tu: segment-loop p t u ∈ loop-space U x0
    by (rule segment-loop-in-U[OF p-path p-imgW t01 u01 ptUV puUV seg-tuU])
  have loop-uv: segment-loop p u v ∈ loop-space U x0
    by (rule segment-loop-in-U[OF p-path p-imgW u01 v01 puUV pvUV seg-uvU])
```

```
  have class-tu-in: loop-class U x0 (segment-loop p t u) ∈ G1
    by (rule loop-class-in-space[OF loop-tu])
  have class-uv-in: loop-class U x0 (segment-loop p u v) ∈ G1
    by (rule loop-class-in-space[OF loop-uv])
  have mult-in:
    mult1 (loop-class U x0 (segment-loop p t u))
      (loop-class U x0 (segment-loop p u v)) ∈ G1
    by (rule fundamental-group-mult-in-space[OF class-tu-in class-uv-in])
  have mult-eq:
    mult1 (loop-class U x0 (segment-loop p t u))
      (loop-class U x0 (segment-loop p u v)) =
      loop-class U x0 (segment-loop p t v)
    by (rule segment-loop-mult-eq-left[OF p-path p-imgW t01 u01 v01 tu uv ptUV
puUV pvUV seg-tvU])
  have step:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop p t u))
        (WordLeft (loop-class U x0 (segment-loop p u v)) rest))
      (WordLeft (mult1 (loop-class U x0 (segment-loop p t u))
        (loop-class U x0 (segment-loop p u v))) rest)
  proof (rule carrier-fpw-reduction-step.combine-left)
    show loop-class U x0 (segment-loop p t u) ∈ G1
      by (rule class-tu-in)
    show loop-class U x0 (segment-loop p u v) ∈ G1
      by (rule class-uv-in)
   show mult1 (loop-class U x0 (segment-loop p t u)) (loop-class U x0 (segment-loop
p u v)) ∈ G1
      by (rule mult-in)
    show fpw-in-space G1 G2 rest
      by (rule rest-in)
  qed
  have red:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop p t u))
        (WordLeft (loop-class U x0 (segment-loop p u v)) rest))
      (WordLeft (mult1 (loop-class U x0 (segment-loop p t u))
        (loop-class U x0 (segment-loop p u v))) rest)
    by (rule carrier-fpw-reduction.step[OF step])
  have rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordLeft (loop-class U x0 (segment-loop p t u))
        (WordLeft (loop-class U x0 (segment-loop p u v)) rest))
      (WordLeft (mult1 (loop-class U x0 (segment-loop p t u))
        (loop-class U x0 (segment-loop p u v))) rest)
    by (rule carrier-full-amalg-equiv.of-reduction[OF red])
  show ?thesis
    using rel mult-eq by simp
qed
```

lemma *segment-word-split-right-equiv*:
  assumes *p-loop*: $p \in$ *loop-space W x0*
    and *t01*: $t \in \{0..1\}$
    and *u01*: $u \in \{0..1\}$
    and *v01*: $v \in \{0..1\}$
    and *tu*: $t < u$
    and *uv*: $u < v$
    and *ptUV*: $p\ t \in U \cap V$
    and *puUV*: $p\ u \in U \cap V$
    and *pvUV*: $p\ v \in U \cap V$
    and *seg-tvV*: *subpathin t v p* ‘ $\{0..1\} \subseteq V$
    and *rest-in*: *fpw-in-space G1 G2 rest*
  shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
      (*WordRight* (*loop-class V x0* (*segment-loop p t u*))
        (*WordRight* (*loop-class V x0* (*segment-loop p u v*)) *rest*))
      (*WordRight* (*loop-class V x0* (*segment-loop p t v*)) *rest*)
proof −
  from *p-loop* **have** *p-path*: *path p* **and** *p-imgW*: *path-image p* $\subseteq W$
    unfolding *loop-space-def* **by** *auto*
  have *seg-tuV*: *subpathin t u p* ‘ $\{0..1\} \subseteq V$
    by (*rule order-trans*[*OF subpathin-image-subset-left*[*OF t01 u01 v01*]]) (*use tu uv seg-tvV* **in** *auto*)
  have *seg-uvV*: *subpathin u v p* ‘ $\{0..1\} \subseteq V$
    by (*rule order-trans*[*OF subpathin-image-subset-right*[*OF t01 u01 v01*]]) (*use tu uv seg-tvV* **in** *auto*)
  have *loop-tu*: *segment-loop p t u* $\in$ *loop-space V x0*
    by (*rule segment-loop-in-V*[*OF p-path p-imgW t01 u01 ptUV puUV seg-tuV*])
  have *loop-uv*: *segment-loop p u v* $\in$ *loop-space V x0*
    by (*rule segment-loop-in-V*[*OF p-path p-imgW u01 v01 puUV pvUV seg-uvV*])
  have *class-tu-in*: *loop-class V x0* (*segment-loop p t u*) $\in$ *G2*
    by (*rule loop-class-in-space*[*OF loop-tu*])
  have *class-uv-in*: *loop-class V x0* (*segment-loop p u v*) $\in$ *G2*
    by (*rule loop-class-in-space*[*OF loop-uv*])
  have *mult-in*:
    mult2 (*loop-class V x0* (*segment-loop p t u*))
      (*loop-class V x0* (*segment-loop p u v*)) $\in$ *G2*
    by (*rule fundamental-group-mult-in-space*[*OF class-tu-in class-uv-in*])
  have *mult-eq*:
    mult2 (*loop-class V x0* (*segment-loop p t u*))
      (*loop-class V x0* (*segment-loop p u v*)) =
      loop-class V x0 (*segment-loop p t v*)
    by (*rule segment-loop-mult-eq-right*[*OF p-path p-imgW t01 u01 v01 tu uv ptUV puUV pvUV seg-tvV*])
  have *step*:
    carrier-fpw-reduction-step G1 G2 mult1 one1 mult2 one2
      (*WordRight* (*loop-class V x0* (*segment-loop p t u*))
        (*WordRight* (*loop-class V x0* (*segment-loop p u v*)) *rest*))
      (*WordRight* (*mult2* (*loop-class V x0* (*segment-loop p t u*))
        (*loop-class V x0* (*segment-loop p u v*))) *rest*)

```
  proof (rule carrier-fpw-reduction-step.combine-right)
    show loop-class V x0 (segment-loop p t u) ∈ G2
      by (rule class-tu-in)
    show loop-class V x0 (segment-loop p u v) ∈ G2
      by (rule class-uv-in)
   show mult2 (loop-class V x0 (segment-loop p t u)) (loop-class V x0 (segment-loop
p u v)) ∈ G2
      by (rule mult-in)
    show fpw-in-space G1 G2 rest
      by (rule rest-in)
  qed
  have red:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordRight (loop-class V x0 (segment-loop p t u))
        (WordRight (loop-class V x0 (segment-loop p u v)) rest))
      (WordRight (mult2 (loop-class V x0 (segment-loop p t u))
        (loop-class V x0 (segment-loop p u v))) rest)
    by (rule carrier-fpw-reduction.step[OF step])
  have rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight (loop-class V x0 (segment-loop p t u))
        (WordRight (loop-class V x0 (segment-loop p u v)) rest))
      (WordRight (mult2 (loop-class V x0 (segment-loop p t u))
        (loop-class V x0 (segment-loop p u v))) rest)
    by (rule carrier-full-amalg-equiv.of-reduction[OF red])
  show ?thesis
    using rel mult-eq by simp
qed

lemma segment-word-switch:
  assumes p-path: path p
    and p-imgW: path-image p ⊆ W
    and t01: t ∈ {0..1}
    and u01: u ∈ {0..1}
    and tu: t < u
    and ptUV: p t ∈ U ∩ V
    and puUV: p u ∈ U ∩ V
    and segUV: subpathin t u p ‘ {0..1} ⊆ U ∩ V
    and rest-in: fpw-in-space G1 G2 rest
    and bc: b ≠ c
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (if b then WordLeft (loop-class U x0 (segment-loop p t u)) rest
       else WordRight (loop-class V x0 (segment-loop p t u)) rest)
      (if c then WordLeft (loop-class U x0 (segment-loop p t u)) rest
       else WordRight (loop-class V x0 (segment-loop p t u)) rest)
proof −
  have conn-t-img: path-image (connector (p t)) ⊆ U ∩ V
    using connector-image-subset[OF ptUV] by blast
  have conn-u-img: path-image (connector (p u)) ⊆ U ∩ V
```

```
    using connector-image-subset[OF puUV] by blast
  have segH: segment-loop p t u ∈ loop-space (U ∩ V) x0
    by (rule segment-loop-in-set[where S = U ∩ V])
        (use p-path p-imgW t01 u01 ptUV puUV segUV x0-in-UV conn-t-img
conn-u-img in auto)
  have h-in: loop-class (U ∩ V) x0 (segment-loop p t u) ∈ H
    by (rule loop-class-in-space[OF segH])
  have base:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (bridge-word b (loop-class (U ∩ V) x0 (segment-loop p t u)) rest)
      (bridge-word c (loop-class (U ∩ V) x0 (segment-loop p t u)) rest)
    by (rule bridge-word-switch[OF h-in bc])
  show ?thesis
    using base
   by (cases b; cases c) (simp-all add: i1-loop-class-eq[OF segH] i2-loop-class-eq[OF
segH])
qed

lemma svk-partition-split-head:
  assumes part: svk-partition p (t # v # us) (b # bs)
    and tu: t < u
    and uv: u < v
    and puUV: p u ∈ U ∩ V
  shows svk-partition p (t # u # v # us) (b # b # bs)
proof −
  have t01: t ∈ {0..1} and v01: v ∈ {0..1}
    and seg-tv:
      (if b then subpathin t v p ‘ {0..1} ⊆ U else subpathin t v p ‘ {0..1} ⊆ V)
    and tail: svk-partition p (v # us) bs
    using part by simp-all
  have ptUV: p t ∈ U ∩ V
    using part by simp
  have pvUV: p v ∈ U ∩ V
    using tail by (cases us; cases bs) simp-all
  have u01: u ∈ {0..1}
    using t01 v01 tu uv by auto
  have seg-tu:
    (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V)
  proof (cases b)
    case True
    have subpathin t u p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}
      by (rule subpathin-image-subset-left[OF t01 u01 v01]) (use tu uv in auto)
    then show ?thesis
      using seg-tv True by auto
  next
    case False
    have subpathin t u p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}
      by (rule subpathin-image-subset-left[OF t01 u01 v01]) (use tu uv in auto)
    then show ?thesis
```

```
    using seg-tv False by auto
  qed
  have seg-uv:
    (if b then subpathin u v p ‘ {0..1} ⊆ U else subpathin u v p ‘ {0..1} ⊆ V)
  proof (cases b)
    case True
    have subpathin u v p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}
      by (rule subpathin-image-subset-right[OF t01 u01 v01]) (use tu uv in auto)
    then show ?thesis
      using seg-tv True by auto
  next
    case False
    have subpathin u v p ‘ {0..1} ⊆ subpathin t v p ‘ {0..1}
      by (rule subpathin-image-subset-right[OF t01 u01 v01]) (use tu uv in auto)
    then show ?thesis
      using seg-tv False by auto
  qed
  show ?thesis
    using t01 ptUV u01 tu seg-tu puUV v01 pvUV uv seg-uv tail by simp
qed

lemma svk-partition-same-start-equiv:
  assumes p-loop: p ∈ loop-space W x0
    and part1: svk-partition p (t # ts) bs
    and part2: svk-partition p (t # us) cs
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word p (t # ts) bs)
      (partition-word p (t # us) cs)
  using assms
proof (induction length bs + length cs arbitrary: t ts bs us cs rule: less-induct)
  case less
  note p-loop = less.prems(1)
  note part1 = less.prems(2)
  note part2 = less.prems(3)

  from p-loop have p-path: path p and p-imgW: path-image p ⊆ W
    unfolding loop-space-def by auto

  show ?case
  proof (cases bs)
    case Nil
    have ts-nil: ts = []
      using part1 Nil by (cases ts) simp-all
    have t1: t = 1
      using svk-partition-last-eq-one[OF part1] ts-nil by simp
    show ?thesis
    proof (cases cs)
      case Nil
      then show ?thesis
```

```
      using ts-nil by simp
    next
      case (Cons c cs')
      obtain v us' where us: us = v # us'
        using part2 Cons by (cases us) auto
      have v01: v ∈ {0..1}
        using part2 Cons us by simp
      have t < v
        using part2 Cons us by simp
      then have False
        using t1 v01 by auto
      then show ?thesis
        by simp
    qed
  next
    case (Cons b bs')
    note bs-cons = Cons
    obtain u ts' where ts: ts = u # ts'
      using part1 Cons by (cases ts) auto
    have t01: t ∈ {0..1} and ptUV: p t ∈ U ∩ V
      and u01: u ∈ {0..1} and tu: t < u
      and seg-tu:
        (if b then subpathin t u p ` {0..1} ⊆ U else subpathin t u p ` {0..1} ⊆ V)
      and tail1: svk-partition p (u # ts') bs'
      using part1 Cons ts by simp-all
    have puUV: p u ∈ U ∩ V
      by (rule svk-partition-next-in-intersection[OF part1[unfolded ts Cons]])
    have first-in1:
      (if b then loop-class U x0 (segment-loop p t u) ∈ G1
       else loop-class V x0 (segment-loop p t u) ∈ G2)
    proof (cases b)
      case True
      have seg-loop: segment-loop p t u ∈ loop-space U x0
        by (rule segment-loop-in-U[OF p-path p-imgW t01 u01 ptUV puUV]) (use
seg-tu True in auto)
      show ?thesis
        using True by (auto intro: loop-class-in-space[OF seg-loop])
    next
      case False
      have seg-loop: segment-loop p t u ∈ loop-space V x0
        by (rule segment-loop-in-V[OF p-path p-imgW t01 u01 ptUV puUV]) (use
seg-tu False in auto)
      show ?thesis
        using False by (auto intro: loop-class-in-space[OF seg-loop])
    qed

    show ?thesis
    proof (cases cs)
      case Nil
```

```
  have us-nil: us = []
    using part2 Nil by (cases us) simp-all
  have t1: t = 1
    using svk-partition-last-eq-one[OF part2] us-nil by simp
  have contradiction: False
    using tu t1 u01 by auto
  from contradiction show ?thesis
    by simp
next
  case (Cons c cs')
  obtain v us' where us: us = v # us'
    using part2 Cons by (cases us) auto
  have v01: v ∈ {0..1} and tv: t < v
    and seg-tv:
      (if c then subpathin t v p ‘ {0..1} ⊆ U else subpathin t v p ‘ {0..1} ⊆ V)
    and tail2: svk-partition p (v # us') cs'
    using part2 Cons us by simp-all
  have pvUV: p v ∈ U ∩ V
    using tail2 by (cases us'; cases cs') simp-all
  have tail2-in:
    fpw-in-space G1 G2 (partition-word p (v # us') cs')
    by (rule svk-partition-partition-word-in-space[OF p-loop tail2])

  show ?thesis
  proof (cases u = v)
    case True
    note uv-eq = True
    have smaller: length bs' + length cs' < length (b # bs') + length (c # cs')
      by simp
    have tail-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (u # ts') bs')
        (partition-word p (u # us') cs')
    proof −
      have tail2': svk-partition p (u # us') cs'
        using tail2 True by simp
      have smaller': length bs' + length cs' < length (b # bs') + length cs
        using smaller Cons by simp
      have bs-eq: bs = b # bs'
        by (rule bs-cons)
      have smaller'': length bs' + length cs' < length bs + length cs
        using smaller' bs-eq by simp
      have ih:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (partition-word p (u # ts') bs')
          (partition-word p (u # us') cs')
        by (rule less.hyps[OF smaller'' p-loop tail1 tail2'])
      show ?thesis
        by (rule ih)
```

qed
have *pref-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word p* (*t # u # ts'*) (*b # bs'*))
(*if b then WordLeft* (*loop-class U x0* (*segment-loop p t u*))
(*partition-word p* (*u # us'*) *cs'*)
else WordRight (*loop-class V x0* (*segment-loop p t u*))
(*partition-word p* (*u # us'*) *cs'*))
proof (*cases b*)
case *True*
have *class-in*: *loop-class U x0* (*segment-loop p t u*) ∈ *G1*
using *first-in1 True* **by** *simp*
have *rel'*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*WordLeft* (*loop-class U x0* (*segment-loop p t u*))
(*partition-word p* (*u # ts'*) *bs'*))
(*WordLeft* (*loop-class U x0* (*segment-loop p t u*))
(*partition-word p* (*u # us'*) *cs'*))
by (*rule carrier-full-amalg-equiv-left-context*[*OF tail-rel class-in*])
then show *?thesis*
using *rel' ts True* **by** *simp*
next
case *False*
have *class-in*: *loop-class V x0* (*segment-loop p t u*) ∈ *G2*
using *first-in1 False* **by** *simp*
have *rel'*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*WordRight* (*loop-class V x0* (*segment-loop p t u*))
(*partition-word p* (*u # ts'*) *bs'*))
(*WordRight* (*loop-class V x0* (*segment-loop p t u*))
(*partition-word p* (*u # us'*) *cs'*))
by (*rule carrier-full-amalg-equiv-right-context*[*OF tail-rel class-in*])
then show *?thesis*
using *rel' ts False* **by** *simp*
qed
show *?thesis*
proof (*cases b = c*)
case *True*
note *bc-eq = True*
show *?thesis*
proof (*cases b*)
case *True*
then show *?thesis*
using *pref-rel bc-eq True ts us uv-eq bs-cons Cons* **by** *simp*
next
case *False*
then show *?thesis*
using *pref-rel bc-eq False ts us uv-eq bs-cons Cons* **by** *simp*
qed

```
      next
        case False
        have segUV: subpathin t u p ‘ {0..1} ⊆ U ∩ V
          using seg-tu seg-tv True False by (cases b; cases c) auto
        have tail2': svk-partition p (u # us') cs'
          using tail2 uv-eq by simp
        have rest-in:
          fpw-in-space G1 G2 (partition-word p (u # us') cs')
          by (rule svk-partition-partition-word-in-space[OF p-loop tail2'])
        have bc-ne: b ≠ c
          by (rule False)
        have switch-raw:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (if b then WordLeft (loop-class U x0 (segment-loop p t u))
              (partition-word p (u # us') cs')
             else WordRight (loop-class V x0 (segment-loop p t u))
              (partition-word p (u # us') cs'))
            (if c then WordLeft (loop-class U x0 (segment-loop p t u))
              (partition-word p (u # us') cs')
             else WordRight (loop-class V x0 (segment-loop p t u))
              (partition-word p (u # us') cs'))
          by (rule segment-word-switch[where rest = partition-word p (u # us')
cs',
                OF p-path p-imgW t01 u01 tu ptUV puUV segUV rest-in bc-ne])
        have switch:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (if b then WordLeft (loop-class U x0 (segment-loop p t u))
              (partition-word p (u # us') cs')
             else WordRight (loop-class V x0 (segment-loop p t u))
              (partition-word p (u # us') cs'))
            (partition-word p (t # u # us') (c # cs'))
          using switch-raw by (cases c) simp-all
        have step:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word p (t # u # ts') (b # bs'))
            (partition-word p (t # u # us') (c # cs'))
          by (rule carrier-full-amalg-equiv.trans[OF pref-rel switch])
        show ?thesis
          using step True ts us uv-eq bs-cons Cons by simp
      qed
    next
      case False
      show ?thesis
      proof (cases u < v)
        case True
        note uv-lt = True
        have part2-tv: svk-partition p (t # v # us') (c # cs')
          using part2 Cons us by simp
        have split2: svk-partition p (t # u # v # us') (c # c # cs')
```

```
      by (rule svk-partition-split-head[OF part2-tv tu True puUV])
    have split-tail2: svk-partition p (u # v # us') (c # cs')
      using split2 by simp
    have split-tail2-in:
      fpw-in-space G1 G2 (partition-word p (u # v # us') (c # cs'))
      by (rule svk-partition-partition-word-in-space[OF p-loop split-tail2])
    have tail2-in:
      fpw-in-space G1 G2 (partition-word p (v # us') cs')
      by (rule svk-partition-partition-word-in-space[OF p-loop tail2])
    have split2-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (t # u # v # us') (c # c # cs'))
        (partition-word p (t # v # us') (c # cs'))
    proof (cases c)
      case True
      have seg-tvU: subpathin t v p ` {0..1} ⊆ U
        using seg-tv True by auto
      have raw:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (WordLeft (loop-class U x0 (segment-loop p t u))
            (WordLeft (loop-class U x0 (segment-loop p u v))
              (partition-word p (v # us') cs')))
          (WordLeft (loop-class U x0 (segment-loop p t v))
            (partition-word p (v # us') cs'))
        by (rule segment-word-split-left-equiv[OF p-loop t01 u01 v01 tu uv-lt
ptUV puUV pvUV seg-tvU tail2-in])
      then show ?thesis
        using True by simp
    next
      case False
      have seg-tvV: subpathin t v p ` {0..1} ⊆ V
        using seg-tv False by auto
      have raw:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (WordRight (loop-class V x0 (segment-loop p t u))
            (WordRight (loop-class V x0 (segment-loop p u v))
              (partition-word p (v # us') cs')))
          (WordRight (loop-class V x0 (segment-loop p t v))
            (partition-word p (v # us') cs'))
        by (rule segment-word-split-right-equiv[OF p-loop t01 u01 v01 tu uv-lt
ptUV puUV pvUV seg-tvV tail2-in])
      then show ?thesis
        using False by simp
    qed
  have smaller: length bs' + length (c # cs') < length (b # bs') + length (c
# cs')
    by simp
  have tail-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
```

```
    (partition-word p (u # ts') bs')
    (partition-word p (u # v # us') (c # cs'))
  proof −
   have smaller': length bs' + length (c # cs') < length (b # bs') + length
cs
    using smaller Cons by simp
   have bs-eq: bs = b # bs'
    by (rule bs-cons)
   have smaller'': length bs' + length (c # cs') < length bs + length cs
    using smaller' bs-eq by simp
   have ih:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (partition-word p (u # ts') bs')
     (partition-word p (u # v # us') (c # cs'))
    by (rule less.hyps[OF smaller'' p-loop tail1 split-tail2])
   show ?thesis
    by (rule ih)
  qed
  have pref-rel:
   carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (partition-word p (t # u # ts') (b # bs'))
    (if b then WordLeft (loop-class U x0 (segment-loop p t u))
      (partition-word p (u # v # us') (c # cs'))
     else WordRight (loop-class V x0 (segment-loop p t u))
      (partition-word p (u # v # us') (c # cs')))
  proof (cases b)
   case True
   have class-in: loop-class U x0 (segment-loop p t u) ∈ G1
    using first-in1 True by simp
   have rel':
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (WordLeft (loop-class U x0 (segment-loop p t u))
      (partition-word p (u # ts') bs'))
     (WordLeft (loop-class U x0 (segment-loop p t u))
      (partition-word p (u # v # us') (c # cs')))
    by (rule carrier-full-amalg-equiv-left-context[OF tail-rel class-in])
   from rel' show ?thesis
    using True ts by simp
  next
   case False
   have class-in: loop-class V x0 (segment-loop p t u) ∈ G2
    using first-in1 False by simp
   have rel':
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (WordRight (loop-class V x0 (segment-loop p t u))
      (partition-word p (u # ts') bs'))
     (WordRight (loop-class V x0 (segment-loop p t u))
      (partition-word p (u # v # us') (c # cs')))
    by (rule carrier-full-amalg-equiv-right-context[OF tail-rel class-in])
```

```
    then show ?thesis
      using False ts by simp
  qed
  show ?thesis
  proof (cases b = c)
    case True
    have pref-to-split:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (if b then WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs'))
         else WordRight (loop-class V x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs')))
        (partition-word p (t # u # v # us') (c # c # cs'))
      using True by simp
    have step:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (t # u # ts') (b # bs'))
        (partition-word p (t # u # v # us') (c # c # cs'))
      by (rule carrier-full-amalg-equiv.trans[OF pref-rel pref-to-split])
    have final:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (t # u # ts') (b # bs'))
        (partition-word p (t # v # us') (c # cs'))
      by (rule carrier-full-amalg-equiv.trans[OF step split2-rel])
    show ?thesis
      using final ts us bs-cons Cons by simp
  next
    case False
    have segUV: subpathin t u p ` {0..1} ⊆ U ∩ V
      using seg-tu split2 False by (cases b; cases c) auto
    have switch-raw:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (if b then WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs'))
         else WordRight (loop-class V x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs')))
        (if c then WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs'))
         else WordRight (loop-class V x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs')))
      by (rule segment-word-switch[where rest = partition-word p (u # v #
us') (c # cs'),
                OF p-path p-imgW t01 u01 tu ptUV puUV segUV split-tail2-in
False])
    have switch:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (if b then WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word p (u # v # us') (c # cs'))
         else WordRight (loop-class V x0 (segment-loop p t u))
```

```
        (partition-word p (u # v # us') (c # cs')))
       (partition-word p (t # u # v # us') (c # c # cs'))
     using switch-raw by (cases c) simp-all
   have step:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (partition-word p (t # u # ts') (b # bs'))
       (partition-word p (t # u # v # us') (c # c # cs'))
     by (rule carrier-full-amalg-equiv.trans[OF pref-rel switch])
   have final:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (partition-word p (t # u # ts') (b # bs'))
       (partition-word p (t # v # us') (c # cs'))
     by (rule carrier-full-amalg-equiv.trans[OF step split2-rel])
   show ?thesis
     using final ts us bs-cons Cons by simp
 qed
next
 case False-lt: False
 have vu: v < u
   using False False-lt by linarith
 have part1-tu: svk-partition p (t # u # ts') (b # bs')
   using part1 bs-cons ts by simp
 have split1: svk-partition p (t # v # u # ts') (b # b # bs')
   by (rule svk-partition-split-head[OF part1-tu tv vu pvUV])
 have split-tail1: svk-partition p (v # u # ts') (b # bs')
   using split1 by simp
 have split1-rel:
   carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (partition-word p (t # v # u # ts') (b # b # bs'))
     (partition-word p (t # u # ts') (b # bs'))
 proof (cases b)
   case True
   have seg-tuU: subpathin t u p ` {0..1} ⊆ U
     using seg-tu True by simp
   have raw:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (WordLeft (loop-class U x0 (segment-loop p t v))
         (WordLeft (loop-class U x0 (segment-loop p v u))
           (partition-word p (u # ts') bs')))
       (WordLeft (loop-class U x0 (segment-loop p t u))
         (partition-word p (u # ts') bs'))
    by (rule segment-word-split-left-equiv[where rest = partition-word p (u
# ts') bs',
         OF p-loop t01 v01 u01 tv vu ptUV pvUV puUV seg-tuU
           svk-partition-partition-word-in-space[OF p-loop tail1]])
   then show ?thesis
     using True by simp
 next
   case False
```

```
      have seg-tuV: subpathin t u p ‘ {0..1} ⊆ V
        using seg-tu False by simp
      have raw:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (WordRight (loop-class V x0 (segment-loop p t v))
            (WordRight (loop-class V x0 (segment-loop p v u))
              (partition-word p (u # ts′) bs′)))
          (WordRight (loop-class V x0 (segment-loop p t u))
            (partition-word p (u # ts′) bs′))
        by (rule segment-word-split-right-equiv[where rest = partition-word p
(u # ts′) bs′,
              OF p-loop t01 v01 u01 tv vu ptUV pvUV puUV seg-tuV
                svk-partition-partition-word-in-space[OF p-loop tail1]])
      then show ?thesis
        using False by simp
    qed
    have smaller: length (b # bs′) + length cs′ < length (b # bs′) + length (c
# cs′)
      by simp
    have tail-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (v # u # ts′) (b # bs′))
        (partition-word p (v # us′) cs′)
    proof −
      have smaller′: length (b # bs′) + length cs′ < length bs + length (c #
cs′)
        using smaller bs-cons by simp
      have cs-eq: cs = c # cs′
        by (rule Cons)
      have smaller′′: length (b # bs′) + length cs′ < length bs + length cs
        using smaller′ cs-eq by simp
      have ih:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (partition-word p (v # u # ts′) (b # bs′))
          (partition-word p (v # us′) cs′)
        by (rule less.hyps[OF smaller′′ p-loop split-tail1 tail2])
      show ?thesis
        by (rule ih)
    qed
    have first-in2:
      (if c then loop-class U x0 (segment-loop p t v) ∈ G1
       else loop-class V x0 (segment-loop p t v) ∈ G2)
    proof (cases c)
      case True
      have seg-loop: segment-loop p t v ∈ loop-space U x0
        by (rule segment-loop-in-U[OF p-path p-imgW t01 v01 ptUV pvUV])
(use seg-tv True in auto)
      show ?thesis
        using True by (auto intro: loop-class-in-space[OF seg-loop])
```

```
      next
        case False
        have seg-loop: segment-loop p t v ∈ loop-space V x0
          by (rule segment-loop-in-V[OF p-path p-imgW t01 v01 ptUV pvUV])
(use seg-tv False in auto)
        show ?thesis
          using False by (auto intro: loop-class-in-space[OF seg-loop])
      qed
      show ?thesis
      proof (cases b = c)
        case True
        from True have bc: b = c .
        have pref-rel:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word p (t # v # u # ts') (b # b # bs'))
            (partition-word p (t # v # us') (c # cs'))
        proof (cases b)
          case True
          have class-in: loop-class U x0 (segment-loop p t v) ∈ G1
            using first-in2 bc True by simp
          from bc True have c-true: c
            by simp
          have tail-rel':
            carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
              (partition-word p (v # u # ts') (True # bs'))
              (partition-word p (v # us') cs')
            using tail-rel True c-true by simp
          have ctx-rel:
            carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
              (WordLeft (loop-class U x0 (segment-loop p t v))
                (partition-word p (v # u # ts') (True # bs')))
              (WordLeft (loop-class U x0 (segment-loop p t v))
                (partition-word p (v # us') cs'))
            by (rule carrier-full-amalg-equiv-left-context[OF tail-rel' class-in])
          show ?thesis
            using bc True ctx-rel by simp
        next
          case False
          have class-in: loop-class V x0 (segment-loop p t v) ∈ G2
            using first-in2 bc False by simp
          from bc False have c-false: ¬ c
            by simp
          have tail-rel':
            carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
              (partition-word p (v # u # ts') (False # bs'))
              (partition-word p (v # us') cs')
            using tail-rel False c-false by simp
          have ctx-rel:
            carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
```

```
      (WordRight (loop-class V x0 (segment-loop p t v))
        (partition-word p (v # u # ts') (False # bs')))
      (WordRight (loop-class V x0 (segment-loop p t v))
        (partition-word p (v # us') cs'))
    by (rule carrier-full-amalg-equiv-right-context[OF tail-rel' class-in])
  show ?thesis
    using bc False ctx-rel by simp
qed
have from-orig:
  carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (partition-word p (t # u # ts') (b # bs'))
    (partition-word p (t # v # u # ts') (b # b # bs'))
  by (rule carrier-full-amalg-equiv.sym[OF split1-rel])
have step:
  carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (partition-word p (t # u # ts') (b # bs'))
    (partition-word p (t # v # us') (c # cs'))
  by (rule carrier-full-amalg-equiv.trans[OF from-orig pref-rel])
show ?thesis
  using step us ts bs-cons Cons by simp
next
case False
have pref-rel:
  carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (partition-word p (t # v # u # ts') (b # b # bs'))
    (if b then WordLeft (loop-class U x0 (segment-loop p t v))
      (partition-word p (v # us') cs')
     else WordRight (loop-class V x0 (segment-loop p t v))
      (partition-word p (v # us') cs'))
proof −
  have first-in-split:
    (if b then loop-class U x0 (segment-loop p t v) ∈ G1
     else loop-class V x0 (segment-loop p t v) ∈ G2)
    using split1 by (cases b) (auto intro: loop-class-in-space
      segment-loop-in-U[OF p-path p-imgW t01 v01 ptUV pvUV]
      segment-loop-in-V[OF p-path p-imgW t01 v01 ptUV pvUV])
  show ?thesis
  proof (cases b)
    case True
    have class-in: loop-class U x0 (segment-loop p t v) ∈ G1
      using first-in-split True by simp
    have tail-rel':
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word p (v # u # ts') (True # bs'))
        (partition-word p (v # us') cs')
      using tail-rel True by simp
    have ctx-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (WordLeft (loop-class U x0 (segment-loop p t v))
```

```
        (partition-word p (v # u # ts') (True # bs')))
      (WordLeft (loop-class U x0 (segment-loop p t v))
        (partition-word p (v # us') cs'))
    by (rule carrier-full-amalg-equiv-left-context[OF tail-rel' class-in])
  show ?thesis
    using True ctx-rel by simp
next
  case False
  have class-in: loop-class V x0 (segment-loop p t v) ∈ G2
    using first-in-split False by simp
  have tail-rel':
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word p (v # u # ts') (False # bs'))
      (partition-word p (v # us') cs')
    using tail-rel False by simp
  have ctx-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (WordRight (loop-class V x0 (segment-loop p t v))
        (partition-word p (v # u # ts') (False # bs')))
      (WordRight (loop-class V x0 (segment-loop p t v))
        (partition-word p (v # us') cs'))
    by (rule carrier-full-amalg-equiv-right-context[OF tail-rel' class-in])
  show ?thesis
    using False ctx-rel by simp
qed
qed
have segUV: subpathin t v p ' {0..1} ⊆ U ∩ V
  using split1 seg-tv False by (cases b; cases c) auto
have switch-raw:
  carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (if b then WordLeft (loop-class U x0 (segment-loop p t v))
      (partition-word p (v # us') cs')
     else WordRight (loop-class V x0 (segment-loop p t v))
      (partition-word p (v # us') cs'))
    (if c then WordLeft (loop-class U x0 (segment-loop p t v))
      (partition-word p (v # us') cs')
     else WordRight (loop-class V x0 (segment-loop p t v))
      (partition-word p (v # us') cs'))
  by (rule segment-word-switch[where rest = partition-word p (v # us')
cs',
        OF p-path p-imgW t01 v01 tv ptUV pvUV segUV tail2-in False])
have switch:
  carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
    (if b then WordLeft (loop-class U x0 (segment-loop p t v))
      (partition-word p (v # us') cs')
     else WordRight (loop-class V x0 (segment-loop p t v))
      (partition-word p (v # us') cs'))
    (partition-word p (t # v # us') (c # cs'))
  using switch-raw by (cases c) simp-all
```

```
        have from-orig:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word p (t # u # ts') (b # bs'))
            (partition-word p (t # v # u # ts') (b # b # bs'))
          by (rule carrier-full-amalg-equiv.sym[OF split1-rel])
        have step1:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word p (t # u # ts') (b # bs'))
            (if b then WordLeft (loop-class U x0 (segment-loop p t v))
              (partition-word p (v # us') cs')
             else WordRight (loop-class V x0 (segment-loop p t v))
              (partition-word p (v # us') cs'))
          by (rule carrier-full-amalg-equiv.trans[OF from-orig pref-rel])
        have final:
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word p (t # u # ts') (b # bs'))
            (partition-word p (t # v # us') (c # cs'))
          by (rule carrier-full-amalg-equiv.trans[OF step1 switch])
        show ?thesis
          using final ts us bs-cons Cons by simp
      qed
    qed
  qed
 qed
qed

lemma valid-partition-same-loop-partition-word-equiv:
 assumes p-loop: p ∈ loop-space W x0
   and part1: valid-partition p ts bs
   and part2: valid-partition p us cs
 shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
     (partition-word p ts bs) (partition-word p us cs)
proof −
 obtain t ts' where ts: ts = t # ts'
   using valid-partition-hd(1)[OF part1] by (cases ts) auto
 obtain u us' where us: us = u # us'
   using valid-partition-hd(1)[OF part2] by (cases us) auto
 have valid-ts: valid-partition p (t # ts') bs
   using part1 unfolding ts by simp
 have t0: t = 0
   by (rule valid-partition-cases(1)[OF valid-ts])
 have part1': svk-partition p (t # ts') bs
   by (rule valid-partition-cases(2)[OF valid-ts])
 have valid-us: valid-partition p (u # us') cs
   using part2 unfolding us by simp
 have u0: u = 0
   by (rule valid-partition-cases(1)[OF valid-us])
 have part2': svk-partition p (u # us') cs
```

```
    by (rule valid-partition-cases(2)[OF valid-us])
  have rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word p (t # ts') bs)
      (partition-word p (u # us') cs)
  proof −
    have part1-0: svk-partition p (0 # ts') bs
      using part1' t0 by simp
    have part2-0: svk-partition p (0 # us') cs
      using part2' u0 by simp
    show ?thesis
      by (subst t0, subst u0, rule svk-partition-same-start-equiv[OF p-loop part1-0
part2-0])
  qed
then show ?thesis
    using ts us t0 u0 by simp
qed

lemma strip-neighbourhood:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
    and a01: a ∈ {0..1}
    and b01: b ∈ {0..1}
    and y0-01: y0 ∈ {0..1}
    and ab: a ≤ b
    and rowS: h ‘ ({a..b} × {y0}) ⊆ S
    and openS: open S
  shows ∃N. openin (top-of-set {0..1}) N ∧ y0 ∈ N ∧ h ‘ ({a..b} × N) ⊆ S
proof −
  have h-on: continuous-on ({0..1} × {0..1}) h
    using h-cont by simp
  have strip-subset: {a..b} × {0..1} ⊆ {0..1} × {0..1}
    using a01 b01 ab by auto
  have strip-cont: continuous-on ({a..b} × {0..1}) h
    by (rule continuous-on-subset[OF h-on strip-subset])
  have strip-open:
    openin (top-of-set ({a..b} × {0..1}))
      (({a..b} × {0..1}) ∩ h −‘ S)
    by (rule continuous-openin-preimage-gen[OF strip-cont openS])
  have line-in:
    (λx. (x, y0)) ‘ {a..b} ⊆ (({a..b} × {0..1}) ∩ h −‘ S)
    using rowS y0-01 by auto
  obtain N where N-open: openin (top-of-set {0..1}) N
    and y0N: y0 ∈ N
    and stripN: {a..b} × N ⊆ (({a..b} × {0..1}) ∩ h −‘ S)
  proof −
    have strip-open-prod:
      openin (prod-topology (top-of-set {a..b}) (top-of-set {0..1}))
```

```
    (({a..b} × {0..1}) ∩ h -` S)
  using strip-open by simp
have line-in-prod:
  {a..b} × {y0} ⊆ (({a..b} × {0..1}) ∩ h -` S)
  using line-in by auto
have compact-ab: compactin (top-of-set {a..b}) {a..b}
  using compact-Icc by simp
obtain M N where M-open: openin (top-of-set {a..b}) M
  and N-open: openin (top-of-set {0..1}) N
  and M-cover: {a..b} ⊆ M
  and y0N: y0 ∈ N
  and MN-strip: M × N ⊆ (({a..b} × {0..1}) ∩ h -` S)
proof -
  have local-boxes:
    ∃M N. openin (top-of-set {a..b}) M ∧ openin (top-of-set {0..1}) N ∧
      x ∈ M ∧ y0 ∈ N ∧ M × N ⊆ (({a..b} × {0..1}) ∩ h -` S)
    if x-in: x ∈ {a..b} for x
  proof -
    have xy-pair: (x, y0) ∈ {a..b} × {y0}
      using x-in by auto
    have xy-in: (x, y0) ∈ (({a..b} × {0..1}) ∩ h -` S)
      by (rule subsetD[OF line-in-prod xy-pair])
    show ?thesis
      using strip-open-prod xy-in by (metis openin-prod-topology-alt)
  qed
  then obtain U V where UV:
    ⋀x. x ∈ {a..b} ⟹
      openin (top-of-set {a..b}) (U x) ∧
      openin (top-of-set {0..1}) (V x) ∧
      x ∈ U x ∧ y0 ∈ V x ∧
      U x × V x ⊆ (({a..b} × {0..1}) ∩ h -` S)
    by metis
  then obtain D where D: finite D D ⊆ {a..b} {a..b} ⊆ ⋃ (U ` D)
    using compactinD[OF compact-ab, of U ` {a..b}]
    by (smt (verit) UN-I finite-subset-image imageE subsetI)
  show ?thesis
  proof (intro that[of ⋃ (U ` D) ⋂ (V ` D)])
    show openin (top-of-set {a..b}) (⋃ (U ` D))
      using D UV by blast
    show openin (top-of-set {0..1}) (⋂ (V ` D))
    proof (rule openin-Inter)
      show finite (V ` D)
        using D by simp
      show V ` D ≠ {}
      proof
        assume V ` D = {}
        then have D = {}
          by auto
        with D(3) ab show False
```

```
        by simp
      qed
      show openin (top-of-set {0..1}) T if T ∈ V ‘ D for T
      proof −
        from that obtain x where xD: x ∈ D and T-def: T = V x
          by auto
        show ?thesis
          using D(2) UV xD T-def by blast
      qed
    qed
    show {a..b} ⊆ ⋃ (U ‘ D)
      using D by blast
    show y0 ∈ ⋂ (V ‘ D)
      using D UV by force
    show (⋃ (U ‘ D)) × (⋂ (V ‘ D)) ⊆ (({a..b} × {0..1}) ∩ h −‘ S)
    proof
      fix xy
      assume xy-in: xy ∈ (⋃ (U ‘ D)) × (⋂ (V ‘ D))
      then obtain x y where xy-eq: xy = (x, y)
        and x-in: x ∈ ⋃ (U ‘ D)
        and y-in: y ∈ ⋂ (V ‘ D)
        by blast
      obtain d where dD: d ∈ D and xUd: x ∈ U d
        using x-in by auto
      have yVd: y ∈ V d
        using y-in dD by auto
      have UdVd-strip: U d × V d ⊆ (({a..b} × {0..1}) ∩ h −‘ S)
        using D(2) UV dD by blast
      have (x, y) ∈ U d × V d
        using xUd yVd by auto
      then have (x, y) ∈ (({a..b} × {0..1}) ∩ h −‘ S)
        using UdVd-strip by blast
      then show xy ∈ (({a..b} × {0..1}) ∩ h −‘ S)
        using xy-eq by simp
    qed
  qed
  have stripN0: {a..b} × N ⊆ (({a..b} × {0..1}) ∩ h −‘ S)
    using M-cover MN-strip by blast
  show thesis
    by (rule that[of N], rule N-open, rule y0N, use stripN0 in auto)
qed
show ?thesis
proof (intro exI conjI)
  show openin (top-of-set {0..1}) N
    by (rule N-open)
  show y0 ∈ N
    by (rule y0N)
  show h ‘ ({a..b} × N) ⊆ S
```

```
    using stripN by auto
  qed
qed

lemma svk-partition-local-neighbourhood:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
    and y0-01: y0 ∈ {0..1}
    and part: svk-partition (λx. h (x, y0)) ts bs
  shows ∃N. openin (top-of-set {0..1}) N ∧ y0 ∈ N ∧
    (∀x∈set ts. h ‘ ({x} × N) ⊆ U ∩ V) ∧
    (∀y z. y ≤ z ⟶ {y..z} ⊆ N ⟶ rectangle-partition h y z ts bs)
  using part
proof (induction ts arbitrary: bs)
  case Nil
  then show ?case
    by simp
next
  case (Cons t ts)
  show ?case
  proof (cases ts)
    case Nil
    have bs-nil: bs = []
      using Cons.prems Nil by (cases bs) simp-all
    have t1: t = 1
      using Cons.prems Nil bs-nil by simp
    have t01: t ∈ {0..1}
      using t1 by simp
    have tt: t ≤ t
      by simp
    have point-row: h ‘ ({t..t} × {y0}) ⊆ U ∩ V
      using Cons.prems Nil bs-nil y0-01 by auto
    have UV-open: open (U ∩ V)
      using U-open V-open by auto
    have point-neigh:
      ∃N. openin (top-of-set {0..1}) N ∧ y0 ∈ N ∧ h ‘ ({t..t} × N) ⊆ U ∩ V
      by (rule strip-neighbourhood[where a = t and b = t and S = U ∩ V, OF
h-cont t01 t01 y0-01 tt point-row UV-open])
    then obtain N where N-open: openin (top-of-set {0..1}) N
      and y0N: y0 ∈ N
      and pointN: h ‘ ({t..t} × N) ⊆ U ∩ V
      by blast
    show ?thesis
    proof (intro exI conjI ballI allI impI)
      show openin (top-of-set {0..1}) N
        by (rule N-open)
      show y0 ∈ N
        by (rule y0N)
```

```
    show h ‘ ({x} × N) ⊆ U ∩ V if x ∈ set (t # ts) for x
      using that Nil pointN by auto
    show rectangle-partition h y z (t # ts) bs if y ≤ z {y..z} ⊆ N for y z
      using t1 Nil bs-nil by simp
  qed
next
  case (Cons u us)
  obtain b bs′ where bs: bs = b # bs′
    using Cons.prems Cons by (cases bs) auto
  have t01: t ∈ {0..1} and u01: u ∈ {0..1} and tu: t < u
    and seg-side:
      (if b
        then subpathin t u (λx. h (x, y0)) ‘ {0..1} ⊆ U
        else subpathin t u (λx. h (x, y0)) ‘ {0..1} ⊆ V)
    and tail: svk-partition (λx. h (x, y0)) (u # us) bs′
    and ptUV: h (t, y0) ∈ U ∩ V
    using Cons.prems Cons bs by simp-all
  have seg-row: h ‘ ({t..u} × {y0}) ⊆ (if b then U else V)
  proof −
    have seg-image: (λx. h (x, y0)) ‘ {t..u} ⊆ (if b then U else V)
      using seg-side tu by (cases b) (auto simp: subpathin-image-eq)
    show ?thesis
    proof
      fix z
      assume z-in: z ∈ h ‘ ({t..u} × {y0})
      then obtain x where x-in: x ∈ {t..u} and z-eq: z = h (x, y0)
        by auto
      have h (x, y0) ∈ (if b then U else V)
      proof (cases b)
        case True
        then show ?thesis
          using seg-image x-in by auto
      next
        case False
        then show ?thesis
          using seg-image x-in by auto
      qed
      then show z ∈ (if b then U else V)
        using z-eq by simp
    qed
  qed
  have point-row: h ‘ ({t..t} × {y0}) ⊆ U ∩ V
    using ptUV by auto
  obtain Nseg where Nseg-open: openin (top-of-set {0..1}) Nseg
    and y0Nseg: y0 ∈ Nseg
    and segN: h ‘ ({t..u} × Nseg) ⊆ (if b then U else V)
  proof (cases b)
    case True
    have seg-rowU: h ‘ ({t..u} × {y0}) ⊆ U
```

```
      using seg-row True by simp
    obtain Nseg where Nseg-open0: openin (top-of-set {0..1}) Nseg
      and y0Nseg0: y0 ∈ Nseg
      and segN0: h ‘ ({t..u} × Nseg) ⊆ U
        using strip-neighbourhood[OF h-cont t01 u01 y0-01 less-imp-le[OF tu]
seg-rowU U-open] by blast
    show thesis
    proof (rule that[of Nseg])
      show openin (top-of-set {0..1}) Nseg
        by (rule Nseg-open0)
      show y0 ∈ Nseg
        by (rule y0Nseg0)
      show h ‘ ({t..u} × Nseg) ⊆ (if b then U else V)
        using True segN0 by simp
    qed
  next
    case False
    have seg-rowV: h ‘ ({t..u} × {y0}) ⊆ V
      using seg-row False by simp
    obtain Nseg where Nseg-open0: openin (top-of-set {0..1}) Nseg
      and y0Nseg0: y0 ∈ Nseg
      and segN0: h ‘ ({t..u} × Nseg) ⊆ V
        using strip-neighbourhood[OF h-cont t01 u01 y0-01 less-imp-le[OF tu]
seg-rowV V-open] by blast
    show thesis
    proof (rule that[of Nseg])
      show openin (top-of-set {0..1}) Nseg
        by (rule Nseg-open0)
      show y0 ∈ Nseg
        by (rule y0Nseg0)
      show h ‘ ({t..u} × Nseg) ⊆ (if b then U else V)
        using False segN0 by simp
    qed
  qed
  obtain Nt where Nt-open: openin (top-of-set {0..1}) Nt
    and y0Nt: y0 ∈ Nt
    and pointN: h ‘ ({t..t} × Nt) ⊆ U ∩ V
   using strip-neighbourhood[OF h-cont t01 t01 y0-01 order-refl point-row] U-open
V-open by blast
  obtain Ntail where Ntail-open: openin (top-of-set {0..1}) Ntail
    and y0Ntail: y0 ∈ Ntail
    and tail-points-raw: ∀ x∈set ts. h ‘ ({x} × Ntail) ⊆ U ∩ V
    and tail-rect-raw:
      ∀ y z. y ≤ z ⟶ {y..z} ⊆ Ntail ⟶ rectangle-partition h y z ts bs′
  proof −
    have tail-ts: svk-partition (λx. h (x, y0)) ts bs′
      using tail Cons by simp
    obtain N where N-open: openin (top-of-set {0..1}) N
      and y0N: y0 ∈ N
```

```
    and pointsN: ∀ x∈set ts. h ‘ ({x} × N) ⊆ U ∩ V
    and rectN:
      ∀ y z. y ≤ z ⟶ {y..z} ⊆ N ⟶ rectangle-partition h y z ts bs'
    using Cons.IH[OF tail-ts] by blast
  show thesis
  proof (rule that[of N])
    show openin (top-of-set {0..1}) N
      by (rule N-open)
    show y0 ∈ N
      by (rule y0N)
    show ∀ x∈set ts. h ‘ ({x} × N) ⊆ U ∩ V
      by (rule pointsN)
    show ∀ y z. y ≤ z ⟶ {y..z} ⊆ N ⟶ rectangle-partition h y z ts bs'
      by (rule rectN)
  qed
qed
have tail-points: ∀ x∈set (u # us). h ‘ ({x} × Ntail) ⊆ U ∩ V
  using tail-points-raw Cons by simp
have tail-rect:
  ∀ y z. y ≤ z ⟶ {y..z} ⊆ Ntail ⟶ rectangle-partition h y z (u # us) bs'
  using tail-rect-raw Cons by simp
let ?N = Nseg ∩ Nt ∩ Ntail
have N-open: openin (top-of-set {0..1}) ?N
  by (intro openin-Int Nseg-open Nt-open Ntail-open)
have y0N: y0 ∈ ?N
  using y0Nseg y0Nt y0Ntail by auto
have points:
  h ‘ ({x} × ?N) ⊆ U ∩ V if x-in: x ∈ set (t # u # us) for x
proof (cases x = t)
  case True
  have point-t: h ‘ ({t..t} × Nt) ⊆ U ∩ V
    by (rule pointN)
  have subset-t: {x} × ?N ⊆ {t..t} × Nt
    using True by auto
  have img-subset-t: h ‘ ({x} × ?N) ⊆ h ‘ ({t..t} × Nt)
    by (rule image-mono[OF subset-t])
  show ?thesis
    using img-subset-t point-t by blast
next
  case False
  then have x ∈ set (u # us)
    using x-in by auto
  then have point-x: h ‘ ({x} × Ntail) ⊆ U ∩ V
    using tail-points by blast
  have subset-x: {x} × ?N ⊆ {x} × Ntail
    by auto
  have img-subset-x: h ‘ ({x} × ?N) ⊆ h ‘ ({x} × Ntail)
    by (rule image-mono[OF subset-x])
  show ?thesis
```

```
      using img-subset-x point-x by blast
    qed
    have rect:
      rectangle-partition h y z (t # u # us) (b # bs')
      if yz: y ≤ z and yzN: {y..z} ⊆ ?N for y z
    proof −
      have yzNseg: {y..z} ⊆ Nseg
        using yzN by auto
      have seg-rect: h ‘ ({t..u} × {y..z}) ⊆ (if b then U else V)
      proof (cases b)
        case True
        have prod-subset: {t..u} × {y..z} ⊆ {t..u} × Nseg
          using yzNseg by auto
        have h ‘ ({t..u} × {y..z}) ⊆ h ‘ ({t..u} × Nseg)
          by (rule image-mono[OF prod-subset])
        also have ... ⊆ U
          using segN True by simp
        finally show ?thesis
          using True by simp
      next
        case False
        have prod-subset: {t..u} × {y..z} ⊆ {t..u} × Nseg
          using yzNseg by auto
        have h ‘ ({t..u} × {y..z}) ⊆ h ‘ ({t..u} × Nseg)
          by (rule image-mono[OF prod-subset])
        also have ... ⊆ V
          using segN False by simp
        finally show ?thesis
          using False by simp
      qed
      have tail-rect': rectangle-partition h y z (u # us) bs'
        by (rule tail-rect[rule-format, OF yz]) (use yzN in auto)
      show ?thesis
        using t01 u01 tu seg-rect tail-rect' by (cases b) simp-all
    qed
    show ?thesis
    proof (intro exI conjI ballI allI impI)
      show openin (top-of-set {0..1}) ?N
        by (rule N-open)
      show y0 ∈ ?N
        by (rule y0N)
      show h ‘ ({x} × ?N) ⊆ U ∩ V if x ∈ set (t # ts) for x
        using that Cons points by simp
      show rectangle-partition h y z (t # ts) bs if y ≤ z {y..z} ⊆ ?N for y z
        using rect[OF that] Cons bs by simp
    qed
  qed
qed
```

lemma *rectangle-partition-svk-partition-row*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *part*: *rectangle-partition h c d ts bs*
and *edgeUV*: $\bigwedge x.\ x \in set\ ts \Longrightarrow h\ `\ (\{x\} \times \{c..d\}) \subseteq U \cap V$
and *y-in*: $y \in \{c..d\}$
shows *svk-partition* $(\lambda x.\ h\ (x,\ y))$ *ts bs*
using *part edgeUV*
proof (*induction ts arbitrary*: *bs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons t ts*)
show *?case*
proof (*cases ts*)
case *Nil*
have *bs-nil*: $bs = []$
using *Cons.prems Nil* **by** (*cases bs*) *simp-all*
have *t1*: $t = 1$
using *Cons.prems Nil bs-nil* **by** *simp*
have *ptUV*: $h\ (t,\ y) \in U \cap V$
proof −
have *t-edge*: $h\ `\ (\{t\} \times \{c..d\}) \subseteq U \cap V$
using *Cons.prems*(*2*)[*of t*] *Nil* **by** *simp*
have $(t,\ y) \in \{t\} \times \{c..d\}$
using *y-in* **by** *auto*
then show *?thesis*
using *t-edge* **by** *auto*
qed
then show *?thesis*
using *Nil bs-nil t1* **by** *simp*
next
case (*Cons u us*)
obtain *b bs′* **where** *bs*: $bs = b\ \#\ bs'$
using *Cons.prems Cons* **by** (*cases bs*) *auto*
have *t01*: $t \in \{0..1\}$ **and** *u01*: $u \in \{0..1\}$ **and** *tu*: $t < u$
and *rect-side*: (*if b then* $h\ `\ (\{t..u\} \times \{c..d\}) \subseteq U$ *else* $h\ `\ (\{t..u\} \times \{c..d\}) \subseteq V$)
and *tail*: *rectangle-partition h c d* $(u\ \#\ us)$ *bs′*
using *Cons.prems Cons bs* **by** *simp-all*
have *edge-t*: $h\ `\ (\{t\} \times \{c..d\}) \subseteq U \cap V$
using *Cons.prems*(*2*)[*of t*] *Cons* **by** *simp*
have *ptUV*: $h\ (t,\ y) \in U \cap V$
proof −
have $(t,\ y) \in \{t\} \times \{c..d\}$
using *y-in* **by** *auto*
then show *?thesis*
using *edge-t* **by** *auto*
qed

```
  have seg-side:
    (if b
      then subpathin t u (λx. h (x, y)) ‘ {0..1} ⊆ U
      else subpathin t u (λx. h (x, y)) ‘ {0..1} ⊆ V)
  proof −
    have row-subset: {t..u} × {y} ⊆ {t..u} × {c..d}
      using y-in by auto
    have row-rect: h ‘ ({t..u} × {y}) ⊆ (if b then U else V)
    proof (cases b)
      case True
      have h ‘ ({t..u} × {y}) ⊆ h ‘ ({t..u} × {c..d})
        by (rule image-mono[OF row-subset])
      also have ... ⊆ U
        using rect-side True by simp
      finally show ?thesis
        using True by simp
    next
      case False
      have h ‘ ({t..u} × {y}) ⊆ h ‘ ({t..u} × {c..d})
        by (rule image-mono[OF row-subset])
      also have ... ⊆ V
        using rect-side False by simp
      finally show ?thesis
        using False by simp
    qed
    then show ?thesis
      using tu by (cases b) (auto simp: subpathin-image-eq)
  qed
  have tail-svk-ts: svk-partition (λx. h (x, y)) ts bs′
  proof (rule Cons.IH[where bs = bs′])
    show rectangle-partition h c d ts bs′
      using tail Cons by simp
    show ⋀x. x ∈ set ts ⟹ h ‘ ({x} × {c..d}) ⊆ U ∩ V
    proof −
      fix x
      assume x-in: x ∈ set ts
      have x ∈ set (t # ts)
        using x-in by simp
      then show h ‘ ({x} × {c..d}) ⊆ U ∩ V
        using Cons.prems(2) by blast
    qed
  qed
  have tail-svk: svk-partition (λx. h (x, y)) (u # us) bs′
    using tail-svk-ts Cons by simp
  have step-svk: svk-partition (λx. h (x, y)) (t # u # us) (b # bs′)
    using t01 u01 tu seg-side ptUV tail-svk by simp
  show ?thesis
    using step-svk Cons bs by simp
qed
```

qed

lemma *rectangle-partition-valid-partition-row*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *part*: *rectangle-partition h c d ts bs*
and *edgeUV*: $\bigwedge x.\ x \in set\ ts \Longrightarrow h\ `\ (\{x\} \times \{c..d\}) \subseteq U \cap V$
and *y-in*: $y \in \{c..d\}$
and *ts-ne*: $ts \neq []$
and *hd0*: $hd\ ts = 0$
shows *valid-partition* $(\lambda x.\ h\ (x,\ y))$ *ts bs*
unfolding *valid-partition-def*
using *rectangle-partition-svk-partition-row*[*OF part edgeUV y-in*] *ts-ne hd0* **by** *simp*

lemma *homotopy-row-in-loop-space*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *h-cont*: *continuous-map* (*top-of-set* $(\{0..1\} \times \{0..1\})$) (*top-of-set W*) *h*
and *y01*: $y \in \{0..1\}$
and *start*: $h\ (0,\ y) = x0$
and *finish*: $h\ (1,\ y) = x0$
shows $(\lambda x.\ h\ (x,\ y)) \in$ *loop-space W x0*
proof −
have *h-on*: *continuous-on* $(\{0..1\} \times \{0..1\})$ *h*
and *h-into*: $h \in (\{0..1\} \times \{0..1\}) \to W$
using *h-cont* **by** *simp-all*
have *row-cont*: *continuous-on* $\{0..1\}$ $(\lambda x.\ h\ (x,\ y))$
proof (*rule continuous-on-compose2*[*OF h-on*])
show *continuous-on* $\{0..1\}$ $(\lambda x.\ (x,\ y))$
by (*intro continuous-intros*)
show $(\lambda x.\ (x,\ y))\ `\ \{0..1\} \subseteq \{0..1\} \times \{0..1\}$
using *y01* **by** *auto*
qed
have *row-path*: *path* $(\lambda x.\ h\ (x,\ y))$
using *row-cont* **by** (*simp add*: *path-def*)
have *row-img*: *path-image* $(\lambda x.\ h\ (x,\ y)) \subseteq W$
using *h-into y01* **by** (*auto simp*: *path-image-def*)
show *?thesis*
unfolding *loop-space-def pathstart-def pathfinish-def*
using *row-path row-img start finish* **by** *simp*
qed

lemma *bridge-word-one-equiv*:
assumes *rest-in*: *fpw-in-space G1 G2 rest*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*bridge-word b* (*fundamental-group-one* $(U \cap V)$ *x0*) *rest*) *rest*
proof (*cases b*)
case *True*
have *i1-one*: *i1* (*fundamental-group-one* $(U \cap V)$ *x0*) = *one1*

```
    by (rule fundamental-group-map-one[OF x0-in-UV]) auto
  have one1-in: one1 ∈ G1
    by (rule fundamental-group-one-in-space[OF x0-in-U])
  have red:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordLeft one1 rest) rest
    by (rule carrier-fpw-reduction.step,
        rule carrier-fpw-reduction-step.remove-left-one[OF one1-in], rule rest-in)
  show ?thesis
    using True i1-one by (simp add: carrier-full-amalg-equiv.of-reduction[OF red])
next
  case False
  have i2-one: i2 (fundamental-group-one (U ∩ V) x0) = one2
    by (rule fundamental-group-map-one[OF x0-in-UV]) auto
  have one2-in: one2 ∈ G2
    by (rule fundamental-group-one-in-space[OF x0-in-V])
  have red:
    carrier-fpw-reduction G1 G2 mult1 one1 mult2 one2
      (WordRight one2 rest) rest
    by (rule carrier-fpw-reduction.step,
        rule carrier-fpw-reduction-step.remove-right-one[OF one2-in], rule rest-in)
  show ?thesis
    using False i2-one by (simp add: carrier-full-amalg-equiv.of-reduction[OF red])
qed

lemma partition-word-with-tail-respects:
  assumes p-loop: p ∈ loop-space W x0
    and part: svk-partition p ts bs
    and rel: carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 r s
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail p ts bs r)
      (partition-word-with-tail p ts bs s)
  using part rel
proof (induction ts arbitrary: bs r s)
  case Nil
  then show ?case
    by simp
next
  case (Cons t ts)
  from p-loop have p-path: path p and p-img: path-image p ⊆ W
    unfolding loop-space-def by auto
  show ?case
  proof (cases ts)
    case Nil
    have bs-nil: bs = []
      using Cons.prems Nil by (cases bs) simp-all
    have pw-r: partition-word-with-tail p (t # ts) bs r = r
      using Nil bs-nil by simp
    have pw-s: partition-word-with-tail p (t # ts) bs s = s
```

```
    using Nil bs-nil by simp
  show ?thesis
    unfolding pw-r pw-s
    by (rule Cons.prems(2))
next
  case (Cons u us)
  obtain b bs' where bs: bs = b # bs'
    using Cons.prems(1) Cons by (cases bs) auto
  have tail: svk-partition p (u # us) bs'
    using Cons.prems(1) Cons bs by simp
  have ts-eq: ts = u # us
    using Cons by simp
  have tail-rel-ts:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail p ts bs' r)
      (partition-word-with-tail p ts bs' s)
    by (rule Cons.IH) (use tail Cons.prems(2) ts-eq in simp-all)
  have tail-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail p (u # us) bs' r)
      (partition-word-with-tail p (u # us) bs' s)
    using tail-rel-ts ts-eq by simp
  have i1-back: fundamental-group-map (U ∩ V) x0 U x0 id = i1
    by simp
  have i2-back: fundamental-group-map (U ∩ V) x0 V x0 id = i2
    by simp
  have t01: t ∈ {0..1} and ptUV: p t ∈ U ∩ V
    using Cons.prems(1) Cons bs by simp-all
  have u01: u ∈ {0..1} and seg-side:
    (if b then subpathin t u p ‘ {0..1} ⊆ U else subpathin t u p ‘ {0..1} ⊆ V)
    using Cons.prems(1) Cons bs by simp-all
  have puUV: p u ∈ U ∩ V
    using Cons.prems(1) Cons bs svk-partition-next-in-intersection[of p t u us b
bs'] by simp
  show ?thesis
  proof (cases b)
    case True
    have segU: segment-loop p t u ∈ loop-space U x0
      by (rule segment-loop-in-U[OF p-path p-img t01 u01 ptUV puUV]) (use
seg-side True in auto)
    have class-in: loop-class U x0 (segment-loop p t u) ∈ G1
      by (rule loop-class-in-space[OF segU])
    have ctx-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word-with-tail p (u # us) bs' r))
        (WordLeft (loop-class U x0 (segment-loop p t u))
          (partition-word-with-tail p (u # us) bs' s))
      by (rule carrier-full-amalg-equiv-left-context[OF tail-rel class-in])
```

```
    show ?thesis
      using True ts-eq ctx-rel
      by (subst i1-back, subst i2-back, simp add: bs)
  next
    case False
    have segV: segment-loop p t u ∈ loop-space V x0
      by (rule segment-loop-in-V[OF p-path p-img t01 u01 ptUV puUV]) (use
seg-side False in auto)
    have class-in: loop-class V x0 (segment-loop p t u) ∈ G2
      by (rule loop-class-in-space[OF segV])
    have ctx-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (WordRight (loop-class V x0 (segment-loop p t u))
          (partition-word-with-tail p (u # us) bs' r))
        (WordRight (loop-class V x0 (segment-loop p t u))
          (partition-word-with-tail p (u # us) bs' s))
      by (rule carrier-full-amalg-equiv-right-context[OF tail-rel class-in])
    show ?thesis
      using False ts-eq ctx-rel
      by (subst i1-back, subst i2-back, simp add: bs)
  qed
 qed
qed

lemma valid-partition-tail-bridge-one-equiv:
  assumes p-loop: p ∈ loop-space W x0
    and part: valid-partition p (t # u # ts) (b # bs)
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail p (t # u # ts) (b # bs)
       (bridge-word (last (b # bs)) (fundamental-group-one (U ∩ V) x0) WordNil))
      (partition-word p (t # u # ts) (b # bs))
proof −
  have svk: svk-partition p (t # u # ts) (b # bs)
    using part by (rule valid-partition-cases(2))
  have tail-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word-with-tail p (t # u # ts) (b # bs)
       (bridge-word (last (b # bs)) (fundamental-group-one (U ∩ V) x0) WordNil))
      (partition-word-with-tail p (t # u # ts) (b # bs) WordNil)
    by (rule partition-word-with-tail-respects[OF p-loop svk bridge-word-one-equiv])
simp
  then show ?thesis
    by simp
qed

lemma bridge-loop-constant-one:
  fixes h :: (real × real) ⇒ 'a
  assumes h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W)
h
```

```
    and t01: t ∈ {0..1}
    and c01: c ∈ {0..1}
    and d01: d ∈ {0..1}
    and cd: c ≤ d
    and const: ∀ y∈{c..d}. h (t, y) = x0
  shows loop-class (U ∩ V) x0 (bridge-loop h t c d) = fundamental-group-one (U
∩ V) x0
proof −
  have edgeUV: h ‘ ({t} × {c..d}) ⊆ U ∩ V
    using const x0-in-UV by auto
  have bridge-loop-in: bridge-loop h t c d ∈ loop-space (U ∩ V) x0
    by (rule vertical-bridge-loop-in-set[OF h-cont t01 c01 d01 cd edgeUV]) simp
  have hc: h (t, c) = x0 and hd: h (t, d) = x0
    using const cd by auto
  have conn-c-img: path-image (connector (h (t, c))) ⊆ {x0}
    using hc by (auto simp: path-image-def connector-def)
  have conn-d-img: path-image (reversepath (connector (h (t, d)))) ⊆ {x0}
    using hd by (auto simp: path-image-def connector-def reversepath-def)
  have vert-img: path-image (vertical-strip-path h t c d) ⊆ {x0}
  proof −
    have vert-img-raw: vertical-strip-path h t c d ‘ {0..1} ⊆ {x0}
    proof
      fix x
      assume x ∈ vertical-strip-path h t c d ‘ {0..1}
      then obtain u where u01: u ∈ {0..1} and x-eq: x = vertical-strip-path h t
c d u
        by blast
      have (d − c) ∗ u + c ∈ {c..d}
        by (rule affine-subinterval-member[OF cd u01])
      then show x ∈ {x0}
        using const x-eq by (auto simp: vertical-strip-path-def)
    qed
    from vert-img-raw show ?thesis
      by (simp add: path-image-def)
  qed
  have join-img:
    path-image (connector (h (t, c)) +++ vertical-strip-path h t c d) ⊆ {x0}
    by (rule subset-path-image-join[OF conn-c-img vert-img])
  have bridge-img-single: path-image (bridge-loop h t c d) ⊆ {x0}
      unfolding bridge-loop-def by (rule subset-path-image-join[OF join-img
conn-d-img])
  have bridge-path: path (bridge-loop h t c d)
    and bridge-img: path-image (bridge-loop h t c d) ⊆ U ∩ V
    using bridge-loop-in unfolding loop-space-def by auto
  have bridge-const:
    homotopic-paths (U ∩ V) (bridge-loop h t c d) (λ-. x0)
  proof (rule homotopic-paths-eq[OF bridge-path bridge-img])
    fix u :: real
    assume u01: u ∈ {0..1}
```

then have *bridge-loop h t c d u* ∈ *path-image* (*bridge-loop h t c d*)
by (*auto simp*: *path-image-def*)
then show *bridge-loop h t c d u* = *x0*
using *bridge-img-single* **by** *auto*
qed
show *?thesis*
unfolding *fundamental-group-one-def*
by (*rule loop-class-eqI*[*OF bridge-loop-in constant-loop-in-space*[*OF x0-in-UV*] *bridge-const*])
qed

lemma *rectangle-partition-partition-word-equiv*:
fixes *h* :: (*real* × *real*) ⇒ *'a*
assumes *h-cont*: *continuous-map* (*top-of-set* ({*0..1*} × {*0..1*})) (*top-of-set W*) *h*
and *c01*: *c* ∈ {*0..1*}
and *d01*: *d* ∈ {*0..1*}
and *cd*: *c* ≤ *d*
and *part*: *rectangle-partition h c d ts bs*
and *ts-ne*: *ts* ≠ []
and *hd0*: *hd ts* = *0*
and *edgeUV*: ⋀*x*. *x* ∈ *set ts* ⟹ *h* ‘ ({*x*} × {*c..d*}) ⊆ *U* ∩ *V*
and *end0*: ∀ *y*∈{*c..d*}. *h* (*0*, *y*) = *x0*
and *end1*: ∀ *y*∈{*c..d*}. *h* (*1*, *y*) = *x0*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*partition-word* (λ*x*. *h* (*x*, *c*)) *ts bs*)
(*partition-word* (λ*x*. *h* (*x*, *d*)) *ts bs*)
proof −
have *rowc-loop*: (λ*x*. *h* (*x*, *c*)) ∈ *loop-space W x0*
by (*rule homotopy-row-in-loop-space*[*OF h-cont c01*]) (*use end0 end1 cd* **in** *auto*)
have *rowd-loop*: (λ*x*. *h* (*x*, *d*)) ∈ *loop-space W x0*
by (*rule homotopy-row-in-loop-space*[*OF h-cont d01*]) (*use end0 end1 cd* **in** *auto*)
have *valid-c*: *valid-partition* (λ*x*. *h* (*x*, *c*)) *ts bs*
by (*rule rectangle-partition-valid-partition-row*[*OF part edgeUV - ts-ne hd0*]) (*use c01 cd* **in** *auto*)
have *valid-d*: *valid-partition* (λ*x*. *h* (*x*, *d*)) *ts bs*
by (*rule rectangle-partition-valid-partition-row*[*OF part edgeUV - ts-ne hd0*]) (*use d01 cd* **in** *auto*)
obtain *t ts'* **where** *ts*: *ts* = *t* # *ts'*
using *ts-ne* **by** (*cases ts*) *auto*
have *valid-c-ts*: *valid-partition* (λ*x*. *h* (*x*, *c*)) *ts bs*
using *valid-c* **by** *simp*
have *t0*: *t* = *0*
using *valid-c-ts* **unfolding** *ts* **by** (*rule valid-partition-cases*(*1*))
have *svk-c*: *svk-partition* (λ*x*. *h* (*x*, *c*)) (*t* # *ts'*) *bs*
using *valid-c-ts* **unfolding** *ts* **by** (*rule valid-partition-cases*(*2*))
have *ts'-ne*: *ts'* ≠ []

proof
assume $ts' = []$
with *svk-c t0* **show** *False*
by (*cases bs*) *auto*
qed
obtain *u us* **where** *ts'*: $ts' = u \# us$
using *ts'-ne* **by** (*cases ts'*) *auto*
obtain *b bs'* **where** *bs*: $bs = b \# bs'$
using *svk-c ts'* **by** (*cases bs*) *auto*
have *part-shape*: *rectangle-partition h c d* $(t \# u \# us)$ $(b \# bs')$
using *part* **unfolding** *ts ts' bs* **by** *simp*
have *last1*: *last* $(t \# u \# us) = 1$
using *valid-partition-last-props*[*OF valid-c*] **unfolding** *ts ts'* **by** *simp*
have *rowd-in*:
fpw-in-space G1 G2 (*partition-word* $(\lambda x.\ h\ (x,\ d))$ $(t \# u \# us)$ $(b \# bs')$)
by (*rule valid-partition-partition-word-in-space*[*OF rowd-loop*]) (*use valid-d ts ts' bs* **in** *simp*)
have *edgeUV-shape*:
$\bigwedge x.\ x \in set\ (t \# u \# us) \Longrightarrow h\ `\ (\{x\} \times \{c..d\}) \subseteq U \cap V$
using *edgeUV* **unfolding** *ts ts'* **by** *simp*
have *wordnil-in*: *fpw-in-space G1 G2 WordNil*
by *simp*
have *rect-rel-raw*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ c))$ $(t \# u \# us)$ $(b \# bs')$
(*bridge-word* (*last* $(b \# bs')$)
(*loop-class* $(U \cap V)$ *x0* (*bridge-loop h* (*last* $(t \# u \# us)$) *c d*)) *WordNil*))
(*bridge-word b* (*loop-class* $(U \cap V)$ *x0* (*bridge-loop h t c d*))
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ d))$ $(t \# u \# us)$ $(b \# bs')$ *WordNil*))
by (*rule rectangle-partition-partition-word-with-tail-equiv*[*OF h-cont c01 d01 cd part-shape edgeUV-shape wordnil-in*])
have *rect-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* $(\lambda x.\ h\ (x,\ c))$ $(t \# u \# us)$ $(b \# bs')$
(*bridge-word* (*last* $(b \# bs')$)
(*loop-class* $(U \cap V)$ *x0* (*bridge-loop h* (*last* $(t \# u \# us)$) *c d*)) *WordNil*))
(*bridge-word b* (*loop-class* $(U \cap V)$ *x0* (*bridge-loop h t c d*))
(*partition-word* $(\lambda x.\ h\ (x,\ d))$ $(t \# u \# us)$ $(b \# bs')$))
using *rect-rel-raw* **by** *simp*
have *start-one*:
loop-class $(U \cap V)$ *x0* (*bridge-loop h t c d*) = *fundamental-group-one* $(U \cap V)$ *x0*
by (*subst t0*, *rule bridge-loop-constant-one*[*OF h-cont*]) (*use c01 d01 cd end0* **in** *auto*)
have *end-one*:
loop-class $(U \cap V)$ *x0* (*bridge-loop h* (*last* $(t \# u \# us)$) *c d*) =
fundamental-group-one $(U \cap V)$ *x0*
by (*subst last1*, *rule bridge-loop-constant-one*[*OF h-cont*]) (*use c01 d01 cd end1* **in** *auto*)

have *rect-rel'*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*)
(*bridge-word* (*last* (*b* # *bs'*)) (*fundamental-group-one* (*U* ∩ *V*) *x0*) *WordNil*))
(*bridge-word b* (*fundamental-group-one* (*U* ∩ *V*) *x0*)
(*partition-word* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *us*) (*b* # *bs'*)))
proof (*cases b*)
case *True*
then show *?thesis*
using *rect-rel start-one end-one* **by** *simp*
next
case *False*
then show *?thesis*
using *rect-rel start-one end-one* **by** *simp*
qed
have *tail-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word-with-tail* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*)
(*bridge-word* (*last* (*b* # *bs'*)) (*fundamental-group-one* (*U* ∩ *V*) *x0*) *WordNil*))
(*partition-word* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*))
by (*rule valid-partition-tail-bridge-one-equiv*[*OF rowc-loop*]) (*use valid-c ts ts' bs* **in** *simp*)
have *prefix-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*bridge-word b* (*fundamental-group-one* (*U* ∩ *V*) *x0*)
(*partition-word* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *us*) (*b* # *bs'*)))
(*partition-word* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *us*) (*b* # *bs'*))
by (*rule bridge-word-one-equiv*[*OF rowd-in*])
have *step1*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*))
(*partition-word-with-tail* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*)
(*bridge-word* (*last* (*b* # *bs'*)) (*fundamental-group-one* (*U* ∩ *V*) *x0*) *WordNil*))
by (*rule carrier-full-amalg-equiv.sym*[*OF tail-rel*])
have *step2*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*))
(*bridge-word b* (*fundamental-group-one* (*U* ∩ *V*) *x0*)
(*partition-word* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *us*) (*b* # *bs'*)))
by (*rule carrier-full-amalg-equiv.trans*[*OF step1 rect-rel'*])
have *step3*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word* (λ*x*. *h* (*x*, *c*)) (*t* # *u* # *us*) (*b* # *bs'*))
(*partition-word* (λ*x*. *h* (*x*, *d*)) (*t* # *u* # *us*) (*b* # *bs'*))
by (*rule carrier-full-amalg-equiv.trans*[*OF step2 prefix-rel*])
then show *?thesis*
using *ts ts' bs* **by** *simp*
qed

lemma *valid-partition-nearby-partition-word-equiv*:
fixes $h :: (real \times real) \Rightarrow 'a$
assumes *h-cont*: *continuous-map* (*top-of-set* ($\{0..1\} \times \{0..1\}$)) (*top-of-set* *W*) *h*
and *end0*: $\forall\, y{\in}\{0..1\}.\ h\ (0,\ y) = x0$
and *end1*: $\forall\, y{\in}\{0..1\}.\ h\ (1,\ y) = x0$
and *y0-01*: $y0 \in \{0..1\}$
and *part0*: *valid-partition* ($\lambda x.\ h\ (x,\ y0)$) *ts bs*
shows $\exists\, e{>}0.\ \forall\, z{\in}\{0..1\}.\ dist\ z\ y0 < e \longrightarrow$
valid-partition ($\lambda x.\ h\ (x,\ z)$) *ts bs* $\wedge$
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word* ($\lambda x.\ h\ (x,\ y0)$) *ts bs*)
(*partition-word* ($\lambda x.\ h\ (x,\ z)$) *ts bs*)
proof −
obtain *t ts'* **where** *ts*: $ts = t\ \#\ ts'$
using *valid-partition-hd*(*1*)[*OF part0*] **by** (*cases ts*) *auto*
have *hd-ts0*: $hd\ ts = 0$
by (*rule valid-partition-hd*(*2*)[*OF part0*])
have *valid-ts*: *valid-partition* ($\lambda x.\ h\ (x,\ y0)$) *ts bs*
using *part0* **by** *simp*
have *t0*: $t = 0$
using *valid-ts* **unfolding** *ts* **by** (*rule valid-partition-cases*(*1*))
have *svk0*: *svk-partition* ($\lambda x.\ h\ (x,\ y0)$) ($t\ \#\ ts'$) *bs*
using *valid-ts* **unfolding** *ts* **by** (*rule valid-partition-cases*(*2*))
have *local-neigh*:
$\exists\, N.$ *openin* (*top-of-set* $\{0..1\}$) $N \wedge y0 \in N \wedge$
$(\forall\, x{\in}set\ (t\ \#\ ts').\ h\ `\ (\{x\} \times N) \subseteq U \cap V) \wedge$
$(\forall\, y\ z.\ y \le z \longrightarrow \{y..z\} \subseteq N \longrightarrow$ *rectangle-partition h y z* ($t\ \#\ ts'$) *bs*)
by (*rule svk-partition-local-neighbourhood*[*OF h-cont y0-01 svk0*])
obtain *N* **where** *N-open*: *openin* (*top-of-set* $\{0..1\}$) *N*
and *y0N*: $y0 \in N$
and *pointN*: $\forall\, x{\in}set\ ts.\ h\ `\ (\{x\} \times N) \subseteq U \cap V$
and *rectN*: $\forall\, y\ z.\ y \le z \longrightarrow \{y..z\} \subseteq N \longrightarrow$ *rectangle-partition h y z ts bs*
using *local-neigh* **unfolding** *ts* **by** *blast*
from *openin-euclidean-subtopology-iff*[*THEN iffD1*, *OF N-open*] *y0N*
obtain *e* **where** *e-pos*: $e > 0$
and *e-ball*: $\forall\, z{\in}\{0..1\}.\ dist\ z\ y0 < e \longrightarrow z \in N$
by *blast*
show *?thesis*
proof (*intro exI conjI ballI impI*)
show $e > 0$
by (*rule e-pos*)
fix *z*
assume *z01*: $z \in \{0..1\}$
and *dz*: $dist\ z\ y0 < e$
have *zN*: $z \in N$
by (*rule e-ball*[*rule-format*, *OF z01 dz*])
have *dyz*: $dist\ y0\ z < e$
using *dz* **by** (*simp add*: *dist-commute*)

have *seg-ball*: *closed-segment y0 z* ⊆ *ball y0 e*
by (*rule closed-segment-subset*) (*use y0-01 z01 e-pos dyz* **in** *auto*)
have *seg-unit*: *closed-segment y0 z* ⊆ {*0..1*}
by (*rule closed-segment-subset*) (*use y0-01 z01* **in** *auto*)
have *segN*: *closed-segment y0 z* ⊆ *N*
proof
fix *w*
assume *wseg*: *w* ∈ *closed-segment y0 z*
have *w01*: *w* ∈ {*0..1*}
using *seg-unit wseg* **by** *auto*
have *w-ball*: *w* ∈ *ball y0 e*
using *seg-ball wseg* **by** *auto*
have *dist w y0* < *e*
using *w-ball* **unfolding** *ball-def* **by** (*simp add*: *dist-commute*)
then show *w* ∈ *N*
by (*rule e-ball*[*rule-format*, *OF w01*])
qed
have *intervalN*: {*min y0 z..max y0 z*} ⊆ *N*
proof
fix *y*
assume *y-in*: *y* ∈ {*min y0 z..max y0 z*}
have *y* ∈ *closed-segment y0 z*
proof (*cases y0* ≤ *z*)
case *True*
with *y-in* **show** *?thesis*
by (*simp add*: *closed-segment-eq-real-ivl*)
next
case *False*
with *y-in* **show** *?thesis*
by (*simp add*: *closed-segment-eq-real-ivl*)
qed
then show *y* ∈ *N*
using *segN* **by** *blast*
qed
have *edgeUV-interval*:
h ‘ ({*x*} × {*min y0 z..max y0 z*}) ⊆ *U* ∩ *V* **if** *x-in*: *x* ∈ *set ts* **for** *x*
proof −
have *edgeN*: *h* ‘ ({*x*} × *N*) ⊆ *U* ∩ *V*
using *pointN x-in* **by** *auto*
have *subset-x*: {*x*} × {*min y0 z..max y0 z*} ⊆ {*x*} × *N*
using *intervalN* **by** *auto*
show *?thesis*
by (*rule order-trans*[*OF image-mono*[*OF subset-x*] *edgeN*])
qed
have *rect*:
rectangle-partition h (*min y0 z*) (*max y0 z*) *ts bs*
by (*rule rectN*[*rule-format*]) (*use intervalN* **in** *auto*)
have *valid-z*: *valid-partition* (λ*x*. *h* (*x*, *z*)) *ts bs*
by (*rule rectangle-partition-valid-partition-row*[*OF rect edgeUV-interval* -

```
valid-partition-hd(1)[OF part0] hd-ts0])
      auto
  have min01: min y0 z ∈ {0..1}
    using y0-01 z01 by auto
  have max01: max y0 z ∈ {0..1}
    using y0-01 z01 by auto
  have interval01: {min y0 z..max y0 z} ⊆ {0..1}
    using y0-01 z01 by auto
  have end0-interval: ∀ y∈{min y0 z..max y0 z}. h (0, y) = x0
  proof
    fix y
    assume y-in: y ∈ {min y0 z..max y0 z}
    then have y01: y ∈ {0..1}
      using interval01 by blast
    show h (0, y) = x0
      using end0 y01 by blast
  qed
  have end1-interval: ∀ y∈{min y0 z..max y0 z}. h (1, y) = x0
  proof
    fix y
    assume y-in: y ∈ {min y0 z..max y0 z}
    then have y01: y ∈ {0..1}
      using interval01 by blast
    show h (1, y) = x0
      using end1 y01 by blast
  qed
  have part-rel-raw:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word (λx. h (x, min y0 z)) ts bs)
      (partition-word (λx. h (x, max y0 z)) ts bs)
    by (rule rectangle-partition-partition-word-equiv[
        OF h-cont min01 max01 - rect valid-partition-hd(1)[OF part0] hd-ts0
          edgeUV-interval end0-interval end1-interval])
       simp
  have part-rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word (λx. h (x, y0)) ts bs)
      (partition-word (λx. h (x, z)) ts bs)
  proof (cases y0 ≤ z)
    case True
    then show ?thesis
      using part-rel-raw by simp
  next
    case False
    then have carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word (λx. h (x, z)) ts bs)
        (partition-word (λx. h (x, y0)) ts bs)
      using part-rel-raw by simp
    then show ?thesis
```

```
      by (rule carrier-full-amalg-equiv.sym)
    qed
    show valid-partition (λx. h (x, z)) ts bs
      using valid-z by simp
    show carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
        (partition-word (λx. h (x, y0)) ts bs)
        (partition-word (λx. h (x, z)) ts bs)
      using part-rel by simp
  qed
qed

definition some-valid-partition :: (real ⇒ 'a) ⇒ real list × bool list where
  some-valid-partition p =
    (SOME tb. case tb of (ts, bs) ⇒ valid-partition p ts bs)

lemma some-valid-partition-spec:
  assumes p-loop: p ∈ loop-space W x0
  shows valid-partition p
    (fst (some-valid-partition p))
    (snd (some-valid-partition p))
proof −
  have ex: ∃ tb. case tb of (ts, bs) ⇒ valid-partition p ts bs
    using loop-has-valid-partition[OF p-loop] by force
  have case some-valid-partition p of (ts, bs) ⇒ valid-partition p ts bs
    unfolding some-valid-partition-def by (rule someI-ex[OF ex])
  then show ?thesis
    by (cases some-valid-partition p) auto
qed
```

15.3 Homotopy invariance of the partition encoding

The central technical step is that different valid partitions of homotopic loops produce equivalent words in the carrier-side amalgamated free product. The proof uses a rectangular decomposition of a homotopy and keeps the word encoding stable while moving from one boundary loop to the other.

```
lemma valid-partition-homotopic-partition-word-equiv:
  assumes p-loop: p ∈ loop-space W x0
    and q-loop: q ∈ loop-space W x0
    and pq: homotopic-paths W p q
    and p-part: valid-partition p ts bs
    and q-part: valid-partition q us cs
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word p ts bs) (partition-word q us cs)
proof −
  obtain h0 :: real × real ⇒ 'a where h0-cont: continuous-on ({0..1} × {0..1})
h0
    and h0-into: h0 ∈ ({0..1} × {0..1}) → W
    and h0-p: ⋀x. h0 (0, x) = p x
    and h0-q: ⋀x. h0 (1, x) = q x
```

```
    and h0-left: ⋀y. y ∈ {0..1} ⟹ h0 (y, 0) = p 0
    and h0-right: ⋀y. y ∈ {0..1} ⟹ h0 (y, 1) = p 1
    using pq
    by (auto simp: homotopic-paths-def homotopic-with-def
                   pathstart-def pathfinish-def image-subset-iff-funcset)
  define h where h = (λxy. h0 (snd xy, fst xy))
  let ?row = λy. λx. h (x, y)
  let ?enc = λy. case some-valid-partition (?row y) of (vs, ds) ⇒ partition-word
(?row y) vs ds
  let ?P =
     {y ∈ {0..1}. carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(?enc 0) (?enc y)}
  have p-start: pathstart p = x0 and p-finish: pathfinish p = x0
    using p-loop unfolding loop-space-def by auto
  have h-cont: continuous-map (top-of-set ({0..1} × {0..1})) (top-of-set W) h
  proof −
    have swap-cont:
      continuous-on ({0..1} × {0..1}) (λxy. (snd xy, fst xy))
      by (intro continuous-intros)
    have swap-in:
      (λxy. (snd xy, fst xy)) ‘ ({0..1} × {0..1}) ⊆ ({0..1} × {0..1})
      by auto
    have h-on: continuous-on ({0..1} × {0..1}) h
    proof −
      have continuous-on ({0..1} × {0..1}) (λxy. h0 (snd xy, fst xy))
        by (rule continuous-on-compose2[OF h0-cont]) (use swap-cont swap-in in
auto)
      then show ?thesis
        unfolding h-def .
    qed
    have h-into: h ∈ ({0..1} × {0..1}) → W
      using h0-into unfolding h-def by auto
    show ?thesis
      using h-on h-into by simp
  qed
  have end0: ∀ y∈{0..1}. h (0, y) = x0
  proof
    fix y :: real
    assume y01: y ∈ {0..1}
    have h0 (y, 0::real) = p 0
      by (rule h0-left[OF y01])
    then show h (0, y) = x0
      using p-start unfolding h-def pathstart-def by simp
  qed
  have end1: ∀ y∈{0..1}. h (1, y) = x0
  proof
    fix y :: real
    assume y01: y ∈ {0..1}
    have h0 (y, 1::real) = p 1
```

```
    by (rule h0-right[OF y01])
  then show h (1, y) = x0
    using p-finish unfolding h-def pathfinish-def by simp
 qed
 have row-loop: ?row y ∈ loop-space W x0 if y01: y ∈ {0..1} for y
   by (rule homotopy-row-in-loop-space[OF h-cont y01]) (use end0 end1 y01 in
auto)
 have openP-local: ∀ y0∈?P. ∃ e>0. ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P
 proof
  fix y0
  assume y0P: y0 ∈ ?P
  then have y0-01: y0 ∈ {0..1}
   by simp
  obtain vs ds where tb0: some-valid-partition (?row y0) = (vs, ds)
   by (cases some-valid-partition (?row y0)) auto
  have part0: valid-partition (?row y0) vs ds
    using some-valid-partition-spec[OF row-loop[OF y0-01]] unfolding tb0 by
simp
  obtain e where e-pos: e > 0
   and near:
    ∀ z∈{0..1}. dist z y0 < e ⟶
     valid-partition (?row z) vs ds ∧
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word (?row y0) vs ds)
      (partition-word (?row z) vs ds)
    using valid-partition-nearby-partition-word-equiv[OF h-cont end0 end1 y0-01
part0]
   by blast
  have nearP: ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P
  proof
   fix z :: real
   assume z01: z ∈ {0..1}
   show dist z y0 < e ⟶ z ∈ ?P
   proof
    assume dz: dist z y0 < e
    have z-part: valid-partition (?row z) vs ds
     and fixed-rel:
      carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (partition-word (?row y0) vs ds)
       (partition-word (?row z) vs ds)
     using near[rule-format, OF z01 dz] by blast+
    obtain us' cs' where tbz: some-valid-partition (?row z) = (us', cs')
     by (cases some-valid-partition (?row z)) auto
    have chosen-z: valid-partition (?row z) us' cs'
      using some-valid-partition-spec[OF row-loop[OF z01]] unfolding tbz by
simp
    have z-rel:
     carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word (?row z) vs ds) (?enc z)
```

```
        using valid-partition-same-loop-partition-word-equiv[OF row-loop[OF z01]
z-part chosen-z]
        unfolding tbz by simp
      have y0-eq: ?enc y0 = partition-word (?row y0) vs ds
        unfolding tb0 by simp
      have base-rel0:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (?enc 0) (?enc y0)
        using y0P by simp
      have base-rel:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (?enc 0) (partition-word (?row y0) vs ds)
        using base-rel0 unfolding y0-eq by simp
      have carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (?enc 0) (?enc z)
        by (rule carrier-full-amalg-equiv.trans[OF base-rel])
           (rule carrier-full-amalg-equiv.trans[OF fixed-rel z-rel])
      then show z ∈ ?P
        using z01 by simp
    qed
  qed
  have witness: e > 0 ∧ (∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P)
  proof
    show e > 0
      by (rule e-pos)
    show ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P
      by (rule nearP)
  qed
  have ex-witness: ∃ eps. eps > 0 ∧ (∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ ?P)
  proof (rule exI[where x = e], rule conjI)
    show e > 0
      using witness by simp
    show ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P
      using witness by simp
  qed
  have ex-witness-set: ∃ eps. eps ∈ {0<..} ∧ (∀ z∈{0..1}. dist z y0 < eps ⟶ z
∈ ?P)
  proof −
    from ex-witness obtain eps where
      eps-pos: eps > 0 and eps-near: ∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ ?P
      by auto
    show ?thesis
    proof (rule exI[where x = eps], rule conjI)
      show eps ∈ {0<..}
        using eps-pos by simp
      show ∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ ?P
        by (rule eps-near)
    qed
  qed
```

```
    show ∃ e>0. ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ ?P
    proof −
      from ex-witness
      show ?thesis
        unfolding Bex-def greaterThan-def
        by blast
    qed
  qed
  have P-subset: ?P ⊆ {0..1}
    by auto
  have openP: openin (top-of-set {0..1}) ?P
    using openP-local P-subset by (auto simp: openin-euclidean-subtopology-iff)
  have open-notP-local:
    ∀ y0∈{0..1} − ?P. ∃ e>0. ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P
  proof
    fix y0
    assume y0-notP: y0 ∈ {0..1} − ?P
    then have y0-01: y0 ∈ {0..1} and y0-not: y0 ∉ ?P
      by auto
    obtain vs ds where tb0: some-valid-partition (?row y0) = (vs, ds)
      by (cases some-valid-partition (?row y0)) auto
    have part0: valid-partition (?row y0) vs ds
      using some-valid-partition-spec[OF row-loop[OF y0-01]] unfolding tb0 by
simp
    obtain e where e-pos: e > 0
      and near:
        ∀ z∈{0..1}. dist z y0 < e ⟶
          valid-partition (?row z) vs ds ∧
          carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
            (partition-word (?row y0) vs ds)
            (partition-word (?row z) vs ds)
      using valid-partition-nearby-partition-word-equiv[OF h-cont end0 end1 y0-01
part0]
      by blast
    have near-notP: ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P
    proof
      fix z :: real
      assume z01: z ∈ {0..1}
      show dist z y0 < e ⟶ z ∈ {0..1} − ?P
      proof
        assume dz: dist z y0 < e
        have z-part: valid-partition (?row z) vs ds
          and fixed-rel:
            carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
              (partition-word (?row y0) vs ds)
              (partition-word (?row z) vs ds)
          using near[rule-format, OF z01 dz] by blast+
        obtain us' cs' where tbz: some-valid-partition (?row z) = (us', cs')
          by (cases some-valid-partition (?row z)) auto
```

```
      have chosen-z: valid-partition (?row z) us' cs'
        using some-valid-partition-spec[OF row-loop[OF z01]] unfolding tbz by
simp
      have z-rel:
        carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
          (partition-word (?row z) vs ds) (?enc z)
        using valid-partition-same-loop-partition-word-equiv[OF row-loop[OF z01]
z-part chosen-z]
        unfolding tbz by simp
      have y0-eq: ?enc y0 = partition-word (?row y0) vs ds
        unfolding tb0 by simp
      show z ∈ {0..1} − ?P
      proof
        show z ∈ {0..1}
          by (rule z01)
        show z ∉ ?P
        proof
          assume zP: z ∈ ?P
          have base-to-z:
           carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (?enc 0)
(?enc z)
            using zP by simp
          have z-to-y0:
           carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (?enc z)
(?enc y0)
          proof −
            have step1:
              carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
                (?enc z) (partition-word (?row z) vs ds)
              by (rule carrier-full-amalg-equiv.sym[OF z-rel])
            show ?thesis
              unfolding y0-eq
              by (rule carrier-full-amalg-equiv.trans[OF step1])
                 (rule carrier-full-amalg-equiv.sym[OF fixed-rel])
          qed
          have carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (?enc
0) (?enc y0)
            by (rule carrier-full-amalg-equiv.trans[OF base-to-z z-to-y0])
          then show False
            using y0-not y0-01 by simp
        qed
      qed
    qed
  qed
  have witness: e > 0 ∧ (∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P)
  proof
    show e > 0
      by (rule e-pos)
    show ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P
```

```
      by (rule near-notP)
    qed
    have ex-witness: ∃ eps. eps > 0 ∧ (∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ {0..1}
− ?P)
    proof (rule exI[where x = e], rule conjI)
      show e > 0
        using witness by simp
      show ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P
        using witness by simp
    qed
    have ex-witness-set: ∃ eps. eps ∈ {0<..} ∧ (∀ z∈{0..1}. dist z y0 < eps ⟶ z
∈ {0..1} − ?P)
    proof −
      from ex-witness obtain eps where
        eps-pos: eps > 0 and eps-near: ∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ {0..1}
− ?P
        by auto
      show ?thesis
      proof (rule exI[where x = eps], rule conjI)
        show eps ∈ {0<..}
          using eps-pos by simp
        show ∀ z∈{0..1}. dist z y0 < eps ⟶ z ∈ {0..1} − ?P
          by (rule eps-near)
      qed
    qed
    show ∃ e>0. ∀ z∈{0..1}. dist z y0 < e ⟶ z ∈ {0..1} − ?P
    proof −
      from ex-witness
      show ?thesis
        unfolding Bex-def greaterThan-def
        by blast
    qed
  qed
  have open-notP: openin (top-of-set {0..1}) ({0..1} − ?P)
    using open-notP-local by (auto simp: openin-euclidean-subtopology-iff)
  have closedP: closedin (top-of-set {0..1}) ?P
  proof −
    have {0..1} − ({0..1} − ?P) = ?P
      using P-subset by auto
    with open-notP show ?thesis
      by (auto simp: openin-closedin-eq)
  qed
  have P-all: ?P = {0..1}
  proof −
    have connected ({0..1} :: real set)
      by simp
    then have P-cases: ?P = {} ∨ ?P = {0..1}
      using openP closedP unfolding connected-clopen by blast
    have (0::real) ∈ ?P
```

```
    by (auto intro: carrier-full-amalg-equiv.refl)
  then have P-nonempty: ?P ≠ {}
    by blast
  from P-cases P-nonempty show ?thesis
    by blast
qed
have row0-eq: ?row 0 = p
proof
  fix x :: real
  show ?row 0 x = p x
    using h0-p[of x] unfolding h-def by simp
qed
have row1-eq: ?row 1 = q
proof
  fix x :: real
  show ?row 1 x = q x
    using h0-q[of x] unfolding h-def by simp
qed
have row0-hom: homotopic-paths W (?row 0) p
proof −
  have row0-path: path (?row 0) and row0-img: path-image (?row 0) ⊆ W
    using row-loop[of 0] unfolding loop-space-def by auto
  show ?thesis
  proof (rule homotopic-paths-eq[OF row0-path row0-img])
    fix t :: real
    assume t01: t ∈ {0..1}
    show ?row 0 t = p t
    proof −
      from row0-eq have ?row 0 t = p t
        by (rule fun-cong)
      then show ?thesis .
    qed
  qed
qed
have row1-hom: homotopic-paths W (?row 1) q
proof −
  have row1-path: path (?row 1) and row1-img: path-image (?row 1) ⊆ W
    using row-loop[of 1] unfolding loop-space-def by auto
  show ?thesis
  proof (rule homotopic-paths-eq[OF row1-path row1-img])
    fix t :: real
    assume t01: t ∈ {0..1}
    show ?row 1 t = q t
    proof −
      from row1-eq have ?row 1 t = q t
        by (rule fun-cong)
      then show ?thesis .
    qed
  qed
```

```
  qed
  obtain vs0 ds0 where tb0: some-valid-partition (?row 0) = (vs0, ds0)
    by (cases some-valid-partition (?row 0)) auto
  obtain vs1 ds1 where tb1: some-valid-partition (?row 1) = (vs1, ds1)
    by (cases some-valid-partition (?row 1)) auto
  have tb0-p: some-valid-partition p = (vs0, ds0)
    using tb0 row0-eq by simp
  have tb1-q: some-valid-partition q = (vs1, ds1)
    using tb1 row1-eq by simp
  have enc0-eq: ?enc 0 = partition-word p vs0 ds0
    using tb0 row0-eq by simp
  have enc1-eq: ?enc 1 = partition-word q vs1 ds1
    using tb1 row1-eq by simp
  have part-row0: valid-partition p vs0 ds0
    using some-valid-partition-spec[OF p-loop] unfolding tb0-p by simp
  have part-row1: valid-partition q vs1 ds1
    using some-valid-partition-spec[OF q-loop] unfolding tb1-q by simp
  have p-to-row0:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (partition-word p ts bs) (?enc 0)
        using valid-partition-same-loop-partition-word-equiv[OF p-loop p-part
part-row0]
    using enc0-eq by simp
  have row1-to-q:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (?enc 1) (partition-word q us cs)
   using valid-partition-same-loop-partition-word-equiv[OF q-loop part-row1 q-part]
    using enc1-eq by simp
  have row0-to-row1:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (?enc 0) (?enc
1)
  proof −
    have (1::real) ∈ ?P
      using P-all by simp
    then show ?thesis
      by simp
  qed
  show ?thesis
    by (rule carrier-full-amalg-equiv.trans[OF p-to-row0])
       (rule carrier-full-amalg-equiv.trans[OF row0-to-row1 row1-to-q])
qed

lemma valid-partition-loop-class-partition-word-equiv:
  assumes p-loop: p ∈ loop-space W x0
    and q-loop: q ∈ loop-space W x0
    and eq: loop-class W x0 p = loop-class W x0 q
    and p-part: valid-partition p ts bs
    and q-part: valid-partition q us cs
  shows carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
```

```
    (partition-word p ts bs) (partition-word q us cs)
proof −
  have hom: homotopic-paths W p q
    using p-loop q-loop eq by (simp add: loop-class-eq-iff)
  show ?thesis
    by (rule valid-partition-homotopic-partition-word-equiv[OF p-loop q-loop hom
p-part q-part])
qed

lemma svk-decode-word-loop:
  assumes w-in: fpw-in-space G1 G2 w
  shows svk-decode w = loop-class W x0 (word-loop w)
proof −
  have w-loop: word-loop w ∈ loop-space W x0
    by (rule word-loop-in-W[OF w-in])
  have w-part:
    valid-partition (word-loop w) (word-partition-times w) (word-partition-bits w)
    by (rule word-loop-valid-partition[OF w-in])
  have decode-part:
          svk-decode (partition-word (word-loop w) (word-partition-times w)
(word-partition-bits w)) =
      loop-class W x0 (word-loop w)
    by (rule valid-partition-decode-partition-word-eq-loop-class[OF w-loop w-part])
  have rel:
    carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
       (partition-word (word-loop w) (word-partition-times w) (word-partition-bits
w)) w
    by (rule partition-word-word-loop-equiv[OF w-in])
    have svk-decode (partition-word (word-loop w) (word-partition-times w)
(word-partition-bits w)) =
      svk-decode w
    by (rule svk-decode-respects[OF rel])
  then show ?thesis
    using decode-part by simp
qed
```

15.4 Encoding, decoding, and the final bijection

With existence, refinement invariance, and homotopy invariance in place, the encode/decode pair can now be defined directly on loop classes. The remaining lemmas verify the round-trip laws required by the abstract interface and turn them into the classical Seifert–van Kampen bijection.

```
definition svk-encode ::
  (real ⇒ 'a) set ⇒ ((real ⇒ 'a) set, (real ⇒ 'a) set) free-product-word
where
  svk-encode A =
    (let p = some-loop W x0 A;
         tb = some-valid-partition p
     in case tb of (ts, bs) ⇒ partition-word p ts bs)
```

lemma *svk-encode-in-space*:
assumes *A-in*: *A* ∈ *fundamental-group-space W x0*
shows *fpw-in-space G1 G2* (*svk-encode A*)
proof −
have *p-loop*: *some-loop W x0 A* ∈ *loop-space W x0*
and *A-eq*: *A* = *loop-class W x0* (*some-loop W x0 A*)
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *part*: *valid-partition* (*some-loop W x0 A*)
(*fst* (*some-valid-partition* (*some-loop W x0 A*)))
(*snd* (*some-valid-partition* (*some-loop W x0 A*)))
by (*rule some-valid-partition-spec*[*OF p-loop*])
show *?thesis*
unfolding *svk-encode-def Let-def*
using *valid-partition-partition-word-in-space*[*OF p-loop part*]
by (*cases some-valid-partition* (*some-loop W x0 A*)) *simp-all*
qed

lemma *svk-decode-encode*:
assumes *A-in*: *A* ∈ *fundamental-group-space W x0*
shows *svk-decode* (*svk-encode A*) = *A*
proof −
have *p-loop*: *some-loop W x0 A* ∈ *loop-space W x0*
and *A-eq*: *A* = *loop-class W x0* (*some-loop W x0 A*)
using *some-loop-spec*[*OF A-in*] **by** *auto*
have *part*: *valid-partition* (*some-loop W x0 A*)
(*fst* (*some-valid-partition* (*some-loop W x0 A*)))
(*snd* (*some-valid-partition* (*some-loop W x0 A*)))
by (*rule some-valid-partition-spec*[*OF p-loop*])
show *?thesis*
unfolding *svk-encode-def Let-def*
using *valid-partition-decode-partition-word-eq-loop-class*[*OF p-loop part*] *A-eq*
by (*cases some-valid-partition* (*some-loop W x0 A*)) *simp-all*
qed

lemma *svk-encode-decode*:
assumes *w-in*: *fpw-in-space G1 G2 w*
shows *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2*
(*svk-encode* (*svk-decode w*)) *w*
proof −
let *?p* = *some-loop W x0* (*svk-decode w*)
let *?tb* = *some-valid-partition ?p*
obtain *ts bs* **where** *tb*: *?tb* = (*ts*, *bs*)
by (*cases ?tb*) *auto*
have *p-loop*: *?p* ∈ *loop-space W x0*
and *p-class*: *svk-decode w* = *loop-class W x0 ?p*
using *some-loop-spec*[*OF svk-decode-in-space*] **by** *auto*
have *p-part*: *valid-partition ?p ts bs*
using *some-valid-partition-spec*[*OF p-loop*] **unfolding** *tb* **by** *simp*

have *w-loop*: *word-loop w* ∈ *loop-space W x0*
by (*rule word-loop-in-W*[*OF w-in*])
have *w-part*:
valid-partition (*word-loop w*) (*word-partition-times w*) (*word-partition-bits w*)
by (*rule word-loop-valid-partition*[*OF w-in*])
have *same-class*: *loop-class W x0 ?p* = *loop-class W x0* (*word-loop w*)
using *p-class svk-decode-word-loop*[*OF w-in*] **by** *simp*
have *part-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word ?p ts bs*)
(*partition-word* (*word-loop w*) (*word-partition-times w*) (*word-partition-bits w*))
by (*rule valid-partition-loop-class-partition-word-equiv*[*OF p-loop w-loop same-class p-part w-part*])
have *word-rel*:
carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
(*partition-word* (*word-loop w*) (*word-partition-times w*) (*word-partition-bits w*)) *w*
by (*rule partition-word-word-loop-equiv*[*OF w-in*])
have *encode-word*: *svk-encode* (*svk-decode w*) = *partition-word ?p ts bs*
unfolding *svk-encode-def Let-def tb* **by** *simp*
show *?thesis*
unfolding *encode-word*
by (*rule carrier-full-amalg-equiv.trans*[*OF part-rel word-rel*])
qed

lemma *svk-decode-surjective*:
assumes *A-in*: *A* ∈ *fundamental-group-space W x0*
shows ∃ *w*. *fpw-in-space G1 G2 w* ∧ *svk-decode w* = *A*
using *svk-encode-in-space*[*OF A-in*] *svk-decode-encode*[*OF A-in*] **by** *blast*

theorem *svk-decode-image*:
svk-decode ‘ {*w*. *fpw-in-space G1 G2 w*} = *fundamental-group-space W x0*
proof
show *svk-decode* ‘ {*w*. *fpw-in-space G1 G2 w*} ⊆ *fundamental-group-space W x0*
using *svk-decode-in-space* **by** *blast*
next
show *fundamental-group-space W x0* ⊆ *svk-decode* ‘ {*w*. *fpw-in-space G1 G2 w*}
using *svk-decode-surjective* **by** *blast*
qed

definition *classical-svk-quotient-map* ::
(*real* ⇒ *'a*) *set* ⇒
(((*real* ⇒ *'a*) *set*, (*real* ⇒ *'a*) *set*) *free-product-word*) *set*
where
classical-svk-quotient-map A =
carrier-full-amalg-class G1 G2 H i1 i2 mult1 one1 mult2 one2 (*svk-encode A*)

lemma *classical-svk-quotient-map-in-space*:

```
  assumes A-in: A ∈ fundamental-group-space W x0
  shows classical-svk-quotient-map A ∈
    carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
  unfolding classical-svk-quotient-map-def
  by (rule carrier-full-amalg-class-in-space[OF svk-encode-in-space[OF A-in]])

lemma classical-svk-quotient-map-injective:
  assumes AB: classical-svk-quotient-map A = classical-svk-quotient-map B
    and A-in: A ∈ fundamental-group-space W x0
    and B-in: B ∈ fundamental-group-space W x0
  shows A = B
proof −
  have carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2
      (svk-encode A) (svk-encode B)
    using AB unfolding classical-svk-quotient-map-def
    by (simp add: carrier-full-amalg-class-eq-iff)
  then have svk-decode (svk-encode A) = svk-decode (svk-encode B)
    by (rule svk-decode-respects)
  then show ?thesis
    using A-in B-in by (simp add: svk-decode-encode)
qed

lemma classical-svk-quotient-map-surjective:
  assumes Q-in:
    Q ∈ carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1
mult2 one2
  shows ∃ A ∈ fundamental-group-space W x0. classical-svk-quotient-map A = Q
proof −
  from Q-in obtain w where w-in: fpw-in-space G1 G2 w
    and Q-rep: Q = carrier-full-amalg-class G1 G2 H i1 i2 mult1 one1 mult2 one2
w
    unfolding carrier-full-amalgamated-free-product-space-def by auto
  have A-in: svk-decode w ∈ fundamental-group-space W x0
    by (rule svk-decode-in-space)
  have classical-svk-quotient-map (svk-decode w) = Q
    unfolding classical-svk-quotient-map-def Q-rep
    by (simp add: carrier-full-amalg-class-eq-iff svk-encode-decode[OF w-in])
  then show ?thesis
    using A-in by blast
qed
```

At this point the encoding and decoding maps satisfy the abstract round-trip laws from the interface theory. The final theorem therefore presents the classical Seifert–van Kampen statement as a bijection between the fundamental group of $U \cup V$ at *x0* and the carrier-based amalgamated free product assembled from U, V, and $U \cap V$.

```
theorem classical-seifert-van-kampen-bij-betw:
  bij-betw classical-svk-quotient-map (fundamental-group-space W x0)
```

```
    (carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2)
  unfolding bij-betw-def
proof
  show inj-on classical-svk-quotient-map (fundamental-group-space W x0)
    unfolding inj-on-def
    using classical-svk-quotient-map-injective by blast
next
  show classical-svk-quotient-map ‘ fundamental-group-space W x0 =
      carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
  proof
    show classical-svk-quotient-map ‘ fundamental-group-space W x0 ⊆
       carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2
one2
      using classical-svk-quotient-map-in-space by blast
  next
    show carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1
mult2 one2
       ⊆ classical-svk-quotient-map ‘ fundamental-group-space W x0
      using classical-svk-quotient-map-surjective by blast
  qed
qed

end

end
theory Binary-Sum-Topology
  imports HOL−Analysis.Abstract-Topology
begin
```

16 Binary coproduct topology

Topological pushouts are constructed by first equipping the disjoint sum with its coproduct topology and only then quotienting by the glue relation. This short theory records that coproduct topology together with the basic continuity lemmas used by the later pushout arguments.

```
definition binary-sum-topology :: 'a topology => 'b topology => ('a + 'b) topology
where
  binary-sum-topology X Y =
    topology (λU.
      U ⊆ Inl ‘ topspace X ∪ Inr ‘ topspace Y ∧
      openin X {x. Inl x ∈ U} ∧
      openin Y {y. Inr y ∈ U})

lemma is-binary-sum-topology:
  istopology (λU.
     U ⊆ Inl ‘ topspace X ∪ Inr ‘ topspace Y ∧
```

openin X $\{x.\ Inl\ x \in U\} \wedge$
openin Y $\{y.\ Inr\ y \in U\}$)
proof −
have *inl-Int*: $\{x.\ Inl\ x \in S \cap T\} = \{x.\ Inl\ x \in S\} \cap \{x.\ Inl\ x \in T\}$ **for** *S T*
by *auto*
have *inr-Int*: $\{y.\ Inr\ y \in S \cap T\} = \{y.\ Inr\ y \in S\} \cap \{y.\ Inr\ y \in T\}$ **for** *S T*
by *auto*
have *inl-Union*: $\{x.\ Inl\ x \in \bigcup \mathcal{K}\} = (\bigcup U{\in}\mathcal{K}.\ \{x.\ Inl\ x \in U\})$ **for** $\mathcal{K}$
by *auto*
have *inr-Union*: $\{y.\ Inr\ y \in \bigcup \mathcal{K}\} = (\bigcup U{\in}\mathcal{K}.\ \{y.\ Inr\ y \in U\})$ **for** $\mathcal{K}$
by *auto*
show *?thesis*
unfolding *istopology-def inl-Int inr-Int inl-Union inr-Union*
by *blast*
qed

lemma *openin-binary-sum-topology*:
openin (*binary-sum-topology X Y*) *U* ⟷
U ⊆ *Inl* ‘ *topspace X* ∪ *Inr* ‘ *topspace Y* ∧
openin X $\{x.\ Inl\ x \in U\} \wedge$
openin Y $\{y.\ Inr\ y \in U\}$
by (*auto simp*: *binary-sum-topology-def is-binary-sum-topology*)

lemma *topspace-binary-sum-topology* [*simp*]:
topspace (*binary-sum-topology X Y*) = *Inl* ‘ *topspace X* ∪ *Inr* ‘ *topspace Y*
proof
have *top-open*: *openin* (*binary-sum-topology X Y*) (*topspace* (*binary-sum-topology X Y*))
by *simp*
show *topspace* (*binary-sum-topology X Y*) ⊆ *Inl* ‘ *topspace X* ∪ *Inr* ‘ *topspace Y*
using *top-open* **unfolding** *openin-binary-sum-topology* **by** *blast*
have *eqX*: $\{x.\ Inl\ x \in Inl\ `\ topspace\ X \cup Inr\ `\ topspace\ Y\} = topspace\ X$
by *auto*
have *eqY*: $\{y.\ Inr\ y \in Inl\ `\ topspace\ X \cup Inr\ `\ topspace\ Y\} = topspace\ Y$
by *auto*
have *openin* (*binary-sum-topology X Y*) (*Inl* ‘ *topspace X* ∪ *Inr* ‘ *topspace Y*)
unfolding *openin-binary-sum-topology eqX eqY* **by** *simp*
then show *Inl* ‘ *topspace X* ∪ *Inr* ‘ *topspace Y* ⊆ *topspace* (*binary-sum-topology X Y*)
by (*rule openin-subset*)
qed

lemma *subset-topspace-binary-sum-inl*:
(*Inl* :: *'a* ⇒ *'a* + *'b*) ‘ *topspace X* ⊆ *topspace* (*binary-sum-topology X Y*)
by *simp*

lemma *subset-topspace-binary-sum-inr*:
(*Inr* :: *'b* ⇒ *'a* + *'b*) ‘ *topspace Y* ⊆ *topspace* (*binary-sum-topology X Y*)
by *simp*

lemma *continuous-map-binary-sum-inl*:
continuous-map X (*binary-sum-topology X Y*) (*Inl* :: *'a* ⇒ *'a* + *'b*)
proof −
have *image*: (*Inl* :: *'a* ⇒ *'a* + *'b*) ‘ *topspace X* ⊆ *topspace* (*binary-sum-topology X Y*)
by *simp*
have *preimage*:
∀ *U*. *openin* (*binary-sum-topology X Y*) *U* ⟶ *openin X* {*x* ∈ *topspace X*. *Inl x* ∈ *U*}
proof (*intro allI impI*)
fix *U*
assume *U*: *openin* (*binary-sum-topology X Y*) *U*
have *openU*: *openin X* {*x*. *Inl x* ∈ *U*}
using *U* **by** (*simp add*: *openin-binary-sum-topology*)
have *subsetX*: {*x*. *Inl x* ∈ *U*} ⊆ *topspace X*
using *openU* **by** (*rule openin-subset*)
have *eq*: {*x* ∈ *topspace X*. *Inl x* ∈ *U*} = {*x*. *Inl x* ∈ *U*}
using *subsetX* **by** *auto*
have *openin X* {*x* ∈ *topspace X*. *Inl x* ∈ *U*}
unfolding *eq* **using** *openU* **by** *simp*
then show *openin X* {*x* ∈ *topspace X*. *Inl x* ∈ *U*} **.**
qed
from *image preimage* **show** *?thesis*
by (*simp add*: *continuous-map*)
qed

lemma *continuous-map-binary-sum-inr*:
continuous-map Y (*binary-sum-topology X Y*) (*Inr* :: *'b* ⇒ *'a* + *'b*)
proof −
have *image*: (*Inr* :: *'b* ⇒ *'a* + *'b*) ‘ *topspace Y* ⊆ *topspace* (*binary-sum-topology X Y*)
by *simp*
have *preimage*:
∀ *U*. *openin* (*binary-sum-topology X Y*) *U* ⟶ *openin Y* {*y* ∈ *topspace Y*. *Inr y* ∈ *U*}
proof (*intro allI impI*)
fix *U*
assume *U*: *openin* (*binary-sum-topology X Y*) *U*
have *openU*: *openin Y* {*y*. *Inr y* ∈ *U*}
using *U* **by** (*simp add*: *openin-binary-sum-topology*)
have *subsetY*: {*y*. *Inr y* ∈ *U*} ⊆ *topspace Y*
using *openU* **by** (*rule openin-subset*)
have *eq*: {*y* ∈ *topspace Y*. *Inr y* ∈ *U*} = {*y*. *Inr y* ∈ *U*}
using *subsetY* **by** *auto*
have *openin Y* {*y* ∈ *topspace Y*. *Inr y* ∈ *U*}
unfolding *eq* **using** *openU* **by** *simp*
then show *openin Y* {*y* ∈ *topspace Y*. *Inr y* ∈ *U*} **.**
qed

```
  from image preimage show ?thesis
    by (simp add: continuous-map)
qed

lemma open-map-binary-sum-inl:
  open-map X (binary-sum-topology X Y) (Inl :: 'a ⇒ 'a + 'b)
  unfolding open-map-def openin-binary-sum-topology
  using openin-subset openin-subopen by (force simp: image-iff)

lemma open-map-binary-sum-inr:
  open-map Y (binary-sum-topology X Y) (Inr :: 'b ⇒ 'a + 'b)
  unfolding open-map-def openin-binary-sum-topology
  using openin-subset openin-subopen by (force simp: image-iff)

lemma continuous-map-from-binary-sum-topology:
  assumes f: continuous-map X Z f
    and g: continuous-map Y Z g
  shows continuous-map (binary-sum-topology X Y) Z (case-sum f g)
proof −
  from f have f-image: f ‘ topspace X ⊆ topspace Z
    by (simp add: continuous-map)
  from g have g-image: g ‘ topspace Y ⊆ topspace Z
    by (simp add: continuous-map)
  have case-image:
     case-sum f g ‘ topspace (binary-sum-topology X Y) = f ‘ topspace X ∪ g ‘
topspace Y
  proof
    show case-sum f g ‘ topspace (binary-sum-topology X Y) ⊆ f ‘ topspace X ∪ g
‘ topspace Y
      by (auto simp: topspace-binary-sum-topology)
       show f ‘ topspace X ∪ g ‘ topspace Y ⊆ case-sum f g ‘ topspace
(binary-sum-topology X Y)
    proof
      fix z
      assume z ∈ f ‘ topspace X ∪ g ‘ topspace Y
      then consider
        (left) x where x ∈ topspace X z = f x
      | (right) y where y ∈ topspace Y z = g y
        by auto
      then show z ∈ case-sum f g ‘ topspace (binary-sum-topology X Y)
      proof cases
        case (left x)
        have Inl x ∈ topspace (binary-sum-topology X Y)
          using left by (auto simp: topspace-binary-sum-topology)
      then have case-sum f g (Inl x) ∈ case-sum f g ‘ topspace (binary-sum-topology
X Y)
          by (rule imageI)
        then show ?thesis
          using left by simp
```

```
      next
        case (right y)
        have Inr y ∈ topspace (binary-sum-topology X Y)
          using right by (auto simp: topspace-binary-sum-topology)
      then have case-sum f g (Inr y) ∈ case-sum f g ‘ topspace (binary-sum-topology
X Y)
          by (rule imageI)
        then show ?thesis
          using right by simp
      qed
    qed
  qed
  have image: case-sum f g ‘ topspace (binary-sum-topology X Y) ⊆ topspace Z
  proof −
    have case-subset: f ‘ topspace X ∪ g ‘ topspace Y ⊆ topspace Z
      using f-image g-image by blast
    from case-image case-subset show ?thesis
      by blast
  qed
  have preimage:
    ∀ U. openin Z U ⟶
      openin (binary-sum-topology X Y) {z ∈ topspace (binary-sum-topology X Y).
case-sum f g z ∈ U}
  proof (intro allI impI)
    fix U
    assume U: openin Z U
    from f U have openX: openin X {x ∈ topspace X. f x ∈ U}
      by (simp add: continuous-map)
    from g U have openY: openin Y {y ∈ topspace Y. g y ∈ U}
      by (simp add: continuous-map)
    let ?W = {z ∈ topspace (binary-sum-topology X Y). case-sum f g z ∈ U}
    have subsetW: ?W ⊆ topspace (binary-sum-topology X Y)
      by auto
    have eqX: {x. Inl x ∈ ?W} = {x ∈ topspace X. f x ∈ U}
      by auto
    have eqY: {y. Inr y ∈ ?W} = {y ∈ topspace Y. g y ∈ U}
      by auto
    show openin (binary-sum-topology X Y) ?W
      unfolding openin-binary-sum-topology eqX eqY
      using subsetW openX openY by simp
  qed
  from image preimage show ?thesis
    by (simp add: continuous-map)
qed

end
theory Topological-Pushout-Scaffold
  imports Binary-Sum-Topology Pushout-Scaffold
begin
```

17 Topological pushouts

The quotient-level pushout relation from *Pushout-Scaffold* becomes a genuine topological pushout here. The theory defines the quotient topology on the glued disjoint sum and proves the universal maps from the two summands into that topological pushout.

definition *pushout-topspace* ::
'a topology => 'b topology => ('c => 'a) => ('c => 'b) => ('a + 'b) set set
where
pushout-topspace X Y f g = pushout-class f g ‘ topspace (binary-sum-topology X Y)

definition *pushout-topology* ::
'a topology => 'b topology => ('c => 'a) => ('c => 'b) => (('a + 'b) set) topology
where
pushout-topology X Y f g =
topology (λU.
U ⊆ pushout-topspace X Y f g ∧
openin (binary-sum-topology X Y)
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ U})

lemma *pushout-topspace-alt*:
pushout-topspace X Y f g = pushout-inl f g ‘ topspace X ∪ pushout-inr f g ‘ topspace Y
unfolding *pushout-topspace-def pushout-inl-def pushout-inr-def topspace-binary-sum-topology*
by *auto*

lemma *pushout-topspace-subset-pushout-space*:
pushout-topspace X Y f g ⊆ pushout-space f g
unfolding *pushout-topspace-def pushout-space-def pushout-class-def quotient-space-def*
by *blast*

lemma *is-pushout-topology*:
istopology (λU.
U ⊆ pushout-topspace X Y f g ∧
openin (binary-sum-topology X Y)
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ U})
proof −
have *Int-eq*:
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ S ∩ T} =
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ S} ∩
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ T} **for** *S T*
by *auto*
have *Union-eq*:
{z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ ⋃K} =
(⋃ U∈K. {z ∈ topspace (binary-sum-topology X Y). pushout-class f g z ∈ U})

for $\mathcal{K}$
by *auto*
show *?thesis*
unfolding *istopology-def pushout-topspace-def Int-eq Union-eq*
by *blast*
qed

lemma *openin-pushout-topology*:
openin (*pushout-topology X Y f g*) *U* $\longleftrightarrow$
U $\subseteq$ *pushout-topspace X Y f g* $\wedge$
openin (*binary-sum-topology X Y*)
{*z* $\in$ *topspace* (*binary-sum-topology X Y*). *pushout-class f g z* $\in$ *U*}
proof −
have *openin* (*pushout-topology X Y f g*) *U* =
(*U* $\subseteq$ *pushout-topspace X Y f g* $\wedge$
openin (*binary-sum-topology X Y*)
{*z* $\in$ *topspace* (*binary-sum-topology X Y*). *pushout-class f g z* $\in$ *U*})
unfolding *pushout-topology-def*
by (*subst topology-inverse'*[*OF is-pushout-topology*]) *simp*
then show *?thesis*
by *simp*
qed

lemma *topspace-pushout-topology* [*simp*]:
topspace (*pushout-topology X Y f g*) = *pushout-topspace X Y f g*
proof
have *top-open*: *openin* (*pushout-topology X Y f g*) (*topspace* (*pushout-topology X Y f g*))
by *simp*
show *topspace* (*pushout-topology X Y f g*) $\subseteq$ *pushout-topspace X Y f g*
using *top-open* **unfolding** *openin-pushout-topology* **by** *blast*
have *preimage-space*:
{*z* $\in$ *topspace* (*binary-sum-topology X Y*). *pushout-class f g z* $\in$ *pushout-topspace X Y f g*} =
topspace (*binary-sum-topology X Y*)
unfolding *pushout-topspace-def* **by** *auto*
have *open-top*: *openin* (*binary-sum-topology X Y*) (*topspace* (*binary-sum-topology X Y*))
by (*rule openin-topspace*)
have *openin* (*pushout-topology X Y f g*) (*pushout-topspace X Y f g*)
unfolding *openin-pushout-topology preimage-space*
using *open-top* **by** *simp*
then show *pushout-topspace X Y f g* $\subseteq$ *topspace* (*pushout-topology X Y f g*)
by (*rule openin-subset*)
qed

lemma *quotient-map-pushout-class*:
quotient-map (*binary-sum-topology X Y*) (*pushout-topology X Y f g*) (*pushout-class f g*)

proof (*unfold quotient-map-def*, *intro conjI allI impI*)
show *pushout-class f g ‘ topspace* (*binary-sum-topology X Y*) = *topspace* (*pushout-topology X Y f g*)
by (*auto simp*: *pushout-topspace-alt pushout-inl-def pushout-inr-def image-Un*)
next
fix *U*
assume *U*: *U* ⊆ *topspace* (*pushout-topology X Y f g*)
then have *U'*: *U* ⊆ *pushout-topspace X Y f g*
by *simp*
show *openin* (*binary-sum-topology X Y*)
{*x* ∈ *topspace* (*binary-sum-topology X Y*). *pushout-class f g x* ∈ *U*} =
openin (*pushout-topology X Y f g*) *U*
using *U'* **by** (*simp add*: *openin-pushout-topology*)
qed

lemma *continuous-map-pushout-class*:
continuous-map (*binary-sum-topology X Y*) (*pushout-topology X Y f g*) (*pushout-class f g*)
using *quotient-map-pushout-class* **by** (*rule quotient-imp-continuous-map*)

lemma *pushout-inl-in-topspace*:
assumes *a* ∈ *topspace X*
shows *pushout-inl f g a* ∈ *pushout-topspace X Y f g*
using *assms*
unfolding *pushout-topspace-def pushout-inl-def topspace-binary-sum-topology*
by *auto*

lemma *pushout-inr-in-topspace*:
assumes *b* ∈ *topspace Y*
shows *pushout-inr f g b* ∈ *pushout-topspace X Y f g*
using *assms*
unfolding *pushout-topspace-def pushout-inr-def topspace-binary-sum-topology*
by *auto*

lemma *continuous-map-pushout-inl* [*continuous-intros*]:
continuous-map X (*pushout-topology X Y f g*) (*pushout-inl f g*)
proof −
have *continuous-map X* (*pushout-topology X Y f g*)
(*pushout-class f g* ∘ (*Inl* :: *'a* ⇒ *'a* + *'b*))
using *continuous-map-binary-sum-inl continuous-map-pushout-class*
by (*rule continuous-map-compose*)
then show *?thesis*
unfolding *pushout-inl-def o-def*
by (*rule continuous-map-eq*) *auto*
qed

lemma *continuous-map-pushout-inr* [*continuous-intros*]:
continuous-map Y (*pushout-topology X Y f g*) (*pushout-inr f g*)
proof −

```
  have inr-cont: continuous-map Y (binary-sum-topology X Y) (λb. Inr b)
    using continuous-map-binary-sum-inr
    by (rule continuous-map-eq) auto
  have continuous-map Y (pushout-topology X Y f g)
         (pushout-class f g ∘ (λb. Inr b))
    using inr-cont continuous-map-pushout-class
    by (rule continuous-map-compose)
  then show ?thesis
    unfolding pushout-inr-def o-def
    by (rule continuous-map-eq) auto
qed

lemma pushout-rec-pushout-class:
  assumes compat: pushout-case-compatible f g left right
    and x-in: x ∈ topspace (binary-sum-topology X Y)
  shows pushout-rec f g left right (pushout-class f g x) = case-sum left right x
proof −
  from x-in have (∃ a∈topspace X. x = Inl a) ∨ (∃ b∈topspace Y. x = Inr b)
    by (auto simp: topspace-binary-sum-topology)
  then show ?thesis
  proof
    assume ∃ a∈topspace X. x = Inl a
    then obtain a where a ∈ topspace X x = Inl a
      by blast
    then show ?thesis
      using pushout-rec-inl[OF compat, of a]
      unfolding pushout-inl-def by simp
  next
    assume ∃ b∈topspace Y. x = Inr b
    then obtain b where b ∈ topspace Y x = Inr b
      by blast
    then show ?thesis
      using pushout-rec-inr[OF compat, of b]
      unfolding pushout-inr-def by simp
  qed
qed

lemma continuous-map-pushout-rec:
  assumes compat: pushout-case-compatible f g left right
    and left-cont: continuous-map X Z left
    and right-cont: continuous-map Y Z right
  shows continuous-map (pushout-topology X Y f g) Z (pushout-rec f g left right)
proof −
   have sum-cont: continuous-map (binary-sum-topology X Y) Z (case-sum left
right)
    using left-cont right-cont by (rule continuous-map-from-binary-sum-topology)
  have comp-eq:
    ((pushout-rec f g left right) ∘ pushout-class f g) x = case-sum left right x
    if x ∈ topspace (binary-sum-topology X Y) for x
```

using *pushout-rec-pushout-class*[*OF compat that*] **by** (*simp add*: *o-def*)
have *comp-eq'*:
case-sum left right x = ((*pushout-rec f g left right*) ∘ *pushout-class f g*) *x*
if *x* ∈ *topspace* (*binary-sum-topology X Y*) **for** *x*
using *comp-eq*[*OF that*] **by** *simp*
have *comp-cont*:
continuous-map (*binary-sum-topology X Y*) *Z* ((*pushout-rec f g left right*) ∘ *pushout-class f g*)
using *sum-cont comp-eq'* **by** (*rule continuous-map-eq*)
show *?thesis*
using *quotient-map-pushout-class comp-cont* **by** (*rule continuous-compose-quotient-map*)
qed

lemma *pushout-rec-universal*:
assumes *compat*: *pushout-case-compatible f g left right*
and *left-cont*: *continuous-map X Z left*
and *right-cont*: *continuous-map Y Z right*
shows *continuous-map* (*pushout-topology X Y f g*) *Z* (*pushout-rec f g left right*)
and ⋀*a*. *pushout-rec f g left right* (*pushout-inl f g a*) = *left a*
and ⋀*b*. *pushout-rec f g left right* (*pushout-inr f g b*) = *right b*
using *assms*
by (*auto intro*: *continuous-map-pushout-rec simp*: *pushout-rec-inl pushout-rec-inr*)

end
theory *Topological-Seifert-Van-Kampen*
imports
Carrier-Amalgamated-Free-Product-Eval
Explicit-Fundamental-Group-Scaffold
Topological-Pushout-Scaffold
Seifert-Van-Kampen-Scaffold
begin

18 Concrete decode data for topological pushouts

Once the pushout topology and the carrier-side amalgamation evaluator are available, the remaining task is to package a concrete decoding map back into the fundamental group of the pushout. This theory shows how such decode data yields the abstract carrier-level Seifert–van Kampen bijection in the topological pushout setting.

lemma *loopin-image-compose* [*simp*]:
loopin-image g (*loopin-image f p*) = *loopin-image* (*g* ∘ *f*) *p*
unfolding *loopin-image-def* **by** (*rule ext*) *simp*

lemma *fundamental-groupin-map-eq*:
assumes *A-in*: *A* ∈ *fundamental-groupin-space X x*
and *f-cont*: *continuous-map X Y f*
and *g-cont*: *continuous-map X Y g*

and *fx*: $f\ x = y$
and *gx*: $g\ x = y$
and *fg*: $\bigwedge u.\ f\ u = g\ u$
shows *fundamental-groupin-map X x Y y f A = fundamental-groupin-map X x Y y g A*
proof −
have *repA*: *some-loopin X x A ∈ loopin-space X x A = loopin-class X x (some-loopin X x A)*
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *left-eq*:
fundamental-groupin-map X x Y y f A =
loopin-class Y y (loopin-image f (some-loopin X x A))
by (*rule fundamental-groupin-map-rep*[*OF A-in repA*(*1*) *repA*(*2*) *f-cont fx*])
have *right-eq*:
fundamental-groupin-map X x Y y g A =
loopin-class Y y (loopin-image g (some-loopin X x A))
by (*rule fundamental-groupin-map-rep*[*OF A-in repA*(*1*) *repA*(*2*) *g-cont gx*])
have *loopin-image f (some-loopin X x A) = loopin-image g (some-loopin X x A)*
using *fg* **by** (*simp add*: *loopin-image-def fun-eq-iff*)
then show *?thesis*
using *left-eq right-eq* **by** *simp*
qed

lemma *fundamental-groupin-map-compose*:
assumes *A-in*: $A \in$ *fundamental-groupin-space X x*
and *f-cont*: *continuous-map X Y f*
and *g-cont*: *continuous-map Y Z g*
and *fx*: $f\ x = y$
and *gy*: $g\ y = z$
shows *fundamental-groupin-map Y y Z z g (fundamental-groupin-map X x Y y f A) =*
fundamental-groupin-map X x Z z $(g \circ f)$ *A*
proof −
have *repA*: *some-loopin X x A ∈ loopin-space X x A = loopin-class X x (some-loopin X x A)*
using *some-loopin-spec*[*OF A-in*] **by** *auto*
have *mapA-in*: *fundamental-groupin-map X x Y y f A ∈ fundamental-groupin-space Y y*
by (*rule fundamental-groupin-map-in-space*[*OF A-in f-cont fx*])
have *mapA-eq*:
fundamental-groupin-map X x Y y f A =
loopin-class Y y (loopin-image f (some-loopin X x A))
by (*rule fundamental-groupin-map-rep*[*OF A-in repA*(*1*) *repA*(*2*) *f-cont fx*])
have *map-loop-in*: *loopin-image f (some-loopin X x A) ∈ loopin-space Y y*
by (*rule loopin-image-in-space*[*OF repA*(*1*) *f-cont fx*])
have *left-eq*:
fundamental-groupin-map Y y Z z g (fundamental-groupin-map X x Y y f A) =
loopin-class Z z (loopin-image g (loopin-image f (some-loopin X x A)))
by (*rule fundamental-groupin-map-rep*[*OF mapA-in map-loop-in mapA-eq g-cont*

gy])
have *comp-cont*: *continuous-map X Z* (*g* ∘ *f*)
using *f-cont g-cont* **by** (*rule continuous-map-compose*)
have *right-eq*:
fundamental-groupin-map X x Z z (*g* ∘ *f*) *A* =
loopin-class Z z (*loopin-image* (*g* ∘ *f*) (*some-loopin X x A*))
by (*rule fundamental-groupin-map-rep*[*OF A-in repA*(*1*) *repA*(*2*) *comp-cont*])
(*simp add*: *fx gy*)
show *?thesis*
using *left-eq right-eq* **by** *simp*
qed

locale *topological-svk-setup* =
fixes *X* :: *'a topology*
and *Y* :: *'b topology*
and *Z* :: *'c topology*
and *f* :: *'c* ⇒ *'a*
and *g* :: *'c* ⇒ *'b*
and *z0* :: *'c*
assumes *z0-in*: *z0* ∈ *topspace Z*
and *f-cont*: *continuous-map Z X f*
and *g-cont*: *continuous-map Z Y g*
begin

abbreviation *x0* **where** *x0* ≡ *f z0*
abbreviation *y0* **where** *y0* ≡ *g z0*
abbreviation *P* **where** *P* ≡ *pushout-topology X Y f g*
abbreviation *p0* **where** *p0* ≡ *pushout-inl f g x0*

abbreviation *G1* **where** *G1* ≡ *fundamental-groupin-space X x0*
abbreviation *G2* **where** *G2* ≡ *fundamental-groupin-space Y y0*
abbreviation *H* **where** *H* ≡ *fundamental-groupin-space Z z0*
abbreviation *mult1* **where** *mult1* ≡ *fundamental-groupin-mult X x0*
abbreviation *one1* **where** *one1* ≡ *fundamental-groupin-one X x0*
abbreviation *mult2* **where** *mult2* ≡ *fundamental-groupin-mult Y y0*
abbreviation *one2* **where** *one2* ≡ *fundamental-groupin-one Y y0*
abbreviation *multP* **where** *multP* ≡ *fundamental-groupin-mult P p0*
abbreviation *oneP* **where** *oneP* ≡ *fundamental-groupin-one P p0*

abbreviation *i1* **where** *i1* ≡ *fundamental-groupin-map Z z0 X x0 f*
abbreviation *i2* **where** *i2* ≡ *fundamental-groupin-map Z z0 Y y0 g*
abbreviation *j1* **where** *j1* ≡ *fundamental-groupin-map X x0 P p0* (*pushout-inl f g*)
abbreviation *j2* **where** *j2* ≡ *fundamental-groupin-map Y y0 P p0* (*pushout-inr f g*)

lemma *x0-in* [*simp*]: *x0* ∈ *topspace X*
using *z0-in f-cont* **unfolding** *continuous-map-def* **by** *blast*

lemma *y0-in* [*simp*]: *y0* ∈ *topspace Y*
using *z0-in g-cont* **unfolding** *continuous-map-def* **by** *blast*

lemma *pushout-basepoint-eq* [*simp*]:
pushout-inr f g y0 = *p0*
by (*simp add*: *pushout-glue*)

lemma *pushout-basepoint-in* [*simp*]:
p0 ∈ *topspace P*
by (*simp add*: *pushout-inl-in-topspace*[*OF x0-in*])

lemma *pushout-fundamental-group-maps-agree*:
assumes *h-in*: *h* ∈ *H*
shows *j1* (*i1 h*) = *j2* (*i2 h*)
proof −
have *left-comp*:
j1 (*i1 h*) = *fundamental-groupin-map Z z0 P p0* ((*pushout-inl f g*) ∘ *f*) *h*
by (*simp add*: *fundamental-groupin-map-compose*[*OF h-in f-cont continuous-map-pushout-inl*])
have *right-comp*:
j2 (*i2 h*) = *fundamental-groupin-map Z z0 P p0* ((*pushout-inr f g*) ∘ *g*) *h*
by (*simp add*: *fundamental-groupin-map-compose*[*OF h-in g-cont continuous-map-pushout-inr*])
have *left-cont*: *continuous-map Z P* ((*pushout-inl f g*) ∘ *f*)
by (*rule continuous-map-compose*[*OF f-cont continuous-map-pushout-inl*])
have *right-cont*: *continuous-map Z P* ((*pushout-inr f g*) ∘ *g*)
by (*rule continuous-map-compose*[*OF g-cont continuous-map-pushout-inr*])
have *left-z0*: ((*pushout-inl f g*) ∘ *f*) *z0* = *p0*
by *simp*
have *right-z0*: ((*pushout-inr f g*) ∘ *g*) *z0* = *p0*
by *simp*
have *comp-fun-eq*: ((*pushout-inl f g*) ∘ *f*) *u* = ((*pushout-inr f g*) ∘ *g*) *u* **for** *u*
by (*simp add*: *pushout-glue*)
have *comp-eq*:
fundamental-groupin-map Z z0 P p0 ((*pushout-inl f g*) ∘ *f*) *h* =
fundamental-groupin-map Z z0 P p0 ((*pushout-inr f g*) ∘ *g*) *h*
by (*rule fundamental-groupin-map-eq*[*OF h-in left-cont right-cont left-z0 right-z0 comp-fun-eq*])
show *?thesis*
using *left-comp right-comp comp-eq* **by** *simp*
qed

lemma *i1-in-G1*:
assumes *h* ∈ *H*
shows *i1 h* ∈ *G1*
by (*rule fundamental-groupin-map-in-space*[*OF assms f-cont*]) *simp*

lemma *i2-in-G2*:
assumes *h* ∈ *H*

shows *i2 h ∈ G2*
by (*rule fundamental-groupin-map-in-space*[*OF assms g-cont*]) *simp*

lemma *decode-locale*:
carrier-full-amalg-word-eval
G1 mult1 one1 (*fundamental-groupin-inv X x0*)
G2 mult2 one2 (*fundamental-groupin-inv Y y0*)
H i1 i2
(*fundamental-groupin-space P p0*) *multP oneP* (*fundamental-groupin-inv P p0*)
j1 j2
proof (*rule carrier-full-amalg-word-eval.intro*)
show *carrier-group*
(*fundamental-groupin-space X x0*)
(*fundamental-groupin-mult X x0*)
(*fundamental-groupin-one X x0*)
(*fundamental-groupin-inv X x0*)
by (*rule fundamental-groupin-carrier-group*[*OF x0-in*])
show *carrier-group*
(*fundamental-groupin-space Y y0*)
(*fundamental-groupin-mult Y y0*)
(*fundamental-groupin-one Y y0*)
(*fundamental-groupin-inv Y y0*)
by (*rule fundamental-groupin-carrier-group*[*OF y0-in*])
show *carrier-group*
(*fundamental-groupin-space P p0*) *multP oneP* (*fundamental-groupin-inv P p0*)
by (*rule fundamental-groupin-carrier-group*[*OF pushout-basepoint-in*])
show *carrier-group-hom*
(*fundamental-groupin-space X x0*)
(*fundamental-groupin-mult X x0*)
(*fundamental-groupin-one X x0*)
(*fundamental-groupin-inv X x0*)
(*fundamental-groupin-space P p0*) *multP oneP* (*fundamental-groupin-inv P p0*)
(*fundamental-groupin-map X x0 P p0* (*pushout-inl f g*))
by (*rule fundamental-groupin-map-carrier-group-hom*[*OF x0-in continuous-map-pushout-inl*]) *simp*
show *carrier-group-hom*
(*fundamental-groupin-space Y y0*)
(*fundamental-groupin-mult Y y0*)
(*fundamental-groupin-one Y y0*)
(*fundamental-groupin-inv Y y0*)
(*fundamental-groupin-space P p0*) *multP oneP* (*fundamental-groupin-inv P p0*)
(*fundamental-groupin-map Y y0 P p0* (*pushout-inr f g*))
by (*rule fundamental-groupin-map-carrier-group-hom*[*OF y0-in continuous-map-pushout-inr*]) *simp*
show *carrier-full-amalg-word-eval-axioms G1 G2 H i1 i2 j1 j2*
proof (*rule carrier-full-amalg-word-eval-axioms.intro*)

show $h \in H \Longrightarrow i1\ h \in G1$ **for** h
by (*rule i1-in-G1*)
show $h \in H \Longrightarrow i2\ h \in G2$ **for** h
by (*rule i2-in-G2*)
show $h \in H \Longrightarrow j1\ (i1\ h) = j2\ (i2\ h)$ **for** h
by (*rule pushout-fundamental-group-maps-agree*)
qed
qed

interpretation *decode*:
carrier-full-amalg-word-eval
G1 mult1 one1 fundamental-groupin-inv X x0
G2 mult2 one2 fundamental-groupin-inv Y y0
H i1 i2
fundamental-groupin-space P p0 multP oneP fundamental-groupin-inv P p0
j1 j2
by (*rule decode-locale*)

abbreviation *svk-word-eval* **where** *svk-word-eval* ≡ *decode.carrier-full-amalg-eval*
abbreviation *svk-decode* **where** *svk-decode* ≡ *decode.carrier-full-amalg-decode*

lemma *svk-decode-in-space*:
svk-decode w ∈ *fundamental-groupin-space P p0*
by (*rule decode.carrier-full-amalg-decode-in-carrier*)

lemma *svk-decode-respects*:
assumes *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v*
shows *svk-decode u* = *svk-decode v*
using *assms* **by** (*rule decode.carrier-full-amalg-decode-respects*)

lemma *svk-decode-eq-eval*:
assumes *fpw-in-space G1 G2 w*
shows *svk-decode w* = *svk-word-eval w*
using *assms* **by** (*rule decode.carrier-full-amalg-decode-eq-eval*)

end

locale *topological-svk-open-cover* =
topological-svk-setup X Y Z f g z0
for $X :: 'a\ topology$
and $Y :: 'b\ topology$
and $Z :: 'c\ topology$
and $f :: 'c \Rightarrow 'a$
and $g :: 'c \Rightarrow 'b$
and $z0 :: 'c +$
assumes *left-open*: *openin P* (*pushout-inl f g* ‘ *topspace X*)
and *right-open*: *openin P* (*pushout-inr f g* ‘ *topspace Y*)
begin

lemma *loopin-subdivision-by-summands*:
assumes *p-loop*: $p \in$ *loopin-space P p0*
shows $\exists\, n{::}nat.\ 0 < n \wedge$
$(\forall\, i{<}n.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\}$
$\subseteq$ *pushout-inl f g* ‘ *topspace X* $\vee$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\}$
$\subseteq$ *pushout-inr f g* ‘ *topspace Y*)
proof −
let *?L* = *pushout-inl f g* ‘ *topspace X*
let *?R* = *pushout-inr f g* ‘ *topspace Y*
from *p-loop* **obtain** *p-path* **where** *p-path*: *pathin P p*
by (*rule loopin-spaceE*)
have *p-img*: *p* ‘ $\{0..1\} \subseteq$ *topspace P*
by (*rule pathin-image-subset-topspace*[*OF p-path*])
have *cover*: *p* ‘ $\{0..1\} \subseteq$ *?L* $\cup$ *?R*
using *p-img* **by** (*auto simp*: *pushout-topspace-alt*)
have *cover′*: *p* ‘ $\{0..1\} \subseteq \bigcup \{$*?L*, *?R*$\}$
using *cover* **by** *auto*
have *open-cover*: *openin P U* **if** $U \in \{$*?L*, *?R*$\}$ **for** *U*
using *that left-open right-open* **by** *auto*
have *subdiv*:
$\exists\, n{::}nat.\ 0 < n \wedge$
$(\forall\, i{<}n.\ \exists\, U{\in}\{$*?L*, *?R*$\}.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U$)
by (*rule pathin-subdivision-open-cover*[*OF p-path cover′ open-cover*])
then obtain n :: *nat* **where** *n-pos*: $0 < n$
and *n-cover*:
$\forall\, i{<}n.\ \exists\, U{\in}\{$*?L*, *?R*$\}.$
subpathin (*real i* / *real n*) (*real* (*Suc i*) / *real n*) *p* ‘ $\{0..1\} \subseteq U$
by *blast*
show *?thesis*
using *n-pos n-cover* **by** *auto*
qed

end

locale *topological-svk* =
topological-svk-setup X Y Z f g z0
for X :: *′a topology*
and Y :: *′b topology*
and Z :: *′c topology*
and f :: $'c \Rightarrow 'a$
and g :: $'c \Rightarrow 'b$
and *z0* :: *′c* +
fixes *encode*
assumes *encode-in-space*: *fpw-in-space G1 G2* (*encode A*)
and *decode-encode*: *svk-decode* (*encode A*) = *A*
and *encode-decode*:

carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 (*encode* (*svk-decode w*)) *w*
begin

lemma *svk-locale*:
carrier-svk-encode-decode-interface
G1 G2 H i1 i2 mult1 one1 mult2 one2 encode svk-decode
proof (*rule carrier-svk-encode-decode-interface.intro*)
show *fpw-in-space G1 G2* (*encode x*) **for** *x*
by (*fact encode-in-space*)
show *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2 u v* ⟹ *svk-decode u* = *svk-decode v* **for** *u v*
by (*rule svk-decode-respects*)
show *svk-decode* (*encode x*) = *x* **for** *x*
by (*fact decode-encode*)
show *carrier-full-amalg-equiv G1 G2 H i1 i2 mult1 one1 mult2 one2* (*encode* (*svk-decode w*)) *w* **for** *w*
by (*fact encode-decode*)
qed

interpretation *svk*: *carrier-svk-encode-decode-interface*
G1 G2 H i1 i2 mult1 one1 mult2 one2 encode svk-decode
by (*rule svk-locale*)

The theorem below isolates the topological conclusion once an encoding map has been supplied and the usual round-trip laws have been verified. The classical open-union theorem later instantiates this interface by constructing that encoding from loop partitions subordinate to U and V.

theorem *topological-seifert-van-kampen-bij-betw*:
bij-betw svk.carrier-svk-quotient-map UNIV
(*carrier-full-amalgamated-free-product-space G1 G2 H i1 i2 mult1 one1 mult2 one2*)
by (*rule svk.carrier-seifert-van-kampen-bij-betw*)

end

end
theory *Seifert-Van-Kampen*
imports
Classical-Seifert-Van-Kampen
Topological-Seifert-Van-Kampen
begin

19 Top-level entry point

This session formalizes a quotient-oriented version of the classical Seifert–van Kampen theorem in Isabelle/HOL. It develops reusable infrastructure for pushout glue relations, free-product words, carrier-based amalgamated

free products, and explicit path/homotopy quotients. The unconditional classical open-union theorem is proved in *Classical-Seifert-Van-Kampen*.

Auxiliary theories package the abstract encode/decode interface and the pushout-level constructions used internally by the proof. The headline theorem exported by this entry is the classical result for open unions.

end

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