Geometric Axioms for Minkowski Spacetime

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Abstract

This is a formalisation of Schutz' system of axioms for Minkowski spacetime [1], as well as the results in his third chapter ("Temporal Order on a Path"), with the exception of the second part of Theorem 12. Many results are proven here that cannot be found in Schutz, either preceding the theorem they are needed for, or in their own thematic section.

Contents

theory *TernaryOrdering*
imports *Util*

begin

Definition of chains using an ordering on sets of events based on natural numbers, plus some proofs.

1 Totally ordered chains

Based on page 110 of Phil Scott's thesis and the following HOL Light definition:

```
let ORDERING = new_definition
  `ORDERING f X <=> (!n. (FINITE X ==> n < CARD X) ==> f n IN X)
                 /\ (!x. x IN X ==> ?n. (FINITE X ==> n < CARD X)
                     /\ f n = x)
                 /\ !n n' n''. (FINITE X ==> n'' < CARD X)
                       /\ n < n' /\ n' < n''
                       ==> between (f n) (f n') (f n'')`;;
```

I've made it strict for simplicity, and because that's how Schutz's ordering is. It could be made more generic by taking in the function corresponding to $<$ as a paramater. Main difference to Schutz: he has local order, not total (cf Theorem 2 and *local-ordering*).

definition *ordering* :: $(nat \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where
$ordering\ f\ ord\ X \equiv (\forall n.\ (finite\ X \longrightarrow n < card\ X) \longrightarrow f\ n \in X)$
$\wedge\ (\forall x{\in}X.\ (\exists n.\ (finite\ X \longrightarrow n < card\ X) \wedge f\ n = x))$
$\wedge\ (\forall n\ n'\ n''.\ (finite\ X \longrightarrow n'' < card\ X) \wedge n < n' \wedge n' < n''$
$\longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n''))$

lemma *finite-ordering-intro*:
assumes *finite X*
and $\forall n < card\ X.\ f\ n \in X$
and $\forall x \in X.\ \exists n < card\ X.\ f\ n = x$
and $\forall n\ n'\ n''.\ n < n' \wedge n' < n'' \wedge n'' < card\ X \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n'')$
shows *ordering f ord X*
unfolding *ordering-def* **by** (*simp add*: *assms*)

lemma *infinite-ordering-intro*:
assumes *infinite X*
and $\forall n{::}nat.\ f\ n \in X$
and $\forall x \in X.\ \exists n{::}nat.\ f\ n = x$

and $\forall\, n\ n'\ n''.\ n < n' \wedge n' < n'' \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n'')$
shows *ordering f ord X*
unfolding *ordering-def* **by** (*simp add*: *assms*)

lemma *ordering-ord-ijk*:
assumes *ordering f ord X*
and $i < j \wedge j < k \wedge (finite\ X \longrightarrow k < card\ X)$
shows *ord* (*f i*) (*f j*) (*f k*)
by (*metis ordering-def assms*)

lemma *empty-ordering* [*simp*]: $\exists f.\ ordering\ f\ ord\ \{\}$
by (*simp add*: *ordering-def*)

lemma *singleton-ordering* [*simp*]: $\exists f.\ ordering\ f\ ord\ \{a\}$
apply (*rule-tac* $x = \lambda n.\ a$ **in** *exI*)
by (*simp add*: *ordering-def*)

lemma *two-ordering* [*simp*]: $\exists f.\ ordering\ f\ ord\ \{a,\ b\}$
proof *cases*
assume $a = b$
thus *?thesis* **using** *singleton-ordering* **by** *simp*
next
assume *a-neq-b*: $a \neq b$
let *?f* = $\lambda n.$ *if n = 0 then a else b*
have *ordering1*: $(\forall\, n.\ (finite\ \{a,b\} \longrightarrow n < card\ \{a,b\}) \longrightarrow ?f\ n \in \{a,b\})$ **by** *simp*
have *local-ordering*: $(\forall\, x{\in}\{a,b\}.\ \exists\, n.\ (finite\ \{a,b\} \longrightarrow n < card\ \{a,b\}) \wedge ?f\ n = x)$
using *a-neq-b all-not-in-conv card-Suc-eq card-0-eq card-gt-0-iff insert-iff lessI* **by** *auto*
have *ordering3*: $(\forall\, n\ n'\ n''.\ (finite\ \{a,b\} \longrightarrow n'' < card\ \{a,b\}) \wedge n < n' \wedge n' < n''$
$\longrightarrow ord\ (?f\ n)\ (?f\ n')\ (?f\ n''))$ **using** *a-neq-b* **by** *auto*
have *ordering ?f ord* $\{a,\ b\}$ **using** *ordering-def ordering1 local-ordering ordering3* **by** *blast*
thus *?thesis* **by** *auto*
qed

lemma *card-le2-ordering*:
assumes *finiteX*: *finite X*
and *card-le2*: *card X* $\leq$ *2*
shows $\exists f.$ *ordering f ord X*
proof −
have *card012*: *card X = 0* $\vee$ *card X = 1* $\vee$ *card X = 2* **using** *card-le2* **by** *auto*
have *card0*: *card X = 0* $\longrightarrow$ *?thesis* **using** *finiteX* **by** *simp*
have *card1*: *card X = 1* $\longrightarrow$ *?thesis* **using** *card-eq-SucD* **by** *fastforce*
have *card2*: *card X = 2* $\longrightarrow$ *?thesis* **by** (*metis two-ordering card-eq-SucD numeral-2-eq-2*)
thus *?thesis* **using** *card012 card0 card1 card2* **by** *auto*

qed

lemma *ord-ordered*:
assumes *abc*: *ord a b c*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$
shows $\exists f$. *ordering f ord* $\{a,b,c\}$
apply (*rule-tac x* = λn. *if n* = *0 then a else if n* = *1 then b else c* **in** *exI*)
apply (*unfold ordering-def*)
using *abc abc-neq* **by** *auto*

lemma *overlap-ordering*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abd*: *ord a b d*
and *acd*: *ord a c d*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
shows $\exists f$. *ordering f ord* $\{a,b,c,d\}$
proof −
let *?X* = $\{a,b,c,d\}$
let *?f* = λn. *if n* = *0 then a else if n* = *1 then b else if n* = *2 then c else d*
have *card4*: *card ?X* = *4* **using** *abc bcd abd abc-neq* **by** *simp*
have *ordering1*: $\forall n$. (*finite ?X* $\longrightarrow$ *n* < *card ?X*) $\longrightarrow$ *?f n* $\in$ *?X* **by** *simp*
have *local-ordering*: $\forall x \in ?X$. $\exists n$. (*finite ?X* $\longrightarrow$ *n* < *card ?X*) $\wedge$ *?f n* = *x*
by (*metis card4 One-nat-def Suc-1 Suc-lessI empty-iff insertE numeral-3-eq-3 numeral-eq-iff*
numeral-eq-one-iff rel-simps(*51*) *semiring-norm*(*85*) *semiring-norm*(*86*) *semiring-norm*(*87*)
semiring-norm(*89*) *zero-neq-numeral*)
have *ordering3*: $(\forall n\ n'\ n''.$ (*finite ?X* $\longrightarrow$ *n''* < *card ?X*) $\wedge$ *n* < *n'* $\wedge$ *n'* < *n''*
$\longrightarrow$ *ord* (*?f n*) (*?f n'*) (*?f n''*))
using *card4 abc bcd abd acd card-0-eq card-insert-if finite.emptyI finite-insert less-antisym*
less-one less-trans-Suc not-less-eq not-one-less-zero numeral-2-eq-2 **by** *auto*
have *ordering ?f ord ?X* **using** *ordering1 local-ordering ordering3 ordering-def* **by** *blast*
thus *?thesis* **by** *auto*
qed

lemma *overlap-ordering-alt1*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abc-bcd-abd*: $\forall$ *a b c d*. *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a b d*
and *abc-bcd-acd*: $\forall$ *a b c d*. *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a c d*
and *ord-distinct*: $\forall a\ b\ c$. (*ord a b c* $\longrightarrow$ $a \neq b \wedge a \neq c \wedge b \neq c$)
shows $\exists f$. *ordering f ord* $\{a,b,c,d\}$
by (*metis* (*full-types*) *assms overlap-ordering*)

lemma *overlap-ordering-alt2*:
assumes *abc*: *ord a b c*

and *bcd*: *ord b c d*
and *abd*: *ord a b d*
and *acd*: *ord a c d*
and *ord-distinct*: $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \wedge a \neq c \wedge b \neq c)$
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$
by (*metis assms overlap-ordering*)

lemma *overlap-ordering-alt*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abc-bcd-abd*: $\forall\ a\ b\ c\ d.\ ord\ a\ b\ c \wedge ord\ b\ c\ d \longrightarrow ord\ a\ b\ d$
and *abc-bcd-acd*: $\forall\ a\ b\ c\ d.\ ord\ a\ b\ c \wedge ord\ b\ c\ d \longrightarrow ord\ a\ c\ d$
and *abc-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$
by (*meson assms overlap-ordering*)

The lemmas below are easy to prove for $X = \{\}$, and if I included that case then I would have to write a conditional definition in place of $\{0..|X| - 1\}$.

lemma *finite-ordering-img*: $[\![X \neq \{\};$ *finite X*; *ordering f ord X*$]\!] \Longrightarrow f\ `\ \{0..card\ X - 1\} = X$
by (*force simp add*: *ordering-def image-def*)

lemma *inf-ordering-img*: $[\![$*infinite X*; *ordering f ord X*$]\!] \Longrightarrow f\ `\ \{0..\} = X$
by (*auto simp add*: *ordering-def image-def*)

lemma *inf-ordering-inv-img*: $[\![$*infinite X*; *ordering f ord X*$]\!] \Longrightarrow f\ -`\ X = \{0..\}$
by (*auto simp add*: *ordering-def image-def*)

lemma *inf-ordering-img-inv-img*: $[\![$*infinite X*; *ordering f ord X*$]\!] \Longrightarrow f\ `\ f\ -`\ X = X$
using *inf-ordering-img* **by** *auto*

lemma *finite-ordering-inj-on*: $[\![$*finite X*; *ordering f ord X*$]\!] \Longrightarrow$ *inj-on f* $\{0..card\ X - 1\}$
by (*metis finite-ordering-img Suc-diff-1 atLeastAtMost-iff card-atLeastAtMost card-eq-0-iff*
diff-0-eq-0 diff-zero eq-card-imp-inj-on gr0I inj-onI le-0-eq)

lemma *finite-ordering-bij*:
assumes *orderingX*: *ordering f ord X*
and *finiteX*: *finite X*
and *non-empty*: $X \neq \{\}$
shows *bij-betw f* $\{0..card\ X - 1\}$ *X*
proof −
have *f-image*: $f\ `\ \{0..card\ X - 1\} = X$ **by** (*metis orderingX finiteX finite-ordering-img non-empty*)
thus *?thesis* **by** (*metis inj-on-imp-bij-betw orderingX finiteX finite-ordering-inj-on*)

qed

lemma *inf-ordering-inj'*:
assumes *infX*: *infinite X*
and *f-ord*: *ordering f ord X*
and *ord-distinct*: $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \wedge a \neq c \wedge b \neq c)$
and *f-eq*: $f\ m = f\ n$
shows $m = n$
proof (*rule ccontr*)
assume *m-not-n*: $m \neq n$
have *betw-3n*: $\forall\, n\ n'\ n''.\ n < n' \wedge n' < n'' \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n'')$
using *f-ord* **by** (*simp add*: *ordering-def infX*)
thus *False*
proof *cases*
assume *m-less-n*: $m < n$
then obtain k **where** $n < k$ **by** *auto*
then have *ord* $(f\ m)\ (f\ n)\ (f\ k)$ **using** *m-less-n betw-3n* **by** *simp*
then have $f\ m \neq f\ n$ **using** *ord-distinct* **by** *simp*
thus *?thesis* **using** *f-eq* **by** *simp*
next
assume $\neg\ m < n$
then have *n-less-m*: $n < m$ **using** *m-not-n* **by** *simp*
then obtain k **where** $m < k$ **by** *auto*
then have *ord* $(f\ n)\ (f\ m)\ (f\ k)$ **using** *n-less-m betw-3n* **by** *simp*
then have $f\ n \neq f\ m$ **using** *ord-distinct* **by** *simp*
thus *?thesis* **using** *f-eq* **by** *simp*
qed
qed

lemma *inf-ordering-inj*:
assumes *infinite X*
and *ordering f ord X*
and $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \wedge a \neq c \wedge b \neq c)$
shows *inj f*
using *inf-ordering-inj' assms* **by** (*metis injI*)

The finite case is a little more difficult as I can't just choose some other natural number to form the third part of the betweenness relation and the initial simplification isn't as nice. Note that I cannot prove *inj f* (over the whole type that f is defined on, i.e. natural numbers), because I need to capture the m and n that obey specific requirements for the finite case. In order to prove *inj f*, I would have to extend the definition for ordering to include m and n beyond *card X*, such that it is still injective. That would probably not be very useful.

lemma *finite-ordering-inj*:
assumes *finiteX*: *finite X*
and *f-ord*: *ordering f ord X*
and *ord-distinct*: $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \wedge a \neq c \wedge b \neq c)$
and *m-less-card*: $m <$ *card X*

```
    and n-less-card: n < card X
    and f-eq: f m = f n
  shows m = n
proof (rule ccontr)
  assume m-not-n: m ≠ n
  have surj-f: ∀ x∈X. ∃ n<card X. f n = x
          using f-ord by (simp add: ordering-def finiteX)
  have betw-3n: ∀ n n' n''. n'' < card X ∧ n < n' ∧ n' < n'' ⟶ ord (f n) (f n')
(f n'')
          using f-ord by (simp add: ordering-def)
  show False
  proof cases
    assume card-le2: card X ≤ 2
    have card0: card X = 0 ⟶ False using m-less-card by simp
    have card1: card X = 1 ⟶ False using m-less-card n-less-card m-not-n by
simp
    have card2: card X = 2 ⟶ False
    proof (rule impI)
      assume card-is-2: card X = 2
      then have mn01: m = 0 ∧ n = 1 ∨ n = 0 ∧ m = 1 using m-less-card
n-less-card m-not-n by auto
    then have f m ≠ f n using card-is-2 surj-f One-nat-def card-eq-SucD insertCI
                              less-2-cases numeral-2-eq-2 by (metis (no-types, lifting))
      thus False using f-eq by simp
    qed
    show False using card0 card1 card2 card-le2 by simp
  next
    assume ¬ card X ≤ 2
    then have card-ge3: card X ≥ 3 by simp
    thus False
    proof cases
      assume m-less-n: m < n
    then obtain k where k-pos: k < m ∨ (m < k ∧ k < n) ∨ (n < k ∧ k < card
X)
        using is-free-nat m-less-n n-less-card card-ge3 by blast
      have k1: k < m ⟶ord (f k) (f m) (f n) using m-less-n n-less-card betw-3n
by simp
      have k2: m < k ∧ k < n ⟶ ord (f m) (f k) (f n) using m-less-n n-less-card
betw-3n by simp
    have k3: n < k ∧ k < card X ⟶ ord (f m) (f n) (f k) using m-less-n betw-3n
by simp
      have f m ≠ f n using k1 k2 k3 k-pos ord-distinct by auto
      thus False using f-eq by simp
    next
      assume ¬ m < n
      then have n-less-m: n < m using m-not-n by simp
    then obtain k where k-pos: k < n ∨ (n < k ∧ k < m) ∨ (m < k ∧ k < card
X)
        using is-free-nat n-less-m m-less-card card-ge3 by blast
```

have *k1*: $k < n \longrightarrow ord\ (f\ k)\ (f\ n)\ (f\ m)$ **using** *n-less-m m-less-card betw-3n* **by** *simp*
have *k2*: $n < k \wedge k < m \longrightarrow ord\ (f\ n)\ (f\ k)\ (f\ m)$ **using** *n-less-m m-less-card betw-3n* **by** *simp*
have *k3*: $m < k \wedge k < card\ X \longrightarrow ord\ (f\ n)\ (f\ m)\ (f\ k)$ **using** *n-less-m betw-3n* **by** *simp*
have $f\ n \neq f\ m$ **using** *k1 k2 k3 k-pos ord-distinct* **by** *auto*
thus *False* **using** *f-eq* **by** *simp*
qed
qed
qed

lemma *ordering-inj*:
assumes *ordering f ord X*
and $\forall\ a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \wedge a \neq c \wedge b \neq c)$
and *finite X* $\longrightarrow m < card\ X$
and *finite X* $\longrightarrow n < card\ X$
and $f\ m = f\ n$
shows $m = n$
using *inf-ordering-inj' finite-ordering-inj assms* **by** *blast*

lemma *ordering-sym*:
assumes *ord-sym*: $\bigwedge a\ b\ c.\ ord\ a\ b\ c \Longrightarrow ord\ c\ b\ a$
and *finite X*
and *ordering f ord X*
shows *ordering* $(\lambda n.\ f\ (card\ X - 1 - n))\ ord\ X$
unfolding *ordering-def* **using** *assms(2)*
apply *auto*
apply (*metis ordering-def assms(3) card-0-eq card-gt-0-iff diff-Suc-less gr-implies-not0*)
proof −
fix *x*
assume *finite X*
assume $x \in X$
obtain *n* **where** *finite X* $\longrightarrow n < card\ X$ **and** $f\ n = x$
by (*metis ordering-def* ‹$x \in X$› *assms(3)*)
have $f\ (card\ X - ((card\ X - 1 - n) + 1)) = x$
by (*simp add*: *Suc-leI* ‹$f\ n = x$› ‹*finite X* $\longrightarrow n < card\ X$› *assms(2)*)
thus $\exists\ n{<}card\ X.\ f\ (card\ X - Suc\ n) = x$
by (*metis* ‹$x \in X$› *add.commute assms(2) card-Diff-singleton card-Suc-Diff1 diff-less-Suc plus-1-eq-Suc*)
next
fix $n\ n'\ n''$
assume *finite X*
assume $n'' < card\ X\ n < n'\ n' < n''$
have $ord\ (f\ (card\ X - Suc\ n''))\ (f\ (card\ X - Suc\ n'))\ (f\ (card\ X - Suc\ n))$
using *assms(3)* **unfolding** *ordering-def*
using ‹$n < n'$› ‹$n' < n''$› ‹$n'' < card\ X$› *diff-less-mono2* **by** *auto*
thus $ord\ (f\ (card\ X - Suc\ n))\ (f\ (card\ X - Suc\ n'))\ (f\ (card\ X - Suc\ n''))$
using *ord-sym* **by** *blast*

qed

lemma *zero-into-ordering*:
assumes *ordering f betw X*
and $X \neq \{\}$
shows $(f\ 0) \in X$
using *ordering-def*
by (*metis assms card-eq-0-iff gr-implies-not0 linorder-neqE-nat*)

2 Locally ordered chains

Definitions for Schutz-like chains, with local order only.

definition *local-ordering* :: $(nat \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where *local-ordering f ord X*
$\equiv (\forall n.\ (finite\ X \longrightarrow n < card\ X) \longrightarrow f\ n \in X) \wedge$
$(\forall x{\in}X.\ \exists n.\ (finite\ X \longrightarrow n < card\ X) \wedge f\ n = x) \wedge$
$(\forall n.\ (finite\ X \longrightarrow Suc\ (Suc\ n) < card\ X) \longrightarrow ord\ (f\ n)\ (f\ (Suc\ n))\ (f\ (Suc\ (Suc\ n))))$

lemma *finite-local-ordering-intro*:
assumes *finite X*
and $\forall n < card\ X.\ f\ n \in X$
and $\forall x \in X.\ \exists n < card\ X.\ f\ n = x$
and $\forall n\ n'\ n''.\ Suc\ n = n' \wedge Suc\ n' = n'' \wedge n'' < card\ X \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n'')$
shows *local-ordering f ord X*
unfolding *local-ordering-def* **by** (*simp add*: *assms*)

lemma *infinite-local-ordering-intro*:
assumes *infinite X*
and $\forall n{::}nat.\ f\ n \in X$
and $\forall x \in X.\ \exists n{::}nat.\ f\ n = x$
and $\forall n\ n'\ n''.\ Suc\ n = n' \wedge Suc\ n' = n'' \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n'')$
shows *local-ordering f ord X*
using *assms* **unfolding** *local-ordering-def* **by** *metis*

lemma *total-implies-local*:
ordering f ord X $\Longrightarrow$ *local-ordering f ord X*
unfolding *ordering-def local-ordering-def*
using *lessI* **by** *presburger*

lemma *ordering-ord-ijk-loc*:
assumes *local-ordering f ord X*
and *finite X* $\longrightarrow$ *Suc* (*Suc i*) $<$ *card X*
shows *ord* (*f i*) (*f* (*Suc i*)) (*f* (*Suc* (*Suc i*)))
by (*metis local-ordering-def assms*)

lemma *empty-ordering-loc* [*simp*]:

$\exists f.$ *local-ordering f ord* $\{\}$
by (*simp add*: *local-ordering-def*)

lemma *singleton-ordered-loc* [*simp*]:
local-ordering f ord $\{f\ 0\}$
unfolding *local-ordering-def* **by** *simp*

lemma *singleton-ordering-loc* [*simp*]:
$\exists f.$ *local-ordering f ord* $\{a\}$
using *singleton-ordered-loc* **by** *fast*

lemma *two-ordered-loc*:
assumes $a = f\ 0$ **and** $b = f\ 1$
shows *local-ordering f ord* $\{a, b\}$
proof *cases*
assume $a = b$
thus *?thesis* **using** *assms singleton-ordered-loc* **by** (*metis insert-absorb2*)
next
assume *a-neq-b*: $a \neq b$
hence $(\forall n.\ (finite\ \{a,b\} \longrightarrow n < card\ \{a,b\}) \longrightarrow f\ n \in \{a,b\})$
using *assms* **by** (*metis One-nat-def card.infinite card-2-iff fact-0 fact-2 insert-iff less-2-cases-iff*)
moreover have $(\forall x{\in}\{a,b\}.\ \exists n.\ (finite\ \{a,b\} \longrightarrow n < card\ \{a,b\}) \wedge f\ n = x)$
using *assms a-neq-b all-not-in-conv card-Suc-eq card-0-eq card-gt-0-iff insert-iff lessI* **by** *auto*
moreover have $(\forall n.\ (finite\ \{a,b\} \longrightarrow Suc\ (Suc\ n) < card\ \{a,b\})$
$\longrightarrow ord\ (f\ n)\ (f\ (Suc\ n))\ (f\ (Suc\ (Suc\ n))))$
using *a-neq-b* **by** *auto*
ultimately have *local-ordering f ord* $\{a, b\}$
using *local-ordering-def* **by** *blast*
thus *?thesis* **by** *auto*
qed

lemma *two-ordering-loc* [*simp*]:
$\exists f.$ *local-ordering f ord* $\{a, b\}$
using *total-implies-local two-ordering* **by** *fastforce*

lemma *card-le2-ordering-loc*:
assumes *finiteX*: *finite X*
and *card-le2*: *card* $X \leq 2$
shows $\exists f.$ *local-ordering f ord X*
using *assms total-implies-local card-le2-ordering* **by** *metis*

lemma *ord-ordered-loc*:
assumes *abc*: *ord a b c*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$
shows $\exists f.$ *local-ordering f ord* $\{a,b,c\}$
using *assms total-implies-local ord-ordered* **by** *metis*

lemma *overlap-ordering-loc*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abd*: *ord a b d*
and *acd*: *ord a c d*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
shows $\exists f$. *local-ordering f ord* $\{a,b,c,d\}$
using *overlap-ordering*[*OF assms*] *total-implies-local* **by** *blast*

lemma *ordering-sym-loc*:
assumes *ord-sym*: $\bigwedge a\ b\ c$. *ord a b c* $\Longrightarrow$ *ord c b a*
and *finite X*
and *local-ordering f ord X*
shows *local-ordering* (λn. *f* (*card X* − *1* − *n*)) *ord X*
unfolding *local-ordering-def* **using** *assms*(*2*) **apply** *auto*
apply (*metis local-ordering-def assms*(*3*) *card-0-eq card-gt-0-iff diff-Suc-less gr-implies-not0*)
proof −
fix *x*
assume *finite X*
assume $x \in X$
obtain *n* **where** *finite X* $\longrightarrow$ $n <$ *card X* **and** *f n* = *x*
by (*metis local-ordering-def* ‹$x \in X$› *assms*(*3*))
have *f* (*card X* − ((*card X* − *1* − *n*) + *1*)) = *x*
by (*simp add*: *Suc-leI* ‹*f n* = *x*› ‹*finite X* $\longrightarrow$ $n <$ *card X*› *assms*(*2*))
thus $\exists n<$*card X*. *f* (*card X* − *Suc n*) = *x*
by (*metis* ‹$x \in X$› *add.commute assms*(*2*) *card-Diff-singleton card-Suc-Diff1*
diff-less-Suc plus-1-eq-Suc)
next
fix *n*
let *?n1* = *Suc n*
let *?n2* = *Suc ?n1*
assume *finite X*
assume *Suc* (*Suc n*) < *card X*
have *ord* (*f* (*card X* − *Suc ?n2*)) (*f* (*card X* − *Suc ?n1*)) (*f* (*card X* − *Suc n*))
using *assms*(*3*) **unfolding** *local-ordering-def*
using ‹*Suc* (*Suc n*) < *card X*› **by** (*metis*
Suc-diff-Suc Suc-lessD card-eq-0-iff card-gt-0-iff diff-less gr-implies-not0
zero-less-Suc)
thus *ord* (*f* (*card X* − *Suc n*)) (*f* (*card X* − *Suc ?n1*)) (*f* (*card X* − *Suc ?n2*))
using *ord-sym* **by** *blast*
qed

lemma *zero-into-ordering-loc*:
assumes *local-ordering f betw X*
and $X \neq \{\}$
shows (*f 0*) $\in X$
using *local-ordering-def* **by** (*metis assms card-eq-0-iff gr-implies-not0 linorder-neqE-nat*)

end

theory *Minkowski*
imports *TernaryOrdering*
begin

Primitives and axioms as given in [1, pp. 9-17].

I've tried to do little to no proofs in this file, and keep that in other files. So, this is mostly locale and other definitions, except where it is nice to prove something about definitional equivalence and the like (plus the intermediate lemmas that are necessary for doing so).

Minkowski spacetime = $(\mathcal{E}, \mathcal{P}, [\ldots])$ except in the notation here I've used $[[\ldots]]$ for $[\ldots]$ as Isabelle uses $[\ldots]$ for lists.
Except where stated otherwise all axioms are exactly as they appear in Schutz97. It is the independent axiomatic system provided in the main body of the book. The axioms O1-O6 are the axioms of order, and largely concern properties of the betweenness relation. I1-I7 are the axioms of incidence. I1-I3 are similar to axioms found in systems for Euclidean geometry. As compared to Hilbert's Foundations (HIn), our incidence axioms (In) are loosely identifiable as I1 $\rightarrow$ HI3, HI8; I2 $\rightarrow$ HI1; I3 $\rightarrow$ HI2. I4 fixes the dimension of the space. I5-I7 are what makes our system non-Galilean, and lead (I think) to Lorentz transforms (together with S?) and the ultimate speed limit. Axioms S and C and the axioms of symmetry and continuity, where the latter is what makes the system second order. Symmetry replaces all of Hilbert's axioms of congruence, when considered in the context of I5-I7.

3 MinkowskiPrimitive: I1-I3

Events $\mathcal{E}$, paths $\mathcal{P}$, and sprays. Sprays only need to refer to $\mathcal{E}$ and $\mathcal{P}$. Axiom *in-path-event* is covered in English by saying "a path is a set of events", but is necessary to have explicitly as an axiom as the types do not force it to be the case.

I think part of why Schutz has I1, together with the trickery $[\![\mathcal{E} \neq \{\}]\!] \Longrightarrow \ldots$ in I4, is that then I4 talks *only* about dimension, and results such as *no-empty-paths* can be proved using only existence of elements and unreachable sets. In our case, it's also a question of ordering the sequence of axiom introductions: dimension should really go at the end, since it is not needed for quite a while; but many earlier proofs rely on the set of events being non-empty. It may be nice to have the existence of paths as a separate axiom too, which currently still relies on the axiom of dimension (Schutz has no such axiom either).

locale *MinkowskiPrimitive* =

fixes $\mathcal{E}$ :: *'a set*
and $\mathcal{P}$ :: (*'a set*) *set*
assumes *in-path-event* [*simp*]: $[\![Q \in \mathcal{P};\ a \in Q]\!] \Longrightarrow a \in \mathcal{E}$

and *nonempty-events* [*simp*]: $\mathcal{E} \neq \{\}$

and *events-paths*: $[\![a \in \mathcal{E};\ b \in \mathcal{E};\ a \neq b]\!] \Longrightarrow \exists\, R{\in}\mathcal{P}.\ \exists\, S{\in}\mathcal{P}.\ a \in R \wedge b \in S \wedge R \cap S \neq \{\}$

and *eq-paths* [*intro*]: $[\![P \in \mathcal{P};\ Q \in \mathcal{P};\ a \in P;\ b \in P;\ a \in Q;\ b \in Q;\ a \neq b]\!] \Longrightarrow P = Q$
begin

This should be ensured by the additional axiom.

lemma *path-sub-events*:
$Q \in \mathcal{P} \Longrightarrow Q \subseteq \mathcal{E}$
by (*simp add*: *subsetI*)

lemma *paths-sub-power*:
$\mathcal{P} \subseteq Pow\ \mathcal{E}$
by (*simp add*: *path-sub-events subsetI*)

Define *path* for more terse statements. $a \neq b$ because a and b are being used to identify the path, and $a = b$ would not do that.

abbreviation *path* :: *'a set* ⇒ *'a* ⇒ *'a* ⇒ *bool* **where**
path ab a b ≡ *ab* ∈ $\mathcal{P}$ ∧ *a* ∈ *ab* ∧ *b* ∈ *ab* ∧ *a* ≠ *b*

abbreviation *path-ex* :: *'a* ⇒ *'a* ⇒ *bool* **where**
path-ex a b ≡ ∃ *Q*. *path Q a b*

lemma *path-permute*:
path ab a b = *path ab b a*
by *auto*

abbreviation *path-of* :: *'a* ⇒ *'a* ⇒ *'a set* **where**
path-of a b ≡ *THE ab*. *path ab a b*

lemma *path-of-ex*: *path* (*path-of a b*) *a b* ⟷ *path-ex a b*
using *theI'* [**where** *P*=λ*x*. *path x a b*] *eq-paths* **by** *blast*

lemma *path-unique*:
assumes *path ab a b* **and** *path ab' a b*
shows *ab* = *ab'*
using *eq-paths assms* **by** *blast*

lemma *paths-cross-once*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *path-R*: $R \in \mathcal{P}$

and *Q-neq-R*: $Q \neq R$
and *QR-nonempty*: $Q \cap R \neq \{\}$
shows $\exists!a{\in}\mathcal{E}.\ Q \cap R = \{a\}$
proof −
have *ab-inQR*: $\exists\, a{\in}\mathcal{E}.\ a{\in}Q \cap R$ **using** *QR-nonempty in-path-event path-Q* **by** *auto*
then obtain *a* **where** *a-event*: $a \in \mathcal{E}$ **and** *a-inQR*: $a \in Q \cap R$ **by** *auto*
have $Q \cap R = \{a\}$
proof (*rule ccontr*)
assume $Q \cap R \neq \{a\}$
then have $\exists\, b{\in}Q \cap R.\ b \neq a$ **using** *a-inQR* **by** *blast*
then have $Q = R$ **using** *eq-paths a-inQR path-Q path-R* **by** *auto*
thus *False* **using** *Q-neq-R* **by** *simp*
qed
thus *?thesis* **using** *a-event* **by** *blast*
qed

4 Primitives: Unreachable Subset (from an Event)

The $Q \in \mathcal{P} \wedge b \in \mathcal{E}$ constraints are necessary as the types as not expressive enough to do it on their own. Schutz's notation is: $Q(b, \emptyset)$.

definition *unreachable-subset* :: *'a set* ⇒ *'a* ⇒ *'a set* (*unreach−on - from -* [*100*, *100*]) **where**
unreach−on Q from b ≡ {*x*∈*Q*. $Q \in \mathcal{P} \wedge b \in \mathcal{E} \wedge b \notin Q \wedge \neg$(*path-ex b x*)}

5 Primitives: Kinematic Triangle

definition *kinematic-triangle* :: *'a* ⇒ *'a* ⇒ *'a* ⇒ *bool* (△ - - - [*100*, *100*, *100*] *100*) **where**
kinematic-triangle a b c ≡
$a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c$
$\wedge$ ($\exists\, Q{\in}\mathcal{P}.\ \exists\, R{\in}\mathcal{P}.\ Q \neq R \wedge (\exists\, S{\in}\mathcal{P}.\ Q \neq S \wedge R \neq S$
$\wedge\ a \in Q \wedge b \in Q$
$\wedge\ a \in R \wedge c \in R$
$\wedge\ b \in S \wedge c \in S$))

A fuller, more explicit equivalent of △, to show that the above definition is sufficient.

lemma *tri-full*:
△ *a b c* = ($a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c$
$\wedge$ ($\exists\, Q{\in}\mathcal{P}.\ \exists\, R{\in}\mathcal{P}.\ Q \neq R \wedge (\exists\, S{\in}\mathcal{P}.\ Q \neq S \wedge R \neq S$
$\wedge\ a \in Q \wedge b \in Q \wedge c \notin Q$
$\wedge\ a \in R \wedge c \in R \wedge b \notin R$
$\wedge\ b \in S \wedge c \in S \wedge a \notin S$)))
unfolding *kinematic-triangle-def* **by** (*meson path-unique*)

6 Primitives: SPRAY

It's okay to not require $x \in \mathcal{E}$ because if $x \notin \mathcal{E}$ the *SPRAY* will be empty anyway, and if it's nonempty then $x \in \mathcal{E}$ is derivable.

definition *SPRAY* :: *'a* ⇒ (*'a set*) *set* **where**
  SPRAY x ≡ {*R*∈*P*. *x* ∈ *R*}

definition *spray* :: *'a* ⇒ *'a set* **where**
  spray x ≡ {*y*. ∃ *R*∈*SPRAY x*. *y* ∈ *R*}

definition *is-SPRAY* :: (*'a set*) *set* ⇒ *bool* **where**
  is-SPRAY S ≡ ∃ *x*∈*E*. *S* = *SPRAY x*

definition *is-spray* :: *'a set* ⇒ *bool* **where**
  is-spray S ≡ ∃ *x*∈*E*. *S* = *spray x*

Some very simple SPRAY and spray lemmas below.

lemma *SPRAY-event*:
  SPRAY x ≠ {} ⟹ *x* ∈ *E*
proof (*unfold SPRAY-def*)
  assume *nonempty-SPRAY*: {*R* ∈ *P*. *x* ∈ *R*} ≠ {}
  then have *x-in-path-R*: ∃ *R* ∈ *P*. *x* ∈ *R* **by** *blast*
  thus *x* ∈ *E* **using** *in-path-event* **by** *blast*
qed

lemma *SPRAY-nonevent*:
  x ∉ *E* ⟹ *SPRAY x* = {}
using *SPRAY-event* **by** *auto*

lemma *SPRAY-path*:
  P ∈ *SPRAY x* ⟹ *P* ∈ *P*
by (*simp add*: *SPRAY-def*)

lemma *in-SPRAY-path*:
  P ∈ *SPRAY x* ⟹ *x* ∈ *P*
by (*simp add*: *SPRAY-def*)

lemma *source-in-SPRAY*:
  SPRAY x ≠ {} ⟹ ∃ *P* ∈ *SPRAY x*. *x* ∈ *P*
using *in-SPRAY-path* **by** *auto*

lemma *spray-event*:
  spray x ≠ {} ⟹ *x* ∈ *E*
proof (*unfold spray-def*)
  assume {*y*. ∃ *R* ∈ *SPRAY x*. *y* ∈ *R*} ≠ {}
  then have ∃ *y*. ∃ *R* ∈ *SPRAY x*. *y* ∈ *R* **by** *simp*
  then have *SPRAY x* ≠ {} **by** *blast*
  thus *x* ∈ *E* **using** *SPRAY-event* **by** *simp*

qed

lemma *spray-nonevent*:
$x \notin \mathcal{E} \Longrightarrow spray\ x = \{\}$
using *spray-event* **by** *auto*

lemma *in-spray-event*:
$y \in spray\ x \Longrightarrow y \in \mathcal{E}$
proof (*unfold spray-def*)
assume $y \in \{y.\ \exists\, R{\in}SPRAY\ x.\ y \in R\}$
then have $\exists\, R{\in}SPRAY\ x.\ y \in R$ **by** (*rule CollectD*)
then obtain R **where** *path-R*: $R \in \mathcal{P}$
and *y-inR*: $y \in R$ **using** *SPRAY-path* **by** *auto*
thus $y \in \mathcal{E}$ **using** *in-path-event* **by** *simp*
qed

lemma *source-in-spray*:
$spray\ x \neq \{\} \Longrightarrow x \in spray\ x$
proof −
assume *nonempty-spray*: $spray\ x \neq \{\}$
have *spray-eq*: $spray\ x = \{y.\ \exists\, R{\in}SPRAY\ x.\ y \in R\}$ **using** *spray-def* **by** *simp*
then have *ex-in-SPRAY-path*: $\exists\, y.\ \exists\, R{\in}SPRAY\ x.\ y \in R$ **using** *nonempty-spray* **by** *simp*
show $x \in spray\ x$ **using** *ex-in-SPRAY-path spray-eq source-in-SPRAY* **by** *auto*
qed

7 Primitives: Path (In)dependence

"A subset of three paths of a SPRAY is dependent if there is a path which does not belong to the SPRAY and which contains one event from each of the three paths: we also say any one of the three paths is dependent on the other two. Otherwise the subset is independent." [Schutz97]

The definition of $SPRAY$ constrains x, Q, R, S to be in $\mathcal{E}$ and $\mathcal{P}$.

definition *dep3-event Q R S x*
$\equiv card\ \{Q,R,S\} = 3 \wedge \{Q,R,S\} \subseteq SPRAY\ x$
$\wedge\ (\exists\, T{\in}\mathcal{P}.\ T \notin SPRAY\ x \wedge\ Q{\cap}T{\neq}\{\} \wedge\ R{\cap}T{\neq}\{\} \wedge\ S{\cap}T{\neq}\{\})$

definition *dep3-spray Q R S SPR* $\equiv \exists\, x.\ SPRAY\ x = SPR \wedge dep3\text{-}event\ Q\ R\ S\ x$

definition *dep3 Q R S* $\equiv \exists\, x.\ dep3\text{-}event\ Q\ R\ S\ x$

Some very simple lemmas related to *dep3-event*.

lemma *dep3-nonspray*:
assumes *dep3-event Q R S x*
shows $\exists\, P{\in}\mathcal{P}.\ P \notin SPRAY\ x$
by (*metis assms dep3-event-def*)

lemma *dep3-path*:
assumes *dep3-QRSx*: *dep3 Q R S*
shows $Q \in \mathcal{P}$ $R \in \mathcal{P}$ $S \in \mathcal{P}$
using *assms dep3-event-def dep3-def SPRAY-path insert-subset* **by** *auto*

lemma *dep3-distinct*:
assumes *dep3-QRSx*: *dep3 Q R S*
shows $Q \neq R$ $Q \neq S$ $R \neq S$
using *assms dep3-def dep3-event-def* **by** (*simp-all add*: *card-3-dist*)

lemma *dep3-is-event*:
dep3-event Q R S x $\Longrightarrow$ $x \in \mathcal{E}$
using *SPRAY-event dep3-event-def* **by** *auto*

lemma *dep3-event-old*:
dep3-event Q R S x $\longleftrightarrow$ $Q \neq R \wedge Q \neq S \wedge R \neq S \wedge Q \in SPRAY\ x \wedge R \in SPRAY\ x \wedge S \in SPRAY\ x$
$\wedge\ (\exists\, T{\in}\mathcal{P}.\ T \notin SPRAY\ x \wedge (\exists\, y{\in}Q.\ y \in T) \wedge (\exists\, y{\in}R.\ y \in T) \wedge (\exists\, y{\in}S.\ y \in T))$
by (*rule iffI*; *unfold dep3-event-def*, (*simp add*: *card-3-dist*), *blast*)

lemma *dep3-event-permute* [*no-atp*]:
assumes *dep3-event Q R S x*
shows *dep3-event Q S R x dep3-event R Q S x dep3-event R S Q x*
dep3-event S Q R x dep3-event S R Q x
using *dep3-event-old assms* **by** *auto*

lemma *dep3-permute* [*no-atp*]:
assumes *dep3 Q R S*
shows *dep3 Q S R dep3 R Q S dep3 R S Q*
and *dep3 S Q R dep3 S R Q*
using *dep3-event-permute dep3-def assms* **by** *meson+*

"We next give recursive definitions of dependence and independence which will be used to characterize the concept of dimension. A path T is dependent on the set of n paths (where $n \geq 3$)

$$S = \{Q_i{:}\, i = 1, 2, ..., n; Q_i \in \mathrm{SPRAY}x\}$$

if it is dependent on two paths S_1 and S_2, where each of these two paths is dependent on some subset of $n-1$ paths from the set S." [Schutz97]

inductive *dep-path* :: *'a set* $\Rightarrow$ (*'a set*) *set* $\Rightarrow$ *bool* **where**
dep-3: *dep3 T A B* $\Longrightarrow$ *dep-path T* {*A*, *B*}
| *dep-n*: ⟦*dep3 T S1 S2*; *dep-path S1 S'*; *dep-path S2 S''*; *S* $\subseteq$ *SPRAY x*; *S'* $\subseteq$ *S*; *S''* $\subseteq$ *S*; *Suc* (*card S'*) = *card S*; *Suc* (*card S''*) = *card S*⟧ $\Longrightarrow$ *dep-path T S*

lemma *card-Suc-ex*:
assumes *card A* = *Suc* (*card B*) *B* $\subseteq$ *A*

shows $\exists\, b.\ A = \mathit{insert}\ b\ B \wedge b \notin B$
proof −
have *finite A* **using** *assms(1) card-ge-0-finite card.infinite* **by** *fastforce*
obtain b **where** $b \in A - B$
by (*metis Diff-eq-empty-iff all-not-in-conv assms n-not-Suc-n subset-antisym*)
show $\exists\, b.\ A = \mathit{insert}\ b\ B \wedge b \notin B$
proof
show $A = \mathit{insert}\ b\ B \wedge b \notin B$
using ‹$b \in A - B$› ‹*finite A*› *assms*
by (*metis DiffD1 DiffD2 Diff-insert-absorb Diff-single-insert card-insert-disjoint card-subset-eq insert-absorb rev-finite-subset*)
qed
qed

lemma *union-of-subsets-by-singleton*:
assumes *Suc* (*card S'*) = *card S Suc* (*card S''*) = *card S*
and $S' \neq S''$ $S' \subseteq S$ $S'' \subseteq S$
shows $S' \cup S'' = S$
proof −
obtain *x y* **where** *x*: *insert x S'* = *S x*$\notin$*S'* **and** *y*: *insert y S''* = *S y*$\notin$*S''*
using *assms(1,2,4,5)* **by** (*metis card-Suc-ex*)
have $x \neq y$ **using** *x y assms(3)* **by** (*metis insert-eq-iff*)
thus *?thesis* **using** *x(1) y(1)* **by** *blast*
qed

lemma *dep-path-card-2*: *dep-path T S* $\Longrightarrow$ *card S* $\geq$ *2*
by (*induct rule*: *dep-path.induct*, *simp add*: *dep3-def dep3-event-old*, *linarith*)

"We also say that the set of $n+1$ paths $S \cup \{T\}$ is a dependent set." [Schutz97] Starting from this constructive definition, the below gives an analytical one.

definition *dep-set* :: (*'a set*) *set* $\Rightarrow$ *bool* **where**
dep-set S $\equiv \exists\, S' \subseteq S.\ \exists\, P \in (S - S').$ *dep-path P S'*

Notice that the relation between *dep-set* and *dep-path* becomes somewhat meaningless in the case where we apply *dep-path* to an element of the set. This is because sets have no duplicate members, and we do not mirror the idea that scalar multiples of vectors linearly depend on those vectors: paths in a SPRAY are (in the $\mathbb{R}^4$ model) already equivalence classes of vectors that are scalar multiples of each other.

lemma *dep-path-imp-dep-set*:
assumes *dep-path P S P*$\notin$*S*
shows *dep-set* (*insert P S*)
using *assms dep-set-def* **by** *auto*

lemma *dep-path-for-set-members*:
assumes $P \in S$
shows *dep-set S* = *dep-set* (*insert P S*)
by (*simp add*: *assms(1) insert-absorb*)

lemma *dependent-superset*:
assumes *dep-set A* **and** *A⊆B*
shows *dep-set B*
using *assms dep-set-def*
by (*meson Diff-mono dual-order.trans in-mono order-refl*)

lemma *path-in-dep-set*:
assumes *dep3 P Q R*
shows *dep-set* {*P*,*Q*,*R*}
using *dep-3 assms dep3-def dep-set-def dep3-event-old*
by (*metis DiffI insert-iff singletonD subset-insertI*)

lemma *path-in-dep-set2a*:
assumes *dep3 P Q R*
shows *dep-path P* {*P*,*Q*,*R*}
proof
let *?S′* = {*P*,*R*}
let *?S″* = {*P*,*Q*}
have *all-neq*: *P≠Q P≠R R≠Q* **using** *assms dep3-def dep3-event-old* **by** *auto*
show *dep3 P Q R* **using** *assms dep3-event-def* **by** (*simp add*: *dep-3*)
show *dep-path Q ?S′* **using** *assms dep3-event-permute*(*2*) *dep-3 dep3-def* **by** *meson*
show *dep-path R ?S″* **using** *assms dep3-event-permute*(*4*) *dep-3 dep3-def* **by** *meson*
show *?S′* ⊆ {*P*, *Q*, *R*} **by** *simp*
show *?S″* ⊆ {*P*, *Q*, *R*} **by** *simp*
show *Suc* (*card ?S′*) = *card* {*P*, *Q*, *R*} *Suc* (*card ?S″*) = *card* {*P*, *Q*, *R*}
using *all-neq card-insert-disjoint* **by** *auto*
show {*P*, *Q*, *R*} ⊆ *SPRAY* (*SOME x*. *dep3-event P Q R x*)
using *assms dep3-def dep3-event-def* **by** (*metis some-eq-ex*)
qed

definition *indep-set* :: ('*a set*) *set* ⇒ *bool* **where**
indep-set S ≡ ¬ *dep-set S*

lemma *no-dep-in-indep*: *indep-set S* ⟹ ¬(∃ *T* ⊆ *S*. *dep-set T*)
using *indep-set-def dependent-superset* **by** *blast*

lemma *indep-set-alt-intro*: ¬(∃ *T* ⊆ *S*. *dep-set T*) ⟹ *indep-set S*
using *indep-set-def* **by** *blast*

lemma *indep-set-alt*: *indep-set S* ⟷ ¬(∃ *S′* ⊆ *S*. *dep-set S′*)
using *no-dep-in-indep indep-set-alt-intro* **by** *blast*

lemma *dep-set S* ∨ *indep-set S*
by (*simp add*: *indep-set-def*)

8 Primitives: 3-SPRAY

"We now make the following definition which enables us to specify the dimensions of Minkowski space-time. A SPRAY is a 3-SPRAY if: i) it contains four independent paths, and ii) all paths of the SPRAY are dependent on these four paths." [Schutz97]

definition *n-SPRAY-basis* :: *nat* ⇒ *'a set set* ⇒ *'a* ⇒ *bool* **where**
n-SPRAY-basis n S x ≡ *S*⊆*SPRAY x* ∧ *card S* = (*Suc n*) ∧ *indep-set S* ∧ (∀ *P*∈*SPRAY x. dep-path P S*)

definition *n-SPRAY* (*-−SPRAY -* [*100*,*100*]) **where**
n−*SPRAY x* ≡ ∃ *S*⊆*SPRAY x. card S* = (*Suc n*) ∧ *indep-set S* ∧ (∀ *P*∈*SPRAY x. dep-path P S*)

abbreviation *three-SPRAY x* ≡ *3*−*SPRAY x*

lemma *n-SPRAY-intro*:
assumes *S*⊆*SPRAY x card S* = (*Suc n*) *indep-set S* ∀ *P*∈*SPRAY x. dep-path P S*
shows *n*−*SPRAY x*
using *assms n-SPRAY-def* **by** *blast*

lemma *three-SPRAY-alt*:
three-SPRAY x = (∃ *S1 S2 S3 S4.*
S1 ≠ *S2* ∧ *S1* ≠ *S3* ∧ *S1* ≠ *S4* ∧ *S2* ≠ *S3* ∧ *S2* ≠ *S4* ∧ *S3* ≠ *S4*
∧ *S1* ∈ *SPRAY x* ∧ *S2* ∈ *SPRAY x* ∧ *S3* ∈ *SPRAY x* ∧ *S4* ∈ *SPRAY x*
∧ (*indep-set* {*S1*, *S2*, *S3*, *S4*})
∧ (∀ *S*∈*SPRAY x. dep-path S* {*S1*,*S2*,*S3*,*S4*}))
(**is** *three-SPRAY x* ⟷ *?three-SPRAY' x*)
proof
assume *three-SPRAY x*
then obtain *S* **where** *ns*: *S*⊆*SPRAY x card S* = *4 indep-set S* ∀ *P*∈*SPRAY x. dep-path P S*
using *n-SPRAY-def* **by** *auto*
then obtain S_1 S_2 S_3 S_4 **where**
$S = \{S_1, S_2, S_3, S_4\}$ **and**
$S_1 \neq S_2 \wedge S_1 \neq S_3 \wedge S_1 \neq S_4 \wedge S_2 \neq S_3 \wedge S_2 \neq S_4 \wedge S_3 \neq S_4$ **and**
$S_1 \in SPRAY\ x \wedge S_2 \in SPRAY\ x \wedge S_3 \in SPRAY\ x \wedge S_4 \in SPRAY\ x$
using *card-4-eq* **by** (*smt* (*verit*) *insert-subset ns*)
thus *?three-SPRAY' x*
by (*metis ns*(*3*,*4*))
next
assume *?three-SPRAY' x*
then obtain S_1 S_2 S_3 S_4 **where** *ns*:
$S_1 \neq S_2 \wedge S_1 \neq S_3 \wedge S_1 \neq S_4 \wedge S_2 \neq S_3 \wedge S_2 \neq S_4 \wedge S_3 \neq S_4$
$S_1 \in SPRAY\ x \wedge S_2 \in SPRAY\ x \wedge S_3 \in SPRAY\ x \wedge S_4 \in SPRAY\ x$
indep-set $\{S_1, S_2, S_3, S_4\}$
∀ *S*∈*SPRAY x. dep-path S* $\{S_1,S_2,S_3,S_4\}$

by *metis*
show *three-SPRAY x*
apply (*intro n-SPRAY-intro*[*of* {S_1, S_2, S_3, S_4}])
by (*simp add*: *ns*)+
qed

lemma *three-SPRAY-intro*:
assumes *S1* $\neq$ *S2* $\wedge$ *S1* $\neq$ *S3* $\wedge$ *S1* $\neq$ *S4* $\wedge$ *S2* $\neq$ *S3* $\wedge$ *S2* $\neq$ *S4* $\wedge$ *S3* $\neq$ *S4*
and *S1* $\in$ *SPRAY x* $\wedge$ *S2* $\in$ *SPRAY x* $\wedge$ *S3* $\in$ *SPRAY x* $\wedge$ *S4* $\in$ *SPRAY x*
and *indep-set* {*S1*, *S2*, *S3*, *S4*}
and $\forall$ *S*$\in$*SPRAY x*. *dep-path S* {*S1*,*S2*,*S3*,*S4*}
shows *three-SPRAY x*
unfolding *three-SPRAY-alt* **by** (*metis assms*)

Lemma *is-three-SPRAY* says "this set of sets of elements is a set of paths which is a 3-SPRAY". Lemma *three-SPRAY-ge4* just extracts a bit of the definition.

definition *is-three-SPRAY* :: ('*a set*) *set* $\Rightarrow$ *bool* **where**
is-three-SPRAY S $\equiv$ $\exists$ *x*. *S* = *SPRAY x* $\wedge$ *3*$-$*SPRAY x*

lemma *three-SPRAY-ge4*:
assumes *three-SPRAY x*
shows $\exists$ *Q1*$\in\mathcal{P}$. $\exists$ *Q2*$\in\mathcal{P}$. $\exists$ *Q3*$\in\mathcal{P}$. $\exists$ *Q4*$\in\mathcal{P}$. *Q1* $\neq$ *Q2* $\wedge$ *Q1* $\neq$ *Q3* $\wedge$ *Q1* $\neq$ *Q4* $\wedge$ *Q2* $\neq$ *Q3* $\wedge$ *Q2* $\neq$ *Q4* $\wedge$ *Q3* $\neq$ *Q4*
using *assms three-SPRAY-alt SPRAY-path* **by** *meson*

end

9 MinkowskiBetweenness: O1-O5

In O4, I have removed the requirement that $a \neq d$ in order to prove negative betweenness statements as Schutz does. For example, if we have [*abc*] and [*bca*] we want to conclude [*aba*] and claim "contradiction!", but we can't as long as we mandate that $a \neq d$.

locale *MinkowskiBetweenness* = *MinkowskiPrimitive* +
fixes *betw* :: '*a* $\Rightarrow$ '*a* $\Rightarrow$ '*a* $\Rightarrow$ *bool* ([-;-;-])

assumes *abc-ex-path*: [*a*;*b*;*c*] $\Longrightarrow$ $\exists$ *Q*$\in\mathcal{P}$. *a* $\in$ *Q* $\wedge$ *b* $\in$ *Q* $\wedge$ *c* $\in$ *Q*

and *abc-sym*: [*a*;*b*;*c*] $\Longrightarrow$ [*c*;*b*;*a*]

and *abc-ac-neq*: [*a*;*b*;*c*] $\Longrightarrow$ *a* $\neq$ *c*

and *abc-bcd-abd* [*intro*]: $[\![$[*a*;*b*;*c*]; [*b*;*c*;*d*]$]\!]$ $\Longrightarrow$ [*a*;*b*;*d*]

and *some-betw*: $[\![$*Q* $\in$ $\mathcal{P}$; *a* $\in$ *Q*; *b* $\in$ *Q*; *c* $\in$ *Q*; *a* $\neq$ *b*; *a* $\neq$ *c*; *b* $\neq$ *c*$]\!]$
$\Longrightarrow$ [*a*;*b*;*c*] $\vee$ [*b*;*c*;*a*] $\vee$ [*c*;*a*;*b*]

begin

The next few lemmas either provide the full axiom from the text derived from a new simpler statement, or provide some very simple fundamental additions which make sense to prove immediately before starting, usually related to set-level things that should be true which fix the type-level ambiguity of 'a.

lemma *betw-events*:
  assumes *abc*: $[a;b;c]$
  shows $a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E}$
proof −
  have $\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$ **using** *abc-ex-path abc* **by** *simp*
  thus *?thesis* **using** *in-path-event* **by** *auto*
qed

This shows the shorter version of O5 is equivalent.

lemma *O5-still-O5* [*no-atp*]:
  $((Q \in \mathcal{P} \wedge \{a,b,c\} \subseteq Q \wedge a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c)$
    $\longrightarrow [a;b;c] \vee [b;c;a] \vee [c;a;b])$
  $=$
  $((Q \in \mathcal{P} \wedge \{a,b,c\} \subseteq Q \wedge a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c)$
    $\longrightarrow [a;b;c] \vee [b;c;a] \vee [c;a;b] \vee [c;b;a] \vee [a;c;b] \vee [b;a;c])$
by (*auto simp add*: *abc-sym*)

lemma *some-betw-xor*:
  $[\![Q \in \mathcal{P};\ a \in Q;\ b \in Q;\ c \in Q;\ a \neq b;\ a \neq c;\ b \neq c]\!]$
    $\Longrightarrow ([a;b;c] \wedge \neg\, [b;c;a] \wedge \neg\, [c;a;b])$
      $\vee ([b;c;a] \wedge \neg\, [a;b;c] \wedge \neg\, [c;a;b])$
      $\vee ([c;a;b] \wedge \neg\, [a;b;c] \wedge \neg\, [b;c;a])$
by (*meson abc-ac-neq abc-bcd-abd some-betw*)

The lemma *abc-abc-neq* is the full O3 as stated by Schutz.

lemma *abc-abc-neq*:
  assumes *abc*: $[a;b;c]$
  shows $a \neq b \wedge a \neq c \wedge b \neq c$
using *abc-sym abc-ac-neq assms abc-bcd-abd* **by** *blast*

lemma *abc-bcd-acd*:
  assumes *abc*: $[a;b;c]$
    and *bcd*: $[b;c;d]$
  shows $[a;c;d]$
proof −
  have *cba*: $[c;b;a]$ **using** *abc-sym abc* **by** *simp*
  have *dcb*: $[d;c;b]$ **using** *abc-sym bcd* **by** *simp*
  have $[d;c;a]$ **using** *abc-bcd-abd dcb cba* **by** *blast*
  thus *?thesis* **using** *abc-sym* **by** *simp*
qed

lemma *abc-only-cba*:
assumes [*a*;*b*;*c*]
shows ¬ [*b*;*a*;*c*] ¬ [*a*;*c*;*b*] ¬ [*b*;*c*;*a*] ¬ [*c*;*a*;*b*]
using *abc-sym abc-abc-neq abc-bcd-abd assms* **by** *blast+*

10 Betweenness: Unreachable Subset Via a Path

definition *unreachable-subset-via* :: *'a set* ⇒ *'a* ⇒ *'a set* ⇒ *'a* ⇒ *'a set* **where**
unreachable-subset-via Q Qa R x ≡ {*Qy*. [*x*;*Qy*;*Qa*] ∧ (∃ *Rw*∈*R*. *Qa* ∈ *unreach*−*on Q from Rw* ∧ *Qy* ∈ *unreach*−*on Q from Rw*)}

definition *unreachable-subset-via-notation* (*unreach*−*via - on - from - to* - [*100*, *100*, *100*, *100*] *100*)
where *unreach*−*via P on Q from a to x* ≡ *unreachable-subset-via Q a P x*

11 Betweenness: Chains

named-theorems *chain-defs*
named-theorems *chain-alts*

11.1 Locally ordered chains with indexing

Definitions for Schutz's chains, with local order only.

A chain can be: (i) a set of two distinct events connected by a path, or ...

definition *short-ch* :: *'a set* ⇒ *bool* **where**
short-ch X ≡ *card X* = *2* ∧ (∃ *P*∈*P*. *X* ⊆ *P*)

lemma *short-ch-alt*[*chain-alts*]:
short-ch X = (∃ *x*∈*X*. ∃ *y*∈*X*. *path-ex x y* ∧ ¬(∃ *z*∈*X*. *z*≠*x* ∧ *z*≠*y*))
short-ch X = (∃ *x y*. *X* = {*x*,*y*} ∧ *path-ex x y*)
unfolding *short-ch-def*
apply (*simp add*: *card-2-iff'*, *smt* (*verit*, *ccfv-SIG*) *in-mono subsetI*)
by (*metis card-2-iff empty-subsetI insert-subset*)

lemma *short-ch-intros*:
⟦*x*∈*X*; *y*∈*X*; *path-ex x y*; ¬(∃ *z*∈*X*. *z*≠*x* ∧ *z*≠*y*)⟧ ⟹ *short-ch X*
⟦*X* = {*x*,*y*}; *path-ex x y*⟧ ⟹ *short-ch X*
by (*auto simp*: *short-ch-alt*)

lemma *short-ch-path*: *short-ch* {*x*,*y*} ⟷ *path-ex x y*
unfolding *short-ch-def* **by** *force*

... a set of at least three events such that any three adjacent events are ordered. Notice infinite sets have card 0, because card gives a natural number always.

definition *local-long-ch-by-ord* :: (*nat* ⇒ *'a*) ⇒ *'a set* ⇒ *bool* **where**

local-long-ch-by-ord f X ≡ (*infinite X* ∨ *card X* ≥ *3*) ∧ *local-ordering f betw X*

lemma *local-long-ch-by-ord-alt* [*chain-alts*]:
local-long-ch-by-ord f X =
(∃ *x*∈*X*. ∃ *y*∈*X*. ∃ *z*∈*X*. *x*≠*y* ∧ *y*≠*z* ∧ *x*≠*z* ∧ *local-ordering f betw X*)
(**is** - = *?ch f X*)
proof
assume *asm*: *local-long-ch-by-ord f X*
{
assume *card X* ≥ *3*
then have ∃ *x y z*. *x*≠*y* ∧ *y*≠*z* ∧ *x*≠*z* ∧ {*x*,*y*,*z*}⊆*X*
apply (*simp add*: *eval-nat-numeral*)
by (*auto simp add*: *card-le-Suc-iff*)
} moreover {
assume *infinite X*
then have ∃ *x y z*. *x*≠*y* ∧ *y*≠*z* ∧ *x*≠*z* ∧ {*x*,*y*,*z*}⊆*X*
using *inf-3-elms bot.extremum* **by** *fastforce*
}
ultimately show *?ch f X* **using** *asm* **unfolding** *local-long-ch-by-ord-def* **by** *auto*
next
assume *asm*: *?ch f X*
then obtain *x y z* **where** *xyz*: {*x*,*y*,*z*}⊆*X* ∧ *x* ≠ *y* ∧ *y* ≠ *z* ∧ *x* ≠ *z*
apply (*simp add*: *eval-nat-numeral*) **by** *auto*
hence *card X* ≥ *3* ∨ *infinite X*
apply (*simp add*: *eval-nat-numeral*)
by (*smt* (*z3*) *xyz card.empty card-insert-if card-subset finite.emptyI finite-insert insertE*
insert-absorb insert-not-empty)
thus *local-long-ch-by-ord f X* **unfolding** *local-long-ch-by-ord-def* **using** *asm* **by** *auto*
qed

lemma *short-xor-long*:
shows *short-ch Q* ⟹ ∄*f*. *local-long-ch-by-ord f Q*
and *local-long-ch-by-ord f Q* ⟹ ¬ *short-ch Q*
unfolding *chain-alts* **by** (*metis*)+

Any short chain can have an "ordering" defined on it: this isn't the ternary ordering *betw* that is used for triplets of elements, but merely an indexing function that fixes the "direction" of the chain, i.e. maps *0* to one element and *1* to the other. We define this in order to be able to unify chain definitions with those for long chains. Thus the indexing function *f* of *short-ch-by-ord f Q* has a similar status to the ordering on a long chain in many regards: e.g. it implies that $f(0 \ldots |Q| - 1) \subseteq Q$.

definition *short-ch-by-ord* :: (*nat*⇒'*a*) ⇒ '*a set* ⇒ *bool*
where *short-ch-by-ord f Q* ≡ *Q* = {*f 0*, *f 1*} ∧ *path-ex* (*f 0*) (*f 1*)

lemma *short-ch-equiv* [*chain-alts*]: $\exists f.$ *short-ch-by-ord* f $Q \longleftrightarrow$ *short-ch* Q
proof −
  { assume *asm*: *short-ch* Q
    obtain x y **where** *xy*: $\{x,y\}\subseteq Q$ *path-ex* x y
      using *asm short-ch-alt*(*2*) **by** (*auto simp*: *short-ch-def*)
    let *?f* = λn::*nat*. *if n=0 then x else y*
    have $\exists f.$ $(\exists x\ y.\ Q = \{x,\ y\} \wedge f\ (0{::}nat) = x \wedge f\ 1 = y \wedge (\exists Q.\ path\ Q\ x\ y))$
      apply (*rule exI*[*of* - *?f*]) **using** *asm xy short-ch-alt*(*2*) **by** *auto*
  } moreover {
    fix f **assume** *asm*: *short-ch-by-ord* f Q
    have *card* Q = *2* $(\exists P{\in}\mathcal{P}.\ Q \subseteq P)$
      using *asm short-ch-by-ord-def* **by** *auto*
  } ultimately show *?thesis* **by** (*metis short-ch-by-ord-def short-ch-def*)
qed

lemma *short-ch-card*:
  short-ch-by-ord f $Q \Longrightarrow$ *card* Q = *2*
  short-ch $Q \Longrightarrow$ *card* Q = *2*
  using *short-ch-by-ord-def short-ch-def short-ch-equiv* **by** *auto*

lemma *short-ch-sym*:
  assumes *short-ch-by-ord* f Q
  shows *short-ch-by-ord* ($\lambda n.$ *if n=0 then f 1 else f 0*) Q
  using *assms* **unfolding** *short-ch-by-ord-def* **by** *auto*

lemma *short-ch-ord-in*:
  assumes *short-ch-by-ord* f Q
  shows *f 0* $\in$ Q *f 1* $\in$ Q
  using *assms* **unfolding** *short-ch-by-ord-def* **by** *auto*

Does this restrict chains to lie on paths? Proven in *TemporalOrderingOnPath*'s Interlude!

definition *ch-by-ord* ([-⇝-]) **where**
  $[f{\rightsquigarrow}X] \equiv$ *short-ch-by-ord* f $X \vee$ *local-long-ch-by-ord* f X

definition *ch* :: *'a set* ⇒ *bool* **where** *ch* $X \equiv \exists f.$ $[f{\rightsquigarrow}X]$

declare *short-ch-def* [*chain-defs*]
  and *local-long-ch-by-ord-def* [*chain-defs*]
  and *ch-by-ord-def* [*chain-defs*]
  and *short-ch-by-ord-def* [*chain-defs*]

We include alternative definitions in the *chain-defs* set, because we do not want arbitrary orderings to appear on short chains. Unless an ordering for a short chain is explicitly written down by the user, we shouldn't introduce a *short-ch-by-ord* when e.g. unfolding.

lemma *ch-alt*[*chain-defs*]: *ch* $X \equiv$ *short-ch* $X \vee (\exists f.$ *local-long-ch-by-ord* f $X)$
  unfolding *ch-def ch-by-ord-def* **using** *chain-defs short-ch-intros*(*2*)
  by (*smt* (*verit*) *short-ch-equiv*)

Since $f(0)$ is always in the chain, and plays a special role particularly for infinite chains (as the 'endpoint', the non-finite edge) let us fix it straight in the definition. Notice we require both *infinite X* and *long-ch-by-ord*, thus circumventing infinite Isabelle sets having cardinality 0.

definition *infinite-chain* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow bool$ **where**
  infinite-chain f Q $\equiv$ *infinite Q* $\wedge$ $[f \rightsquigarrow Q]$

declare *infinite-chain-def* [*chain-defs*]

lemma *infinite-chain-alt*[*chain-alts*]:
  infinite-chain f Q $\longleftrightarrow$ *infinite Q* $\wedge$ *local-ordering f betw Q*
  unfolding *chain-defs* **by** *fastforce*

definition *infinite-chain-with* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow bool$ ([-$\rightsquigarrow$-|- ..]) **where**
  infinite-chain-with f Q x $\equiv$ *infinite-chain f Q* $\wedge$ *f 0* $=$ *x*

declare *infinite-chain-with-def* [*chain-defs*]

lemma *infinite-chain f Q* $\longleftrightarrow$ $[f \rightsquigarrow Q|f\ 0..]$
  by (*simp add*: *infinite-chain-with-def*)

definition *finite-chain* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow bool$ **where**
  finite-chain f Q $\equiv$ *finite Q* $\wedge$ $[f \rightsquigarrow Q]$

declare *finite-chain-def* [*chain-defs*]

lemma *finite-chain-alt*[*chain-alts*]: *finite-chain f Q* $\longleftrightarrow$ *short-ch-by-ord f Q* $\vee$ (*finite Q* $\wedge$ *local-long-ch-by-ord f Q*)
  unfolding *chain-defs* **by** *auto*

definition *finite-chain-with* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([-$\rightsquigarrow$-|- .. -]) **where**
  $[f \rightsquigarrow Q|x..y]$ $\equiv$ *finite-chain f Q* $\wedge$ *f 0* $=$ *x* $\wedge$ *f* (*card Q* $-$ *1*) $=$ *y*

declare *finite-chain-with-def* [*chain-defs*]

lemma *finite-chain f Q* $\longleftrightarrow$ $[f \rightsquigarrow Q|f\ 0\ ..\ f\ (card\ Q - 1)]$
  by (*simp add*: *finite-chain-with-def*)

lemma *finite-chain-with-alt* [*chain-alts*]:
  $[f \rightsquigarrow Q|x..z] \longleftrightarrow$ (*short-ch-by-ord f Q* $\vee$ (*card Q* $\geq$ *3* $\wedge$ *local-ordering f betw Q*)) $\wedge$
    x $=$ *f 0* $\wedge$ *z* $=$ *f* (*card Q* $-$ *1*)
  unfolding *chain-defs*
  by (*metis card.infinite finite.emptyI finite.insertI not-numeral-le-zero*)

lemma *finite-chain-with-cases*:
  assumes $[f \rightsquigarrow Q|x..z]$
  obtains

(*short*) $x = f\ 0\ z = f\ (card\ Q - 1)$ *short-ch-by-ord f Q*
| (*long*) $x = f\ 0\ z = f\ (card\ Q - 1)$ $card\ Q \geq 3$ *local-long-ch-by-ord f Q*
using *assms finite-chain-with-alt* **by** (*meson local-long-ch-by-ord-def*)

definition *finite-long-chain-with*:: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([-⇝-|-..-..-])
where $[f \rightsquigarrow Q|x..y..z] \equiv [f \rightsquigarrow Q|x..z] \wedge x \neq y \wedge y \neq z \wedge y \in Q$

declare *finite-long-chain-with-def* [*chain-defs*]

lemma *points-in-chain*:
assumes $[f \rightsquigarrow Q|x..z]$
shows $x \in Q \wedge z \in Q$
apply (*cases rule*: *finite-chain-with-cases*[*OF assms*])
using *short-ch-card*(*1*) *short-ch-ord-in* **by** (*simp add*: *chain-defs local-ordering-def*[*of f betw Q*])+

lemma *points-in-long-chain*:
assumes $[f \rightsquigarrow Q|x..y..z]$
shows $x \in Q$ **and** $y \in Q$ **and** $z \in Q$
using *points-in-chain finite-long-chain-with-def assms* **by** *meson*+

lemma *finite-chain-with-card-less3*:
assumes $[f \rightsquigarrow Q|x..z]$
and $card\ Q < 3$
shows *short-ch-by-ord f Q z* = *f 1*
proof −
show *1*: *short-ch-by-ord f Q*
using *finite-chain-with-alt assms* **by** *simp*
thus $z = f\ 1$
using *assms*(*1*) **by** (*auto simp*: *eval-nat-numeral chain-defs*)
qed

lemma *ch-long-if-card-geq3*:
assumes *ch X*
and $card\ X \geq 3$
shows ∃*f*. *local-long-ch-by-ord f X*
proof −
show ∃*f*. *local-long-ch-by-ord f X*
proof (*rule ccontr*)
assume ∄*f*. *local-long-ch-by-ord f X*
hence *short-ch X*
using *assms*(*1*) **unfolding** *chain-defs* **by** *auto*
obtain *x y z* **where** $x \in X \wedge y \in X \wedge z \in X$ **and** $x \neq y \wedge y \neq z \wedge x \neq z$
using *assms*(*2*) **by** (*auto simp add*: *card-le-Suc-iff numeral-3-eq-3*)
thus *False*
using ‹*short-ch X*› **by** (*metis short-ch-alt*(*1*))
qed

qed

lemma *ch-short-if-card-less3*:
assumes *ch Q*
and *card Q* < *3*
and *finite Q*
shows ∃*f*. *short-ch-by-ord f Q*
using *short-ch-equiv finite-chain-with-card-less3*
by (*metis assms ch-alt diff-is-0-eq' less-irrefl-nat local-long-ch-by-ord-def zero-less-diff*)

lemma *three-in-long-chain*:
assumes *local-long-ch-by-ord f X*
obtains *x y z* **where** *x*∈*X* **and** *y*∈*X* **and** *z*∈*X* **and** *x*≠*y* **and** *x*≠*z* **and** *y*≠*z*
using *assms*(*1*) *local-long-ch-by-ord-alt* **by** *auto*

lemma *short-ch-card-2*:
assumes *ch-by-ord f X*
shows *short-ch X* ⟷ *card X* = *2*
using *assms* **unfolding** *chain-defs* **using** *card-2-iff' card-gt-0-iff* **by** *fastforce*

lemma *long-chain-card-geq*:
assumes *local-long-ch-by-ord f X* **and** *fin*: *finite X*
shows *card X* ≥ *3*
proof −
obtain *x y z* **where** *xyz*: *x*∈*X y*∈*X z*∈*X* **and** *neq*: *x*≠*y x*≠*z y*≠*z*
using *three-in-long-chain assms* **by** *blast*
let *?S* = {*x*,*y*,*z*}
have *?S* ⊆ *X*
by (*simp add*: *xyz*)
moreover have *card ?S* ≥ *3*
using *antisym* ‹*x* ≠ *y*› ‹*x* ≠ *z*› ‹*y* ≠ *z*› **by** *auto*
ultimately show *?thesis*
by (*meson neq fin three-subset*)
qed

lemma *fin-chain-card-geq-2*:
assumes [*f*⇝*X*|*a*..*b*]
shows *card X* ≥ *2*
using *finite-chain-with-def* **apply** (*cases short-ch X*)
using *short-ch-card-2*
apply (*metis dual-order.eq-iff short-ch-def*)
using *assms chain-defs not-less* **by** *fastforce*

12 Betweenness: Rays and Intervals

"Given any two distinct events a, b of a path we define the segment $(ab) = \{x : [a\ x\ b],\ x \in ab\}$" [Schutz97] Our version is a little different, because it is defined for any a, b of type $'a$. Thus we can have empty set segments, while Schutz can prove (once he proves path density) that segments are never empty.

definition *segment* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$
  where $segment\ a\ b \equiv \{x::'a.\ \exists\, ab.\ [a;x;b] \wedge x \in ab \wedge path\ ab\ a\ b\}$

abbreviation *is-segment* :: $'a\ set \Rightarrow bool$
  where $is\text{-}segment\ ab \equiv (\exists\, a\ b.\ ab = segment\ a\ b)$

definition *interval* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$
  where $interval\ a\ b \equiv insert\ b\ (insert\ a\ (segment\ a\ b))$

abbreviation *is-interval* :: $'a\ set \Rightarrow bool$
  where $is\text{-}interval\ ab \equiv (\exists\, a\ b.\ ab = interval\ a\ b)$

definition *prolongation* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$
  where $prolongation\ a\ b \equiv \{x::'a.\ \exists\, ab.\ [a;b;x] \wedge x \in ab \wedge path\ ab\ a\ b\}$

abbreviation *is-prolongation* :: $'a\ set \Rightarrow bool$
  where $is\text{-}prolongation\ ab \equiv \exists\, a\ b.\ ab = prolongation\ a\ b$

I think this is what Schutz actually meant, maybe there is a typo in the text? Notice that $b \in ray\ a\ b$ for any a, always. Cf the comment on *segment-def*. Thus $\exists\, ray\ a\ b \neq \{\}$ is no guarantee that a path ab exists.

definition *ray* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$
  where $ray\ a\ b \equiv insert\ b\ (segment\ a\ b \cup prolongation\ a\ b)$

abbreviation *is-ray* :: $'a\ set \Rightarrow bool$
  where $is\text{-}ray\ R \equiv \exists\, a\ b.\ R = ray\ a\ b$

definition *is-ray-on* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$
  where $is\text{-}ray\text{-}on\ R\ P \equiv P \in \mathcal{P} \wedge R \subseteq P \wedge is\text{-}ray\ R$

This is as in Schutz. Notice b is not in the ray through b?

definition *ray-Schutz* :: $'a \Rightarrow 'a \Rightarrow 'a\ set$
  where $ray\text{-}Schutz\ a\ b \equiv insert\ a\ (segment\ a\ b \cup prolongation\ a\ b)$

lemma *ends-notin-segment*: $a \notin segment\ a\ b \wedge b \notin segment\ a\ b$
  using *abc-abc-neq segment-def* **by** *fastforce*

lemma *ends-in-int*: $a \in interval\ a\ b \wedge b \in interval\ a\ b$
  using *interval-def* **by** *auto*

lemma *seg-betw*: $x \in segment\ a\ b \longleftrightarrow [a;x;b]$

using *segment-def abc-abc-neq abc-ex-path* **by** *fastforce*

lemma *pro-betw*: $x \in$ *prolongation a b* $\longleftrightarrow$ [*a*;*b*;*x*]
using *prolongation-def abc-abc-neq abc-ex-path* **by** *fastforce*

lemma *seg-sym*: *segment a b* = *segment b a*
using *abc-sym segment-def* **by** *auto*

lemma *empty-segment*: *segment a a* = {}
by (*simp add*: *segment-def*)

lemma *int-sym*: *interval a b* = *interval b a*
by (*simp add*: *insert-commute interval-def seg-sym*)

lemma *seg-path*:
assumes $x \in$ *segment a b*
obtains *ab* **where** *path ab a b segment a b* $\subseteq$ *ab*
proof −
obtain *ab* **where** *path ab a b*
using *abc-abc-neq abc-ex-path assms seg-betw*
by *meson*
have *segment a b* $\subseteq$ *ab*
using ‹*path ab a b*› *abc-ex-path path-unique seg-betw*
by *fastforce*
thus *?thesis*
using ‹*path ab a b*› *that* **by** *blast*
qed

lemma *seg-path2*:
assumes *segment a b* ≠ {}
obtains *ab* **where** *path ab a b segment a b* $\subseteq$ *ab*
using *assms seg-path* **by** *force*

Path density (theorem 17) will extend this by weakening the assumptions to *segment a b* ≠ {}.

lemma *seg-endpoints-on-path*:
assumes *card* (*segment a b*) $\geq$ *2 segment a b* $\subseteq$ *P P*∈$\mathcal{P}$
shows *path P a b*
proof −
have *non-empty*: *segment a b* ≠ {} **using** *assms*(*1*) *numeral-2-eq-2* **by** *auto*
then obtain *ab* **where** *path ab a b segment a b* $\subseteq$ *ab*
using *seg-path2* **by** *force*
have *a*≠*b* **by** (*simp add*: ‹*path ab a b*›)
obtain *x y* **where** *x*∈*segment a b y*∈*segment a b x*≠*y*
using *assms*(*1*) *numeral-2-eq-2*
by (*metis card.infinite card-le-Suc0-iff-eq not-less-eq-eq not-numeral-le-zero*)
have [*a*;*x*;*b*]
using ‹*x* ∈ *segment a b*› *seg-betw* **by** *auto*
have [*a*;*y*;*b*]

```
    using ‹y ∈ segment a b› seg-betw by auto
  have x∈P ∧ y∈P
    using ‹x ∈ segment a b› ‹y ∈ segment a b› assms(2) by blast
  have x∈ab ∧ y∈ab
    using ‹segment a b ⊆ ab› ‹x ∈ segment a b› ‹y ∈ segment a b› by blast
  have ab=P
     using ‹path ab a b› ‹x ∈ P ∧ y ∈ P› ‹x ∈ ab ∧ y ∈ ab› ‹x ≠ y› assms(3)
path-unique by auto
  thus ?thesis
    using ‹path ab a b› by auto
qed

lemma pro-path:
  assumes x ∈ prolongation a b
  obtains ab where path ab a b prolongation a b ⊆ ab
proof −
  obtain ab where path ab a b
    using abc-abc-neq abc-ex-path assms pro-betw
    by meson
  have prolongation a b ⊆ ab
    using ‹path ab a b› abc-ex-path path-unique pro-betw
    by fastforce
  thus ?thesis
    using ‹path ab a b› that by blast
qed

lemma ray-cases:
  assumes x ∈ ray a b
  shows [a;x;b] ∨ [a;b;x] ∨ x = b
proof −
  have x∈segment a b ∨ x∈ prolongation a b ∨ x=b
    using assms ray-def by auto
  thus [a;x;b] ∨ [a;b;x] ∨ x = b
    using pro-betw seg-betw by auto
qed

lemma ray-path:
  assumes x ∈ ray a b x≠b
  obtains ab where path ab a b ∧ ray a b ⊆ ab
proof −
  let ?r = ray a b
  have ?r ≠ {b}
    using assms by blast
  have ∃ ab. path ab a b ∧ ray a b ⊆ ab
  proof −
    have betw-cases: [a;x;b] ∨ [a;b;x] using ray-cases assms
      by blast
    then obtain ab where path ab a b
      using abc-abc-neq abc-ex-path by blast
```

have *?r* ⊆ *ab* **using** *betw-cases*
proof (*rule disjE*)
assume [*a*;*x*;*b*]
show *?r* ⊆ *ab*
proof
fix *x* **assume** *x*∈*?r*
show *x*∈*ab*
by (*metis* ‹*path ab a b*› ‹*x* ∈ *ray a b*› *abc-ex-path eq-paths ray-cases*)
qed
next assume [*a*;*b*;*x*]
show *?r* ⊆ *ab*
proof
fix *x* **assume** *x*∈*?r*
show *x*∈*ab*
by (*metis* ‹*path ab a b*› ‹*x* ∈ *ray a b*› *abc-ex-path eq-paths ray-cases*)
qed
qed
thus *?thesis*
using ‹*path ab a b*› **by** *blast*
qed
thus *?thesis*
using *that* **by** *blast*
qed

end

13 MinkowskiChain: O6

O6 supposedly serves the same purpose as Pasch's axiom.

locale *MinkowskiChain* = *MinkowskiBetweenness* +
assumes *O6*: ⟦{ *Q*,*R*,*S*,*T*} ⊆ 𝒫; *card*{ *Q*,*R*,*S*} = *3*; *a* ∈ *Q*∩*R*; *b* ∈ *Q*∩*S*; *c* ∈ *R*∩*S*; *d*∈*S*∩*T*; *e*∈*R*∩*T*; [*b*;*c*;*d*]; [*c*;*e*;*a*]⟧
⟹ ∃ *f*∈*T*∩*Q*. ∃ *g X*. [*g*⇝*X*|*a*..*f*..*b*]
begin

lemma *O6-old*: ⟦*Q* ∈ 𝒫; *R* ∈ 𝒫; *S* ∈ 𝒫; *T* ∈ 𝒫; *Q* ≠ *R*; *Q* ≠ *S*; *R* ≠ *S*; *a* ∈ *Q*∩*R* ∧ *b* ∈ *Q*∩*S* ∧ *c* ∈ *R*∩*S*;
∃ *d*∈*S*. [*b*;*c*;*d*] ∧ (∃ *e*∈*R*. *d* ∈ *T* ∧ *e* ∈ *T* ∧ [*c*;*e*;*a*])⟧
⟹ ∃ *f*∈*T*∩*Q*. ∃ *g X*. [*g*⇝*X*|*a*..*f*..*b*]
using *O6*[*of Q R S T a b c*] **by** (*metis IntI card-3-dist empty-subsetI insert-subset*)

14 Chains: (Closest) Bounds

definition *is-bound-f* :: *'a* ⇒ *'a set* ⇒ (*nat*⇒*'a*) ⇒ *bool* **where**
is-bound-f Q$_b$ *Q f* ≡
∀ *i j* ::*nat*. [*f*⇝*Q*|(*f* 0)..] ∧ (*i*<*j* ⟶ [*f i*; *f j*; *Q*$_b$])

definition *is-bound* **where**
  is-bound Q_b $Q \equiv$
    $\exists f{::}(nat \Rightarrow 'a).$ *is-bound-f* Q_b Q f

Q_b has to be on the same path as the chain Q. This is left implicit in the betweenness condition (as is $Q_b \in \mathcal{E}$). So this is equivalent to Schutz only if we also assume his axioms, i.e. the statement of the continuity axiom is no longer independent of other axioms.

definition *all-bounds* **where**
  all-bounds $Q = \{Q_b.$ *is-bound* Q_b $Q\}$

definition *bounded* **where**
  bounded $Q \equiv \exists$ $Q_b.$ *is-bound* Q_b Q

lemma *bounded-imp-inf*:
  assumes *bounded* Q
  shows *infinite* Q
  using *assms bounded-def is-bound-def is-bound-f-def chain-defs* **by** *meson*

definition *closest-bound-f* **where**
  closest-bound-f Q_b Q $f \equiv$
~~Q is an infinite chain indexed by f bound by Q_b~~
    is-bound-f Q_b Q $f \wedge$
~~Any other bound must be further from the start of the chain than the closest bound~~
    $(\forall$ $Q_b'.$ (*is-bound* Q_b' $Q \wedge Q_b' \neq Q_b) \longrightarrow [f\ 0;\ Q_b;\ Q_b'])$

definition *closest-bound* **where**
  closest-bound Q_b $Q \equiv$
    $\exists f.$ *is-bound-f* Q_b Q f
     $\wedge (\forall$ $Q_b'.$ (*is-bound* Q_b' $Q \wedge Q_b' \neq Q_b) \longrightarrow [f\ 0;\ Q_b;\ Q_b'])$

lemma *closest-bound* Q_b $Q = (\exists f.$ *closest-bound-f* Q_b Q $f)$
  unfolding *closest-bound-f-def closest-bound-def* **by** *simp*

end

15 MinkowskiUnreachable: I5-I7

locale *MinkowskiUnreachable* = *MinkowskiChain* +
  assumes *I5*: $[\![Q \in \mathcal{P};\ b \in \mathcal{E} - Q]\!] \Longrightarrow \exists x\ y.\ \{x,y\} \subseteq$ *unreach−on* Q *from* $b \wedge x \neq y$
    and *I6*: $[\![Q \in \mathcal{P};\ b \in \mathcal{E} - Q;\ \{Qx,Qz\} \subseteq$ *unreach−on* Q *from* $b;\ Qx \neq Qz]\!]$
    $\Longrightarrow \exists X\ f.\ [f \rightsquigarrow X | Qx..Qz]$
      $\wedge\ (\forall\ i \in \{1\ ..$ *card* $X - 1\}.\ (f\ i) \in$ *unreach−on* Q *from* b
        $\wedge\ (\forall\ Qy \in \mathcal{E}.\ [f(i-1);\ Qy;\ f\ i] \longrightarrow Qy \in$ *unreach−on* Q *from* $b))$

and *I7*: $[\![Q \in \mathcal{P};\ b \in \mathcal{E}-Q;\ Qx \in Q - unreach{-}on\ Q\ from\ b;\ Qy \in unreach{-}on\ Q\ from\ b]\!]$
$\Longrightarrow \exists\, g\ X\ Qn.\ [g{\rightsquigarrow}X|\, Qx..Qy..Qn] \wedge Qn \in Q - unreach{-}on\ Q\ from\ b$
begin

lemma *two-in-unreach*:
$[\![Q \in \mathcal{P};\ b \in \mathcal{E};\ b \notin Q]\!] \Longrightarrow \exists\, x{\in}unreach{-}on\ Q\ from\ b.\ \exists\, y{\in}unreach{-}on\ Q\ from\ b.\ x \neq y$
using *I5* **by** *fastforce*

lemma *I6-old*:
assumes $Q \in \mathcal{P}\ b \notin Q\ b \in \mathcal{E}\ Qx \in (unreach{-}on\ Q\ from\ b)\ Qz \in (unreach{-}on\ Q\ from\ b)\ Qx{\neq}Qz$
shows $\exists\, X.\ \exists f.\ ch{-}by{-}ord\ f\ X \wedge f\ 0 = Qx \wedge f\ (card\ X - 1) = Qz \wedge$
$(\forall\, i{\in}\{1..card\ X - 1\}.\ (f\ i) \in unreach{-}on\ Q\ from\ b \wedge (\forall\, Qy{\in}\mathcal{E}.\ [f(i{-}1);\ Qy;\ f\ i] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b)) \wedge$
$(short{-}ch\ X \longrightarrow Qx{\in}X \wedge Qz{\in}X \wedge (\forall\, Qy{\in}\mathcal{E}.\ [Qx;Qy;Qz] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b))$
proof −
from *assms I6*[*of Q b Qx Qz*] **obtain** *f X*
where *fX*: $[f{\rightsquigarrow}X|\, Qx..Qz]$
$(\forall\, i{\in}\{1\ ..\ card\ X - 1\}.\ (f\ i) \in unreach{-}on\ Q\ from\ b \wedge (\forall\, Qy{\in}\mathcal{E}.\ [f(i{-}1);\ Qy;\ f\ i] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b))$
using *DiffI Un-Diff-cancel* **by** *blast*
show *?thesis*
proof ((*rule exI*)+, *intro conjI*, *rule-tac*[*4*] *ballI*, *rule-tac*[*5*] *impI*; (*intro conjI*)*?*)
show *1*: $[f{\rightsquigarrow}X]\ f\ 0 = Qx\ f\ (card\ X - 1) = Qz$
using *fX*(*1*) *chain-defs* **by** *meson*+
{
fix *i* **assume** *i-asm*: $i{\in}\{1..card\ X - 1\}$
show *2*: $f\ i \in unreach{-}on\ Q\ from\ b$
using *fX*(*2*) *i-asm* **by** *fastforce*
show *3*: $\forall\, Qy{\in}\mathcal{E}.\ [f\ (i - 1);Qy;f\ i] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b$
using *fX*(*2*) *i-asm* **by** *blast*
} {
assume *X-asm*: *short-ch X*
show *4*: $Qx \in X\ Qz \in X$
using *fX*(*1*) *points-in-chain* **by** *auto*
have $\{1..card\ X{-}1\} = \{1\}$
using *X-asm short-ch-alt*(*2*) **by** *force*
thus *5*: $\forall\, Qy{\in}\mathcal{E}.\ [Qx;Qy;Qz] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b$
using *fX*(*2*) *1*(*2*,*3*) **by** *auto*
}
qed
qed

lemma *I7-old*:
assumes $Q \in \mathcal{P}\ b \notin Q\ b \in \mathcal{E}\ Qx \in Q - unreach{-}on\ Q\ from\ b\ Qy \in unreach{-}on\ Q\ from\ b$

shows $\exists\, g\ X\ Qn.\ [g \rightsquigarrow X | Qx..Qy..Qn] \wedge Qn \in Q - unreach{-}on\ Q\ from\ b$
using *I7 assms* **by** *auto*

lemma *card-unreach-geq-2*:
assumes $Q{\in}\mathcal{P}\ b{\in}\mathcal{E}{-}Q$
shows $2 \leq card\ (unreach{-}on\ Q\ from\ b) \vee (infinite\ (unreach{-}on\ Q\ from\ b))$
using *DiffD1 assms(1) assms(2) card-le-Suc0-iff-eq two-in-unreach* **by** *fastforce*

In order to more faithfully capture Schutz' definition of unreachable subsets via a path, we show that intersections of distinct paths are unique, and then define a new notation that doesn't carry the intersection of two paths around.

lemma *unreach-empty-on-same-path*:
assumes $P{\in}\mathcal{P}\ Q{\in}\mathcal{P}\ P{=}Q$
shows $\forall\, x.\ unreach{-}via\ P\ on\ Q\ from\ a\ to\ x = \{\}$
unfolding *unreachable-subset-via-notation-def unreachable-subset-via-def unreachable-subset-def*
by (*simp add*: *assms(3)*)

definition *unreachable-subset-via-notation-2* (*unreach−via - on - from -* [*100*, *100*, *100*] *100*)
where $unreach{-}via\ P\ on\ Q\ from\ a \equiv unreachable\text{-}subset\text{-}via\ Q\ a\ P\ (THE\ x.\ x{\in}Q{\cap}P)$

lemma *unreach-via-for-crossing-paths*:
assumes $P{\in}\mathcal{P}\ Q{\in}\mathcal{P}\ P{\cap}Q = \{x\}$
shows $unreach{-}via\ P\ on\ Q\ from\ a\ to\ x = unreach{-}via\ P\ on\ Q\ from\ a$
unfolding *unreachable-subset-via-notation-2-def is-singleton-def unreachable-subset-via-notation-def*
using *the-equality assms* **by** (*metis Int-commute empty-iff insert-iff*)

end

16 MinkowskiSymmetry: Symmetry

locale *MinkowskiSymmetry* = *MinkowskiUnreachable* +
assumes *Symmetry*: $[\![\{Q,R,S\} \subseteq \mathcal{P};\ card\ \{Q,R,S\} = 3;$
$x \in Q{\cap}R{\cap}S;\ Q_a \in Q;\ Q_a \neq x;$
$unreach{-}via\ R\ on\ Q\ from\ Q_a = unreach{-}via\ S\ on\ Q\ from\ Q_a]\!]$
$\Longrightarrow \exists\, \vartheta{::}'a{\Rightarrow}'a.$ *(* there is a map *)*
bij-betw $(\lambda P.\ \{\vartheta\ y \mid y.\ y{\in}P\})\ \mathcal{P}\ \mathcal{P}$ *(* which induces a bijection *)*
(*)
$\wedge\ (y{\in}Q \longrightarrow \vartheta\ y = y)$ *(* that leaves Q invariant *)*
$\wedge\ (\lambda P.\ \{\vartheta\ y \mid y.\ y{\in}P\})\ R = S$ *(* that maps R to S *)*
begin

lemma *Symmetry-old*:
assumes $Q \in \mathcal{P}\ R \in \mathcal{P}\ S \in \mathcal{P}\ Q \neq R\ Q \neq S\ R \neq S$
and $x \in Q{\cap}R{\cap}S\ Q_a \in Q\ Q_a \neq x$

and *unreach−via R on Q from* Q_a *to* x = *unreach−via S on Q from* Q_a *to* x
shows $\exists\,\vartheta::'a\Rightarrow'a.$ *bij-betw* $(\lambda P.\ \{\vartheta\ y \mid y.\ y\in P\})\ \mathcal{P}\ \mathcal{P}$
$\wedge\ (y\in Q \longrightarrow \vartheta\ y = y)$
$\wedge\ (\lambda P.\ \{\vartheta\ y \mid y.\ y\in P\})\ R = S$
proof −
have *QS*: $Q\cap S = \{x\}$ **and** *QR*: $Q\cap R = \{x\}$
using *assms*(*1−7*) *paths-cross-once* **by** (*metis Int-iff empty-iff insertE*)+
have *unreach−via R on Q from* Q_a = *unreach−via R on Q from* Q_a *to* x
using *unreach-via-for-crossing-paths QR* **by** (*simp add*: *Int-commute assms*(*1*,*2*))
moreover have *unreach−via S on Q from* Q_a = *unreach−via S on Q from* Q_a *to* x
using *unreach-via-for-crossing-paths QS* **by** (*simp add*: *Int-commute assms*(*1*,*3*))
ultimately show *?thesis*
using *Symmetry assms* **by** *simp*
qed

end

17 MinkowskiContinuity: Continuity

locale *MinkowskiContinuity* = *MinkowskiSymmetry* +
assumes *Continuity*: *bounded* $Q \Longrightarrow \exists\, Q_b.$ *closest-bound* Q_b Q

18 MinkowskiSpacetime: Dimension (I4)

locale *MinkowskiSpacetime* = *MinkowskiContinuity* +

assumes *ex-3SPRAY* [*simp*]: $[\![\mathcal{E} \neq \{\}]\!] \Longrightarrow \exists\, x\in\mathcal{E}.$ *3−SPRAY x*
begin

There exists an event by *nonempty-events*, and by *ex-3SPRAY* there is a three-SPRAY, which by *three-SPRAY-ge4* means that there are at least four paths.

lemma *four-paths*:
$\exists\, Q1\in\mathcal{P}.\ \exists\, Q2\in\mathcal{P}.\ \exists\, Q3\in\mathcal{P}.\ \exists\, Q4\in\mathcal{P}.\ Q1 \neq Q2 \wedge Q1 \neq Q3 \wedge Q1 \neq Q4 \wedge Q2 \neq Q3 \wedge Q2 \neq Q4 \wedge Q3 \neq Q4$
using *nonempty-events ex-3SPRAY three-SPRAY-ge4* **by** *blast*

end

end

theory *TemporalOrderOnPath*
imports *Minkowski HOL−Library.Disjoint-Sets*
begin

In Schutz [1, pp. 18-30], this is "Chapter 3: Temporal order on a path". All theorems are from Schutz, all lemmas are either parts of the Schutz proofs extracted, or additional lemmas which needed to be added, with the exception of the three transitivity lemmas leading to Theorem 9, which are given by Schutz as well. Much of what we'd like to prove about chains with respect to injectivity, surjectivity, bijectivity, is proved in *TernaryOrdering.thy*. Some more things are proved in interlude sections.

19 Preliminary Results for Primitives

First some proofs that belong in this section but aren't proved in the book or are covered but in a different form or off-handed remark.

context *MinkowskiPrimitive* **begin**

lemma *cross-once-notin*:
  assumes $Q \in \mathcal{P}$
    and $R \in \mathcal{P}$
    and $a \in Q$
    and $b \in Q$
    and $b \in R$
    and $a \neq b$
    and $Q \neq R$
  shows $a \notin R$
using *assms paths-cross-once eq-paths* **by** *meson*

lemma *paths-cross-at*:
  assumes *path-Q*: $Q \in \mathcal{P}$ **and** *path-R*: $R \in \mathcal{P}$
    and *Q-neq-R*: $Q \neq R$
    and *QR-nonempty*: $Q \cap R \neq \{\}$
    and *x-inQ*: $x \in Q$ **and** *x-inR*: $x \in R$
  shows $Q \cap R = \{x\}$
proof (*rule equalityI*)
  show $Q \cap R \subseteq \{x\}$
  proof (*rule subsetI*, *rule ccontr*)
    fix y
    assume *y-in-QR*: $y \in Q \cap R$
      and *y-not-in-just-x*: $y \notin \{x\}$
    then have *y-neq-x*: $y \neq x$ **by** *simp*
    then have $\neg\ (\exists z.\ Q \cap R = \{z\})$
      by (*meson Q-neq-R path-Q path-R x-inQ x-inR y-in-QR cross-once-notin IntD1 IntD2*)

thus *False* **using** *paths-cross-once* **by** (*meson QR-nonempty Q-neq-R path-Q path-R*)
qed
show $\{x\} \subseteq Q \cap R$ **using** *x-inQ x-inR* **by** *simp*
qed

lemma *events-distinct-paths*:
assumes *a-event*: $a \in \mathcal{E}$
and *b-event*: $b \in \mathcal{E}$
and *a-neq-b*: $a \neq b$
shows $\exists\, R{\in}\mathcal{P}.\ \exists\, S{\in}\mathcal{P}.\ a \in R \wedge b \in S \wedge (R \neq S \longrightarrow (\exists!c{\in}\mathcal{E}.\ R \cap S = \{c\}))$
by (*metis events-paths assms paths-cross-once*)

end
context *MinkowskiBetweenness* **begin**

lemma assumes $[a;b;c]$ **shows** $\exists f.$ *local-long-ch-by-ord f* $\{a,b,c\}$
using *abc-abc-neq*[*OF assms*] **unfolding** *chain-defs*
by (*simp add*: *assms ord-ordered-loc*)

lemma *between-chain*: $[a;b;c] \Longrightarrow$ *ch* $\{a,b,c\}$
proof −
assume $[a;b;c]$
hence $\exists f.$ *local-ordering f betw* $\{a,b,c\}$
by (*simp add*: *abc-abc-neq ord-ordered-loc*)
hence $\exists f.$ *local-long-ch-by-ord f* $\{a,b,c\}$
using ‹$[a;b;c]$› *abc-abc-neq local-long-ch-by-ord-def* **by** *auto*
thus *?thesis*
by (*simp add*: *chain-defs*)
qed

end

20 3.1 Order on a finite chain

context *MinkowskiBetweenness* **begin**

20.1 Theorem 1

See *Minkowski.abc-only-cba*. Proving it again here to show it can be done following the prose in Schutz.

theorem *theorem1* [*no-atp*]:
assumes *abc*: $[a;b;c]$
shows $[c;b;a] \wedge \neg\, [b;c;a] \wedge \neg\, [c;a;b]$
proof −

have *part-i*: $[c;b;a]$ **using** *abc abc-sym* **by** *simp*

have *part-ii*: ¬ [*b*;*c*;*a*]
proof (*rule notI*)
assume [*b*;*c*;*a*]
then have [*a*;*b*;*a*] **using** *abc abc-bcd-abd* **by** *blast*
thus *False* **using** *abc-ac-neq* **by** *blast*
qed

have *part-iii*: ¬ [*c*;*a*;*b*]
proof (*rule notI*)
assume [*c*;*a*;*b*]
then have [*c*;*a*;*c*] **using** *abc abc-bcd-abd* **by** *blast*
thus *False* **using** *abc-ac-neq* **by** *blast*
qed
thus *?thesis* **using** *part-i part-ii part-iii* **by** *auto*
qed

20.2 Theorem 2

The lemma *abc-bcd-acd*, equal to the start of Schutz's proof, is given in *Minkowski* in order to prove some equivalences. We're splitting up Theorem 2 into two named results:

order-finite-chain there is a betweenness relation for each triple of adjacent events, and

index-injective all events of a chain are distinct.

We will be following Schutz' proof for both. Distinctness of chain events is interpreted as injectivity of the indexing function (see *index-injective*): we assume that this corresponds to what Schutz means by distinctness of elements in a sequence.

For the case of two-element chains: the elements are distinct by definition, and the statement on *local-ordering* is void (respectively, $False \Longrightarrow P$ for any P). We exclude this case from our proof of *order-finite-chain*. Two helper lemmas are provided, each capturing one of the proofs by induction in Schutz' writing.

lemma *thm2-ind1*:
assumes *chX*: *local-long-ch-by-ord f X*
and *finiteX*: *finite X*
shows ∀ *j i*. ((*i*::*nat*) < *j* ∧ *j* < *card X* − *1*) ⟶ [*f i*; *f j*; *f* (*j* + *1*)]
proof (*rule allI*)+
let *?P* = λ *i j*. [*f i*; *f j*; *f* (*j*+*1*)]
fix *i j*
show (*i*<*j* ∧ *j*<*card X* −*1*) ⟶ *?P i j*
proof (*induct j*)
case *0*
show *?case* **by** *blast*
next

```
    case (Suc j)
    show ?case
    proof (clarify)
      assume asm: i<Suc j Suc j<card X −1
      have pj: ?P j (Suc j)
        using asm(2) chX less-diff-conv local-long-ch-by-ord-def local-ordering-def
        by (metis Suc-eq-plus1)
      have i<j ∨ i=j using asm(1)
        by linarith
      thus ?P i (Suc j)
      proof
        assume i=j
        hence Suc i = Suc j ∧ Suc (Suc j) = Suc (Suc j)
          by simp
        thus ?P i (Suc j)
          using pj by auto
      next
        assume i<j
        have j < card X − 1
          using asm(2) by linarith
        thus ?P i (Suc j)
          using ‹i<j› Suc.hyps asm(1) asm(2) chX finiteX Suc-eq-plus1 abc-bcd-acd
pj
          by presburger
      qed
    qed
  qed
qed

lemma thm2-ind2:
  assumes chX: local-long-ch-by-ord f X
      and finiteX: finite X
    shows ∀ m l. (0<(l−m) ∧ (l−m) < l ∧ l < card X) ⟶ [f (l−m−1); f (l−m);
(f l)]
proof (rule allI)+
  fix l m
  let ?P = λ k l. [f (k−1); f k; f l]
  let ?n = card X
  let ?k = (l::nat)−m
  show 0 < ?k ∧ ?k < l ∧ l < ?n ⟶ ?P ?k l
  proof (induct m)
    case 0
    show ?case by simp
  next
    case (Suc m)
    show ?case
    proof (clarify)
      assume asm: 0 < l − Suc m l − Suc m < l l < ?n
      have Suc m = 1 ∨ Suc m > 1 by linarith
```

```
    thus [f (l − Suc m − 1); f (l − Suc m); f l] (is ?goal)
    proof
      assume Suc m = 1
      show ?goal
      proof −
        have l − Suc m < card X
          using asm(2) asm(3) less-trans by blast
        then show ?thesis
          using ‹Suc m = 1› asm finiteX thm2-ind1 chX
          using Suc-eq-plus1 add-diff-inverse-nat diff-Suc-less
                gr-implies-not-zero less-one plus-1-eq-Suc
          by (smt local-long-ch-by-ord-def ordering-ord-ijk-loc)
      qed
    next
      assume Suc m > 1
      show ?goal
        apply (rule-tac a=f l and c=f(l − Suc m − 1) in abc-sym)
          apply (rule-tac a=f l and c=f(l−Suc m) and d=f(l−Suc m−1) and
b=f(l−m) in abc-bcd-acd)
      proof −
        have [f(l−m−1); f(l−m); f l]
          using Suc.hyps ‹1 < Suc m› asm(1,3) by force
        thus [f l; f(l − m); f(l − Suc m)]
          using abc-sym One-nat-def diff-zero minus-nat.simps(2)
          by metis
        have Suc(l − Suc m − 1) = l − Suc m Suc(l − Suc m) = l−m
          using Suc-pred asm(1) by presburger+
        hence [f(l − Suc m − 1); f(l − Suc m); f(l − m)]
          using chX unfolding local-long-ch-by-ord-def local-ordering-def
          by (metis asm(2,3) less-trans-Suc)
        thus [f(l − m); f(l − Suc m); f(l − Suc m − 1)]
          using abc-sym by blast
      qed
    qed
  qed
qed

lemma thm2-ind2b:
  assumes chX: local-long-ch-by-ord f X
      and finiteX: finite X
      and ordered-nats: 0<k ∧ k<l ∧ l < card X
    shows [f (k−1); f k; f l]
  using thm2-ind2 finiteX chX ordered-nats
  by (metis diff-diff-cancel less-imp-le)
```

This is Theorem 2 properly speaking, except for the "chain elements are distinct" part (which is proved as injectivity of the index later). Follows Schutz fairly well! The statement Schutz proves under (i) is given in *Minkowski-*

Betweenness.abc-bcd-acd instead.

theorem *order-finite-chain*:
assumes *chX*: *local-long-ch-by-ord f X*
and *finiteX*: *finite X*
and *ordered-nats*: $0 \leq (i::nat) \wedge i < j \wedge j < l \wedge l < card\ X$
shows [*f i*; *f j*; *f l*]
proof −
let *?n* = *card X* − *1*
have *ord1*: *0*≤*i* ∧ *i*<*j* ∧ *j*<*?n*
using *ordered-nats* **by** *linarith*
have *e2*: [*f i*; *f j*; *f* (*j*+*1*)] **using** *thm2-ind1*
using *Suc-eq-plus1 chX finiteX ord1*
by *presburger*
have *e3*: ∀ *k*. *0*<*k* ∧ *k*<*l* ⟶ [*f* (*k*−*1*); *f k*; *f l*]
using *thm2-ind2b chX finiteX ordered-nats*
by *blast*
have *j*<*l*−*1* ∨ *j*=*l*−*1*
using *ordered-nats* **by** *linarith*
thus *?thesis*
proof
assume *j*<*l*−*1*
have [*f j*; *f* (*j*+*1*); *f l*]
using *e3 abc-abc-neq ordered-nats*
using ‹*j* < *l* − *1*› *less-diff-conv* **by** *auto*
thus *?thesis*
using *e2 abc-bcd-abd*
by *blast*
next
assume *j*=*l*−*1*
thus *?thesis* **using** *e2*
using *ordered-nats* **by** *auto*
qed
qed

corollary *order-finite-chain2*:
assumes *chX*: [*f*⇝*X*]
and *finiteX*: *finite X*
and *ordered-nats*: $0 \leq (i::nat) \wedge i < j \wedge j < l \wedge l < card\ X$
shows [*f i*; *f j*; *f l*]
proof −
have *card X* > *2* **using** *ordered-nats* **by** (*simp add*: *eval-nat-numeral*)
thus *?thesis* **using** *order-finite-chain chain-defs short-ch-card*(*1*) **by** (*metis assms nat-neq-iff*)
qed

theorem *index-injective*:
fixes *i*::*nat* **and** *j*::*nat*

```
  assumes chX: local-long-ch-by-ord f X
      and finiteX: finite X
      and indices: i<j j<card X
    shows f i ≠ f j
proof (cases)
  assume Suc i < j
  then have [f i; f (Suc(i)); f j]
    using order-finite-chain chX finiteX indices(2) by blast
  then show ?thesis
    using abc-abc-neq by blast
next
  assume ¬Suc i < j
  hence Suc i = j
    using Suc-lessI indices(1) by blast
  show ?thesis
  proof (cases)
    assume Suc j = card X
    then have 0<i
    proof −
      have card X ≥ 3
        using assms(1) finiteX long-chain-card-geq by blast
      thus ?thesis
        using ‹Suc i = j› ‹Suc j = card X› by linarith
    qed
    then have [f 0; f i; f j]
      using assms order-finite-chain by blast
    thus ?thesis
      using abc-abc-neq by blast
  next
    assume ¬Suc j = card X
    then have Suc j < card X
      using Suc-lessI indices(2) by blast
    then have [f i; f j; f(Suc j)]
      using chX finiteX indices(1) order-finite-chain by blast
    thus ?thesis
      using abc-abc-neq by blast
  qed
qed

theorem index-injective2:
  fixes i::nat and j::nat
  assumes chX: [f⇝X]
      and finiteX: finite X
      and indices: i<j j<card X
    shows f i ≠ f j
  using assms(1) unfolding ch-by-ord-def
proof (rule disjE)
  assume asm: short-ch-by-ord f X
  hence card X = 2 using short-ch-card(1) by simp
```

hence *j=1 i=0* **using** *indices plus-1-eq-Suc* **by** *auto*
thus *?thesis* **using** *asm* **unfolding** *chain-defs* **by** *force*
next
assume *local-long-ch-by-ord f X* **thus** *?thesis* **using** *index-injective assms* **by** *presburger*
qed

Surjectivity of the index function is easily derived from the definition of *local-ordering*, so we obtain bijectivity as an easy corollary to the second part of Theorem 2.

corollary *index-bij-betw*:
assumes *chX*: *local-long-ch-by-ord f X*
and *finiteX*: *finite X*
shows *bij-betw f* {*0..<card X*} *X*
proof (*unfold bij-betw-def*, (*rule conjI*))
show *inj-on f* {*0..<card X*}
using *index-injective*[*OF assms*] **by** (*metis* (*mono-tags*) *atLeastLessThan-iff inj-onI nat-neq-iff*)
{
fix *n* **assume** *n* ∈ {*0..<card X*}
then have *f n* ∈ *X*
using *assms* **unfolding** *chain-defs local-ordering-def* **by** *auto*
} moreover {
fix *x* **assume** *x* ∈ *X*
then have ∃ *n* ∈ {*0..<card X*}. *f n* = *x*
using *assms* **unfolding** *chain-defs local-ordering-def*
using *atLeastLessThan-iff bot-nat-0.extremum* **by** *blast*
} ultimately show *f* ‘ {*0..<card X*} = *X* **by** *blast*
qed

corollary *index-bij-betw2*:
assumes *chX*: [*f*⇝*X*]
and *finiteX*: *finite X*
shows *bij-betw f* {*0..<card X*} *X*
using *assms*(*1*) **unfolding** *ch-by-ord-def*
proof (*rule disjE*)
assume *local-long-ch-by-ord f X*
thus *bij-betw f* {*0..<card X*} *X* **using** *index-bij-betw assms* **by** *presburger*
next
assume *asm*: *short-ch-by-ord f X*
show *bij-betw f* {*0..<card X*} *X*
proof (*unfold bij-betw-def*, (*rule conjI*))
show *inj-on f* {*0..<card X*}
using *index-injective2*[*OF assms*] **by** (*metis* (*mono-tags*) *atLeastLessThan-iff inj-onI nat-neq-iff*)
{
fix *n* **assume** *asm2*: *n* ∈ {*0..<card X*}
have *f n* ∈ *X*
using *asm asm2 short-ch-card*(*1*) **apply** (*simp add*: *eval-nat-numeral*)

by (*metis One-nat-def less-Suc0 less-antisym short-ch-ord-in*)
} moreover {
fix x **assume** *asm2*: $x \in X$
have $\exists\, n \in \{0..<card\ X\}.\ f\ n = x$
using *short-ch-card*(*1*) *short-ch-by-ord-def asm asm2 atLeast0-lessThan-Suc*
by (*auto simp*: *eval-nat-numeral*)[*1*]
} ultimately show $f\ `\ \{0..<card\ X\} = X$ **by** *blast*
qed
qed

20.3 Additional lemmas about chains

lemma *first-neq-last*:
assumes $[f \rightsquigarrow Q|x..z]$
shows $x \neq z$
apply (*cases rule*: *finite-chain-with-cases*[*OF assms*])
using *chain-defs* **apply** (*metis Suc-1 card-2-iff diff-Suc-1*)
using *index-injective*[*of f Q 0 card Q − 1*]
by (*metis card.infinite diff-is-0-eq diff-less gr0I le-trans less-imp-le-nat*
less-numeral-extra(*1*) *numeral-le-one-iff semiring-norm*(*70*))

lemma *index-middle-element*:
assumes $[f \rightsquigarrow X|a..b..c]$
shows $\exists\, n.\ 0<n \wedge n<(card\ X - 1) \wedge f\ n = b$
proof −
obtain n **where** *n-def*: $n < card\ X\ f\ n = b$
using *local-ordering-def assms chain-defs* **by** (*metis two-ordered-loc*)
have $0<n \wedge n<(card\ X - 1) \wedge f\ n = b$
using *assms chain-defs n-def*
by (*metis* (*no-types*, *lifting*) *Suc-pred′ gr-implies-not0 less-SucE not-gr-zero*)
thus *?thesis* **by** *blast*
qed

Another corollary to Theorem 2, without mentioning indices.

corollary *fin-ch-betw*: $[f \rightsquigarrow X|a..b..c] \Longrightarrow [a;b;c]$
using *order-finite-chain2 index-middle-element*
using *finite-chain-def finite-chain-with-def finite-long-chain-with-def*
by (*metis* (*no-types*, *lifting*) *card-gt-0-iff diff-less empty-iff le-eq-less-or-eq less-one*)

lemma *long-chain-2-imp-3*: $[\![[f \rightsquigarrow X|a..c];\ card\ X > 2]\!] \Longrightarrow \exists\, b.\ [f \rightsquigarrow X|a..b..c]$
using *points-in-chain first-neq-last finite-long-chain-with-def*
by (*metis card-2-iff′ numeral-less-iff semiring-norm*(*75*,*78*))

lemma *finite-chain2-betw*: $[\![[f \rightsquigarrow X|a..c];\ card\ X > 2]\!] \Longrightarrow \exists\, b.\ [a;b;c]$
using *fin-ch-betw long-chain-2-imp-3* **by** *metis*

lemma *finite-long-chain-with-alt* [*chain-alts*]: $[f{\rightsquigarrow}Q|x..y..z] \longleftrightarrow [f{\rightsquigarrow}Q|x..z] \wedge [x;y;z] \wedge y{\in}Q$
proof
  {
    assume $[f{\rightsquigarrow}Q|x\ ..\ z] \wedge [x;y;z] \wedge y{\in}Q$
    thus $[f{\rightsquigarrow}Q|x..y..z]$
      using *abc-abc-neq finite-long-chain-with-def* **by** *blast*
  } {
    assume *asm*: $[f{\rightsquigarrow}Q|x..y..z]$
    show $[f{\rightsquigarrow}Q|x\ ..\ z] \wedge [x;y;z] \wedge y{\in}Q$
    using *asm fin-ch-betw finite-long-chain-with-def* **by** *blast*
  }
qed

lemma *finite-long-chain-with-card*: $[f{\rightsquigarrow}Q|x..y..z] \Longrightarrow card\ Q \geq 3$
  unfolding *chain-defs numeral-3-eq-3* **by** *fastforce*

lemma *finite-long-chain-with-alt2*:
  assumes *finite Q local-long-ch-by-ord f Q f 0* $= x$ *f* $(card\ Q - 1) = z$ $[x;y;z] \wedge y{\in}Q$
  shows $[f{\rightsquigarrow}Q|x..y..z]$
   using *assms finite-chain-alt finite-chain-with-def finite-long-chain-with-alt* **by** *blast*

lemma *finite-long-chain-with-alt3*:
  assumes *finite Q local-long-ch-by-ord f Q f 0* $= x$ *f* $(card\ Q - 1) = z$ $y{\neq}x \wedge y{\neq}z \wedge y{\in}Q$
  shows $[f{\rightsquigarrow}Q|x..y..z]$
   using *assms finite-chain-alt finite-chain-with-def finite-long-chain-with-def* **by** *auto*

lemma *chain-sym-obtain*:
  assumes $[f{\rightsquigarrow}X|a..b..c]$
  obtains *g* **where** $[g{\rightsquigarrow}X|c..b..a]$ **and** $g{=}(\lambda n.\ f\ (card\ X - 1 - n))$
  using *ordering-sym-loc*[*of betw X f*] *abc-sym assms* **unfolding** *chain-defs*
  using *first-neq-last points-in-long-chain*(*3*)
  by (*metis assms diff-self-eq-0 empty-iff finite-long-chain-with-def insert-iff minus-nat.diff-0*)

lemma *chain-sym*:
  assumes $[f{\rightsquigarrow}X|a..b..c]$
    shows $[\lambda n.\ f\ (card\ X - 1 - n){\rightsquigarrow}X|c..b..a]$
  using *chain-sym-obtain* [**where** $f{=}f$ **and** $a{=}a$ **and** $b{=}b$ **and** $c{=}c$ **and** $X{=}X$]
  using *assms*(*1*) **by** *blast*

lemma *chain-sym2*:
assumes $[f{\rightsquigarrow}X|a..c]$
shows $[\lambda n.\ f\ (card\ X\ -\ 1\ -\ n){\rightsquigarrow}X|c..a]$
proof −
{
assume *asm*: $a = f\ 0\ c = f\ (card\ X\ -\ 1)$
and *asm-short*: *short-ch-by-ord f X*
hence *cardX*: $card\ X = 2$
using *short-ch-card*(*1*) **by** *auto*
hence *ac*: $f\ 0 = a\ f\ 1 = c$
by (*simp add*: *asm*)+
have $n{=}1 \vee n{=}0$ **if** $n{<}card\ X$ **for** n
using *cardX that* **by** *linarith*
hence *fn-eq*: $(\lambda n.\ if\ n = 0\ then\ f\ 1\ else\ f\ 0) = (\lambda n.\ f\ (card\ X\ -\ Suc\ n))$ **if** $n{<}card\ X$ **for** n
by (*metis One-nat-def Zero-not-Suc ac*(*2*) *asm*(*2*) *not-gr-zero old.nat.inject zero-less-diff*)
have $c = f\ (card\ X\ -\ 1\ -\ 0)$ **and** $a = f\ (card\ X\ -\ 1\ -\ (card\ X\ -\ 1))$
and *short-ch-by-ord* $(\lambda n.\ f\ (card\ X\ -\ 1\ -\ n))\ X$
apply (*simp add*: *asm*)+
using *short-ch-sym*[*OF asm-short*] *fn-eq* ‹$f\ 1 = c$› *asm*(*2*) *short-ch-by-ord-def* **by** *fastforce*
}
consider *short-ch-by-ord f X*|$\exists\, b.\ [f{\rightsquigarrow}X|a..b..c]$
using *assms long-chain-2-imp-3 finite-chain-with-alt* **by** *fastforce*
thus *?thesis*
apply *cases*
using ‹$[\![a{=}f\ 0;\ c{=}f\ (card\ X{-}1);\ short\text{-}ch\text{-}by\text{-}ord\ f\ X]\!] \Longrightarrow short\text{-}ch\text{-}by\text{-}ord\ (\lambda n.\ f\ (card\ X\ -1{-}n))\ X$›
assms finite-chain-alt finite-chain-with-def **apply** *auto*[*1*]
using *chain-sym finite-long-chain-with-alt* **by** *blast*
qed

lemma *chain-sym-obtain2*:
assumes $[f{\rightsquigarrow}X|a..c]$
obtains g **where** $[g{\rightsquigarrow}X|c..a]$ **and** $g{=}(\lambda n.\ f\ (card\ X\ -\ 1\ -\ n))$
using *assms chain-sym2* **by** *auto*

end

21 Preliminary Results for Kinematic Triangles and Paths/Betweenness

Theorem 3 (collinearity) First we prove some lemmas that will be very helpful.

context *MinkowskiPrimitive* **begin**

lemma *triangle-permutes* [*no-atp*]:
assumes $\triangle\ a\ b\ c$
shows $\triangle\ a\ c\ b\ \triangle\ b\ a\ c\ \triangle\ b\ c\ a\ \triangle\ c\ a\ b\ \triangle\ c\ b\ a$
using *assms* **by** (*auto simp add*: *kinematic-triangle-def*)+

lemma *triangle-paths* [*no-atp*]:
assumes *tri-abc*: $\triangle\ a\ b\ c$
shows *path-ex a b path-ex a c path-ex b c*
using *tri-abc* **by** (*auto simp add*: *kinematic-triangle-def*)+

lemma *triangle-paths-unique*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
shows $\exists!ab.$ *path ab a b*
using *path-unique tri-abc triangle-paths*(*1*) **by** *auto*

The definition of the kinematic triangle says that there exist paths that a and b pass through, and a and c pass through etc that are not equal. But we can show there is a *unique* ab that a and b pass through, and assuming there is a path abc that a, b, c pass through, it must be unique. Therefore $ab = abc$ and $ac = abc$, but $ab \neq ac$, therefore *False*. Lemma *tri-three-paths* is not in the books but might simplify some path obtaining.

lemma *triangle-diff-paths*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
shows $\neg\ (\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q)$
proof (*rule notI*)
assume *not-thesis*: $\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$

then obtain *abc* **where** *path-abc*: $abc \in \mathcal{P} \wedge a \in abc \wedge b \in abc \wedge c \in abc$ **by** *auto*
have *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$ **using** *tri-abc kinematic-triangle-def* **by** *simp*

have $\exists\, ab{\in}\mathcal{P}.\ \exists\, ac{\in}\mathcal{P}.\ ab \neq ac \wedge a \in ab \wedge b \in ab \wedge a \in ac \wedge c \in ac$
using *tri-abc kinematic-triangle-def* **by** *metis*
then obtain *ab ac* **where** *ab-ac-relate*: $ab \in \mathcal{P} \wedge ac \in \mathcal{P} \wedge ab \neq ac \wedge \{a,b\} \subseteq ab \wedge \{a,c\} \subseteq ac$
by *blast*
have $\exists!ab{\in}\mathcal{P}.\ a \in ab \wedge b \in ab$ **using** *tri-abc triangle-paths-unique* **by** *blast*
then have *ab-eq-abc*: $ab = abc$ **using** *path-abc ab-ac-relate* **by** *auto*
have $\exists!ac{\in}\mathcal{P}.\ a \in ac \wedge b \in ac$ **using** *tri-abc triangle-paths-unique* **by** *blast*
then have *ac-eq-abc*: $ac = abc$ **using** *path-abc ab-ac-relate eq-paths abc-neq* **by** *auto*
have $ab = ac$ **using** *ab-eq-abc ac-eq-abc* **by** *simp*
thus *False* **using** *ab-ac-relate* **by** *simp*
qed

lemma *tri-three-paths* [*elim*]:

assumes *tri-abc*: $\triangle\ a\ b\ c$
shows $\exists\, ab\ bc\ ca.\ path\ ab\ a\ b \wedge path\ bc\ b\ c \wedge path\ ca\ c\ a \wedge ab \neq bc \wedge ab \neq ca \wedge bc \neq ca$
using *tri-abc triangle-diff-paths triangle-paths*(*2*,*3*) *triangle-paths-unique*
by *fastforce*

lemma *triangle-paths-neq*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
and *path-ab*: *path ab a b*
and *path-ac*: *path ac a c*
shows $ab \neq ac$
using *assms triangle-diff-paths* **by** *blast*

end
context *MinkowskiBetweenness* **begin**

lemma *abc-ex-path-unique*:
assumes *abc*: $[a;b;c]$
shows $\exists!Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$
proof −
have *a-neq-c*: $a \neq c$ **using** *abc-ac-neq abc* **by** *simp*
have $\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$ **using** *abc-ex-path abc* **by** *simp*
then obtain *P Q* **where** *path-P*: $P \in \mathcal{P}$ **and** *abc-inP*: $a \in P \wedge b \in P \wedge c \in P$
and *path-Q*: $Q \in \mathcal{P}$ **and** *abc-in-Q*: $a \in Q \wedge b \in Q \wedge c \in Q$ **by** *auto*
then have $P = Q$ **using** *a-neq-c eq-paths* **by** *blast*
thus *?thesis* **using** *eq-paths a-neq-c* **using** *abc-inP path-P* **by** *auto*
qed

lemma *betw-c-in-path*:
assumes *abc*: $[a;b;c]$
and *path-ab*: *path ab a b*
shows $c \in ab$

using *eq-paths abc-ex-path assms* **by** *blast*

lemma *betw-b-in-path*:
assumes *abc*: $[a;b;c]$
and *path-ab*: *path ac a c*
shows $b \in ac$
using *assms abc-ex-path-unique path-unique* **by** *blast*

lemma *betw-a-in-path*:
assumes *abc*: $[a;b;c]$
and *path-ab*: *path bc b c*
shows $a \in bc$
using *assms abc-ex-path-unique path-unique* **by** *blast*

lemma *triangle-not-betw-abc*:
assumes *tri-abc*: $\triangle\ a\ b\ c$

shows $\neg$ [a;b;c]
using *tri-abc abc-ex-path triangle-diff-paths* **by** *blast*

lemma *triangle-not-betw-acb*:
assumes *tri-abc*: $\triangle$ a b c
shows $\neg$ [a;c;b]
by (*simp add*: *tri-abc triangle-not-betw-abc triangle-permutes*(*1*))

lemma *triangle-not-betw-bac*:
assumes *tri-abc*: $\triangle$ a b c
shows $\neg$ [b;a;c]
by (*simp add*: *tri-abc triangle-not-betw-abc triangle-permutes*(*2*))

lemma *triangle-not-betw-any*:
assumes *tri-abc*: $\triangle$ a b c
shows $\neg$ ($\exists$ d∈{a,b,c}. $\exists$ e∈{a,b,c}. $\exists$ f∈{a,b,c}. [d;e;f])
by (*metis abc-ex-path abc-abc-neq empty-iff insertE tri-abc triangle-diff-paths*)

end

22 3.2 First collinearity theorem

theorem (**in** *MinkowskiChain*) *collinearity-alt2*:
assumes *tri-abc*: $\triangle$ a b c
and *path-de*: *path de d e*

and *path-ab*: *path ab a b*
and *bcd*: [b;c;d]
and *cea*: [c;e;a]
shows $\exists$ f∈de∩ab. [a;f;b]
proof −
have $\exists$ f∈ab ∩ de. $\exists$ X *ord*. [*ord*↝X|a..f..b]
proof −
have *path-ex a c* **using** *tri-abc triangle-paths*(*2*) **by** *auto*
then obtain *ac* **where** *path-ac*: *path ac a c* **by** *auto*
have *path-ex b c* **using** *tri-abc triangle-paths*(*3*) **by** *auto*
then obtain *bc* **where** *path-bc*: *path bc b c* **by** *auto*
have *ab-neq-ac*: *ab* ≠ *ac* **using** *triangle-paths-neq path-ab path-ac tri-abc* **by** *fastforce*
have *ab-neq-bc*: *ab* ≠ *bc* **using** *eq-paths ab-neq-ac path-ab path-ac path-bc* **by** *blast*
have *ac-neq-bc*: *ac* ≠ *bc* **using** *eq-paths ab-neq-bc path-ab path-ac path-bc* **by** *blast*
have *d-in-bc*: *d* ∈ *bc* **using** *bcd betw-c-in-path path-bc* **by** *blast*
have *e-in-ac*: *e* ∈ *ac* **using** *betw-b-in-path cea path-ac* **by** *blast*
show *?thesis*
using *O6-old* [**where** Q = *ab* **and** R = *ac* **and** S = *bc* **and** T = *de* **and** a = a **and** b = b **and** c = c]
ab-neq-ac ab-neq-bc ac-neq-bc path-ab path-bc path-ac path-de bcd cea

d-in-bc e-in-ac
by *auto*
qed
thus *?thesis* **using** *fin-ch-betw* **by** *blast*
qed

theorem (**in** *MinkowskiChain*) *collinearity-alt*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
and *path-de*: *path de d e*
and *bcd*: $[b;c;d]$
and *cea*: $[c;e;a]$
shows $\exists\, ab.\ path\ ab\ a\ b \wedge (\exists\, f{\in}de{\cap}ab.\ [a;f;b])$
proof −
have *ex-path-ab*: *path-ex a b*
using *tri-abc triangle-paths-unique* **by** *blast*
then obtain *ab* **where** *path-ab*: *path ab a b*
by *blast*
have $\exists\, f{\in}ab \cap de.\ \exists\, X\ g.\ [g{\rightsquigarrow}X|a..f..b]$
proof −
have *path-ex a c* **using** *tri-abc triangle-paths*(*2*) **by** *auto*
then obtain *ac* **where** *path-ac*: *path ac a c* **by** *auto*
have *path-ex b c* **using** *tri-abc triangle-paths*(*3*) **by** *auto*
then obtain *bc* **where** *path-bc*: *path bc b c* **by** *auto*
have *ab-neq-ac*: $ab \neq ac$ **using** *triangle-paths-neq path-ab path-ac tri-abc* **by** *fastforce*
have *ab-neq-bc*: $ab \neq bc$ **using** *eq-paths ab-neq-ac path-ab path-ac path-bc* **by** *blast*
have *ac-neq-bc*: $ac \neq bc$ **using** *eq-paths ab-neq-bc path-ab path-ac path-bc* **by** *blast*
have *d-in-bc*: $d \in bc$ **using** *bcd betw-c-in-path path-bc* **by** *blast*
have *e-in-ac*: $e \in ac$ **using** *betw-b-in-path cea path-ac* **by** *blast*
show *?thesis*
using *O6-old* [**where** $Q = ab$ **and** $R = ac$ **and** $S = bc$ **and** $T = de$ **and** $a = a$ **and** $b = b$ **and** $c = c$]
ab-neq-ac ab-neq-bc ac-neq-bc path-ab path-bc path-ac path-de bcd cea d-in-bc e-in-ac
by *auto*
qed
thus *?thesis* **using** *fin-ch-betw path-ab* **by** *fastforce*
qed

theorem (**in** *MinkowskiChain*) *collinearity*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
and *path-de*: *path de d e*
and *bcd*: $[b;c;d]$
and *cea*: $[c;e;a]$
shows $(\exists\, f{\in}de{\cap}(path\text{-}of\ a\ b).\ [a;f;b])$

proof −
let *?ab* = *path-of a b*
have *path-ab*: *path ?ab a b*
using *tri-abc theI'* [*OF triangle-paths-unique*] **by** *blast*
have ∃ *f*∈*?ab* ∩ *de*. ∃ *X ord*. [*ord*⇝*X*|*a..f..b*]
proof −
have *path-ex a c* **using** *tri-abc triangle-paths*(*2*) **by** *auto*
then obtain *ac* **where** *path-ac*: *path ac a c* **by** *auto*
have *path-ex b c* **using** *tri-abc triangle-paths*(*3*) **by** *auto*
then obtain *bc* **where** *path-bc*: *path bc b c* **by** *auto*
have *ab-neq-ac*: *?ab* ≠ *ac* **using** *triangle-paths-neq path-ab path-ac tri-abc* **by** *fastforce*
have *ab-neq-bc*: *?ab* ≠ *bc* **using** *eq-paths ab-neq-ac path-ab path-ac path-bc* **by** *blast*
have *ac-neq-bc*: *ac* ≠ *bc* **using** *eq-paths ab-neq-bc path-ab path-ac path-bc* **by** *blast*
have *d-in-bc*: *d* ∈ *bc* **using** *bcd betw-c-in-path path-bc* **by** *blast*
have *e-in-ac*: *e* ∈ *ac* **using** *betw-b-in-path cea path-ac* **by** *blast*
show *?thesis*
using *O6-old* [**where** *Q* = *?ab* **and** *R* = *ac* **and** *S* = *bc* **and** *T* = *de* **and** *a* = *a* **and** *b* = *b* **and** *c* = *c*]
ab-neq-ac ab-neq-bc ac-neq-bc path-ab path-bc path-ac path-de bcd cea d-in-bc e-in-ac
IntI Int-commute
by (*metis* (*no-types*, *lifting*))
qed
thus *?thesis* **using** *fin-ch-betw* **by** *blast*
qed

23 Additional results for Paths and Unreachables

context *MinkowskiPrimitive* **begin**

The degenerate case.

lemma *big-bang*:
assumes *no-paths*: $\mathcal{P} = \{\}$
shows ∃ *a*. $\mathcal{E} = \{a\}$
proof −
have ∃ *a*. *a* ∈ $\mathcal{E}$ **using** *nonempty-events* **by** *blast*
then obtain *a* **where** *a-event*: *a* ∈ $\mathcal{E}$ **by** *auto*
have ¬ (∃ *b*∈$\mathcal{E}$. *b* ≠ *a*)
proof (*rule notI*)
assume ∃ *b*∈$\mathcal{E}$. *b* ≠ *a*
then have ∃ *Q*. *Q* ∈ $\mathcal{P}$ **using** *events-paths a-event* **by** *auto*
thus *False* **using** *no-paths* **by** *simp*
qed
then have ∀ *b*∈$\mathcal{E}$. *b* = *a* **by** *simp*
thus *?thesis* **using** *a-event* **by** *auto*
qed

lemma *two-events-then-path*:
assumes *two-events*: $\exists\, a{\in}\mathcal{E}.\ \exists\, b{\in}\mathcal{E}.\ a \neq b$
shows $\exists\, Q.\ Q \in \mathcal{P}$
proof −
have $(\forall\, a.\ \mathcal{E} \neq \{a\}) \longrightarrow \mathcal{P} \neq \{\}$ **using** *big-bang* **by** *blast*
then have $\mathcal{P} \neq \{\}$ **using** *two-events* **by** *blast*
thus *?thesis* **by** *blast*
qed

lemma *paths-are-events*: $\forall\, Q{\in}\mathcal{P}.\ \forall\, a{\in}Q.\ a \in \mathcal{E}$
by *simp*

lemma *same-empty-unreach*:
$[\![Q \in \mathcal{P};\ a \in Q]\!] \Longrightarrow$ *unreach−on Q from a* $= \{\}$
apply (*unfold unreachable-subset-def*)
by *simp*

lemma *same-path-reachable*:
$[\![Q \in \mathcal{P};\ a \in Q;\ b \in Q]\!] \Longrightarrow a \in Q -$ *unreach−on Q from b*
by (*simp add*: *same-empty-unreach*)

If we have two paths crossing and a is on the crossing point, and b is on one of the paths, then a is in the reachable part of the path b is on.

lemma *same-path-reachable2*:
$[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ a \in R;\ b \in Q]\!] \Longrightarrow a \in R -$ *unreach−on R from b*
unfolding *unreachable-subset-def* **by** *blast*

lemma *cross-in-reachable*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *b-inR*: $b \in R$
shows $b \in R -$ *unreach−on R from a*
unfolding *unreachable-subset-def* **using** *a-inQ b-inQ b-inR path-Q* **by** *auto*

lemma *reachable-path*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *b-event*: $b \in \mathcal{E}$
and *a-reachable*: $a \in Q -$ *unreach−on Q from b*
shows $\exists\, R{\in}\mathcal{P}.\ a \in R \wedge b \in R$
proof −
have *a-inQ*: $a \in Q$ **using** *a-reachable* **by** *simp*
have $Q \notin \mathcal{P} \vee b \notin \mathcal{E} \vee b \in Q \vee (\exists\, R{\in}\mathcal{P}.\ b \in R \wedge a \in R)$
using *a-reachable unreachable-subset-def* **by** *auto*
then have $b \in Q \vee (\exists\, R{\in}\mathcal{P}.\ b \in R \wedge a \in R)$ **using** *path-Q b-event* **by** *simp*
thus *?thesis*
proof (*rule disjE*)

```
    assume b ∈ Q
    thus ?thesis using a-inQ path-Q by auto
  next
    assume ∃ R∈𝒫. b ∈ R ∧ a ∈ R
    thus ?thesis using conj-commute by simp
  qed
qed

end
context MinkowskiBetweenness begin


lemma ord-path-of:
  assumes [a;b;c]
  shows a ∈ path-of b c b ∈ path-of a c c ∈ path-of a b
    and path-of a b = path-of a c path-of a b = path-of b c
proof −
  show a ∈ path-of b c
   using betw-a-in-path[of a b c path-of b c] path-of-ex abc-ex-path-unique abc-abc-neq
assms
    by (smt (z3) betw-a-in-path the1-equality)
  show b ∈ path-of a c
   using betw-b-in-path[of a b c path-of a c] path-of-ex abc-ex-path-unique abc-abc-neq
assms
    by (smt (z3) betw-b-in-path the1-equality)
  show c ∈ path-of a b
   using betw-c-in-path[of a b c path-of a b] path-of-ex abc-ex-path-unique abc-abc-neq
assms
    by (smt (z3) betw-c-in-path the1-equality)
  show path-of a b = path-of a c
   by (metis (mono-tags) abc-ac-neq assms betw-b-in-path betw-c-in-path ends-notin-segment
seg-betw)
  show path-of a b = path-of b c
    by (metis (mono-tags) assms betw-a-in-path betw-c-in-path ends-notin-segment
seg-betw)
qed
```

Schutz defines chains as subsets of paths. The result below proves that even though we do not include this fact in our definition, it still holds, at least for finite chains.

Notice that this whole proof would be unnecessary if including path-belongingness in the definition, as Schutz does. This would also keep path-belongingness independent of axiom O1 and O4, thus enabling an independent statement of axiom O6, which perhaps we now lose. In exchange, our definition is slightly weaker (for *card X* $\geq$ *3* and *infinite X*).

```
lemma obtain-index-fin-chain:
  assumes [f⇝X] x∈X finite X
  obtains i where f i = x i<card X
```

```
proof −
  have ∃ i<card X. f i = x
    using assms(1) unfolding ch-by-ord-def
  proof (rule disjE)
    assume asm: short-ch-by-ord f X
    hence card X = 2
      using short-ch-card(1) by auto
    thus ∃ i<card X. f i = x
      using asm assms(2) unfolding chain-defs by force
  next
    assume asm: local-long-ch-by-ord f X
    thus ∃ i<card X. f i = x
      using asm assms(2,3) unfolding chain-defs local-ordering-def by blast
  qed
  thus ?thesis using that by blast
qed

lemma obtain-index-inf-chain:
  assumes [f⇝X] x∈X infinite X
  obtains i where f i = x
  using assms unfolding chain-defs local-ordering-def by blast


lemma fin-chain-on-path2:
  assumes [f⇝X] finite X
  shows ∃ P∈𝒫. X⊆P
  using assms(1) unfolding ch-by-ord-def
proof (rule disjE)
  assume short-ch-by-ord f X
  thus ?thesis
    using short-ch-by-ord-def by auto
next
  assume asm: local-long-ch-by-ord f X
  have [f 0;f 1;f 2]
    using order-finite-chain asm assms(2) local-long-ch-by-ord-def by auto
  then obtain P where P∈𝒫 {f 0,f 1,f 2} ⊆ P
    by (meson abc-ex-path empty-subsetI insert-subset)
  then have path P (f 0) (f 1)
    using ‹[f 0;f 1;f 2]› by (simp add: abc-abc-neq)
  {
    fix x assume x∈X
    then obtain i where i: f i = x i<card X
      using obtain-index-fin-chain assms by blast
    consider i=0∨i=1|i>1 by linarith
    hence x∈P
    proof (cases)
      case 1 thus ?thesis
      using i(1) ‹{f 0, f 1, f 2} ⊆ P› by auto
    next
```

```
    case 2
    hence [f 0;f 1;f i]
      using assms i(2) order-finite-chain2 by auto
    hence {f 0,f 1,f i}⊆P
      using ‹path P (f 0) (f 1)› betw-c-in-path by blast
    thus ?thesis by (simp add: i(1))
   qed
  }
  thus ?thesis
   using ‹P∈𝒫› by auto
qed


lemma fin-chain-on-path:
  assumes [f⇝X] finite X
  shows ∃!P∈𝒫. X⊆P
proof −
  obtain P where P: P∈𝒫 X⊆P
    using fin-chain-on-path2[OF assms] by auto
  obtain a b where ab: a∈X b∈X a≠b
   using assms(1) unfolding chain-defs by (metis assms(2) insertCI three-in-set3)
  thus ?thesis using P ab by (meson eq-paths in-mono)
qed


lemma fin-chain-on-path3:
  assumes [f⇝X] finite X a∈X b∈X a≠b
  shows X ⊆ path-of a b
proof −
  let ?ab = path-of a b
  obtain P where P: P∈𝒫 X⊆P using fin-chain-on-path2[OF assms(1,2)] by
auto
  have path P a b using P assms(3−5) by auto
  then have path ?ab a b using path-of-ex by blast
  hence ?ab = P using eq-paths ‹path P a b› by auto
  thus X ⊆ path-of a b using P by simp
qed


end
context MinkowskiUnreachable begin
```

First some basic facts about the primitive notions, which seem to belong here. I don't think any/all of these are explicitly proved in Schutz.

```
lemma no-empty-paths [simp]:
  assumes Q∈𝒫
  shows Q≠{}

proof −
```

```
  obtain a where a∈ℰ using nonempty-events by blast
  have a∈Q ∨ a∉Q by auto
  thus ?thesis
  proof
    assume a∈Q
    thus ?thesis by blast
  next
    assume a∉Q
    then obtain b where b∈unreach−on Q from a
      using two-in-unreach ‹a ∈ ℰ› assms
      by blast
    thus ?thesis
      using unreachable-subset-def by auto
  qed
qed

lemma events-ex-path:
  assumes ge1-path: 𝒫 ≠ {}
  shows ∀ x∈ℰ. ∃ Q∈𝒫. x ∈ Q

proof
  fix x
  assume x-event: x ∈ ℰ
  have ∃ Q. Q ∈ 𝒫 using ge1-path using ex-in-conv by blast
  then obtain Q where path-Q: Q ∈ 𝒫 by auto
  then have ∃ y. y ∈ Q using no-empty-paths by blast
  then obtain y where y-inQ: y ∈ Q by auto
  then have y-event: y ∈ ℰ using in-path-event path-Q by simp
  have ∃ P∈𝒫. x ∈ P
  proof cases
    assume x = y
    thus ?thesis using y-inQ path-Q by auto
  next
    assume x ≠ y
    thus ?thesis using events-paths x-event y-event by auto
  qed
  thus ∃ Q∈𝒫. x ∈ Q by simp
qed

lemma unreach-ge2-then-ge2:
  assumes ∃ x∈unreach−on Q from b. ∃ y∈unreach−on Q from b. x ≠ y
  shows ∃ x∈Q. ∃ y∈Q. x ≠ y
using assms unreachable-subset-def by auto
```

This lemma just proves that the chain obtained to bound the unreachable set of a path is indeed on that path. Extends I6; requires Theorem 2; used in Theorem 13. Seems to be assumed in Schutz' chain notation in I6.

```
lemma chain-on-path-I6:
  assumes path-Q: Q∈𝒫
```

and *event-b*: $b \notin Q\ b \in \mathcal{E}$
and *unreach*: $Q_x \in$ *unreach−on* Q *from* $b\ Q_z \in$ *unreach−on* Q *from* $b\ Q_x \neq Q_z$
and *X-def*: $[f \rightsquigarrow X | Q_x .. Q_z]$
$(\forall\, i \in \{1\ ..\ card\ X - 1\}.\ (f\ i) \in$ *unreach−on* Q *from* $b \wedge (\forall\, Q_y \in \mathcal{E}.\ [(f(i-1)); Q_y; f\ i] \longrightarrow Q_y \in$ *unreach−on* Q *from* $b))$
shows $X \subseteq Q$
proof −
have *1*: *path* $Q\ Q_x\ Q_z$ **using** *unreachable-subset-def unreach path-Q* **by** *simp*
then have *2*: $Q =$ *path-of* $Q_x\ Q_z$ **using** *path-of-ex*[*of* $Q_x\ Q_z$] **by** (*meson eq-paths*)
have $X \subseteq$*path-of* $Q_x\ Q_z$
proof (*rule fin-chain-on-path3*[*of f*])
from *unreach*(*3*) **show** $Q_x \neq Q_z$ **by** *simp*
from *X-def chain-defs* **show** $[f \rightsquigarrow X]$ *finite X* **by** *metis+*
from *assms*(*7*) *points-in-chain* **show** $Q_x \in X\ Q_z \in X$ **by** *auto*
qed
thus *?thesis* **using** *2* **by** *simp*
qed

end

24 Results about Paths as Sets

Note several of the following don't need MinkowskiPrimitive, they are just Set lemmas; nevertheless I'm naming them and writing them this way for clarity.

context *MinkowskiPrimitive* **begin**

lemma *distinct-paths*:
assumes $Q \in \mathcal{P}$
and $R \in \mathcal{P}$
and $d \notin Q$
and $d \in R$
shows $R \neq Q$
using *assms* **by** *auto*

lemma *distinct-paths2*:
assumes $Q \in \mathcal{P}$
and $R \in \mathcal{P}$
and $\exists\, d.\ d \notin Q \wedge d \in R$
shows $R \neq Q$
using *assms* **by** *auto*

lemma *external-events-neq*:
$[\![Q \in \mathcal{P};\ a \in Q;\ b \in \mathcal{E};\ b \notin Q]\!] \Longrightarrow a \neq b$
by *auto*

lemma *notin-cross-events-neq*:

$[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ Q \neq R;\ a \in Q;\ b \in R;\ a \notin R \cap Q]\!] \Longrightarrow a \neq b$
by *blast*

lemma *nocross-events-neq*:
$[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ b \in R;\ R \cap Q = \{\}]\!] \Longrightarrow a \neq b$
by *auto*

Given a nonempty path Q, and an external point d, we can find another path R passing through d (by I2 aka *events-paths*). This path is distinct from Q, as it passes through a point external to it.

lemma *external-path*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *d-notinQ*: $d \notin Q$
and *d-event*: $d \in \mathcal{E}$
shows $\exists\, R{\in}\mathcal{P}.\ d \in R$
proof −
have *a-neq-d*: $a \neq d$ **using** *a-inQ d-notinQ* **by** *auto*
thus $\exists\, R{\in}\mathcal{P}.\ d \in R$ **using** *events-paths* **by** (*meson a-inQ d-event in-path-event path-Q*)
qed

lemma *distinct-path*:
assumes $Q \in \mathcal{P}$
and $a \in Q$
and $d \notin Q$
and $d \in \mathcal{E}$
shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q$
using *assms external-path* **by** *metis*

lemma *external-distinct-path*:
assumes $Q \in \mathcal{P}$
and $a \in Q$
and $d \notin Q$
and $d \in \mathcal{E}$
shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q \wedge d \in R$
using *assms external-path* **by** *fastforce*

end

25 3.3 Boundedness of the unreachable set

25.1 Theorem 4 (boundedness of the unreachable set)

The same assumptions as I7, different conclusion. This doesn't just give us boundedness, it gives us another event outside of the unreachable set, as long as we have one already. I7 conclusion: $\exists\, g\ X\ Qn.\ [g{\rightsquigarrow}X|Qx..Qy..Qn] \wedge Qn \in Q - unreach{-}on\ Q\ from\ b$

theorem (**in** *MinkowskiUnreachable*) *unreachable-set-bounded*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *b-nin-Q*: $b \notin Q$
and *b-event*: $b \in \mathcal{E}$
and *Qx-reachable*: $Qx \in Q - unreach{-}on\ Q\ from\ b$
and *Qy-unreachable*: $Qy \in unreach{-}on\ Q\ from\ b$
shows $\exists\, Qz{\in}Q - unreach{-}on\ Q\ from\ b.\ [Qx;Qy;Qz] \wedge Qx \neq Qz$
using *assms I7-old finite-long-chain-with-def fin-ch-betw*
by (*metis first-neq-last*)

25.2 Theorem 5 (first existence theorem)

The lemma below is used in the contradiction in *external-event*, which is the essential part to Theorem 5(i).

lemma (**in** *MinkowskiUnreachable*) *only-one-path*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *all-inQ*: $\forall\, a{\in}\mathcal{E}.\ a \in Q$
and *path-R*: $R \in \mathcal{P}$
shows $R = Q$
proof (*rule ccontr*)
assume $\neg\ R = Q$
then have *R-neq-Q*: $R \neq Q$ **by** *simp*
have $\mathcal{E} = Q$
by (*simp add*: *all-inQ antisym path-Q path-sub-events subsetI*)
hence $R{\subset}Q$
using *R-neq-Q path-R path-sub-events* **by** *auto*
obtain *c* **where** $c{\notin}R\ c{\in}Q$
using ‹$R \subset Q$› **by** *blast*
then obtain *a b* **where** *path R a b*
using ‹$\mathcal{E} = Q$› *path-R two-in-unreach unreach-ge2-then-ge2* **by** *blast*
have $a{\in}Q\ b{\in}Q$
using ‹$\mathcal{E} = Q$› ‹*path R a b*› *in-path-event* **by** *blast+*
thus *False* **using** *eq-paths*
using *R-neq-Q* ‹*path R a b*› *path-Q* **by** *blast*
qed

context *MinkowskiSpacetime* **begin**

Unfortunately, we cannot assume that a path exists without the axiom of dimension.

lemma *external-event*:
assumes *path-Q*: $Q \in \mathcal{P}$
shows $\exists\, d{\in}\mathcal{E}.\ d \notin Q$
proof (*rule ccontr*)
assume $\neg\ (\exists\, d{\in}\mathcal{E}.\ d \notin Q)$
then have *all-inQ*: $\forall\, d{\in}\mathcal{E}.\ d \in Q$ **by** *simp*
then have *only-one-path*: $\forall\, P{\in}\mathcal{P}.\ P = Q$ **by** (*simp add*: *only-one-path path-Q*)
thus *False* **using** *ex-3SPRAY three-SPRAY-ge4 four-paths* **by** *auto*

qed

Now we can prove the first part of the theorem's conjunction. This follows pretty much exactly the same pattern as the book, except it relies on more intermediate lemmas.

theorem *ge2-events*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
shows $\exists\, b{\in}Q.\ b \neq a$
proof −
have *d-notinQ*: $\exists\, d{\in}\mathcal{E}.\ d \notin Q$ **using** *path-Q external-event* **by** *blast*
then obtain d **where** $d \in \mathcal{E}$ **and** $d \notin Q$ **by** *auto*
thus *?thesis* **using** *two-in-unreach* [**where** $Q = Q$ **and** $b = d$] *path-Q unreach-ge2-then-ge2* **by** *metis*
qed

Simple corollary which is easier to use when we don't have one event on a path yet. Anything which uses this implicitly used *no-empty-paths* on top of *ge2-events*.

lemma *ge2-events-lax*:
assumes *path-Q*: $Q \in \mathcal{P}$
shows $\exists\, a{\in}Q.\ \exists\, b{\in}Q.\ a \neq b$
proof −
have $\exists\, a{\in}\mathcal{E}.\ a \in Q$ **using** *path-Q no-empty-paths* **by** (*meson ex-in-conv in-path-event*)
thus *?thesis* **using** *path-Q ge2-events* **by** *blast*
qed

lemma *ex-crossing-path*:
assumes *path-Q*: $Q \in \mathcal{P}$
shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q \wedge (\exists\, c.\ c \in R \wedge c \in Q)$
proof −
obtain a **where** *a-inQ*: $a \in Q$ **using** *ge2-events-lax path-Q* **by** *blast*
obtain d **where** *d-event*: $d \in \mathcal{E}$
and *d-notinQ*: $d \notin Q$ **using** *external-event path-Q* **by** *auto*
then have $a \neq d$ **using** *a-inQ* **by** *auto*
then have *ex-through-d*: $\exists\, R{\in}\mathcal{P}.\ \exists\, S{\in}\mathcal{P}.\ a \in R \wedge d \in S \wedge R \cap S \neq \{\}$
using *events-paths* [**where** $a = a$ **and** $b = d$]
path-Q a-inQ in-path-event d-event **by** *simp*
then obtain R S **where** *path-R*: $R \in \mathcal{P}$
and *path-S*: $S \in \mathcal{P}$
and *a-inR*: $a \in R$
and *d-inS*: $d \in S$
and *R-crosses-S*: $R \cap S \neq \{\}$ **by** *auto*
have *S-neq-Q*: $S \neq Q$ **using** *d-notinQ d-inS* **by** *auto*
show *?thesis*
proof *cases*
assume $R = Q$
then have $Q \cap S \neq \{\}$ **using** *R-crosses-S* **by** *simp*
thus *?thesis* **using** *S-neq-Q path-S* **by** *blast*

next
assume $R \neq Q$
thus *?thesis* **using** *a-inQ a-inR path-R* **by** *blast*
qed
qed

If we have two paths Q and R with a on Q and b at the intersection of Q and R, then by *two-in-unreach* (I5) and Theorem 4 (boundedness of the unreachable set), there is an unreachable set from a on one side of b on R, and on the other side of that there is an event which is reachable from a by some path, which is the path we want.

lemma *path-past-unreach*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *path-R*: $R \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *b-inR*: $b \in R$
and *Q-neq-R*: $Q \neq R$
and *a-neq-b*: $a \neq b$
shows $\exists\, S{\in}\mathcal{P}.\ S \neq Q \wedge a \in S \wedge (\exists\, c.\ c \in S \wedge c \in R)$
proof −
obtain *d* **where** *d-event*: $d \in \mathcal{E}$
and *d-notinR*: $d \notin R$ **using** *external-event path-R* **by** *blast*
have *b-reachable*: $b \in R - \mathit{unreach}{-}\mathit{on}\ R\ \mathit{from}\ a$ **using** *cross-in-reachable path-R a-inQ b-inQ b-inR path-Q* **by** *simp*
have *a-notinR*: $a \notin R$ **using** *cross-once-notin*
Q-neq-R a-inQ a-neq-b b-inQ b-inR path-Q path-R **by** *blast*
then obtain *u* **where** $u \in \mathit{unreach}{-}\mathit{on}\ R\ \mathit{from}\ a$
using *two-in-unreach a-inQ in-path-event path-Q path-R* **by** *blast*
then obtain *c* **where** *c-reachable*: $c \in R - \mathit{unreach}{-}\mathit{on}\ R\ \mathit{from}\ a$
and *c-neq-b*: $b \neq c$ **using** *unreachable-set-bounded*
[**where** $Q = R$ **and** $Qx = b$ **and** $b = a$ **and** $Qy = u$]
path-R d-event d-notinR
using *a-inQ a-notinR b-reachable in-path-event path-Q* **by** *blast*
then obtain *S* **where** *S-facts*: $S \in \mathcal{P} \wedge a \in S \wedge (c \in S \wedge c \in R)$ **using** *reachable-path*
by (*metis Diff-iff a-inQ in-path-event path-Q path-R*)
then have $S \neq Q$ **using** *Q-neq-R b-inQ b-inR c-neq-b eq-paths path-R* **by** *blast*
thus *?thesis* **using** *S-facts* **by** *auto*
qed

theorem *ex-crossing-at*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
shows $\exists\, ac{\in}\mathcal{P}.\ ac \neq Q \wedge (\exists\, c.\ c \notin Q \wedge a \in ac \wedge c \in ac)$
proof −
obtain *b* **where** *b-inQ*: $b \in Q$
and *a-neq-b*: $a \neq b$ **using** *a-inQ ge2-events path-Q* **by** *blast*
have $\exists\, R{\in}\mathcal{P}.\ R \neq Q \wedge (\exists\, e.\ e \in R \wedge e \in Q)$ **by** (*simp add*: *ex-crossing-path*

path-Q)
then obtain *R e* **where** *path-R*: $R \in \mathcal{P}$
and *R-neq-Q*: $R \neq Q$
and *e-inR*: $e \in R$
and *e-inQ*: $e \in Q$ **by** *auto*
thus *?thesis*
proof *cases*
assume *e-eq-a*: $e = a$
then have $\exists\, c.\ c \in$ *unreach−on R from b* **using** *R-neq-Q a-inQ a-neq-b b-inQ e-inR path-Q path-R*
two-in-unreach path-unique in-path-event **by** *metis*
thus *?thesis* **using** *R-neq-Q e-eq-a e-inR path-Q path-R*
eq-paths ge2-events-lax **by** *metis*
next
assume *e-neq-a*: $e \neq a$

then have $\exists\, S{\in}\mathcal{P}.\ S \neq Q \wedge a \in S \wedge (\exists\, c.\ c \in S \wedge c \in R)$
using *path-past-unreach*
R-neq-Q a-inQ e-inQ e-inR path-Q path-R **by** *auto*
thus *?thesis* **by** (*metis R-neq-Q e-inR e-neq-a eq-paths path-Q path-R*)
qed
qed

lemma *ex-crossing-at-alt*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
shows $\exists\, ac.\ \exists\, c.$ *path ac a c* $\wedge\ ac \neq Q \wedge c \notin Q$
using *ex-crossing-at assms* **by** *fastforce*

end

26 3.4 Prolongation

context *MinkowskiSpacetime* **begin**

lemma (**in** *MinkowskiPrimitive*) *unreach-on-path*:
$a \in$ *unreach−on Q from b* $\Longrightarrow a \in Q$
using *unreachable-subset-def* **by** *simp*

lemma (**in** *MinkowskiUnreachable*) *unreach-equiv*:
$[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ b \in R;\ a \in$ *unreach−on Q from b*$]\!] \Longrightarrow b \in$ *unreach−on R from a*
unfolding *unreachable-subset-def* **by** *auto*

theorem *prolong-betw*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$

and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}\mathcal{E}.\ [a;b;c]$
proof −
obtain *e ae* **where** *e-event*: $e \in \mathcal{E}$
and *e-notinQ*: $e \notin Q$
and *path-ae*: *path ae a e*
using *ex-crossing-at a-inQ path-Q in-path-event* **by** *blast*
have $b \notin ae$ **using** *a-inQ ab-neq b-inQ e-notinQ eq-paths path-Q path-ae* **by** *blast*
then obtain *f* **where** *f-unreachable*: $f \in unreach{-}on\ ae\ from\ b$
using *two-in-unreach b-inQ in-path-event path-Q path-ae* **by** *blast*
then have *b-unreachable*: $b \in unreach{-}on\ Q\ from\ f$ **using** *unreach-equiv*
by (*metis* (*mono-tags*, *lifting*) *CollectD b-inQ path-Q unreachable-subset-def*)
have *a-reachable*: $a \in Q - unreach{-}on\ Q\ from\ f$
using *same-path-reachable2* [**where** $Q = ae$ **and** $R = Q$ **and** $a = a$ **and** $b = f$]
path-ae a-inQ path-Q f-unreachable unreach-on-path **by** *blast*
thus *?thesis*
using *unreachable-set-bounded* [**where** $Qy = b$ **and** $Q = Q$ **and** $b = f$ **and** $Qx = a$]
b-unreachable unreachable-subset-def **by** *auto*
qed

lemma (**in** *MinkowskiSpacetime*) *prolong-betw2*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}Q.\ [a;b;c]$
by (*metis assms betw-c-in-path prolong-betw*)

lemma (**in** *MinkowskiSpacetime*) *prolong-betw3*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}Q.\ \exists\, d{\in}Q.\ [a;b;c] \wedge [a;b;d] \wedge c{\neq}d$
by (*metis* (*full-types*) *abc-abc-neq abc-bcd-abd a-inQ ab-neq b-inQ path-Q prolong-betw2*)

lemma *finite-path-has-ends*:
assumes $Q \in \mathcal{P}$
and $X \subseteq Q$
and *finite X*
and *card* $X \geq 3$
shows $\exists\, a{\in}X.\ \exists\, b{\in}X.\ a \neq b \wedge (\forall\, c{\in}X.\ a \neq c \wedge b \neq c \longrightarrow [a;c;b])$
using *assms*
proof (*induct card X − 3 arbitrary*: *X*)
case *0*

```
  then have card X = 3
    by linarith
  then obtain a b c where X-eq: X = {a, b, c}
    by (metis card-Suc-eq numeral-3-eq-3)
  then have abc-neq: a ≠ b a ≠ c b ≠ c
    by (metis ‹card X = 3› empty-iff insert-iff order-refl three-in-set3)+
  then consider [a;b;c] | [b;c;a] | [c;a;b]
    using some-betw [of Q a b c] 0.prems(1) 0.prems(2) X-eq by auto
  thus ?case
  proof (cases)
    assume [a;b;c]
    thus ?thesis — All d not equal to a or c is just d = b, so it immediately follows.
      using X-eq abc-neq(2) by blast
  next
    assume [b;c;a]
    thus ?thesis
      by (simp add: X-eq abc-neq(1))
  next
    assume [c;a;b]
    thus ?thesis
      using X-eq abc-neq(3) by blast
  qed
next
  case IH: (Suc n)
  obtain Y x where X-eq: X = insert x Y and x ∉ Y
    by (meson IH.prems(4) Set.set-insert three-in-set3)
  then have card Y − 3 = n card Y ≥ 3
    using IH.hyps(2) IH.prems(3) X-eq ‹x ∉ Y› by auto
  then obtain a b where ab-Y: a ∈ Y b ∈ Y a ≠ b
                    and Y-ends: ∀ c∈Y. (a ≠ c ∧ b ≠ c) ⟶ [a;c;b]
    using IH(1) [of Y] IH.prems(1−3) X-eq by auto
  consider [a;x;b] | [x;b;a] | [b;a;x]
    using some-betw [of Q a x b] ab-Y IH.prems(1,2) X-eq ‹x ∉ Y› by auto
  thus ?case
  proof (cases)
    assume [a;x;b]
    thus ?thesis
      using Y-ends X-eq ab-Y by auto
  next
    assume [x;b;a]
    { fix c
      assume c ∈ X x ≠ c a ≠ c
      then have [x;c;a]
        by (smt IH.prems(2) X-eq Y-ends ‹[x;b;a]› ab-Y(1) abc-abc-neq abc-bcd-abd
abc-only-cba(3) abc-sym ‹Q ∈ 𝒫› betw-b-in-path insert-iff some-betw subsetD)
    }
    thus ?thesis
      using X-eq ‹[x;b;a]› ab-Y(1) abc-abc-neq insert-iff by force
  next
```

```
    assume [b;a;x]
    { fix c
      assume c ∈ X b ≠ c x ≠ c
      then have [b;c;x]
        by (smt IH.prems(2) X-eq Y-ends ‹[b;a;x]› ab-Y(1) abc-abc-neq abc-bcd-acd
abc-only-cba(1)
            abc-sym ‹Q ∈ P› betw-a-in-path insert-iff some-betw subsetD)
    }
    thus ?thesis
      using X-eq ‹x ∉ Y› ab-Y(2) by fastforce
  qed
qed


lemma obtain-fin-path-ends:
  assumes path-X: X∈P
      and fin-Q: finite Q
      and card-Q: card Q ≥ 3
      and events-Q: Q⊆X
  obtains a b where a≠b and a∈Q and b∈Q and ∀ c∈Q. (a≠c ∧ b≠c) ⟶ [a;c;b]
proof −
  obtain n where n≥0 and card Q = n+3
    using card-Q nat-le-iff-add
    by auto
  then obtain a b where a≠b and a∈Q and b∈Q and ∀ c∈Q. (a≠c ∧ b≠c) ⟶
[a;c;b]
    using finite-path-has-ends assms ‹n≥0›
    by metis
  thus ?thesis
    using that by auto
qed


lemma path-card-nil:
  assumes Q∈P
  shows card Q = 0
proof (rule ccontr)
  assume card Q ≠ 0
  obtain n where n = card Q
    by auto
  hence n≥1
    using ‹card Q ≠ 0› by linarith
  then consider (n1) n=1 | (n2) n=2 | (n3) n≥3
    by linarith
  thus False
  proof (cases)
    case n1
    thus ?thesis
      using One-nat-def card-Suc-eq ge2-events-lax singletonD assms(1)
```

```
    by (metis ‹n = card Q›)
  next
    case n2
    then obtain a b where a≠b and a∈Q and b∈Q
      using ge2-events-lax assms(1) by blast
    then obtain c where c∈Q and c≠a and c≠b
      using prolong-betw2 by (metis abc-abc-neq assms(1))
    hence card Q ≠ 2
      by (metis ‹a ∈ Q› ‹a ≠ b› ‹b ∈ Q› card-2-iff')
    thus False
      using ‹n = card Q› ‹n = 2› by blast
  next
    case n3
    have fin-Q: finite Q
    proof −
      have (0::nat) ≠ 1
        by simp
      then show ?thesis
        by (meson ‹card Q ≠ 0› card.infinite)
    qed
    have card-Q: card Q ≥ 3
      using ‹n = card Q› n3 by blast
    have Q⊆Q by simp
    then obtain a b where a∈Q and b∈Q and a≠b
       and acb: ∀ c∈Q. (c≠a ∧ c≠b) ⟶ [a;c;b]
      using obtain-fin-path-ends card-Q fin-Q assms(1)
      by metis
    then obtain x where [a;b;x] and x∈Q
      using prolong-betw2 assms(1) by blast
    thus False
      by (metis acb abc-abc-neq abc-only-cba(2))
  qed
qed


theorem infinite-paths:
  assumes P∈𝒫
  shows infinite P
proof
  assume fin-P: finite P
  have P≠{}
    by (simp add: assms)
  hence card P ≠ 0
    by (simp add: fin-P)
  moreover have ¬(card P ≥ 1)
    using path-card-nil
    by (simp add: assms)
  ultimately show False
    by simp
```

qed

end

27 3.5 Second collinearity theorem

We start with a useful betweenness lemma.

lemma (**in** *MinkowskiBetweenness*) *some-betw2*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *c-inQ*: $c \in Q$
shows $a = b \vee a = c \vee b = c \vee [a;b;c] \vee [b;c;a] \vee [c;a;b]$
using *a-inQ b-inQ c-inQ path-Q some-betw* **by** *blast*

lemma (**in** *MinkowskiPrimitive*) *paths-tri*:
assumes *path-ab*: *path ab a b*
and *path-bc*: *path bc b c*
and *path-ca*: *path ca c a*
and *a-notin-bc*: $a \notin bc$
shows $\triangle\ a\ b\ c$
proof −
have *abc-events*: $a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E}$
using *path-ab path-bc path-ca in-path-event* **by** *auto*
have *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$
using *path-ab path-bc path-ca* **by** *auto*
have *paths-neq*: $ab \neq bc \wedge ab \neq ca \wedge bc \neq ca$
using *a-notin-bc cross-once-notin path-ab path-bc path-ca* **by** *blast*
show *?thesis*
unfolding *kinematic-triangle-def*
using *abc-events abc-neq paths-neq path-ab path-bc path-ca*
by *auto*
qed

lemma (**in** *MinkowskiPrimitive*) *paths-tri2*:
assumes *path-ab*: *path ab a b*
and *path-bc*: *path bc b c*
and *path-ca*: *path ca c a*
and *ab-neq-bc*: $ab \neq bc$
shows $\triangle\ a\ b\ c$
by (*meson ab-neq-bc cross-once-notin path-ab path-bc path-ca paths-tri*)

Schutz states it more like $[\![tri\text{-}abc;\ bcd;\ cea]\!] \Longrightarrow (path\ de\ d\ e \longrightarrow \exists f{\in}de.\ [a;f;b]{\wedge}[d;e;f])$. Equivalent up to usage of *impI*.

theorem (**in** *MinkowskiChain*) *collinearity2*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
and *bcd*: $[b;c;d]$

and *cea*: $[c;e;a]$
and *path-de*: *path de d e*
shows $\exists f.\ [a;f;b] \wedge [d;e;f]$
proof −
obtain *ab* **where** *path-ab*: *path ab a b* **using** *tri-abc triangle-paths-unique* **by** *blast*
then obtain *f* **where** *afb*: $[a;f;b]$
and *f-in-de*: $f \in de$ **using** *collinearity tri-abc path-de path-ab bcd cea* **by** *blast*

obtain *af* **where** *path-af*: *path af a f* **using** *abc-abc-neq afb betw-b-in-path path-ab* **by** *blast*
have $[d;e;f]$
proof −
have *def-in-de*: $d \in de \wedge e \in de \wedge f \in de$ **using** *path-de f-in-de* **by** *simp*
then have *five-poss*:$f = d \vee f = e \vee [e;f;d] \vee [f;d;e] \vee [d;e;f]$
using *path-de some-betw2* **by** *blast*
have $f = d \vee f = e \longrightarrow (\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q)$
by (*metis abc-abc-neq afb bcd betw-a-in-path betw-b-in-path cea path-ab*)
then have *f-neq-d-e*: $f \neq d \wedge f \neq e$ **using** *tri-abc*
using *triangle-diff-paths* **by** *simp*
then consider $[e;f;d] \mid [f;d;e] \mid [d;e;f]$ **using** *five-poss* **by** *linarith*
thus *?thesis*
proof (*cases*)
assume *efd*: $[e;f;d]$
obtain *dc* **where** *path-dc*: *path dc d c* **using** *abc-abc-neq abc-ex-path bcd* **by** *blast*
obtain *ce* **where** *path-ce*: *path ce c e* **using** *abc-abc-neq abc-ex-path cea* **by** *blast*
have $dc{\neq}ce$
using *bcd betw-a-in-path betw-c-in-path cea path-ce path-dc tri-abc triangle-diff-paths*
by *blast*
hence $\triangle\ d\ c\ e$
using *paths-tri2 path-ce path-dc path-de* **by** *blast*
then obtain *x* **where** *x-in-af*: $x \in af$
and *dxc*: $[d;x;c]$
using *collinearity*
[**where** $a = d$ **and** $b = c$ **and** $c = e$ **and** $d = a$ **and** $e = f$ **and** $de = af$]
cea efd path-dc path-af **by** *blast*
then have *x-in-dc*: $x \in dc$ **using** *betw-b-in-path path-dc* **by** *blast*
then have $x = b$ **using** *eq-paths* **by** (*metis path-af path-dc afb bcd tri-abc x-in-af*
betw-a-in-path betw-c-in-path triangle-diff-paths)
then have $[d;b;c]$ **using** *dxc* **by** *simp*
then have *False* **using** *bcd abc-only-cba* [**where** $a = b$ **and** $b = c$ **and** $c = d$] **by** *simp*
thus *?thesis* **by** *simp*

next
assume *fde*: [*f*;*d*;*e*]
obtain *bd* **where** *path-bd*: *path bd b d* **using** *abc-abc-neq abc-ex-path bcd* **by** *blast*
obtain *ea* **where** *path-ea*: *path ea e a* **using** *abc-abc-neq abc-ex-path-unique cea* **by** *blast*
obtain *fe* **where** *path-fe*: *path fe f e* **using** *f-in-de f-neq-d-e path-de* **by** *blast*
have *fe*≠*ea*
using *tri-abc afb cea path-ea path-fe*
by (*metis abc-abc-neq betw-a-in-path betw-c-in-path triangle-paths-neq*)
hence △ *e a f*
by (*metis path-unique path-af path-ea path-fe paths-tri2*)
then obtain *y* **where** *y-in-bd*: *y* ∈ *bd*
and *eya*: [*e*;*y*;*a*] **thm** *collinearity*
using *collinearity*
[**where** *a* = *e* **and** *b* = *a* **and** *c* = *f* **and** *d* = *b* **and** *e* = *d* **and** *de* = *bd*]
afb fde path-bd path-ea **by** *blast*
then have *y* = *c* **by** (*metis* (*mono-tags*, *lifting*)
afb bcd cea path-bd tri-abc
abc-ac-neq betw-b-in-path path-unique triangle-paths(*2*)
triangle-paths-neq)
then have [*e*;*c*;*a*] **using** *eya* **by** *simp*
then have *False* **using** *cea abc-only-cba* [**where** *a* = *c* **and** *b* = *e* **and** *c* = *a*] **by** *simp*
thus *?thesis* **by** *simp*
next
assume [*d*;*e*;*f*]
thus *?thesis* **by** *assumption*
qed
qed
thus *?thesis* **using** *afb f-in-de* **by** *blast*
qed

28 3.6 Order on a path - Theorems 8 and 9

context *MinkowskiSpacetime* **begin**

28.1 Theorem 8 (as in Veblen (1911) Theorem 6)

Note $a'b'c'$ don't necessarily form a triangle, as there still needs to be paths between them.

theorem (**in** *MinkowskiChain*) *tri-betw-no-path*:
assumes *tri-abc*: △ *a b c*
and *ab'c*: [*a*; *b'*; *c*]
and *bc'a*: [*b*; *c'*; *a*]
and *ca'b*: [*c*; *a'*; *b*]
shows ¬ (∃ *Q*∈*P*. *a'* ∈ *Q* ∧ *b'* ∈ *Q* ∧ *c'* ∈ *Q*)

proof −
have *abc-a′b′c′-neq*: $a \neq a' \wedge a \neq b' \wedge a \neq c'$
$\wedge\ b \neq a' \wedge b \neq b' \wedge b \neq c'$
$\wedge\ c \neq a' \wedge c \neq b' \wedge c \neq c'$
using *abc-ac-neq*
by (*metis ab′c abc-abc-neq bc′a ca′b tri-abc triangle-not-betw-abc triangle-permutes*(*4*))

have *tri-betw-no-path-single-case*: *False*
if $a'b'c'$: $[a';\ b';\ c']$ **and** *tri-abc*: $\triangle\ a\ b\ c$
and $ab'c$: $[a;\ b';\ c]$ **and** $bc'a$: $[b;\ c';\ a]$ **and** $ca'b$: $[c;\ a';\ b]$
for $a\ b\ c\ a'\ b'\ c'$
proof −
have *abc-a′b′c′-neq*: $a \neq a' \wedge a \neq b' \wedge a \neq c'$
$\wedge\ b \neq a' \wedge b \neq b' \wedge b \neq c'$
$\wedge\ c \neq a' \wedge c \neq b' \wedge c \neq c'$
using *abc-abc-neq that* **by** (*metis triangle-not-betw-abc triangle-permutes*(*4*))
have $c'b'a'$: $[c';\ b';\ a']$ **using** *abc-sym a′b′c′* **by** *simp*
have *nopath-a′c′b*: $\neg\ (\exists\ Q{\in}\mathcal{P}.\ a' \in Q \wedge c' \in Q \wedge b \in Q)$
proof (*rule notI*)
assume $\exists\ Q{\in}\mathcal{P}.\ a' \in Q \wedge c' \in Q \wedge b \in Q$
then obtain Q **where** *path-Q*: $Q \in \mathcal{P}$
and *a′-inQ*: $a' \in Q$
and *c′-inQ*: $c' \in Q$
and *b-inQ*: $b \in Q$ **by** *blast*
then have *ac-inQ*: $a \in Q \wedge c \in Q$ **using** *eq-paths*
by (*metis abc-a′b′c′-neq ca′b bc′a betw-a-in-path betw-c-in-path*)
thus *False* **using** *b-inQ path-Q tri-abc triangle-diff-paths* **by** *blast*
qed
then have *tri-a′bc′*: $\triangle\ a'\ b\ c'$
by (*smt bc′a ca′b a′b′c′ paths-tri abc-ex-path-unique*)
obtain ab' **where** *path-ab′*: *path ab′ a b′* **using** *ab′c abc-a′b′c′-neq abc-ex-path* **by** *blast*
obtain $a'b$ **where** *path-a′b*: *path a′b a′ b* **using** *tri-a′bc′ triangle-paths*(*1*) **by** *blast*
then have $\exists\ x{\in}a'b.\ [a';\ x;\ b] \wedge [a;\ b';\ x]$
using *collinearity2* [**where** $a = a'$ **and** $b = b$ **and** $c = c'$ **and** $e = b'$ **and** $d = a$ **and** $de = ab'$]
bc′a betw-b-in-path c′b′a′ path-ab′ tri-a′bc′ **by** *blast*
then obtain x **where** *x-in-a′b*: $x \in a'b$
and $a'xb$: $[a';\ x;\ b]$
and $ab'x$: $[a;\ b';\ x]$ **by** *blast*

have *c-in-ab′*: $c \in ab'$ **using** *ab′c betw-c-in-path path-ab′* **by** *auto*
have *c-in-a′b*: $c \in a'b$ **using** *ca′b betw-a-in-path path-a′b* **by** *auto*
have *ab′-a′b-distinct*: $ab' \neq a'b$
using *c-in-a′b path-a′b path-ab′ tri-abc triangle-diff-paths* **by** *blast*
have $ab' \cap a'b = \{c\}$
using *paths-cross-at ab′-a′b-distinct c-in-a′b c-in-ab′ path-a′b path-ab′* **by**

auto
  then have $x = c$ **using** *ab'x path-ab' x-in-a'b betw-c-in-path* **by** *auto*
  then have $[a'; c; b]$ **using** *a'xb* **by** *auto*
  thus *?thesis* **using** *ca'b abc-only-cba* **by** *blast*
 qed

 show *?thesis*
 proof (*rule notI*)
  assume *path-a'b'c'*: $\exists\, Q{\in}\mathcal{P}.\ a' \in Q \wedge b' \in Q \wedge c' \in Q$
  consider $[a'; b'; c'] \mid [b'; c'; a'] \mid [c'; a'; b']$ **using** *some-betw*
    by (*smt abc-a'b'c'-neq path-a'b'c' bc'a ca'b ab'c tri-abc*
        abc-ex-path cross-once-notin triangle-diff-paths)
  thus *False*
   apply (*cases*)
   using *tri-betw-no-path-single-case*[*of a' b' c'*] *ab'c bc'a ca'b tri-abc* **apply** *blast*
    using *tri-betw-no-path-single-case ab'c bc'a ca'b tri-abc triangle-permutes*
abc-sym **by** *blast+*
 qed
qed

28.2 Theorem 9

We now begin working on the transitivity lemmas needed to prove Theorem 9. Multiple lemmas below obtain primed variables (e.g. d'). These are starred in Schutz (e.g. $d*$), but that notation is already reserved in Isabelle.

lemma *unreachable-bounded-path-only*:
 assumes *d'-def*: $d' \notin$ *unreach−on ab from e* $d' \in ab$ $d' \neq e$
   and *e-event*: $e \in \mathcal{E}$
   and *path-ab*: $ab \in \mathcal{P}$
   and *e-notin-S*: $e \notin ab$
 shows $\exists\, d'e.$ *path* $d'e$ d' e
proof (*rule ccontr*)
 assume $\neg(\exists\, d'e.$ *path* $d'e$ d' $e)$
 hence $\neg(\exists\, R{\in}\mathcal{P}.\ d' \in R \wedge e \in R \wedge d' \neq e)$
  by *blast*
 hence $\neg(\exists\, R{\in}\mathcal{P}.\ \ e \in R \wedge d' \in R)$
  using *d'-def(3)* **by** *blast*
 moreover have $ab \in \mathcal{P} \wedge e \in \mathcal{E} \wedge e \notin ab$
  by (*simp add*: *e-event e-notin-S path-ab*)
 ultimately have $d' \in$ *unreach−on ab from e*
  unfolding *unreachable-subset-def* **using** *d'-def(2)*
  by *blast*
 thus *False*
  using *d'-def(1)* **by** *auto*
qed

lemma *unreachable-bounded-path*:
 assumes *S-neq-ab*: $S \neq ab$
   and *a-inS*: $a \in S$

```
    and e-inS: e ∈ S
    and e-neq-a: e ≠ a
    and path-S: S ∈ 𝒫
    and path-ab: path ab a b
    and path-be: path be b e
    and no-de: ¬(∃ de. path de d e)
    and abd:[a;b;d]
  obtains d' d'e where d'∈ab ∧ path d'e d' e ∧ [b; d; d']
proof −
  have e-event: e∈ℰ
    using e-inS path-S by auto
  have e∉ab
    using S-neq-ab a-inS e-inS e-neq-a eq-paths path-S path-ab by auto
  have ab∈𝒫 ∧ e∉ab
    using S-neq-ab a-inS e-inS e-neq-a eq-paths path-S path-ab
    by auto
  have b ∈ ab − unreach−on ab from e
    using cross-in-reachable path-ab path-be
    by blast
  have d ∈ unreach−on ab from e
    using no-de abd path-ab e-event ‹e∉ab›
      betw-c-in-path unreachable-bounded-path-only
    by blast
  have ∃ d' d'e. d'∈ab ∧ path d'e d' e ∧ [b; d; d']
  proof −
    obtain d' where [b; d; d'] d'∈ab d'∉ unreach−on ab from e b≠d' e≠d'
        using unreachable-set-bounded ‹b ∈ ab − unreach−on ab from e› ‹d ∈
unreach−on ab from e› e-event ‹e∉ab› path-ab
      by (metis DiffE)
    then obtain d'e where path d'e d' e
      using unreachable-bounded-path-only e-event ‹e∉ab› path-ab
      by blast
    thus ?thesis
      using ‹[b; d; d']› ‹d' ∈ ab›
      by blast
  qed
  thus ?thesis
    using that by blast
qed
```

This lemma collects the first three paragraphs of Schutz' proof of Theorem 9 - Lemma 1. Several case splits need to be considered, but have no further importance outside of this lemma: thus we parcel them away from the main proof.

```
lemma exist-c'd'-alt:
  assumes abc: [a;b;c]
    and abd: [a;b;d]
    and dbc: [d;b;c]
    and c-neq-d: c ≠ d
```

```
      and path-ab: path ab a b
      and path-S: S ∈ 𝒫
      and a-inS: a ∈ S
      and e-inS: e ∈ S
      and e-neq-a: e ≠ a
      and S-neq-ab: S ≠ ab
      and path-be: path be b e
  shows ∃ c' d'. ∃ d'e c'e. c'∈ab ∧ d'∈ab
                        ∧ [a; b; d'] ∧ [c'; b; a] ∧ [c'; b; d']
                        ∧ path d'e d' e ∧ path c'e c' e
proof (cases)
  assume ∃ de. path de d e
  then obtain de where path de d e
    by blast
  hence [a;b;d] ∧ d∈ab
    using abd betw-c-in-path path-ab by blast
  thus ?thesis
  proof (cases)
    assume ∃ ce. path ce c e
    then obtain ce where path ce c e by blast
    have c ∈ ab
      using abc betw-c-in-path path-ab by blast
    thus ?thesis
     using ‹[a;b;d] ∧ d ∈ ab› ‹∃ ce. path ce c e› ‹c ∈ ab› ‹path de d e› abc abc-sym
dbc
      by blast
  next
    assume ¬(∃ ce. path ce c e)
    obtain c' c'e where c'∈ab ∧ path c'e c' e ∧ [b; c; c']
      using unreachable-bounded-path [where ab=ab and e=e and b=b and d=c
and a=a and S=S and be=be]
       S-neq-ab ‹¬(∃ ce. path ce c e)› a-inS abc e-inS e-neq-a path-S path-ab path-be
    by (metis (mono-tags, lifting))
    hence [a; b; c'] ∧ [d; b; c']
      using abc dbc by blast
    hence [c'; b; a] ∧ [c'; b; d]
      using theorem1 by blast
    thus ?thesis
      using ‹[a;b;d] ∧ d ∈ ab› ‹c' ∈ ab ∧ path c'e c' e ∧ [b; c; c']› ‹path de d e›
      by blast
  qed
next
  assume ¬ (∃ de. path de d e)
  obtain d' d'e where d'-in-ab: d' ∈ ab
                  and bdd': [b; d; d']
                  and path d'e d' e
    using unreachable-bounded-path [where ab=ab and e=e and b=b and d=d
and a=a and S=S and be=be]
     S-neq-ab ‹∄ de. path de d e› a-inS abd e-inS e-neq-a path-S path-ab path-be
```

by (*metis* (*mono-tags*, *lifting*))
hence [*a*; *b*; *d'*] **using** *abd* **by** *blast*
thus *?thesis*
proof (*cases*)
assume ∃ *ce*. *path ce c e*
then obtain *ce* **where** *path ce c e* **by** *blast*
have *c* ∈ *ab*
using *abc betw-c-in-path path-ab* **by** *blast*
thus *?thesis*
using ‹[*a*; *b*; *d'*]› ‹*d'* ∈ *ab*› ‹*path ce c e*› ‹*c* ∈ *ab*› ‹*path d'e d' e*› *abc abc-sym dbc*
by (*meson abc-bcd-acd bdd'*)
next
assume ¬(∃ *ce*. *path ce c e*)
obtain *c' c'e* **where** *c'*∈*ab* ∧ *path c'e c' e* ∧ [*b*; *c*; *c'*]
using *unreachable-bounded-path* [**where** *ab*=*ab* **and** *e*=*e* **and** *b*=*b* **and** *d*=*c* **and** *a*=*a* **and** *S*=*S* **and** *be*=*be*]
S-neq-ab ‹¬(∃ *ce*. *path ce c e*)› *a-inS abc e-inS e-neq-a path-S path-ab path-be*
by (*metis* (*mono-tags*, *lifting*))
hence [*a*; *b*; *c'*] ∧ [*d*; *b*; *c'*]
using *abc dbc* **by** *blast*
hence [*c'*; *b*; *a*] ∧ [*c'*; *b*; *d*]
using *theorem1* **by** *blast*
thus *?thesis*
using ‹[*a*; *b*; *d'*]› ‹*c'* ∈ *ab* ∧ *path c'e c' e* ∧ [*b*; *c*; *c'*]› ‹*path d'e d' e*› *bdd' d'-in-ab*
by *blast*
qed
qed

lemma *exist-c'd'*:
assumes *abc*: [*a*;*b*;*c*]
and *abd*: [*a*;*b*;*d*]
and *dbc*: [*d*;*b*;*c*]
and *path-S*: *path S a e*
and *path-be*: *path be b e*
and *S-neq-ab*: *S* ≠ *path-of a b*
shows ∃ *c' d'*. [*a*; *b*; *d'*] ∧ [*c'*; *b*; *a*] ∧ [*c'*; *b*; *d'*] ∧
path-ex d' e ∧ *path-ex c' e*
proof (*cases path-ex d e*)
let *?ab* = *path-of a b*
have *path-ex a b*
using *abc abc-abc-neq abc-ex-path* **by** *blast*
hence *path-ab*: *path ?ab a b* **using** *path-of-ex* **by** *simp*
have *c*≠*d* **using** *abc-ac-neq dbc* **by** *blast*
{
case *True*
then obtain *de* **where** *path de d e*
by *blast*

hence $[a;b;d] \wedge d \in ?ab$
using *abd betw-c-in-path path-ab* **by** *blast*
thus *?thesis*
proof (*cases path-ex c e*)
case *True*
then obtain *ce* **where** *path ce c e* **by** *blast*
have $c \in ?ab$
using *abc betw-c-in-path path-ab* **by** *blast*
thus *?thesis*
using ‹$[a;b;d] \wedge d \in ?ab$› ‹$\exists ce.$ *path ce c e*› ‹$c \in ?ab$› ‹*path de d e*› *abc abc-sym dbc*
by *blast*
next
case *False*
obtain c' $c'e$ **where** $c' \in ?ab \wedge$ *path* $c'e$ c' $e \wedge [b;\ c;\ c']$
using *unreachable-bounded-path* [**where** $ab=?ab$ **and** $e=e$ **and** $b=b$ **and** $d=c$ **and** $a=a$ **and** $S=S$ **and** *be=be*]
S-neq-ab ‹$\neg(\exists ce.$ *path ce c e*)› *abc path-S path-ab path-be*
by (*metis* (*mono-tags*, *lifting*))
hence $[a;\ b;\ c'] \wedge [d;\ b;\ c']$
using *abc dbc* **by** *blast*
hence $[c';\ b;\ a] \wedge [c';\ b;\ d]$
using *theorem1* **by** *blast*
thus *?thesis*
using ‹$[a;b;d] \wedge d \in ?ab$› ‹$c' \in ?ab \wedge$ *path* $c'e$ c' $e \wedge [b;\ c;\ c']$› ‹*path de d e*›
by *blast*
qed
} {
case *False*
obtain d' $d'e$ **where** d'*-in-ab*: $d' \in ?ab$
and *bdd'*: $[b;\ d;\ d']$
and *path* $d'e$ d' e
using *unreachable-bounded-path* [**where** $ab=?ab$ **and** $e=e$ **and** $b=b$ **and** $d=d$ **and** $a=a$ **and** $S=S$ **and** *be=be*]
S-neq-ab ‹$\neg$*path-ex d e*› *abd path-S path-ab path-be*
by (*metis* (*mono-tags*, *lifting*))
hence $[a;\ b;\ d']$ **using** *abd* **by** *blast*
thus *?thesis*
proof (*cases path-ex c e*)
case *True*
then obtain *ce* **where** *path ce c e* **by** *blast*
have $c \in ?ab$
using *abc betw-c-in-path path-ab* **by** *blast*
thus *?thesis*
using ‹$[a;\ b;\ d']$› ‹$d' \in ?ab$› ‹*path ce c e*› ‹$c \in ?ab$› ‹*path* $d'e$ d' *e*› *abc abc-sym dbc*
by (*meson abc-bcd-acd bdd'*)
next
case *False*

obtain c' $c'e$ **where** $c'\in ?ab \wedge path\ c'e\ c'\ e \wedge [b;\ c;\ c']$
using *unreachable-bounded-path* [**where** $ab=?ab$ **and** $e=e$ **and** $b=b$ **and** $d=c$ **and** $a=a$ **and** $S=S$ **and** $be=be$]
S-neq-ab ‹¬(*path-ex c e*)› *abc path-S path-ab path-be*
by (*metis* (*mono-tags*, *lifting*))
hence $[a;\ b;\ c'] \wedge [d;\ b;\ c']$
using *abc dbc* **by** *blast*
hence $[c';\ b;\ a] \wedge [c';\ b;\ d]$
using *theorem1* **by** *blast*
thus *?thesis*
using ‹$[a;\ b;\ d']$› ‹$c' \in ?ab \wedge path\ c'e\ c'\ e \wedge [b;\ c;\ c']$› ‹*path d'e d' e*› *bdd' d'-in-ab*
by *blast*
qed
}
qed

lemma *exist-f'-alt*:
assumes *path-ab*: *path ab a b*
and *path-S*: $S \in \mathcal{P}$
and *a-inS*: $a \in S$
and *e-inS*: $e \in S$
and *e-neq-a*: $e \neq a$
and *f-def*: $[e;\ c';\ f]\ f\in c'e$
and *S-neq-ab*: $S \neq ab$
and *c'd'-def*: $c'\in ab \wedge d'\in ab$
$\wedge\ [a;\ b;\ d'] \wedge [c';\ b;\ a] \wedge [c';\ b;\ d']$
$\wedge$ *path d'e d' e* $\wedge$ *path c'e c' e*
shows $\exists f'.\ \exists f'b.\ [e;\ c';\ f'] \wedge path\ f'b\ f'\ b$
proof (*cases*)
assume $\exists\, bf.$ *path bf b f*
thus *?thesis*
using ‹$[e;\ c';\ f]$› **by** *blast*
next
assume $\neg(\exists\, bf.$ *path bf b f*$)$
hence $f \in$ *unreach−on c'e from b*
using *assms*(*1−5*,*7−9*) *abc-abc-neq betw-events eq-paths unreachable-bounded-path-only*
by *metis*
moreover have $c' \in c'e -$ *unreach−on c'e from b*
using *c'd'-def cross-in-reachable path-ab* **by** *blast*
moreover have $b\in\mathcal{E} \wedge b\notin c'e$
using ‹$f \in$ *unreach−on c'e from b*› *betw-events c'd'-def same-empty-unreach*
by *auto*
ultimately obtain f' **where** *f'-def*: $[c';\ f;\ f']\ f'\in c'e\ f'\notin$ *unreach−on c'e from b* $c'\neq f'$ $b\neq f'$
using *unreachable-set-bounded c'd'-def*
by (*metis DiffE*)
hence $[e;\ c';\ f']$

using ‹$[e;\ c';\ f]$› **by** *blast*
moreover obtain $f'b$ **where** *path* $f'b\ f'\ b$
using ‹$b \in \mathcal{E} \wedge b \notin c'e$› $c'd'$-*def* f'-*def*(*2,3*) *unreachable-bounded-path-only*
by *blast*
ultimately show *?thesis* **by** *blast*
qed

lemma *exist-f'*:
assumes *path-ab*: *path ab a b*
and *path-S*: *path S a e*
and *f-def*: $[e;\ c';\ f]$
and *S-neq-ab*: $S \neq ab$
and $c'd'$-*def*: $[a;\ b;\ d']\ [c';\ b;\ a]\ [c';\ b;\ d']$
path $d'e\ d'\ e$ *path* $c'e\ c'\ e$
shows $\exists f'.\ [e;\ c';\ f'] \wedge$ *path-ex* $f'\ b$
proof (*cases*)
assume *path-ex b f*
thus *?thesis*
using *f-def* **by** *blast*
next
assume *no-path*: $\neg$(*path-ex b f*)
have *path-S-2*: $S \in \mathcal{P}\ a \in S\ e \in S\ e \neq a$
using *path-S* **by** *auto*
have $f \in c'e$
using *betw-c-in-path f-def* $c'd'$-*def*(*5*) **by** *blast*
have $c' \in ab\ d' \in ab$
using *betw-a-in-path betw-c-in-path* $c'd'$-*def*(*1,2*) *path-ab* **by** *blast+*
have $f \in$ *unreach−on* $c'e$ *from b*
using *no-path assms*(*1,4−9*) *path-S-2* ‹$f \in c'e$› ‹$c' \in ab$› ‹$d' \in ab$›
abc-abc-neq betw-events eq-paths unreachable-bounded-path-only
by *metis*
moreover have $c' \in c'e -$ *unreach−on* $c'e$ *from b*
using $c'd'$-*def cross-in-reachable path-ab* ‹$c' \in ab$› **by** *blast*
moreover have $b \in \mathcal{E} \wedge b \notin c'e$
using ‹$f \in$ *unreach−on* $c'e$ *from b*› *betw-events* $c'd'$-*def same-empty-unreach*
by *auto*
ultimately obtain f' **where** f'-*def*: $[c';\ f;\ f']\ f' \in c'e\ f' \notin$ *unreach−on* $c'e$ *from b* $c' \neq f'\ b \neq f'$
using *unreachable-set-bounded* $c'd'$-*def*
by (*metis DiffE*)
hence $[e;\ c';\ f']$
using ‹$[e;\ c';\ f]$› **by** *blast*
moreover obtain $f'b$ **where** *path* $f'b\ f'\ b$
using ‹$b \in \mathcal{E} \wedge b \notin c'e$› $c'd'$-*def* f'-*def*(*2,3*) *unreachable-bounded-path-only*
by *blast*
ultimately show *?thesis* **by** *blast*
qed

lemma *abc-abd-bcdbdc*:
assumes *abc*: $[a;b;c]$
and *abd*: $[a;b;d]$
and *c-neq-d*: $c \neq d$
shows $[b;c;d] \vee [b;d;c]$
proof −
have $\neg\ [d;b;c]$
proof (*rule notI*)
assume *dbc*: $[d;b;c]$
obtain *ab* **where** *path-ab*: *path ab a b*
using *abc-abc-neq abc-ex-path-unique abc* **by** *blast*
obtain *S* **where** *path-S*: $S \in \mathcal{P}$
and *S-neq-ab*: $S \neq ab$
and *a-inS*: $a \in S$
using *ex-crossing-at path-ab*
by *auto*

have $\exists\, e{\in}S.\ e \neq a \wedge (\exists\, be{\in}\mathcal{P}.\ path\ be\ b\ e)$
proof −
have *b-notinS*: $b \notin S$ **using** *S-neq-ab a-inS path-S path-ab path-unique* **by** *blast*
then obtain *x y z* **where** *x-in-unreach*: $x \in unreach{-}on\ S\ from\ b$
and *y-in-unreach*: $y \in unreach{-}on\ S\ from\ b$
and *x-neq-y*: $x \neq y$
and *z-in-reach*: $z \in S - unreach{-}on\ S\ from\ b$
using *two-in-unreach* [**where** $Q = S$ **and** $b = b$]
in-path-event path-S path-ab a-inS cross-in-reachable
by *blast*
then obtain *w* **where** *w-in-reach*: $w \in S - unreach{-}on\ S\ from\ b$
and *w-neq-z*: $w \neq z$
using *unreachable-set-bounded* [**where** $Q = S$ **and** $b = b$ **and** $Qx = z$ **and** $Qy = x$]
b-notinS in-path-event path-S path-ab **by** *blast*
thus *?thesis* **by** (*metis DiffD1 b-notinS in-path-event path-S path-ab reachable-path z-in-reach*)
qed
then obtain *e be* **where** *e-inS*: $e \in S$
and *e-neq-a*: $e \neq a$
and *path-be*: *path be b e*
by *blast*
have *path-ae*: *path S a e*
using *a-inS e-inS e-neq-a path-S* **by** *auto*
have *S-neq-ab-2*: $S \neq path\text{-}of\ a\ b$
using *S-neq-ab cross-once-notin path-ab path-of-ex* **by** *blast*

have $\exists\, c'\ d'.$
$c'{\in}ab \wedge d'{\in}ab$
$\wedge\ [a;\ b;\ d'] \wedge [c';\ b;\ a] \wedge [c';\ b;\ d']$

$\wedge$ *path-ex* d' e $\wedge$ *path-ex* c' e
using *exist-c'd'* [**where** a=a **and** b=b **and** c=c **and** d=d **and** e=e **and** *be*=*be* **and** S=S]
using *assms*(*1*−*2*) *dbc e-neq-a path-ae path-be S-neq-ab-2*
using *abc-sym betw-a-in-path path-ab* **by** *blast*
then obtain c' d' *d'e c'e*
where *c'd'-def*: c'∈*ab* $\wedge$ d'∈*ab*
$\wedge$ [a; b; d'] $\wedge$ [c'; b; a] $\wedge$ [c'; b; d']
$\wedge$ *path d'e d' e* $\wedge$ *path c'e c' e*
by *blast*

obtain f **where** *f-def*: f∈*c'e* [e; c'; f]
using *c'd'-def prolong-betw2* **by** *blast*
then obtain f' *f'b* **where** *f'-def*: [e; c'; f'] $\wedge$ *path f'b f' b*
using *exist-f'*
[**where** e=e **and** c'=c' **and** b=b **and** f=f **and** S=S **and** *ab*=*ab* **and** d'=d' **and** a=a **and** *c'e*=*c'e*]
using *path-ab path-S a-inS e-inS e-neq-a f-def S-neq-ab c'd'-def*
by *blast*

obtain *ae* **where** *path-ae*: *path ae a e* **using** *a-inS e-inS e-neq-a path-S* **by** *blast*
have *tri-aec*: △ *a e c'*
by (*smt cross-once-notin S-neq-ab a-inS abc abc-abc-neq abc-ex-path e-inS e-neq-a path-S path-ab c'd'-def paths-tri*)

then obtain h **where** *h-in-f'b*: h ∈ *f'b*
and *ahe*: [*a*;*h*;*e*]
and *f'bh*: [*f'*; *b*; *h*]
using *collinearity2* [**where** a = a **and** b = e **and** c = c' **and** d = f' **and** e = b **and** *de* = *f'b*]
f'-def c'd'-def f'-def betw-c-in-path **by** *blast*
have *tri-dec*: △ d' e c'
using *cross-once-notin S-neq-ab a-inS abc abc-abc-neq abc-ex-path e-inS e-neq-a path-S path-ab c'd'-def paths-tri* **by** *smt*
then obtain g **where** *g-in-f'b*: g ∈ *f'b*
and *d'ge*: [*d'*; *g*; *e*]
and *f'bg*: [*f'*; *b*; *g*]
using *collinearity2* [**where** a = d' **and** b = e **and** c = c' **and** d = f' **and** e = b **and** *de* = *f'b*]
f'-def c'd'-def betw-c-in-path **by** *blast*
have △ *e a d'* **by** (*smt betw-c-in-path paths-tri2 S-neq-ab a-inS abc-ac-neq abd e-inS e-neq-a c'd'-def path-S path-ab*)
thus *False*
using *tri-betw-no-path* [**where** a = e **and** b = a **and** c = d' **and** b' = g **and** a' = b **and** c' = h]
f'-def c'd'-def h-in-f'b g-in-f'b abd d'ge ahe abc-sym

by *blast*
qed
thus *?thesis*
by (*smt abc abc-abc-neq abc-ex-path abc-sym abd c-neq-d cross-once-notin some-betw*)
qed

lemma *abc-abd-acdadc*:
assumes *abc*: $[a;b;c]$
and *abd*: $[a;b;d]$
and *c-neq-d*: $c \neq d$
shows $[a;c;d] \vee [a;d;c]$
proof −
have *cba*: $[c;b;a]$ **using** *abc-sym abc* **by** *simp*
have *dba*: $[d;b;a]$ **using** *abc-sym abd* **by** *simp*
have *dcb-over-cba*: $[d;c;b] \wedge [c;b;a] \Longrightarrow [d;c;a]$ **by** *auto*
have *cdb-over-dba*: $[c;d;b] \wedge [d;b;a] \Longrightarrow [c;d;a]$ **by** *auto*

have *bcdbdc*: $[b;c;d] \vee [b;d;c]$ **using** *abc abc-abd-bcdbdc abd c-neq-d* **by** *auto*
then have *dcb-or-cdb*: $[d;c;b] \vee [c;d;b]$ **using** *abc-sym* **by** *blast*
then have $[d;c;a] \vee [c;d;a]$ **using** *abc-only-cba dcb-over-cba cdb-over-dba cba dba* **by** *blast*
thus *?thesis* **using** *abc-sym* **by** *auto*
qed

lemma *abc-acd-bcd*:
assumes *abc*: $[a;b;c]$
and *acd*: $[a;c;d]$
shows $[b;c;d]$
proof −
have *path-abc*: $\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$ **using** *abc* **by** (*simp add*: *abc-ex-path*)
have *path-acd*: $\exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge c \in Q \wedge d \in Q$ **using** *acd* **by** (*simp add*: *abc-ex-path*)
then have $\exists\, Q{\in}\mathcal{P}.\ b \in Q \wedge c \in Q \wedge d \in Q$ **using** *path-abc abc-abc-neq acd cross-once-notin* **by** *metis*
then have *bcd3*: $[b;c;d] \vee [b;d;c] \vee [c;b;d]$ **by** (*metis abc abc-only-cba*(*1*,*2*) *acd some-betw2*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ $[b;c;d]$
then have $[b;d;c] \vee [c;b;d]$ **using** *bcd3* **by** *simp*
thus *False*
proof (*rule disjE*)
assume $[b;d;c]$
then have $[c;d;b]$ **using** *abc-sym* **by** *simp*

```
    then have [a;c;b] using acd abc-bcd-abd by blast
    thus False using abc abc-only-cba by blast
  next
    assume cbd: [c;b;d]
    have cba: [c;b;a] using abc abc-sym by blast
    have a-neq-d: a ≠ d using abc-ac-neq acd by auto
    then have [c;a;d] ∨ [c;d;a] using abc-abd-acdadc cbd cba by simp
    thus False using abc-only-cba acd by blast
  qed
 qed
qed
```

A few lemmas that don't seem to be proved by Schutz, but can be proven now, after Lemma 3. These sometimes avoid us having to construct a chain explicitly.

```
lemma abd-bcd-abc:
 assumes abd: [a;b;d]
    and bcd: [b;c;d]
 shows [a;b;c]
proof −
 have dcb: [d;c;b] using abc-sym bcd by simp
 have dba: [d;b;a] using abc-sym abd by simp
 have [c;b;a] using abc-acd-bcd dcb dba by blast
 thus ?thesis using abc-sym by simp
qed

lemma abc-acd-abd:
 assumes abc: [a;b;c]
    and acd: [a;c;d]
  shows [a;b;d]
 using abc abc-acd-bcd acd by blast

lemma abd-acd-abcacb:
 assumes abd: [a;b;d]
    and acd: [a;c;d]
    and bc: b≠c
  shows [a;b;c] ∨ [a;c;b]
proof −
 obtain P where P-def: P∈𝒫 a∈P b∈P d∈P
   using abd abc-ex-path by blast
 hence c∈P
   using acd abc-abc-neq betw-b-in-path by blast
 have ¬[b;a;c]
   using abc-only-cba abd acd by blast
 thus ?thesis
   by (metis P-def(1−3) ‹c ∈ P› abc-abc-neq abc-sym abd acd bc some-betw)
qed

lemma abe-ade-bcd-ace:
```

assumes *abe*: $[a;b;e]$
and *ade*: $[a;d;e]$
and *bcd*: $[b;c;d]$
shows $[a;c;e]$
proof −
have *abdadb*: $[a;b;d] \vee [a;d;b]$
using *abc-ac-neq abd-acd-abcacb abe ade bcd* **by** *auto*
thus *?thesis*
proof
assume $[a;b;d]$ **thus** *?thesis*
by (*meson abc-acd-abd abc-sym ade bcd*)
next assume $[a;d;b]$ **thus** *?thesis*
by (*meson abc-acd-abd abc-sym abe bcd*)
qed
qed

Now we start on Theorem 9. Based on Veblen (1904) Lemma 2 p357.

lemma (**in** *MinkowskiBetweenness*) *chain3*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *c-inQ*: $c \in Q$
and *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$
shows *ch* $\{a,b,c\}$
proof −
have *abc-betw*: $[a;b;c] \vee [a;c;b] \vee [b;a;c]$
using *assms* **by** (*meson in-path-event abc-sym some-betw insert-subset*)
have *ch1*: $[a;b;c] \longrightarrow ch\ \{a,b,c\}$
using *abc-abc-neq ch-by-ord-def ch-def ord-ordered-loc between-chain* **by** *auto*
have *ch2*: $[a;c;b] \longrightarrow ch\ \{a,c,b\}$
using *abc-abc-neq ch-by-ord-def ch-def ord-ordered-loc between-chain* **by** *auto*
have *ch3*: $[b;a;c] \longrightarrow ch\ \{b,a,c\}$
using *abc-abc-neq ch-by-ord-def ch-def ord-ordered-loc between-chain* **by** *auto*
show *?thesis*
using *abc-betw ch1 ch2 ch3* **by** (*metis insert-commute*)
qed

lemma *overlap-chain*: $[\![[a;b;c];\ [b;c;d]]\!] \Longrightarrow ch\ \{a,b,c,d\}$
proof −
assume $[a;b;c]$ **and** $[b;c;d]$
have $\exists f.$ *local-ordering f betw* $\{a,b,c,d\}$
proof −
have *f1*: $[a;b;d]$
using ‹$[a;b;c]$› ‹$[b;c;d]$› **by** *blast*
have $[a;c;d]$
using ‹$[a;b;c]$› ‹$[b;c;d]$› *abc-bcd-acd* **by** *blast*
then show *?thesis*
using *f1* **by** (*metis* (*no-types*) ‹$[a;b;c]$› ‹$[b;c;d]$› *abc-abc-neq overlap-ordering-loc*)
qed

```
  hence ∃f. local-long-ch-by-ord f {a,b,c,d}
    apply (simp add: chain-defs eval-nat-numeral)
    using ‹[a;b;c]› abc-abc-neq
   by (smt (z3) ‹[b;c;d]› card.empty card-insert-disjoint card-insert-le finite.emptyI
      finite.insertI insertE insert-absorb insert-not-empty)
  thus ?thesis
    by (simp add: chain-defs)
qed
```

The book introduces Theorem 9 before the above three lemmas but can only complete the proof once they are proven. This doesn't exactly say it the same way as the book, as the book gives the *local-ordering* (abcd) explicitly (for arbitrarly named events), but is equivalent.

```
theorem chain4:
  assumes path-Q: Q ∈ 𝒫
      and inQ: a ∈ Q b ∈ Q c ∈ Q d ∈ Q
      and abcd-neq: a ≠ b ∧ a ≠ c ∧ a ≠ d ∧ b ≠ c ∧ b ≠ d ∧ c ≠ d
    shows ch {a,b,c,d}
proof −
  obtain a' b' c' where a'-pick: a' ∈ {a,b,c,d}
                    and b'-pick: b' ∈ {a,b,c,d}
                    and c'-pick: c' ∈ {a,b,c,d}
                    and a'b'c': [a'; b'; c']
      using some-betw by (metis inQ(1,2,4) abcd-neq insert-iff path-Q)
  then obtain d' where d'-neq: d' ≠ a' ∧ d' ≠ b' ∧ d' ≠ c'
                   and d'-pick: d' ∈ {a,b,c,d}
    using insert-iff abcd-neq by metis
  have all-picked-on-path: a'∈Q b'∈Q c'∈Q d'∈Q
    using a'-pick b'-pick c'-pick d'-pick inQ by blast+
  consider [d'; a'; b'] | [a'; d'; b'] | [a'; b'; d']
    using some-betw abc-only-cba all-picked-on-path(1,2,4)
    by (metis a'b'c' d'-neq path-Q)
  then have picked-chain: ch {a',b',c',d'}
  proof (cases)
    assume [d'; a'; b']
    thus ?thesis using a'b'c' overlap-chain by (metis (full-types) insert-commute)
  next
    assume a'd'b': [a'; d'; b']
    then have [d'; b'; c'] using abc-acd-bcd a'b'c' by blast
    thus ?thesis using a'd'b' overlap-chain by (metis (full-types) insert-commute)
  next
    assume a'b'd': [a'; b'; d']
   then have two-cases: [b'; c'; d'] ∨ [b'; d'; c'] using abc-abd-bcdbdc a'b'c' d'-neq
by blast

    have case1: [b'; c'; d'] ⟹ ?thesis using a'b'c' overlap-chain by blast
    have case2: [b'; d'; c'] ⟹ ?thesis
        using abc-only-cba abc-acd-bcd a'b'd' overlap-chain
        by (metis (full-types) insert-commute)
```

show *?thesis* **using** *two-cases case1 case2* **by** *blast*
qed
have $\{a',b',c',d'\} = \{a,b,c,d\}$
proof (*rule Set.set-eqI*, *rule iffI*)
fix x
assume $x \in \{a',b',c',d'\}$
thus $x \in \{a,b,c,d\}$ **using** *a'-pick b'-pick c'-pick d'-pick* **by** *auto*
next
fix x
assume *x-pick*: $x \in \{a,b,c,d\}$
have $a' \neq b' \wedge a' \neq c' \wedge a' \neq d' \wedge b' \neq c' \wedge c' \neq d'$
using *a'b'c' abc-abc-neq d'-neq* **by** *blast*
thus $x \in \{a',b',c',d'\}$
using *a'-pick b'-pick c'-pick d'-pick x-pick d'-neq* **by** *auto*
qed
thus *?thesis* **using** *picked-chain* **by** *simp*
qed

theorem *chain4-alt*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *abcd-inQ*: $\{a,b,c,d\} \subseteq Q$
and *abcd-distinct*: *card* $\{a,b,c,d\} = 4$
shows *ch* $\{a,b,c,d\}$
proof −
have *abcd-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
using *abcd-distinct numeral-3-eq-3*
by (*smt* (*z3*) *card-1-singleton-iff card-2-iff card-3-dist insert-absorb2 insert-commute numeral-1-eq-Suc-0 numeral-eq-iff semiring-norm*(*85*) *semiring-norm*(*88*) *verit-eq-simplify*(*8*))
have *inQ*: $a \in Q$ $b \in Q$ $c \in Q$ $d \in Q$
using *abcd-inQ* **by** *auto*
show *?thesis* **using** *chain4*[*OF assms*(*1*) *inQ*] *abcd-neq* **by** *simp*
qed

end

29 Interlude - Chains, segments, rays

context *MinkowskiBetweenness* **begin**

29.1 General results for chains

lemma *inf-chain-is-long*:
assumes $[f \rightsquigarrow X|x..]$
shows *local-long-ch-by-ord f X* $\wedge$ *f 0* $= x \wedge$ *infinite X*
using *chain-defs* **by** (*metis assms infinite-chain-alt*)

A reassurance that the starting point x is implied.

lemma *long-inf-chain-is-semifin*:

assumes *local-long-ch-by-ord f X* ∧ *infinite X*
shows ∃ *x*. [*f*⇝*X*|*x*..]
using *assms infinite-chain-with-def chain-alts* **by** *auto*

lemma *endpoint-in-semifin*:
assumes [*f*⇝*X*|*x*..]
shows *x*∈*X*
using *zero-into-ordering-loc* **by** (*metis assms empty-iff inf-chain-is-long local-long-ch-by-ord-alt*)

Yet another corollary to Theorem 2, without indices, for arbitrary events on the chain.

corollary *all-aligned-on-fin-chain*:
assumes [*f*⇝*X*] *finite X*
and *x*: *x*∈*X* **and** *y*: *y*∈*X* **and** *z*:*z*∈*X* **and** *xy*: *x*≠*y* **and** *xz*: *x*≠*z* **and** *yz*: *y*≠*z*
shows [*x*;*y*;*z*] ∨ [*x*;*z*;*y*] ∨ [*y*;*x*;*z*]
proof −
have *card X* ≥ *3* **using** *assms*(*2*−*5*) *three-subset*[*OF xy xz yz*] **by** *blast*
hence *1*: *local-long-ch-by-ord f X*
using *assms*(*1*,*3*−) *chain-defs* **by** (*metis short-ch-alt*(*1*) *short-ch-card*(*1*) *short-ch-card-2*)
obtain *i j k* **where** *ijk*: *x*=*f i i*<*card X y*=*f j j*<*card X z*=*f k k*<*card X*
using *obtain-index-fin-chain assms*(*1*−*5*) **by** *metis*
have *2*: [*f i*;*f j*;*f k*] **if** *i*<*j* ∧ *j*<*k k*<*card X* **for** *i j k*
using *assms order-finite-chain2 that*(*1*,*2*) **by** *auto*
consider *i*<*j* ∧ *j*<*k*|*i*<*k* ∧ *k*<*j*|*j*<*i* ∧ *i*<*k*|*i*>*j* ∧ *j*>*k*|*i*>*k* ∧ *k*>*j*|*j*>*i* ∧ *i*>*k*
using *xy xz yz ijk*(*1*,*3*,*5*) **by** (*metis linorder-neqE-nat*)
thus *?thesis*
apply *cases* **using** *2 abc-sym ijk* **by** *presburger*+
qed

lemma (**in** *MinkowskiPrimitive*) *card2-either-elt1-or-elt2*:
assumes *card X* = *2* **and** *x*∈*X* **and** *y*∈*X* **and** *x*≠*y*
and *z*∈*X* **and** *z*≠*x*
shows *z*=*y*
by (*metis assms card-2-iff'*)

lemma *get-fin-long-ch-bounds*:
assumes *local-long-ch-by-ord f X*
and *finite X*
shows ∃ *x*∈*X*. ∃ *y*∈*X*. ∃ *z*∈*X*. [*f*⇝*X*|*x*..*y*..*z*]
proof (*rule bexI*)+
show *1*:[*f*⇝*X*|*f 0*..*f 1*..*f* (*card X* − *1*)]
using *assms* **unfolding** *finite-long-chain-with-def* **using** *index-injective*
by (*auto simp*: *finite-chain-with-alt local-long-ch-by-ord-def local-ordering-def*)
show *f* (*card X* − *1*) ∈ *X*
using *1 points-in-long-chain*(*3*) **by** *auto*
show *f 0* ∈ *X f 1* ∈ *X*
using *1 points-in-long-chain* **by** *auto*
qed

lemma *get-fin-long-ch-bounds2*:
assumes *local-long-ch-by-ord f X*
and *finite X*
obtains *x y z* n_x n_y n_z
where $x{\in}X$ $y{\in}X$ $z{\in}X$ $[f{\rightsquigarrow}X|x..y..z]$ $f\ n_x = x\ f\ n_y = y\ f\ n_z = z$
using *get-fin-long-ch-bounds assms*
by (*meson finite-chain-with-def finite-long-chain-with-alt index-middle-element*)

lemma *long-ch-card-ge3*:
assumes *ch-by-ord f X finite X*
shows *local-long-ch-by-ord f X* $\longleftrightarrow$ *card X* $\geq$ *3*
using *assms ch-by-ord-def local-long-ch-by-ord-def short-ch-card*(*1*) **by** *auto*

lemma *fin-ch-betw2*:
assumes $[f{\rightsquigarrow}X|a..c]$ **and** $b{\in}X$
obtains $b{=}a|b{=}c|[a;b;c]$
by (*metis assms finite-long-chain-with-alt finite-long-chain-with-def*)

lemma *chain-bounds-unique*:
assumes $[f{\rightsquigarrow}X|a..c]$ $[g{\rightsquigarrow}X|x..z]$
shows $(a{=}x \wedge c{=}z) \vee (a{=}z \wedge c{=}x)$
using *assms points-in-chain abc-abc-neq abc-bcd-acd abc-sym*
by (*metis* (*full-types*) *fin-ch-betw2*)

end

29.2 Results for segments, rays and (sub)chains

context *MinkowskiBetweenness* **begin**

lemma *inside-not-bound*:
assumes $[f{\rightsquigarrow}X|a..c]$
and $j{<}card\ X$
shows $j{>}0 \Longrightarrow f\ j \neq a\ j{<}card\ X - 1 \Longrightarrow f\ j \neq c$
using *index-injective2 assms finite-chain-def finite-chain-with-def* **apply** *metis*
using *index-injective2 assms finite-chain-def finite-chain-with-def* **by** *auto*

Converse to Theorem 2(i).

lemma (**in** *MinkowskiBetweenness*) *order-finite-chain-indices*:
assumes *chX*: *local-long-ch-by-ord f X finite X*
and *abc*: $[a;b;c]$
and *ijk*: $f\ i = a\ f\ j = b\ f\ k = c$ $i{<}card\ X$ $j{<}card\ X$ $k{<}card\ X$
shows $i{<}j \wedge j{<}k \vee k{<}j \wedge j{<}i$
by (*metis abc-abc-neq abc-only-cba*(*1,2,3*) *assms bot-nat-0.extremum linorder-neqE-nat order-finite-chain*)

lemma *order-finite-chain-indices2*:
assumes [$f \rightsquigarrow X | a..c$]
and $f\ j = b\ j < card\ X$
obtains $0<j \wedge j<(card\ X - 1) | j=(card\ X - 1) \wedge b=c | j=0 \wedge b=a$
proof −
have *finX*: *finite X*
using *assms(3) card.infinite gr-implies-not0* **by** *blast*
have $b \in X$
using *assms* **unfolding** *chain-defs local-ordering-def*
by (*metis One-nat-def card-2-iff insertI1 insert-commute less-2-cases*)
have *a*: $f\ 0 = a$ **and** *c*: $f\ (card\ X - 1) = c$
using *assms(1) finite-chain-with-def* **by** *auto*

have $0<j \wedge j<(card\ X - 1) \vee j=(card\ X - 1) \wedge b=c \vee j=0 \wedge b=a$
proof (*cases short-ch-by-ord f X*)
case *True*
hence $X=\{a,c\}$
using *a assms(1) first-neq-last points-in-chain short-ch-by-ord-def* **by** *fastforce*
then consider $b=a | b=c$
using ‹$b \in X$› **by** *fastforce*
thus *?thesis*
apply *cases* **using** *assms(2,3) a c le-less* **by** *fastforce+*
next
case *False*
hence *chX*: *local-long-ch-by-ord f X*
using *assms(1)* **unfolding** *finite-chain-with-alt* **using** *chain-defs* **by** *meson*
consider $[a;b;c] | b=a | b=c$
using ‹$b \in X$› *assms(1) fin-ch-betw2* **by** *blast*
thus *?thesis* **apply** *cases*
using ‹$f\ 0 = a$› *chX finX assms(2,3) a c order-finite-chain-indices* **apply** *fastforce*
using ‹$f\ 0 = a$› *chX finX assms(2,3) index-injective* **apply** *blast*
using *a c assms chX finX index-injective linorder-neqE-nat inside-not-bound(2)*
by *metis*
qed
thus *?thesis* **using** *that* **by** *blast*
qed

lemma *index-bij-betw-subset*:
assumes *chX*: [$f \rightsquigarrow X | a..b..c$] $f\ i = b\ card\ X > i$
shows *bij-betw f* $\{0<..<i\}$ $\{e \in X.\ [a;e;b]\}$
proof (*unfold bij-betw-def*, *intro conjI*)
have *chX2*: *local-long-ch-by-ord f X finite X*
using *assms* **unfolding** *chain-defs* **apply** (*metis chX(1)*
abc-ac-neq fin-ch-betw points-in-long-chain(1,3) short-ch-alt(1) short-ch-path)
using *assms* **unfolding** *chain-defs* **by** *simp*
from *index-bij-betw*[*OF this*] **have** *1*: *bij-betw f* $\{0..<card\ X\}\ X$ **.**
have $\{0<..<i\} \subset \{0..<card\ X\}$

using *assms(1,3)* **unfolding** *chain-defs* **by** *fastforce*
show *inj-on f {0<..<i}*
using *1 assms(3)* **unfolding** *bij-betw-def*
by (*smt (z3) atLeastLessThan-empty-iff2 atLeastLessThan-iff empty-iff greaterThanLessThan-iff inj-on-def less-or-eq-imp-le*)
show *f ‘ {0<..<i} = {e ∈ X. [a;e;b]}*
proof
show *f ‘ {0<..<i} ⊆ {e ∈ X. [a;e;b]}*
proof (*auto simp add*: *image-subset-iff conjI*)
fix *j* **assume** *asm*: *j>0 j<i*
hence *j < card X* **using** *chX(3) less-trans* **by** *blast*
thus *f j ∈ X [a;f j;b]*
using *chX(1) asm(1)* **unfolding** *chain-defs local-ordering-def*
apply (*metis chX2(1) chX(1) fin-chain-card-geq-2 short-ch-card-2 short-xor-long(2) le-antisym set-le-two finite-chain-def finite-chain-with-def finite-long-chain-with-alt*)
using *chX asm chX2(1) order-finite-chain* **unfolding** *chain-defs local-ordering-def* **by** *force*
qed
show *{e ∈ X. [a;e;b]} ⊆ f ‘ {0<..<i}*
proof (*auto*)
fix *e* **assume** *e*: *e ∈ X [a;e;b]*
obtain *j* **where** *f j = e j<card X*
using *e chX2* **unfolding** *chain-defs local-ordering-def* **by** *blast*
show *e ∈ f ‘ {0<..<i}*
proof
have *0<j∧j<i ∨ i<j∧j<0*
using *order-finite-chain-indices chX chain-defs*
by (*smt (z3) ‹f j = e› ‹j < card X› chX2(1) e(2) gr-implies-not-zero linorder-neqE-nat*)
hence *j<i* **by** *simp*
thus *j∈{0<..<i} e = f j*
using *‹0 < j ∧ j < i ∨ i < j ∧ j < 0› greaterThanLessThan-iff*
by (*blast,*(*simp add*: *‹f j = e›*))
qed
qed
qed
qed

lemma *bij-betw-extend*:
assumes *bij-betw f A B*
and *f a = b a∉A b∉B*
shows *bij-betw f* (*insert a A*) (*insert b B*)
by (*smt* (*verit*, *ccfv-SIG*) *assms(1) assms(2) assms(4) bij-betwI' bij-betw-iff-bijections insert-iff*)

lemma *insert-iff2*:
assumes *a∈X* **shows** *insert a {x∈X. P x} = {x∈X. P x ∨ x=a}*

using *insert-iff assms* **by** *blast*

lemma *index-bij-betw-subset2*:
assumes *chX*: $[f\rightsquigarrow X|a..b..c]$ $f\ i = b$ *card* $X > i$
shows *bij-betw* f $\{0..i\}$ $\{e{\in}X.\ [a;e;b]{\vee}a{=}e{\vee}b{=}e\}$
proof −
have *bij-betw* f $\{0{<}..{<}i\}$ $\{e{\in}X.\ [a;e;b]\}$ **using** *index-bij-betw-subset*[*OF assms*] **.**
moreover have $0{\notin}\{0{<}..{<}i\}$ $i{\notin}\{0{<}..{<}i\}$ **by** *simp+*
moreover have $a{\notin}\{e{\in}X.\ [a;e;b]\}$ $b{\notin}\{e{\in}X.\ [a;e;b]\}$ **using** *abc-abc-neq* **by** *auto+*
moreover have $f\ 0 = a$ $f\ i = b$ **using** *assms* **unfolding** *chain-defs* **by** *simp+*
moreover have (*insert b* (*insert a* $\{e{\in}X.\ [a;e;b]\}$)) $= \{e{\in}X.\ [a;e;b]{\vee}a{=}e{\vee}b{=}e\}$
proof −
have *1*: (*insert a* $\{e{\in}X.\ [a;e;b]\}$) $= \{e{\in}X.\ [a;e;b]{\vee}a{=}e\}$
using *insert-iff2*[*OF points-in-long-chain*(*1*)[*OF chX*(*1*)]] **by** *auto*
have $b{\notin}\{e{\in}X.\ [a;e;b]{\vee}a{=}e\}$
using *abc-abc-neq chX*(*1*) *fin-ch-betw* **by** *fastforce*
thus (*insert b* (*insert a* $\{e{\in}X.\ [a;e;b]\}$)) $= \{e{\in}X.\ [a;e;b]{\vee}a{=}e{\vee}b{=}e\}$
using *1 insert-iff2 points-in-long-chain*(*2*)[*OF chX*(*1*)] **by** *auto*
qed
moreover have (*insert i* (*insert 0* $\{0{<}..{<}i\}$)) $= \{0..i\}$ **using** *image-Suc-lessThan*
by *auto*
ultimately show *?thesis* **using** *bij-betw-extend*[*of f*]
by (*metis* (*no-types*, *lifting*) *chX*(*1*) *finite-long-chain-with-def insert-iff*)
qed

lemma *chain-shortening*:
assumes $[f\rightsquigarrow X|a..b..c]$
shows $[f \rightsquigarrow \{e{\in}X.\ [a;e;b] \vee e{=}a \vee e{=}b\}\ |a..b]$
proof (*unfold finite-chain-with-def finite-chain-def*, (*intro conjI*))

Different forms of assumptions for compatibility with needed antecedents later.

show $f\ 0 = a$ **using** *assms* **unfolding** *chain-defs* **by** *simp*
have *chX*: *local-long-ch-by-ord f X*
using *assms first-neq-last points-in-long-chain*(*1*,*3*) *short-ch-card*(*1*) *chain-defs*
by (*metis card2-either-elt1-or-elt2*)
have *finX*: *finite X*
by (*meson assms chain-defs*)

General facts about the shortened set, which we will call Y.

let *?Y* $= \{e{\in}X.\ [a;e;b] \vee e{=}a \vee e{=}b\}$
show *finY*: *finite ?Y*
using *assms finite-chain-def finite-chain-with-def finite-long-chain-with-alt* **by**
auto
have $a{\neq}b$ $a{\in}?Y$ $b{\in}?Y$ $c{\notin}?Y$
using *assms finite-long-chain-with-def* **apply** *simp*
using *assms points-in-long-chain*(*1*,*2*) **apply** *auto*[*1*]

using *assms points-in-long-chain*(*2*) **apply** *auto*[*1*]
using *abc-ac-neq abc-only-cba*(*2*) *assms fin-ch-betw* **by** *fastforce*
from *this*(*1−3*) *finY* **have** *cardY*: *card ?Y* ≥ *2*
by (*metis* (*no-types*, *lifting*) *card-le-Suc0-iff-eq not-less-eq-eq numeral-2-eq-2*)

Obtain index for b (a is at index *0*): this index i is *card ?Y − 1*.

obtain *i* **where** *i*: *i*<*card X f i*=*b*
using *assms* **unfolding** *chain-defs local-ordering-def* **using** *Suc-leI diff-le-self*
by *force*
hence *i*<*card X − 1*
using *assms* **unfolding** *chain-defs*
by (*metis Suc-lessI diff-Suc-Suc diff-Suc-eq-diff-pred minus-nat.diff-0 zero-less-diff*)
have *card01*: *i*+*1* = *card* {*0..i*} **by** *simp*
have *bb*: *bij-betw f* {*0..i*} *?Y* **using** *index-bij-betw-subset2*[*OF assms i*(*2*,*1*)]
Collect-cong **by** *smt*
hence *i-eq*: *i* = *card ?Y − 1* **using** *bij-betw-same-card* **by** *force*
thus *f* (*card ?Y − 1*) = *b* **using** *i*(*2*) **by** *simp*

The path P on which X lies. If *?Y* has two arguments, P makes it a short chain.

obtain *P* **where** *P-def*: *P*∈*𝒫 X*⊆*P* ⋀*Q*. *Q*∈*𝒫* ∧ *X*⊆*Q* ⟹ *Q*=*P*
using *fin-chain-on-path*[*of f X*] *assms* **unfolding** *chain-defs* **by** *force*
have *a*∈*P b*∈*P* **using** *P-def* **by** (*meson assms in-mono points-in-long-chain*)+

consider (*eq-1*)*i*=*1*|(*gt-1*)*i*>*1* **using** ‹*a* ≠ *b*› ‹*f 0* = *a*› *i*(*2*) *less-linear* **by** *blast*
thus [*f*⇝*?Y*]
proof (*cases*)
case *eq-1*
hence {*0..i*}={*0*,*1*} **by** *auto*
hence *bij-betw f* {*0*,*1*} *?Y* **using** *bb* **by** *auto*
from *bij-betw-imp-surj-on*[*OF this*] **show** *?thesis*
unfolding *chain-defs* **using** *P-def eq-1* ‹*a* ≠ *b*› ‹*f 0* = *a*› *i*(*2*) **by** *blast*
next
case *gt-1*
have *1*: *3*≤*card ?Y* **using** *gt-1 cardY i-eq* **by** *linarith*
{
fix *n* **assume** *n* < *card ?Y*
hence *n*<*card X*
using ‹*i*<*card X − 1*› *add-diff-inverse-nat i-eq nat-diff-split-asm* **by** *linarith*
have *f n* ∈ *?Y*
proof (*simp*, *intro conjI*)
show *f n* ∈ *X*
using ‹*n*<*card X*› *assms chX chain-defs local-ordering-def* **by** *metis*
consider *0*<*n* ∧ *n*<*card ?Y − 1*|*n*=*card ?Y − 1*|*n*=*0*
using ‹*n*<*card ?Y*› *nat-less-le zero-less-diff* **by** *linarith*
thus [*a*;*f n*;*b*] ∨ *f n* = *a* ∨ *f n* = *b*
using *i i-eq* ‹*f 0* = *a*› *chX finX le-numeral-extra*(*3*) *order-finite-chain* **by**
fastforce
qed

```
    } moreover {
      fix x assume x∈?Y hence x∈X by simp
      obtain i_x where i_x: i_x < card X f i_x = x
        using assms obtain-index-fin-chain chain-defs ‹x∈X› by metis
      have i_x < card ?Y
      proof −
        consider [a;x;b]|x=a|x=b using ‹x∈?Y› by auto
        hence (i_x<i ∨ i_x<0) ∨ i_x=0 ∨ i_x=i
          apply cases
            apply (metis ‹f 0=a› chX finX i i_x less-nat-zero-code neq0-conv or-
der-finite-chain-indices)
          using ‹f 0 = a› chX finX i_x index-injective apply blast
          by (metis chX finX i(2) i_x index-injective linorder-neqE-nat)
        thus ?thesis using gt-1 i-eq by linarith
      qed
      hence ∃ n. n < card ?Y ∧ f n = x using i_x(2) by blast
    } moreover {
      fix n assume Suc (Suc n) < card ?Y
      hence Suc (Suc n) < card X
        using i(1) i-eq by linarith
      hence [f n; f (Suc n); f (Suc (Suc n))]
        using assms unfolding chain-defs local-ordering-def by auto
    }
    ultimately have 2: local-ordering f betw ?Y
      by (simp add: local-ordering-def finY)
    show ?thesis using 1 2 chain-defs by blast
  qed
qed


corollary ord-fin-ch-right:
  assumes [f⇝X|a..f i..c] j≥i j<card X
  shows [f i;f j;c] ∨ j = card X − 1 ∨ j = i
proof −
  consider (inter)j>i ∧ j<card X − 1|(left)j=i|(right)j=card X − 1
    using assms(3,2) by linarith
  thus ?thesis
    apply cases
    using assms(1) chain-defs order-finite-chain2 apply force
    by simp+
qed

lemma f-img-is-subset:
  assumes [f⇝X|(f 0) ..] i≥0 j>i Y=f‘{i..j}
  shows Y⊆X
proof
  fix x assume x∈Y
  then obtain n where n∈{i..j} f n = x
    using assms(4) by blast
```

hence $f\ n \in X$
by (*metis local-ordering-def assms(1) inf-chain-is-long local-long-ch-by-ord-def*)
thus $x \in X$
using ‹$f\ n = x$› **by** *blast*
qed

lemma *i-le-j-events-neq*:
assumes $[f \rightsquigarrow X | a..b..c]$
and $i<j\ j<card\ X$
shows $f\ i \neq f\ j$
using *chain-defs* **by** (*meson assms index-injective2*)

lemma *indices-neq-imp-events-neq*:
assumes $[f \rightsquigarrow X | a..b..c]$
and $i \neq j\ j<card\ X\ i<card\ X$
shows $f\ i \neq f\ j$
by (*metis assms i-le-j-events-neq less-linear*)

end

context *MinkowskiSpacetime* **begin**

lemma *bound-on-path*:
assumes $Q \in \mathcal{P}\ [f \rightsquigarrow X | (f\ 0)..]\ X \subseteq Q$ *is-bound-f b X f*
shows $b \in Q$
proof −
obtain $a\ c$ **where** $a \in X\ c \in X\ [a;c;b]$
using *assms(4)*
by (*metis local-ordering-def inf-chain-is-long is-bound-f-def local-long-ch-by-ord-def zero-less-one*)
thus *?thesis*
using *abc-abc-neq assms(1) assms(3) betw-c-in-path* **by** *blast*
qed

lemma *pro-basis-change*:
assumes $[a;b;c]$
shows *prolongation a c* = *prolongation b c* (**is** *?ac=?bc*)
proof
show *?ac* $\subseteq$ *?bc*
proof
fix x **assume** $x \in$ *?ac*
hence $[a;c;x]$
by (*simp add*: *pro-betw*)
hence $[b;c;x]$
using *assms abc-acd-bcd* **by** *blast*
thus $x \in$ *?bc*
using *abc-abc-neq pro-betw* **by** *blast*
qed

```
  show ?bc ⊆ ?ac
  proof
    fix x assume x∈?bc
    hence [b;c;x]
      by (simp add: pro-betw)
    hence [a;c;x]
      using assms abc-bcd-acd by blast
    thus x∈?ac
      using abc-abc-neq pro-betw by blast
  qed
qed

lemma adjoining-segs-exclusive:
  assumes [a;b;c]
  shows segment a b ∩ segment b c = {}
proof (cases)
  assume segment a b = {} thus ?thesis by blast
next
  assume segment a b ≠ {}
  have x∈segment a b ⟶ x∉segment b c for x
  proof
    fix x assume x∈segment a b
    hence [a;x;b] by (simp add: seg-betw)
    have ¬[a;b;x] by (meson ‹[a;x;b]› abc-only-cba)
    have ¬[b;x;c]
      using ‹¬ [a;b;x]› abd-bcd-abc assms by blast
    thus x∉segment b c
      by (simp add: seg-betw)
  qed
  thus ?thesis by blast
qed

end
```

30 3.6 Order on a path - Theorems 10 and 11

```
context MinkowskiSpacetime begin
```

30.1 Theorem 10 (based on Veblen (1904) theorem 10).

```
lemma (in MinkowskiBetweenness) two-event-chain:
  assumes finiteX: finite X
      and path-Q: Q ∈ 𝒫
      and events-X: X ⊆ Q
      and card-X: card X = 2
    shows ch X
proof −
  obtain a b where X-is: X={a,b}
    using card-le-Suc-iff numeral-2-eq-2
```

by (*meson card-2-iff card-X*)
have *no-c*: $\neg(\exists\, c{\in}\{a,b\}.\ c{\neq}a \wedge c{\neq}b)$
by *blast*
have $a{\neq}b \wedge a{\in}Q$ & $b{\in}Q$
using *X-is card-X events-X* **by** *force*
hence *short-ch* $\{a,b\}$
using *path-Q no-c* **by** (*meson short-ch-intros*(*2*))
thus *?thesis*
by (*simp add*: *X-is chain-defs*)
qed

lemma (**in** *MinkowskiBetweenness*) *three-event-chain*:
assumes *finiteX*: *finite X*
and *path-Q*: $Q \in \mathcal{P}$
and *events-X*: $X \subseteq Q$
and *card-X*: *card X* = *3*
shows *ch X*
proof −
obtain *a b c* **where** *X-is*: $X{=}\{a,b,c\}$
using *numeral-3-eq-3 card-X* **by** (*metis card-Suc-eq*)
then have *all-neq*: $a{\neq}b \wedge a{\neq}c \wedge b{\neq}c$
using *card-X numeral-2-eq-2 numeral-3-eq-3*
by (*metis Suc-n-not-le-n insert-absorb2 insert-commute set-le-two*)
have *in-path*: $a{\in}Q \wedge b{\in}Q \wedge c{\in}Q$
using *X-is events-X* **by** *blast*
hence $[a;b;c] \vee [b;c;a] \vee [c;a;b]$
using *some-betw all-neq path-Q* **by** *auto*
thus *ch X*
using *between-chain X-is all-neq chain3 in-path path-Q* **by** *auto*
qed

This is case (i) of the induction in Theorem 10.

lemma *chain-append-at-left-edge*:
assumes *long-ch-Y*: $[f{\rightsquigarrow}Y|a_1..a..a_n]$
and bY: $[b;\ a_1;\ a_n]$
fixes *g* **defines** *g-def*: $g \equiv (\lambda j{::}nat.$ *if* $j{\geq}1$ *then f* $(j{-}1)$ *else b*)
shows $[g{\rightsquigarrow}(insert\ b\ Y)|b\ ..\ a_1\ ..\ a_n]$
proof −
let *?X* = *insert b Y*
have *ord-fY*: *local-ordering f betw Y* **using** *long-ch-Y finite-long-chain-with-card chain-defs*
by (*meson long-ch-card-ge3*)
have $b{\notin}Y$
using *abc-ac-neq abc-only-cba*(*1*) *assms* **by** (*metis fin-ch-betw2 finite-long-chain-with-alt*)
have *bound-indices*: *f 0* = $a_1 \wedge f$ (*card Y* − *1*) = a_n
using *long-ch-Y* **by** (*simp add*: *chain-defs*)
have *fin-Y*: *card Y* ≥ *3*
using *finite-long-chain-with-def long-ch-Y numeral-2-eq-2 points-in-long-chain*
by (*metis abc-abc-neq bY card2-either-elt1-or-elt2 fin-chain-card-geq-2 leI le-less-Suc-eq*

numeral-3-eq-3)
hence *num-ord*: *0* ≤ (*0*::*nat*) ∧ *0*<(*1*::*nat*) ∧ *1* < *card Y* − *1* ∧ *card Y* − *1* < *card Y*
by *linarith*
hence [a_1; *f 1*; a_n]
using *order-finite-chain chain-defs long-ch-Y*
by *auto*

Schutz has a step here that says [$ba_1a_2a_n$] is a chain (using Theorem 9). We have no easy way (yet) of denoting an ordered 4-element chain, so we skip this step using a *local-ordering* lemma from our script for 3.6, which Schutz doesn't list.

hence [*b*; a_1; *f 1*]
using *bY abd-bcd-abc* **by** *blast*
have *local-ordering g betw ?X*
proof −
{
fix *n* **assume** *finite ?X* ⟶ *n*<*card ?X*
have *g n* ∈ *?X*
apply (*cases n*≥*1*)
prefer *2* **apply** (*simp add*: *g-def*)
proof
assume *1*≤*n g n* ∉ *Y*
hence *g n* = *f*(*n*−*1*) **unfolding** *g-def* **by** *auto*
hence *g n* ∈ *Y*
proof (*cases n* = *card ?X* − *1*)
case *True*
thus *?thesis*
using ‹*b*∉*Y*› *card.insert diff-Suc-1 long-ch-Y points-in-long-chain chain-defs*
by (*metis* ‹*g n* = *f* (*n* − *1*)›)
next
case *False*
hence *n* < *card Y*
using *points-in-long-chain* ‹*finite ?X* ⟶ *n* < *card ?X*› ‹*g n* = *f* (*n* − *1*)› ‹*g n* ∉ *Y*› ‹*b*∉*Y*› *chain-defs*
by (*metis card.insert finite-insert long-ch-Y not-less-simps*(*1*))
hence *n*−*1* < *card Y* − *1*
using ‹*1* ≤ *n*› *diff-less-mono* **by** *blast*
hence *f*(*n*−*1*)∈*Y*
using *long-ch-Y fin-Y* **unfolding** *chain-defs local-ordering-def*
by (*metis Suc-le-D card-3-dist diff-Suc-1 insert-absorb2 le-antisym less-SucI numeral-3-eq-3 set-le-three*)
thus *?thesis*
using ‹*g n* = *f* (*n* − *1*)› **by** *presburger*
qed
hence *False* **using** ‹*g n* ∉ *Y*› **by** *auto*
thus *g n* = *b* **by** *simp*
qed

```
    } moreover {
      fix n assume (finite ?X ⟶ Suc(Suc n) < card ?X)
      hence [g n; g (Suc n); g (Suc(Suc n))]
        apply (cases n≥1)
        using ‹b∉Y› ‹[b; a₁; f 1]› g-def ordering-ord-ijk-loc[OF ord-fY] fin-Y
       apply (metis Suc-diff-le card-insert-disjoint diff-Suc-1 finite-insert le-Suc-eq
not-less-eq)
        by (metis One-nat-def Suc-leI ‹[b;a₁;f 1]› bound-indices diff-Suc-1 g-def
          not-less-less-Suc-eq zero-less-Suc)
    } moreover {
      fix x assume x∈?X x=b
      have (finite ?X ⟶ 0 < card ?X) ∧ g 0 = x
        by (simp add: ‹b∉Y› ‹x = b› g-def)
    } moreover {
      fix x assume x∈?X x≠b
      hence ∃ n. (finite ?X ⟶ n < card ?X) ∧ g n = x
      proof −
        obtain n where f n = x n < card Y
          using ‹x∈?X› ‹x≠b› local-ordering-def insert-iff long-ch-Y chain-defs by
(metis ord-fY)
        have (finite ?X ⟶ n+1 < card ?X) g(n+1) = x
          apply (simp add: ‹b∉Y› ‹n < card Y›)
          by (simp add: ‹f n = x› g-def)
        thus ?thesis by auto
      qed
    }
    ultimately show ?thesis
      unfolding local-ordering-def
      by smt
  qed
  hence local-long-ch-by-ord g ?X
    unfolding local-long-ch-by-ord-def
    using fin-Y ‹b∉Y›
    by (meson card-insert-le finite-insert le-trans)
  show ?thesis
  proof (intro finite-long-chain-with-alt2)
    show local-long-ch-by-ord g ?X using ‹local-long-ch-by-ord g ?X› by simp
   show [b;a₁;aₙ] ∧ a₁ ∈ ?X using bY long-ch-Y points-in-long-chain(1) by auto
    show g 0 = b using g-def by simp
    show finite ?X
       using fin-Y ‹b∉Y› eval-nat-numeral by (metis card.infinite finite.insertI
not-numeral-le-zero)
    show g (card ?X − 1) = aₙ
      using g-def ‹b∉Y› bound-indices eval-nat-numeral
      by (metis One-nat-def card.infinite card-insert-disjoint diff-Suc-Suc
        diff-is-0-eq′ less-nat-zero-code minus-nat.diff-0 nat-le-linear num-ord)
  qed
qed
```

This is case (iii) of the induction in Theorem 10. Schutz says merely "The

proof for this case is similar to that for Case (i)." Thus I feel free to use a result on symmetry, rather than going through the pain of Case (i) (*chain-append-at-left-edge*) again.

lemma *chain-append-at-right-edge*:
assumes *long-ch-Y*: $[f \rightsquigarrow Y | a_1..a..a_n]$
and *Yb*: $[a_1;\ a_n;\ b]$
fixes *g* **defines** *g-def*: $g \equiv (\lambda j{::}nat.\ if\ j \leq (card\ Y - 1)\ then\ f\ j\ else\ b)$
shows $[g \rightsquigarrow (insert\ b\ Y) | a_1\ ..\ a_n\ ..\ b]$
proof −
let *?X* = *insert b Y*
have $b \notin Y$
using *Yb abc-abc-neq abc-only-cba*(*2*) *long-ch-Y*
by (*metis fin-ch-betw2 finite-long-chain-with-def*)
have *fin-Y*: *card Y* ≥ *3*
using *finite-long-chain-with-card long-ch-Y* **by** *auto*
hence *fin-X*: *finite ?X*
by (*metis card.infinite finite.insertI not-numeral-le-zero*)
have $a_1 \in Y \wedge a_n \in Y \wedge a \in Y$
using *long-ch-Y points-in-long-chain* **by** *meson*
have $a_1 \neq a \wedge a \neq a_n \wedge a_1 \neq a_n$
using *Yb abc-abc-neq finite-long-chain-with-def long-ch-Y* **by** *auto*
have *Suc* (*card Y*) = *card ?X*
using ‹$b \notin Y$› *fin-X finite-long-chain-with-def long-ch-Y* **by** *auto*
obtain *f2* **where** *f2-def*: $[f2 \rightsquigarrow Y | a_n..a..a_1]$ $f2 = (\lambda n.\ f\ (card\ Y - 1 - n))$
using *chain-sym long-ch-Y* **by** *blast*
obtain *g2* **where** *g2-def*: *g2* = (λ*j*::*nat*. *if j*≥*1 then f2* (*j*−*1*) *else b*)
by *simp*
have $[b;\ a_n;\ a_1]$
using *abc-sym Yb* **by** *blast*
hence *g2-ord-X*: $[g2 \rightsquigarrow ?X | b\ ..\ a_n\ ..\ a_1]$
using *chain-append-at-left-edge* [**where** $a_1 = a_n$ **and** $a_n = a_1$ **and** *f*=*f2*]
fin-X ‹$b \notin Y$› *f2-def g2-def*
by *blast*
then obtain *g1* **where** *g1-def*: $[g1 \rightsquigarrow ?X | a_1..a_n..b]$ $g1 = (\lambda n.\ g2\ (card\ ?X - 1 - n))$
using *chain-sym* **by** *blast*
have *sYX*: (*card Y*) = (*card ?X*) − *1*
using *assms*(*2*,*3*) *finite-long-chain-with-def long-ch-Y* ‹*Suc* (*card Y*) = *card ?X*› **by** *linarith*
have *g1*=*g*
unfolding *g1-def g2-def f2-def g-def*
proof
fix *n*
show (
if 1 ≤ *card ?X* − 1 − *n then*
f (*card Y* − 1 − (*card ?X* − 1 − *n* − 1))
else b
) = (
if n ≤ *card Y* − *1 then*

```
          f n
         else b
       ) (is ?lhs=?rhs)
   proof (cases)
     assume n ≤ card ?X − 2
     show ?lhs=?rhs
       using ‹n ≤ card ?X − 2› finite-long-chain-with-def long-ch-Y sYX ‹Suc
(card Y) = card ?X›
     by (metis (mono-tags, opaque-lifting) Suc-1 Suc-leD diff-Suc-Suc diff-commute
diff-diff-cancel
         diff-le-mono2 fin-chain-card-geq-2)
   next
     assume ¬ n ≤ card ?X − 2
     thus ?lhs=?rhs
       by (metis ‹Suc (card Y) = card ?X› Suc-1 diff-Suc-1 diff-Suc-eq-diff-pred
diff-diff-cancel
           diff-is-0-eq' nat-le-linear not-less-eq-eq)
   qed
 qed
 thus ?thesis
   using g1-def(1) by blast
qed


lemma S-is-dense:
 assumes long-ch-Y: [f⇝Y|a1..a..an]
     and S-def: S = {k::nat. [a1; f k; b] ∧ k < card Y}
     and k-def: S≠{} k = Max S
     and k'-def: k'>0 k'<k
 shows k' ∈ S
proof −
```

We will prove this by contradiction. We can obtain the path that Y lies on, and show b is on it too. Then since $f`S$ must be on this path, there must be an ordering involving b, $f\ k$ and $f\ k'$ that leads to contradiction with the definition of S and $k \notin S$. Notice we need no knowledge about b except how it relates to S.

```
 have [f⇝Y] using long-ch-Y chain-defs by meson
 have card Y ≥ 3 using finite-long-chain-with-card long-ch-Y by blast
 hence finite Y by (metis card.infinite not-numeral-le-zero)
have k∈S using k-def Max-in S-def by (metis finite-Collect-conjI finite-Collect-less-nat)
 hence k<card Y using S-def by auto
 have k'<card Y using S-def k'-def ‹k∈S› by auto
 show k' ∈ S
 proof (rule ccontr)
   assume asm: ¬k'∈S
   have 1: [f 0;f k;f k']
   proof −
     have [a1; b; f k']
```

using *order-finite-chain2 long-ch-Y* ‹$k \in S$› ‹$k' < card\ Y$› *chain-defs*
by (*smt* (*z3*) *abc-acd-abd asm le-numeral-extra*(*3*) *assms mem-Collect-eq*)
have $[a_1;\ f\ k;\ b]$
using *S-def* ‹$k \in S$› **by** *blast*
have $[f\ k;\ b;\ f\ k']$
using *abc-acd-bcd* ‹$[a_1;\ b;\ f\ k']$› ‹$[a_1;\ f\ k;\ b]$› **by** *blast*
thus *?thesis*
using ‹$[a_1;f\ k;b]$› *long-ch-Y* **unfolding** *finite-long-chain-with-def finite-chain-with-def*
by *blast*
qed
have *2*: $[f\ 0;f\ k';f\ k]$
apply (*intro order-finite-chain2*[*OF* ‹$[f \rightsquigarrow Y]$› ‹*finite Y*›]) **by** (*simp add*: ‹$k <$ *card Y*› *k'-def*)
show *False* **using** *1 2 abc-only-cba*(*2*) **by** *blast*
qed
qed

lemma *smallest-k-ex*:
assumes *long-ch-Y*: $[f \rightsquigarrow Y | a_1..a..a_n]$
and *Y-def*: $b \notin Y$
and *Yb*: $[a_1;\ b;\ a_n]$
shows $\exists\, k{>}0.\ [a_1;\ b;\ f\ k] \wedge k < card\ Y \wedge \neg(\exists\, k'{<}k.\ [a_1;\ b;\ f\ k'])$
proof −

have *bound-indices*: $f\ 0 = a_1 \wedge f\ (card\ Y - 1) = a_n$
using *chain-defs long-ch-Y* **by** *auto*
have *fin-Y*: *finite Y*
using *chain-defs long-ch-Y* **by** *presburger*
have *card-Y*: $card\ Y \geq 3$
using *long-ch-Y points-in-long-chain finite-long-chain-with-card* **by** *blast*

We consider all indices of chain elements between a_1 and b, and find the maximal one.

let *?S* = $\{k{::}nat.\ [a_1;\ f\ k;\ b] \wedge k < card\ Y\}$
obtain *S* **where** *S-def*: *S*=*?S*
by *simp*
have $S{\subseteq}\{0..card\ Y\}$
using *S-def* **by** *auto*
hence *finite S*
using *finite-subset* **by** *blast*

show *?thesis*
proof (*cases*)
assume $S{=}\{\}$
show *?thesis*
proof
show $(0{::}nat){<}1 \wedge [a_1;\ b;\ f\ 1] \wedge 1 < card\ Y \wedge \neg\ (\exists\, k'{::}nat.\ k' < 1 \wedge [a_1;\ b;\ f\ k'])$

```
    proof (intro conjI)
      show (0::nat)<1 by simp
      show 1 < card Y
        using Yb abc-ac-neq bound-indices not-le by fastforce
      show ¬ (∃ k'::nat. k' < 1 ∧ [a₁; b; f k'])
        using abc-abc-neq bound-indices
        by blast
      show [a₁; b; f 1]
      proof −
        have f 1 ∈ Y
          using long-ch-Y chain-defs local-ordering-def by (metis ‹1 < card Y›
short-ch-ord-in(2))
        hence [a₁; f 1; aₙ]
          using bound-indices long-ch-Y chain-defs local-ordering-def card-Y
            by (smt (z3) Nat.lessE One-nat-def Suc-le-lessD Suc-lessD diff-Suc-1
diff-Suc-less
              fin-ch-betw2 i-le-j-events-neq less-numeral-extra(1) numeral-3-eq-3)
        hence [a₁; b; f 1] ∨ [a₁; f 1; b] ∨ [b; a₁; f 1]
          using abc-ex-path-unique some-betw abc-sym
          by (smt Y-def Yb ‹f 1 ∈ Y› abc-abc-neq cross-once-notin)
        thus [a₁; b; f 1]
        proof −
          have ∀ n. ¬ ([a₁; f n; b] ∧ n < card Y)
            using S-def ‹S = {}›
            by blast
          then have [a₁; b; f 1] ∨ ¬ [aₙ; f 1; b] ∧ ¬ [a₁; f 1; b]
            using bound-indices abc-sym abd-bcd-abc Yb
            by (metis (no-types) diff-is-0-eq' nat-le-linear nat-less-le)
          then show ?thesis
            using abc-bcd-abd abc-sym
            by (meson ‹[a₁; b; f 1] ∨ [a₁; f 1; b] ∨ [b; a₁; f 1]› ‹[a₁; f 1; aₙ]›)
        qed
      qed
    qed
  next assume ¬S={}
    obtain k where k = Max S
      by simp
    hence  k ∈ S using Max-in
      by (simp add: ‹S ≠ {}› ‹finite S›)
    have k≥1
    proof (rule ccontr)
      assume ¬ 1 ≤ k
      hence k=0 by simp
      have [a₁; f k; b]
        using ‹k∈S› S-def
        by blast
      thus False
        using bound-indices ‹k = 0› abc-abc-neq
```

```
      by blast
    qed

    show ?thesis
    proof
      let ?k = k+1
      show 0<?k ∧ [a₁; b; f ?k] ∧ ?k < card Y ∧ ¬ (∃ k'::nat. k' < ?k ∧ [a₁; b; f
k'])
      proof (intro conjI)
        show (0::nat)<?k by simp
        show ?k < card Y
         by (metis (no-types, lifting) S-def Yb ‹k ∈ S› abc-only-cba(2) add.commute
          add-diff-cancel-right' bound-indices less-SucE mem-Collect-eq nat-add-left-cancel-less
              plus-1-eq-Suc)
        show [a₁; b; f ?k]
        proof −
          have f ?k ∈ Y
            using ‹k + 1 < card Y› long-ch-Y card-Y unfolding local-ordering-def
chain-defs
           by (metis One-nat-def Suc-numeral not-less-eq-eq numeral-3-eq-3 numer-
als(1) semiring-norm(2) set-le-two)
          have [a₁; f ?k; aₙ] ∨ f ?k = aₙ
            using fin-ch-betw2 inside-not-bound(1) long-ch-Y chain-defs
            by (metis ‹0 < k + 1› ‹k + 1 < card Y› ‹f (k + 1) ∈ Y›)
          thus [a₁; b; f ?k]
          proof (rule disjE)
            assume [a₁; f ?k; aₙ]
            hence f ?k ≠ aₙ
              by (simp add: abc-abc-neq)
            hence [a₁; b; f ?k] ∨ [a₁; f ?k; b] ∨ [b; a₁; f ?k]
              using abc-ex-path-unique some-betw abc-sym ‹[a₁; f ?k; aₙ]›
                ‹f ?k ∈ Y› Yb abc-abc-neq assms(3) cross-once-notin
              by (smt Y-def)
            moreover have ¬ [a₁; f ?k; b]
            proof
              assume [a₁; f ?k; b]
              hence ?k ∈ S
                using S-def ‹[a₁; f ?k; b]› ‹k + 1 < card Y› by blast
              hence ?k ≤ k
                by (simp add: ‹finite S› ‹k = Max S›)
              thus False
                by linarith
            qed
            moreover have ¬ [b; a₁; f ?k]
              using Yb ‹[a₁; f ?k; aₙ]› abc-only-cba
              by blast
            ultimately show [a₁; b; f ?k]
              by blast
          next assume f ?k = aₙ
```

```
        show ?thesis
          using Yb ‹f (k + 1) = a_n› by blast
      qed
    qed
    show ¬(∃k'::nat. k' < k + 1 ∧ [a_1; b; f k'])
    proof
      assume ∃k'::nat. k' < k + 1 ∧ [a_1; b; f k']
      then obtain k' where k'-def: k'>0 k' < k + 1 [a_1; b; f k']
        using abc-ac-neq bound-indices neq0-conv
        by blast
      hence k'<k
        using S-def ‹k ∈ S› abc-only-cba(2) less-SucE by fastforce
      hence k'∈S
        using S-is-dense long-ch-Y S-def ‹¬S={}› ‹k = Max S› ‹k'>0›
        by blast
      thus False
        using S-def abc-only-cba(2) k'-def(3) by blast
    qed
  qed
 qed
qed
```

```
lemma greatest-k-ex:
  assumes long-ch-Y: [f⇝Y|a_1..a..a_n]
      and Y-def: b∉Y
      and Yb: [a_1; b; a_n]
    shows ∃k. [f k; b; a_n] ∧ k < card Y − 1 ∧ ¬(∃k'<card Y. k'>k ∧ [f k'; b; a_n])
proof −
  have bound-indices: f 0 = a_1 ∧ f (card Y − 1) = a_n
    using chain-defs long-ch-Y by simp
  have fin-Y: finite Y
    using chain-defs long-ch-Y by presburger
  have card-Y: card Y ≥ 3
    using long-ch-Y points-in-long-chain finite-long-chain-with-card by blast
  have chY2: local-long-ch-by-ord f Y
    using long-ch-Y chain-defs by (meson card-Y long-ch-card-ge3)
```

Again we consider all indices of chain elements between a_1 and b.

```
  let ?S = {k::nat. [a_n; f k; b] ∧ k < card Y}
  obtain S where S-def: S=?S
    by simp
  have S⊆{0..card Y}
    using S-def by auto
  hence finite S
    using finite-subset by blast
```

```
  show ?thesis
  proof (cases)
    assume S={}
    show ?thesis
    proof
      let ?n = card Y − 2
     show [f ?n; b; a_n] ∧ ?n < card Y − 1 ∧ ¬(∃ k'<card Y. k'>?n ∧ [f k'; b; a_n])
      proof (intro conjI)
        show ?n < card Y − 1
          using Yb abc-ac-neq bound-indices not-le by fastforce
      next show ¬(∃ k'<card Y. k'>?n ∧ [f k'; b; a_n])
          using abc-abc-neq bound-indices
            by (metis One-nat-def Suc-diff-le Suc-leD Suc-lessI card-Y diff-Suc-1
diff-Suc-Suc
              not-less-eq numeral-2-eq-2 numeral-3-eq-3)
      next show [f ?n; b; a_n]
        proof −
          have [f 0;f ?n; f (card Y − 1)]
            apply (intro order-finite-chain[of f Y], (simp-all add: chY2 fin-Y))
            using card-Y by linarith
          hence [a_1; f ?n; a_n]
            using long-ch-Y unfolding chain-defs by simp
          have f ?n ∈ Y
          using long-ch-Y eval-nat-numeral unfolding local-ordering-def chain-defs
                by (metis card-1-singleton-iff card-Suc-eq card-gt-0-iff diff-Suc-less
diff-self-eq-0 insert-iff numeral-2-eq-2)
          hence [a_n; b; f ?n] ∨ [a_n; f ?n; b] ∨ [b; a_n; f ?n]
            using abc-ex-path-unique some-betw abc-sym ‹[a_1; f ?n; a_n]›
            by (smt Y-def Yb ‹f ?n ∈ Y› abc-abc-neq cross-once-notin)
          thus [f ?n; b; a_n]
          proof −
            have ∀ n. ¬ ([a_n; f n; b] ∧ n < card Y)
              using S-def ‹S = {}›
              by blast
            then have [a_n; b; f ?n] ∨ ¬ [a_1; f ?n; b] ∧ ¬ [a_n; f ?n; b]
              using bound-indices abc-sym abd-bcd-abc Yb
             by (metis (no-types, lifting) ‹f (card Y − 2) ∈ Y› card-gt-0-iff diff-less
empty-iff fin-Y zero-less-numeral)
            then show ?thesis
              using abc-bcd-abd abc-sym
              by (meson ‹[a_n; b; f ?n] ∨ [a_n; f ?n; b] ∨ [b; a_n; f ?n]› ‹[a_1; f ?n; a_n]›)
          qed
        qed
      qed
    qed
  next assume ¬S={}
    obtain k where k = Min S
      by simp
    hence  k ∈ S
```

by (*simp add*: ‹$S \neq \{\}$› ‹*finite S*›)

show *?thesis*
proof
let *?k* = *k−1*
show [*f ?k*; *b*; a_n] ∧ *?k* < *card Y* − *1* ∧ ¬ (∃ *k′*<*card Y*. *?k* < *k′* ∧ [*f k′*; *b*; a_n])
proof (*intro conjI*)
show *?k* < *card Y* − *1*
using *S-def* ‹*k* ∈ *S*› *less-imp-diff-less card-Y*
by (*metis* (*no-types*, *lifting*) *One-nat-def diff-is-0-eq′ diff-less-mono lessI less-le-trans*
mem-Collect-eq nat-le-linear numeral-3-eq-3 zero-less-diff)
show [*f ?k*; *b*; a_n]
proof −
have *f ?k* ∈ *Y*
using ‹*k* − *1* < *card Y* − *1*› *long-ch-Y card-Y eval-nat-numeral* **unfolding** *local-ordering-def chain-defs*
by (*metis Suc-pred′ less-Suc-eq less-nat-zero-code not-less-eq not-less-eq-eq set-le-two*)
have [a_1; *f ?k*; a_n] ∨ *f ?k* = a_1
using *bound-indices long-ch-Y* ‹*k* − *1* < *card Y* − *1*› *chain-defs*
unfolding *finite-long-chain-with-alt*
by (*metis* ‹*f* (*k* − *1*) ∈ *Y*› *card-Diff1-less card-Diff-singleton-if chY2 index-injective*)
thus [*f ?k*; *b*; a_n]
proof (*rule disjE*)
assume [a_1; *f ?k*; a_n]
hence *f ?k* ≠ a_1
using *abc-abc-neq* **by** *blast*
hence [a_n; *b*; *f ?k*] ∨ [a_n; *f ?k*; *b*] ∨ [*b*; a_n; *f ?k*]
using *abc-ex-path-unique some-betw abc-sym* ‹[a_1; *f ?k*; a_n]›
‹*f ?k* ∈ *Y*› *Yb abc-abc-neq assms*(*3*) *cross-once-notin*
by (*smt Y-def*)
moreover have ¬ [a_n; *f ?k*; *b*]
proof
assume [a_n; *f ?k*; *b*]
hence *?k* ∈ *S*
using *S-def* ‹[a_n; *f ?k*; *b*]› ‹*k* − *1* < *card Y* − *1*›
by *simp*
hence *?k* ≥ *k*
by (*simp add*: ‹*finite S*› ‹*k* = *Min S*›)
thus *False*
using ‹*f* (*k* − *1*) ≠ a_1› *chain-defs long-ch-Y*
by *auto*
qed
moreover have ¬ [*b*; a_n; *f ?k*]
using *Yb* ‹[a_1; *f ?k*; a_n]› *abc-only-cba*(*2*) *abc-bcd-acd*
by *blast*

ultimately show [f *?k*; b; a_n]
using *abc-sym* **by** *auto*
next assume f *?k* = a_1
show *?thesis*
using *Yb* ‹f (k − *1*) = a_1› **by** *blast*
qed
qed
show ¬(∃ k'<*card Y*. k−*1* < k' ∧ [f k'; b; a_n])
proof
assume ∃ k'<*card Y*. k−*1* < k' ∧ [f k'; b; a_n]
then obtain k' **where** k'*-def*: k'<*card Y* −*1* k' > k − *1* [a_n; b; f k']
using *abc-ac-neq bound-indices neq0-conv*
by (*metis Suc-diff-1 abc-sym gr-implies-not0 less-SucE*)
hence k'>k
using *S-def* ‹k ∈ S› *abc-only-cba*(*2*) *less-SucE*
by (*metis* (*no-types*, *lifting*) *add-diff-inverse-nat less-one mem-Collect-eq not-less-eq plus-1-eq-Suc*)**thm** *S-is-dense*
hence k'∈S
apply (*intro S-is-dense*[*of f Y* a_1 *a* a_n *- b Max S*])
apply (*simp add*: *long-ch-Y*)
apply (*smt* (*verit*, *ccfv-SIG*) *S-def* ‹k ∈ S› *abc-acd-abd abc-only-cba*(*4*) *add-diff-inverse-nat bound-indices chY2 diff-add-zero diff-is-0-eq fin-Y k'-def*(*1*,*3*) *less-add-one less-diff-conv2 less-nat-zero-code mem-Collect-eq nat-diff-split order-finite-chain*)
apply (*simp add*: ‹S ≠ {}›, *simp*, *simp*)
using k'*-def S-def*
by (*smt* (*verit*, *ccfv-SIG*) ‹k ∈ S› *abc-acd-abd abc-only-cba*(*4*) *add-diff-cancel-right' add-diff-inverse-nat bound-indices chY2 fin-Y le-eq-less-or-eq less-nat-zero-code mem-Collect-eq nat-diff-split nat-neq-iff order-finite-chain zero-less-diff zero-less-one*)
thus *False*
using *S-def abc-only-cba*(*2*) k'*-def*(*3*)
by *blast*
qed
qed
qed
qed
qed

lemma *get-closest-chain-events*:
assumes *long-ch-Y*: [f⇝Y|a_0..a..a_n]
and *x-def*: x∉Y [a_0; x; a_n]
obtains n_b n_c b c
where b=f n_b c=f n_c [b;x;c] b∈Y c∈Y n_b = n_c − *1* n_c<*card Y* n_c>*0*
¬(∃ k < *card Y*. [f k; x; a_n] ∧ k>n_b) ¬(∃ k<n_c. [a_0; x; f k])
proof −
have ∃ n_b n_c b c. b=f n_b ∧ c=f n_c ∧ [b;x;c] ∧ b∈Y ∧ c∈Y ∧ n_b = n_c − *1* ∧

n_c<*card Y* ∧ n_c>*0*
∧ ¬(∃ *k* < *card Y*. [*f k*; *x*; a_n] ∧ *k*>n_b) ∧ ¬(∃ *k* < n_c. [a_0; *x*; *f k*])
proof −
have *bound-indices*: *f 0* = a_0 ∧ *f* (*card Y* − *1*) = a_n
using *chain-defs long-ch-Y* **by** *simp*
have *fin-Y*: *finite Y*
using *chain-defs long-ch-Y* **by** *presburger*
have *card-Y*: *card Y* ≥ *3*
using *long-ch-Y points-in-long-chain finite-long-chain-with-card* **by** *blast*
have *chY2*: *local-long-ch-by-ord f Y*
using *long-ch-Y chain-defs* **by** (*meson card-Y long-ch-card-ge3*)
obtain *P* **where** *P-def*: *P*∈*𝒫 Y*⊆*P*
using *fin-chain-on-path long-ch-Y fin-Y chain-defs* **by** *meson*
hence *x*∈*P*
using *betw-b-in-path x-def*(*2*) *long-ch-Y points-in-long-chain*
by (*metis abc-abc-neq in-mono*)
obtain n_c **where** *nc-def*: ¬(∃ *k*. [a_0; *x*; *f k*] ∧ *k*<n_c) [a_0; *x*; *f* n_c] n_c<*card Y* n_c>*0*
using *smallest-k-ex* [**where** a_1=a_0 **and** *a*=*a* **and** a_n=a_n **and** *b*=*x* **and** *f*=*f* **and** *Y*=*Y*]
long-ch-Y x-def
by *blast*
then obtain *c* **where** *c-def*: *c*=*f* n_c *c*∈*Y*
using *chain-defs local-ordering-def* **by** (*metis chY2*)
have *c-goal*: *c*=*f* n_c ∧ *c*∈*Y* ∧ n_c<*card Y* ∧ n_c>*0* ∧ ¬(∃ *k* < *card Y*. [a_0; *x*; *f k*] ∧ *k*<n_c)
using *c-def nc-def*(*1*,*3*,*4*) **by** *blast*
obtain n_b **where** *nb-def*: ¬(∃ *k* < *card Y*. [*f k*; *x*; a_n] ∧ *k*>n_b) [*f* n_b; *x*; a_n] n_b<*card Y*−*1*
using *greatest-k-ex* [**where** a_1=a_0 **and** *a*=*a* **and** a_n=a_n **and** *b*=*x* **and** *f*=*f* **and** *Y*=*Y*]
long-ch-Y x-def
by *blast*
hence n_b<*card Y*
by *linarith*
then obtain *b* **where** *b-def*: *b*=*f* n_b *b*∈*Y*
using *nb-def chY2 local-ordering-def* **by** (*metis local-long-ch-by-ord-alt*)
have [*b*;*x*;*c*]
proof −
have [*b*; *x*; a_n]
using *b-def*(*1*) *nb-def*(*2*) **by** *blast*
have [a_0; *x*; *c*]
using *c-def*(*1*) *nc-def*(*2*) **by** *blast*
moreover have ∀ *a*. [*a*;*x*;*b*] ∨ ¬ [*a*; a_n; *x*]
using ‹[*b*; *x*; a_n]› *abc-bcd-acd*
by (*metis* (*full-types*) *abc-sym*)
moreover have ∀ *a*. [*a*;*x*;*b*] ∨ ¬ [a_n; *a*; *x*]
using ‹[*b*; *x*; a_n]› **by** (*meson abc-acd-bcd abc-sym*)
moreover have a_n = *c* ⟶ [*b*;*x*;*c*]

using ‹$[b;\ x;\ a_n]$› **by** *meson*
ultimately show *?thesis*
using *abc-abd-bcdbdc abc-sym x-def(2)*
by *meson*
qed
have $n_b{<}n_c$
using ‹$[b;x;c]$› ‹$n_c{<}card\ Y$› ‹$n_b{<}card\ Y$› ‹$c = f\ n_c$› ‹$b = f\ n_b$›
by (*smt* (*z3*) *abc-abd-bcdbdc abc-ac-neq abc-acd-abd abc-only-cba(4) abc-sym bot-nat-0.extremum*
bound-indices chY2 fin-Y nat-neq-iff nc-def(2) nc-def(4) order-finite-chain)
have $n_b = n_c - 1$
proof (*rule ccontr*)
assume $n_b \neq n_c - 1$
have $n_b{<}n_c{-}1$
using ‹$n_b \neq n_c - 1$› ‹$n_b{<}n_c$› **by** *linarith*
hence $[f\ n_b;\ (f(n_c{-}1));\ f\ n_c]$
using ‹$n_b \neq n_c - 1$› *long-ch-Y nc-def(3) order-finite-chain chain-defs*
by *auto*
have $\neg[a_0;\ x;\ (f(n_c{-}1))]$
using *nc-def(1,4) diff-less less-numeral-extra(1)*
by *blast*
have $n_c{-}1{\neq}0$
using ‹$n_b < n_c$› ‹$n_b \neq n_c - 1$› **by** *linarith*
hence $f(n_c{-}1){\neq}a_0 \wedge a_0{\neq}x$
using *bound-indices* ‹$n_b < n_c - 1$› *abc-abc-neq less-imp-diff-less nb-def(1) nc-def(3) x-def(2)*
by *blast*
have $x{\neq}f(n_c{-}1)$
using *x-def(1) nc-def(3) chY2* **unfolding** *chain-defs local-ordering-def*
by (*metis One-nat-def Suc-pred less-Suc-eq nc-def(4) not-less-eq*)
hence $[a_0;\ f\ (n_c{-}1);\ x]$
using *long-ch-Y nc-def c-def chain-defs*
by (*metis* ‹$[f\ n_b;f\ (n_c - 1);f\ n_c]$› ‹$\neg\ [a_0;x;f\ (n_c - 1)]$› *abc-ac-neq abc-acd-abd abc-bcd-acd*
abd-acd-abcacb abd-bcd-abc b-def(1) b-def(2) fin-ch-betw2 nb-def(2))
hence $[(f(n_c{-}1));\ x;\ a_n]$
using *abc-acd-bcd x-def(2)* **by** *blast*
thus *False* **using** *nb-def(1)*
using ‹$n_b < n_c - 1$› *less-imp-diff-less nc-def(3)*
by *blast*
qed
have *b-goal*: $b{=}f\ n_b \wedge b{\in}Y \wedge n_b{=}n_c{-}1 \wedge \neg(\exists\, k < card\ Y.\ [f\ k;\ x;\ a_n] \wedge k{>}n_b)$
using *b-def nb-def(1) nb-def(3)* ‹$n_b{=}n_c{-}1$› **by** *blast*
thus *?thesis*
using ‹$[b;x;c]$› *c-goal*
using ‹$n_b < card\ Y$› *nc-def(1)* **by** *auto*
qed
thus *?thesis*
using *that* **by** *auto*

qed

This is case (ii) of the induction in Theorem 10.

lemma *chain-append-inside*:
assumes *long-ch-Y*: $[f \rightsquigarrow Y | a_1 .. a .. a_n]$
and *Y-def*: $b \notin Y$
and *Yb*: $[a_1;\ b;\ a_n]$
and *k-def*: $[a_1;\ b;\ f\ k]\ k < card\ Y\ \neg(\exists\, k'.\ (0::nat) < k' \wedge k' < k \wedge [a_1;\ b;\ f\ k'])$
fixes *g*
defines *g-def*: $g \equiv (\lambda j::nat.\ if\ (j \leq k-1)\ then\ f\ j\ else\ (if\ (j=k)\ then\ b\ else\ f\ (j-1)))$
shows $[g \rightsquigarrow insert\ b\ Y | a_1\ ..\ b\ ..\ a_n]$
proof −
let *?X* = *insert b Y*
have *fin-X*: *finite ?X*
by (*meson chain-defs finite.insertI long-ch-Y*)
have *bound-indices*: $f\ 0 = a_1 \wedge f\ (card\ Y - 1) = a_n$
using *chain-defs long-ch-Y*
by *auto*
have *fin-Y*: *finite Y*
using *chain-defs long-ch-Y* **by** *presburger*
have *f-def*: *local-long-ch-by-ord f Y*
using *chain-defs long-ch-Y* **by** (*meson finite-long-chain-with-card long-ch-card-ge3*)
have $\langle a_1 \neq a_n \wedge a_1 \neq b \wedge b \neq a_n \rangle$
using *Yb abc-abc-neq* **by** *blast*
have $k \neq 0$
using *abc-abc-neq bound-indices k-def*
by *metis*

have *b-middle*: $[f(k-1);\ b;\ f\ k]$
proof (*cases*)
assume *k=1* **show** $[f(k-1);\ b;\ f\ k]$
using $\langle [a_1;\ b;\ f\ k] \rangle$ $\langle k = 1 \rangle$ *bound-indices* **by** *auto*
next assume *k≠1* **show** $[f(k-1);\ b;\ f\ k]$
proof −
have *a1k*: $[a_1;\ f\ (k-1);\ f\ k]$ **using** *bound-indices*
using $\langle k < card\ Y \rangle$ $\langle k \neq 0 \rangle$ $\langle k \neq 1 \rangle$ *long-ch-Y fin-Y order-finite-chain*
unfolding *chain-defs* **by** *auto*

In fact, the comprehension below gives the order of elements too. Our notation and Theorem 9 are too weak to say that just now.

have *ch-with-b*: *ch* $\{a_1, (f\ (k-1)), b, (f\ k)\}$ **using** *chain4*
using *k-def*(*1*) *abc-ex-path-unique between-chain cross-once-notin*
by (*smt* $\langle [a_1;\ f\ (k-1);\ f\ k] \rangle$ *abc-abc-neq insert-absorb2*)
have $f\ (k-1) \neq b \wedge (f\ k) \neq (f\ (k-1)) \wedge b \neq (f\ k)$
using *abc-abc-neq f-def k-def*(*2*) *Y-def*
by (*metis local-ordering-def* $\langle [a_1;\ f\ (k-1);\ f\ k] \rangle$ *less-imp-diff-less local-long-ch-by-ord-def*)
hence *some-ord-bk*: $[f(k-1);\ b;\ f\ k] \vee [b;\ f\ (k-1);\ f\ k] \vee [f\ (k-1);\ f\ k;\ b]$
using *fin-chain-on-path ch-with-b some-betw Y-def chain-defs*

```
      by (metis a1k abc-acd-bcd abd-acd-abcacb k-def(1))
    thus [f(k−1); b; f k]
    proof −
      have ¬ [a1; f k; b]
        by (simp add: ‹[a1; b; f k]› abc-only-cba(2))
      thus ?thesis
        using some-ord-bk k-def abc-bcd-acd abd-bcd-abc bound-indices
        by (metis diff-is-0-eq' diff-less less-imp-diff-less less-irrefl-nat not-less
            zero-less-diff zero-less-one ‹[a1; b; f k]› a1k)
    qed
  qed
qed

let ?case1 ∨ ?case2 = k−2 ≥ 0 ∨ k+1 ≤ card Y −1

have b-right: [f (k−2); f (k−1); b] if k ≥ 2
proof −
  have k−1 < (k::nat)
    using ‹k ≠ 0› diff-less zero-less-one by blast
  hence k−2 < k−1
    using ‹2 ≤ k› by linarith
  have [f (k−2); f (k−1); b]
    using abd-bcd-abc b-middle f-def k-def(2) fin-Y ‹k−2 < k−1› ‹k−1 < k›
thm2-ind2 chain-defs
    by (metis Suc-1 Suc-le-lessD diff-Suc-eq-diff-pred that zero-less-diff)
  thus [f (k−2); f (k−1); b]
    using ‹[f(k − 1); b; f k]› abd-bcd-abc
    by blast
qed

have b-left: [b; f k; f (k+1)] if k+1 ≤ card Y −1
proof −
  have [f (k−1); f k; f (k+1)]
    using ‹k ≠ 0› f-def fin-Y order-finite-chain that
    by auto
  thus [b; f k; f (k+1)]
    using ‹[f (k − 1); b; f k]› abc-acd-bcd
    by blast
qed

have local-ordering g betw ?X
proof −
  have ∀ n. (finite ?X ⟶ n < card ?X) ⟶ g n ∈ ?X
  proof (clarify)
    fix n assume finite ?X ⟶ n < card ?X g n ∉ Y
    consider n≤k−1 | n≥k+1 | n=k
      by linarith
    thus g n = b
    proof (cases)
```

```
      assume n ≤ k − 1
      thus g n = b
        using f-def k-def(2) Y-def(1) chain-defs local-ordering-def g-def
        by (metis ‹g n ∉ Y› ‹k ≠ 0› diff-less le-less less-one less-trans not-le)
    next
      assume k + 1 ≤ n
      show g n = b
      proof −
        have f n ∈ Y ∨ ¬(n < card Y) for n
          using chain-defs by (metis local-ordering-def f-def)
        then show g n = b
          using ‹finite ?X ⟶ n < card ?X› fin-Y g-def Y-def ‹g n ∉ Y› ‹k + 1
≤ n›
            not-less not-less-simps(1) not-one-le-zero
          by fastforce
      qed
    next
      assume n=k
      thus g n = b
        using Y-def ‹k ≠ 0› g-def
        by auto
    qed
  qed
  moreover have ∀ x∈?X. ∃ n. (finite ?X ⟶ n < card ?X) ∧ g n = x
  proof
    fix x assume x∈?X
    show ∃ n. (finite ?X ⟶ n < card ?X) ∧ g n = x
    proof (cases)
      assume x∈Y
      show ?thesis
      proof −
        obtain ix where f ix = x ix < card Y
          using ‹x ∈ Y› f-def fin-Y
          unfolding chain-defs local-ordering-def
          by auto
        have ix≤k−1 ∨ ix≥k
          by linarith
        thus ?thesis
        proof
          assume ix≤k−1
          hence g ix = x
            using ‹f ix = x› g-def by auto
          moreover have finite ?X ⟶ ix < card ?X
            using Y-def ‹ix < card Y› by auto
          ultimately show ?thesis by metis
        next assume ix≥k
          hence g (ix+1) = x
            using ‹f ix = x› g-def by auto
          moreover have finite ?X ⟶ ix+1 < card ?X
```

```
          using Y-def ‹ix < card Y› by auto
        ultimately show ?thesis by metis
      qed
    qed
  next assume x∉Y
    hence x=b
      using Y-def ‹x ∈ ?X› by blast
    thus ?thesis
   using Y-def ‹k ≠ 0› k-def(2) ordered-cancel-comm-monoid-diff-class.le-diff-conv2
g-def
      by auto
  qed
 qed
 moreover have ∀ n n' n''. (finite ?X ⟶ n'' < card ?X) ∧ Suc n = n' ∧ Suc
n' = n''
      ⟶ [g n; g (Suc n); g (Suc (Suc n))]
 proof (clarify)
  fix n n' n'' assume  a: (finite ?X ⟶ (Suc (Suc n)) < card ?X)
```

Introduce the two-case splits used later.

```
  have  cases-sn: Suc n≤k−1 ∨ Suc n=k if n≤k−1
    using ‹k ≠ 0› that by linarith
  have cases-ssn: Suc(Suc n)≤k−1 ∨ Suc(Suc n)=k if n≤k−1 Suc n≤k−1
    using that(2) by linarith

  consider n≤k−1 | n≥k+1 | n=k
    by linarith
  then show [g n; g (Suc n); g (Suc (Suc n))]
  proof (cases)
    assume n≤k−1 show ?thesis
      using cases-sn
    proof (rule disjE)
      assume Suc n ≤ k − 1
      show ?thesis using cases-ssn
      proof (rule disjE)
        show  n ≤ k − 1 using ‹n ≤ k − 1› by blast
        show ‹Suc n ≤ k − 1› using ‹Suc n ≤ k − 1› by blast
      next
        assume Suc (Suc n) ≤ k − 1
        thus ?thesis
          using  ‹Suc n ≤ k − 1› ‹k ≠ 0› ‹n ≤ k − 1› ordering-ord-ijk-loc f-def
g-def k-def(2)
          by (metis (no-types, lifting) add-diff-inverse-nat less-Suc-eq-le
       less-imp-le-nat less-le-trans less-one local-long-ch-by-ord-def plus-1-eq-Suc)
      next
        assume Suc (Suc n) = k
        thus ?thesis
          using b-right g-def by force
      qed
```

```
      next
        assume Suc n = k
        show ?thesis
          using b-middle ‹Suc n = k› ‹n ≤ k − 1› g-def
          by auto
      next show n ≤ k−1 using ‹n ≤ k − 1› by blast
      qed
    next assume n≥k+1 show ?thesis
      proof −
        have g n = f (n−1)
          using ‹k + 1 ≤ n› less-imp-diff-less g-def
          by auto
        moreover have g (Suc n) = f (n)
          using ‹k + 1 ≤ n› g-def by auto
        moreover have g (Suc (Suc n)) = f (Suc n)
          using ‹k + 1 ≤ n› g-def by auto
        moreover have n−1<n ∧ n<Suc n
          using ‹k + 1 ≤ n› by auto
        moreover have finite Y ⟶ Suc n < card Y
          using Y-def a by auto
        ultimately show ?thesis
          using f-def unfolding chain-defs local-ordering-def
          by (metis ‹k + 1 ≤ n› add-leD2 le-add-diff-inverse plus-1-eq-Suc)
      qed
    next assume n=k
      show ?thesis
        using ‹k ≠ 0› ‹n = k› b-left g-def Y-def(1) a assms(3) fin-Y
        by auto
    qed
  qed
  ultimately show local-ordering g betw ?X
    unfolding local-ordering-def
    by presburger
qed
hence local-long-ch-by-ord g ?X
  using Y-def f-def local-long-ch-by-ord-def local-long-ch-by-ord-def
  by auto
thus [g⇝?X|a1..b..an]
    using fin-X ‹a1 ≠ an ∧ a1 ≠ b ∧ b ≠ an› bound-indices k-def(2) Y-def g-def
chain-defs
    by simp
qed


lemma card4-eq:
  assumes card X = 4
  shows ∃ a b c d. a ≠ b ∧ a ≠ c ∧ a ≠ d ∧ b ≠ c ∧ b ≠ d ∧ c ≠ d ∧ X = {a,
b, c, d}
proof −
```

obtain $a\ X'$ **where** $X = \mathit{insert}\ a\ X'$ **and** $a \notin X'$
by (*metis Suc-eq-numeral assms card-Suc-eq*)
then have $\mathit{card}\ X' = 3$
by (*metis add-2-eq-Suc' assms card-eq-0-iff card-insert-if diff-Suc-1 finite-insert numeral-3-eq-3 numeral-Bit0 plus-nat.add-0 zero-neq-numeral*)
then obtain $b\ X''$ **where** $X' = \mathit{insert}\ b\ X''$ **and** $b \notin X''$
by (*metis card-Suc-eq numeral-3-eq-3*)
then have $\mathit{card}\ X'' = 2$
by (*metis Suc-eq-numeral ‹card X' = 3› card.infinite card-insert-if finite-insert pred-numeral-simps(3) zero-neq-numeral*)
then have $\exists\, c\ d.\ c \neq d \wedge X'' = \{c, d\}$
by (*meson card-2-iff*)
thus *?thesis*
using ‹$X = \mathit{insert}\ a\ X'$› ‹$X' = \mathit{insert}\ b\ X''$› ‹$a \notin X'$› ‹$b \notin X''$› **by** *blast*
qed

theorem *path-finsubset-chain*:
assumes $Q \in \mathcal{P}$
and $X \subseteq Q$
and $\mathit{card}\ X \geq 2$
shows $\mathit{ch}\ X$
proof −
have *finite X*
using *assms(3) not-numeral-le-zero* **by** *fastforce*
consider $\mathit{card}\ X = 2 \mid \mathit{card}\ X = 3 \mid \mathit{card}\ X \geq 4$
using ‹*card X ≥ 2*› **by** *linarith*
thus *?thesis*
proof (*cases*)
assume $\mathit{card}\ X = 2$
thus *?thesis*
using ‹*finite X*› *assms two-event-chain* **by** *blast*
next
assume $\mathit{card}\ X = 3$
thus *?thesis*
using ‹*finite X*› *assms three-event-chain* **by** *blast*
next
assume $\mathit{card}\ X \geq 4$
thus *?thesis*
using *assms(1,2)* ‹*finite X*›
proof (*induct card X − 4 arbitrary*: X)
case *0*
then have $\mathit{card}\ X = 4$
by *auto*
then have $\exists\, a\ b\ c\ d.\ a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge X = \{a, b, c, d\}$
using *card4-eq* **by** *fastforce*
thus *?case*
using *0.prems(3) assms(1) chain4* **by** *auto*

next
case *IH*: (*Suc n*)

then obtain *Y b* **where** *X-eq*: $X = insert\ b\ Y$ **and** $b \notin Y$
by (*metis Diff-iff card-eq-0-iff finite.cases insertI1 insert-Diff-single not-numeral-le-zero*)
have *card Y* $\geq$ *4 n* = *card Y* − *4*
using *IH.hyps*(*2*) *IH.prems*(*4*) *X-eq* ‹$b \notin Y$› **by** *auto*
then have *ch Y*
using *IH*(*1*) [*of Y*] *IH.prems*(*3*,*4*) *X-eq assms*(*1*) **by** *auto*

then obtain *f* **where** *f-ords*: *local-long-ch-by-ord f Y*
using ‹*4* $\leq$ *card Y*› *ch-alt short-ch-card*(*2*) **by** *auto*
then obtain a_1 *a* a_n **where** *long-ch-Y*: [$f \rightsquigarrow Y | a_1..a..a_n$]
using ‹*4* $\leq$ *card Y*› *get-fin-long-ch-bounds* **by** *fastforce*
hence *bound-indices*: *f 0* = $a_1 \wedge f\ (card\ Y - 1) = a_n$
by (*simp add*: *chain-defs*)
have $a_1 \neq a_n \wedge a_1 \neq b \wedge b \neq a_n$
using ‹$b \notin Y$› *abc-abc-neq fin-ch-betw long-ch-Y points-in-long-chain* **by** *metis*
moreover have $a_1 \in Q \wedge a_n \in Q \wedge b \in Q$
using *IH.prems*(*3*) *X-eq long-ch-Y points-in-long-chain* **by** *auto*
ultimately consider [*b*; a_1; a_n] | [a_1; a_n; *b*] | [a_n; *b*; a_1]
using *some-betw* [*of Q b* a_1 a_n] ‹$Q \in \mathcal{P}$› **by** *blast*
thus *ch X*
proof (*cases*)

assume [*b*; a_1; a_n]
have *X-eq'*: $X = Y \cup \{b\}$
using *X-eq* **by** *auto*
let *?g* = $\lambda j.$ *if j* $\geq$ *1 then f* (*j* − *1*) *else b*
have [*?g*$\rightsquigarrow X | b..a_1..a_n$]
using *chain-append-at-left-edge IH.prems*(*4*) *X-eq'* ‹[*b*; a_1; a_n]› ‹$b \notin Y$› *long-ch-Y X-eq*
by *presburger*
thus *ch X*
using *chain-defs* **by** *auto*
next

assume [a_1; a_n; *b*]
let *?g* = $\lambda j.$ *if j* $\leq$ (*card X* − *2*) *then f j else b*
have [*?g*$\rightsquigarrow X | a_1..a_n..b$]
using *chain-append-at-right-edge IH.prems*(*4*) *X-eq* ‹[a_1; a_n; *b*]› ‹$b \notin Y$› *long-ch-Y*
by *auto*
thus *ch X* **using** *chain-defs* **by** (*meson ch-def*)
next

assume [a_n; *b*; a_1]
then have [a_1; *b*; a_n]

```
      by (simp add: abc-sym)
    obtain k where
      k-def: [a₁; b; f k] k < card Y ¬ (∃k'. 0 < k' ∧ k' < k ∧ [a₁; b; f k'])
      using ‹[a₁; b; aₙ]› ‹b ∉ Y› long-ch-Y smallest-k-ex by blast
    obtain g where g = (λj::nat. if j ≤ k − 1
                                 then f j
                                 else if j = k
                                   then b else f (j − 1))
      by simp
    hence [g⇝X|a₁..b..aₙ]
      using chain-append-inside [of f Y a₁ a aₙ b k] IH.prems(4) X-eq
        ‹[a₁; b; aₙ]› ‹b ∉ Y› k-def long-ch-Y
      by auto
    thus ch X
      using chain-defs ch-def by auto
  qed
 qed
qed


lemma path-finsubset-chain2:
 assumes Q ∈ 𝒫 and X ⊆ Q and card X ≥ 2
 obtains f a b where [f⇝X|a..b]
proof −
 have finX: finite X
   by (metis assms(3) card.infinite rel-simps(28))
 have ch-X: ch X
   using path-finsubset-chain[OF assms] by blast
 obtain f a b where f-def: [f⇝X|a..b] a∈X ∧ b∈X
   using assms finX ch-X get-fin-long-ch-bounds chain-defs
   by (metis ch-def points-in-chain)
 thus ?thesis
   using that by auto
qed
```

30.2 Theorem 11

Notice this case is so simple, it doesn't even require the path density larger sets of segments rely on for fixing their cardinality.

```
lemma segmentation-ex-N2:
 assumes path-P: P∈𝒫
    and Q-def: finite (Q::'a set) card Q = N Q⊆P N=2
    and f-def: [f⇝Q|a..b]
    and S-def: S = {segment a b}
    and P1-def: P1 = prolongation b a
    and P2-def: P2 = prolongation a b
  shows P = ((⋃S) ∪ P1 ∪ P2 ∪ Q) ∧
        card S = (N−1) ∧ (∀x∈S. is-segment x) ∧
```

```
        P1∩P2={} ∧ (∀ x∈S. (x∩P1={} ∧ x∩P2={} ∧ (∀ y∈S. x≠y ⟶ x∩y={})))
proof −
  have a∈Q ∧ b∈Q ∧ a≠b
    using chain-defs f-def points-in-chain first-neq-last
    by (metis)
  hence Q={a,b}
    using assms(3,5)
   by (smt card-2-iff insert-absorb insert-commute insert-iff singleton-insert-inj-eq)
  have a∈P ∧ b∈P
    using ‹Q={a,b}› assms(4) by auto
  have a≠b using ‹Q={a,b}›
    using ‹N = 2› assms(3) by force
  obtain s where s-def: s = segment a b by simp
  let ?S = {s}
  have P = ((⋃{s}) ∪ P1 ∪ P2 ∪ Q) ∧
          card {s} = (N−1) ∧ (∀ x∈{s}. is-segment x) ∧
            P1∩P2={} ∧ (∀ x∈{s}. (x∩P1={} ∧ x∩P2={} ∧ (∀ y∈{s}. x≠y ⟶
x∩y={})))
  proof (rule conjI)
    { fix x assume x∈P
      have [a;x;b] ∨ [b;a;x] ∨ [a;b;x] ∨ x=a ∨ x=b
        using ‹a∈P ∧ b∈P› some-betw path-P ‹a≠b›
        by (meson ‹x ∈ P› abc-sym)
      then have x∈s ∨ x∈P1 ∨ x∈P2 ∨ x=a ∨ x=b
        using pro-betw seg-betw P1-def P2-def s-def ‹Q = {a, b}›
        by auto
      hence x ∈ (⋃{s}) ∪ P1 ∪ P2 ∪ Q
        using ‹Q = {a, b}› by auto
    } moreover {
      fix x assume x ∈ (⋃{s}) ∪ P1 ∪ P2 ∪ Q
      hence x∈s ∨ x∈P1 ∨ x∈P2 ∨ x=a ∨ x=b
        using ‹Q = {a, b}› by blast
      hence [a;x;b] ∨ [b;a;x] ∨ [a;b;x] ∨ x=a ∨ x=b
        using s-def P1-def P2-def
        unfolding segment-def prolongation-def
        by auto
      hence x∈P
        using ‹a ∈ P ∧ b ∈ P› ‹a ≠ b› betw-b-in-path betw-c-in-path path-P
        by blast
    }
    ultimately show union-P: P = ((⋃{s}) ∪ P1 ∪ P2 ∪ Q)
      by blast
    show card {s} = (N−1) ∧ (∀ x∈{s}. is-segment x) ∧ P1∩P2={} ∧
          (∀ x∈{s}. (x∩P1={} ∧ x∩P2={} ∧ (∀ y∈{s}. x≠y ⟶ x∩y={})))
    proof (safe)
      show card {s} = N − 1
        using ‹Q = {a, b}› ‹a ≠ b› assms(3) by auto
      show is-segment s
        using s-def by blast
```

```
    show ⋀x. x ∈ P1 ⟹ x ∈ P2 ⟹ x ∈ {}
    proof −
      fix x assume x∈P1 x∈P2
      show x∈{}
        using P1-def P2-def ‹x ∈ P1› ‹x ∈ P2› abc-only-cba pro-betw
        by metis
    qed
    show ⋀x xa. xa ∈ s ⟹ xa ∈ P1 ⟹ xa ∈ {}
    proof −
      fix x xa assume xa∈s xa∈P1
      show xa∈{}
        using abc-only-cba seg-betw pro-betw  P1-def ‹xa ∈ P1› ‹xa ∈ s› s-def
        by (metis)
    qed
    show ⋀x xa. xa ∈ s ⟹ xa ∈ P2 ⟹ xa ∈ {}
    proof −
      fix x xa assume xa∈s xa∈P2
      show xa∈{}
        using abc-only-cba seg-betw pro-betw
        by (metis P2-def ‹xa ∈ P2› ‹xa ∈ s› s-def)
    qed
  qed
 qed
 thus ?thesis
  by (simp add: S-def s-def)
qed



lemma int-split-to-segs:
 assumes f-def: [f⇝Q|a..b..c]
 fixes S defines S-def: S ≡ {segment (f i) (f(i+1)) | i. i<card Q−1}
 shows interval a c = (⋃S) ∪ Q
proof
 let ?N = card Q
 have f-def-2: a∈Q ∧ b∈Q ∧ c∈Q
  using f-def points-in-long-chain by blast
 hence ?N ≥ 3
  using f-def long-ch-card-ge3 chain-defs
  by (meson finite-long-chain-with-card)
 have bound-indices: f 0 = a ∧ f (card Q − 1) = c
  using f-def chain-defs by auto
 let ?i = ?u = interval a c = (⋃S) ∪ Q
 show ?i⊆?u
 proof
  fix p assume p ∈ ?i
  show p∈?u
  proof (cases)
   assume p∈Q thus ?thesis by blast
```

```
    next assume p∉Q
      hence p≠a ∧ p≠c
        using f-def f-def-2 by blast
      hence [a;p;c]
        using seg-betw ‹p ∈ interval a c› interval-def
        by auto
      then obtain n_y n_z y z
        where yz-def: y=f n_y z=f n_z [y;p;z] y∈Q z∈Q n_y=n_z−1 n_z<card Q
          ¬(∃ k < card Q. [f k; p; c] ∧ k>n_y) ¬(∃ k<n_z. [a; p; f k])
        using get-closest-chain-events [where f=f and x=p and Y=Q and a_n=c
and a_0=a and a=b]
          f-def ‹p∉Q›
        by metis
      have n_y<card Q−1
        using yz-def(6,7) f-def index-middle-element
        by fastforce
      let ?s = segment (f n_y) (f n_z)
      have p∈?s
        using ‹[y;p;z]› abc-abc-neq seg-betw yz-def(1,2)
        by blast
      have n_z = n_y + 1
        using yz-def(6)
      by (metis abc-abc-neq add.commute add-diff-inverse-nat less-one yz-def(1,2,3)
zero-diff)
      hence ?s∈S
        using S-def ‹n_y<card Q−1› assms(2)
        by blast
      hence p∈⋃ S
        using ‹p ∈ ?s› by blast
      thus ?thesis by blast
    qed
  qed
  show ?u⊆?i
  proof
    fix p assume p ∈ ?u
    hence p∈⋃ S ∨ p∈Q by blast
    thus p∈?i
    proof
      assume p∈Q
      then consider p=a|p=c|[a;p;c]
        using f-def by (meson fin-ch-betw2 finite-long-chain-with-alt)
      thus ?thesis
      proof (cases)
        assume p=a
        thus ?thesis by (simp add: interval-def)
      next assume p=c
        thus ?thesis by (simp add: interval-def)
      next assume [a;p;c]
        thus ?thesis using interval-def seg-betw by auto
```

qed
next assume $p \in \bigcup S$
then obtain s **where** $p \in s$ $s \in S$
by *blast*
then obtain y **where** $s = segment\ (f\ y)\ (f\ (y+1))$ $y < ?N-1$
using *S-def* **by** *blast*
hence $y+1 < ?N$ **by** (*simp add*: *assms*(2))
hence *fy-in-Q*: $(f\ y) \in Q \wedge f\ (y+1) \in Q$
using *f-def add-lessD1* **unfolding** *chain-defs local-ordering-def*
by (*metis One-nat-def Suc-eq-plus1 Zero-not-Suc* ‹3≤*card Q*› *card-1-singleton-iff card-gt-0-iff*
card-insert-if diff-add-inverse2 diff-is-0-eq' less-numeral-extra(1) *numeral-3-eq-3 plus-1-eq-Suc*)
have $[a;\ f\ y;\ c] \vee y = 0$
using ‹$y < ?N - 1$› *assms*(2) *f-def chain-defs order-finite-chain* **by** *auto*
moreover have $[a;\ f\ (y+1);\ c] \vee y = ?N-2$
using ‹$y+1 < card\ Q$› *assms*(2) *f-def chain-defs order-finite-chain i-le-j-events-neq*
using *indices-neq-imp-events-neq fin-ch-betw2 fy-in-Q*
by (*smt* (*z3*) *Nat.add-0-right Nat.add-diff-assoc add-gr-0 card-Diff1-less card-Diff-singleton-if*
diff-diff-left diff-is-0-eq' le-numeral-extra(4) *less-numeral-extra*(1) *nat-1-add-1*)
ultimately consider $y=0 | y=?N-2 | ([a;\ f\ y;\ c] \wedge [a;\ f\ (y+1);\ c])$
by *linarith*
hence $[a;p;c]$
proof (*cases*)
assume $y=0$
hence $f\ y = a$
by (*simp add*: *bound-indices*)
hence $[a;\ p;\ (f(y+1))]$
using ‹$p \in s$› ‹$s = segment\ (f\ y)\ (f\ (y+1))$› *seg-betw*
by *auto*
moreover have $[a;\ (f(y+1));\ c]$
using ‹$[a;\ (f(y+1));\ c] \vee y = ?N - 2$› ‹$y = 0$› ‹$?N \geq 3$›
by *linarith*
ultimately show $[a;p;c]$
using *abc-acd-abd* **by** *blast*
next
assume $y=?N-2$
hence $f\ (y+1) = c$
using *bound-indices* ‹$?N \geq 3$› *numeral-2-eq-2 numeral-3-eq-3*
by (*metis One-nat-def Suc-diff-le add.commute add-leD2 diff-Suc-Suc plus-1-eq-Suc*)
hence $[f\ y;\ p;\ c]$
using ‹$p \in s$› ‹$s = segment\ (f\ y)\ (f\ (y+1))$› *seg-betw*
by *auto*
moreover have $[a;\ f\ y;\ c]$
using ‹$[a;\ f\ y;\ c] \vee y = 0$› ‹$y = ?N - 2$› ‹$?N \geq 3$›
by *linarith*
ultimately show $[a;p;c]$

```
        by (meson abc-acd-abd abc-sym)
      next
       assume [a; f y; c] ∧ [a; (f(y+1)); c]
       thus [a;p;c]
        using abe-ade-bcd-ace [where a=a and b=f y and d=f (y+1) and e=c
and c=p]
        using ‹p ∈ s› ‹s = segment (f y) (f(y+1))› seg-betw
        by auto
      qed
      thus ?thesis
       using interval-def seg-betw by auto
    qed
  qed
qed


lemma path-is-union:
  assumes path-P: P∈𝒫
      and Q-def: finite (Q::'a set) card Q = N Q⊆P N≥3
      and f-def: a∈Q ∧ b∈Q ∧ c∈Q [f⇝Q|a..b..c]
      and S-def: S = {s. ∃ i<(N−1). s = segment (f i) (f (i+1))}
      and P1-def: P1 = prolongation b a
      and P2-def: P2 = prolongation b c
    shows P = ((⋃ S) ∪ P1 ∪ P2 ∪ Q)
proof −

  have in-P: a∈P ∧ b∈P ∧ c∈P
    using assms(4) f-def by blast
  have bound-indices: f 0 = a ∧ f (card Q − 1) = c
    using f-def chain-defs by auto
  have points-neq: a≠b ∧ b≠c ∧ a≠c
    using f-def chain-defs by (metis first-neq-last)
```

The proof in two parts: subset inclusion one way, then the other.

```
  { fix x assume x∈P
    have [a;x;c] ∨ [b;a;x] ∨ [b;c;x] ∨ x=a ∨ x=c
      using in-P some-betw path-P points-neq ‹x ∈ P› abc-sym
      by (metis (full-types) abc-acd-bcd fin-ch-betw f-def(2))
    then have (∃ s∈S. x∈s) ∨ x∈P1 ∨ x∈P2 ∨ x∈Q
    proof (cases)
      assume [a;x;c]
      hence only-axc: ¬([b;a;x] ∨ [b;c;x] ∨ x=a ∨ x=c)
        using abc-only-cba
        by (meson abc-bcd-abd abc-sym f-def fin-ch-betw)
      have x ∈ interval a c
        using ‹[a;x;c]› interval-def seg-betw by auto
      hence x∈Q ∨ x∈⋃ S
        using int-split-to-segs S-def assms(2,3,5) f-def
        by blast
```

```
    thus ?thesis by blast
   next assume ¬[a;x;c]
    hence [b;a;x] ∨ [b;c;x] ∨ x=a ∨ x=c
     using ‹[a;x;c] ∨ [b;a;x] ∨ [b;c;x] ∨ x = a ∨ x = c› by blast
    hence  x∈P1 ∨ x∈P2 ∨ x∈Q
     using P1-def P2-def f-def pro-betw by auto
    thus ?thesis by blast
   qed
   hence x ∈ (⋃S) ∪ P1 ∪ P2 ∪ Q by blast
  } moreover {
   fix x assume x ∈ (⋃S) ∪ P1 ∪ P2 ∪ Q
   hence (∃ s∈S. x∈s) ∨ x∈P1 ∨ x∈P2 ∨ x∈Q
    by blast
   hence x∈⋃S ∨ [b;a;x] ∨ [b;c;x] ∨ x∈Q
    using S-def P1-def P2-def
    unfolding segment-def prolongation-def
    by auto
   hence x∈P
   proof (cases)
    assume x∈⋃S
    have S = {segment (f i) (f(i+1)) | i. i<N−1}
     using S-def by blast
    hence x∈interval a c
     using int-split-to-segs [OF f-def(2)] assms ‹x∈⋃S›
     by (simp add: UnCI)
    hence [a;x;c] ∨ x=a ∨ x=c
     using interval-def seg-betw by auto
    thus ?thesis
    proof (rule disjE)
     assume x=a ∨ x=c
     thus ?thesis
      using in-P by blast
    next
     assume [a;x;c]
     thus ?thesis
      using betw-b-in-path in-P path-P points-neq by blast
    qed
   next assume x∉⋃S
    hence [b;a;x] ∨ [b;c;x] ∨ x∈Q
     using ‹x ∈ ⋃ S ∨ [b;a;x] ∨ [b;c;x] ∨ x ∈ Q›
     by blast
    thus ?thesis
     using assms(4) betw-c-in-path in-P path-P points-neq
     by blast
   qed
  }
  ultimately show P = ((⋃S) ∪ P1 ∪ P2 ∪ Q)
   by blast
qed
```

lemma *inseg-axc*:
assumes *path-P*: $P\in\mathcal{P}$
and *Q-def*: *finite* $(Q::'a\ set)$ *card* $Q = N$ $Q\subseteq P$ $N\geq 3$
and *f-def*: $a\in Q \wedge b\in Q \wedge c\in Q$ $[f\rightsquigarrow Q|a..b..c]$
and *S-def*: $S = \{s.\ \exists i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
and *x-def*: $x\in s$ $s\in S$
shows $[a;x;c]$
proof −
have *fQ*: *local-long-ch-by-ord f Q*
using *f-def Q-def chain-defs* **by** (*metis ch-long-if-card-geq3 path-P short-ch-card*(*1*) *short-xor-long*(*2*))
have *inseg-neq-ac*: $x\neq a \wedge x\neq c$ **if** $x\in s$ $s\in S$ **for** x s
proof
show $x\neq a$
proof (*rule notI*)
assume $x=a$
obtain n **where** *s-def*: $s = segment\ (f\ n)\ (f\ (n+1))$ $n<N-1$
using *S-def* ‹$s \in S$› **by** *blast*
hence $n<card\ Q$ **using** *assms*(*3*) **by** *linarith*
hence $f\ n \in Q$
using *fQ* **unfolding** *chain-defs local-ordering-def* **by** *blast*
hence $[a;\ f\ n;\ c]$
using *f-def finite-long-chain-with-def assms*(*3*) *order-finite-chain seg-betw that*(*1*)
using ‹$n < N - 1$› ‹$s = segment\ (f\ n)\ (f\ (n + 1))$› ‹$x = a$›
by (*metis abc-abc-neq add-lessD1 fin-ch-betw inside-not-bound*(*2*) *less-diff-conv*)
moreover have $[(f(n));\ x;\ (f(n+1))]$
using ‹$x\in s$› *seg-betw s-def*(*1*) **by** *simp*
ultimately show *False*
using ‹$x=a$› *abc-only-cba*(*1*) *assms*(*3*) *fQ chain-defs s-def*(*2*)
by (*smt* (*z3*) ‹$n < card\ Q$› *f-def*(*2*) *order-finite-chain-indices2 thm2-ind1*)
qed

show $x\neq c$
proof (*rule notI*)
assume $x=c$
obtain n **where** *s-def*: $s = segment\ (f\ n)\ (f\ (n+1))$ $n<N-1$
using *S-def* ‹$s \in S$› **by** *blast*
hence $n+1<N$ **by** *simp*
have $[(f(n));\ x;\ (f(n+1))]$
using ‹$x\in s$› *seg-betw s-def*(*1*) **by** *simp*
have $f\ (n) \in Q$
using *fQ* ‹$n+1 < N$› *chain-defs local-ordering-def*
by (*metis add-lessD1 assms*(*3*))
have $f\ (n+1) \in Q$
using ‹$n+1 < N$› *fQ chain-defs local-ordering-def*
by (*metis assms*(*3*))

```
    have f(n+1) ≠ c
      using ‹x=c› ‹[(f(n)); x; (f(n+1))]› abc-abc-neq
      by blast
    hence [a; (f(n+1)); c]
      using f-def finite-long-chain-with-def assms(3) order-finite-chain seg-betw
that(1)
       abc-abc-neq abc-only-cba fin-ch-betw
      by (metis ‹[f n; x; f (n + 1)]› ‹f (n + 1) ∈ Q› ‹f n ∈ Q› ‹x = c›)
    thus False
      using ‹x=c› ‹[(f(n)); x; (f(n+1))]› assms(3) f-def s-def(2)
        abc-only-cba(1) finite-long-chain-with-def order-finite-chain
      by (metis ‹f n ∈ Q› abc-bcd-acd abc-only-cba(1,2) fin-ch-betw)
  qed
 qed

 show [a;x;c]
 proof −
  have x∈interval a c
    using int-split-to-segs [OF f-def(2)] S-def assms(2,3,5) x-def
    by blast
  have x≠a ∧ x≠c using inseg-neq-ac
    using x-def by auto
  thus ?thesis
    using seg-betw ‹x ∈ interval a c› interval-def
    by auto
 qed
qed


lemma disjoint-segmentation:
 assumes path-P: P∈𝒫
    and Q-def: finite (Q::'a set) card Q = N Q⊆P N≥3
    and f-def: a∈Q ∧ b∈Q ∧ c∈Q [f⇝Q|a..b..c]
    and S-def: S = {s. ∃ i<(N−1). s = segment (f i) (f (i+1))}
    and P1-def: P1 = prolongation b a
    and P2-def: P2 = prolongation b c
    shows P1∩P2={} ∧ (∀ x∈S. (x∩P1={} ∧ x∩P2={} ∧ (∀ y∈S. x≠y ⟶
x∩y={})))
proof (rule conjI)
 have fQ: local-long-ch-by-ord f Q
  using f-def Q-def chain-defs by (metis ch-long-if-card-geq3 path-P short-ch-card(1)
short-xor-long(2))
 show P1 ∩ P2 = {}
 proof (safe)
  fix x assume x∈P1 x∈P2
  show x∈{}
    using abc-only-cba pro-betw P1-def P2-def
    by (metis ‹x ∈ P1› ‹x ∈ P2› abc-bcd-abd f-def(2) fin-ch-betw)
 qed
```

show $\forall x{\in}S.\ (x{\cap}P1{=}\{\} \wedge x{\cap}P2{=}\{\} \wedge (\forall y{\in}S.\ x{\neq}y \longrightarrow x{\cap}y{=}\{\}))$
proof (*rule ballI*)
fix s **assume** $s{\in}S$
show $s \cap P1 = \{\} \wedge s \cap P2 = \{\} \wedge (\forall y{\in}S.\ s \neq y \longrightarrow s \cap y = \{\})$
proof (*intro conjI ballI impI*)
show $s{\cap}P1{=}\{\}$
proof (*safe*)
fix x **assume** $x{\in}s\ x{\in}P1$
hence $[a;x;c]$
using *inseg-axc* ‹$s \in S$› *assms* **by** *blast*
thus $x{\in}\{\}$
by (*metis P1-def* ‹$x \in P1$› *abc-bcd-abd abc-only-cba*(*1*) *f-def*(*2*) *fin-ch-betw pro-betw*)
qed
show $s{\cap}P2{=}\{\}$
proof (*safe*)
fix x **assume** $x{\in}s\ x{\in}P2$
hence $[a;x;c]$
using *inseg-axc* ‹$s \in S$› *assms* **by** *blast*
thus $x{\in}\{\}$
by (*metis P2-def* ‹$x \in P2$› *abc-bcd-acd abc-only-cba*(*2*) *f-def*(*2*) *fin-ch-betw pro-betw*)
qed
fix r **assume** $r{\in}S\ s{\neq}r$
show $s{\cap}r{=}\{\}$
proof (*safe*)
fix y **assume** $y \in r\ y \in s$
obtain $n\ m$ **where** *rs-def*: $r = segment\ (f\ n)\ (f(n{+}1))\ s = segment\ (f\ m)\ (f(m{+}1))$
$n{\neq}m\ n{<}N{-}1\ m{<}N{-}1$
using *S-def* ‹$r \in S$› ‹$s \neq r$› ‹$s \in S$› **by** *blast*
have *y-betw*: $[f\ n;\ y;\ (f(n{+}1))] \wedge [f\ m;\ y;\ (f(m{+}1))]$
using *seg-betw* ‹$y{\in}r$› ‹$y{\in}s$› *rs-def*(*1,2*) **by** *simp*
have *False*
proof (*cases*)
assume $n{<}m$
have $[f\ n;\ f\ m;\ (f(m{+}1))]$
using ‹$n < m$› *assms*(*3*) *fQ chain-defs order-finite-chain rs-def*(*5*) **by** (*metis assms*(*2*) *thm2-ind1*)
have $n{+}1{<}m$
using ‹$[f\ n;\ f\ m;\ f(m\ +\ 1)]$› ‹$n < m$› *abc-only-cba*(*2*) *abd-bcd-abc y-betw*
by (*metis Suc-eq-plus1 Suc-leI le-eq-less-or-eq*)
hence $[f\ n;\ (f(n{+}1));\ f\ m]$
using *fQ assms*(*3*) *rs-def*(*5*) **unfolding** *chain-defs local-ordering-def*
by (*metis* (*full-types*) ‹$[f\ n;f\ m;f\ (m\ +\ 1)]$› *abc-only-cba*(*1*) *abc-sym abd-bcd-abc assms*(*2*) *fQ thm2-ind1 y-betw*)
hence $[f\ n;\ (f(n{+}1));\ y]$
using ‹$[f\ n;\ f\ m;\ f(m\ +\ 1)]$› *abc-acd-abd abd-bcd-abc y-betw*

by *blast*
thus *?thesis*
using *abc-only-cba y-betw* **by** *blast*
next
assume $\neg n<m$
hence $n>m$ **using** *nat-neq-iff rs-def(3)* **by** *blast*
have [*f m*; *f n*; (*f*(*n*+*1*))]
using ‹$n > m$› *assms(3) fQ chain-defs rs-def(4)* **by** (*metis assms(2) thm2-ind1*)
hence $m+1<n$
using ‹$n > m$› *abc-only-cba(2) abd-bcd-abc y-betw*
by (*metis Suc-eq-plus1 Suc-leI le-eq-less-or-eq*)
hence [*f m*; (*f*(*m*+*1*)); *f n*]
using *fQ assms(2,3) rs-def(4)* **unfolding** *chain-defs local-ordering-def*
by (*metis* (*no-types*, *lifting*) ‹[*f m*;*f n*;*f* (*n* + *1*)]› *abc-only-cba(1) abc-sym abd-bcd-abc fQ thm2-ind1 y-betw*)
hence [*f m*; (*f*(*m*+*1*)); *y*]
using ‹[*f m*; *f n*; *f*(*n* + *1*)]› *abc-acd-abd abd-bcd-abc y-betw*
by *blast*
thus *?thesis*
using *abc-only-cba y-betw* **by** *blast*
qed
thus $y\in\{\}$ **by** *blast*
qed
qed
qed
qed

lemma *segmentation-ex-Nge3*:
assumes *path-P*: $P\in\mathcal{P}$
and *Q-def*: *finite* (*Q*::*'a set*) *card Q* = *N* $Q\subseteq P$ $N\geq 3$
and *f-def*: $a\in Q \wedge b\in Q \wedge c\in Q$ [$f\rightsquigarrow Q|a..b..c$]
and *S-def*: $S = \{s.\ \exists\, i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
and *P1-def*: *P1* = *prolongation b a*
and *P2-def*: *P2* = *prolongation b c*
shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q) \wedge$
$(\forall\, x\in S.\ is\text{-}segment\ x) \wedge$
$P1\cap P2=\{\} \wedge (\forall\, x\in S.\ (x\cap P1=\{\} \wedge x\cap P2=\{\} \wedge (\forall\, y\in S.\ x\neq y \longrightarrow x\cap y=\{\})))$
proof (*intro disjoint-segmentation conjI*)
show $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$
using *path-is-union assms*
by *blast*
show $\forall\, x\in S.\ is\text{-}segment\ x$
proof
fix *s* **assume** $s\in S$
thus *is-segment s* **using** *S-def* **by** *auto*
qed
qed (*use assms disjoint-segmentation* **in** *auto*)

Some unfolding of the definition for a finite chain that happens to be short.

lemma *finite-chain-with-card-2*:
assumes *f-def*: $[f \rightsquigarrow Q|a..b]$
and *card-Q*: *card Q = 2*
shows *finite Q f 0 = a f (card Q − 1) = b Q = {f 0, f 1} ∃ Q. path Q (f 0) (f 1)*
using *assms* **unfolding** *chain-defs* **by** *auto*

Schutz says "As in the proof of the previous theorem [...]" - does he mean to imply that this should really be proved as induction? I can see that quite easily, induct on N, and add a segment by either splitting up a segment or taking a piece out of a prolongation. But I think that might be too much trouble.

theorem *show-segmentation*:
assumes *path-P*: $P \in \mathcal{P}$
and *Q-def*: $Q \subseteq P$
and *f-def*: $[f \rightsquigarrow Q|a..b]$
fixes *P1* **defines** *P1-def*: *P1 ≡ prolongation b a*
fixes *P2* **defines** *P2-def*: *P2 ≡ prolongation a b*
fixes *S* **defines** *S-def*: $S \equiv \{segment\ (f\ i)\ (f\ (i+1)) \mid i.\ i<card\ Q-1\}$
shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$ $(\forall x \in S.\ is\text{-}segment\ x)$
disjoint $(S \cup \{P1,P2\})$ $P1 \neq P2$ $P1 \notin S$ $P2 \notin S$
proof −
have *card-Q*: *card Q ≥ 2*
using *fin-chain-card-geq-2 f-def* **by** *blast*
have *finite Q*
by (*metis card.infinite card-Q rel-simps*(*28*))
have *f-def-2*: $a \in Q \wedge b \in Q$
using *f-def points-in-chain finite-chain-with-def* **by** *auto*
have $a \neq b$
using *f-def chain-defs* **by** (*metis first-neq-last*)
{
assume *card Q = 2*
hence *card Q − 1 = Suc 0* **by** *simp*
have *Q = {f 0, f 1} ∃ Q. path Q (f 0) (f 1) f 0 = a f (card Q − 1) = b*
using ‹*card Q = 2*› *finite-chain-with-card-2 f-def* **by** *auto*
hence *S={segment a b}*
unfolding *S-def* **using** ‹*card Q − 1 = Suc 0*› **by** (*simp add*: *eval-nat-numeral*)
hence $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$ $(\forall x \in S.\ is\text{-}segment\ x)$ $P1 \cap P2 = \{\}$
$(\forall x \in S.\ (x \cap P1 = \{\} \wedge x \cap P2 = \{\} \wedge (\forall y \in S.\ x \neq y \longrightarrow x \cap y = \{\})))$
using *assms f-def* ‹*finite Q*› *segmentation-ex-N2*
[**where** *P=P* **and** *Q=Q* **and** *N=card Q*]
by (*metis* (*no-types*, *lifting*) ‹*card Q = 2*›)+
} moreover {
assume *card Q ≠ 2*
hence *card Q ≥ 3*
using *card-Q* **by** *auto*
then obtain *c* **where** *c-def*: $[f \rightsquigarrow Q|a..c..b]$
using *assms*(*3*,*5*) ‹$a \neq b$› *chain-defs*

by (*metis f-def three-in-set3*)
have *pro-equiv*: *P1 = prolongation c a* ∧ *P2 = prolongation c b*
using *pro-basis-change*
using *P1-def P2-def abc-sym c-def fin-ch-betw* **by** *auto*
have *S-def2*: *S* = {*s*. ∃ *i*<(*card Q*−*1*). *s* = *segment* (*f i*) (*f* (*i*+*1*))}
using *S-def* ‹*card Q* ≥ *3*› **by** *auto*
have *P* = ((⋃ *S*) ∪ *P1* ∪ *P2* ∪ *Q*) (∀ *x*∈*S*. *is-segment x*) *P1*∩*P2*={}
(∀ *x*∈*S*. (*x*∩*P1*={} ∧ *x*∩*P2*={} ∧ (∀ *y*∈*S*. *x*≠*y* ⟶ *x*∩*y*={})))
using *f-def-2 assms f-def* ‹*card Q* ≥ *3*› *c-def pro-equiv*
segmentation-ex-Nge3 [**where** *P*=*P* **and** *Q*=*Q* **and** *N*=*card Q* **and** *S*=*S* **and** *a*=*a* **and** *b*=*c* **and** *c*=*b* **and** *f*=*f*]
using *points-in-long-chain* ‹*finite Q*› *S-def2* **by** *metis+*
}
ultimately have *old-thesis*: *P* = ((⋃ *S*) ∪ *P1* ∪ *P2* ∪ *Q*) (∀ *x*∈*S*. *is-segment x*) *P1*∩*P2*={}
(∀ *x*∈*S*. (*x*∩*P1*={} ∧ *x*∩*P2*={} ∧ (∀ *y*∈*S*. *x*≠*y* ⟶ *x*∩*y*={}))) **by** *meson+*
thus *disjoint* (*S*∪{*P1*,*P2*}) *P1*≠*P2* *P1*∉*S* *P2*∉*S*
P = ((⋃ *S*) ∪ *P1* ∪ *P2* ∪ *Q*) (∀ *x*∈*S*. *is-segment x*)
unfolding *disjoint-def* **apply** (*simp add*: *Int-commute*)
apply (*metis P2-def Un-iff old-thesis*(*1*,*3*) ‹*a* ≠ *b*› *disjoint-iff f-def-2 path-P pro-betw prolong-betw2*)
apply (*metis P1-def Un-iff old-thesis*(*1*,*4*) ‹*a* ≠ *b*› *disjoint-iff f-def-2 path-P pro-betw prolong-betw3*)
apply (*metis P2-def Un-iff old-thesis*(*1*,*4*) ‹*a* ≠ *b*› *disjoint-iff f-def-2 path-P pro-betw prolong-betw*)
using *old-thesis*(*1*,*2*) **by** *linarith+*
qed

theorem *segmentation*:
assumes *path-P*: *P*∈*𝒫*
and *Q-def*: *card Q*≥*2* *Q*⊆*P*
shows ∃ *S P1 P2*. *P* = ((⋃ *S*) ∪ *P1* ∪ *P2* ∪ *Q*) ∧
disjoint (*S*∪{*P1*,*P2*}) ∧ *P1*≠*P2* ∧ *P1*∉*S* ∧ *P2*∉*S* ∧
(∀ *x*∈*S*. *is-segment x*) ∧ *is-prolongation P1* ∧ *is-prolongation P2*
proof −
let *?N* = *card Q*

obtain *f a b* **where** *f-def*: [*f*⇝*Q*|*a*..*b*]
using *path-finsubset-chain2*[*OF path-P Q-def*(*2*,*1*)]
by *metis*
let *?S* = {*segment* (*f i*) (*f* (*i*+*1*)) | *i*. *i*<*card Q*−*1*}
let *?P1* = *prolongation b a*
let *?P2* = *prolongation a b*
have *from-seg*: *P* = ((⋃ *?S*) ∪ *?P1* ∪ *?P2* ∪ *Q*) (∀ *x*∈*?S*. *is-segment x*)
disjoint (*?S*∪{*?P1*,*?P2*}) *?P1*≠*?P2* *?P1*∉*?S* *?P2*∉*?S*
using *show-segmentation*[*OF path-P Q-def*(*2*) ‹[*f*⇝*Q*|*a*..*b*]›]
by *force+*

thus *?thesis*
by *blast*
qed

end

31 Chains are unique up to reversal

context *MinkowskiSpacetime* **begin**

lemma *chain-remove-at-right-edge*:
assumes $[f{\rightsquigarrow}X|a..c]$ $f\ (card\ X - 2) = p$ $3 \leq card\ X$ $X = insert\ c\ Y$ $c \notin Y$
shows $[f{\rightsquigarrow}Y|a..p]$
proof −
have *lch-X*: *local-long-ch-by-ord f X*
using *assms*(*1*,*3*) *chain-defs short-ch-card-2*
by *fastforce*
have $p{\in}X$
by (*metis local-ordering-def assms*(*2*) *card.empty card-gt-0-iff diff-less lch-X*
local-long-ch-by-ord-def not-numeral-le-zero zero-less-numeral)
have *bound-ind*: $f\ 0 = a \wedge f\ (card\ X - 1) = c$
using *lch-X assms*(*1*,*3*) **unfolding** *finite-chain-with-def finite-long-chain-with-def*
by *metis*

have $[a;p;c]$
proof −
have $card\ X - 2 < card\ X - 1$
using ‹$3 \leq card\ X$› **by** *auto*
moreover have $card\ X - 2 > 0$
using ‹$3 \leq card\ X$› **by** *linarith*
ultimately show *?thesis*
using *order-finite-chain*[*OF lch-X*] ‹$3 \leq card\ X$› *assms*(*2*) *bound-ind*
by (*metis card.infinite diff-less le-numeral-extra*(*3*) *less-numeral-extra*(*1*)
not-gr-zero not-numeral-le-zero)
qed

have $[f{\rightsquigarrow}X|a..p..c]$
unfolding *finite-long-chain-with-alt* **by** (*simp add*: *assms*(*1*) ‹$[a;p;c]$› ‹$p{\in}X$›)

have *1*: $x = a$ **if** $x \in Y$ $\neg\ [a;x;p]$ $x \neq p$ **for** x
proof −
have $x{\in}X$
using *that*(*1*) *assms*(*4*) **by** *simp*
hence *01*: $x{=}a \vee [a;p;x]$
by (*metis that*(*2*,*3*) ‹$[a;p;c]$› *abd-acd-abcacb assms*(*1*) *fin-ch-betw2*)
have *02*: $x{=}c$ **if** $[a;p;x]$
proof −

```
    obtain i where i-def: f i = x i<card X
      using ‹x∈X› chain-defs by (meson assms(1) obtain-index-fin-chain)
    have f 0 = a
      by (simp add: bound-ind)
    have card X − 2 < i
     using order-finite-chain-indices[OF lch-X - that ‹f 0 = a› assms(2) i-def(1)
- - i-def(2)]
         by (metis card-eq-0-iff card-gt-0-iff diff-less i-def(2) less-nat-zero-code
zero-less-numeral)
    hence i = card X − 1 using i-def(2) by linarith
    thus ?thesis using bound-ind i-def(1) by blast
  qed
  show ?thesis using 01 02 assms(5) that(1) by auto
 qed

 have Y = {e ∈ X. [a;e;p] ∨ e = a ∨ e = p}
  apply (safe, simp-all add: assms(4) 1)
  using ‹[a;p;c]› abc-only-cba(2) abc-abc-neq assms(4) by blast+

 thus ?thesis using chain-shortening[OF ‹[f⇝X|a..p..c]›] by simp
qed


lemma (in MinkowskiChain) fin-long-ch-imp-fin-ch:
 assumes [f⇝X|a..b..c]
 shows [f⇝X|a..c]
 using assms by (simp add: finite-long-chain-with-alt)
```

If we ever want to have chains less strongly identified by endpoints, this result should generalise - a, c, x, z are only used to identify reversal/no-reversal cases.

```
lemma chain-unique-induction-ax:
 assumes card X ≥ 3
    and i < card X
    and [f⇝X|a..c]
    and [g⇝X|x..z]
    and a = x ∨ c = z
  shows f i = g i
using assms
proof (induct card X − 3 arbitrary: X a c x z)
 case Nil: 0
 have card X = 3
  using Nil.hyps Nil.prems(1) by auto

 obtain b where f-ch: [f⇝X|a..b..c]
  using chain-defs by (metis Nil.prems(1,3) three-in-set3)
 obtain y where g-ch: [g⇝X|x..y..z]
  using Nil.prems chain-defs by (metis three-in-set3)
```

```
  have i=1 ∨ i=0 ∨ i=2
    using ‹card X = 3› Nil.prems(2) by linarith
  thus ?case
  proof (rule disjE)
    assume i=1
    hence f i = b ∧ g i = y
      using index-middle-element f-ch g-ch ‹card X = 3› numeral-3-eq-3
     by (metis One-nat-def add-diff-cancel-left' less-SucE not-less-eq plus-1-eq-Suc)
    have f i = g i
    proof (rule ccontr)
      assume f i ≠ g i
      hence g i ≠ b
        by (simp add: ‹f i = b ∧ g i = y›)
      have g i ∈ X
        using ‹f i = b ∧ g i = y›  g-ch points-in-long-chain by blast
      have X = {a,b,c}
        using f-ch unfolding finite-long-chain-with-alt
        using ‹card X = 3› points-in-long-chain[OF f-ch] abc-abc-neq[of a b c]
        by (simp add: card-3-eq'(2))
      hence (g i = a ∨ g i = c)
        using ‹g i ≠ b› ‹g i ∈ X› by blast
      hence ¬ [a; g i; c]
        using abc-abc-neq by blast
      hence g i ∉ X
         using ‹f i=b ∧ g i=y› ‹g i=a ∨ g i=c›  f-ch g-ch chain-bounds-unique
finite-long-chain-with-def
        by blast
      thus False
        by (simp add: ‹g i ∈ X›)
    qed
    thus ?thesis
      by (simp add: ‹card X = 3› ‹i = 1›)
  next
    assume i = 0 ∨ i = 2
    show ?thesis
      using Nil.prems(5) ‹card X = 3› ‹i = 0 ∨ i = 2› chain-bounds-unique f-ch
g-ch chain-defs
      by (metis diff-Suc-1 numeral-2-eq-2 numeral-3-eq-3)
  qed
next
  case IH: (Suc n)
  have lch-fX: local-long-ch-by-ord f X
    using chain-defs long-ch-card-ge3 IH(3,5)
    by fastforce
  have lch-gX: local-long-ch-by-ord g X
    using IH(3,6) chain-defs long-ch-card-ge3
    by fastforce
  have fin-X: finite X
    using IH(4) le-0-eq by fastforce
```

have *ch-by-ord f X*
using *lch-fX* **unfolding** *ch-by-ord-def* **by** *blast*
have *card X ≥ 4*
using *IH.hyps(2)* **by** *linarith*

obtain *b* **where** *f-ch*: *[f⇝X|a..b..c]*
using *IH(3,5) chain-defs* **by** (*metis three-in-set3*)
obtain *y* **where** *g-ch*: *[g⇝X|x..y..z]*
using *IH.prems(1,4) chain-defs* **by** (*metis three-in-set3*)

obtain *p* **where** *p-def*: *p = f (card X − 2)* **by** *simp*
have *[a;p;c]*
proof −
have *card X − 2 < card X − 1*
using ‹*4 ≤ card X*› **by** *auto*
moreover have *card X − 2 > 0*
using ‹*3 ≤ card X*› **by** *linarith*
ultimately show *?thesis*
using *f-ch p-def chain-defs* ‹*[f⇝X]*› *order-finite-chain2* **by** *force*
qed
hence *p≠c ∧ p≠a*
using *abc-abc-neq* **by** *blast*

obtain *Y* **where** *Y-def*: *X = insert c Y c∉Y*
using *f-ch points-in-long-chain*
by (*meson mk-disjoint-insert*)
hence *fin-Y*: *finite Y*
using *f-ch chain-defs* **by** *auto*
hence *n = card Y − 3*
using ‹*Suc n = card X − 3*› ‹*X = insert c Y*› ‹*c∉Y*› *card-insert-if*
by *auto*
hence *card-Y*: *card Y = n + 3*
using *Y-def(1) Y-def(2) fin-Y IH.hyps(2)* **by** *fastforce*
have *card Y = card X − 1*
using *Y-def(1,2) fin-X* **by** *auto*
have *p∈Y*
using ‹*X = insert c Y*› ‹*[a;p;c]*› *abc-abc-neq lch-fX p-def IH.prems(1,3) Y-def(2)*
by (*metis chain-remove-at-right-edge points-in-chain*)
have *[f⇝Y|a..p]*
using *chain-remove-at-right-edge* [**where** *f=f* **and** *a=a* **and** *c=c* **and** *X=X* **and** *p=p* **and** *Y=Y*]
using *fin-long-ch-imp-fin-ch* [**where** *f=f* **and** *a=a* **and** *c=c* **and** *b=b* **and** *X=X*]
using *f-ch p-def* ‹*card X ≥ 3*› *Y-def*
by *blast*
hence *ch-fY*: *local-long-ch-by-ord f Y*
using *card-Y fin-Y chain-defs long-ch-card-ge3*
by *force*

```
  have p-closest: ¬ (∃ q∈X. [p;q;c])
  proof
    assume (∃ q∈X. [p;q;c])
    then obtain q where q∈X [p;q;c] by blast
    then obtain j where j < card X f j = q
      using lch-fX lch-gX fin-X points-in-chain ‹p≠c ∧ p≠a› chain-defs
      by (metis local-ordering-def)
    have j > card X − 2 ∧ j < card X − 1
    proof −
      have j > card X − 2 ∧ j < card X − 1 ∨ j > card X − 1 ∧ j < card X − 2
        apply (intro order-finite-chain-indices[OF lch-fX ‹finite X› ‹[p;q;c]›])
         using p-def ‹f j = q› IH.prems(3) finite-chain-with-def ‹j < card X› by
auto
      thus ?thesis by linarith
    qed
    thus False by linarith
  qed

  have g (card X − 2) = p
  proof (rule ccontr)
    assume asm-false: g (card X − 2) ≠ p
    obtain j where g j = p j < card X − 1 j>0
      using ‹X = insert c Y› ‹p∈Y›  points-in-chain ‹p≠c ∧ p≠a›
      by (metis (no-types) chain-bounds-unique f-ch
          finite-long-chain-with-def g-ch index-middle-element insert-iff)
    hence j < card X − 2
      using asm-false le-eq-less-or-eq by fastforce
    hence j < card Y − 1
      by (simp add: Y-def(1,2) fin-Y)
    obtain d where d = g (card X − 2) by simp
    have [p;d;z]
    proof −
      have card X − 1 > card X − 2
        using ‹j < card X − 1› by linarith
      thus ?thesis
        using lch-gX ‹j < card Y − 1› ‹card Y = card X − 1› ‹d = g (card X −
2)› ‹g j = p›
          order-finite-chain[OF lch-gX] chain-defs local-ordering-def
       by (smt (z3) IH.prems(3−5) asm-false chain-bounds-unique chain-remove-at-right-edge
p-def
            ‹⋀thesis. (⋀Y. ⟦X = insert c Y; c ∉ Y⟧ ⟹ thesis) ⟹ thesis›)
    qed
    moreover have d∈X
        using lch-gX ‹d = g (card X − 2)› unfolding local-long-ch-by-ord-def
local-ordering-def
      by auto
    ultimately show False
      using p-closest abc-sym IH.prems(3−5) chain-bounds-unique f-ch g-ch
```

```
    by blast
  qed

  hence ch-gY: local-long-ch-by-ord g Y
    using IH.prems(1,4,5) g-ch f-ch ch-fY card-Y chain-remove-at-right-edge fin-Y
chain-defs
    by (metis Y-def chain-bounds-unique long-ch-card-ge3)

  have f i ∈ Y ∨ f i = c
    by (metis local-ordering-def ‹X = insert c Y› ‹i < card X› lch-fX insert-iff
local-long-ch-by-ord-def)
  thus f i = g i
  proof (rule disjE)
    assume f i ∈ Y
    hence f i ≠ c
      using ‹c ∉ Y› by blast
    hence i < card Y
    using ‹X = insert c Y› ‹c∉Y› IH(3,4) f-ch fin-Y chain-defs not-less-less-Suc-eq
      by (metis ‹card Y = card X − 1› card-insert-disjoint)
    hence 3 ≤ card Y
      using card-Y le-add2 by presburger
    show f i = g i
      using IH(1) [of Y]
      using ‹n = card Y − 3› ‹3 ≤ card Y› ‹i < card Y›
      using Y-def card-Y chain-remove-at-right-edge le-add2
      by (metis IH.prems(1,3,4,5) chain-bounds-unique)
  next
    assume f i = c
    show ?thesis
    using IH.prems(2,5) ‹f i = c› chain-bounds-unique f-ch g-ch indices-neq-imp-events-neq
chain-defs
      by (metis ‹card Y = card X − 1› Y-def card-insert-disjoint fin-Y lessI)
  qed
qed
```

I'm really impressed *sledgehammer/smt* can solve this if I just tell them "Use symmetry!".

```
lemma chain-unique-induction-cx:
  assumes card X ≥ 3
      and i < card X
      and [f⇝X|a..c]
      and [g⇝X|x..z]
      and c = x ∨ a = z
    shows f i = g (card X − i − 1)
  using chain-sym-obtain2 chain-unique-induction-ax assms diff-right-commute by
smt
```

This lemma has to exclude two-element chains again, because no order exists within them. Alternatively, the result is trivial: any function that assigns one

element to index 0 and the other to 1 can be replaced with the (unique) other assignment, without destroying any (trivial, since ternary) *local-ordering* of the chain. This could be made generic over the *local-ordering* similar to $[?f{\rightsquigarrow}?X|?a..?b..?c] \Longrightarrow [\lambda n.\ ?f\ (card\ ?X - 1 - n){\rightsquigarrow}?X|?c..?b..?a]$ relying on $[\![\bigwedge a\ b\ c.\ ?ord\ a\ b\ c \Longrightarrow ?ord\ c\ b\ a;\ finite\ ?X;\ local\text{-}ordering\ ?f\ ?ord\ ?X]\!] \Longrightarrow local\text{-}ordering\ (\lambda n.\ ?f\ (card\ ?X - 1 - n))\ ?ord\ ?X$.

lemma *chain-unique-upto-rev-cases*:
  assumes *ch-f*: $[f{\rightsquigarrow}X|a..c]$
    and *ch-g*: $[g{\rightsquigarrow}X|x..z]$
    and *card-X*: $card\ X \geq 3$
    and *valid-index*: $i < card\ X$
  shows $((a{=}x \vee c{=}z) \longrightarrow (f\ i = g\ i))\ ((a{=}z \vee c{=}x) \longrightarrow (f\ i = g\ (card\ X - i - 1)))$
proof −
  obtain *n* **where** *n-def*: $n = card\ X - 3$
    by *blast*
  hence *valid-index-2*: $i < n + 3$
    by (*simp add*: *card-X valid-index*)

  show $((a{=}x \vee c{=}z) \longrightarrow (f\ i = g\ i))$
    using *card-X ch-f ch-g chain-unique-induction-ax valid-index* **by** *blast*
  show $((a{=}z \vee c{=}x) \longrightarrow (f\ i = g\ (card\ X - i - 1)))$
    using *assms*(*3*) *ch-f ch-g chain-unique-induction-cx valid-index* **by** *blast*
qed

lemma *chain-unique-upto-rev*:
  assumes $[f{\rightsquigarrow}X|a..c]$ $[g{\rightsquigarrow}X|x..z]$ $card\ X \geq 3$ $i < card\ X$
  shows $f\ i = g\ i \vee f\ i = g\ (card\ X - i - 1)$ $a{=}x{\wedge}c{=}z \vee c{=}x{\wedge}a{=}z$
proof −
  have $(a{=}x \vee c{=}z) \vee (a{=}z \vee c{=}x)$
    using *chain-bounds-unique* **by** (*metis assms*(*1,2*))
  thus $f\ i = g\ i \vee f\ i = g\ (card\ X - i - 1)$
    using *assms*(*3*) ‹$i < card\ X$› *assms chain-unique-upto-rev-cases* **by** *blast*
  thus $(a{=}x{\wedge}c{=}z) \vee (c{=}x{\wedge}a{=}z)$
    by (*meson assms*(*1−3*) *chain-bounds-unique*)
    qed

end

32 Interlude: betw4 and WLOG

32.1 betw4 - strict and non-strict, basic lemmas

context *MinkowskiBetweenness* **begin**

Define additional notation for non-strict *local-ordering* - cf Schutz' monograph [1, p. 27].

abbreviation *nonstrict-betw-right* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([-;-;-⟧) **where**
nonstrict-betw-right a b c ≡ $[a;b;c] \vee b = c$

abbreviation *nonstrict-betw-left* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ (⟦-;-;-]) **where**
nonstrict-betw-left a b c ≡ $[a;b;c] \vee b = a$

abbreviation *nonstrict-betw-both* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
nonstrict-betw-both a b c ≡ *nonstrict-betw-left a b c* ∨ *nonstrict-betw-right a b c*

abbreviation *betw4* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([-;-;-;-]) **where**
betw4 a b c d ≡ $[a;b;c] \wedge [b;c;d]$

abbreviation *nonstrict-betw-right4* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([-;-;-;-⟧) **where**
nonstrict-betw-right4 a b c d ≡ *betw4 a b c d* ∨ *c* = *d*

abbreviation *nonstrict-betw-left4* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ (⟦-;-;-;-]) **where**
nonstrict-betw-left4 a b c d ≡ *betw4 a b c d* ∨ *a* = *b*

abbreviation *nonstrict-betw-both4* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
nonstrict-betw-both4 a b c d ≡ *nonstrict-betw-left4 a b c d* ∨ *nonstrict-betw-right4 a b c d*

lemma *betw4-strong*:
assumes *betw4 a b c d*
shows $[a;b;d] \wedge [a;c;d]$
using *abc-bcd-acd assms* **by** *blast*

lemma *betw4-imp-neq*:
assumes *betw4 a b c d*
shows *a*≠*b* ∧ *a*≠*c* ∧ *a*≠*d* ∧ *b*≠*c* ∧ *b*≠*d* ∧ *c*≠*d*
using *abc-only-cba assms* **by** *blast*

end
context *MinkowskiSpacetime* **begin**

lemma *betw4-weak*:
fixes *a b c d* :: $'a$
assumes $[a;b;c] \wedge [a;c;d]$
$\vee [a;b;c] \wedge [b;c;d]$
$\vee [a;b;d] \wedge [b;c;d]$
$\vee [a;b;d] \wedge [b;c;d]$
shows *betw4 a b c d*
using *abc-acd-bcd abd-bcd-abc assms* **by** *blast*

lemma *betw4-sym*:
fixes *a*::$'a$ **and** *b*::$'a$ **and** *c*::$'a$ **and** *d*::$'a$
shows *betw4 a b c d* ⟷ *betw4 d c b a*

using *abc-sym* **by** *blast*

lemma *abcd-dcba-only*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$
assumes $[a;b;c;d]$
shows $\neg[a;b;d;c]$ $\neg[a;c;b;d]$ $\neg[a;c;d;b]$ $\neg[a;d;b;c]$ $\neg[a;d;c;b]$
$\neg[b;a;c;d]$ $\neg[b;a;d;c]$ $\neg[b;c;a;d]$ $\neg[b;c;d;a]$ $\neg[b;d;c;a]$ $\neg[b;d;a;c]$
$\neg[c;a;b;d]$ $\neg[c;a;d;b]$ $\neg[c;b;a;d]$ $\neg[c;b;d;a]$ $\neg[c;d;a;b]$ $\neg[c;d;b;a]$
$\neg[d;a;b;c]$ $\neg[d;a;c;b]$ $\neg[d;b;a;c]$ $\neg[d;b;c;a]$ $\neg[d;c;a;b]$
using *abc-only-cba assms* **by** *blast+*

lemma *some-betw4a*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$ **and** P
assumes $P\in\mathcal{P}$ $a\in P$ $b\in P$ $c\in P$ $d\in P$ $a\neq b \wedge a\neq c \wedge a\neq d \wedge b\neq c \wedge b\neq d \wedge c\neq d$
and $\neg([a;b;c;d] \vee [a;b;d;c] \vee [a;c;b;d] \vee [a;c;d;b] \vee [a;d;b;c] \vee [a;d;c;b])$
shows $[b;a;c;d] \vee [b;a;d;c] \vee [b;c;a;d] \vee [b;d;a;c] \vee [c;a;b;d] \vee [c;b;a;d]$
by (*smt abc-bcd-acd abc-sym abd-bcd-abc assms some-betw-xor*)

lemma *some-betw4b*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$ **and** P
assumes $P\in\mathcal{P}$ $a\in P$ $b\in P$ $c\in P$ $d\in P$ $a\neq b \wedge a\neq c \wedge a\neq d \wedge b\neq c \wedge b\neq d \wedge c\neq d$
and $\neg([b;a;c;d] \vee [b;a;d;c] \vee [b;c;a;d] \vee [b;d;a;c] \vee [c;a;b;d] \vee [c;b;a;d])$
shows $[a;b;c;d] \vee [a;b;d;c] \vee [a;c;b;d] \vee [a;c;d;b] \vee [a;d;b;c] \vee [a;d;c;b]$
by (*smt abc-bcd-acd abc-sym abd-bcd-abc assms some-betw-xor*)

lemma *abd-acd-abcdacbd*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$
assumes *abd*: $[a;b;d]$ **and** *acd*: $[a;c;d]$ **and** $b\neq c$
shows $[a;b;c;d] \vee [a;c;b;d]$
proof −
obtain P **where** $P\in\mathcal{P}$ $a\in P$ $b\in P$ $d\in P$
using *abc-ex-path abd* **by** *blast*
have $c\in P$
using ‹$P \in \mathcal{P}$› ‹$a \in P$› ‹$d \in P$› *abc-abc-neq acd betw-b-in-path* **by** *blast*
have $\neg[b;d;c]$
using *abc-sym abcd-dcba-only*(*5*) *abd acd* **by** *blast*
hence $[b;c;d] \vee [c;b;d]$
using *abc-abc-neq abc-sym abd acd assms*(*3*) *some-betw*
by (*metis* ‹$P \in \mathcal{P}$› ‹$b \in P$› ‹$c \in P$› ‹$d \in P$›)
thus *?thesis*
using *abd acd betw4-weak* **by** *blast*
qed

end

32.2 WLOG for two general symmetric relations of two elements on a single path

context *MinkowskiBetweenness* **begin**

This first one is really just trying to get a hang of how to write these things. If you have a relation that does not care which way round the "endpoints" (if Q is the interval-relation) go, then anything you want to prove about both undistinguished endpoints, follows from a proof involving a single endpoint.

lemma *wlog-sym-element*:
  assumes *symmetric-rel*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *one-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{=}a]\!] \Longrightarrow P\ x\ I$
   shows *other-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{=}b]\!] \Longrightarrow P\ x\ I$
  using *assms* **by** *fastforce*

This one gives the most pertinent case split: a proof involving e.g. an element of an interval must consider the edge case and the inside case.

lemma *wlog-element*:
  assumes *symmetric-rel*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *one-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{=}a]\!] \Longrightarrow P\ x\ I$
    and *neither-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{\in}I;\ (x{\neq}a \wedge x{\neq}b)]\!] \Longrightarrow P\ x\ I$
   shows *any-element*: $\bigwedge x\ I.\ [\![x{\in}I;\ (\exists\ a\ b.\ Q\ I\ a\ b)]\!] \Longrightarrow P\ x\ I$
  by (*metis assms*)

Summary of the two above. Use for early case splitting in proofs. Doesn't need P to be symmetric - the context in the conclusion is explicitly symmetric.

lemma *wlog-two-sets-element*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *case-split*: $\bigwedge a\ b\ c\ d\ x\ I\ J.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d]\!] \Longrightarrow$
        $(x{=}a \vee x{=}c \longrightarrow P\ x\ I\ J) \wedge (\neg(x{=}a \vee x{=}b \vee x{=}c \vee x{=}d) \longrightarrow P\ x\ I\ J)$
   shows $\bigwedge x\ I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ J\ a\ b]\!] \Longrightarrow P\ x\ I\ J$
  by (*smt case-split symmetric-Q*)

Now we start on the actual result of interest. First we assume the events are all distinct, and we deal with the degenerate possibilities after.

lemma *wlog-endpoints-distinct1*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ [a;b;c;d]]\!] \Longrightarrow P\ I\ J$
   shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
        $[b;a;c;d] \vee [a;b;d;c] \vee [b;a;d;c] \vee [d;c;b;a]]\!] \Longrightarrow P\ I\ J$
  by (*meson abc-sym assms*(*2*) *symmetric-Q*)

lemma *wlog-endpoints-distinct2*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ [a;c;b;d]]\!] \Longrightarrow P\ I\ J$
   shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
        $[b;c;a;d] \vee [a;d;b;c] \vee [b;d;a;c] \vee [d;b;c;a]]\!] \Longrightarrow P\ I\ J$

by (*meson abc-sym assms*(*2*) *symmetric-Q*)

lemma *wlog-endpoints-distinct3*:
assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ [a;c;d;b]]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$[a;d;c;b] \vee [b;c;d;a] \vee [b;d;c;a] \vee [c;a;b;d]]\!] \Longrightarrow P\ I\ J$
by (*meson assms*)

lemma (**in** *MinkowskiSpacetime*) *wlog-endpoints-distinct4*:
fixes *Q*:: ('*a set*) ⇒ '*a* ⇒ '*a* ⇒ *bool*
and *P*:: ('*a set*) ⇒ ('*a set*) ⇒ *bool*
and *A*:: ('*a set*)
assumes *path-A*: $A \in \mathcal{P}$
and *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I \subseteq A;\ Q\ I\ a\ b]\!] \Longrightarrow b \in A \wedge a \in A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.$
$[\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I \subseteq A;\ J \subseteq A;\ [a;b;c;d] \vee [a;c;b;d] \vee [a;c;d;b]]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I \subseteq A;\ J \subseteq A;$
$a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d]\!] \Longrightarrow P\ I\ J$
proof −
fix *I J a b c d*
assume *asm*: $Q\ I\ a\ b\ Q\ J\ c\ d\ I \subseteq A\ J \subseteq A$
$a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
have *endpoints-on-path*: $a \in A\ b \in A\ c \in A\ d \in A$
using *Q-implies-path asm* **by** *blast*+
show *P I J*
proof (*cases*)
assume $[b;a;c;d] \vee [b;a;d;c] \vee [b;c;a;d] \vee$
$[b;d;a;c] \vee [c;a;b;d] \vee [c;b;a;d]$
then consider $[b;a;c;d] \mid\mid [b;a;d;c] \mid\mid [b;c;a;d] \mid$
$[b;d;a;c] \mid\mid [c;a;b;d] \mid\mid [c;b;a;d]$
by *linarith*
thus *P I J*
apply (*cases*)
apply (*metis*(*mono-tags*) *asm*(*1−4*) *assms*(*5*) *symmetric-Q*)+
apply (*metis asm*(*1−4*) *assms*(*4,5*))
by (*metis asm*(*1−4*) *assms*(*2,4,5*) *symmetric-Q*)
next
assume $\neg([b;a;c;d] \vee [b;a;d;c] \vee [b;c;a;d] \vee$
$[b;d;a;c] \vee [c;a;b;d] \vee [c;b;a;d])$
hence $[a;b;c;d] \vee [a;b;d;c] \vee [a;c;b;d] \vee$
$[a;c;d;b] \vee [a;d;b;c] \vee [a;d;c;b]$
using *some-betw4b* [**where** *P*=*A* **and** *a*=*a* **and** *b*=*b* **and** *c*=*c* **and** *d*=*d*]
using *endpoints-on-path asm path-A* **by** *simp*
then consider $[a;b;c;d] \mid\mid [a;b;d;c] \mid\mid [a;c;b;d] \mid$
$[a;c;d;b] \mid\mid [a;d;b;c] \mid\mid [a;d;c;b]$

```
    by linarith
  thus P I J
    apply (cases)
    by (metis asm(1−4) assms(5) symmetric-Q)+
 qed
qed


lemma (in MinkowskiSpacetime) wlog-endpoints-distinct':
 assumes A ∈ 𝒫
    and ⋀a b I. Q I a b ⟹ Q I b a
    and ⋀a b I. ⟦I ⊆ A; Q I a b⟧ ⟹ a ∈ A
    and ⋀I J. ⟦∃ a b. Q I a b; ∃ a b. Q J a b; P I J⟧ ⟹ P J I
    and ⋀I J a b c d.
        ⟦Q I a b; Q J c d; I⊆A; J⊆A; betw4 a b c d ∨ betw4 a c b d ∨ betw4 a c d
b⟧ ⟹ P I J
    and Q I a b
    and Q J c d
    and I ⊆ A
    and J ⊆ A
    and a ≠ b a ≠ c a ≠ d b ≠ c b ≠ d c ≠ d
 shows P I J
proof −
 {
  let ?R = (λI. (∃ a b. Q I a b))
  have ⋀I J. ⟦?R I; ?R J; P I J⟧ ⟹ P J I
    using assms(4) by blast
 }
 thus ?thesis
  using wlog-endpoints-distinct4
    [where P=P and Q=Q and A=A and I=I and J=J and a=a and b=b
and c=c and d=d]
  by (smt assms(1−3,5−))
qed

lemma (in MinkowskiSpacetime) wlog-endpoints-distinct:
 assumes path-A: A∈𝒫
    and symmetric-Q: ⋀a b I. Q I a b ⟹ Q I b a
    and Q-implies-path: ⋀a b I. ⟦I⊆A; Q I a b⟧ ⟹ b∈A ∧ a∈A
    and symmetric-P: ⋀I J. ⟦∃ a b. Q I a b; ∃ a b. Q J a b; P I J⟧ ⟹ P J I
    and ⋀I J a b c d.
        ⟦Q I a b; Q J c d; I⊆A; J⊆A; [a;b;c;d] ∨ [a;c;b;d] ∨ [a;c;d;b]⟧ ⟹ P I J
 shows ⋀I J a b c d. ⟦Q I a b; Q J c d; I⊆A; J⊆A;
           a≠b ∧ a≠c ∧ a≠d ∧ b≠c ∧ b≠d ∧ c≠d⟧ ⟹ P I J
 by (smt (verit, ccfv-SIG) assms some-betw4b)


lemma wlog-endpoints-degenerate1:
 assumes symmetric-Q: ⋀a b I. Q I a b ⟹ Q I b a
```

and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ I\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$

and *two*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{=}b \wedge b{=}c \wedge c{=}d) \vee (a{=}b \wedge b{\neq}c \wedge c{=}d)]\!] \Longrightarrow P\ I\ J$

and *one*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{=}b \wedge b{=}c \wedge c{\neq}d) \vee (a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d)]\!] \Longrightarrow P\ I\ J$

and *no*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{=}d) \vee (a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d)]\!] \Longrightarrow P\ I\ J$

shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ \neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$

by (*metis assms*)

lemma *wlog-endpoints-degenerate2*:

assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$

and *Q-implies-path*: $\bigwedge a\ b\ I\ A.\ [\![I{\subseteq}A;\ A{\in}\mathcal{P};\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$

and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$

and $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$[a;b;c] \wedge a{=}d]\!] \Longrightarrow P\ I\ J$

and $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$[b;a;c] \wedge a{=}d]\!] \Longrightarrow P\ I\ J$

shows $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{=}d]\!] \Longrightarrow P\ I\ J$

proof −

have *last-case*: $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$[b;c;a] \wedge a{=}d]\!] \Longrightarrow P\ I\ J$

using *assms*(*1*,*3*−*5*) **by** (*metis abc-sym*)

thus $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{=}d]\!] \Longrightarrow P\ I\ J$

by (*smt* (*z3*) *abc-sym assms*(*2*,*4*,*5*) *some-betw*)

qed

lemma *wlog-endpoints-degenerate*:

assumes *path-A*: $A{\in}\mathcal{P}$

and *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$

and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I{\subseteq}A;\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$

and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$

and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A]\!]$
$\Longrightarrow ((a{=}b \wedge b{=}c \wedge c{=}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge ((a{=}b \wedge b{=}c \wedge c{\neq}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d) \longrightarrow P\ I\ J)$
$\wedge ((a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge (([a;b;c] \wedge a{=}d) \longrightarrow P\ I\ J) \wedge (([b;a;c] \wedge a{=}d) \longrightarrow P\ I\ J)$

shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;$
$\neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$

proof −

We first extract some of the assumptions of this lemma into the form of

other WLOG lemmas' assumptions.

```
  have ord1: ⋀I J a b c d. ⟦Q I a b; Q J c d; I⊆A; J⊆A;
            [a;b;c] ∧ a=d⟧ ⟹ P I J
    using assms(5) by auto
  have ord2: ⋀I J a b c d. ⟦Q I a b; Q J c d; I⊆A; J⊆A;
            [b;a;c] ∧ a=d⟧ ⟹ P I J
    using assms(5) by auto
  have last-case: ⋀I J a b c d. ⟦Q I a b; Q J c d; I⊆A; J⊆A;
            a≠b ∧ b≠c ∧ c≠d ∧ a=d⟧ ⟹ P I J
   using ord1 ord2 wlog-endpoints-degenerate2 symmetric-P symmetric-Q Q-implies-path
path-A
    by (metis abc-sym some-betw)
  show ⋀I J a b c d. ⟦Q I a b; Q J c d; I⊆A; J⊆A;
          ¬(a≠b ∧ b≠c ∧ c≠d ∧ a≠d ∧ a≠c ∧ b≠d)⟧ ⟹ P I J
  proof −
```

Fix the sets on the path, and obtain the assumptions of *wlog-endpoints-degenerate1*.

```
    fix I J
    assume asm1: I⊆A J⊆A
    have two: ⋀a b c d. ⟦Q I a b; Q J c d; a=b ∧ b=c ∧ c=d⟧ ⟹ P I J
          ⋀a b c d. ⟦Q I a b; Q J c d; a=b ∧ b≠c ∧ c=d⟧ ⟹ P I J
      using ‹J ⊆ A› ‹I ⊆ A› path-A assms(5) by blast+
    have one: ⋀ a b c d. ⟦Q I a b; Q J c d; a=b ∧ b=c ∧ c≠d⟧ ⟹ P I J
         ⋀ a b c d. ⟦Q I a b; Q J c d; a=b ∧ b≠c ∧ c≠d ∧ a≠d⟧ ⟹ P I J
      using ‹I ⊆ A› ‹J ⊆ A› path-A assms(5) by blast+
    have no: ⋀ a b c d. ⟦Q I a b; Q J c d; a≠b ∧ b≠c ∧ c≠d ∧ a=d⟧ ⟹ P I J
         ⋀ a b c d. ⟦Q I a b; Q J c d; a≠b ∧ b=c ∧ c≠d ∧ a=d⟧ ⟹ P I J
      using ‹I ⊆ A› ‹J ⊆ A› path-A last-case apply blast
      using ‹I ⊆ A› ‹J ⊆ A› path-A assms(5) by auto
```

Now unwrap the remaining object logic and finish the proof.

```
    fix a b c d
    assume asm2: Q I a b Q J c d ¬(a≠b ∧ b≠c ∧ c≠d ∧ a≠d ∧ a≠c ∧ b≠d)
    show P I J
      using two [where a=a and b=b and c=c and d=d]
      using one [where a=a and b=b and c=c and d=d]
      using no [where a=a and b=b and c=c and d=d]
      using wlog-endpoints-degenerate1
        [where I=I and J=J and a=a and b=b and c=c and d=d and P=P
and Q=Q]
      using asm1 asm2 symmetric-P last-case assms(5) symmetric-Q
      by smt
  qed
qed


lemma (in MinkowskiSpacetime) wlog-intro:
  assumes path-A: A∈𝒫
      and symmetric-Q: ⋀a b I. Q I a b ⟹ Q I b a
```

and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I{\subseteq}A;\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, c\ d.\ Q\ J\ c\ d;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and *essential-cases*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A]\!]$
$\Longrightarrow ((a{=}b \wedge b{=}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge\ ((a{=}b \wedge b{\neq}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge\ ((a{=}b \wedge b{=}c \wedge c{\neq}d) \longrightarrow P\ I\ J)$
$\wedge\ ((a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d) \longrightarrow P\ I\ J)$
$\wedge\ ((a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge\ (([a;b;c] \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge\ (([b;a;c] \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge\ ([a;b;c;d] \longrightarrow P\ I\ J)$
$\wedge\ ([a;c;b;d] \longrightarrow P\ I\ J)$
$\wedge\ ([a;c;d;b] \longrightarrow P\ I\ J)$
and *antecedants*: $Q\ I\ a\ b\ Q\ J\ c\ d\ I{\subseteq}A\ J{\subseteq}A$
shows $P\ I\ J$
using *essential-cases antecedants*
and *wlog-endpoints-degenerate*[*OF path-A symmetric-Q Q-implies-path symmetric-P*]
and *wlog-endpoints-distinct*[*OF path-A symmetric-Q Q-implies-path symmetric-P*]
by (*smt* (*z3*) *Q-implies-path path-A symmetric-P symmetric-Q some-betw2 some-betw4b abc-only-cba*(*1*))

end

32.3 WLOG for two intervals

context *MinkowskiBetweenness* **begin**

This section just specifies the results for a generic relation Q in the previous section to the interval relation.

lemma *wlog-two-interval-element*:
assumes $\bigwedge x\ I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J;\ P\ x\ J\ I]\!] \Longrightarrow P\ x\ I\ J$
and $\bigwedge a\ b\ c\ d\ x\ I\ J.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d]\!] \Longrightarrow$
$(x{=}a \vee x{=}c \longrightarrow P\ x\ I\ J) \wedge (\neg(x{=}a \vee x{=}b \vee x{=}c \vee x{=}d) \longrightarrow P\ x\ I\ J)$
shows $\bigwedge x\ I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J]\!] \Longrightarrow P\ x\ I\ J$
by (*metis assms*(*2*) *int-sym*)

lemma (**in** *MinkowskiSpacetime*) *wlog-interval-endpoints-distinct*:
assumes $\bigwedge I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
$\bigwedge I\ J\ a\ b\ c\ d.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d]\!]$
$\Longrightarrow ([a;b;c;d] \longrightarrow P\ I\ J) \wedge ([a;c;b;d] \longrightarrow P\ I\ J) \wedge ([a;c;d;b] \longrightarrow P\ I\ J)$
shows $\bigwedge I\ J\ Q\ a\ b\ c\ d.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P};$
$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d]\!] \Longrightarrow P\ I\ J$
proof −
let *?Q* = $\lambda\ I\ a\ b.\ I = \textit{interval}\ a\ b$

fix $I\ J\ A\ a\ b\ c\ d$
assume *asm*: $?Q\ I\ a\ b\ ?Q\ J\ c\ d\ I{\subseteq}A\ J{\subseteq}A\ A{\in}\mathcal{P}\ a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d$
show $P\ I\ J$
proof (*rule wlog-endpoints-distinct*)
show $\bigwedge a\ b\ I.\ ?Q\ I\ a\ b \Longrightarrow ?Q\ I\ b\ a$
by (*simp add*: *int-sym*)
show $\bigwedge a\ b\ I.\ I \subseteq A \Longrightarrow ?Q\ I\ a\ b \Longrightarrow b \in A \wedge a \in A$
by (*simp add*: *ends-in-int subset-iff*)
show $\bigwedge I\ J.\ is\text{-}interval\ I \Longrightarrow is\text{-}interval\ J \Longrightarrow P\ I\ J \Longrightarrow P\ J\ I$
using *assms*(*1*) **by** *blast*
show $\bigwedge I\ J\ a\ b\ c\ d.\ [\![?Q\ I\ a\ b;\ ?Q\ J\ c\ d;\ [a;b;c;d] \vee [a;c;b;d] \vee [a;c;d;b]]\!]$
$\Longrightarrow P\ I\ J$
by (*meson assms*(*2*))
show $I = interval\ a\ b\ J = interval\ c\ d\ I{\subseteq}A\ J{\subseteq}A\ A{\in}\mathcal{P}$
$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d$
using *asm* **by** *simp+*
qed
qed

lemma *wlog-interval-endpoints-degenerate*:
assumes *symmetry*: $\bigwedge I\ J.\ [\![is\text{-}interval\ I;\ is\text{-}interval\ J;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d\ Q.\ [\![I = interval\ a\ b;\ J = interval\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P}]\!]$
$\Longrightarrow ((a{=}b \wedge b{=}c \wedge c{=}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge ((a{=}b \wedge b{=}c \wedge c{\neq}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d) \longrightarrow P\ I\ J)$
$\wedge ((a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge (([a;b;c] \wedge a{=}d) \longrightarrow P\ I\ J) \wedge (([b;a;c] \wedge a{=}d) \longrightarrow P\ I\ J)$
shows $\bigwedge I\ J\ a\ b\ c\ d\ Q.\ [\![I = interval\ a\ b;\ J = interval\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P};$
$\neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$
proof −
let $?Q = \lambda\ I\ a\ b.\ I = interval\ a\ b$

fix $I\ J\ a\ b\ c\ d\ A$
assume *asm*: $?Q\ I\ a\ b\ ?Q\ J\ c\ d\ I{\subseteq}A\ J{\subseteq}A\ A{\in}\mathcal{P}\ \neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)$
show $P\ I\ J$
proof (*rule wlog-endpoints-degenerate*)
show $\bigwedge a\ b\ I.\ ?Q\ I\ a\ b \Longrightarrow ?Q\ I\ b\ a$
by (*simp add*: *int-sym*)
show $\bigwedge a\ b\ I.\ I \subseteq A \Longrightarrow ?Q\ I\ a\ b \Longrightarrow b \in A \wedge a \in A$
by (*simp add*: *ends-in-int subset-iff*)
show $\bigwedge I\ J.\ is\text{-}interval\ I \Longrightarrow is\text{-}interval\ J \Longrightarrow P\ I\ J \Longrightarrow P\ J\ I$
using *symmetry* **by** *blast*
show $I = interval\ a\ b\ J = interval\ c\ d\ I{\subseteq}A\ J{\subseteq}A\ A{\in}\mathcal{P}$
$\neg\ (a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)$
using *asm* **by** *auto+*
show $\bigwedge I\ J\ a\ b\ c\ d.\ [\![?Q\ I\ a\ b;\ ?Q\ J\ c\ d;\ I \subseteq A;\ J \subseteq A]\!] \Longrightarrow$

$(a = b \wedge b = c \wedge c = d \longrightarrow P\ I\ J) \wedge$
$(a = b \wedge b \neq c \wedge c = d \longrightarrow P\ I\ J) \wedge$
$(a = b \wedge b = c \wedge c \neq d \longrightarrow P\ I\ J) \wedge$
$(a = b \wedge b \neq c \wedge c \neq d \wedge a \neq d \longrightarrow P\ I\ J) \wedge$
$(a \neq b \wedge b = c \wedge c \neq d \wedge a = d \longrightarrow P\ I\ J) \wedge$
$([a;b;c] \wedge a = d \longrightarrow P\ I\ J) \wedge ([b;a;c] \wedge a = d \longrightarrow P\ I\ J)$
using *assms(2)* ‹$A \in \mathcal{P}$› **by** *auto*
qed
qed

end

33 Interlude: Intervals, Segments, Connectedness

context *MinkowskiSpacetime* **begin**

In this secion, we apply the WLOG lemmas from the previous section in order to reduce the number of cases we need to consider when thinking about two arbitrary intervals on a path. This is used to prove that the (countable) intersection of intervals is an interval. These results cannot be found in Schutz, but he does use them (without justification) in his proof of Theorem 12 (even for uncountable intersections).

lemma *int-of-ints-is-interval-neq*:
assumes *I1 = interval a b I2 = interval c d I1⊆P I2⊆P P∈𝒫 I1∩I2 ≠ {}*
and *events-neq*: *a≠b a≠c a≠d b≠c b≠d c≠d*
shows *is-interval (I1 ∩ I2)*
proof −
have *on-path*: *a∈P ∧ b∈P ∧ c∈P ∧ d∈P*
using *assms(1−4) interval-def* **by** *auto*

let *?prop* = *λ I J. is-interval (I∩J) ∨ (I∩J) = {}*

have *symmetry*: (*⋀I J. is-interval I ⟹ is-interval J ⟹ ?prop I J ⟹ ?prop J I*)
by (*simp add*: *Int-commute*)

{
fix *I J a b c d*
assume *I = interval a b J = interval c d*
have ([*a;b;c;d*] ⟶ *?prop I J*)
([*a;c;b;d*] ⟶ *?prop I J*)
([*a;c;d;b*] ⟶ *?prop I J*)
proof (*rule-tac* [!] *impI*)
assume *betw4 a b c d*
have *I∩J = {}*
proof (*rule ccontr*)
assume *I∩J≠{}*
then obtain *x* **where** *x∈I∩J*

by *blast*
show *False*
proof (*cases*)
assume $x{\neq}a \wedge x{\neq}b \wedge x{\neq}c \wedge x{\neq}d$
hence $[a;x;b]$ $[c;x;d]$
using ‹$I{=}$*interval a b*› ‹$x{\in}I{\cap}J$› ‹$J{=}$*interval c d*› ‹$x{\in}I{\cap}J$›
by (*simp add*: *interval-def seg-betw*)+
thus *False*
by (*meson* ‹*betw4 a b c d*› *abc-only-cba*(*3*) *abc-sym abd-bcd-abc*)
next
assume $\neg(x{\neq}a \wedge x{\neq}b \wedge x{\neq}c \wedge x{\neq}d)$
thus *False*
using *interval-def seg-betw* ‹I = *interval a b*› ‹J = *interval c d*› *abcd-dcba-only*(*21*)
‹$x \in I \cap J$› ‹*betw4 a b c d*› *abc-bcd-abd abc-bcd-acd abc-only-cba*(*1,2*)
by (*metis* (*full-types*) *insert-iff Int-iff*)
qed
qed
thus *?prop I J* **by** *simp*
next
assume $[a;c;b;d]$
then have $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
using *betw4-imp-neq* **by** *blast*
have $I{\cap}J$ = *interval c b*
proof (*safe*)
fix x
assume $x \in$ *interval c b*
{
assume $x{=}b \vee x{=}c$
hence $x{\in}I$
using ‹$[a;c;b;d]$› ‹I = *interval a b*› *interval-def seg-betw* **by** *auto*
have $x{\in}J$
using ‹$x{=}b \vee x{=}c$›
using ‹$[a;c;b;d]$› ‹J = *interval c d*› *interval-def seg-betw* **by** *auto*
hence $x{\in}I \wedge x{\in}J$ **using** ‹$x \in I$› **by** *blast*
} moreover {
assume $\neg(x{=}b \vee x{=}c)$
hence $[c;x;b]$
using ‹$x{\in}$*interval c b*› **unfolding** *interval-def segment-def* **by** *simp*
hence $[a;x;b]$
by (*meson* ‹$[a;c;b;d]$› *abc-acd-abd abc-sym*)
have $[c;x;d]$
using ‹$[a;c;b;d]$› ‹$[c;x;b]$› *abc-acd-abd* **by** *blast*
have $x{\in}I$ $x{\in}J$
using ‹I = *interval a b*› ‹$[a;x;b]$› ‹J = *interval c d*› ‹$[c;x;d]$›
interval-def seg-betw **by** *auto*
}
ultimately show $x{\in}I$ $x{\in}J$ **by** *blast*+
next

fix x
assume $x{\in}I$ $x{\in}J$
show $x \in$ *interval c b*
proof (*cases*)
assume *not-eq*: $x{\neq}a \wedge x{\neq}b \wedge x{\neq}c \wedge x{\neq}d$
have $[a;x;b]$ $[c;x;d]$
using ‹$x{\in}I$› ‹$I =$ *interval a b*› ‹$x{\in}J$› ‹$J =$ *interval c d*›
not-eq **unfolding** *interval-def segment-def* **by** *blast+*
hence $[c;x;b]$
by (*meson* ‹$[a;c;b;d]$› *abc-bcd-acd betw4-weak*)
thus *?thesis*
unfolding *interval-def segment-def* **using** *seg-betw segment-def* **by** *auto*
next
assume *not-not-eq*: $\neg(x{\neq}a \wedge x{\neq}b \wedge x{\neq}c \wedge x{\neq}d)$
{
assume $x{=}a$
have $\neg[d;a;c]$
using ‹$[a;c;b;d]$› *abcd-dcba-only*(*9*) **by** *blast*
hence $a \notin$ *interval c d* **unfolding** *interval-def segment-def*
using *abc-sym* ‹$a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$› **by** *blast*
hence *False* **using** ‹$x{\in}J$› ‹$J =$ *interval c d*› ‹$x{=}a$› **by** *blast*
} moreover {
assume $x{=}d$
have $\neg[a;d;b]$ **using** ‹*betw4 a c b d*› *abc-sym abcd-dcba-only*(*9*) **by** *blast*
hence $d{\notin}$*interval a b* **unfolding** *interval-def segment-def*
using ‹$a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$› **by** *blast*
hence *False* **using** ‹$x{\in}I$› ‹$x{=}d$› ‹$I =$ *interval a b*› **by** *blast*
}
ultimately show *?thesis*
using *interval-def not-not-eq* **by** *auto*
qed
qed
thus *?prop I J* **by** *auto*
next
assume $[a;c;d;b]$
have $I{\cap}J =$ *interval c d*
proof (*safe*)
fix x
assume $x \in$ *interval c d*
{
assume $x{\neq}c \wedge x{\neq}d$
have $x \in J$
by (*simp add*: ‹$J =$ *interval c d*› ‹$x \in$ *interval c d*›)
have $[c;x;d]$
using ‹$x \in$ *interval c d*› ‹$x \neq c \wedge x \neq d$› *interval-def seg-betw* **by** *auto*
have $[a;x;b]$
by (*meson* ‹*betw4 a c d b*› ‹$[c;x;d]$› *abc-bcd-abd abc-sym abe-ade-bcd-ace*)
have $x \in I$

```
        using ‹I = interval a b› ‹[a;x;b]› interval-def seg-betw by auto
      hence x∈I ∧ x∈J by (simp add: ‹x ∈ J›)
    } moreover {
      assume ¬ (x≠c ∧ x≠d)
      hence x∈I ∧ x∈J
        by (metis ‹I = interval a b› ‹J = interval c d› ‹[a;c;d;b]› ‹x ∈ interval c
d›
            abc-bcd-abd abc-bcd-acd insertI2 interval-def seg-betw)
    }
    ultimately show x∈I x∈J by blast+
  next
    fix x
    assume x∈I x∈J
    show x ∈ interval c d
      using ‹J = interval c d› ‹x ∈ J› by auto
  qed
  thus ?prop I J by auto
 qed
}

then show is-interval (I1∩I2)
 using wlog-interval-endpoints-distinct
   [where P=?prop and I=I1 and J=I2 and Q=P and a=a and b=b and
c=c and d=d]
 using symmetry assms by simp
qed


lemma int-of-ints-is-interval-deg:
 assumes I = interval a b J = interval c d I∩J ≠ {} I⊆P J⊆P P∈𝒫
   and events-deg: ¬(a≠b ∧ b≠c ∧ c≠d ∧ a≠d ∧ a≠c ∧ b≠d)
 shows is-interval (I ∩ J)
proof −

 let ?p = λ I J. (is-interval (I ∩ J) ∨ I∩J = {})

 have symmetry: ⋀I J. ⟦is-interval I; is-interval J; ?p I J⟧ ⟹ ?p J I
  by (simp add: inf-commute)

 have degen-cases: ⋀I J a b c d Q. ⟦I = interval a b; J = interval c d; I⊆Q;
J⊆Q; Q∈𝒫⟧
       ⟹ ((a=b ∧ b=c ∧ c=d) ⟶ ?p I J) ∧ ((a=b ∧ b≠c ∧ c=d) ⟶ ?p I J)
        ∧ ((a=b ∧ b=c ∧ c≠d) ⟶ ?p I J) ∧ ((a=b ∧ b≠c ∧ c≠d ∧ a≠d) ⟶
?p I J)
        ∧ ((a≠b ∧ b=c ∧ c≠d ∧ a=d) ⟶ ?p I J)
        ∧ (([a;b;c] ∧ a=d) ⟶ ?p I J) ∧ (([b;a;c] ∧ a=d) ⟶ ?p I J)
 proof −
  fix I J a b c d Q
  assume I = interval a b J = interval c d I⊆Q J⊆Q Q∈𝒫
```

```
  show ((a=b ∧ b=c ∧ c=d) ⟶ ?p I J) ∧ ((a=b ∧ b≠c ∧ c=d) ⟶ ?p I J)
        ∧ ((a=b ∧ b=c ∧ c≠d) ⟶ ?p I J) ∧ ((a=b ∧ b≠c ∧ c≠d ∧ a≠d) ⟶
?p I J)
         ∧ ((a≠b ∧ b=c ∧ c≠d ∧ a=d) ⟶ ?p I J)
         ∧ (([a;b;c] ∧ a=d) ⟶ ?p I J) ∧ (([b;a;c] ∧ a=d) ⟶ ?p I J)
  proof (intro conjI impI)
    assume a = b ∧ b = c ∧ c = d thus ?p I J
      using ‹I = interval a b› ‹J = interval c d› by auto
  next
    assume a = b ∧ b ≠ c ∧ c = d thus ?p I J
      using ‹J = interval c d› empty-segment interval-def by auto
  next
    assume a = b ∧ b = c ∧ c ≠ d thus ?p I J
      using ‹I = interval a b› empty-segment interval-def by auto
  next
    assume a = b ∧ b ≠ c ∧ c ≠ d ∧ a ≠ d thus ?p I J
      using ‹I = interval a b› empty-segment interval-def by auto
  next
    assume a ≠ b ∧ b = c ∧ c ≠ d ∧ a = d thus ?p I J
      using ‹I = interval a b› ‹J = interval c d› int-sym by auto
  next
    assume [a;b;c] ∧ a = d show ?p I J
    proof (cases)
      assume I∩J = {} thus ?thesis by simp
    next
      assume I∩J ≠ {}
      have I∩J = interval a b
      proof (safe)
        fix x assume x∈I x∈J
        thus x∈interval a b
          using ‹I = interval a b› by blast
      next
        fix x assume x∈interval a b
        show x∈I
          by (simp add: ‹I = interval a b› ‹x ∈ interval a b›)
        have [d;b;c]
          using ‹[a;b;c] ∧ a = d› by blast
        have [a;x;b] ∨ x=a ∨ x=b
          using ‹I = interval a b› ‹x ∈ I› interval-def seg-betw by auto
        consider [d;x;c]|x=a ∨ x=b
          using ‹[a;b;c] ∧ a = d› ‹[a;x;b] ∨ x = a ∨ x = b› abc-acd-abd by blast
        thus x∈J
        proof (cases)
          case 1
          then show ?thesis
              by (simp add: ‹J = interval c d› abc-abc-neq abc-sym interval-def
seg-betw)
        next
          case 2
```

```
      then have x ∈ interval c d
        using ‹[a;b;c] ∧ a = d› int-sym interval-def seg-betw
        by force
      then show ?thesis
        using ‹J = interval c d› by blast
    qed
  qed
  thus ?p I J by blast
 qed
next
 assume [b;a;c] ∧ a = d show ?p I J
 proof (cases)
  assume I∩J = {} thus ?thesis by simp
 next
  assume I∩J ≠ {}
  have I∩J = {a}
  proof (safe)
   fix x assume x∈I x∈J x∉{}
   have cxd: [c;x;d] ∨ x=c ∨ x=d
    using ‹J = interval c d› ‹x ∈ J› interval-def seg-betw by auto
   consider [a;x;b]|x=a|x=b
    using ‹I = interval a b› ‹x ∈ I› interval-def seg-betw by auto
   then show x=a
   proof (cases)
    assume [a;x;b]
    hence [b;x;d;c]
     using ‹[b;a;c] ∧ a = d› abc-acd-bcd abc-sym by meson
    hence False
     using cxd abc-abc-neq by blast
    thus ?thesis by simp
   next
    assume x=b
    hence [b;d;c]
     using ‹[b;a;c] ∧ a = d› by blast
    hence False
     using cxd ‹x = b› abc-abc-neq by blast
    thus ?thesis
     by simp
   next
    assume x=a thus x=a by simp
   qed
  next
   show a∈I
    by (simp add: ‹I = interval a b› ends-in-int)
   show a∈J
    by (simp add: ‹J = interval c d› ‹[b;a;c] ∧ a = d› ends-in-int)
  qed
  thus ?p I J
   by (simp add: empty-segment interval-def)
```

```
    qed
  qed
 qed

 have ?p I J
  using wlog-interval-endpoints-degenerate
    [where P=?p and I=I and J=J and a=a and b=b and c=c and d=d
and Q=P]
  using degen-cases
  using symmetry assms
  by smt

 thus ?thesis
  using assms(3) by blast
qed


lemma int-of-ints-is-interval:
 assumes is-interval I is-interval J I⊆P J⊆P P∈𝒫 I∩J ≠ {}
 shows is-interval (I ∩ J)
 using int-of-ints-is-interval-neq int-of-ints-is-interval-deg
 by (meson assms)


lemma int-of-ints-is-interval2:
 assumes ∀x∈S. (is-interval x ∧ x⊆P) P∈𝒫 ⋂S ≠ {} finite S S≠{}
 shows is-interval (⋂S)
proof −
 obtain n where n = card S
  by simp
 consider n=0|n=1|n≥2
  by linarith
 thus ?thesis
 proof (cases)
  assume n=0
  then have False
   using ‹n = card S› assms(4,5) by simp
  thus ?thesis
   by simp
 next
  assume n=1
  then obtain I where S = {I}
   using ‹n = card S› card-1-singletonE by auto
  then have ⋂S = I
   by simp
  moreover have is-interval I
   by (simp add: ‹S = {I}› assms(1))
  ultimately show ?thesis
   by blast
```

```
  next
    assume 2≤n
    obtain m where m+2=n
      using ‹2 ≤ n› le-add-diff-inverse2 by blast
    have ind: ⋀S. ⟦∀x∈S. (is-interval x ∧ x⊆P); P∈𝒫; ⋂S ≠ {}; finite S; S≠{};
m+2=card S⟧
      ⟹ is-interval (⋂S)
    proof (induct m)
      case 0
      then have card S = 2
        by auto
      then obtain I J where S={I,J} I≠J
        by (meson card-2-iff)
      then have I∈S J∈S
        by blast+
      then have is-interval I is-interval J I⊆P J⊆P
          by (simp add: 0.prems(1))+
      also have I∩J ≠ {}
        using ‹S={I,J}› 0.prems(3) by force
      then have is-interval(I∩J)
       using assms(2) calculation int-of-ints-is-interval[where I=I and J=J and
P=P]
        by fastforce
      then show ?case
        by (simp add: ‹S = {I, J}›)
    next
      case (Suc m)
      obtain S' I where I∈S S = insert I S' I∉S'
        using Suc.prems(4,5) by (metis Set.set-insert finite.simps insertI1)
      then have is-interval (⋂S')
      proof −
        have m+2 = card S'
          using Suc.prems(4,6) ‹S = insert I S'› ‹I∉S'› by auto
        moreover have ∀x∈S'. is-interval x ∧ x ⊆ P
          by (simp add: Suc.prems(1) ‹S = insert I S'›)
        moreover have ⋂ S' ≠ {}
          using Suc.prems(3) ‹S = insert I S'› by auto
        moreover have finite S'
          using Suc.prems(4) ‹S = insert I S'› by auto
        ultimately show ?thesis
          using assms(2) Suc(1) [where S=S'] by fastforce
      qed
      then have is-interval ((⋂S')∩I)
      proof (rule int-of-ints-is-interval)
        show is-interval I
          by (simp add: Suc.prems(1) ‹I ∈ S›)
        show ⋂S' ⊆ P
          using ‹I ∉ S'› ‹S = insert I S'› Suc.prems(1,4,6) Inter-subset
          by (metis Suc-n-not-le-n card.empty card-insert-disjoint finite-insert
```

le-add2 numeral-2-eq-2 subset-eq subset-insertI)
show $I \subseteq P$
by (*simp add*: *Suc.prems*(*1*) ‹$I \in S$›)
show $P \in \mathcal{P}$
using *assms*(*2*) **by** *auto*
show $\bigcap S' \cap I \neq \{\}$
using *Suc.prems*(*3*) ‹$S = insert\ I\ S'$› **by** *auto*
qed
thus *?case*
using ‹$S = insert\ I\ S'$› **by** (*simp add*: *inf.commute*)
qed
then show *?thesis*
using ‹$m + 2 = n$› ‹$n = card\ S$› *assms* **by** *blast*
qed
qed

end

34 3.7 Continuity and the monotonic sequence property

context *MinkowskiSpacetime* **begin**

This section only includes a proof of the first part of Theorem 12, as well as some results that would be useful in proving part (ii).

theorem *two-rays*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *event-a*: $a \in Q$
shows $\exists R\ L.$ (*is-ray-on R Q* $\wedge$ *is-ray-on L Q*
$\wedge\ Q - \{a\} \subseteq (R \cup L)$ *events of Q except a belong to two rays*
$\wedge\ (\forall r \in R.\ \forall l \in L.\ [l;a;r])$ *a is between any l of one ray and any r a of the other*
$\wedge\ (\forall x \in R.\ \forall y \in R.\ \neg\ [x;a;y])$ *but a is not between any two events*
$\wedge\ (\forall x \in L.\ \forall y \in L.\ \neg\ [x;a;y]))$ *on the same ray*
proof −

Schutz here uses Theorem 6, but we don't need it.

obtain b **where** $b \in \mathcal{E}$ **and** $b \in Q$ **and** $b \neq a$
using *event-a ge2-events in-path-event path-Q* **by** *blast*
let *?L* = $\{x.\ [x;a;b]\}$
let *?R* = $\{y.\ [a;y;b] \vee [a;b;y]\!]\}$
have Q = *?L* $\cup$ $\{a\}$ $\cup$ *?R*
proof −
have *inQ*: $\forall x \in Q.\ [x;a;b] \vee x = a \vee [a;x;b] \vee [a;b;x]\!]$
by (*meson* ‹$b \in Q$› ‹$b \neq a$› *abc-sym event-a path-Q some-betw*)
show *?thesis*
proof (*safe*)

fix x
assume $x \in Q\ x \neq a \neg [x;a;b] \neg [a;x;b]\ b \neq x$
then show $[a;b;x]$
using *inQ* **by** *blast*
next
fix x
assume $[x;a;b]$
then show $x \in Q$
by (*simp add*: ‹$b \in Q$› *abc-abc-neq betw-a-in-path event-a path-Q*)
next
show $a \in Q$
by (*simp add*: *event-a*)
next
fix x
assume $[a;x;b]$
then show $x \in Q$
by (*simp add*: ‹$b \in Q$› *abc-abc-neq betw-b-in-path event-a path-Q*)
next
fix x
assume $[a;b;x]$
then show $x \in Q$
by (*simp add*: ‹$b \in Q$› *abc-abc-neq betw-c-in-path event-a path-Q*)
next
show $b \in Q$ **using** ‹$b \in Q$› **.**
qed
qed
have *disjointLR*: $?L \cap ?R = \{\}$
using *abc-abc-neq abc-only-cba* **by** *blast*

have *wxyz-ord*: $[x;a;y;b]\!] \vee [x;a;b;y]\!]$
$\wedge$ (([*w*;*x*;*a*] $\wedge$ [*x*;*a*;*y*]) $\vee$ ([*x*;*w*;*a*] $\wedge$ [*w*;*a*;*y*]))
$\wedge$ (([*x*;*a*;*y*] $\wedge$ [*a*;*y*;*z*]) $\vee$ ([*x*;*a*;*z*] $\wedge$ [*a*;*z*;*y*]))
if $x \in ?L\ w \in ?L\ y \in ?R\ z \in ?R\ w \neq x\ y \neq z$ **for** $x\ w\ y\ z$
using *path-finsubset-chain order-finite-chain*
by (*smt abc-abd-bcdbdc abc-bcd-abd abc-sym abd-bcd-abc mem-Collect-eq that*)

obtain $x\ y$ **where** $x \in ?L\ y \in ?R$
by (*metis* (*mono-tags*) ‹$b \in Q$› ‹$b \neq a$› *abc-sym event-a mem-Collect-eq path-Q prolong-betw2*)
obtain w **where** $w \in ?L\ w \neq x$
by (*metis* ‹$b \in Q$› ‹$b \neq a$› *abc-sym event-a mem-Collect-eq path-Q prolong-betw3*)

obtain z **where** $z \in ?R\ y \neq z$
by (*metis* (*mono-tags*) ‹$b \in Q$› ‹$b \neq a$› *event-a mem-Collect-eq path-Q prolong-betw3*)

have *is-ray-on* $?R\ Q \wedge$
is-ray-on $?L\ Q \wedge$
$Q - \{a\} \subseteq ?R \cup ?L \wedge$

```
        (∀ r∈?R. ∀ l∈?L. [l;a;r]) ∧
        (∀ x∈?R. ∀ y∈?R. ¬ [x;a;y]) ∧
        (∀ x∈?L. ∀ y∈?L. ¬ [x;a;y])
proof (intro conjI)
  show is-ray-on ?L Q
  proof (unfold is-ray-on-def, safe)
    show Q ∈ 𝒫
      by (simp add: path-Q)
  next
    fix x
    assume [x;a;b]
    then show x ∈ Q
      using ‹b ∈ Q› ‹b ≠ a› betw-a-in-path event-a path-Q by blast
  next
    show is-ray {x. [x;a;b]}
  proof −
    have [x;a;b]
      using ‹x∈?L› by simp
    have ?L = ray a x
    proof
      show ray a x ⊆ ?L
      proof
        fix e assume e∈ray a x
        show e∈?L
          using wxyz-ord ray-cases abc-bcd-abd abd-bcd-abc abc-sym
          by (metis ‹[x;a;b]› ‹e ∈ ray a x› mem-Collect-eq)
      qed
      show ?L ⊆ ray a x
      proof
        fix e assume e∈?L
        hence [e;a;b]
          by simp
        show e∈ray a x
        proof (cases)
          assume e=x
          thus ?thesis
            by (simp add: ray-def)
        next
          assume e≠x
          hence [e;x;a] ∨ [x;e;a] using wxyz-ord
            by (meson ‹[e;a;b]› ‹[x;a;b]› abc-abd-bcdbdc abc-sym)
          thus e∈ray a x
            by (metis Un-iff abc-sym insertCI pro-betw ray-def seg-betw)
        qed
      qed
    qed
    thus is-ray ?L by auto
  qed
qed
```

show *is-ray-on ?R Q*
proof (*unfold is-ray-on-def*, *safe*)
show $Q \in \mathcal{P}$
by (*simp add*: *path-Q*)
next
fix x
assume $[a;x;b]$
then show $x \in Q$
by (*simp add*: ‹$b \in Q$› *abc-abc-neq betw-b-in-path event-a path-Q*)
next
fix x
assume $[a;b;x]$
then show $x \in Q$
by (*simp add*: ‹$b \in Q$› *abc-abc-neq betw-c-in-path event-a path-Q*)
next
show $b \in Q$ **using** ‹$b \in Q$› **.**
next
show *is-ray* $\{y.\ [a;y;b] \vee [a;b;y]\!]\}$
proof −
have $[a;y;b] \vee [a;b;y] \vee y{=}b$
using ‹$y{\in}?R$› **by** *blast*
have $?R = ray\ a\ y$
proof
show $ray\ a\ y \subseteq ?R$
proof
fix e **assume** $e{\in}ray\ a\ y$
hence $[a;e;y] \vee [a;y;e] \vee y{=}e$
using *ray-cases* **by** *auto*
show $e{\in}?R$
proof −
{ assume $e \neq b$
have $(e \neq y \wedge e \neq b) \wedge [w;a;y] \vee [a;e;b] \vee [a;b;e]\!]$
using ‹$[a;y;b] \vee [a;b;y] \vee y = b$› ‹$w \in \{x.\ [x;a;b]\}$› *abd-bcd-abc* **by** *blast*
hence $[a;e;b] \vee [a;b;e]\!]$
using *abc-abd-bcdbdc abc-bcd-abd abd-bcd-abc*
by (*metis* ‹$[a;e;y] \vee [a;y;e]\!]$› ‹$w \in ?L$› *mem-Collect-eq*)
}
thus *?thesis*
by *blast*
qed
qed
show $?R \subseteq ray\ a\ y$
proof
fix e **assume** $e{\in}?R$
hence *aeb-cases*: $[a;e;b] \vee [a;b;e] \vee e{=}b$
by *blast*
hence *aey-cases*: $[a;e;y] \vee [a;y;e] \vee e{=}y$
using *abc-abd-bcdbdc abc-bcd-abd abd-bcd-abc*

```
          by (metis ‹[a;y;b] ∨ [a;b;y] ∨ y = b› ‹x ∈ {x. [x;a;b]}› mem-Collect-eq)
        show e∈ray a y
        proof −
          {
            assume e=b
            hence ?thesis
             using ‹[a;y;b] ∨ [a;b;y] ∨ y = b› ‹b ≠ a› pro-betw ray-def seg-betw by
auto
          } moreover {
            assume [a;e;b] ∨ [a;b;e]
            assume y≠e
            hence [a;e;y] ∨ [a;y;e]
              using aey-cases by auto
            hence e∈ray a y
              unfolding ray-def using abc-abc-neq pro-betw seg-betw by auto
          } moreover {
            assume [a;e;b] ∨ [a;b;e]
            assume y=e
            have e∈ray a y
              unfolding ray-def by (simp add: ‹y = e›)
          }
          ultimately show ?thesis
            using aeb-cases by blast
        qed
       qed
     qed
     thus is-ray ?R by auto
   qed
  qed
    show (∀ r∈?R. ∀ l∈?L. [l;a;r])
      using abd-bcd-abc by blast
    show ∀ x∈?R. ∀ y∈?R. ¬ [x;a;y]
      by (smt abc-ac-neq abc-bcd-abd abd-bcd-abc mem-Collect-eq)
    show ∀ x∈?L. ∀ y∈?L. ¬ [x;a;y]
      using abc-abc-neq abc-abd-bcdbdc abc-only-cba by blast
    show Q−{a} ⊆ ?R ∪ ?L
      using ‹Q = {x. [x;a;b]} ∪ {a} ∪ {y. [a;y;b] ∨ [a;b;y]}› by blast
  qed
  thus ?thesis
    by (metis (mono-tags, lifting))
qed
```

The definition *closest-to* in prose: Pick any $r \in R$. The closest event c is such that there is no closer event in L, i.e. all other events of L are further away from r. Thus in L, c is the element closest to R.

```
definition closest-to :: ('a set) ⇒ 'a ⇒ ('a set) ⇒ bool
  where closest-to L c R ≡ c∈L ∧ (∀ r∈R. ∀ l∈L−{c}. [l;c;r])
```

lemma *int-on-path*:
assumes $l{\in}L$ $r{\in}R$ $Q{\in}\mathcal{P}$
and *partition*: $L{\subseteq}Q$ $L{\neq}\{\}$ $R{\subseteq}Q$ $R{\neq}\{\}$ $L{\cup}R{=}Q$
shows *interval* l r $\subseteq$ Q
proof
fix x **assume** $x{\in}$*interval* l r
thus $x{\in}Q$
unfolding *interval-def segment-def*
using *betw-b-in-path partition*(5) ‹$Q{\in}\mathcal{P}$› *seg-betw* ‹$l \in L$› ‹$r \in R$›
by *blast*
qed

lemma *ray-of-bounds1*:
assumes $Q{\in}\mathcal{P}$ $[f{\rightsquigarrow}X|(f\ 0)..]$ $X{\subseteq}Q$ *closest-bound* c X *is-bound-f* b X f $b{\neq}c$
assumes *is-bound-f* x X f
shows $x{=}b \vee x{=}c \vee [c;x;b] \vee [c;b;x]$
proof −
have $x{\in}Q$
using *bound-on-path assms*(1,3,7) **unfolding** *all-bounds-def is-bound-def is-bound-f-def*
by *auto*
{
assume $x{=}b$
hence *?thesis* **by** *blast*
} moreover {
assume $x{=}c$
hence *?thesis* **by** *blast*
} moreover {
assume $x{\neq}b$ $x{\neq}c$
hence *?thesis*
by (*meson abc-abd-bcdbdc assms*(4,5,6,7) *closest-bound-def is-bound-def*)
}
ultimately show *?thesis* **by** *blast*
qed

lemma *ray-of-bounds2*:
assumes $Q{\in}\mathcal{P}$ $[f{\rightsquigarrow}X|(f\ 0)..]$ $X{\subseteq}Q$ *closest-bound-f* c X f *is-bound-f* b X f $b{\neq}c$
assumes $x{=}b \vee x{=}c \vee [c;x;b] \vee [c;b;x]$
shows *is-bound-f* x X f
proof −
have $x{\in}Q$
using *assms*(1,3,4,5,6,7) *betw-b-in-path betw-c-in-path bound-on-path*
using *closest-bound-f-def is-bound-f-def* **by** *metis*
{
assume $x{=}b$
hence *?thesis*
by (*simp add*: *assms*(5))
} moreover {

```
    assume x=c
    hence ?thesis using assms(4)
      by (simp add: closest-bound-f-def)
  } moreover {
    assume [c;x;b]
    hence ?thesis unfolding is-bound-f-def
    proof (safe)
      fix i j::nat
      show [f⇝X|f 0..]
        by (simp add: assms(2))
      assume i<j
      hence [f i; f j; b]
        using assms(5) is-bound-f-def by blast
      hence [f j; b; c] ∨ [f j; c; b]
        using ‹i < j› abc-abd-bcdbdc assms(4,6) closest-bound-f-def is-bound-f-def
by auto
      thus [f i; f j; x]
        by (meson ‹[c;x;b]› ‹[f i; f j; b]› abc-bcd-acd abc-sym abd-bcd-abc)
    qed
  } moreover {
    assume [c;b;x]
    hence ?thesis unfolding is-bound-f-def
    proof (safe)
      fix i j::nat
      show [f⇝X|f 0..]
        by (simp add: assms(2))
      assume i<j
      hence [f i; f j; b]
        using assms(5) is-bound-f-def by blast
      hence [f j; b; c] ∨ [f j; c; b]
        using ‹i < j› abc-abd-bcdbdc assms(4,6) closest-bound-f-def is-bound-f-def
by auto
      thus [f i; f j; x]
      proof −
        have (c = b) ∨ [f 0; c; b]
          using assms(4,5) closest-bound-f-def is-bound-def by auto
        hence [f j; b; c] ⟶ [x; f j; f i]
          by (metis abc-bcd-acd abc-only-cba(2) assms(5) is-bound-f-def neq0-conv)
        thus ?thesis
          using ‹[c;b;x]› ‹[f i; f j; b]› ‹[f j; b; c] ∨ [f j; c; b]› abc-bcd-acd abc-sym
          by blast
      qed
    qed
  }
  ultimately show ?thesis using assms(7) by blast
qed


lemma ray-of-bounds3:
```

assumes $Q{\in}\mathcal{P}$ $[f{\rightsquigarrow}X|(f\ 0)..]$ $X{\subseteq}Q$ *closest-bound-f c X f is-bound-f b X f* $b{\neq}c$
shows *all-bounds X = insert c* (*ray c b*)
proof
let *?B = all-bounds X*
let *?C = insert c* (*ray c b*)
show *?B* $\subseteq$ *?C*
proof
fix *x* **assume** $x{\in}$*?B*
hence *is-bound x X*
by (*simp add*: *all-bounds-def*)
hence $x{=}b \vee x{=}c \vee [c;x;b] \vee [c;b;x]$
using *ray-of-bounds1 abc-abd-bcdbdc assms*(*4*,*5*,*6*)
by (*meson closest-bound-f-def is-bound-def*)
thus $x{\in}$*?C*
using *pro-betw ray-def seg-betw* **by** *auto*
qed
show *?C* $\subseteq$ *?B*
proof
fix *x* **assume** $x{\in}$*?C*
hence $x{=}b \vee x{=}c \vee [c;x;b] \vee [c;b;x]$
using *pro-betw ray-def seg-betw* **by** *auto*
hence *is-bound x X*
unfolding *is-bound-def* **using** *ray-of-bounds2 assms*
by *blast*
thus $x{\in}$*?B*
by (*simp add*: *all-bounds-def*)
qed
qed

lemma *int-in-closed-ray*:
assumes *path ab a b*
shows *interval a b* $\subset$ *insert a* (*ray a b*)
proof
let *?i = interval a b*
show *interval a b* $\neq$ *insert a* (*ray a b*)
proof −
obtain *c* **where** $[a;b;c]$ **using** *prolong-betw2*
using *assms* **by** *blast*
hence $c{\in}$*ray a b*
using *abc-abc-neq pro-betw ray-def* **by** *auto*
have $c{\notin}$*interval a b*
using ‹$[a;b;c]$› *abc-abc-neq abc-only-cba*(*2*) *interval-def seg-betw* **by** *auto*
thus *?thesis*
using ‹$c \in$ *ray a b*› **by** *blast*
qed
show *interval a b* $\subseteq$ *insert a* (*ray a b*)
using *interval-def ray-def* **by** *auto*
qed

end

35 3.8 Connectedness of the unreachable set

context *MinkowskiSpacetime* **begin**

35.1 Theorem 13 (Connectedness of the Unreachable Set)

theorem *unreach-connected*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *event-b*: $b \notin Q$ $b \in \mathcal{E}$
and *unreach*: $Q_x \in unreach{-}on\ Q\ from\ b$ $Q_z \in unreach{-}on\ Q\ from\ b$
and *xyz*: $[Q_x;\ Q_y;\ Q_z]$
shows $Q_y \in unreach{-}on\ Q\ from\ b$
proof −
have *xz*: $Q_x \neq Q_z$ **using** *abc-ac-neq xyz* **by** *blast*

First we obtain the chain from $[\![?Q \in \mathcal{P};\ ?b \in \mathcal{E} - ?Q;\ \{?Qx,\ ?Qz\} \subseteq unreach{-}on\ ?Q\ from\ ?b;\ ?Qx \neq ?Qz]\!] \Longrightarrow \exists X\ f.\ [f{\rightsquigarrow}X|?Qx\ ..\ ?Qz] \wedge (\forall i{\in}\{1..card\ X - 1\}.\ f\ i \in unreach{-}on\ ?Q\ from\ ?b \wedge (\forall Qy{\in}\mathcal{E}.\ [f\ (i - 1);Qy;f\ i] \longrightarrow Qy \in unreach{-}on\ ?Q\ from\ ?b))$.

have *in-Q*: $Q_x{\in}Q \wedge Q_y{\in}Q \wedge Q_z{\in}Q$
using *betw-b-in-path path-Q unreach(1,2) xz unreach-on-path xyz* **by** *blast*
hence *event-y*: $Q_y{\in}\mathcal{E}$
using *in-path-event path-Q* **by** *blast*

legacy: $[\![?Q \in \mathcal{P};\ ?b \notin ?Q;\ ?b \in \mathcal{E};\ ?Qx \in unreach{-}on\ ?Q\ from\ ?b;\ ?Qz \in unreach{-}on\ ?Q\ from\ ?b;\ ?Qx \neq ?Qz]\!] \Longrightarrow \exists X\ f.\ [f{\rightsquigarrow}X] \wedge f\ 0 = ?Qx \wedge f\ (card\ X - 1) = ?Qz \wedge (\forall i{\in}\{1..card\ X - 1\}.\ f\ i \in unreach{-}on\ ?Q\ from\ ?b \wedge (\forall Qy{\in}\mathcal{E}.\ [f\ (i - 1);Qy;f\ i] \longrightarrow Qy \in unreach{-}on\ ?Q\ from\ ?b)) \wedge (short\text{-}ch\ X \longrightarrow ?Qx \in X \wedge ?Qz \in X \wedge (\forall Qy{\in}\mathcal{E}.\ [?Qx;Qy;?Qz] \longrightarrow Qy \in unreach{-}on\ ?Q\ from\ ?b))$ instead of $[\![?Q \in \mathcal{P};\ ?b \in \mathcal{E} - ?Q;\ \{?Qx,\ ?Qz\} \subseteq unreach{-}on\ ?Q\ from\ ?b;\ ?Qx \neq ?Qz]\!] \Longrightarrow \exists X\ f.\ [f{\rightsquigarrow}X|?Qx\ ..\ ?Qz] \wedge (\forall i{\in}\{1..card\ X - 1\}.\ f\ i \in unreach{-}on\ ?Q\ from\ ?b \wedge (\forall Qy{\in}\mathcal{E}.\ [f\ (i - 1);Qy;f\ i] \longrightarrow Qy \in unreach{-}on\ ?Q\ from\ ?b))$

obtain *X f* **where** *X-def*: *ch-by-ord f X* $f\ 0 = Q_x$ $f\ (card\ X - 1) = Q_z$
$(\forall i{\in}\{1\ ..\ card\ X - 1\}.\ (f\ i) \in unreach{-}on\ Q\ from\ b \wedge (\forall Qy{\in}\mathcal{E}.\ [f\ (i - 1);\ Qy;\ f\ i] \longrightarrow Qy \in unreach{-}on\ Q\ from\ b))$
$short\text{-}ch\ X \longrightarrow Q_x \in X \wedge Q_z \in X \wedge (\forall Q_y{\in}\mathcal{E}.\ [Q_x;\ Q_y;\ Q_z] \longrightarrow Q_y \in unreach{-}on\ Q\ from\ b)$
using *I6-old* [*OF assms(1–5) xz*] **by** *blast*
hence *fin-X*: *finite X*
using *xz not-less* **by** *fastforce*
obtain *N* **where** *N*=*card X* $N{\geq}2$
using *X-def(2,3) xz* **by** *fastforce*

Then we have to manually show the bounds, defined via indices only, are in the obtained chain.

let *?a = f 0*
let *?d = f (card X − 1)*
{
assume *card X = 2*
hence *short-ch X ?a ∈ X ∧ ?d ∈ X ?a ≠ ?d*
using *X-def ‹card X = 2› short-ch-card-2 xz* **by** *blast+*
}
hence *[f⇝X|Q_x..Q_z]*
using *chain-defs* **by** *(metis X-def(1−3) fin-X)*

Further on, we split the proof into two cases, namely the split Schutz absorbs into his non-strict *local-ordering*. Just below is the statement we use ⟦*?P ∨ ?Q*; *?P ⟹ ?R*; *?Q ⟹ ?R*⟧ ⟹ *?R* with.

have *y-cases*: *Q_y∈X ∨ Q_y∉X* **by** *blast*
have *y-int*: *Q_y∈interval Q_x Q_z*
using *interval-def seg-betw xyz* **by** *auto*
have *X-in-Q*: *X⊆Q*
using *chain-on-path-I6* [**where** *Q=Q* **and** *X=X*] *X-def event-b path-Q unreach xz ‹[f⇝X|Q_x .. Q_z]›* **by** *blast*

show *?thesis*
proof (*cases*)

We treat short chains separately. (Legacy: they used to have a separate clause in ⟦*?Q ∈ 𝒫*; *?b ∈ ℰ − ?Q*; {*?Qx*, *?Qz*} ⊆ *unreach−on ?Q from ?b*; *?Qx ≠ ?Qz*⟧ ⟹ ∃ *X f*. [*f⇝X|?Qx .. ?Qz*] ∧ (∀ *i*∈{*1..card X − 1*}. *f i ∈ unreach−on ?Q from ?b* ∧ (∀ *Qy*∈ℰ. [*f (i − 1);Qy;f i*] ⟶ *Qy ∈ unreach−on ?Q from ?b*)), now ⟦*?Q ∈ 𝒫*; *?b ∉ ?Q*; *?b ∈ ℰ*; *?Qx ∈ unreach−on ?Q from ?b*; *?Qz ∈ unreach−on ?Q from ?b*; *?Qx ≠ ?Qz*⟧ ⟹ ∃ *X f*. [*f⇝X*] ∧ *f 0 = ?Qx* ∧ *f (card X − 1) = ?Qz* ∧ (∀ *i*∈{*1..card X − 1*}. *f i ∈ unreach−on ?Q from ?b* ∧ (∀ *Qy*∈ℰ. [*f (i − 1);Qy;f i*] ⟶ *Qy ∈ unreach−on ?Q from ?b*)) ∧ (*short-ch X* ⟶ *?Qx ∈ X* ∧ *?Qz ∈ X* ∧ (∀ *Qy*∈ℰ. [*?Qx;Qy;?Qz*] ⟶ *Qy ∈ unreach−on ?Q from ?b*)))

assume *N=2*
thus *?thesis*
using *X-def(1,5) xyz ‹N = card X› event-y short-ch-card-2* **by** *auto*
next

This is where Schutz obtains the chain from Theorem 11. We instead use the chain we already have with only a part of Theorem 11, namely [*?f⇝?Q|?a..?b..?c*] ⟹ *interval ?a ?c* = ⋃ {*segment (?f i) (?f (i + 1)) |i. i < card ?Q − 1*} ∪ *?Q*. *?S* is defined like in ⟦*?P ∈ 𝒫*; *2 ≤ card ?Q*; *?Q ⊆ ?P*⟧ ⟹ ∃ *S P1 P2*. *?P* = ⋃ *S* ∪ *P1* ∪ *P2* ∪ *?Q* ∧ *disjoint* (*S* ∪ {*P1*, *P2*}) ∧ *P1 ≠ P2* ∧ *P1 ∉ S* ∧ *P2 ∉ S* ∧ (∀ *x*∈*S. is-segment x*) ∧ *is-prolongation P1* ∧ *is-prolongation P2*.

assume $N{\neq}2$
hence $N{\geq}3$ **using** ‹$2 \leq N$› **by** *auto*
have $2{\leq}card\ X$ **using** ‹$2 \leq N$› ‹$N = card\ X$› **by** *blast*
show *?thesis* **using** *y-cases*
proof (*rule disjE*)
assume $Q_y{\in}X$
then obtain i **where** *i-def*: $i{<}card\ X\ Q_y = f\ i$
using *X-def*(*1*) **by** (*metis fin-X obtain-index-fin-chain*)
have $i{\neq}0 \wedge i{\neq}card\ X - 1$
using *X-def*(*2,3*)
by (*metis abc-abc-neq i-def*(*2*) *xyz*)
hence $i{\in}\{1..card\ X - 1\}$
using *i-def*(*1*) **by** *fastforce*
thus *?thesis* **using** *X-def*(*4*) *i-def*(*2*) **by** *metis*
next
assume $Q_y{\notin}X$

let *?S* = *if card X = 2 then* {*segment ?a ?d*} *else* {*segment* (*f i*) (*f*(*i*+*1*)) | *i*. *i*<*card X* − *1*}

have $Q_y{\in}\bigcup$ *?S*
proof −
obtain *c* **where** $[f{\rightsquigarrow}X|Q_x..c..Q_z]$
using *X-def*(*1*) ‹*N = card X*› ‹$N{\neq}2$› ‹$[f{\rightsquigarrow}X|Q_x..Q_z]$› *short-ch-card-2*
by (*metis* ‹$2 \leq N$› *le-neq-implies-less long-chain-2-imp-3*)
have *interval* $Q_x\ Q_z = \bigcup$ *?S* ∪ *X*
using *int-split-to-segs* [*OF* ‹$[f{\rightsquigarrow}X|Q_x..c..Q_z]$›] **by** *auto*
thus *?thesis*
using ‹$Q_y{\notin}X$› *y-int* **by** *blast*
qed
then obtain *s* **where** $s{\in}$*?S* $Q_y{\in}s$ **by** *blast*

have $\exists\, i.\ i{\in}\{1..(card\ X){-}1\} \wedge [(f(i{-}1));\ Q_y;\ f\ i]$
proof −
obtain i' **where** *i'-def*: $i' < N{-}1\ s = segment\ (f\ i')\ (f\ (i' + 1))$
using ‹$Q_y{\in}s$› ‹$s{\in}$*?S*› ‹$N{=}card\ X$›
by (*smt* ‹$2 \leq N$› ‹$N \neq 2$› *le-antisym mem-Collect-eq not-less*)
show *?thesis*
proof (*rule exI*, *rule conjI*)
show (i'+*1*) ∈ {*1..card X* − *1*}
using *i'-def*(*1*)
by (*simp add*: ‹*N = card X*›)
show [*f*((*i'*+*1*) − *1*); *Q*$_y$; *f*(*i'*+*1*)]
using *i'-def*(*2*) ‹$Q_y{\in}s$› *seg-betw* **by** *simp*
qed
qed
then obtain *i* **where** *i-def*: $i{\in}\{1..(card\ X){-}1\}$ $[(f(i{-}1));\ Q_y;\ f\ i]$
by *blast*

show *?thesis*
by (*meson X-def(4) i-def event-y*)
qed
qed
qed

35.2 Theorem 14 (Second Existence Theorem)

lemma *union-of-bounded-sets-is-bounded*:
assumes $\forall\, x{\in}A.\ [a;x;b]\ \forall\, x{\in}B.\ [c;x;d]\ A{\subseteq}Q\ B{\subseteq}Q\ Q{\in}\mathcal{P}$
card A > 1 ∨ infinite A card B > 1 ∨ infinite B
shows $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}A{\cup}B.\ [l;x;u]$
proof −
let *?P* = $\lambda\ A\ B.\ \exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}A{\cup}B.\ [l;x;u]$
let *?I* = $\lambda\ A\ a\ b.$ (*card A > 1 ∨ infinite A*) ∧ $(\forall\, x{\in}A.\ [a;x;b])$
let *?R* = $\lambda A.\ \exists\, a\ b.$ *?I A a b*

have *on-path*: $\bigwedge a\ b\ A.\ A \subseteq Q \Longrightarrow$ *?I A a b* $\Longrightarrow b \in Q \wedge a \in Q$
proof −
fix *a b A* **assume** $A{\subseteq}Q$ *?I A a b*
show $b{\in}Q{\wedge}a{\in}Q$
proof (*cases*)
assume *card A* $\leq$ *1* ∧ *finite A*
thus *?thesis*
using ‹*?I A a b*› **by** *auto*
next
assume ¬ (*card A* $\leq$ *1* ∧ *finite A*)
hence *asmA*: *card A > 1 ∨ infinite A*
by *linarith*
then obtain *x y* **where** $x{\in}A\ y{\in}A\ x{\neq}y$
proof
assume *1 < card A* $\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ x \neq y]\!] \Longrightarrow$ *thesis*
then show *?thesis*
by (*metis One-nat-def Suc-le-eq card-le-Suc-iff insert-iff*)
next
assume *infinite A* $\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ x \neq y]\!] \Longrightarrow$ *thesis*
then show *?thesis*
using *infinite-imp-nonempty* **by** (*metis finite-insert finite-subset singletonI subsetI*)
qed
have $x{\in}Q\ y{\in}Q$
using ‹$A \subseteq Q$› ‹$x \in A$› ‹$y \in A$› **by** *auto*
have [*a;x;b*] [*a;y;b*]
by (*simp add*: ‹(*1 < card A ∨ infinite A*) ∧ $(\forall\, x{\in}A.\ [a;x;b])$› ‹$x \in A$› ‹$y \in A$›)+
hence *betw4 a x y b ∨ betw4 a y x b*
using ‹$x \neq y$› *abd-acd-abcdacbd* **by** *blast*
hence $a{\in}Q \wedge b{\in}Q$
using ‹$Q{\in}\mathcal{P}$› ‹$x{\in}Q$› ‹$x{\neq}y$› ‹$x{\in}Q$› ‹$y{\in}Q$› *betw-a-in-path betw-c-in-path* **by**

blast
thus *?thesis* **by** *simp*
qed
qed

show *?thesis*
proof (*cases*)
assume $a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d$
show *?P A B*
proof (*rule-tac P=?P* **and** *A=Q* **in** *wlog-endpoints-distinct*)

First, some technicalities: the relations P, I, R have the symmetry required.

show $\bigwedge a\ b\ I.$ *?I I a b* $\Longrightarrow$ *?I I b a* **using** *abc-sym* **by** *blast*
show $\bigwedge a\ b\ A.\ A \subseteq Q \Longrightarrow$ *?I A a b* $\Longrightarrow b \in Q \wedge a \in Q$ **using** *on-path assms*(*5*) **by** *blast*
show $\bigwedge I\ J.$ *?R I* $\Longrightarrow$ *?R J* $\Longrightarrow$ *?P I J* $\Longrightarrow$ *?P J I* **by** (*simp add*: *Un-commute*)

Next, the lemma/case assumptions have to be repeated for Isabelle.

show *?I A a b ?I B c d* $A{\subseteq}Q\ B{\subseteq}Q\ Q{\in}\mathcal{P}$
using *assms* **by** *simp+*
show $a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d$
using ‹$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d$› **by** *simp*

Finally, the important bit: proofs for the necessary cases of betweenness.

show *?P I J*
if *?I I a b ?I J c d* $I{\subseteq}Q\ J{\subseteq}Q$
and $[a;b;c;d] \vee [a;c;b;d] \vee [a;c;d;b]$
for *I J a b c d*
proof −
consider $[a;b;c;d]|[a;c;b;d]|[a;c;d;b]$
using ‹$[a;b;c;d] \vee [a;c;b;d] \vee [a;c;d;b]$› **by** *fastforce*
thus *?thesis*
proof (*cases*)
assume *asm*: $[a;b;c;d]$ **show** *?P I J*
proof −
have $\forall\, x{\in}\ I{\cup}J.\ [a;x;d]$
by (*metis Un-iff asm betw4-strong betw4-weak that*(*1*) *that*(*2*))
moreover have $a{\in}Q\ d{\in}Q$
using *assms*(*5*) *on-path that*(*1−4*) **by** *blast+*
ultimately show *?thesis* **by** *blast*
qed
next
assume $[a;c;b;d]$ **show** *?P I J*
proof −
have $\forall\, x{\in}\ I{\cup}J.\ [a;x;d]$
by (*metis Un-iff* ‹*betw4 a c b d*› *abc-bcd-abd abc-bcd-acd betw4-weak that*(*1,2*))
moreover have $a{\in}Q\ d{\in}Q$
using *assms*(*5*) *on-path that*(*1−4*) **by** *blast+*

```
        ultimately show ?thesis by blast
      qed
    next
      assume [a;c;d;b] show ?P I J
      proof −
        have ∀ x∈ I∪J. [a;x;b]
          using ‹betw4 a c d b› abc-bcd-abd abc-bcd-acd abe-ade-bcd-ace
          by (meson UnE that(1,2))
        moreover have a∈Q b∈Q
          using assms(5) on-path that(1−4) by blast+
        ultimately show ?thesis by blast
      qed
    qed
  qed
 next
  assume ¬(a≠b ∧ a≠c ∧ a≠d ∧ b≠c ∧ b≠d ∧ c≠d)

  show ?P A B
  proof (rule-tac P=?P and A=Q in wlog-endpoints-degenerate)
```

This case follows the same pattern as above: the next five *show* statements are effectively bookkeeping.

```
    show ⋀a b I. ?I I a b ⟹ ?I I b a using abc-sym by blast
    show ⋀a b A. A ⊆ Q ⟹ ?I A a b ⟹ b ∈ Q ∧ a ∈ Q using on-path ‹Q∈𝒫› by blast
    show ⋀I J. ?R I ⟹ ?R J ⟹ ?P I J ⟹ ?P J I by (simp add: Un-commute)

    show ?I A a b ?I B c d A⊆Q B⊆Q Q∈𝒫
      using assms by simp+
    show ¬ (a ≠ b ∧ b ≠ c ∧ c ≠ d ∧ a ≠ d ∧ a ≠ c ∧ b ≠ d)
      using ‹¬ (a ≠ b ∧ a ≠ c ∧ a ≠ d ∧ b ≠ c ∧ b ≠ d ∧ c ≠ d)› by blast
```

Again, this is the important bit: proofs for the necessary cases of degeneracy.

```
    show (a = b ∧ b = c ∧ c = d ⟶ ?P I J) ∧ (a = b ∧ b ≠ c ∧ c = d ⟶ ?P I J) ∧
        (a = b ∧ b = c ∧ c ≠ d ⟶ ?P I J) ∧ (a = b ∧ b ≠ c ∧ c ≠ d ∧ a ≠ d ⟶ ?P I J) ∧
        (a ≠ b ∧ b = c ∧ c ≠ d ∧ a = d ⟶ ?P I J) ∧
        ([a;b;c] ∧ a = d ⟶ ?P I J) ∧ ([b;a;c] ∧ a = d ⟶ ?P I J)
    if ?I I a b ?I J c d I ⊆ Q J ⊆ Q
    for I J a b c d
    proof (intro conjI impI)
      assume a = b ∧ b = c ∧ c = d
      show ∃ l∈Q. ∃ u∈Q. ∀ x∈I ∪ J. [l;x;u]
        using ‹a = b ∧ b = c ∧ c = d› abc-ac-neq assms(5) ex-crossing-path that(1,2)
        by fastforce
    next
```

assume $a = b \wedge b \neq c \wedge c = d$
show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$
using ‹$a = b \wedge b \neq c \wedge c = d$› *abc-ac-neq assms(5) ex-crossing-path that(1,2)*
by (*metis Un-iff*)
next
assume $a = b \wedge b = c \wedge c \neq d$
hence $\forall\, x{\in}\ I{\cup}J.\ [c;x;d]$
using *abc-abc-neq that(1,2)* **by** *fastforce*
moreover have $c{\in}Q\ d{\in}Q$
using *on-path* ‹$a = b \wedge b = c \wedge c \neq d$› *that(1,3) abc-abc-neq* **by** *metis+*
ultimately show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$ **by** *blast*
next
assume $a = b \wedge b \neq c \wedge c \neq d \wedge a \neq d$
hence $\forall\, x{\in}\ I{\cup}J.\ [c;x;d]$
using *abc-abc-neq that(1,2)* **by** *fastforce*
moreover have $c{\in}Q\ d{\in}Q$
using *on-path* ‹$a = b \wedge b \neq c \wedge c \neq d \wedge a \neq d$› *that(1,3) abc-abc-neq* **by** *metis+*
ultimately show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$ **by** *blast*
next
assume $a \neq b \wedge b = c \wedge c \neq d \wedge a = d$
hence $\forall\, x{\in}\ I{\cup}J.\ [c;x;d]$
using *abc-sym that(1,2)* **by** *auto*
moreover have $c{\in}Q\ d{\in}Q$
using *on-path* ‹$a \neq b \wedge b = c \wedge c \neq d \wedge a = d$› *that(1,3) abc-abc-neq* **by** *metis+*
ultimately show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$ **by** *blast*
next
assume $[a;b;c] \wedge a = d$
hence $\forall\, x{\in}\ I{\cup}J.\ [c;x;d]$
by (*metis UnE abc-acd-abd abc-sym that(1,2)*)
moreover have $c{\in}Q\ d{\in}Q$
using *on-path that(2,4)* **by** *blast+*
ultimately show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$ **by** *blast*
next
assume $[b;a;c] \wedge a = d$
hence $\forall\, x{\in}\ I{\cup}J.\ [c;x;b]$
using *abc-sym abd-bcd-abc betw4-strong that(1,2)* **by** (*metis Un-iff*)
moreover have $c{\in}Q\ b{\in}Q$
using *on-path that* **by** *blast+*
ultimately show $\exists\, l{\in}Q.\ \exists\, u{\in}Q.\ \forall\, x{\in}I \cup J.\ [l;x;u]$ **by** *blast*
qed
qed
qed
qed

lemma *union-of-bounded-sets-is-bounded2*:

assumes $\forall x{\in}A.\ [a;x;b]\ \forall x{\in}B.\ [c;x;d]\ A{\subseteq}Q\ B{\subseteq}Q\ Q{\in}\mathcal{P}$
$1{<}card\ A \vee infinite\ A\ 1{<}card\ B \vee infinite\ B$
shows $\exists l{\in}Q{-}(A{\cup}B).\ \exists u{\in}Q{-}(A{\cup}B).\ \forall x{\in}A{\cup}B.\ [l;x;u]$
using *assms union-of-bounded-sets-is-bounded*
[**where** $A{=}A$ **and** $a{=}a$ **and** $b{=}b$ **and** $B{=}B$ **and** $c{=}c$ **and** $d{=}d$ **and** $Q{=}Q$]
by (*metis Diff-iff abc-abc-neq*)

Schutz proves a mildly stronger version of this theorem than he states. Namely, he gives an additional condition that has to be fulfilled by the bounds y, z in the proof ($y,z{\notin}unreach{-}on\ Q\ from\ ab$). This condition is trivial given *abc-abc-neq*. His stating it in the proof makes me wonder whether his (strictly speaking) undefined notion of bounded set is somehow weaker than the version using strict betweenness in his theorem statement and used here in Isabelle. This would make sense, given the obvious analogy with sets on the real line.

theorem *second-existence-thm-1*:
assumes *path-Q*: $Q{\in}\mathcal{P}$
and *events*: $a{\notin}Q\ b{\notin}Q$
and *reachable*: *path-ex a q1 path-ex b q2* $q1{\in}Q\ q2{\in}Q$
shows $\exists y{\in}Q.\ \exists z{\in}Q.\ (\forall x{\in}unreach{-}on\ Q\ from\ a.\ [y;x;z]) \wedge (\forall x{\in}unreach{-}on\ Q\ from\ b.\ [y;x;z])$
proof −

Slightly annoying: Schutz implicitly extends *bounded* to sets, so his statements are neater.

have $\exists q{\in}Q.\ q{\notin}(unreach{-}on\ Q\ from\ a)\ \exists q{\in}Q.\ q{\notin}(unreach{-}on\ Q\ from\ b)$
using *cross-in-reachable reachable* **by** *blast+*

This is a helper statement for obtaining bounds in both directions of both unreachable sets. Notice this needs Theorem 13 right now, Schutz claims only Theorem 4. I think this is necessary?

have *get-bds*: $\exists la{\in}Q.\ \exists ua{\in}Q.\ la{\notin}unreach{-}on\ Q\ from\ a \wedge ua{\notin}unreach{-}on\ Q\ from\ a \wedge (\forall x{\in}unreach{-}on\ Q\ from\ a.\ [la;x;ua])$
if *asm*: $a{\notin}Q$ *path-ex a q* $q{\in}Q$
for *a q*
proof −
obtain *Qy* **where** $Qy{\in}unreach{-}on\ Q\ from\ a$
using *asm*(*2*) ‹$a \notin Q$› *in-path-event path-Q two-in-unreach* **by** *blast*
then obtain *la* **where** $la \in Q - unreach{-}on\ Q\ from\ a$
using *asm*(*2,3*) *cross-in-reachable* **by** *blast*
then obtain *ua* **where** $ua \in Q - unreach{-}on\ Q\ from\ a\ [la;Qy;ua]\ la \neq ua$
using *unreachable-set-bounded* [**where** $Q{=}Q$ **and** $b{=}a$ **and** $Qx{=}la$ **and** $Qy{=}Qy$]
using ‹$Qy \in unreach{-}on\ Q\ from\ a$› *asm in-path-event path-Q* **by** *blast*
have $la \notin unreach{-}on\ Q\ from\ a \wedge ua \notin unreach{-}on\ Q\ from\ a \wedge (\forall x{\in}unreach{-}on\ Q\ from\ a.\ (x{\neq}la \wedge x{\neq}ua) \longrightarrow [la;x;ua])$
proof (*intro conjI*)

```
    show la ∉ unreach−on Q from a
      using ‹la ∈ Q − unreach−on Q from a› by force
   next
    show ua ∉ unreach−on Q from a
      using ‹ua ∈ Q − unreach−on Q from a› by force
   next show ∀ x∈unreach−on Q from a. x ≠ la ∧ x ≠ ua ⟶ [la;x;ua]
   proof (safe)
    fix x assume x∈unreach−on Q from a x≠la x≠ua
    {
     assume x=Qy hence [la;x;ua] by (simp add: ‹[la;Qy;ua]›)
    } moreover {
     assume x≠Qy
     have [Qy;x;la] ∨ [la;Qy;x]
     proof −
      { assume [x;la;Qy]
       hence la∈unreach−on Q from a
         using unreach-connected ‹Qy∈unreach−on Q from a›‹x∈unreach−on
Q from a›‹x≠Qy› in-path-event path-Q that by blast
       hence False
        using ‹la ∈ Q − unreach−on Q from a› by blast }
      thus [Qy;x;la] ∨ [la;Qy;x]
       using some-betw [where Q=Q and a=x and b=la and c=Qy] path-Q
unreach-on-path
       using ‹Qy ∈ unreach−on Q from a› ‹la ∈ Q − unreach−on Q from a›
‹x ∈ unreach−on Q from a› ‹x ≠ Qy› ‹x ≠ la› by force
     qed
     hence [la;x;ua]
     proof
      assume [Qy;x;la]
      thus ?thesis using ‹[la;Qy;ua]› abc-acd-abd abc-sym by blast
     next
      assume [la;Qy;x]
      hence [la;x;ua] ∨ [la;ua;x]
       using ‹[la;Qy;ua]› ‹x ≠ ua› abc-abd-acdadc by auto
      have ¬[la;ua;x]
      using unreach-connected that abc-abc-neq abc-acd-bcd in-path-event path-Q
       by (metis DiffD2 ‹Qy ∈ unreach−on Q from a› ‹[la;Qy;ua]› ‹ua ∈ Q −
unreach−on Q from a› ‹x ∈ unreach−on Q from a›)
      show ?thesis
       using ‹[la;x;ua] ∨ [la;ua;x]› ‹¬ [la;ua;x]› by linarith
     qed
    }
    ultimately show [la;x;ua] by blast
   qed
  qed
   thus ?thesis using ‹la ∈ Q − unreach−on Q from a› ‹ua ∈ Q − unreach−on
Q from a› by force
  qed
```

have $\exists y{\in}Q.\ \exists z{\in}Q.\ (\forall x{\in}(\mathit{unreach{-}on}\ Q\ \mathit{from}\ a){\cup}(\mathit{unreach{-}on}\ Q\ \mathit{from}\ b).\ [y;x;z])$
proof −
obtain *la ua* **where** $\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a.\ [la;x;ua]$
using *events*(*1*) *get-bds reachable*(*1*,*3*) **by** *blast*
obtain *lb ub* **where** $\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ b.\ [lb;x;ub]$
using *events*(*2*) *get-bds reachable*(*2*,*4*) **by** *blast*
have $\mathit{unreach{-}on}\ Q\ \mathit{from}\ a \subseteq Q\ \mathit{unreach{-}on}\ Q\ \mathit{from}\ b \subseteq Q$
by (*simp add*: *subsetI unreach-on-path*)+
moreover have $1 < \mathit{card}\ (\mathit{unreach{-}on}\ Q\ \mathit{from}\ a) \vee \mathit{infinite}\ (\mathit{unreach{-}on}\ Q\ \mathit{from}\ a)$
using *two-in-unreach events*(*1*) *in-path-event path-Q reachable*(*1*)
by (*metis One-nat-def card-le-Suc0-iff-eq not-less*)
moreover have $1 < \mathit{card}\ (\mathit{unreach{-}on}\ Q\ \mathit{from}\ b) \vee \mathit{infinite}\ (\mathit{unreach{-}on}\ Q\ \mathit{from}\ b)$
using *two-in-unreach events*(*2*) *in-path-event path-Q reachable*(*2*)
by (*metis One-nat-def card-le-Suc0-iff-eq not-less*)
ultimately show *?thesis*
using *union-of-bounded-sets-is-bounded* [**where** $Q{=}Q$ **and** $A{=}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a$ **and** $B{=}\mathit{unreach{-}on}\ Q\ \mathit{from}\ b$]
using *get-bds assms* ‹$\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a.\ [la;x;ua]$› ‹$\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ b.\ [lb;x;ub]$›
by *blast*
qed

then obtain *y z* **where** $y{\in}Q\ z{\in}Q\ (\forall x{\in}(\mathit{unreach{-}on}\ Q\ \mathit{from}\ a){\cup}(\mathit{unreach{-}on}\ Q\ \mathit{from}\ b).\ [y;x;z])$
by *blast*
show *?thesis*
proof (*rule bexI*)+
show $y{\in}Q$ **by** (*simp add*: ‹$y \in Q$›)
show $z{\in}Q$ **by** (*simp add*: ‹$z \in Q$›)
show $(\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a.\ [z;x;y]) \wedge (\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ b.\ [z;x;y])$
by (*simp add*: ‹$\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a \cup \mathit{unreach{-}on}\ Q\ \mathit{from}\ b.\ [y;x;z]$› *abc-sym*)
qed
qed

theorem *second-existence-thm-2*:
assumes *path-Q*: $Q{\in}\mathcal{P}$
and *events*: $a{\notin}Q\ b{\notin}Q\ c{\in}Q\ d{\in}Q\ c{\neq}d$
and *reachable*: $\exists P{\in}\mathcal{P}.\ \exists q{\in}Q.\ \mathit{path}\ P\ a\ q\ \exists P{\in}\mathcal{P}.\ \exists q{\in}Q.\ \mathit{path}\ P\ b\ q$
shows $\exists e{\in}Q.\ \exists ae{\in}\mathcal{P}.\ \exists be{\in}\mathcal{P}.\ \mathit{path}\ ae\ a\ e \wedge \mathit{path}\ be\ b\ e \wedge [c;d;e]$
proof −
obtain *y z* **where** *bounds-yz*: $(\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ a.\ [z;x;y]) \wedge (\forall x{\in}\mathit{unreach{-}on}\ Q\ \mathit{from}\ b.\ [z;x;y])$
and *yz-inQ*: $y{\in}Q\ z{\in}Q$
using *second-existence-thm-1* [**where** $Q{=}Q$ **and** $a{=}a$ **and** $b{=}b$]

using *path-Q events(1,2) reachable* **by** *blast*
have $y \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ a) \cup (\mathit{unreach-on}\ Q\ \mathit{from}\ b)$ $z \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ a) \cup (\mathit{unreach-on}\ Q\ \mathit{from}\ b)$
by (*meson Un-iff* ‹$(\forall x \in \mathit{unreach-on}\ Q\ \mathit{from}\ a.\ [z;x;y]) \wedge (\forall x \in \mathit{unreach-on}\ Q\ \mathit{from}\ b.\ [z;x;y])$› *abc-abc-neq*)+
let *?P* = $\lambda e\ ae\ be.\ (e \in Q \wedge \mathit{path}\ ae\ a\ e \wedge \mathit{path}\ be\ b\ e \wedge [c;d;e])$

have *exist-ay*: $\exists\, ay.\ \mathit{path}\ ay\ a\ y$
if $a \notin Q$ $\exists\, P \in \mathcal{P}.\ \exists\, q \in Q.\ \mathit{path}\ P\ a\ q\ y \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ a)\ y \in Q$
for *a y*
using *in-path-event path-Q that unreachable-bounded-path-only*
by *blast*

have $[c;d;y] \vee [\![y;c;d] \vee [c;y;d]\!]$
by (*meson* ‹$y \in Q$› *abc-sym events(3−5) path-Q some-betw*)
moreover have $[c;d;z] \vee [\![z;c;d] \vee [c;z;d]\!]$
by (*meson* ‹$z \in Q$› *abc-sym events(3−5) path-Q some-betw*)
ultimately consider $[c;d;y] \mid [c;d;z] \mid ([\![y;c;d] \vee [c;y;d]\!]) \wedge ([\![z;c;d] \vee [c;z;d]\!])$
by *auto*
thus *?thesis*
proof (*cases*)
assume $[c;d;y]$
have $y \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ a)$ $y \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ b)$
using ‹$y \notin \mathit{unreach-on}\ Q\ \mathit{from}\ a \cup \mathit{unreach-on}\ Q\ \mathit{from}\ b$› **by** *blast*+
then obtain *ay yb* **where** *path ay a y path yb b y*
using ‹$y \in Q$› *exist-ay events(1,2) reachable(1,2)* **by** *blast*
have *?P y ay yb*
using ‹$[c;d;y]$› ‹*path ay a y*› ‹*path yb b y*› ‹$y \in Q$› **by** *blast*
thus *?thesis* **by** *blast*
next
assume $[c;d;z]$
have $z \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ a)$ $z \notin (\mathit{unreach-on}\ Q\ \mathit{from}\ b)$
using ‹$z \notin \mathit{unreach-on}\ Q\ \mathit{from}\ a \cup \mathit{unreach-on}\ Q\ \mathit{from}\ b$› **by** *blast*+
then obtain *az bz* **where** *path az a z path bz b z*
using ‹$z \in Q$› *exist-ay events(1,2) reachable(1,2)* **by** *blast*
have *?P z az bz*
using ‹$[c;d;z]$› ‹*path az a z*› ‹*path bz b z*› ‹$z \in Q$› **by** *blast*
thus *?thesis* **by** *blast*
next
assume $([\![y;c;d] \vee [c;y;d]\!]) \wedge ([\![z;c;d] \vee [c;z;d]\!])$
have $\exists\, e.\ [c;d;e]$
using *prolong-betw*
using *events(3−5) path-Q* **by** *blast*
then obtain *e* **where** $[c;d;e]$ **by** *auto*
have $\neg[y;e;z]$
proof (*rule notI*)

Notice Theorem 10 is not needed for this proof, and does not seem to help

sledgehammer. I think this is because it cannot be easily/automatically reconciled with non-strict notation.

```
    assume [y;e;z]
    moreover consider ([[y;c;d] ∧ [[z;c;d]) | ([[y;c;d] ∧ [c;z;d]]) |
          ([c;y;d]] ∧ [[z;c;d]) | ([c;y;d]] ∧ [c;z;d]]) |
      using ‹([[y;c;d] ∨ [c;y;d]]) ∧ ([[z;c;d] ∨ [c;z;d]])› by linarith
    ultimately show False
      by (smt ‹[c;d;e]› abc-ac-neq betw4-strong betw4-weak)
  qed
  have e∈Q
    using ‹[c;d;e]› betw-c-in-path events(3−5) path-Q by blast
  have e∉ unreach−on Q from a e∉ unreach−on Q from b
    using bounds-yz ‹¬ [y;e;z]› abc-sym by blast+
  hence ex-aebe: ∃ ae be. path ae a e ∧ path be b e
      using ‹e ∈ Q› events(1,2) in-path-event path-Q reachable(1,2) unreach-
able-bounded-path-only
    by metis
  thus ?thesis
    using ‹[c;d;e]› ‹e ∈ Q› by blast
 qed
qed
```

The assumption $Q{\neq}R$ in Theorem 14(iii) is somewhat implicit in Schutz. If $Q{=}R$, *unreach−on Q from a* is empty, so the third conjunct of the conclusion is meaningless.

```
theorem second-existence-thm-3:
 assumes paths: Q∈P R∈P Q≠R
    and events: x∈Q x∈R a∈R a≠x b∉Q
    and reachable: ∃ P∈P. ∃ q∈Q. path P b q
  shows ∃ e∈E. ∃ ae∈P. ∃ be∈P. path ae a e ∧ path be b e ∧ (∀ y∈unreach−on Q
from a. [x;y;e])
proof −
 have a∉Q
   using events(1−4) paths eq-paths by blast
 hence unreach−on Q from a ≠ {}
   by (metis events(3) ex-in-conv in-path-event paths(1,2) two-in-unreach)

 then obtain d where d∈ unreach−on Q from a
   by blast
 have x≠d
    using ‹d ∈ unreach−on Q from a› cross-in-reachable events(1) events(2)
events(3) paths(2) by auto
 have d∈Q
   using ‹d ∈ unreach−on Q from a› unreach-on-path by blast

 have ∃ e∈Q. ∃ ae be. [x;d;e] ∧ path ae a e ∧ path be b e
   using second-existence-thm-2 [where c=x and Q=Q and a=a and b=b and
d=d]
   using ‹a ∉ Q› ‹d ∈ Q› ‹x ≠ d› events(1−3,5) paths(1,2) reachable by blast
```

then obtain *e ae be* **where** *conds*: $[x;d;e] \wedge$ *path ae a e* $\wedge$ *path be b e* **by** *blast*
have $\forall\, y{\in}$(*unreach*$-$*on Q from a*). $[x;y;e]$
proof
fix *y* **assume** $y{\in}$(*unreach*$-$*on Q from a*)
hence $y{\in}Q$
using *unreach-on-path* **by** *blast*
show $[x;y;e]$
proof (*rule ccontr*)
assume $\neg[x;y;e]$
then consider $y{=}x \mid y{=}e \mid [y;x;e] \mid [x;e;y]$
by (*metis* ‹$d{\in}Q$› ‹$y{\in}Q$› *abc-abc-neq abc-sym betw-c-in-path conds events*(*1*) *paths*(*1*) *some-betw*)
thus *False*
proof (*cases*)
assume $y{=}x$ **thus** *False*
using ‹*y* ∈ *unreach*$-$*on Q from a*› *events*(*2*,*3*) *paths*(*1*,*2*) *same-empty-unreach unreach-equiv unreach-on-path*
by *blast*
next
assume $y{=}e$ **thus** *False*
by (*metis* ‹$y{\in}Q$› *assms*(*1*) *conds empty-iff same-empty-unreach unreach-equiv* ‹*y* ∈ *unreach*$-$*on Q from a*›)
next
assume $[y;x;e]$
hence $[y;x;d]$
using *abd-bcd-abc conds* **by** *blast*
hence $x{\in}$(*unreach*$-$*on Q from a*)
using *unreach-connected* [**where** $Q{=}Q$ **and** $Q_x{=}y$ **and** $Q_y{=}x$ **and** $Q_z{=}d$ **and** $b{=}a$]
using ‹$\neg[x;y;e]$› ‹$a{\notin}Q$› ‹$d{\in}$*unreach*$-$*on Q from a*› ‹$y{\in}$*unreach*$-$*on Q from a*› *conds in-path-event paths*(*1*) **by** *blast*
thus *False*
using *empty-iff events*(*2*,*3*) *paths*(*1*,*2*) *same-empty-unreach unreach-equiv unreach-on-path*
by *metis*
next
assume $[x;e;y]$
hence $[d;e;y]$
using *abc-acd-bcd conds* **by** *blast*
hence $e{\in}$(*unreach*$-$*on Q from a*)
using *unreach-connected* [**where** $Q{=}Q$ **and** $Q_x{=}y$ **and** $Q_y{=}e$ **and** $Q_z{=}d$ **and** $b{=}a$]
using ‹*a* ∉ *Q*› ‹*d* ∈ *unreach*$-$*on Q from a*› ‹*y* ∈ *unreach*$-$*on Q from a*› *abc-abc-neq abc-sym events*(*3*) *in-path-event paths*(*1*,*2*)
by *blast*
thus *False*
by (*metis conds empty-iff paths*(*1*) *same-empty-unreach unreach-equiv unreach-on-path*)
qed

qed
qed
thus *?thesis*
using *conds in-path-event* **by** *blast*
qed

end

36 Theorem 11 - with path density assumed

locale *MinkowskiDense* = *MinkowskiSpacetime* +
assumes *path-dense*: *path ab a b* $\Longrightarrow \exists\, x.\ [a;x;b]$
begin

Path density: if a and b are connected by a path, then the segment between them is nonempty. Since Schutz insists on the number of segments in his segmentation (Theorem 11), we prove it here, showcasing where his missing assumption of path density fits in (it is used three times in *number-of-segments*, once in each separate meaningful *local-ordering* case).

lemma *segment-nonempty*:
assumes *path ab a b*
obtains *x* **where** $x \in$ *segment a b*
using *path-dense* **by** (*metis seg-betw assms*)

lemma *number-of-segments*:
assumes *path-P*: $P \in \mathcal{P}$
and *Q-def*: $Q \subseteq P$
and *f-def*: $[f \rightsquigarrow Q | a..b..c]$
shows *card* $\{$*segment* (*f i*) (*f* (*i*+*1*)) | *i*. *i*<(*card Q*−*1*)$\}$ = *card Q* − *1*
proof −
let *?S* = $\{$*segment* (*f i*) (*f* (*i*+*1*)) | *i*. *i*<(*card Q*−*1*)$\}$
let *?N* = *card Q*
let *?g* = λ *i*. *segment* (*f i*) (*f* (*i*+*1*))
have *?N* $\geq$ *3* **using** *chain-defs f-def* **by** (*meson finite-long-chain-with-card*)
have *?g* ‘ $\{$*0*..*?N*−*2*$\}$ = *?S*
proof (*safe*)
fix *i* **assume** *i*∈$\{$(*0*::*nat*)..*?N*−*2*$\}$
show ∃ *ia*. *segment* (*f i*) (*f* (*i*+*1*)) = *segment* (*f ia*) (*f* (*ia*+*1*)) ∧ *ia*<*card Q* − *1*
proof
have *i*<*?N*−*1*
using *assms* ‹*i*∈$\{$(*0*::*nat*)..*?N*−*2*$\}$› ‹*?N*≥*3*›
by (*metis One-nat-def Suc-diff-Suc atLeastAtMost-iff le-less-trans lessI less-le-trans*
less-trans numeral-2-eq-2 numeral-3-eq-3)
then show *segment* (*f i*) (*f* (*i* + *1*)) = *segment* (*f i*) (*f* (*i* + *1*)) ∧ *i*<*?N*−*1*

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      by blast
    qed
  next
    fix x i assume i < card Q − 1
    let ?s = segment (f i) (f (i + 1))
    show ?s ∈ ?g ‘ {0..?N − 2}
    proof −
      have i∈{0..?N−2}
        using ‹i < card Q − 1› by force
      thus ?thesis by blast
    qed
  qed
  moreover have inj-on ?g {0..?N−2}
  proof
    fix i j assume asm: i∈{0..?N−2} j∈{0..?N−2} ?g i = ?g j
    show i=j
    proof (rule ccontr)
      assume i≠j
      hence f i ≠ f j
        using asm(1,2) f-def assms(3) indices-neq-imp-events-neq
          [where X=Q and f=f and a=a and b=b and c=c and i=i and j=j]
        by auto
      show False
      proof (cases)
        assume j=i+1 hence j=Suc i by linarith
         have Suc(Suc i) < ?N using asm(1,2) eval-nat-numeral ‹j = Suc i› by
auto
        hence [f i; f (Suc i); f (Suc (Suc i))]
          using assms short-ch-card ‹?N≥3› chain-defs local-ordering-def
          by (metis short-ch-alt(1) three-in-set3)
        hence [f i; f j; f (j+1)] by (simp add: ‹j = i + 1›)
        obtain e where e∈?g j using segment-nonempty abc-ex-path asm(3)
          by (metis ‹[f i; f j; f (j+1)]› ‹f i ≠ f j› ‹j = i + 1›)
        hence e∈?g i
          using asm(3) by blast
        have [f i; f j; e]
          using abd-bcd-abc ‹[f i; f j; f (j+1)]›
          by (meson ‹e ∈ segment (f j) (f (j + 1))› seg-betw)
        thus False
          using ‹e ∈ segment (f i) (f (i + 1))› ‹j = i + 1› abc-only-cba(2) seg-betw
          by auto
      next assume j≠i+1
        have i < card Q ∧ j < card Q ∧ (i+1) < card Q
         using add-mono-thms-linordered-field(3) asm(1,2) assms ‹?N≥3› by auto
        hence f i ∈ Q ∧ f j ∈ Q ∧ f (i+1) ∈ Q
          using f-def unfolding chain-defs local-ordering-def
            by (metis One-nat-def Suc-diff-le Suc-eq-plus1 ‹3 ≤ card Q› add-Suc
card-1-singleton-iff
        card-gt-0-iff card-insert-if diff-Suc-1 diff-Suc-Suc less-natE less-numeral-extra(1)
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        nat.discI numeral-3-eq-3)
      hence f i ∈ P ∧ f j ∈ P ∧ f (i+1) ∈ P
        using path-is-union assms
        by (simp add: subset-iff)
      then consider [f i; (f(i+1)); f j] | [f i; f j; (f(i+1))] |
                    [(f(i+1)); f i; f j]
        using some-betw path-P f-def indices-neq-imp-events-neq
          ‹f i ≠ f j› ‹i < card Q ∧ j < card Q ∧ i + 1 < card Q› ‹j ≠ i + 1›
        by (metis abc-sym less-add-one less-irrefl-nat)
      thus False
      proof (cases)
        assume [(f(i+1)); f i; f j]
        then obtain e where e∈?g i using segment-nonempty
          by (metis ‹f i ∈ P ∧ f j ∈ P ∧ f (i + 1) ∈ P› abc-abc-neq path-P)
        hence [e; f j; (f(j+1))]
          using ‹[(f(i+1)); f i; f j]›
          by (smt abc-acd-abd abc-acd-bcd abc-only-cba abc-sym asm(3) seg-betw)
        moreover have e∈?g j
          using ‹e ∈ ?g i› asm(3) by blast
        ultimately show False
          by (simp add: abc-only-cba(1) seg-betw)
      next
        assume [f i; f j; (f(i+1))]
        thus False
            using abc-abc-neq [where b=f j and a=f i and c=f(i+1)] asm(3)
seg-betw [where x=f j]
          using ends-notin-segment by blast
      next
        assume [f i; (f(i+1)); f j]
        then obtain e where e∈?g i using segment-nonempty
          by (metis ‹f i ∈ P ∧ f j ∈ P ∧ f (i + 1) ∈ P› abc-abc-neq path-P)
        hence [e; f j; (f(j+1))]
        proof −
          have f (i+1) ≠ f j
            using ‹[f i; (f(i+1)); f j]› abc-abc-neq by presburger
          then show ?thesis
           using ‹e ∈ segment (f i) (f (i+1))› ‹[f i; (f(i+1)); f j]› asm(3) seg-betw
            by (metis (no-types) abc-abc-neq abc-acd-abd abc-acd-bcd abc-sym)
        qed
        moreover have e∈?g j
          using ‹e ∈ ?g i› asm(3) by blast
        ultimately show False
          by (simp add: abc-only-cba(1) seg-betw)
      qed
    qed
  qed
 qed
 ultimately have bij-betw ?g {0..?N−2} ?S
   using inj-on-imp-bij-betw by fastforce
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thus *?thesis*
using *assms(2) bij-betw-same-card numeral-2-eq-2 numeral-3-eq-3* ‹*?N≥3*›
by (*metis* (*no-types*, *lifting*) *One-nat-def Suc-diff-Suc card-atLeastAtMost le-less-trans less-Suc-eq-le minus-nat.diff-0 not-less not-numeral-le-zero*)
qed

theorem *segmentation-card*:
assumes *path-P*: $P \in \mathcal{P}$
and *Q-def*: $Q \subseteq P$
and *f-def*: $[f \rightsquigarrow Q|a..b]$
fixes *P1* **defines** *P1-def*: $P1 \equiv$ *prolongation b a*
fixes *P2* **defines** *P2-def*: $P2 \equiv$ *prolongation a b*
fixes *S* **defines** *S-def*: $S \equiv \{segment\ (f\ i)\ (f\ (i+1))\ |\ i.\ i<card\ Q-1\}$
shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$

$card\ S = (card\ Q-1) \wedge (\forall x \in S.\ is\text{-}segment\ x)$

$disjoint\ (S \cup \{P1,P2\})\ P1 \neq P2\ P1 \notin S\ P2 \notin S$

proof −
let *?N* = *card Q*
have $2 \leq card\ Q$
using *f-def fin-chain-card-geq-2* **by** *blast*
have *seg-facts*: $P = (\bigcup S \cup P1 \cup P2 \cup Q)\ (\forall x \in S.\ is\text{-}segment\ x)$
disjoint $(S \cup \{P1,P2\})\ P1 \neq P2\ P1 \notin S\ P2 \notin S$
using *show-segmentation* [*OF path-P Q-def f-def*]
using *P1-def P2-def S-def* **by** *fastforce+*
show $P = \bigcup S \cup P1 \cup P2 \cup Q$ **by** (*simp add*: *seg-facts*(*1*))
show *disjoint* $(S \cup \{P1,P2\})\ P1 \neq P2\ P1 \notin S\ P2 \notin S$
using *seg-facts*(*3−6*) **by** *blast+*
have *card S* = (*?N−1*)
proof (*cases*)
assume *?N=2*
hence *card S* = *1*
by (*simp add*: *S-def*)
thus *?thesis*
by (*simp add*: ‹*?N = 2*›)
next
assume *?N≠2*
hence *?N≥3*
using ‹$2 \leq$ *card Q*› **by** *linarith*
then obtain *c* **where** $[f \rightsquigarrow Q|a..c..b]$
using *assms chain-defs short-ch-card-2* ‹$2 \leq$ *card Q*› ‹*card Q* ≠ *2*›
by (*metis three-in-set3*)
show *?thesis*
using *number-of-segments* [*OF assms*(*1,2*) ‹$[f \rightsquigarrow Q|a..c..b]$›]
using *S-def* ‹*card Q* ≠ *2*› **by** *presburger*
qed

thus *card S = card Q − 1 ∧ Ball S is-segment*
using *seg-facts*(*2*) **by** *blast*
qed

end

end

References

[1] J. W. Schutz. *Independent Axioms for Minkowski Space-Time.* CRC Press, Oct. 1997.