S-Finite Measure Monad on Quasi-Borel Spaces

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Abstract

The s-finite measure monad on quasi-Borel spaces provides a suitable denotational model for higher-order probabilistic programs with conditioning. This entry is a formalization of the s-finite measure monad and related notions, including s-finite measures, s-finite kernels, and a proof automation for quasi-Borel spaces which is an extension of our previous entry *quasi-Borel spaces*. We also implement several examples of probabilistic programs in previous works and prove their property.

This work is a part of the work by Hirata, Minamide, and Sato, *Semantic Foundations of Higher-Order Probabilistic Programs in Isabelle/HOL* which will be presented at the 14th Conference on Interactive Theorem Proving (ITP2023).

Contents

For the terminology of s-finite measures/kernels, we refer to the work by Staton [4]. For the definition of the s-finite measure monad, we refer to the lecture note by Yang [6]. The construction of the s-finite measure monad is based on the detailed pencil-and-paper proof by Tetsuya Sato.

1 Lemmas

theory *Lemmas-S-Finite-Measure-Monad*
  imports *HOL−Probability.Probability Standard-Borel-Spaces.StandardBorel*
begin

lemma *integrable-mono-measure*:
  fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
  assumes [*measurable-cong*,*measurable*]:*sets M = sets N M* $\leq$ *N integrable N f*
  shows *integrable M f*
  ⟨*proof*⟩

lemma *AE-mono-measure*:
  assumes *sets M = sets N M* $\leq$ *N AE x in N. P x*
  shows *AE x in M. P x*
  ⟨*proof*⟩

lemma *finite-measure-return*:*finite-measure* (*return M x*)
  ⟨*proof*⟩

lemma *nn-integral-return'*:
  assumes $x \notin$ *space M*
  shows ($\int^+ x.\ g\ x\ \partial return\ M\ x$) = *0*
⟨*proof*⟩

lemma *pair-measure-return*: *return M l* $\bigotimes_M$ *return N r* = *return* (*M* $\bigotimes_M$ *N*) (*l*,*r*)
⟨*proof*⟩

lemma *null-measure-distr*: *distr* (*null-measure M*) *N f* = *null-measure N*

⟨*proof*⟩

lemma *integral-measurable-subprob-algebra2*:
fixes f :: - ⇒ - ⇒ -::{*banach*,*second-countable-topology*}
assumes [*measurable*]: $(\lambda(x,\ y).\ f\ x\ y) \in$ *borel-measurable* $(M \bigotimes_M N)$ $L \in$ *measurable* M (*subprob-algebra* N)
shows $(\lambda x.\ integral^L\ (L\ x)\ (f\ x)) \in$ *borel-measurable* M
⟨*proof*⟩

lemma *distr-id'*:
assumes *sets* N = *sets* M
and $\bigwedge x.\ x \in space\ N \Longrightarrow f\ x = x$
shows *distr* $N\ M\ f = N$
⟨*proof*⟩

lemma *measure-density-times*:
assumes [*measurable*]:$S \in sets\ M\ X \in sets\ M\ r \neq \infty$
shows *measure* (*density* M $(\lambda x.\ indicator\ S\ x * r)$) X = *enn2real* r ∗ *measure* M $(S \cap X)$
⟨*proof*⟩

lemma *complete-the-square*:
fixes $a\ b\ c\ x$ :: *real*
assumes $a \neq 0$
shows $a*x^2 + b * x + c = a * (x + (b\ /\ (2*a)))^2 - ((b^2 - 4* a * c)/(4*a))$
⟨*proof*⟩

lemma *complete-the-square2'*:
fixes $a\ b\ c\ x$ :: *real*
assumes $a \neq 0$
shows $a*x^2 - 2 * b * x + c = a * (x - (b\ /\ a))^2 - ((b^2 - a*c)/a)$
⟨*proof*⟩

lemma *normal-density-mu-x-swap*:
normal-density $\mu\ \sigma\ x$ = *normal-density* $x\ \sigma\ \mu$
⟨*proof*⟩

lemma *normal-density-plus-shift*: *normal-density* $\mu\ \sigma\ (x + y)$ = *normal-density* $(\mu - x)\ \sigma\ y$
⟨*proof*⟩

lemma *normal-density-times*:
assumes $\sigma > 0\ \sigma' > 0$
shows *normal-density* $\mu\ \sigma\ x$ ∗ *normal-density* $\mu'\ \sigma'\ x$ = $(1\ /\ sqrt\ (2 * pi * (\sigma^2 + \sigma'^2))) * exp\ (-\ (\mu - \mu')^2\ /\ (2 * (\sigma^2 + \sigma'^2)))$ ∗ *normal-density* $((\mu*\sigma'^2 + \mu'*\sigma^2)/(\sigma^2 + \sigma'^2))\ (\sigma * \sigma'\ /\ sqrt\ (\sigma^2 + \sigma'^2))\ x$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *KL-normal-density*:
assumes [*arith*]: $b > 0$ $d > 0$
shows *KL-divergence* (*exp 1*) (*density lborel* (*normal-density a b*)) (*density lborel* (*normal-density c d*)) $= ln\ (b\ /\ d) + (d^2 + (c - a)^2)\ /\ (2 * b^2) - 1\ /\ 2$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *count-space-prod*:*count-space* (*UNIV* :: ('*a* :: *countable*) *set*) $\bigotimes_M$ *count-space* (*UNIV* :: ('*b* :: *countable*) *set*) = *count-space UNIV*
⟨*proof*⟩

lemma *measure-pair-pmf*:
fixes *p* :: ('*a* :: *countable*) *pmf* **and** *q* :: ('*b* :: *countable*) *pmf*
shows *measure-pmf p* $\bigotimes_M$ *measure-pmf q* = *measure-pmf* (*pair-pmf p q*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *distr-PiM-distr*:
assumes *finite I* $\bigwedge i.\ i \in I \Longrightarrow$ *sigma-finite-measure* (*distr* (*M i*) (*N i*) (*f i*))
and $\bigwedge i.\ i \in I \Longrightarrow f\ i \in M\ i \rightarrow_M N\ i$
shows *distr* $(\Pi_M\ i{\in}I.\ M\ i)\ (\Pi_M\ i{\in}I.\ N\ i)\ (\lambda xi.\ \lambda i{\in}I.\ f\ i\ (xi\ i)) = (\Pi_M\ i{\in}I.$ *distr* (*M i*) (*N i*) (*f i*))
⟨*proof*⟩

lemma *distr-PiM-distr-prob*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* (*M i*)
and $\bigwedge i.\ i \in I \Longrightarrow f\ i \in M\ i \rightarrow_M N\ i$
shows *distr* $(\Pi_M\ i{\in}I.\ M\ i)\ (\Pi_M\ i{\in}I.\ N\ i)\ (\lambda xi.\ \lambda i{\in}I.\ f\ i\ (xi\ i)) = (\Pi_M\ i{\in}I.$ *distr* (*M i*) (*N i*) (*f i*))
⟨*proof*⟩

end

2 Kernels

theory *Kernels*
imports *Lemmas-S-Finite-Measure-Monad*
begin

2.1 S-Finite Measures

locale *s-finite-measure* =
fixes *M* :: '*a measure*
assumes *s-finite-sum*: $\exists\,Mi$ :: *nat* ⇒ '*a measure*. $(\forall\,i.$ *sets* $(Mi\ i) =$ *sets* $M) \wedge (\forall\,i.$ *finite-measure* $(Mi\ i)) \wedge (\forall\,A{\in}$*sets* $M.\ M\ A = (\sum i.\ Mi\ i\ A))$

lemma(**in** *sigma-finite-measure*) *s-finite-measure*: *s-finite-measure M*
⟨*proof*⟩

lemmas(**in** *finite-measure*) *s-finite-measure-finite-measure* = *s-finite-measure*

lemmas(**in** *subprob-space*) *s-finite-measure-subprob* = *s-finite-measure*

lemmas(**in** *prob-space*) *s-finite-measure-prob* = *s-finite-measure*

sublocale *sigma-finite-measure* ⊆ *s-finite-measure*
⟨*proof*⟩

lemma *s-finite-measureI*:
assumes $\bigwedge i.\ sets\ (Mi\ i) = sets\ M$ $\bigwedge i.\ finite\text{-}measure\ (Mi\ i)$ $\bigwedge A.\ A{\in}sets\ M \Longrightarrow M\ A = (\sum i.\ Mi\ i\ A)$
shows *s-finite-measure M*
⟨*proof*⟩

lemma *s-finite-measure-prodI*:
assumes $\bigwedge i\ j.\ sets\ (Mij\ i\ j) = sets\ M$ $\bigwedge i\ j.\ Mij\ i\ j\ (space\ M) < \infty$ $\bigwedge A.\ A \in sets\ M \Longrightarrow M\ A = (\sum i.\ (\sum j.\ Mij\ i\ j\ A))$
shows *s-finite-measure M*
⟨*proof*⟩

corollary *s-finite-measure-s-finite-sumI*:
assumes $\bigwedge i.\ sets\ (Mi\ i) = sets\ M$ $\bigwedge i.\ s\text{-}finite\text{-}measure\ (Mi\ i)$ $\bigwedge A.\ A \in sets\ M \Longrightarrow M\ A = (\sum i.\ Mi\ i\ A)$
shows *s-finite-measure M*
⟨*proof*⟩

lemma *s-finite-measure-finite-sumI*:
assumes $finite\ I$ $\bigwedge i.\ i \in I \Longrightarrow s\text{-}finite\text{-}measure\ (Mi\ i)$ $\bigwedge i.\ i \in I \Longrightarrow sets\ (Mi\ i) = sets\ M$
and $\bigwedge A.\ A \in sets\ M \Longrightarrow M\ A = (\sum i{\in}I.\ Mi\ i\ A)$
shows *s-finite-measure M*
⟨*proof*⟩

lemma *countable-space-s-finite-measure*:
assumes $countable\ (space\ M)$ $sets\ M = Pow\ (space\ M)$
shows *s-finite-measure M*
⟨*proof*⟩

lemma *s-finite-measure-subprob-space*:
$s\text{-}finite\text{-}measure\ M \longleftrightarrow (\exists\, Mi :: nat \Rightarrow 'a\ measure.\ (\forall\, i.\ sets\ (Mi\ i) = sets\ M) \wedge (\forall\, i.\ (Mi\ i)\ (space\ M) \le 1) \wedge (\forall\, A{\in}sets\ M.\ M\ A = (\sum i.\ Mi\ i\ A)))$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *finite-measures*:
obtains *Mi* **where** $\bigwedge i.\ sets\ (Mi\ i) = sets\ M$ $\bigwedge i.\ (Mi\ i)\ (space\ M) \le 1$ $\bigwedge A.\ M\ A = (\sum i.\ Mi\ i\ A)$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *finite-measures-ne*:
assumes *space* $M \neq \{\}$
obtains *Mi* **where** $\bigwedge i.$ *sets* (*Mi i*) = *sets M* $\bigwedge i.$ *subprob-space* (*Mi i*) $\bigwedge A.$ M $A = (\sum i.\ Mi\ i\ A)$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *finite-measures*′:
obtains *Mi* **where** $\bigwedge i.$ *sets* (*Mi i*) = *sets M* $\bigwedge i.$ *finite-measure* (*Mi i*) $\bigwedge A.$ M $A = (\sum i.\ Mi\ i\ A)$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *s-finite-measure-distr*:
assumes *f*[*measurable*]:$f \in M \rightarrow_M N$
shows *s-finite-measure* (*distr M N f*)
⟨*proof*⟩

lemma *nn-integral-measure-suminf*:
assumes [*measurable-cong*]:$\bigwedge i.$ *sets* (*Mi i*) = *sets M* **and** $\bigwedge A.$ $A \in$*sets M* $\Longrightarrow M$ $A = (\sum i.\ Mi\ i\ A)$ $f \in$ *borel-measurable M*
shows $(\sum i. \int^+ x.\ f\ x\ \partial(Mi\ i)) = (\int^+ x.\ f\ x\ \partial M)$
⟨*proof*⟩

A *density M f* of *s*-finite measure M and $f \in$ *borel-measurable M* is again s-finite. We do not require additional assumption, unlike σ-finite measures.

lemma(**in** *s-finite-measure*) *s-finite-measure-density*:
assumes *f*[*measurable*]:$f \in$ *borel-measurable M*
shows *s-finite-measure* (*density M f*)
⟨*proof*⟩

lemma
fixes f :: ′$a \Rightarrow$ ′b::{*banach*, *second-countable-topology*}
assumes [*measurable-cong*]:$\bigwedge i.$ *sets* (*Mi i*) = *sets M* **and** $\bigwedge A.$ $A \in$*sets M* $\Longrightarrow M$ $A = (\sum i.\ Mi\ i\ A)$ *integrable M f*
shows *lebesgue-integral-measure-suminf*:$(\sum i. \int x.\ f\ x\ \partial(Mi\ i)) = (\int x.\ f\ x\ \partial M)$ (**is** *?suminf*)
and *lebesgue-integral-measure-suminf-summable-norm*: *summable* ($\lambda i.$ *norm* $(\int x.\ f\ x\ \partial(Mi\ i))$) (**is** *?summable2*)
and *lebesgue-integral-measure-suminf-summable-norm-in*: *summable* ($\lambda i. \int x.$ *norm* $(f\ x)\ \partial(Mi\ i)$) (**is** *?summable*)
⟨*proof*⟩

lemma (**in** *s-finite-measure*) *measurable-emeasure-Pair*′:
assumes $Q \in$ *sets* $(N \bigotimes_M M)$
shows ($\lambda x.$ *emeasure M* (*Pair x* −‘ *Q*)) $\in$ *borel-measurable N* (**is** *?s Q* ∈ -)
⟨*proof*⟩

lemma (**in** *s-finite-measure*) *measurable-emeasure*′[*measurable* (*raw*)]:
assumes *space*: $\bigwedge x.\ x \in$ *space N* $\Longrightarrow A\ x \subseteq$ *space M*

assumes *A*: $\{x{\in}space\ (N \bigotimes_M M).\ snd\ x \in A\ (fst\ x)\} \in sets\ (N \bigotimes_M M)$
shows $(\lambda x.\ emeasure\ M\ (A\ x)) \in borel\text{-}measurable\ N$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *emeasure-pair-measure′*:
assumes $X \in sets\ (N \bigotimes_M M)$
shows $emeasure\ (N \bigotimes_M M)\ X = (\int^+ x.\ \int^+ y.\ indicator\ X\ (x, y)\ \partial M\ \partial N)$
(**is** - = *?μ X*)
⟨*proof*⟩

lemma (**in** *s-finite-measure*) *emeasure-pair-measure-alt′*:
assumes *X*: $X \in sets\ (N \bigotimes_M M)$
shows $emeasure\ (N \bigotimes_M M)\ X = (\int^+ x.\ emeasure\ M\ (Pair\ x -` X)\ \partial N)$
⟨*proof*⟩

proposition (**in** *s-finite-measure*) *emeasure-pair-measure-Times′*:
assumes *A*: $A \in sets\ N$ **and** *B*: $B \in sets\ M$
shows $emeasure\ (N \bigotimes_M M)\ (A \times B) = emeasure\ N\ A * emeasure\ M\ B$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *measure-times*:
assumes[*measurable*]: $A \in sets\ N\ B \in sets\ M$
shows $measure\ (N \bigotimes_M M)\ (A \times B) = measure\ N\ A * measure\ M\ B$
⟨*proof*⟩

lemma *pair-measure-s-finite-measure-suminf*:
assumes *Mi*[*measurable-cong*]:$\bigwedge i.\ sets\ (Mi\ i) = sets\ M\ \bigwedge i.\ finite\text{-}measure\ (Mi\ i)\ \bigwedge A.\ M\ A = (\sum i.\ Mi\ i\ A)$
and *Ni*[*measurable-cong*]:$\bigwedge i.\ sets\ (Ni\ i) = sets\ N\ \bigwedge i.\ finite\text{-}measure\ (Ni\ i)\ \bigwedge A.\ N\ A = (\sum i.\ Ni\ i\ A)$
shows $(M \bigotimes_M N)\ A = (\sum i\ j.\ (Mi\ i \bigotimes_M Ni\ j)\ A)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *pair-measure-s-finite-measure-suminf′*:
assumes *Mi*[*measurable-cong*]:$\bigwedge i.\ sets\ (Mi\ i) = sets\ M\ \bigwedge i.\ finite\text{-}measure\ (Mi\ i)\ \bigwedge A.\ M\ A = (\sum i.\ Mi\ i\ A)$
and *Ni*[*measurable-cong*]:$\bigwedge i.\ sets\ (Ni\ i) = sets\ N\ \bigwedge i.\ finite\text{-}measure\ (Ni\ i)\ \bigwedge A.\ N\ A = (\sum i.\ Ni\ i\ A)$
shows $(M \bigotimes_M N)\ A = (\sum i\ j.\ (Mi\ j \bigotimes_M Ni\ i)\ A)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *pair-measure-s-finite-measure*:
assumes *s-finite-measure M* **and** *s-finite-measure N*
shows $s\text{-}finite\text{-}measure\ (M \bigotimes_M N)$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *borel-measurable-nn-integral-fst′*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ (N \bigotimes_M M)$

shows $(\lambda x.\ \int^{+} y.\ f\ (x,\ y)\ \partial M) \in$ *borel-measurable N*
$\langle proof \rangle$

lemma (**in** *s-finite-measure*) *nn-integral-fst'*:
assumes f: $f \in$ *borel-measurable* $(M1 \bigotimes_M M)$
shows $(\int^{+} x.\ \int^{+} y.\ f\ (x,\ y)\ \partial M\ \partial M1) = integral^N\ (M1 \bigotimes_M M)\ f$ (**is** *?I f* = -)
$\langle proof \rangle$

lemma (**in** *s-finite-measure*) *borel-measurable-nn-integral'*[*measurable* (*raw*)]:
case-prod f $\in$ *borel-measurable* $(N \bigotimes_M M) \Longrightarrow (\lambda x.\ \int^{+} y.\ f\ x\ y\ \partial M) \in$ *borel-measurable N*
$\langle proof \rangle$

lemma *distr-pair-swap-s-finite*:
assumes *s-finite-measure M1* **and** *s-finite-measure M2*
shows $M1 \bigotimes_M M2 = distr\ (M2 \bigotimes_M M1)\ (M1 \bigotimes_M M2)\ (\lambda(x,\ y).\ (y,\ x))$ (**is** *?P* = *?D*)
$\langle proof \rangle$

proposition *nn-integral-snd'*:
assumes *s-finite-measure M1 s-finite-measure M2*
and f[*measurable*]: $f \in$ *borel-measurable* $(M1 \bigotimes_M M2)$
shows $(\int^{+} y.\ (\int^{+} x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = integral^N\ (M1 \bigotimes_M M2)\ f$
$\langle proof \rangle$

lemma (**in** *s-finite-measure*) *borel-measurable-lebesgue-integrable'*[*measurable* (*raw*)]:
fixes f :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]: *case-prod f* $\in$ *borel-measurable* $(N \bigotimes_M M)$
shows *Measurable.pred N* $(\lambda x.$ *integrable M* $(f\ x))$
$\langle proof \rangle$

lemma (**in** *s-finite-measure*) *measurable-measure'*[*measurable* (*raw*)]:
$(\bigwedge x.\ x \in space\ N \Longrightarrow A\ x \subseteq space\ M) \Longrightarrow$
$\{x \in space\ (N \bigotimes_M M).\ snd\ x \in A\ (fst\ x)\} \in sets\ (N \bigotimes_M M) \Longrightarrow$
$(\lambda x.\ measure\ M\ (A\ x)) \in$ *borel-measurable N*
$\langle proof \rangle$

proposition (**in** *s-finite-measure*) *borel-measurable-lebesgue-integral'*[*measurable* (*raw*)]:
fixes f :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes f[*measurable*]: *case-prod f* $\in$ *borel-measurable* $(N \bigotimes_M M)$
shows $(\lambda x.\ \int y.\ f\ x\ y\ \partial M) \in$ *borel-measurable N*
$\langle proof \rangle$

lemma *integrable-product-swap-s-finite*:
fixes f :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes *M1*:*s-finite-measure M1* **and** *M2*:*s-finite-measure M2*
and *integrable* $(M1 \bigotimes_M M2)\ f$
shows *integrable* $(M2 \bigotimes_M M1)\ (\lambda(x,y).\ f\ (y,x))$

⟨*proof*⟩

lemma *integrable-product-swap-iff-s-finite*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes *M1*:*s-finite-measure M1* **and** *M2*:*s-finite-measure M2*
  shows *integrable* (*M2* $\bigotimes_M$ *M1*) ($\lambda(x,y)$. f (y,x)) ⟷ *integrable* (*M1* $\bigotimes_M$ *M2*)
f
⟨*proof*⟩

lemma *integral-product-swap-s-finite*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes *M1*:*s-finite-measure M1* **and** *M2*:*s-finite-measure M2*
    and f: $f \in$ *borel-measurable* (*M1* $\bigotimes_M$ *M2*)
  shows ($\int$ (x,y). f (y,x) ∂(*M2* $\bigotimes_M$ *M1*)) = *integral*^{L} (*M1* $\bigotimes_M$ *M2*) f
⟨*proof*⟩

theorem(**in** *s-finite-measure*) *Fubini-integrable'*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes f[*measurable*]: $f \in$ *borel-measurable* (*M1* $\bigotimes_M$ *M*)
    and *integ1*: *integrable M1* (λx. $\int$ y. *norm* (f $(x,\ y)$) ∂M)
    and *integ2*: *AE x in M1*. *integrable M* (λy. f $(x,\ y)$)
  shows *integrable* (*M1* $\bigotimes_M$ *M*) f
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *emeasure-pair-measure-finite'*:
  assumes A: $A \in$ *sets* (*M1* $\bigotimes_M$ *M*) **and** *finite*: *emeasure* (*M1* $\bigotimes_M$ *M*) $A < \infty$
  shows *AE x in M1*. *emeasure M* {y∈*space M*. $(x,\ y) \in A$} $< \infty$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *AE-integrable-fst'''*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes f[*measurable*]: *integrable* (*M1* $\bigotimes_M$ *M*) f
  shows *AE x in M1*. *integrable M* (λy. f $(x,\ y)$)
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *integrable-fst-norm'*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes f[*measurable*]: *integrable* (*M1* $\bigotimes_M$ *M*) f
  shows *integrable M1* (λx. $\int$ y. *norm* (f $(x,\ y)$) ∂M)
  ⟨*proof*⟩

lemma(**in** *s-finite-measure*) *integrable-fst'''*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes f[*measurable*]: *integrable* (*M1* $\bigotimes_M$ *M*) f
  shows *integrable M1* (λx. $\int$ y. f $(x,\ y)$ ∂M)
  ⟨*proof*⟩

proposition(**in** *s-finite-measure*) *integral-fst'''*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}

assumes f: *integrable* $(M1 \bigotimes_M M)$ f
shows $(\int x.\ (\int y.\ f\ (x,\ y)\ \partial M)\ \partial M1) = integral^L\ (M1 \bigotimes_M M)\ f$
$\langle proof \rangle$

lemma (**in** *s-finite-measure*)
fixes f :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes f: *integrable* $(M1 \bigotimes_M M)$ (*case-prod* f)
shows *AE-integrable-fst*$''$: *AE* x *in* $M1$. *integrable* M $(\lambda y.\ f\ x\ y)$
and *integrable-fst*$''$: *integrable* $M1$ $(\lambda x.\ \int y.\ f\ x\ y\ \partial M)$
and *integrable-fst-norm*: *integrable* $M1$ $(\lambda x.\ \int y.\ norm\ (f\ x\ y)\ \partial M)$
and *integral-fst*$''$: $(\int x.\ (\int y.\ f\ x\ y\ \partial M)\ \partial M1) = integral^L\ (M1 \bigotimes_M M)\ (\lambda(x, y).\ f\ x\ y)$
$\langle proof \rangle$

lemma
fixes f :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes $M1$:*s-finite-measure* $M1$ **and** $M2$:*s-finite-measure* $M2$
and f[*measurable*]: *integrable* $(M1 \bigotimes_M M2)$ (*case-prod* f)
shows *AE-integrable-snd-s-finite*: *AE* y *in* $M2$. *integrable* $M1$ $(\lambda x.\ f\ x\ y)$ (**is** *?AE*)
and *integrable-snd-s-finite*: *integrable* $M2$ $(\lambda y.\ \int x.\ f\ x\ y\ \partial M1)$ (**is** *?INT*)
and *integrable-snd-norm-s-finite*: *integrable* $M2$ $(\lambda y.\ \int x.\ norm\ (f\ x\ y)\ \partial M1)$ (**is** *?INT2*)
and *integral-snd-s-finite*: $(\int y.\ (\int x.\ f\ x\ y\ \partial M1)\ \partial M2) = integral^L\ (M1 \bigotimes_M M2)$ (*case-prod* f) (**is** *?EQ*)
$\langle proof \rangle$

proposition *Fubini-integral*$'$:
fixes f :: - $\Rightarrow$ - $\Rightarrow$ - :: {*banach*, *second-countable-topology*}
assumes $M1$:*s-finite-measure* $M1$ **and** $M2$:*s-finite-measure* $M2$
and f: *integrable* $(M1 \bigotimes_M M2)$ (*case-prod* f)
shows $(\int y.\ (\int x.\ f\ x\ y\ \partial M1)\ \partial M2) = (\int x.\ (\int y.\ f\ x\ y\ \partial M2)\ \partial M1)$
$\langle proof \rangle$

locale *product-s-finite* =
fixes M :: $'i \Rightarrow 'a$ *measure*
assumes *s-finite-measures*: $\bigwedge i.$ *s-finite-measure* $(M\ i)$

sublocale *product-s-finite* $\subseteq$ *M?*: *s-finite-measure* M i **for** i
$\langle proof \rangle$

locale *finite-product-s-finite* = *product-s-finite* M **for** M :: $'i \Rightarrow 'a$ *measure* +
fixes I :: $'i$ *set*
assumes *finite-index*: *finite* I

lemma (**in** *product-s-finite*) *emeasure-PiM*:
finite $I \Longrightarrow (\bigwedge i.\ i{\in}I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow emeasure\ (PiM\ I\ M)\ (Pi_E\ I\ A) = (\prod i{\in}I.\ emeasure\ (M\ i)\ (A\ i))$
$\langle proof \rangle$

lemma (**in** *finite-product-s-finite*) *measure-times*:
$(\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow emeasure\ (Pi_M\ I\ M)\ (Pi_E\ I\ A) = (\prod i{\in}I.\ emeasure\ (M\ i)\ (A\ i))$
⟨*proof*⟩

lemma (**in** *product-s-finite*) *nn-integral-empty*:
$0 \le f\ (\lambda k.\ undefined) \Longrightarrow integral^N\ (Pi_M\ \{\}\ M)\ f = f\ (\lambda k.\ undefined)$
⟨*proof*⟩

Every s-finite measure is represented as the push-forward measure of a σ-finite measure.

definition *Mi-to-NM* :: $(nat \Rightarrow 'a\ measure) \Rightarrow 'a\ measure \Rightarrow (nat \times 'a)\ measure$
where
$Mi\text{-}to\text{-}NM\ Mi\ M \equiv measure\text{-}of\ (space\ (count\text{-}space\ UNIV \bigotimes_M M))\ (sets\ (count\text{-}space\ UNIV \bigotimes_M M))\ (\lambda A.\ \sum i.\ distr\ (Mi\ i)\ (count\text{-}space\ UNIV \bigotimes_M M)\ (\lambda x.\ (i,x))\ A)$

lemma
shows *sets-Mi-to-NM*[*measurable-cong*,*simp*]: $sets\ (Mi\text{-}to\text{-}NM\ Mi\ M) = sets\ (count\text{-}space\ UNIV \bigotimes_M M)$
and *space-Mi-to-NM*[*simp*]: $space\ (Mi\text{-}to\text{-}NM\ Mi\ M) = space\ (count\text{-}space\ UNIV \bigotimes_M M)$
⟨*proof*⟩

context
fixes $M :: 'a\ measure$ **and** $Mi :: nat \Rightarrow 'a\ measure$
assumes *sets-Mi*[*measurable-cong*,*simp*]: $\bigwedge i.\ sets\ (Mi\ i) = sets\ M$
and *emeasure-Mi*: $\bigwedge A.\ A \in sets\ M \Longrightarrow M\ A = (\sum i.\ Mi\ i\ A)$
begin

lemma *emeasure-Mi-to-NM*:
assumes [*measurable*]: $A \in sets\ (count\text{-}space\ UNIV \bigotimes_M M)$
shows $emeasure\ (Mi\text{-}to\text{-}NM\ Mi\ M)\ A = (\sum i.\ distr\ (Mi\ i)\ (count\text{-}space\ UNIV \bigotimes_M M)\ (\lambda x.\ (i,x))\ A)$
⟨*proof*⟩

lemma *sigma-finite-Mi-to-NM-measure*:
assumes $\bigwedge i.\ finite\text{-}measure\ (Mi\ i)$
shows $sigma\text{-}finite\text{-}measure\ (Mi\text{-}to\text{-}NM\ Mi\ M)$
⟨*proof*⟩

lemma *distr-Mi-to-NM-M*: $distr\ (Mi\text{-}to\text{-}NM\ Mi\ M)\ M\ snd = M$
⟨*proof*⟩

end

context

fixes μ :: *'a measure*
assumes *standard-borel-ne*: *standard-borel-ne* μ
and *s-finite*: *s-finite-measure* μ
begin

interpretation μ : *s-finite-measure* μ $\langle proof \rangle$

interpretation *n-*μ: *standard-borel-ne count-space* (*UNIV* :: *nat set*) $\bigotimes_M$ μ
$\langle proof \rangle$

lemma *exists-push-forward*:
$\exists (\mu' :: real\ measure)\ f.\ f \in borel \rightarrow_M \mu \wedge sets\ \mu' = sets\ borel \wedge sigma\text{-}finite\text{-}measure\ \mu'$
$\wedge\ distr\ \mu'\ \mu\ f = \mu$
$\langle proof \rangle$

abbreviation μ'*-and-f* $\equiv$ (*SOME* (μ'::*real measure*,*f*). $f \in borel \rightarrow_M \mu \wedge sets\ \mu' = sets\ borel \wedge sigma\text{-}finite\text{-}measure\ \mu' \wedge distr\ \mu'\ \mu\ f = \mu$)

definition *sigma-pair-*μ $\equiv$ *fst* μ'*-and-f*
definition *sigma-pair-f* $\equiv$ *snd* μ'*-and-f*

lemma
shows *sigma-pair-f-measurable* : *sigma-pair-f* $\in$ *borel* $\rightarrow_M$ μ (**is** *?g1*)
and *sets-sigma-pair-*μ: *sets sigma-pair-*μ = *sets borel* (**is** *?g2*)
and *sigma-finite-sigma-pair-*μ: *sigma-finite-measure sigma-pair-*μ (**is** *?g3*)
and *distr-sigma-pair*: *distr sigma-pair-*μ μ *sigma-pair-f* = μ (**is** *?g4*)
$\langle proof \rangle$

end

definition *s-finite-measure-algebra* :: *'a measure* $\Rightarrow$ *'a measure measure* **where**
s-finite-measure-algebra K =
(*SUP* $A \in$ *sets* K. *vimage-algebra* $\{M.$ *s-finite-measure* $M \wedge$ *sets* M = *sets* $K\}$ ($\lambda M.$ *emeasure* M A) *borel*)

lemma *space-s-finite-measure-algebra*:
space (*s-finite-measure-algebra* K) = $\{M.$ *s-finite-measure* $M \wedge$ *sets* M = *sets* $K\}$
$\langle proof \rangle$

lemma *s-finite-measure-algebra-cong*: *sets* M = *sets* $N \Longrightarrow$ *s-finite-measure-algebra* M = *s-finite-measure-algebra* N
$\langle proof \rangle$

lemma *measurable-emeasure-s-finite-measure-algebra*[*measurable*]:
$a \in$ *sets* $A \Longrightarrow$ ($\lambda M.$ *emeasure* M a) $\in$ *borel-measurable* (*s-finite-measure-algebra* A)
$\langle proof \rangle$

lemma *measurable-measure-s-finite-measure-algebra*[*measurable*]:
$a \in sets\ A \Longrightarrow (\lambda M.\ measure\ M\ a) \in borel\text{-}measurable\ (s\text{-}finite\text{-}measure\text{-}algebra\ A)$
⟨*proof*⟩

lemma *s-finite-measure-algebra-measurableD*:
assumes N: $N \in measurable\ M\ (s\text{-}finite\text{-}measure\text{-}algebra\ S)$ **and** x: $x \in space\ M$
shows $space\ (N\ x) = space\ S$
and $sets\ (N\ x) = sets\ S$
and $measurable\ (N\ x)\ K = measurable\ S\ K$
and $measurable\ K\ (N\ x) = measurable\ K\ S$
⟨*proof*⟩

context
fixes $K\ M\ N$ **assumes** K: $K \in measurable\ M\ (s\text{-}finite\text{-}measure\text{-}algebra\ N)$
begin

lemma *s-finite-measure-algebra-kernel*: $a \in space\ M \Longrightarrow s\text{-}finite\text{-}measure\ (K\ a)$
⟨*proof*⟩

lemma *s-finite-measure-algebra-sets-kernel*: $a \in space\ M \Longrightarrow sets\ (K\ a) = sets\ N$
⟨*proof*⟩

lemma *measurable-emeasure-kernel-s-finite-measure-algebra*[*measurable*]:
$A \in sets\ N \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in borel\text{-}measurable\ M$
⟨*proof*⟩

end

lemma *measurable-s-finite-measure-algebra*:
$(\bigwedge a.\ a \in space\ M \Longrightarrow s\text{-}finite\text{-}measure\ (K\ a)) \Longrightarrow$
$(\bigwedge a.\ a \in space\ M \Longrightarrow sets\ (K\ a) = sets\ N) \Longrightarrow$
$(\bigwedge A.\ A \in sets\ N \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in borel\text{-}measurable\ M) \Longrightarrow$
$K \in measurable\ M\ (s\text{-}finite\text{-}measure\text{-}algebra\ N)$
⟨*proof*⟩

definition *bind-kernel* :: $'a\ measure \Rightarrow ('a \Rightarrow 'b\ measure) \Rightarrow 'b\ measure$ (**infixl** $\gg\!=_k$ *54*) **where**
$bind\text{-}kernel\ M\ k = (if\ space\ M = \{\}\ then\ count\text{-}space\ \{\}\ else$
$let\ Y = k\ (SOME\ x.\ x \in space\ M)\ in$
$measure\text{-}of\ (space\ Y)\ (sets\ Y)\ (\lambda B.\ \int^+ x.\ (k\ x\ B)\ \partial M))$

lemma *bind-kernel-cong-All*:
assumes $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x$
shows $M \gg\!=_k f = M \gg\!=_k g$
⟨*proof*⟩

lemma *sets-bind-kernel*:
assumes $\bigwedge x.\ x \in space\ M \Longrightarrow sets\ (k\ x) = sets\ N\ space\ M \neq \{\}$

shows *sets* $(M \ggg_k k) = $ *sets* N
$\langle proof \rangle$

2.2 Measure Kernel

locale *measure-kernel* =
fixes X :: $'a$ *measure* **and** Y :: $'b$ *measure* **and** κ :: $'a \Rightarrow 'b$ *measure*
assumes *kernel-sets*[*measurable-cong*]: $\bigwedge x.\ x \in$ *space* $X \Longrightarrow$ *sets* $(\kappa\ x) =$ *sets* Y
and *emeasure-measurable*[*measurable*]: $\bigwedge B.\ B \in$ *sets* $Y \Longrightarrow (\lambda x.$ *emeasure* $(\kappa\ x)\ B) \in$ *borel-measurable* X
and *Y-not-empty*: *space* $X \neq \{\} \Longrightarrow$ *space* $Y \neq \{\}$
begin

lemma *kernel-space* :$\bigwedge x.\ x \in$ *space* $X \Longrightarrow$ *space* $(\kappa\ x) =$ *space* Y
$\langle proof \rangle$

lemma *measure-measurable*:
assumes $B \in$ *sets* Y
shows $(\lambda x.$ *measure* $(\kappa\ x)\ B) \in$ *borel-measurable* X
$\langle proof \rangle$

lemma *set-nn-integral-measure*:
assumes [*measurable-cong*]: *sets* $\mu =$ *sets* X **and** [*measurable*]: $A \in$ *sets* X $B \in$ *sets* Y
defines $\nu \equiv$ *measure-of* (*space* Y) (*sets* Y) $(\lambda B.\ \int^+ x \in A.\ (\kappa\ x\ B)\ \partial\mu)$
shows $\nu\ B = (\int^+ x \in A.\ (\kappa\ x\ B)\ \partial\mu)$
$\langle proof \rangle$

corollary *nn-integral-measure*:
assumes *sets* $\mu =$ *sets* X $B \in$ *sets* Y
defines $\nu \equiv$ *measure-of* (*space* Y) (*sets* Y) $(\lambda B.\ \int^+ x.\ (\kappa\ x\ B)\ \partial\mu)$
shows $\nu\ B = (\int^+ x.\ (\kappa\ x\ B)\ \partial\mu)$
$\langle proof \rangle$

lemma *distr-measure-kernel*:
assumes [*measurable*]:$f \in Y \rightarrow_M Z$
shows *measure-kernel* X Z $(\lambda x.$ *distr* $(\kappa\ x)\ Z\ f)$
$\langle proof \rangle$

lemma *measure-kernel-comp*:
assumes [*measurable*]: $f \in W \rightarrow_M X$
shows *measure-kernel* W Y $(\lambda x.\ \kappa\ (f\ x))$
$\langle proof \rangle$

lemma *emeasure-bind-kernel*:
assumes *sets* $\mu =$ *sets* X $B \in$ *sets* Y *space* $X \neq \{\}$
shows $(\mu \ggg_k \kappa)\ B = (\int^+ x.\ (\kappa\ x\ B)\ \partial\mu)$
$\langle proof \rangle$

lemma *measure-bind-kernel*:
  assumes [*measurable-cong*]:*sets* μ = *sets* X **and** [*measurable*]:$B \in$ *sets* Y *space* $X \neq \{\}$ *AE* x *in* μ. $\kappa\ x\ B < \infty$
  shows *measure* $(\mu \gg=_k \kappa)\ B = (\int x.\ measure\ (\kappa\ x)\ B\ \partial\mu)$
  ⟨*proof*⟩

lemma *sets-bind-kernel*:
  assumes *space* $X \neq \{\}$ *sets* μ = *sets* X
  shows *sets* $(\mu \gg=_k \kappa)$ = *sets* Y
  ⟨*proof*⟩

lemma *distr-bind-kernel*:
  assumes *space* $X \neq \{\}$ **and** [*measurable-cong*]:*sets* μ = *sets* X **and** [*measurable*]: $f \in Y \to_M Z$
  shows *distr* $(\mu \gg=_k \kappa)\ Z\ f = \mu \gg=_k (\lambda x.\ distr\ (\kappa\ x)\ Z\ f)$
⟨*proof*⟩

lemma *bind-kernel-distr*:
  assumes [*measurable*]: $f \in W \to_M X$ **and** *space* $W \neq \{\}$
  shows *distr* $W\ X\ f \gg=_k \kappa = W \gg=_k (\lambda x.\ \kappa\ (f\ x))$
⟨*proof*⟩

lemma *bind-kernel-return*:
  assumes $x \in$ *space* X
  shows *return* $X\ x \gg=_k \kappa = \kappa\ x$
⟨*proof*⟩

lemma *kernel-nn-integral-measurable*:
  assumes $f \in$ *borel-measurable* Y
  shows $(\lambda x.\ \int^+ y.\ f\ y\ \partial(\kappa\ x)) \in$ *borel-measurable* X
  ⟨*proof*⟩

lemma *bind-kernel-measure-kernel*:
  assumes *measure-kernel* $Y\ Z\ k'$
  shows *measure-kernel* $X\ Z\ (\lambda x.\ \kappa\ x \gg=_k k')$
⟨*proof*⟩

lemma *restrict-measure-kernel*: *measure-kernel* (*restrict-space* X A) Y κ
⟨*proof*⟩

end

lemma *measure-kernel-cong-sets*:
  assumes *sets* X = *sets* X' *sets* Y = *sets* Y'
  shows *measure-kernel* $X\ Y$ = *measure-kernel* $X'\ Y'$
  ⟨*proof*⟩

lemma *measure-kernel-pair-countble1*:
  assumes *countable* A $\bigwedge i.\ i \in A \Longrightarrow$ *measure-kernel* $X\ Y\ (\lambda x.\ k\ (i,x))$

shows *measure-kernel* (*count-space* $A \bigotimes_M X$) Y k
⟨*proof*⟩

lemma *measure-kernel-empty-trivial*:
assumes *space* $X = \{\}$
shows *measure-kernel* X Y k
⟨*proof*⟩

2.3 Finite Kernel

locale *finite-kernel* = *measure-kernel* +
assumes *finite-measure-spaces*: $\exists\, r<\infty.\ \forall\, x\in$ *space* $X.\ \kappa\ x$ (*space* Y) $< r$
begin

lemma *finite-measures*:
assumes $x \in$ *space* X
shows *finite-measure* $(\kappa\ x)$
⟨*proof*⟩

end

lemma *finite-kernel-empty-trivial*: *space* $X = \{\} \Longrightarrow$ *finite-kernel* X Y f
⟨*proof*⟩

lemma *finite-kernel-cong-sets*:
assumes *sets* X = *sets* X' *sets* Y = *sets* Y'
shows *finite-kernel* X Y = *finite-kernel* X' Y'
⟨*proof*⟩

2.4 Sub-Probability Kernel

locale *subprob-kernel* = *measure-kernel* +
assumes *subprob-spaces*: $\bigwedge x.\ x \in$ *space* $X \Longrightarrow$ *subprob-space* $(\kappa\ x)$
begin
lemma *subprob-space*:
$\bigwedge x.\ x \in$ *space* $X \Longrightarrow \kappa\ x$ (*space* Y) ≤ 1
⟨*proof*⟩

lemma *subprob-measurable*[*measurable*]:
$\kappa \in X \to_M$ *subprob-algebra* Y
⟨*proof*⟩

lemma *finite-kernel*: *finite-kernel* X Y κ
⟨*proof*⟩

sublocale *finite-kernel*
⟨*proof*⟩

end

lemma *subprob-kernel-def'*:
$subprob\text{-}kernel\ X\ Y\ \kappa \longleftrightarrow \kappa \in X \to_M subprob\text{-}algebra\ Y$
⟨*proof*⟩

lemmas *subprob-kernelI* = *measurable-subprob-algebra*[*simplified subprob-kernel-def'*[*symmetric*]]

lemma *subprob-kernel-cong-sets*:
assumes $sets\ X = sets\ X'$ $sets\ Y = sets\ Y'$
shows $subprob\text{-}kernel\ X\ Y = subprob\text{-}kernel\ X'\ Y'$
⟨*proof*⟩

lemma *subprob-kernel-empty-trivial*:
assumes $space\ X = \{\}$
shows $subprob\text{-}kernel\ X\ Y\ k$
⟨*proof*⟩

lemma *bind-kernel-bind*:
assumes $f \in M \to_M subprob\text{-}algebra\ N$
shows $M \gg\!=_k f = M \gg\!= f$
⟨*proof*⟩

lemma(**in** *measure-kernel*) *subprob-kernel-sum*:
assumes $\bigwedge x.\ x \in space\ X \Longrightarrow finite\text{-}measure\ (\kappa\ x)$
obtains *ki* **where** $\bigwedge i.\ subprob\text{-}kernel\ X\ Y\ (ki\ i)$ $\bigwedge A\ x.\ x \in space\ X \Longrightarrow \kappa\ x\ A$
$= (\sum i.\ ki\ i\ x\ A)$
⟨*proof*⟩

2.5 Probability Kernel

locale *prob-kernel* = *measure-kernel* +
assumes *prob-spaces*: $\bigwedge x.\ x \in space\ X \Longrightarrow prob\text{-}space\ (\kappa\ x)$
begin

lemma *prob-space*:
$\bigwedge x.\ x \in space\ X \Longrightarrow \kappa\ x\ (space\ Y) = 1$
⟨*proof*⟩

lemma *prob-measurable*[*measurable*]:
$\kappa \in X \to_M prob\text{-}algebra\ Y$
⟨*proof*⟩

lemma *subprob-kernel*: $subprob\text{-}kernel\ X\ Y\ \kappa$
⟨*proof*⟩

sublocale *subprob-kernel*
⟨*proof*⟩

lemma *restrict-probability-kernel*:
$prob\text{-}kernel\ (restrict\text{-}space\ X\ A)\ Y\ \kappa$

$\langle proof \rangle$

end

lemma *prob-kernel-def'*:
prob-kernel $X\ Y\ \kappa \longleftrightarrow \kappa \in X \to_M$ *prob-algebra* Y
$\langle proof \rangle$

lemma *bind-kernel-return''*:
assumes *sets* $M =$ *sets* N
shows $M \gg\!=_k$ *return* $N = M$
$\langle proof \rangle$

2.6 S-Finite Kernel

locale *s-finite-kernel* = *measure-kernel* +
assumes *s-finite-kernel-sum*: $\exists\, ki.\ (\forall\, i.$ *finite-kernel* $X\ Y\ (ki\ i) \wedge (\forall\, x{\in}$*space* $X.$ $\forall\, A{\in}$*sets* $Y.\ \kappa\ x\ A = (\sum i.\ ki\ i\ x\ A)))$

lemma *s-finite-kernel-subI*:
assumes $\bigwedge x.\ x \in$ *space* $X \Longrightarrow$ *sets* $(\kappa\ x) =$ *sets* Y $\bigwedge i.$ *subprob-kernel* $X\ Y\ (ki\ i)$ $\bigwedge x\ A.\ x \in$ *space* $X \Longrightarrow A \in$ *sets* $Y \Longrightarrow$ *emeasure* $(\kappa\ x)\ A = (\sum i.\ ki\ i\ x\ A)$
shows *s-finite-kernel* $X\ Y\ \kappa$
$\langle proof \rangle$

context *s-finite-kernel*
begin

lemma *s-finite-kernels-fin*:
obtains *ki* **where** $\bigwedge i.$ *finite-kernel* $X\ Y\ (ki\ i)$ $\bigwedge x\ A.\ x \in$ *space* $X \Longrightarrow \kappa\ x\ A = (\sum i.\ ki\ i\ x\ A)$
$\langle proof \rangle$

lemma *s-finite-kernels*:
obtains *ki* **where** $\bigwedge i.$ *subprob-kernel* $X\ Y\ (ki\ i)$ $\bigwedge x\ A.\ x \in$ *space* $X \Longrightarrow \kappa\ x\ A = (\sum i.\ ki\ i\ x\ A)$
$\langle proof \rangle$

lemma *image-s-finite-measure*:
assumes $x \in$ *space* X
shows *s-finite-measure* $(\kappa\ x)$
$\langle proof \rangle$

corollary *kernel-measurable-s-finite*[*measurable*]:$\kappa \in X \to_M$ *s-finite-measure-algebra* Y
$\langle proof \rangle$

lemma *comp-measurable*:

assumes $f[measurable]{:}f \in M \rightarrow_M X$
shows *s-finite-kernel* $M\ Y\ (\lambda x.\ \kappa\ (f\ x))$
⟨*proof*⟩

lemma *distr-s-finite-kernel*:
assumes $f[measurable]{:}\ f \in Y \rightarrow_M Z$
shows *s-finite-kernel* $X\ Z\ (\lambda x.\ distr\ (\kappa\ x)\ Z\ f)$
⟨*proof*⟩

lemma *comp-s-finite-measure*:
assumes *s-finite-measure* μ **and** [*measurable-cong*]: *sets* μ = *sets* X
shows *s-finite-measure* $(\mu \gg\!=_k \kappa)$
⟨*proof*⟩

end

lemma *s-finite-kernel-empty-trivial*:
assumes *space* $X = \{\}$
shows *s-finite-kernel* $X\ Y\ k$
⟨*proof*⟩

lemma *s-finite-kernel-def'*: *s-finite-kernel* $X\ Y\ \kappa \longleftrightarrow ((\forall x.\ x \in space\ X \longrightarrow sets\ (\kappa\ x) = sets\ Y) \wedge (\exists ki.\ (\forall i.\ subprob\text{-}kernel\ X\ Y\ (ki\ i)) \wedge (\forall x\ A.\ x \in space\ X \longrightarrow A \in sets\ Y \longrightarrow emeasure\ (\kappa\ x)\ A = (\sum i.\ ki\ i\ x\ A))))$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

lemma(**in** *finite-kernel*) *s-finite-kernel-finite-kernel*: *s-finite-kernel* $X\ Y\ \kappa$
⟨*proof*⟩

lemmas(**in** *subprob-kernel*) *s-finite-kernel-subprob-kernel* = *s-finite-kernel-finite-kernel*
lemmas(**in** *prob-kernel*) *s-finite-kernel-prob-kernel* = *s-finite-kernel-subprob-kernel*

sublocale *finite-kernel* ⊆ *s-finite-kernel*
⟨*proof*⟩

lemma *s-finite-kernel-cong-sets*:
assumes *sets* X = *sets* X' *sets* Y = *sets* Y'
shows *s-finite-kernel* $X\ Y$ = *s-finite-kernel* $X'\ Y'$
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *s-finite-kernel-cong*:
assumes $\bigwedge x.\ x \in space\ X \Longrightarrow \kappa\ x = g\ x$
shows *s-finite-kernel* $X\ Y\ g$
⟨*proof*⟩

lemma(**in** *s-finite-measure*) *s-finite-kernel-const*:
assumes *space* $M \neq \{\}$
shows *s-finite-kernel* $X\ M\ (\lambda x.\ M)$
⟨*proof*⟩

lemma *s-finite-kernel-pair-countble1*:
assumes *countable A* $\bigwedge i.\ i \in A \Longrightarrow$ *s-finite-kernel X Y* $(\lambda x.\ k\ (i,x))$
shows *s-finite-kernel* (*count-space A* $\bigotimes_M$ *X*) *Y k*
⟨*proof*⟩

lemma *s-finite-kernel-s-finite-kernel*:
assumes $\bigwedge i.$ *s-finite-kernel X Y* (*ki i*) $\bigwedge x.\ x \in$ *space X* $\Longrightarrow$ *sets* $(k\ x)$ = *sets Y* $\bigwedge x\ A.\ x \in$ *space X* $\Longrightarrow A \in$ *sets Y* $\Longrightarrow$ *emeasure* $(k\ x)\ A = (\sum i.$ (*ki i*) $x\ A)$
shows *s-finite-kernel X Y k*
⟨*proof*⟩

lemma *s-finite-kernel-finite-sumI*:
assumes [*measurable-cong*]: $\bigwedge x.\ x \in$ *space X* $\Longrightarrow$ *sets* $(\kappa\ x)$ = *sets Y*
and $\bigwedge i.\ i \in I \Longrightarrow$ *subprob-kernel X Y* (*ki i*) $\bigwedge x\ A.\ x \in$ *space X* $\Longrightarrow A \in$ *sets Y* $\Longrightarrow$ *emeasure* $(\kappa\ x)\ A = (\sum i{\in}I.$ *ki i x A*) *finite I* $I \neq \{\}$
shows *s-finite-kernel X Y* κ
⟨*proof*⟩

Each kernel does not need to be bounded by a uniform upper-bound in the definition of *s-finite-kernel*

lemma *s-finite-kernel-finite-bounded-sum*:
assumes [*measurable-cong*]: $\bigwedge x.\ x \in$ *space X* $\Longrightarrow$ *sets* $(\kappa\ x)$ = *sets Y*
and $\bigwedge i.$ *measure-kernel X Y* (*ki i*) $\bigwedge x\ A.\ x \in$ *space X* $\Longrightarrow A \in$ *sets Y* $\Longrightarrow$ $\kappa\ x\ A = (\sum i.$ *ki i x A*) $\bigwedge i\ x.\ x \in$ *space X* $\Longrightarrow$ *ki i x* (*space Y*) $< \infty$
shows *s-finite-kernel X Y* κ
⟨*proof*⟩

lemma(**in** *measure-kernel*) *s-finite-kernel-finite-bounded*:
assumes $\bigwedge x.\ x \in$ *space X* $\Longrightarrow \kappa\ x$ (*space Y*) $< \infty$
shows *s-finite-kernel X Y* κ
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *density-s-finite-kernel*:
assumes *f*[*measurable*]: *case-prod f* $\in X \bigotimes_M Y \rightarrow_M$ *borel*
shows *s-finite-kernel X Y* $(\lambda x.$ *density* $(\kappa\ x)\ (f\ x))$
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *nn-integral-measurable-f*:
assumes [*measurable*]:$(\lambda(x,y).\ f\ x\ y) \in$ *borel-measurable* $(X \bigotimes_M Y)$
shows $(\lambda x.\ \int^+ y.\ f\ x\ y\ \partial(\kappa\ x)) \in$ *borel-measurable X*
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *nn-integral-measurable-f'*:
assumes $f \in$ *borel-measurable* $(X \bigotimes_M Y)$
shows $(\lambda x.\ \int^+ y.\ f\ (x,\ y)\ \partial(\kappa\ x)) \in$ *borel-measurable X*
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *bind-kernel-s-finite-kernel'*:

assumes *s-finite-kernel* $(X \bigotimes_M Y)$ Z (*case-prod* g)
shows *s-finite-kernel* X Z $(\lambda x.\ \kappa\ x \gg\!=_k g\ x)$
⟨*proof*⟩

corollary(**in** *s-finite-kernel*) *bind-kernel-s-finite-kernel*:
assumes *s-finite-kernel* Y Z k'
shows *s-finite-kernel* X Z $(\lambda x.\ \kappa\ x \gg\!=_k k')$
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *nn-integral-bind-kernel*:
assumes $f \in$ *borel-measurable* Y *sets* $\mu =$ *sets* X
shows $(\int^+ y.\ f\ y\ \partial(\mu \gg\!=_k \kappa)) = (\int^+ x.\ (\int^+ y.\ f\ y\ \partial(\kappa\ x))\ \partial\mu)$
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *bind-kernel-assoc*:
assumes *s-finite-kernel* Y Z k' *sets* $\mu =$ *sets* X
shows $\mu \gg\!=_k (\lambda x.\ \kappa\ x \gg\!=_k k') = \mu \gg\!=_k \kappa \gg\!=_k k'$
⟨*proof*⟩

lemma *s-finite-kernel-pair-measure*:
assumes *s-finite-kernel* X Y k **and** *s-finite-kernel* X Z k'
shows *s-finite-kernel* X $(Y \bigotimes_M Z)$ $(\lambda x.\ k\ x \bigotimes_M k'\ x)$
⟨*proof*⟩

lemma *pair-measure-eq-bind-s-finite*:
assumes *s-finite-measure* μ *s-finite-measure* ν
shows $\mu \bigotimes_M \nu = \mu \gg\!=_k (\lambda x.\ \nu \gg\!=_k (\lambda y.\ \mathit{return}\ (\mu \bigotimes_M \nu)\ (x,y)))$
⟨*proof*⟩

lemma *bind-kernel-rotate-return*:
assumes *s-finite-measure* μ *s-finite-measure* ν
shows $\mu \gg\!=_k (\lambda x.\ \nu \gg\!=_k (\lambda y.\ \mathit{return}\ (\mu \bigotimes_M \nu)\ (x,y))) = \nu \gg\!=_k (\lambda y.\ \mu \gg\!=_k (\lambda x.\ \mathit{return}\ (\mu \bigotimes_M \nu)\ (x,y)))$
⟨*proof*⟩

lemma *bind-kernel-rotate'*:
assumes *s-finite-measure* μ *s-finite-measure* ν *s-finite-kernel* $(\mu \bigotimes_M \nu)$ Z (*case-prod* f)
shows $\mu \gg\!=_k (\lambda x.\ \nu \gg\!=_k (\lambda y.\ f\ x\ y)) = \nu \gg\!=_k (\lambda y.\ \mu \gg\!=_k (\lambda x.\ f\ x\ y))$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-kernel-rotate*:
assumes *sets* $\mu =$ *sets* X **and** *sets* $\nu =$ *sets* Y
and *s-finite-measure* μ *s-finite-measure* ν *s-finite-kernel* $(X \bigotimes_M Y)$ Z $(\lambda(x,y).\ f\ x\ y)$
shows $\mu \gg\!=_k (\lambda x.\ \nu \gg\!=_k (\lambda y.\ f\ x\ y)) = \nu \gg\!=_k (\lambda y.\ \mu \gg\!=_k (\lambda x.\ f\ x\ y))$
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *emeasure-measurable′*:
assumes *A*[*measurable*]: (*SIGMA x*:*space X*. *A x*) $\in$ *sets* ($X \bigotimes_M Y$)
shows (λx. *emeasure* (κ *x*) (*A x*)) $\in$ *borel-measurable X*
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *measure-measurable′*:
assumes (*SIGMA x*:*space X*. *A x*) $\in$ *sets* ($X \bigotimes_M Y$)
shows (λx. *measure* (κ *x*) (*A x*)) $\in$ *borel-measurable X*
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *AE-pred*:
assumes *P*[*measurable*]:*Measurable.pred* ($X \bigotimes_M Y$) (*case-prod P*)
shows *Measurable.pred X* (λx. *AE y in* κ *x*. *P x y*)
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *integrable-probability-kernel-pred*:
fixes *f* :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:(λ(*x*,*y*). *f x y*) $\in$ *borel-measurable* ($X \bigotimes_M Y$)
shows *Measurable.pred X* (λx. *integrable* (κ *x*) (*f x*))
⟨*proof*⟩

corollary *integrable-measurable-subprob′*:
fixes *f* :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:(λ(*x*,*y*). *f x y*) $\in$ *borel-measurable* ($X \bigotimes_M Y$) *k* $\in X \rightarrow_M$ *subprob-algebra Y*
shows *Measurable.pred X* (λx. *integrable* (*k x*) (*f x*))
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *integrable-probability-kernel-pred′*:
fixes *f* :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes *f* $\in$ *borel-measurable* ($X \bigotimes_M Y$)
shows *Measurable.pred X* (λx. *integrable* (κ *x*) (*curry f x*))
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *lebesgue-integral-measurable-f-subprob*:
fixes *f* :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:*f* $\in$ *borel-measurable* ($X \bigotimes_M Y$)
shows (λx. $\int$ *y*. *f* (*x*,*y*) ∂(κ *x*)) $\in$ *borel-measurable X*
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *integrable-measurable-pred*[*measurable* (*raw*)]:
fixes *f* :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:*case-prod f* $\in$ *borel-measurable* ($X \bigotimes_M Y$)
shows *Measurable.pred X* (λx. *integrable* (κ *x*) (*f x*))
⟨*proof*⟩

lemma(**in** *s-finite-kernel*) *integral-measurable-f*:
fixes *f* :: - $\Rightarrow$ - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:*case-prod f* $\in$ *borel-measurable* ($X \bigotimes_M Y$)

shows $(\lambda x.\ \int y.\ f\ x\ y\ \partial(\kappa\ x)) \in$ *borel-measurable X*
$\langle proof \rangle$

lemma(**in** *s-finite-kernel*) *integral-measurable-f'*:
fixes f :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:$f \in$ *borel-measurable* $(X \bigotimes_M Y)$
shows $(\lambda x.\ \int y.\ f\ (x,y)\ \partial(\kappa\ x)) \in$ *borel-measurable X*
$\langle proof \rangle$

lemma(**in** *s-finite-kernel*)
fixes f :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable-cong*]: *sets* μ = *sets X*
and *integrable* $(\mu \ggg_k \kappa)$ f
shows *integrable-bind-kernelD1*: *integrable* μ $(\lambda x.\ \int y.$ *norm* $(f\ y)\ \partial\kappa\ x)$ (**is** *?g1*)
and *integrable-bind-kernelD1'*: *integrable* μ $(\lambda x.\ \int y.\ f\ y\ \partial\kappa\ x)$ (**is** *?g1'*)
and *integrable-bind-kernelD2*: *AE x in* μ. *integrable* $(\kappa\ x)$ f (**is** *?g2*)
and *integrable-bind-kernelD3*: *space X* $\neq \{\} \Longrightarrow f \in$ *borel-measurable Y* (**is** - $\Longrightarrow$ *?g3*)
$\langle proof \rangle$

lemma(**in** *s-finite-kernel*)
fixes f :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
assumes [*measurable-cong*]: *sets* μ = *sets X*
and [*measurable*]:*AE x in* μ. *integrable* $(\kappa\ x)$ f *integrable* μ $(\lambda x.\ \int y.$ *norm* $(f\ y)\ \partial\kappa\ x)$ $f \in$ *borel-measurable Y*
shows *integrable-bind-kernel*: *integrable* $(\mu \ggg_k \kappa)$ f
and *integral-bind-kernel*: $(\int y.\ f\ y\ \partial(\mu \ggg_k \kappa)) = (\int x.\ (\int y.\ f\ y \partial\kappa\ x)\partial\ \mu)$ (**is** *?eq*)
$\langle proof \rangle$

end

3 Quasi-Borel Spaces

theory *QuasiBorel*
imports *HOL−Probability.Probability*
begin

3.1 Definitions

3.1.1 Quasi-Borel Spaces

definition *qbs-closed1* :: $(real \Rightarrow 'a)$ *set* $\Rightarrow$ *bool*
where *qbs-closed1 Mx* $\equiv (\forall\, a \in Mx.\ \forall f \in (borel :: real\ measure) \rightarrow_M (borel :: real\ measure).\ a \circ f \in Mx)$

definition *qbs-closed2* :: $['a\ set, (real \Rightarrow 'a)\ set] \Rightarrow$ *bool*
where *qbs-closed2 X Mx* $\equiv (\forall\, x \in X.\ (\lambda r.\ x) \in Mx)$

definition *qbs-closed3* :: $(real \Rightarrow 'a)$ *set* $\Rightarrow$ *bool*
where *qbs-closed3 Mx* $\equiv (\forall P{::}real \Rightarrow nat.\ \forall Fi{::}nat \Rightarrow real \Rightarrow 'a.$
$(P \in borel \rightarrow_M count\text{-}space\ UNIV) \longrightarrow (\forall i.\ Fi\ i \in Mx) \longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx)$

lemma *separate-measurable*:
fixes $P :: real \Rightarrow nat$
assumes $\bigwedge i.\ P -` \{i\} \in sets\ borel$
shows $P \in borel \rightarrow_M count\text{-}space\ UNIV$
⟨*proof*⟩

lemma *measurable-separate*:
fixes $P :: real \Rightarrow nat$
assumes $P \in borel \rightarrow_M count\text{-}space\ UNIV$
shows $P -` \{i\} \in sets\ borel$
⟨*proof*⟩

definition *is-quasi-borel X Mx* $\longleftrightarrow Mx \subseteq UNIV \rightarrow X \wedge$ *qbs-closed1 Mx* $\wedge$ *qbs-closed2 X Mx* $\wedge$ *qbs-closed3 Mx*

lemma *is-quasi-borel-intro*[*simp*]:
assumes $Mx \subseteq UNIV \rightarrow X$
and *qbs-closed1 Mx qbs-closed2 X Mx qbs-closed3 Mx*
shows *is-quasi-borel X Mx*
⟨*proof*⟩

typedef *'a quasi-borel* = $\{(X{::}'a\ set,\ Mx).\ is\text{-}quasi\text{-}borel\ X\ Mx\}$
⟨*proof*⟩

definition *qbs-space* :: *'a quasi-borel* $\Rightarrow$ *'a set* **where**
qbs-space X $\equiv$ *fst* (*Rep-quasi-borel X*)

definition *qbs-Mx* :: *'a quasi-borel* $\Rightarrow (real \Rightarrow 'a)$ *set* **where**
qbs-Mx X $\equiv$ *snd* (*Rep-quasi-borel X*)

declare [[*coercion qbs-space*]]

lemma *qbs-decomp* : (*qbs-space X*,*qbs-Mx X*) $\in \{(X{::}'a\ set,\ Mx).\ is\text{-}quasi\text{-}borel\ X\ Mx\}$
⟨*proof*⟩

lemma *qbs-Mx-to-X*:
assumes $\alpha \in$ *qbs-Mx X*
shows $\alpha\ r \in$ *qbs-space X*
⟨*proof*⟩

lemma *qbs-closed1I*:
assumes $\bigwedge \alpha\ f.\ \alpha \in Mx \Longrightarrow f \in borel \rightarrow_M borel \Longrightarrow \alpha \circ f \in Mx$

shows *qbs-closed1 Mx*
⟨*proof*⟩

lemma *qbs-closed1-dest*[*simp*]:
assumes $\alpha \in$ *qbs-Mx X*
and $f \in$ *borel* $\rightarrow_M$ *borel*
shows $\alpha \circ f \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed1-dest'*[*simp*]:
assumes $\alpha \in$ *qbs-Mx X*
and $f \in$ *borel* $\rightarrow_M$ *borel*
shows $(\lambda r.\ \alpha\ (f\ r)) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed2I*:
assumes $\bigwedge x.\ x \in X \Longrightarrow (\lambda r.\ x) \in Mx$
shows *qbs-closed2 X Mx*
⟨*proof*⟩

lemma *qbs-closed2-dest*[*simp*]:
assumes $x \in$ *qbs-space X*
shows $(\lambda r.\ x) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed3I*:
assumes $\bigwedge(P :: real \Rightarrow nat)\ Fi.\ P \in borel \rightarrow_M count\text{-}space\ UNIV \Longrightarrow (\bigwedge i.\ Fi\ i \in Mx)$
$\Longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx$
shows *qbs-closed3 Mx*
⟨*proof*⟩

lemma *qbs-closed3I'*:
assumes $\bigwedge(P :: real \Rightarrow nat)\ Fi.\ (\bigwedge i.\ P -` \{i\} \in sets\ borel) \Longrightarrow (\bigwedge i.\ Fi\ i \in Mx)$
$\Longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx$
shows *qbs-closed3 Mx*
⟨*proof*⟩

lemma *qbs-closed3-dest*[*simp*]:
fixes $P{::}real \Rightarrow nat$ **and** $Fi :: nat \Rightarrow real \Rightarrow$ -
assumes $P \in$ *borel* $\rightarrow_M$ *count-space UNIV*
and $\bigwedge i.\ Fi\ i \in$ *qbs-Mx X*
shows $(\lambda r.\ Fi\ (P\ r)\ r) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed3-dest'*:
fixes $P{::}real \Rightarrow nat$ **and** $Fi :: nat \Rightarrow real \Rightarrow$ -
assumes $\bigwedge i.\ P -` \{i\} \in$ *sets borel*

and $\bigwedge i.\ Fi\ i \in qbs\text{-}Mx\ X$
shows $(\lambda r.\ Fi\ (P\ r)\ r) \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-closed3-dest2*:
assumes *countable I*
and [*measurable*]: $P \in borel \rightarrow_M count\text{-}space\ I$
and $\bigwedge i.\ i \in I \Longrightarrow Fi\ i \in qbs\text{-}Mx\ X$
shows $(\lambda r.\ Fi\ (P\ r)\ r) \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-closed3-dest2'*:
assumes *countable I*
and [*measurable*]: $P \in borel \rightarrow_M count\text{-}space\ I$
and $\bigwedge i.\ i \in range\ P \Longrightarrow Fi\ i \in qbs\text{-}Mx\ X$
shows $(\lambda r.\ Fi\ (P\ r)\ r) \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-Mx-indicat*:
assumes $S \in sets\ borel\ \alpha \in qbs\text{-}Mx\ X\ \beta \in qbs\text{-}Mx\ X$
shows $(\lambda r.\ if\ r \in S\ then\ \alpha\ r\ else\ \beta\ r) \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-space-Mx*: $qbs\text{-}space\ X = \{\alpha\ x\ |x\ \alpha.\ \alpha \in qbs\text{-}Mx\ X\}$
⟨*proof*⟩

lemma *qbs-space-eq-Mx*:
assumes $qbs\text{-}Mx\ X = qbs\text{-}Mx\ Y$
shows $qbs\text{-}space\ X = qbs\text{-}space\ Y$
⟨*proof*⟩

lemma *qbs-eqI*:
assumes $qbs\text{-}Mx\ X = qbs\text{-}Mx\ Y$
shows $X = Y$
⟨*proof*⟩

3.1.2 Empty Space

definition *empty-quasi-borel* :: *'a quasi-borel* **where**
$empty\text{-}quasi\text{-}borel \equiv Abs\text{-}quasi\text{-}borel\ (\{\},\{\})$

lemma
shows *eqb-space*[*simp*]: $qbs\text{-}space\ empty\text{-}quasi\text{-}borel = (\{\} :: 'a\ set)$
and *eqb-Mx*[*simp*]: $qbs\text{-}Mx\ empty\text{-}quasi\text{-}borel = (\{\} :: (real \Rightarrow 'a)\ set)$
⟨*proof*⟩

lemma *qbs-empty-equiv* :$qbs\text{-}space\ X = \{\} \longleftrightarrow qbs\text{-}Mx\ X = \{\}$
⟨*proof*⟩

lemma *empty-quasi-borel-iff*:
qbs-space $X = \{\} \longleftrightarrow X =$ *empty-quasi-borel*
⟨*proof*⟩

3.1.3 Unit Space

definition *unit-quasi-borel* :: *unit quasi-borel* (1_Q) **where**
unit-quasi-borel ≡ *Abs-quasi-borel* (*UNIV*,*UNIV*)

lemma
shows *unit-qbs-space*[*simp*]: *qbs-space unit-quasi-borel* = {()}
and *unit-qbs-Mx*[*simp*]: *qbs-Mx unit-quasi-borel* = $\{\lambda r.\ ()\}$
⟨*proof*⟩

3.1.4 Sub-Spaces

definition *sub-qbs* :: [*'a quasi-borel*, *'a set*] ⇒ *'a quasi-borel* **where**
sub-qbs X U ≡ *Abs-quasi-borel* (*qbs-space X* ∩ *U*,$\{\alpha.\ \alpha \in$ *qbs-Mx X* $\wedge (\forall r.\ \alpha\ r \in U)\}$)

lemma
shows *sub-qbs-space*: *qbs-space* (*sub-qbs X U*) = *qbs-space X* ∩ *U*
and *sub-qbs-Mx*: *qbs-Mx* (*sub-qbs X U*) = $\{\alpha.\ \alpha \in$ *qbs-Mx X* $\wedge (\forall r.\ \alpha\ r \in U)\}$
⟨*proof*⟩

lemma *sub-qbs*:
assumes *U* ⊆ *qbs-space X*
shows (*qbs-space* (*sub-qbs X U*), *qbs-Mx* (*sub-qbs X U*)) = (*U*, $\{f \in UNIV \rightarrow U.\ f \in$ *qbs-Mx X*$\}$)
⟨*proof*⟩

lemma *sub-qbs-ident*: *sub-qbs X* (*qbs-space X*) = *X*
⟨*proof*⟩

lemma *sub-qbs-sub-qbs*: *sub-qbs* (*sub-qbs X A*) *B* = *sub-qbs X* (*A* ∩ *B*)
⟨*proof*⟩

3.1.5 Image Spaces

definition *map-qbs* :: [*'a* ⇒ *'b*] ⇒ *'a quasi-borel* ⇒ *'b quasi-borel* **where**
map-qbs f X = *Abs-quasi-borel* (*f* ' (*qbs-space X*),$\{f \circ \alpha\ |\alpha.\ \alpha \in$ *qbs-Mx X*$\}$)

lemma
shows *map-qbs-space*: *qbs-space* (*map-qbs f X*) = *f* ' (*qbs-space X*)
and *map-qbs-Mx*: *qbs-Mx* (*map-qbs f X*) = $\{f \circ \alpha\ |\alpha.\ \alpha \in$ *qbs-Mx X*$\}$
⟨*proof*⟩

3.1.6 Binary Product Spaces

definition *pair-qbs* :: $['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel] \Rightarrow ('a \times 'b)\ quasi\text{-}borel$ (**infixr** $\bigotimes_Q$ *80*) **where**
pair-qbs X Y = *Abs-quasi-borel* (*qbs-space X* × *qbs-space Y*, {*f*. *fst* ∘ *f* ∈ *qbs-Mx X* ∧ *snd* ∘ *f* ∈ *qbs-Mx Y*})

lemma
shows *pair-qbs-space*: *qbs-space* ($X \bigotimes_Q Y$) = *qbs-space X* × *qbs-space Y*
and *pair-qbs-Mx*: *qbs-Mx* ($X \bigotimes_Q Y$) = {*f*. *fst* ∘ *f* ∈ *qbs-Mx X* ∧ *snd* ∘ *f* ∈ *qbs-Mx Y*}
⟨*proof*⟩

lemma *pair-qbs-fst*:
assumes *qbs-space Y* ≠ {}
shows *map-qbs fst* ($X \bigotimes_Q Y$) = *X*
⟨*proof*⟩

lemma *pair-qbs-snd*:
assumes *qbs-space X* ≠ {}
shows *map-qbs snd* ($X \bigotimes_Q Y$) = *Y*
⟨*proof*⟩

3.1.7 Binary Coproduct Spaces

definition *copair-qbs-Mx* :: ['*a quasi-borel*, '*b quasi-borel*] ⇒ (*real* => '*a* + '*b*) *set*
where
copair-qbs-Mx X Y ≡
{*g*. ∃ *S* ∈ *sets borel*.
(*S* = {} ⟶ (∃ *α1*∈ *qbs-Mx X*. *g* = (λ*r*. *Inl* (*α1 r*)))) ∧
(*S* = *UNIV* ⟶ (∃ *α2*∈ *qbs-Mx Y*. *g* = (λ*r*. *Inr* (*α2 r*)))) ∧
((*S* ≠ {} ∧ *S* ≠ *UNIV*) ⟶
(∃ *α1*∈ *qbs-Mx X*. ∃ *α2*∈ *qbs-Mx Y*.
g = (λ*r*::*real*. (*if* (*r* ∈ *S*) *then Inl* (*α1 r*) *else Inr* (*α2 r*)))))}

definition *copair-qbs* :: ['*a quasi-borel*, '*b quasi-borel*] ⇒ ('*a* + '*b*) *quasi-borel* (**infixr** $\bigoplus_Q$ *65*) **where**
copair-qbs X Y ≡ *Abs-quasi-borel* (*qbs-space X* <+> *qbs-space Y*, *copair-qbs-Mx X Y*)

The following is an equivalent definition of *copair-qbs-Mx*.

definition *copair-qbs-Mx2* :: ['*a quasi-borel*, '*b quasi-borel*] ⇒ (*real* => '*a* + '*b*) *set* **where**
copair-qbs-Mx2 X Y ≡
{*g*. (*if qbs-space X* = {} ∧ *qbs-space Y* = {} *then False*
else if qbs-space X ≠ {} ∧ *qbs-space Y* = {} *then*
(∃ *α1*∈ *qbs-Mx X*. *g* = (λ*r*. *Inl* (*α1 r*)))
else if qbs-space X = {} ∧ *qbs-space Y* ≠ {} *then*
(∃ *α2*∈ *qbs-Mx Y*. *g* = (λ*r*. *Inr* (*α2 r*)))
else

$(\exists\, S \in \textit{sets borel}.\ \exists\, \alpha 1 \in \textit{qbs-Mx}\ X.\ \exists\, \alpha 2 \in \textit{qbs-Mx}\ Y.$
$g = (\lambda r{::}\textit{real}.\ (\textit{if}\ (r \in S)\ \textit{then Inl}\ (\alpha 1\ r)\ \textit{else Inr}\ (\alpha 2\ r))))\ \}$

lemma *copair-qbs-Mx-equiv* :*copair-qbs-Mx* $(X :: {}'a\ \textit{quasi-borel})$ $(Y :: {}'b\ \textit{quasi-borel})$ $=$ *copair-qbs-Mx2* X Y
⟨*proof*⟩

lemma
shows *copair-qbs-space*: *qbs-space* $(X \bigoplus_Q Y) =$ *qbs-space* X <+> *qbs-space* Y (**is** *?goal1*)
and *copair-qbs-Mx*: *qbs-Mx* $(X \bigoplus_Q Y) =$ *copair-qbs-Mx* X Y (**is** *?goal2*)
⟨*proof*⟩

lemma *copair-qbs-MxD*:
assumes $g \in \textit{qbs-Mx}\ (X \bigoplus_Q Y)$
and $\bigwedge\alpha.\ \alpha \in \textit{qbs-Mx}\ X \Longrightarrow g = (\lambda r.\ \textit{Inl}\ (\alpha\ r)) \Longrightarrow P\ g$
and $\bigwedge\beta.\ \beta \in \textit{qbs-Mx}\ Y \Longrightarrow g = (\lambda r.\ \textit{Inr}\ (\beta\ r)) \Longrightarrow P\ g$
and $\bigwedge S\ \alpha\ \beta.\ (S :: \textit{real set}) \in \textit{sets borel} \Longrightarrow S \neq \{\} \Longrightarrow S \neq \textit{UNIV} \Longrightarrow \alpha \in \textit{qbs-Mx}\ X \Longrightarrow \beta \in \textit{qbs-Mx}\ Y \Longrightarrow g = (\lambda r.\ \textit{if}\ r \in S\ \textit{then Inl}\ (\alpha\ r)\ \textit{else Inr}\ (\beta\ r)) \Longrightarrow P\ g$
shows $P\ g$
⟨*proof*⟩

3.1.8 Product Spaces

definition *PiQ* :: $'a\ \textit{set} \Rightarrow ('a \Rightarrow {}'b\ \textit{quasi-borel}) \Rightarrow ('a \Rightarrow {}'b)\ \textit{quasi-borel}$ **where**
$\textit{PiQ}\ I\ X \equiv \textit{Abs-quasi-borel}\ (\Pi_E\ i{\in}I.\ \textit{qbs-space}\ (X\ i),\ \{\alpha.\ \forall\, i.\ (i \in I \longrightarrow (\lambda r.\ \alpha\ r\ i) \in \textit{qbs-Mx}\ (X\ i)) \wedge (i \notin I \longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ \textit{undefined}))\})$

syntax
-PiQ :: $\textit{pttrn} \Rightarrow {}'i\ \textit{set} \Rightarrow {}'a\ \textit{quasi-borel} \Rightarrow ('i => {}'a)\ \textit{quasi-borel}$ $((3\Pi_Q\ \text{-}{\in}\text{-}./\ \text{-})\ 10)$
translations
$\Pi_Q\ x{\in}I.\ X == \textit{CONST PiQ}\ I\ (\lambda x.\ X)$

lemma
shows *PiQ-space*: $\textit{qbs-space}\ (\textit{PiQ}\ I\ X) = (\Pi_E\ i{\in}I.\ \textit{qbs-space}\ (X\ i))$ (**is** *?goal1*)
and *PiQ-Mx*: $\textit{qbs-Mx}\ (\textit{PiQ}\ I\ X) = \{\alpha.\ \forall\, i.\ (i \in I \longrightarrow (\lambda r.\ \alpha\ r\ i) \in \textit{qbs-Mx}\ (X\ i)) \wedge (i \notin I \longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ \textit{undefined}))\}$ (**is** - = *?Mx*)
⟨*proof*⟩

lemma *prod-qbs-MxI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow (\lambda r.\ \alpha\ r\ i) \in \textit{qbs-Mx}\ (X\ i)$
and $\bigwedge i.\ i \notin I \Longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ \textit{undefined})$
shows $\alpha \in \textit{qbs-Mx}\ (\textit{PiQ}\ I\ X)$
⟨*proof*⟩

lemma *prod-qbs-MxD*:
assumes $\alpha \in \textit{qbs-Mx}\ (\textit{PiQ}\ I\ X)$

shows $\bigwedge i.\ i \in I \Longrightarrow (\lambda r.\ \alpha\ r\ i) \in$ *qbs-Mx* $(X\ i)$
and $\bigwedge i.\ i \notin I \Longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.$ *undefined*$)$
and $\bigwedge i\ r.\ i \notin I \Longrightarrow \alpha\ r\ i =$ *undefined*
⟨*proof*⟩

lemma *PiQ-eqI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow X\ i = Y\ i$
shows *PiQ* $I\ X =$ *PiQ* $I\ Y$
⟨*proof*⟩

lemma *PiQ-empty*: *qbs-space* (*PiQ* {} X) = {$\lambda i.$ *undefined*}
⟨*proof*⟩

lemma *PiQ-empty-Mx*: *qbs-Mx* (*PiQ* {} X) = {$\lambda r\ i.$ *undefined*}
⟨*proof*⟩

3.1.9 Coproduct Spaces

definition *coPiQ-Mx* :: [*'a set*, *'a* ⇒ *'b quasi-borel*] ⇒ (*real* ⇒ *'a* × *'b*) *set* **where**
coPiQ-Mx $I\ X \equiv$ { $\lambda r.$ ($f\ r$, α ($f\ r$) r) |$f\ \alpha.\ f \in$ *borel* $\rightarrow_M$ *count-space* $I \wedge (\forall\ i \in$*range* $f.\ \alpha\ i \in$ *qbs-Mx* $(X\ i))$}

definition *coPiQ-Mx'* :: [*'a set*, *'a* ⇒ *'b quasi-borel*] ⇒ (*real* ⇒ *'a* × *'b*) *set* **where**
coPiQ-Mx' $I\ X \equiv$ { $\lambda r.$ ($f\ r$, α ($f\ r$) r) |$f\ \alpha.\ f \in$ *borel* $\rightarrow_M$ *count-space* $I \wedge (\forall\ i.$ $(i \in$ *range* $f \vee$ *qbs-space* $(X\ i) \neq \{\}) \longrightarrow \alpha\ i \in$ *qbs-Mx* $(X\ i))$}

lemma *coPiQ-Mx-eq*:
coPiQ-Mx $I\ X =$ *coPiQ-Mx'* $I\ X$
⟨*proof*⟩

definition *coPiQ* :: [*'a set*, *'a* ⇒ *'b quasi-borel*] ⇒ (*'a* × *'b*) *quasi-borel* **where**
coPiQ $I\ X \equiv$ *Abs-quasi-borel* (*SIGMA* i:I. *qbs-space* $(X\ i)$, *coPiQ-Mx* $I\ X$)

syntax
-coPiQ :: *pttrn* ⇒ *'i set* ⇒ *'a quasi-borel* ⇒ (*'i* × *'a*) *quasi-borel* ((*3*$\amalg_Q$ -∈-./ -) *10*)
translations
$\amalg_Q$ x∈I. X ⇌ *CONST coPiQ* I ($\lambda x.\ X$)

lemma
shows *coPiQ-space*: *qbs-space* (*coPiQ* $I\ X$) = (*SIGMA* i:I. *qbs-space* $(X\ i)$) (**is** *?goal1*)
and *coPiQ-Mx*: *qbs-Mx* (*coPiQ* $I\ X$) = *coPiQ-Mx* $I\ X$ (**is** *?goal2*)
⟨*proof*⟩

lemma *coPiQ-MxI*:
assumes $f \in$ *borel* $\rightarrow_M$ *count-space* I
and $\bigwedge i.\ i \in$ *range* $f \Longrightarrow \alpha\ i \in$ *qbs-Mx* $(X\ i)$
shows ($\lambda r.$ ($f\ r$, α ($f\ r$) r)) ∈ *qbs-Mx* (*coPiQ* $I\ X$)

$\langle proof \rangle$

lemma *coPiQ-eqI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow X\ i = Y\ i$
shows $coPiQ\ I\ X = coPiQ\ I\ Y$
$\langle proof \rangle$

3.1.10 List Spaces

We define the quasi-Borel spaces on list using the following isomorphism.

$$List(X) \cong \coprod_{n \in \mathbb{N}} \prod_{0 \leq i < n} X$$

definition *list-nil* :: $nat \times (nat \Rightarrow 'a)$ **where**
$list\text{-}nil \equiv (0,\ \lambda n.\ undefined)$
definition *list-cons* :: $['a,\ nat \times (nat \Rightarrow 'a)] \Rightarrow nat \times (nat \Rightarrow 'a)$ **where**
$list\text{-}cons\ x\ l \equiv (Suc\ (fst\ l),\ (\lambda n.\ if\ n = 0\ then\ x\ else\ (snd\ l)\ (n - 1)))$

fun *from-list* :: $'a\ list \Rightarrow nat \times (nat \Rightarrow 'a)$ **where**
$from\text{-}list\ [] = list\text{-}nil\ |$
$from\text{-}list\ (a\#l) = list\text{-}cons\ a\ (from\text{-}list\ l)$

fun *to-list'* :: $nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a\ list$ **where**
$to\text{-}list'\ 0\ \text{-} = []\ |$
$to\text{-}list'\ (Suc\ n)\ f = f\ 0\ \#\ to\text{-}list'\ n\ (\lambda n.\ f\ (Suc\ n))$

definition *to-list* :: $nat \times (nat \Rightarrow 'a) \Rightarrow 'a\ list$ **where**
$to\text{-}list \equiv case\text{-}prod\ to\text{-}list'$

lemma *inj-on-to-list*: $inj\text{-}on\ (to\text{-}list :: nat \times (nat \Rightarrow 'a) \Rightarrow 'a\ list)\ (SIGMA\ n{:}UNIV.\ PiE\ \{..<n\}\ A)$
$\langle proof \rangle$

Definition

definition *list-qbs* :: $'a\ quasi\text{-}borel \Rightarrow 'a\ list\ quasi\text{-}borel$ **where**
$list\text{-}qbs\ X \equiv map\text{-}qbs\ to\text{-}list\ (\Pi_Q\ n{\in}(UNIV :: nat\ set).\Pi_Q\ i{\in}\{..<n\}.\ X)$

definition *list-head* :: $nat \times (nat \Rightarrow 'a) \Rightarrow 'a$ **where**
$list\text{-}head\ l = snd\ l\ 0$
definition *list-tail* :: $nat \times (nat \Rightarrow 'a) \Rightarrow nat \times (nat \Rightarrow 'a)$ **where**
$list\text{-}tail\ l = (fst\ l - 1,\ \lambda m.\ (snd\ l)\ (Suc\ m))$

lemma *list-simp1*: $list\text{-}nil \neq list\text{-}cons\ x\ l$
$\langle proof \rangle$

lemma *list-simp2*:
assumes $list\text{-}cons\ a\ al = list\text{-}cons\ b\ bl$
shows $a = b\ al = bl$

⟨*proof*⟩

lemma
shows *list-simp3*:*list-head* (*list-cons a l*) = *a*
and *list-simp4*:*list-tail* (*list-cons a l*) = *l*
⟨*proof*⟩

lemma *list-decomp1*:
assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
shows $l =$ *list-nil* $\vee$
($\exists\, a\ l'$. $a \in$ *qbs-space X* $\wedge$ $l' \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*) $\wedge$ $l =$ *list-cons a* l')
⟨*proof*⟩

lemma *list-simp5*:
assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
and $l \neq$ *list-nil*
shows $l =$ *list-cons* (*list-head l*) (*list-tail l*)
⟨*proof*⟩

lemma *list-simp6*:
list-nil $\in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
⟨*proof*⟩

lemma *list-simp7*:
assumes $a \in$ *qbs-space X*
and $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
shows *list-cons a l* $\in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
⟨*proof*⟩

lemma *list-destruct-rule*:
assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
P list-nil
and $\bigwedge a\ l'$. $a \in$ *qbs-space X* $\Longrightarrow$ $l' \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*) $\Longrightarrow$ *P* (*list-cons a* l')
shows *P l*
⟨*proof*⟩

lemma *list-induct-rule*:
assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*)
P list-nil
and $\bigwedge a\ l'$. $a \in$ *qbs-space X* $\Longrightarrow$ $l' \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *X*) $\Longrightarrow$ *P* l' $\Longrightarrow$ *P* (*list-cons a* l')
shows *P l*
⟨*proof*⟩

lemma *to-list-simp1*: *to-list list-nil* = []
⟨*proof*⟩

lemma *to-list-simp2*:
  assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  shows *to-list* (*list-cons* a l) = a # *to-list* l
  ⟨*proof*⟩

lemma *to-list-set*:
  assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  shows *set* (*to-list* l) $\subseteq$ *qbs-space* X
  ⟨*proof*⟩

lemma *from-list-length*: *fst* (*from-list* l) = *length* l
  ⟨*proof*⟩

lemma *from-list-in-list-of*:
  assumes *set* $l \subseteq$ *qbs-space* X
  shows *from-list* $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  ⟨*proof*⟩

lemma *from-list-in-list-of'*: *from-list* $l \in$ *qbs-space* ((Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. *Abs-quasi-borel* (*UNIV*,*UNIV*)))
⟨*proof*⟩

lemma *list-cons-in-list-of*:
  assumes *set* (a#l) $\subseteq$ *qbs-space* X
  shows *list-cons* a (*from-list* l) $\in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  ⟨*proof*⟩

lemma *from-list-to-list-ident*:
  to-list (*from-list* l) = l
  ⟨*proof*⟩

lemma *to-list-from-list-ident*:
  assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  shows *from-list* (*to-list* l) = l
⟨*proof*⟩

definition *rec-list'* :: $'b \Rightarrow ('a \Rightarrow (nat \times (nat \Rightarrow 'a)) \Rightarrow 'b \Rightarrow 'b) \Rightarrow (nat \times (nat \Rightarrow 'a)) \Rightarrow 'b$ **where**
rec-list' *t0* f l $\equiv$ (*rec-list* *t0* (λx l'. f x (*from-list* l')) (*to-list* l))

lemma *rec-list'-simp1*:
  rec-list' t f *list-nil* = t
  ⟨*proof*⟩

lemma *rec-list'-simp2*:
  assumes $l \in$ *qbs-space* (Π_Q $n \in$(*UNIV* :: *nat set*).Π_Q $i \in \{..<n\}$. X)
  shows *rec-list'* t f (*list-cons* x l) = f x l (*rec-list'* t f l)
  ⟨*proof*⟩

lemma *list-qbs-space*: $qbs\text{-}space\ (list\text{-}qbs\ X) = lists\ (qbs\text{-}space\ X)$
⟨*proof*⟩

3.1.11 Option Spaces

The option spaces is defined using the following isomorphism.

$$Option(X) \cong X + 1$$

definition *option-qbs* :: $'a\ quasi\text{-}borel \Rightarrow 'a\ option\ quasi\text{-}borel$ **where**
$option\text{-}qbs\ X = map\text{-}qbs\ (\lambda x.\ case\ x\ of\ Inl\ y \Rightarrow Some\ y \mid Inr\ y \Rightarrow None)\ (X \bigoplus_Q 1_Q)$

lemma *option-qbs-space*: $qbs\text{-}space\ (option\text{-}qbs\ X) = \{Some\ x|x.\ x \in qbs\text{-}space\ X\} \cup \{None\}$
⟨*proof*⟩

3.1.12 Function Spaces

definition *exp-qbs* :: $['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel] \Rightarrow ('a \Rightarrow 'b)\ quasi\text{-}borel$ (**infixr** $\Rightarrow_Q$ *61*) **where**
$X \Rightarrow_Q Y \equiv Abs\text{-}quasi\text{-}borel\ (\{f.\ \forall\alpha \in qbs\text{-}Mx\ X.\ f \circ \alpha \in qbs\text{-}Mx\ Y\}, \{g.\ \forall\alpha\in borel\text{-}measurable\ borel.\ \forall\beta\in qbs\text{-}Mx\ X.\ (\lambda r.\ g\ (\alpha\ r)\ (\beta\ r)) \in qbs\text{-}Mx\ Y\})$

lemma
shows *exp-qbs-space*: $qbs\text{-}space\ (exp\text{-}qbs\ X\ Y) = \{f.\ \forall\alpha \in qbs\text{-}Mx\ X.\ f \circ \alpha \in qbs\text{-}Mx\ Y\}$
and *exp-qbs-Mx*: $qbs\text{-}Mx\ (exp\text{-}qbs\ X\ Y) = \{g.\ \forall\alpha\in borel\text{-}measurable\ borel.\ \forall\beta\in qbs\text{-}Mx\ X.\ (\lambda r.\ g\ (\alpha\ r)\ (\beta\ r)) \in qbs\text{-}Mx\ Y\}$
⟨*proof*⟩

3.1.13 Ordering on Quasi-Borel Spaces

inductive-set *generating-Mx* :: $'a\ set \Rightarrow (real \Rightarrow 'a)\ set \Rightarrow (real \Rightarrow 'a)\ set$
for X :: $'a\ set$ **and** Mx :: $(real \Rightarrow 'a)\ set$
where
Basic: $\alpha \in Mx \Longrightarrow \alpha \in generating\text{-}Mx\ X\ Mx$
| *Const*: $x \in X \Longrightarrow (\lambda r.\ x) \in generating\text{-}Mx\ X\ Mx$
| *Comp* : $f \in (borel :: real\ measure) \rightarrow_M (borel :: real\ measure) \Longrightarrow \alpha \in generating\text{-}Mx\ X\ Mx \Longrightarrow \alpha \circ f \in generating\text{-}Mx\ X\ Mx$
| *Part* : $(\bigwedge i.\ Fi\ i \in generating\text{-}Mx\ X\ Mx) \Longrightarrow P \in borel \rightarrow_M count\text{-}space\ (UNIV :: nat\ set) \Longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in generating\text{-}Mx\ X\ Mx$

lemma *generating-Mx-to-space*:
assumes $Mx \subseteq UNIV \rightarrow X$
shows $generating\text{-}Mx\ X\ Mx \subseteq UNIV \rightarrow X$
⟨*proof*⟩

lemma *generating-Mx-closed1*:
qbs-closed1 (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-closed2*:
qbs-closed2 X (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-closed3*:
qbs-closed3 (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-Mx*:
generating-Mx (*qbs-space X*) (*qbs-Mx X*) = *qbs-Mx X*
⟨*proof*⟩

instantiation *quasi-borel* :: (*type*) *order-bot*
begin

inductive *less-eq-quasi-borel* :: *'a quasi-borel* ⇒ *'a quasi-borel* ⇒ *bool* **where**
qbs-space X ⊂ *qbs-space Y* ⟹ *less-eq-quasi-borel X Y*
| *qbs-space X* = *qbs-space Y* ⟹ *qbs-Mx Y* ⊆ *qbs-Mx X* ⟹ *less-eq-quasi-borel X Y*

lemma *le-quasi-borel-iff*:
X ≤ *Y* ⟷ (*if qbs-space X* = *qbs-space Y then qbs-Mx Y* ⊆ *qbs-Mx X else qbs-space X* ⊂ *qbs-space Y*)
⟨*proof*⟩

definition *less-quasi-borel* :: *'a quasi-borel* ⇒ *'a quasi-borel* ⇒ *bool* **where**
less-quasi-borel X Y ⟷ (*X* ≤ *Y* ∧ ¬ *Y* ≤ *X*)

definition *bot-quasi-borel* :: *'a quasi-borel* **where**
bot-quasi-borel = *empty-quasi-borel*

instance
⟨*proof*⟩
end

definition *inf-quasi-borel* :: [*'a quasi-borel*, *'a quasi-borel*] ⇒ *'a quasi-borel* **where**
inf-quasi-borel X X' = *Abs-quasi-borel* (*qbs-space X* ∩ *qbs-space X'*, *qbs-Mx X* ∩ *qbs-Mx X'*)

lemma *inf-quasi-borel-correct*: *Rep-quasi-borel* (*inf-quasi-borel X X'*) = (*qbs-space X* ∩ *qbs-space X'*, *qbs-Mx X* ∩ *qbs-Mx X'*)
⟨*proof*⟩

lemma *inf-qbs-space*[*simp*]: *qbs-space* (*inf-quasi-borel X X'*) = *qbs-space X* ∩ *qbs-space X'*

⟨*proof*⟩

lemma *inf-qbs-Mx*[*simp*]: *qbs-Mx* (*inf-quasi-borel X X′*) = *qbs-Mx X* ∩ *qbs-Mx X′*
⟨*proof*⟩

definition *max-quasi-borel* :: *′a set* ⇒ *′a quasi-borel* **where**
max-quasi-borel X = *Abs-quasi-borel* (*X*, *UNIV* → *X*)

lemma *max-quasi-borel-correct*: *Rep-quasi-borel* (*max-quasi-borel X*) = (*X*, *UNIV* → *X*)
⟨*proof*⟩

lemma *max-qbs-space*[*simp*]: *qbs-space* (*max-quasi-borel X*) = *X*
⟨*proof*⟩

lemma *max-qbs-Mx*[*simp*]: *qbs-Mx* (*max-quasi-borel X*) = *UNIV* → *X*
⟨*proof*⟩

instantiation *quasi-borel* :: (*type*) *semilattice-sup*
begin

definition *sup-quasi-borel* :: *′a quasi-borel* ⇒ *′a quasi-borel* ⇒ *′a quasi-borel* **where**
sup-quasi-borel X Y ≡ (*if qbs-space X* = *qbs-space Y* *then inf-quasi-borel X Y*
else if qbs-space X ⊂ *qbs-space Y then Y*
else if qbs-space Y ⊂ *qbs-space X then X*
else max-quasi-borel (*qbs-space X* ∪ *qbs-space Y*))

instance
⟨*proof*⟩

end

end

3.2 Morphisms of Quasi-Borel Spaces

theory *QBS-Morphism*

imports
QuasiBorel

begin

abbreviation *qbs-morphism* :: [*′a quasi-borel*, *′b quasi-borel*] ⇒ (*′a* ⇒ *′b*) *set* (**infixr** $\rightarrow_Q$ *60*) **where**
$X \rightarrow_Q Y$ ≡ *qbs-space* ($X \Rightarrow_Q Y$)

lemma *qbs-morphismI*: ($\bigwedge\alpha$. α ∈ *qbs-Mx X* ⟹ *f* ∘ α ∈ *qbs-Mx Y*) ⟹ *f* ∈ *X*

$\to_Q$ Y
⟨*proof*⟩

lemma *qbs-morphism-def*: $X \to_Q Y = \{f{\in}\mathit{qbs\text{-}space}\ X \to \mathit{qbs\text{-}space}\ Y.\ \forall\,\alpha \in \mathit{qbs\text{-}Mx}\ X.\ f \circ \alpha \in \mathit{qbs\text{-}Mx}\ Y\}$
⟨*proof*⟩

lemma *qbs-morphism-Mx*:
assumes $f \in X \to_Q Y$ $\alpha \in \mathit{qbs\text{-}Mx}\ X$
shows $f \circ \alpha \in \mathit{qbs\text{-}Mx}\ Y$
⟨*proof*⟩

lemma *qbs-morphism-space*:
assumes $f \in X \to_Q Y$ $x \in \mathit{qbs\text{-}space}\ X$
shows $f\ x \in \mathit{qbs\text{-}space}\ Y$
⟨*proof*⟩

lemma *qbs-morphism-ident*[*simp*]:
$\mathit{id} \in X \to_Q X$
⟨*proof*⟩

lemma *qbs-morphism-ident'*[*simp*]:
$(\lambda x.\ x) \in X \to_Q X$
⟨*proof*⟩

lemma *qbs-morphism-comp*:
assumes $f \in X \to_Q Y$ $g \in Y \to_Q Z$
shows $g \circ f \in X \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-compose-rev*:
assumes $f \in Y \to_Q Z$ **and** $g \in X \to_Q Y$
shows $(\lambda x.\ f\ (g\ x)) \in X \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-compose*:
assumes $g \in X \to_Q Y$ **and** $f \in Y \to_Q Z$
shows $(\lambda x.\ f\ (g\ x)) \in X \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-cong'*:
assumes $\bigwedge x.\ x \in \mathit{qbs\text{-}space}\ X \Longrightarrow f\ x = g\ x$
and $f \in X \to_Q Y$
shows $g \in X \to_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-cong*:
assumes $\bigwedge x.\ x \in \mathit{qbs\text{-}space}\ X \Longrightarrow f\ x = g\ x$
shows $f \in X \to_Q Y \longleftrightarrow g \in X \to_Q Y$

$\langle proof \rangle$

lemma *qbs-morphism-const*:
  assumes $y \in$ *qbs-space* Y
  shows $(\lambda x.\ y) \in X \rightarrow_Q Y$
  $\langle proof \rangle$

lemma *qbs-morphism-from-empty*: *qbs-space* $X = \{\} \Longrightarrow f \in X \rightarrow_Q Y$
  $\langle proof \rangle$

lemma *unit-quasi-borel-terminal*: $\exists!\ f.\ f \in X \rightarrow_Q$ *unit-quasi-borel*
  $\langle proof \rangle$

definition *to-unit-quasi-borel* :: $'a \Rightarrow$ *unit* $(!_Q)$ **where**
to-unit-quasi-borel $\equiv (\lambda r.())$

lemma *to-unit-quasi-borel-morphism*:
$!_Q \in X \rightarrow_Q$ *unit-quasi-borel*
  $\langle proof \rangle$

lemma *qbs-morphism-subD*:
  assumes $f \in X \rightarrow_Q$ *sub-qbs* Y A
  shows $f \in X \rightarrow_Q Y$
  $\langle proof \rangle$

lemma *qbs-morphism-subI1*:
  assumes $f \in X \rightarrow_Q Y$ $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\,x \in A$
  shows $f \in X \rightarrow_Q$ *sub-qbs* Y A
  $\langle proof \rangle$

lemma *qbs-morphism-subI2*:
  assumes $f \in X \rightarrow_Q Y$
  shows $f \in$ *sub-qbs* X $A \rightarrow_Q Y$
  $\langle proof \rangle$

corollary *qbs-morphism-subsubI*:
  assumes $f \in X \rightarrow_Q Y$ $\bigwedge x.\ x \in A \Longrightarrow f\,x \in B$
  shows $f \in$ *sub-qbs* X $A \rightarrow_Q$ *sub-qbs* Y B
  $\langle proof \rangle$

lemma *map-qbs-morphism-f*: $f \in X \rightarrow_Q$ *map-qbs* f X
  $\langle proof \rangle$

lemma *map-qbs-morphism-inverse-f*:
  assumes $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow g\ (f\,x) = x$
  shows $g \in$ *map-qbs* f $X \rightarrow_Q X$
$\langle proof \rangle$

lemma *pair-qbs-morphismI*:

assumes $\bigwedge\alpha\ \beta.\ \alpha \in \mathit{qbs\text{-}Mx}\ X \Longrightarrow \beta \in \mathit{qbs\text{-}Mx}\ Y$
$\Longrightarrow (\lambda r.\ f\ (\alpha\ r,\ \beta\ r)) \in \mathit{qbs\text{-}Mx}\ Z$
shows $f \in (X \bigotimes_Q\ Y) \rightarrow_Q Z$
⟨*proof*⟩

lemma *pair-qbs-MxD*:
assumes $\gamma \in \mathit{qbs\text{-}Mx}\ (X \bigotimes_Q\ Y)$
obtains $\alpha\ \beta$ **where** $\alpha \in \mathit{qbs\text{-}Mx}\ X\ \beta \in \mathit{qbs\text{-}Mx}\ Y\ \gamma = (\lambda x.\ (\alpha\ x,\ \beta\ x))$
⟨*proof*⟩

lemma *pair-qbs-MxI*:
assumes $(\lambda x.\ \mathit{fst}\ (\gamma\ x)) \in \mathit{qbs\text{-}Mx}\ X$ **and** $(\lambda x.\ \mathit{snd}\ (\gamma\ x)) \in \mathit{qbs\text{-}Mx}\ Y$
shows $\gamma \in \mathit{qbs\text{-}Mx}\ (X \bigotimes_Q\ Y)$
⟨*proof*⟩

lemma
shows *fst-qbs-morphism*: $\mathit{fst} \in X \bigotimes_Q\ Y \rightarrow_Q X$
and *snd-qbs-morphism*: $\mathit{snd} \in X \bigotimes_Q\ Y \rightarrow_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-pair-iff*:
$f \in X \rightarrow_Q Y \bigotimes_Q Z \longleftrightarrow \mathit{fst} \circ f \in X \rightarrow_Q Y \wedge \mathit{snd} \circ f \in X \rightarrow_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-Pair*:
assumes $f \in Z \rightarrow_Q X$
and $g \in Z \rightarrow_Q Y$
shows $(\lambda z.\ (f\ z,\ g\ z)) \in Z \rightarrow_Q X \bigotimes_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-curry*: $\mathit{curry} \in \mathit{exp\text{-}qbs}\ (X \bigotimes_Q\ Y)\ Z \rightarrow_Q \mathit{exp\text{-}qbs}\ X\ (\mathit{exp\text{-}qbs}\ Y\ Z)$
⟨*proof*⟩

corollary *curry-preserves-morphisms*:
assumes $(\lambda xy.\ f\ (\mathit{fst}\ xy)\ (\mathit{snd}\ xy)) \in X \bigotimes_Q\ Y \rightarrow_Q Z$
shows $f \in X \rightarrow_Q Y \Rightarrow_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-eval*:
$(\lambda fx.\ (\mathit{fst}\ fx)\ (\mathit{snd}\ fx)) \in (X \Rightarrow_Q Y) \bigotimes_Q X \rightarrow_Q Y$
⟨*proof*⟩

corollary *qbs-morphism-app*:
assumes $f \in X \rightarrow_Q (Y \Rightarrow_Q Z)\ g \in X \rightarrow_Q Y$
shows $(\lambda x.\ (f\ x)\ (g\ x)) \in X \rightarrow_Q Z$
⟨*proof*⟩

⟨*ML*⟩

declare
fst-qbs-morphism[*qbs*]
snd-qbs-morphism[*qbs*]
qbs-morphism-const[*qbs*]
qbs-morphism-ident[*qbs*]
qbs-morphism-ident'[*qbs*]
qbs-morphism-curry[*qbs*]

lemma [*qbs*]:
shows *qbs-morphism-Pair1*: $Pair \in X \to_Q Y \Rightarrow_Q (X \bigotimes_Q Y)$
⟨*proof*⟩

lemma *qbs-morphism-case-prod*[*qbs*]: $case\text{-}prod \in exp\text{-}qbs\ X\ (exp\text{-}qbs\ Y\ Z) \to_Q exp\text{-}qbs\ (X \bigotimes_Q Y)\ Z$
⟨*proof*⟩

lemma *uncurry-preserves-morphisms*:
assumes [*qbs*]:$(\lambda x\ y.\ f\ (x,y)) \in X \to_Q Y \Rightarrow_Q Z$
shows $f \in X \bigotimes_Q Y \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-comp'*[*qbs*]:$comp \in Y \Rightarrow_Q Z \to_Q (X \Rightarrow_Q Y) \Rightarrow_Q X \Rightarrow_Q Z$
⟨*proof*⟩

lemma *arg-swap-morphism*:
assumes $f \in X \to_Q exp\text{-}qbs\ Y\ Z$
shows $(\lambda y\ x.\ f\ x\ y) \in Y \to_Q exp\text{-}qbs\ X\ Z$
⟨*proof*⟩

lemma *exp-qbs-comp-morphism*:
assumes $f \in W \to_Q exp\text{-}qbs\ X\ Y$
and $g \in W \to_Q exp\text{-}qbs\ Y\ Z$
shows $(\lambda w.\ g\ w \circ f\ w) \in W \to_Q exp\text{-}qbs\ X\ Z$
⟨*proof*⟩

lemma *arg-swap-morphism-map-qbs1*:
assumes $g \in exp\text{-}qbs\ W\ (exp\text{-}qbs\ X\ Y) \to_Q Z$
shows $(\lambda k.\ g\ (k \circ f)) \in exp\text{-}qbs\ (map\text{-}qbs\ f\ W)\ (exp\text{-}qbs\ X\ Y) \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-map-prod*[*qbs*]: $map\text{-}prod \in X \Rightarrow_Q Y \to_Q (W \Rightarrow_Q Z) \Rightarrow_Q (X \bigotimes_Q W) \Rightarrow_Q (Y \bigotimes_Q Z)$
⟨*proof*⟩

lemma *qbs-morphism-pair-swap*:
assumes $f \in X \bigotimes_Q Y \to_Q Z$
shows $(\lambda(x,y).\ f\ (y,x)) \in Y \bigotimes_Q X \to_Q Z$
⟨*proof*⟩

lemma
shows *qbs-morphism-pair-assoc1*: $(\lambda((x,y),z).\ (x,(y,z))) \in (X \bigotimes_Q Y) \bigotimes_Q Z \to_Q X \bigotimes_Q (Y \bigotimes_Q Z)$
and *qbs-morphism-pair-assoc2*: $(\lambda(x,(y,z)).\ ((x,y),z)) \in X \bigotimes_Q (Y \bigotimes_Q Z) \to_Q (X \bigotimes_Q Y) \bigotimes_Q Z$
⟨*proof*⟩

lemma *Inl-qbs-morphism*[*qbs*]: $Inl \in X \to_Q X \bigoplus_Q Y$
⟨*proof*⟩

lemma *Inr-qbs-morphism*[*qbs*]: $Inr \in Y \to_Q X \bigoplus_Q Y$
⟨*proof*⟩

lemma *case-sum-qbs-morphism*[*qbs*]: $\textit{case-sum} \in X \Rightarrow_Q Z \to_Q (Y \Rightarrow_Q Z) \Rightarrow_Q (X \bigoplus_Q Y \Rightarrow_Q Z)$
⟨*proof*⟩

lemma *map-sum-qbs-morphism*[*qbs*]: $\textit{map-sum} \in X \Rightarrow_Q Y \to_Q (X' \Rightarrow_Q Y') \Rightarrow_Q (X \bigoplus_Q X' \Rightarrow_Q Y \bigoplus_Q Y')$
⟨*proof*⟩

lemma *qbs-morphism-component-singleton*[*qbs*]:
assumes $i \in I$
shows $(\lambda x.\ x\ i) \in (\Pi_Q\ i{\in}I.\ (M\ i)) \to_Q M\ i$
⟨*proof*⟩

lemma *qbs-morphism-component-singleton'*:
assumes $f \in Y \to_Q (\Pi_Q\ i{\in}I.\ X\ i)$ $g \in Z \to_Q Y$ $i \in I$
shows $(\lambda x.\ f\ (g\ x)\ i) \in Z \to_Q X\ i$
⟨*proof*⟩

lemma *product-qbs-canonical1*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in Y \to_Q X\ i$
and $\bigwedge i.\ i \notin I \Longrightarrow f\ i = (\lambda y.\ \textit{undefined})$
shows $(\lambda y\ i.\ f\ i\ y) \in Y \to_Q (\Pi_Q\ i{\in}I.\ X\ i)$
⟨*proof*⟩

lemma *product-qbs-canonical2*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in Y \to_Q X\ i$
$\bigwedge i.\ i \notin I \Longrightarrow f\ i = (\lambda y.\ \textit{undefined})$
$g \in Y \to_Q (\Pi_Q\ i{\in}I.\ X\ i)$
$\bigwedge i.\ i \in I \Longrightarrow f\ i = (\lambda x.\ x\ i) \circ g$
and $y \in \textit{qbs-space}\ Y$
shows $g\ y = (\lambda i.\ f\ i\ y)$
⟨*proof*⟩

lemma *merge-qbs-morphism*:
$\textit{merge}\ I\ J \in (\Pi_Q\ i{\in}I.\ (M\ i)) \bigotimes_Q (\Pi_Q\ j{\in}J.\ (M\ j)) \to_Q (\Pi_Q\ i{\in}I{\cup}J.\ (M\ i))$

⟨*proof*⟩

lemma *ini-morphism*[*qbs*]:
assumes $j \in I$
shows $(\lambda x.\ (j,x)) \in X\ j \to_Q (\amalg_Q\ i{\in}I.\ X\ i)$
⟨*proof*⟩

lemma *coPiQ-canonical1*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow f\ i \in X\ i \to_Q Y$
shows $(\lambda(i,x).\ f\ i\ x) \in (\amalg_Q\ i \in I.\ X\ i) \to_Q Y$
⟨*proof*⟩

lemma *coPiQ-canonical1'*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow (\lambda x.\ f\ (i,x)) \in X\ i \to_Q Y$
shows $f \in (\amalg_Q\ i \in I.\ X\ i) \to_Q Y$
⟨*proof*⟩

lemma *None-qbs*[*qbs*]: *None* $\in$ *qbs-space* (*option-qbs X*)
⟨*proof*⟩

lemma *Some-qbs*[*qbs*]: *Some* $\in X \to_Q$ *option-qbs X*
⟨*proof*⟩

lemma *case-option-qbs-morphism*[*qbs*]: *case-option* $\in$ *qbs-space* $(Y \Rightarrow_Q (X \Rightarrow_Q Y) \Rightarrow_Q$ *option-qbs* $X \Rightarrow_Q Y)$
⟨*proof*⟩

lemma *rec-option-qbs-morphism*[*qbs*]: *rec-option* $\in$ *qbs-space* $(Y \Rightarrow_Q (X \Rightarrow_Q Y) \Rightarrow_Q$ *option-qbs* $X \Rightarrow_Q Y)$
⟨*proof*⟩

lemma *bind-option-qbs-morphism*[*qbs*]: $(\ggg=) \in$ *qbs-space* (*option-qbs* $X \Rightarrow_Q (X \Rightarrow_Q$ *option-qbs* $Y) \Rightarrow_Q$ *option-qbs* Y)
⟨*proof*⟩

lemma *Let-qbs-morphism*[*qbs*]: *Let* $\in X \Rightarrow_Q (X \Rightarrow_Q Y) \Rightarrow_Q Y$
⟨*proof*⟩

end

3.3 Relation to Measurable Spaces

theory *Measure-QuasiBorel-Adjunction*
imports *QuasiBorel QBS-Morphism Lemmas-S-Finite-Measure-Monad*
begin

We construct the adjunction between **Meas** and **QBS**, where **Meas** is the category of measurable spaces and measurable functions, and **QBS** is the category of quasi-Borel spaces and morphisms.

3.3.1 The Functor R

definition *measure-to-qbs* :: *'a measure* $\Rightarrow$ *'a quasi-borel* **where**
measure-to-qbs $X \equiv$ *Abs-quasi-borel* (*space* X, *borel* $\rightarrow_M$ X)

declare [[*coercion measure-to-qbs*]]

lemma
shows *qbs-space-R*: *qbs-space* (*measure-to-qbs* X) = *space* X (**is** *?goal1*)
and *qbs-Mx-R*: *qbs-Mx* (*measure-to-qbs* X) = *borel* $\rightarrow_M$ X (**is** *?goal2*)
⟨*proof*⟩

The following lemma says that *measure-to-qbs* is a functor from **Meas** to **QBS**.

lemma *r-preserves-morphisms*:
$X \rightarrow_M Y \subseteq$ (*measure-to-qbs* X) $\rightarrow_Q$ (*measure-to-qbs* Y)
⟨*proof*⟩

lemma *measurable-imp-qbs-morphism*: $f \in M \rightarrow_M N \Longrightarrow f \in M \rightarrow_Q N$
⟨*proof*⟩

3.3.2 The Functor L

definition *sigma-Mx* :: *'a quasi-borel* $\Rightarrow$ *'a set set* **where**
sigma-Mx $X \equiv \{U \cap$ *qbs-space* X $|U.\ \forall \alpha \in$ *qbs-Mx* $X.\ \alpha$ −‘ $U \in$ *sets borel*$\}$

definition *qbs-to-measure* :: *'a quasi-borel* $\Rightarrow$ *'a measure* **where**
qbs-to-measure $X \equiv$ *Abs-measure* (*qbs-space* X, *sigma-Mx* X, λA. (*if* $A = \{\}$ *then* *0 else if* $A \in -$ *sigma-Mx* X *then 0 else* ∞))

lemma *measure-space-L*: *measure-space* (*qbs-space* X) (*sigma-Mx* X) (λA. (*if* $A = \{\}$ *then 0 else if* $A \in -$ *sigma-Mx* X *then 0 else* ∞))
⟨*proof*⟩

lemma
shows *space-L*: *space* (*qbs-to-measure* X) = *qbs-space* X (**is** *?goal1*)
and *sets-L*: *sets* (*qbs-to-measure* X) = *sigma-Mx* X (**is** *?goal2*)
and *emeasure-L*: *emeasure* (*qbs-to-measure* X) = (λA. *if* $A = \{\} \vee A \notin$ *sigma-Mx* X *then 0 else* ∞) (**is** *?goal3*)
⟨*proof*⟩

lemma *qbs-Mx-sigma-Mx-contra*:
assumes *qbs-space* X = *qbs-space* Y
and *qbs-Mx* $X \subseteq$ *qbs-Mx* Y
shows *sigma-Mx* $Y \subseteq$ *sigma-Mx* X
⟨*proof*⟩

The following lemma says that *qbs-to-measure* is a functor from **QBS** to **Meas**.

lemma *l-preserves-morphisms*:
  $X \to_Q Y \subseteq$ (*qbs-to-measure* X) $\to_M$ (*qbs-to-measure* Y)
⟨*proof*⟩

lemma *qbs-morphism-imp-measurable*: $f \in X \to_Q Y \Longrightarrow f \in$ *qbs-to-measure* X $\to_M$ *qbs-to-measure* Y
  ⟨*proof*⟩

abbreviation *qbs-borel* ($borel_Q$) **where** $borel_Q \equiv$ *measure-to-qbs borel*
abbreviation *qbs-count-space* (*count′-space*$_Q$) **where** *qbs-count-space* $I \equiv$ *measure-to-qbs* (*count-space* I)

lemma
  shows *qbs-space-qbs-borel*[*simp*]: *qbs-space* $borel_Q$ = *UNIV*
    and *qbs-space-count-space*[*simp*]: *qbs-space* (*qbs-count-space* I) = I
    and *qbs-Mx-qbs-borel*: *qbs-Mx* $borel_Q$ = *borel-measurable borel*
    and *qbs-Mx-count-space*: *qbs-Mx* (*qbs-count-space* I) = *borel* $\to_M$ *count-space* I
  ⟨*proof*⟩

lemma
  shows *qbs-space-qbs-borel′*[*qbs*]: $r \in$ *qbs-space* $borel_Q$
    and *qbs-space-count-space-UNIV′*[*qbs*]: $x \in$ *qbs-space* (*qbs-count-space* (*UNIV* :: (- :: *countable*) *set*))
  ⟨*proof*⟩

lemma *qbs-Mx-is-morphisms*: *qbs-Mx* X = $borel_Q \to_Q X$
⟨*proof*⟩

lemma *exp-qbs-Mx′*: *qbs-Mx* (*exp-qbs* X Y) = $\{g.$ *case-prod* $g \in borel_Q \bigotimes_Q X \to_Q Y\}$
  ⟨*proof*⟩

lemma *arg-swap-morphism′*:
  assumes $(\lambda g.\ f\ (\lambda w\ x.\ g\ x\ w)) \in$ *exp-qbs* X (*exp-qbs* W Y) $\to_Q Z$
  shows $f \in$ *exp-qbs* W (*exp-qbs* X Y) $\to_Q Z$
⟨*proof*⟩

lemma *qbs-Mx-subset-of-measurable*: *qbs-Mx* $X \subseteq$ *borel* $\to_M$ *qbs-to-measure* X
⟨*proof*⟩

lemma *L-max-of-measurables*:
  assumes *space* M = *qbs-space* X
    and *qbs-Mx* $X \subseteq$ *borel* $\to_M M$
  shows *sets* $M \subseteq$ *sets* (*qbs-to-measure* X)
⟨*proof*⟩

lemma *qbs-Mx-are-measurable*[*simp*,*measurable*]:
assumes $\alpha \in$ *qbs-Mx X*
shows $\alpha \in$ *borel* $\rightarrow_M$ *qbs-to-measure X*
⟨*proof*⟩

lemma *measure-to-qbs-cong-sets*:
assumes *sets M* = *sets N*
shows *measure-to-qbs M* = *measure-to-qbs N*
⟨*proof*⟩

lemma *lr-sets*[*simp*]:
sets X $\subseteq$ *sets* (*qbs-to-measure* (*measure-to-qbs X*))
⟨*proof*⟩

lemma(**in** *standard-borel*) *lr-sets-ident*[*simp*, *measurable-cong*]:
sets (*qbs-to-measure* (*measure-to-qbs M*)) = *sets M*
⟨*proof*⟩

corollary *sets-lr-polish-borel*[*simp*, *measurable-cong*]: *sets* (*qbs-to-measure qbs-borel*) = *sets* (*borel* :: (- :: *polish-space*) *measure*)
⟨*proof*⟩

corollary *sets-lr-count-space*[*simp*, *measurable-cong*]: *sets* (*qbs-to-measure* (*qbs-count-space* (*UNIV* :: (- :: *countable*) *set*))) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *map-qbs-embed-measure1*:
assumes *inj-on f* (*space M*)
shows *map-qbs f* (*measure-to-qbs M*) = *measure-to-qbs* (*embed-measure M f*)
⟨*proof*⟩

lemma *map-qbs-embed-measure2*:
assumes *inj-on f* (*qbs-space X*)
shows *sets* (*qbs-to-measure* (*map-qbs f X*)) = *sets* (*embed-measure* (*qbs-to-measure X*) *f*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *map-qbs-embed-measure2'*:
assumes *inj-on f* (*space M*)
shows *sets* (*qbs-to-measure* (*map-qbs f* (*measure-to-qbs M*))) = *sets* (*embed-measure M f*)
⟨*proof*⟩

3.3.3 The Adjunction

lemma *lr-adjunction-correspondence* :
$X \rightarrow_Q$ (*measure-to-qbs Y*) = (*qbs-to-measure X*) $\rightarrow_M$ *Y*
⟨*proof*⟩

lemma(**in** *standard-borel*) *standard-borel-r-full-faithful*:
$M \rightarrow_M Y = \textit{measure-to-qbs}\ M \rightarrow_Q \textit{measure-to-qbs}\ Y$
⟨*proof*⟩

lemma *qbs-morphism-dest*:
assumes $f \in X \rightarrow_Q \textit{measure-to-qbs}\ Y$
shows $f \in \textit{qbs-to-measure}\ X \rightarrow_M Y$
⟨*proof*⟩

lemma(**in** *standard-borel*) *qbs-morphism-dest*:
assumes $k \in \textit{measure-to-qbs}\ M \rightarrow_Q \textit{measure-to-qbs}\ Y$
shows $k \in M \rightarrow_M Y$
⟨*proof*⟩

lemma *qbs-morphism-measurable-intro*:
assumes $f \in \textit{qbs-to-measure}\ X \rightarrow_M Y$
shows $f \in X \rightarrow_Q \textit{measure-to-qbs}\ Y$
⟨*proof*⟩

lemma(**in** *standard-borel*) *qbs-morphism-measurable-intro*:
assumes $k \in M \rightarrow_M Y$
shows $k \in \textit{measure-to-qbs}\ M \rightarrow_Q \textit{measure-to-qbs}\ Y$
⟨*proof*⟩

lemma *r-preserves-product* :
$\textit{measure-to-qbs}\ (X \bigotimes_M Y) = \textit{measure-to-qbs}\ X \bigotimes_Q \textit{measure-to-qbs}\ Y$
⟨*proof*⟩

lemma *l-product-sets*:
$\textit{sets}\ (\textit{qbs-to-measure}\ X \bigotimes_M \textit{qbs-to-measure}\ Y) \subseteq \textit{sets}\ (\textit{qbs-to-measure}\ (X \bigotimes_Q Y))$
⟨*proof*⟩

corollary *qbs-borel-prod*: $\textit{qbs-borel} \bigotimes_Q \textit{qbs-borel} = (\textit{qbs-borel} :: ('a::\textit{second-countable-topology} \times 'b::\textit{second-countable-topology})\ \textit{quasi-borel})$
⟨*proof*⟩

corollary *qbs-count-space-prod*: $\textit{qbs-count-space}\ (\textit{UNIV} :: ('a :: \textit{countable})\ \textit{set}) \bigotimes_Q \textit{qbs-count-space}\ (\textit{UNIV} :: ('b :: \textit{countable})\ \textit{set}) = \textit{qbs-count-space}\ \textit{UNIV}$
⟨*proof*⟩

lemma *r-preserves-product'*: $\textit{measure-to-qbs}\ (\Pi_M\ i{\in}I.\ M\ i) = (\Pi_Q\ i{\in}I.\ \textit{measure-to-qbs}\ (M\ i))$
⟨*proof*⟩

lemma *PiQ-qbs-borel*:
$(\Pi_Q\ i::('a::\ \textit{countable}){\in}\textit{UNIV}.\ (\textit{qbs-borel} :: ('b::\textit{second-countable-topology}\ \textit{quasi-borel}))) = \textit{qbs-borel}$
⟨*proof*⟩

lemma *qbs-morphism-from-countable*:
fixes X :: *'a quasi-borel*
assumes *countable* (*qbs-space* X)
qbs-Mx $X \subseteq$ *borel* $\rightarrow_M$ *count-space* (*qbs-space* X)
and $\bigwedge i.\ i \in$ *qbs-space* $X \Longrightarrow f\ i \in$ *qbs-space* Y
shows $f \in X \rightarrow_Q Y$
⟨*proof*⟩

corollary *qbs-morphism-count-space'*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in$ *qbs-space* Y *countable* I
shows $f \in$ *qbs-count-space* $I \rightarrow_Q Y$
⟨*proof*⟩

corollary *qbs-morphism-count-space*:
assumes $\bigwedge i.\ f\ i \in$ *qbs-space* Y
shows $f \in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q Y$
⟨*proof*⟩

lemma [*qbs*]:
shows *not-qbs-pred*: *Not* $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *qbs-count-space UNIV*
and *or-qbs-pred*: ($\vee$) $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *and-qbs-pred*: ($\wedge$) $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *implies-qbs-pred*: ($\longrightarrow$) $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *iff-qbs-pred*: ($\longleftrightarrow$) $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
⟨*proof*⟩

lemma [*qbs*]:
shows *less-count-qbs-pred*: ($<$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *le-count-qbs-pred*: ($\leq$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *eq-count-qbs-pred*: ($=$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *plus-count-qbs-morphism*: ($+$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *minus-count-qbs-morphism*: ($-$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *mult-count-qbs-morphism*: ($*$) $\in$ *qbs-count-space* (*UNIV* :: (- :: *countable*) *set*) $\rightarrow_Q$ *exp-qbs* (*qbs-count-space UNIV*) (*qbs-count-space UNIV*)
and *Suc-qbs-morphism*: *Suc* $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *qbs-count-space UNIV*
⟨*proof*⟩

lemma *qbs-morphism-product-iff*:
$f \in X \rightarrow_Q (\Pi_Q\ i$ :: (- :: *countable*)$\in$*UNIV*. $Y) \longleftrightarrow f \in X \rightarrow_Q$ *qbs-count-space*

$UNIV \Rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-pair-countable1*:
assumes $countable\ (qbs\text{-}space\ X)$
$qbs\text{-}Mx\ X \subseteq borel \rightarrow_M count\text{-}space\ (qbs\text{-}space\ X)$
and $\bigwedge i.\ i \in qbs\text{-}space\ X \Longrightarrow f\ i \in Y \rightarrow_Q Z$
shows $(\lambda(x,y).\ f\ x\ y) \in X \bigotimes_Q Y \rightarrow_Q Z$
$\langle proof \rangle$

lemma *qbs-morphism-pair-countable2*:
assumes $countable\ (qbs\text{-}space\ Y)$
$qbs\text{-}Mx\ Y \subseteq borel \rightarrow_M count\text{-}space\ (qbs\text{-}space\ Y)$
and $\bigwedge i.\ i \in qbs\text{-}space\ Y \Longrightarrow (\lambda x.\ f\ x\ i) \in X \rightarrow_Q Z$
shows $(\lambda(x,y).\ f\ x\ y) \in X \bigotimes_Q Y \rightarrow_Q Z$
$\langle proof \rangle$

corollary *qbs-morphism-pair-count-space1*:
assumes $\bigwedge i.\ f\ i \in Y \rightarrow_Q Z$
shows $(\lambda(x,y).\ f\ x\ y) \in qbs\text{-}count\text{-}space\ (UNIV :: ('a :: countable)\ set) \bigotimes_Q Y \rightarrow_Q Z$
$\langle proof \rangle$

corollary *qbs-morphism-pair-count-space2*:
assumes $\bigwedge i.\ (\lambda x.\ f\ x\ i) \in X \rightarrow_Q Z$
shows $(\lambda(x,y).\ f\ x\ y) \in X \bigotimes_Q qbs\text{-}count\text{-}space\ (UNIV :: ('a :: countable)\ set) \rightarrow_Q Z$
$\langle proof \rangle$

lemma *qbs-morphism-compose-countable'*:
assumes [*qbs*]:$\bigwedge i.\ i \in I \Longrightarrow (\lambda x.\ f\ i\ x) \in X \rightarrow_Q Y\ g \in X \rightarrow_Q qbs\text{-}count\text{-}space\ I\ countable\ I$
shows $(\lambda x.\ f\ (g\ x)\ x) \in X \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-compose-countable*:
assumes [*simp*]:$\bigwedge i{::}'i{::}countable.\ (\lambda x.\ f\ i\ x) \in X \rightarrow_Q Y\ g \in X \rightarrow_Q (qbs\text{-}count\text{-}space\ UNIV)$
shows $(\lambda x.\ f\ (g\ x)\ x) \in X \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-op*:
assumes $case\text{-}prod\ f \in X \bigotimes_M Y \rightarrow_M Z$
shows $f \in measure\text{-}to\text{-}qbs\ X \rightarrow_Q measure\text{-}to\text{-}qbs\ Y \Rightarrow_Q measure\text{-}to\text{-}qbs\ Z$
$\langle proof \rangle$

lemma [*qbs*]:
shows *plus-qbs-morphism*: $(+) \in (qbs\text{-}borel :: (\text{-}{::}\{second\text{-}countable\text{-}topology, topological\text{-}monoid\text{-}add\})\ quasi\text{-}borel) \rightarrow_Q qbs\text{-}borel \Rightarrow_Q qbs\text{-}borel$

and *plus-ereal-qbs-morphism*: $(+) \in$ (*qbs-borel* :: *ereal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *diff-qbs-morphism*: $(-) \in$ (*qbs-borel* :: (-::{*second-countable-topology*, *real-normed-vector*}) *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *diff-ennreal-qbs-morphism*: $(-) \in$ (*qbs-borel* :: *ennreal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *diff-ereal-qbs-morphism*: $(-) \in$ (*qbs-borel* :: *ereal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *times-qbs-morphism*: $(*) \in$ (*qbs-borel* :: (-::{*second-countable-topology*, *real-normed-algebra*}) *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *times-ennreal-qbs-morphism*: $(*) \in$ (*qbs-borel* :: *ennreal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *times-ereal-qbs-morphism*: $(*) \in$ (*qbs-borel* :: *ereal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *divide-qbs-morphism*: $(/) \in$ (*qbs-borel* :: (-::{*second-countable-topology*, *real-normed-div-algebra*}) *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *divide-ennreal-qbs-morphism*: $(/) \in$ (*qbs-borel* :: *ennreal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *divide-ereal-qbs-morphism*: $(/) \in$ (*qbs-borel* :: *ereal quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *log-qbs-morphism*: *log* $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *root-qbs-morphism*: *root* $\in$ *qbs-count-space UNIV* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *scaleR-qbs-morphism*: $(*_R) \in$ *qbs-borel* $\rightarrow_Q$ (*qbs-borel* :: (-::{*second-countable-topology*, *real-normed-vector*}) *quasi-borel*) $\Rightarrow_Q$ *qbs-borel*
and *qbs-morphism-inner*: $(\cdot) \in$ *qbs-borel* $\rightarrow_Q$ (*qbs-borel* :: (-::{*second-countable-topology*, *real-inner*}) *quasi-borel*) $\Rightarrow_Q$ *qbs-borel*
and *dist-qbs-morphism*: *dist* $\in$ (*qbs-borel* :: (-::{*second-countable-topology*, *metric-space*}) *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *powr-qbs-morphism*: (*powr*) $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ (*qbs-borel* :: *real quasi-borel*)
and *max-qbs-morphism*: (*max* :: (- :: {*second-countable-topology*, *linorder-topology*}) $\Rightarrow$ - $\Rightarrow$ -) $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *min-qbs-morphism*: (*min* :: (- :: {*second-countable-topology*, *linorder-topology*}) $\Rightarrow$ - $\Rightarrow$ -) $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *sup-qbs-morphism*: (*sup* :: (- :: {*lattice*,*second-countable-topology*, *linorder-topology*}) $\Rightarrow$ - $\Rightarrow$ -) $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *inf-qbs-morphism*: (*inf* :: (- :: {*lattice*,*second-countable-topology*, *linorder-topology*}) $\Rightarrow$ - $\Rightarrow$ -) $\in$ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
and *less-qbs-pred*: $(<) \in$ (*qbs-borel* :: - ::{*second-countable-topology*, *linorder-topology*} *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-count-space UNIV*
and *eq-qbs-pred*: $(=) \in$ (*qbs-borel* :: - ::{*second-countable-topology*, *linorder-topology*} *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-count-space UNIV*
and *le-qbs-pred*: $(\leq) \in$ (*qbs-borel* :: - ::{*second-countable-topology*, *linorder-topology*} *quasi-borel*) $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-count-space UNIV*
⟨*proof*⟩

lemma [*qbs*]:
shows *abs-real-qbs-morphism*: *abs* $\in$ (*qbs-borel* :: *real quasi-borel*) $\rightarrow_Q$ *qbs-borel*
and *abs-ereal-qbs-morphism*: *abs* $\in$ (*qbs-borel* :: *ereal quasi-borel*) $\rightarrow_Q$ *qbs-borel*

and *real-floor-qbs-morphism*: (*floor* :: *real* ⇒ *int*) ∈ *qbs-borel* $\to_Q$ *qbs-count-space UNIV*
and *real-ceiling-qbs-morphism*: (*ceiling* :: *real* ⇒ *int*) ∈ *qbs-borel* $\to_Q$ *qbs-count-space UNIV*
and *exp-qbs-morphism*: (*exp*::'*a*::{*real-normed-field*,*banach*}⇒'*a*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *ln-qbs-morphism*: *ln* ∈ (*qbs-borel* :: *real quasi-borel*) $\to_Q$ *qbs-borel*
and *sqrt-qbs-morphism*: *sqrt* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *of-real-qbs-morphism*: (*of-real* :: - ⇒ (-::*real-normed-algebra*)) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *sin-qbs-morphism*: (*sin* :: - ⇒ (-::{*real-normed-field*,*banach*})) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *cos-qbs-morphism*: (*cos* :: - ⇒ (-::{*real-normed-field*,*banach*})) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *arctan-qbs-morphism*: *arctan* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *Re-qbs-morphism*: *Re* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *Im-qbs-morphism*: *Im* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *sgn-qbs-morphism*: (*sgn*::-::*real-normed-vector* ⇒ -) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *norm-qbs-morphism*: *norm* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *invers-qbs-morphism*: (*inverse* :: - ⇒ (- ::*real-normed-div-algebra*)) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *invers-ennreal-qbs-morphism*: (*inverse* :: - ⇒ *ennreal*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *invers-ereal-qbs-morphism*: (*inverse* :: - ⇒ *ereal*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *uminus-qbs-morphism*: (*uminus* :: - ⇒ (-::{*second-countable-topology*, *real-normed-vector*})) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *ereal-qbs-morphism*: *ereal* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *real-of-ereal-qbs-morphism*: *real-of-ereal* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *enn2ereal-qbs-morphism*: *enn2ereal* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *e2ennreal-qbs-morphism*: *e2ennreal* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *ennreal-qbs-morphism*: *ennreal* ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *qbs-morphism-nth*: (λ*x*::*real*^'*n*. *x* $ *i*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *qbs-morphism-product-candidate*: ⋀*i*. (λ*x*. *x i*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
and *uminus-ereal-qbs-morphism*: (*uminus* :: - ⇒ *ereal*) ∈ *qbs-borel* $\to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-sum*:
fixes *f* :: '*c* ⇒ '*a* ⇒ '*b*::{*second-countable-topology*, *topological-comm-monoid-add*}
assumes ⋀*i*. *i* ∈ *S* ⟹ *f i* ∈ *X* $\to_Q$ *qbs-borel*
shows (λ*x*. ∑ *i*∈*S*. *f i x*) ∈ *X* $\to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-suminf-order*:
fixes *f* :: *nat* ⇒ '*a* ⇒ '*b*::{*complete-linorder*, *second-countable-topology*, *linorder-topology*, *topological-comm-monoid-add*}
assumes ⋀*i*. *f i* ∈ *X* $\to_Q$ *qbs-borel*
shows (λ*x*. ∑ *i*. *f i x*) ∈ *X* $\to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-prod*:
fixes f :: $'c \Rightarrow 'a \Rightarrow 'b$::{*second-countable-topology*, *real-normed-field*}
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in X \rightarrow_Q$ *qbs-borel*
shows $(\lambda x.\ \prod i{\in}S.\ f\ i\ x) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-Min*:
finite $I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i \in X \rightarrow_Q$ *qbs-borel*$) \Longrightarrow (\lambda x.\ Min\ ((\lambda i.\ f\ i\ x)\ `I)$ :: $'b$::{*second-countable-topology*, *linorder-topology*}$) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-Max*:
finite $I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i \in X \rightarrow_Q$ *qbs-borel*$) \Longrightarrow (\lambda x.\ Max\ ((\lambda i.\ f\ i\ x)\ `I)$ :: $'b$::{*second-countable-topology*, *linorder-topology*}$) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-Max2*:
fixes f::- $\Rightarrow$ - $\Rightarrow$ $'a$::{*second-countable-topology*, *dense-linorder*, *linorder-topology*}
shows *finite* $I \Longrightarrow (\bigwedge i.\ f\ i \in X \rightarrow_Q$ *qbs-borel*$) \Longrightarrow (\lambda x.\ Max\{f\ i\ x\ |i.\ i \in I\}) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma [*qbs*]:
shows *qbs-morphism-liminf*: *liminf* $\in$ (*qbs-count-space UNIV* $\Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ (*qbs-borel* :: $'a$ :: {*complete-linorder*, *second-countable-topology*, *linorder-topology*} *quasi-borel*)
and *qbs-morphism-limsup*: *limsup* $\in$ (*qbs-count-space UNIV* $\Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ (*qbs-borel* :: $'a$ :: {*complete-linorder*, *second-countable-topology*, *linorder-topology*} *quasi-borel*)
and *qbs-morphism-lim*: *lim* $\in$ (*qbs-count-space UNIV* $\Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ (*qbs-borel* :: $'a$ :: {*complete-linorder*, *second-countable-topology*, *linorder-topology*} *quasi-borel*)
⟨*proof*⟩

lemma *qbs-morphism-SUP*:
fixes F :: - $\Rightarrow$ - $\Rightarrow$ -::{*complete-linorder*, *linorder-topology*, *second-countable-topology*}
assumes *countable* I $\bigwedge i.\ i \in I \Longrightarrow F\ i \in X \rightarrow_Q$ *qbs-borel*
shows $(\lambda x.\ \bigsqcup\ i{\in}I.\ F\ i\ x) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-INF*:
fixes F :: - $\Rightarrow$ - $\Rightarrow$ -::{*complete-linorder*, *linorder-topology*, *second-countable-topology*}
assumes *countable* I $\bigwedge i.\ i \in I \Longrightarrow F\ i \in X \rightarrow_Q$ *qbs-borel*
shows $(\lambda x.\ \bigsqcap\ i{\in}I.\ F\ i\ x) \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-cSUP*:
fixes F :: - $\Rightarrow$ - $\Rightarrow$ $'a$::{*conditionally-complete-linorder*, *linorder-topology*, *second-countable-topology*}
assumes *countable* I $\bigwedge i.\ i \in I \Longrightarrow F\ i \in X \rightarrow_Q$ *qbs-borel* $\bigwedge x.\ x \in$ *qbs-space* X

$\Longrightarrow$ *bdd-above* $((\lambda i.\ F\ i\ x)$ ‘ $I)$
shows $(\lambda x.\ \bigsqcup\ i{\in}I.\ F\ i\ x) \in X \to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-cINF*:
fixes F :: - $\Rightarrow$ - $\Rightarrow$ $'a$::{*conditionally-complete-linorder*, *linorder-topology*, *second-countable-topology*}
assumes *countable* I $\bigwedge i.\ i \in I \Longrightarrow F\ i \in X \to_Q$ *qbs-borel* $\bigwedge x.\ x \in$ *qbs-space* X $\Longrightarrow$ *bdd-below* $((\lambda i.\ F\ i\ x)$ ‘ $I)$
shows $(\lambda x.\ \bigsqcap\ i{\in}I.\ F\ i\ x) \in X \to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-lim-metric*:
fixes f :: *nat* $\Rightarrow$ $'a$ $\Rightarrow$ $'b$::{*banach*, *second-countable-topology*}
assumes $\bigwedge i.\ f\ i \in X \to_Q$ *qbs-borel*
shows $(\lambda x.\ lim\ (\lambda i.\ f\ i\ x)) \in X \to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-LIMSEQ-metric*:
fixes f :: *nat* $\Rightarrow$ $'a$ $\Rightarrow$ $'b$ :: *metric-space*
assumes $\bigwedge i.\ f\ i \in X \to_Q$ *qbs-borel* $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow (\lambda i.\ f\ i\ x) \longrightarrow g\ x$
shows $g \in X \to_Q$ *qbs-borel*
⟨*proof*⟩

lemma *power-qbs-morphism*[*qbs*]:
(*power* :: (- ::{*power*,*real-normed-algebra*}) $\Rightarrow$ *nat* $\Rightarrow$ -) $\in$ *qbs-borel* $\to_Q$ *qbs-count-space UNIV* $\Rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *power-ennreal-qbs-morphism*[*qbs*]:
(*power* :: *ennreal* $\Rightarrow$ *nat* $\Rightarrow$ -) $\in$ *qbs-borel* $\to_Q$ *qbs-count-space UNIV* $\Rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-morphism-compw*: $(\frown\frown) \in (X \Rightarrow_Q X) \to_Q$ *qbs-count-space UNIV* $\Rightarrow_Q (X \Rightarrow_Q X)$
⟨*proof*⟩

lemma *qbs-morphism-compose-n*[*qbs*]:
assumes [*qbs*]: $f \in X \to_Q X$
shows $(\lambda n.\ f\frown\frown n) \in$ *qbs-count-space UNIV* $\to_Q X \Rightarrow_Q X$
⟨*proof*⟩

lemma *qbs-morphism-compose-n'*:
assumes $f \in X \to_Q X$
shows $f\frown\frown n \in X \to_Q X$
⟨*proof*⟩

lemma *qbs-morphism-uminus-eq-ereal*[*simp*]:

$(\lambda x. - f\ x :: ereal) \in X \to_Q qbs\text{-}borel \longleftrightarrow f \in X \to_Q qbs\text{-}borel$ (**is** $?l = ?r$)
$\langle proof \rangle$

lemma *qbs-morphism-ereal-iff*:
shows $(\lambda x.\ ereal\ (f\ x)) \in X \to_Q qbs\text{-}borel \longleftrightarrow f \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-morphism-ereal-sum*:
fixes $f :: 'c \Rightarrow 'a \Rightarrow ereal$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in X \to_Q qbs\text{-}borel$
shows $(\lambda x. \sum i{\in}S.\ f\ i\ x) \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-morphism-ereal-prod*:
fixes $f :: 'c \Rightarrow 'a \Rightarrow ereal$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in X \to_Q qbs\text{-}borel$
shows $(\lambda x. \prod i{\in}S.\ f\ i\ x) \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-morphism-extreal-suminf*:
fixes $f :: nat \Rightarrow 'a \Rightarrow ereal$
assumes $\bigwedge i.\ f\ i \in X \to_Q qbs\text{-}borel$
shows $(\lambda x. (\sum i.\ f\ i\ x)) \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-morphism-ennreal-iff*:
assumes $\bigwedge x.\ x \in qbs\text{-}space\ X \Longrightarrow 0 \le f\ x$
shows $(\lambda x.\ ennreal\ (f\ x)) \in X \to_Q qbs\text{-}borel \longleftrightarrow f \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-morphism-prod-ennreal*:
fixes $f :: 'c \Rightarrow 'a \Rightarrow ennreal$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in X \to_Q qbs\text{-}borel$
shows $(\lambda x. \prod i{\in}S.\ f\ i\ x) \in X \to_Q qbs\text{-}borel$
$\langle proof \rangle$

lemma *count-space-qbs-morphism*:
$f \in qbs\text{-}count\text{-}space\ (UNIV :: 'a\ set) \to_Q qbs\text{-}borel$
$\langle proof \rangle$

declare *count-space-qbs-morphism*[**where** $'a$=- :: *countable,qbs*]

lemma *count-space-count-space-qbs-morphism*:
$f \in qbs\text{-}count\text{-}space\ (UNIV :: (- :: countable)\ set) \to_Q qbs\text{-}count\text{-}space\ (UNIV :: (- :: countable)\ set)$
$\langle proof \rangle$

lemma *qbs-morphism-case-nat'*:
assumes [*qbs*]: $i = 0 \Longrightarrow f \in X \to_Q Y$

$\bigwedge j.\ i = Suc\ j \Longrightarrow (\lambda x.\ g\ x\ j) \in X \rightarrow_Q Y$
shows $(\lambda x.\ case\text{-}nat\ (f\ x)\ (g\ x)\ i) \in X \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-case-nat*[*qbs*]:
$case\text{-}nat \in X \rightarrow_Q (qbs\text{-}count\text{-}space\ UNIV \Rightarrow_Q X) \Rightarrow_Q qbs\text{-}count\text{-}space\ UNIV \Rightarrow_Q X$
$\langle proof \rangle$

lemma *qbs-morphism-case-nat''*:
assumes $f \in X \rightarrow_Q Y\ g \in X \rightarrow_Q (\Pi_Q\ i{\in}UNIV.\ Y)$
shows $(\lambda x.\ case\text{-}nat\ (f\ x)\ (g\ x)) \in X \rightarrow_Q (\Pi_Q\ i{\in}UNIV.\ Y)$
$\langle proof \rangle$

lemma *qbs-morphism-rec-nat*[*qbs*]: $rec\text{-}nat \in X \rightarrow_Q (count\text{-}space\ UNIV \Rightarrow_Q X \Rightarrow_Q X) \Rightarrow_Q count\text{-}space\ UNIV \Rightarrow_Q X$
$\langle proof \rangle$

lemma *qbs-morphism-Max-nat*:
fixes $P :: nat \Rightarrow 'a \Rightarrow bool$
assumes $\bigwedge i.\ P\ i \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
shows $(\lambda x.\ Max\ \{i.\ P\ i\ x\}) \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
$\langle proof \rangle$

lemma *qbs-morphism-Min-nat*:
fixes $P :: nat \Rightarrow 'a \Rightarrow bool$
assumes $\bigwedge i.\ P\ i \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
shows $(\lambda x.\ Min\ \{i.\ P\ i\ x\}) \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
$\langle proof \rangle$

lemma *qbs-morphism-sum-nat*:
fixes $f :: 'c \Rightarrow 'a \Rightarrow nat$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
shows $(\lambda x.\ \sum i{\in}S.\ f\ i\ x) \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$
$\langle proof \rangle$

lemma *qbs-morphism-case-enat'*:
assumes f[*qbs*]: $f \in X \rightarrow_Q qbs\text{-}count\text{-}space\ UNIV$ **and** [*qbs*]: $\bigwedge i.\ g\ i \in X \rightarrow_Q Y\ h \in X \rightarrow_Q Y$
shows $(\lambda x.\ case\ f\ x\ of\ enat\ i \Rightarrow g\ i\ x \mid \infty \Rightarrow h\ x) \in X \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-case-enat*[*qbs*]: $case\text{-}enat \in qbs\text{-}space\ ((qbs\text{-}count\text{-}space\ UNIV \Rightarrow_Q X) \Rightarrow_Q X \Rightarrow_Q qbs\text{-}count\text{-}space\ UNIV \Rightarrow_Q X)$
$\langle proof \rangle$

lemma *qbs-morphism-restrict*[*qbs*]:

assumes X: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in X \rightarrow_Q (Y\ i)$
shows $(\lambda x.\ \lambda i{\in}I.\ f\ i\ x) \in X \rightarrow_Q (\Pi_Q\ i{\in}I.\ Y\ i)$
⟨*proof*⟩

lemma *If-qbs-morphism*[*qbs*]: *If* ∈ *qbs-count-space UNIV* $\rightarrow_Q$ $X \Rightarrow_Q X \Rightarrow_Q X$
⟨*proof*⟩

lemma *normal-density-qbs*[*qbs*]: *normal-density* ∈ *qbs-borel* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *erlang-density-qbs*[*qbs*]: *erlang-density* ∈ *qbs-count-space UNIV* $\rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel* $\Rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *list-nil-qbs*[*qbs*]: [] ∈ *qbs-space* (*list-qbs* X)
⟨*proof*⟩

lemma *list-cons-qbs-morphism*: *list-cons* ∈ $X \rightarrow_Q (\Pi_Q\ n{\in}(UNIV :: nat\ set).\Pi_Q\ i{\in}\{..{<}n\}.\ X) \Rightarrow_Q (\Pi_Q\ n{\in}(UNIV :: nat\ set).\Pi_Q\ i{\in}\{..{<}n\}.\ X)$
⟨*proof*⟩

corollary *cons-qbs-morphism*[*qbs*]: *Cons* ∈ $X \rightarrow_Q$ (*list-qbs* X) $\Rightarrow_Q$ *list-qbs* X
⟨*proof*⟩

lemma *rec-list-morphism′*:
rec-list′ ∈ *qbs-space* $(Y \Rightarrow_Q (X \Rightarrow_Q (\Pi_Q\ n{\in}(UNIV :: nat\ set).\Pi_Q\ i{\in}\{..{<}n\}.\ X) \Rightarrow_Q Y \Rightarrow_Q Y) \Rightarrow_Q (\Pi_Q\ n{\in}(UNIV :: nat\ set).\Pi_Q\ i{\in}\{..{<}n\}.\ X) \Rightarrow_Q Y)$
⟨*proof*⟩

lemma *rec-list-morphism*[*qbs*]: *rec-list* ∈ *qbs-space* ($Y \Rightarrow_Q$ ($X \Rightarrow_Q$ *list-qbs* $X \Rightarrow_Q Y \Rightarrow_Q Y) \Rightarrow_Q$ *list-qbs* $X \Rightarrow_Q Y$)
⟨*proof*⟩

hide-const (**open**) *list-nil list-cons list-head list-tail from-list rec-list′ to-list′*

hide-fact (**open**) *list-simp1 list-simp2 list-simp3 list-simp4 list-simp5 list-simp6 list-simp7 from-list-in-list-of′ list-cons-qbs-morphism rec-list′-simp1 to-list-from-list-ident from-list-in-list-of to-list-set to-list-simp1 to-list-simp2 list-head-def list-tail-def from-list-length list-cons-in-list-of rec-list-morphism′ rec-list′-simp2 list-decomp1 list-destruct-rule list-induct-rule from-list-to-list-ident*

corollary *case-list-morphism*[*qbs*]: *case-list* ∈ *qbs-space* ((Y :: *′b quasi-borel*) $\Rightarrow_Q$ ((X :: *′a quasi-borel*) $\Rightarrow_Q$ *list-qbs* $X \Rightarrow_Q Y) \Rightarrow_Q$ *list-qbs* $X \Rightarrow_Q Y$)
⟨*proof*⟩

lemma *fold-qbs-morphism*[*qbs*]: *fold* ∈ *qbs-space* (($X \Rightarrow_Q Y \Rightarrow_Q Y) \Rightarrow_Q$ *list-qbs* $X \Rightarrow_Q Y \Rightarrow_Q Y$)

⟨*proof*⟩

lemma [*qbs*]:
shows *foldr-qbs-morphism*: *foldr* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *Y* $\Rightarrow_Q$ *Y*) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *Y* $\Rightarrow_Q$ *Y*)
and *foldl-qbs-morphism*: *foldl* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *Y* $\Rightarrow_Q$ *X*) $\Rightarrow_Q$ *X* $\Rightarrow_Q$ *list-qbs Y* $\Rightarrow_Q$ *X*)
and *zip-qbs-morphism*: *zip* $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *list-qbs Y* $\Rightarrow_Q$ *list-qbs* (*pair-qbs X Y*))
and *append-qbs-morphism*: *append* $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
and *concat-qbs-morphism*: *concat* $\in$ *qbs-space* (*list-qbs* (*list-qbs X*) $\Rightarrow_Q$ *list-qbs X*)
and *drop-qbs-morphism*: *drop* $\in$ *qbs-space* (*qbs-count-space UNIV* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
and *take-qbs-morphism*: *take* $\in$ *qbs-space* (*qbs-count-space UNIV* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
and *rev-qbs-morphism*: *rev* $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
⟨*proof*⟩

lemma [*qbs*]:
fixes *X* :: *'a quasi-borel* **and** *Y* :: *'b quasi-borel*
shows *map-qbs-morphism*: *map* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *Y*) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs Y*) (**is** *?map*)
and *fileter-qbs-morphism*: *filter* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *count-space*$_Q$ *UNIV*) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*) (**is** *?filter*)
and *length-qbs-morphism*: *length* $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *qbs-count-space UNIV*) (**is** *?length*)
and *tl-qbs-morphism*: *tl* $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *list-qbs X*) (**is** *?tl*)
and *list-all-qbs-morphism*: *list-all* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *qbs-count-space UNIV*) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *qbs-count-space UNIV*) (**is** *?list-all*)
and *bind-list-qbs-morphism*: ($\gg=$) $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ (*X* $\Rightarrow_Q$ *list-qbs Y*) $\Rightarrow_Q$ *list-qbs Y*) (**is** *?bind*)
⟨*proof*⟩

lemma *list-eq-qbs-morphism*[*qbs*]:
assumes [*qbs*]: (=) $\in$ *qbs-space* (*X* $\Rightarrow_Q$ *X* $\Rightarrow_Q$ *count-space UNIV*)
shows (=) $\in$ *qbs-space* (*list-qbs X* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *count-space UNIV*)
⟨*proof*⟩

lemma *insort-key-qbs-morphism*[*qbs*]:
shows *insort-key* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ (*borel*$_Q$ ::*'b* :: {*second-countable-topology*, *linorder-topology*} *quasi-borel*)) $\Rightarrow_Q$ *X* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*) (**is** *?g1*)
and *insort-key* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *count-space*$_Q$ (*UNIV* :: (- :: *countable*) *set*)) $\Rightarrow_Q$ *X* $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*) (**is** *?g2*)
⟨*proof*⟩

lemma *sort-key-qbs-morphism*[*qbs*]:
shows *sort-key* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ (*borel*$_Q$ ::*'b* :: {*second-countable-topology*,

linorder-topology} *quasi-borel*)) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
and *sort-key* $\in$ *qbs-space* ((*X* $\Rightarrow_Q$ *count-space*$_Q$ (*UNIV* :: (- :: *countable*) *set*)) $\Rightarrow_Q$ *list-qbs X* $\Rightarrow_Q$ *list-qbs X*)
⟨*proof*⟩

lemma *sort-qbs-morphism*[*qbs*]:
shows *sort* $\in$ *list-qbs* (*borel*$_Q$ ::'*b* :: {*second-countable-topology*, *linorder-topology*} *quasi-borel*) $\rightarrow_Q$ *list-qbs borel*$_Q$
and *sort* $\in$ *list-qbs* (*count-space*$_Q$ (*UNIV* :: (- :: *countable*) *set*)) $\rightarrow_Q$ *list-qbs* (*count-space*$_Q$ *UNIV*)
⟨*proof*⟩

3.3.4 Morphism Pred

abbreviation *qbs-pred X P* $\equiv$ *P* $\in$ *X* $\rightarrow_Q$ *qbs-count-space* (*UNIV* :: *bool set*)

lemma *qbs-pred-iff-measurable-pred*:
qbs-pred X P = *Measurable.pred* (*qbs-to-measure X*) *P*
⟨*proof*⟩

lemma(**in** *standard-borel*) *qbs-pred-iff-measurable-pred*:
qbs-pred (*measure-to-qbs M*) *P* = *Measurable.pred M P*
⟨*proof*⟩

lemma *qbs-pred-iff-sets*:
{*x* $\in$*space* (*qbs-to-measure X*). *P x*} $\in$ *sets* (*qbs-to-measure X*) $\longleftrightarrow$ *qbs-pred X P*
⟨*proof*⟩

lemma
assumes [*qbs*]:*P* $\in$ *X* $\rightarrow_Q$ *Y* $\Rightarrow_Q$ *qbs-count-space UNIV f* $\in$ *X* $\rightarrow_Q$ *Y*
shows *indicator-qbs-morphism'''*: (λ*x*. *indicator* {*y*. *P x y*} (*f x*)) $\in$ *X* $\rightarrow_Q$ *qbs-borel* (**is** *?g1*)
and *indicator-qbs-morphism''*: (λ*x*. *indicator* {*y*$\in$*qbs-space Y*. *P x y*} (*f x*)) $\in$ *X* $\rightarrow_Q$ *qbs-borel* (**is** *?g2*)
⟨*proof*⟩

lemma
assumes [*qbs*]:*P* $\in$ *X* $\rightarrow_Q$ *Y* $\Rightarrow_Q$ *qbs-count-space UNIV*
shows *indicator-qbs-morphism*[*qbs*]:(λ*x*. *indicator* {*y* $\in$ *qbs-space Y*. *P x y*}) $\in$ *X* $\rightarrow_Q$ *Y* $\Rightarrow_Q$ *qbs-borel* (**is** *?g1*)
and *indicator-qbs-morphism'*:(λ*x*. *indicator* {*y*. *P x y*}) $\in$ *X* $\rightarrow_Q$ *Y* $\Rightarrow_Q$ *qbs-borel* (**is** *?g2*)
⟨*proof*⟩

lemma *indicator-qbs*[*qbs*]:
assumes *qbs-pred X P*
shows *indicator* {*x*. *P x*} $\in$ *X* $\rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *All-qbs-pred*[*qbs*]: *qbs-pred* (*count-space*$_Q$ (*UNIV* :: ($'a$ :: *countable*) *set*) $\Rightarrow_Q$ *count-space*$_Q$ *UNIV*) *All*
⟨*proof*⟩

lemma *Ex-qbs-pred*[*qbs*]: *qbs-pred* (*count-space*$_Q$ (*UNIV* :: ($'a$ :: *countable*) *set*) $\Rightarrow_Q$ *count-space*$_Q$ *UNIV*) *Ex*
⟨*proof*⟩

lemma *Ball-qbs-pred-countable*:
assumes $\bigwedge i::'a$ :: *countable*. $i \in I \Longrightarrow$ *qbs-pred* X (P i)
shows *qbs-pred* X ($\lambda x.$ $\forall x \in I.$ P i x)
⟨*proof*⟩

lemma *Ball-qbs-pred*:
assumes *finite* I $\bigwedge i.$ $i \in I \Longrightarrow$ *qbs-pred* X (P i)
shows *qbs-pred* X ($\lambda x.$ $\forall x \in I.$ P i x)
⟨*proof*⟩

lemma *Bex-qbs-pred-countable*:
assumes $\bigwedge i::'a$ :: *countable*. $i \in I \Longrightarrow$ *qbs-pred* X (P i)
shows *qbs-pred* X ($\lambda x.$ $\exists x \in I.$ P i x)
⟨*proof*⟩

lemma *Bex-qbs-pred*:
assumes *finite* I $\bigwedge i.$ $i \in I \Longrightarrow$ *qbs-pred* X (P i)
shows *qbs-pred* X ($\lambda x.$ $\exists x \in I.$ P i x)
⟨*proof*⟩

lemma *qbs-morphism-If-sub-qbs*:
assumes [*qbs*]: *qbs-pred* X P
and [*qbs*]: $f \in$ *sub-qbs* X $\{x \in$ *qbs-space* $X.$ P $x\}$ $\rightarrow_Q$ Y $g \in$ *sub-qbs* X $\{x \in$ *qbs-space* $X.$ $\neg$ P $x\}$ $\rightarrow_Q$ Y
shows ($\lambda x.$ *if* P x *then* f x *else* g x) $\in X \rightarrow_Q Y$
⟨*proof*⟩

3.3.5 The Adjunction w.r.t. Ordering

lemma *l-mono*: *mono qbs-to-measure*
⟨*proof*⟩

lemma *r-mono*: *mono measure-to-qbs*
⟨*proof*⟩

lemma *rl-order-adjunction*:
$X \le$ *qbs-to-measure* $Y \longleftrightarrow$ *measure-to-qbs* $X \le Y$
⟨*proof*⟩

end

4 The S-Finite Measure Monad

theory *Monad-QuasiBorel*
imports
Measure-QuasiBorel-Adjunction
Kernels

begin

4.1 The S-Finite Measure Monad

4.1.1 Space of S-Finite Measures

locale *in-Mx* =
fixes X :: *'a quasi-borel*
and α :: *real* ⇒ *'a*
assumes *in-Mx*[*simp*]:$\alpha \in$ *qbs-Mx X*
begin

lemma *α-measurable*[*measurable*]: $\alpha \in$ *borel* $\rightarrow_M$ *qbs-to-measure X*
⟨*proof*⟩

lemma *α-qbs-morphism*[*qbs*]: $\alpha \in$ *qbs-borel* $\rightarrow_Q$ *X*
⟨*proof*⟩

lemma *X-not-empty*: *qbs-space X* $\neq \{\}$
⟨*proof*⟩

lemma *inverse-UNIV*[*simp*]: α −‘ (*qbs-space X*) = *UNIV*
⟨*proof*⟩

end

locale *qbs-s-finite* = *in-Mx X* α + *s-finite-measure* μ
for X :: *'a quasi-borel* **and** α **and** μ :: *real measure* +
assumes *mu-sets*[*measurable-cong*]: *sets* μ = *sets borel*
begin

lemma *mu-not-empty*: *space* $\mu \neq \{\}$
⟨*proof*⟩

end

lemma *qbs-s-finite-All*:
assumes $\alpha \in$ *qbs-Mx X s-finite-kernel M borel k x* $\in$ *space M*
shows *qbs-s-finite X* α (*k x*)
⟨*proof*⟩

locale *qbs-prob* = *in-Mx X* α + *real-distribution* μ
for X :: *'a quasi-borel* **and** α μ

begin

lemma *qbs-s-finite*: *qbs-s-finite* X α μ
⟨*proof*⟩

sublocale *qbs-s-finite* ⟨*proof*⟩

end

lemma(**in** *qbs-s-finite*) *qbs-probI*: *prob-space* $\mu \Longrightarrow$ *qbs-prob* X α μ
⟨*proof*⟩

locale *pair-qbs-s-finites* = *pq1*: *qbs-s-finite* X α μ + *pq2*: *qbs-s-finite* Y β ν
for X :: *'a quasi-borel* **and** α μ **and** Y :: *'b quasi-borel* **and** β ν
begin

lemma *ab-measurable*[*measurable*]: *map-prod* α $\beta \in$ *borel* $\bigotimes_M$ *borel* $\rightarrow_M$ *qbs-to-measure* ($X \bigotimes_Q Y$)
⟨*proof*⟩

end

locale *pair-qbs-probs* = *pq1*: *qbs-prob* X α μ + *pq2*: *qbs-prob* Y β ν
for X :: *'a quasi-borel* **and** α μ **and** Y :: *'b quasi-borel* **and** β ν
begin
sublocale *pair-qbs-s-finites*
⟨*proof*⟩
end

locale *pair-qbs-s-finite* = *pq1*: *qbs-s-finite* X α μ + *pq2*: *qbs-s-finite* X β ν
for X :: *'a quasi-borel* **and** α μ **and** β ν
begin
sublocale *pair-qbs-s-finites* X α μ X β ν
⟨*proof*⟩
end

locale *pair-qbs-prob* = *pq1*: *qbs-prob* X α μ + *pq2*: *qbs-prob* X β ν
for X :: *'a quasi-borel* **and** α μ **and** β ν
begin

sublocale *pair-qbs-s-finite* X α μ β ν
⟨*proof*⟩

sublocale *pair-qbs-probs* X α μ X β μ
⟨*proof*⟩

end

type-synonym *'a qbs-s-finite-t* = *'a quasi-borel* $*$ (*real* $\Rightarrow$ *'a*) $*$ *real measure*

definition *qbs-s-finite-eq* :: $['a$ *qbs-s-finite-t*, $'a$ *qbs-s-finite-t*$] \Rightarrow$ *bool* **where**
qbs-s-finite-eq p1 p2 $\equiv$
(*let* $(X, \alpha, \mu) = p1$;
$(Y, \beta, \nu) = p2$ *in*
qbs-s-finite $X\ \alpha\ \mu \wedge$ *qbs-s-finite* $Y\ \beta\ \nu \wedge X = Y \wedge$
distr μ (*qbs-to-measure* X) $\alpha =$ *distr* ν (*qbs-to-measure* Y) β)

definition *qbs-s-finite-eq'* :: $['a$ *qbs-s-finite-t*, $'a$ *qbs-s-finite-t*$] \Rightarrow$ *bool* **where**
qbs-s-finite-eq' p1 p2 $\equiv$
(*let* $(X, \alpha, \mu) = p1$;
$(Y, \beta, \nu) = p2$ *in*
qbs-s-finite $X\ \alpha\ \mu \wedge$ *qbs-s-finite* $Y\ \beta\ \nu \wedge X = Y \wedge$
$(\forall f \in X \rightarrow_Q$ (*qbs-borel* :: *ennreal quasi-borel*). $(\int^+ x.\ f\ (\alpha\ x)\ \partial\mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\nu)))$

lemma(**in** *qbs-s-finite*)
shows *qbs-s-finite-eq-refl*[*simp*]: *qbs-s-finite-eq* (X,α,μ) (X,α,μ)
and *qbs-s-finite-eq'-refl*[*simp*]: *qbs-s-finite-eq'* (X,α,μ) (X,α,μ)
⟨*proof*⟩

lemma(**in** *pair-qbs-s-finite*)
shows *qbs-s-finite-eq-intro*: *distr* μ (*qbs-to-measure* X) $\alpha =$ *distr* ν (*qbs-to-measure* X) $\beta \Longrightarrow$ *qbs-s-finite-eq* (X,α,μ) (X,β,ν)
and *qbs-s-finite-eq'-intro*: $(\bigwedge f.\ f \in X \rightarrow_Q$ *qbs-borel* $\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)) \Longrightarrow$ *qbs-s-finite-eq'* (X,α,μ) (X,β,ν)
⟨*proof*⟩

lemma *qbs-s-finite-eq-dest*:
assumes *qbs-s-finite-eq* (X,α,μ) (Y,β,ν)
shows *qbs-s-finite* $X\ \alpha\ \mu$ *qbs-s-finite* $Y\ \beta\ \nu$ $Y = X$ *distr* μ (*qbs-to-measure* X) $\alpha =$ *distr* ν (*qbs-to-measure* X) β
⟨*proof*⟩

lemma *qbs-s-finite-eq'-dest*:
assumes *qbs-s-finite-eq'* (X,α,μ) (Y,β,ν)
shows *qbs-s-finite* $X\ \alpha\ \mu$ *qbs-s-finite* $Y\ \beta\ \nu$ $Y = X$ $\bigwedge f.\ f \in X \rightarrow_Q$ *qbs-borel* $\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)$
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-s-finite-eq-qbs-prob-cong*:
assumes *qbs-s-finite-eq* (X,α,μ) (Y,β,ν)
shows *qbs-prob* $Y\ \beta\ \nu$
⟨*proof*⟩

lemma
shows *qbs-s-finite-eq-symp*: *symp qbs-s-finite-eq*
and *qbs-s-finite-eq-transp*: *transp qbs-s-finite-eq*
⟨*proof*⟩

quotient-type *'a qbs-measure* = *'a qbs-s-finite-t* / *partial*: *qbs-s-finite-eq*
  morphisms *rep-qbs-measure qbs-measure*
⟨*proof*⟩

interpretation *qbs-measure* : *quot-type qbs-s-finite-eq Abs-qbs-measure Rep-qbs-measure*
  ⟨*proof*⟩

syntax
 -qbs-measure :: *'a quasi-borel* ⇒ (*real* ⇒ *'a*) ⇒ *real measure* ⇒ *'a qbs-measure*
($[\![$-,/ -,/ -$]\!]_{sfin}$)
translations
 $[\![X,\ \alpha,\ \mu]\!]_{sfin}$ ⇌ *CONST qbs-measure* (X, α, μ)

lemma *rep-qbs-s-finite-measure'*: $\exists X\ \alpha\ \mu.\ p = [\![X,\ \alpha,\ \mu]\!]_{sfin} \wedge$ *qbs-s-finite* $X\ \alpha\ \mu$
  ⟨*proof*⟩

lemma *rep-qbs-s-finite-measure*:
  obtains $X\ \alpha\ \mu$ **where** $p = [\![X,\ \alpha,\ \mu]\!]_{sfin}$ *qbs-s-finite* $X\ \alpha\ \mu$
  ⟨*proof*⟩

definition *qbs-null-measure* :: *'a quasi-borel* ⇒ *'a qbs-measure* **where**
qbs-null-measure $X \equiv [\![X,$ *SOME a. a* ∈ *qbs-Mx X*, *null-measure borel*$]\!]_{sfin}$

lemma *qbs-null-measure-s-finite*: *qbs-space X* ≠ {} ⟹ *qbs-s-finite X* (*SOME a. a* ∈ *qbs-Mx X*) (*null-measure borel*)
  ⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *in-Rep-qbs-measure'*:
  assumes *qbs-s-finite-eq* (X,α,μ) (X',α',μ')
  shows (X',α',μ') ∈ *Rep-qbs-measure* $[\![X,\ \alpha,\ \mu]\!]_{sfin}$
  ⟨*proof*⟩

lemmas(**in** *qbs-s-finite*) *in-Rep-qbs-measure* = *in-Rep-qbs-measure'*[*OF qbs-s-finite-eq-refl*]

lemma(**in** *qbs-s-finite*) *if-in-Rep-qbs-measure*:
  assumes (X',α',μ') ∈ *Rep-qbs-measure* $[\![X,\ \alpha,\ \mu]\!]_{sfin}$
  shows $X' = X$
    qbs-s-finite $X'\ \alpha'\ \mu'$
    qbs-s-finite-eq (X,α,μ) (X',α',μ')
⟨*proof*⟩

lemma *qbs-s-finite-eq-1-imp-2*:
  assumes *qbs-s-finite-eq* (X,α,μ) (Y,β,ν) $f \in X \rightarrow_Q$ (*qbs-borel* :: (- :: {*banach*}) *quasi-borel*)
  shows ($\int x.\ f\ (\alpha\ x)\ \partial\mu$) = ($\int x.\ f\ (\beta\ x)\ \partial\nu$) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-s-finite-eq-equiv*: *qbs-s-finite-eq* = *qbs-s-finite-eq'*
⟨*proof*⟩

lemma *qbs-s-finite-measure-eq*: *qbs-s-finite-eq* (X,α,μ) $(Y,\beta,\nu) \Longrightarrow [\![X,\ \alpha,\ \mu]\!]_{sfin} = [\![Y,\ \beta,\ \nu]\!]_{sfin}$
⟨*proof*⟩

lemma(**in** *pair-qbs-s-finite*) *qbs-s-finite-measure-eq*:
distr μ (*qbs-to-measure* X) α = *distr* ν (*qbs-to-measure* X) $\beta \Longrightarrow [\![X,\ \alpha,\ \mu]\!]_{sfin} = [\![X,\ \beta,\ \nu]\!]_{sfin}$
⟨*proof*⟩

lemma(**in** *pair-qbs-s-finite*) *qbs-s-finite-measure-eq'*:
$(\bigwedge f.\ f \in X \to_Q$ *qbs-borel* $\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)) \Longrightarrow [\![X,\ \alpha,\ \mu]\!]_{sfin} = [\![X,\ \beta,\ \nu]\!]_{sfin}$
⟨*proof*⟩

lemma(**in** *pair-qbs-s-finite*) *qbs-s-finite-measure-eq-inverse*:
assumes $[\![X,\ \alpha,\ \mu]\!]_{sfin} = [\![X,\ \beta,\ \nu]\!]_{sfin}$
shows *qbs-s-finite-eq* (X,α,μ) (X,β,ν) *qbs-s-finite-eq'* (X,α,μ) (X,β,ν)
⟨*proof*⟩

lift-definition *qbs-space-of* :: *'a qbs-measure* ⇒ *'a quasi-borel*
is *fst* ⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-space-of*[*simp*]:
qbs-space-of $[\![X,\ \alpha,\ \mu]\!]_{sfin} = X$ ⟨*proof*⟩

lemma *rep-qbs-space-of*:
assumes *qbs-space-of* $s = X$
shows $\exists\,\alpha\ \mu.\ s = [\![X,\ \alpha,\ \mu]\!]_{sfin} \land$ *qbs-s-finite* $X\ \alpha\ \mu$
⟨*proof*⟩

corollary *qbs-s-space-of-not-empty*: *qbs-space* (*qbs-space-of* X) $\neq \{\}$
⟨*proof*⟩

4.1.2 The S-Finite Measure Monad

definition *monadM-qbs* :: *'a quasi-borel* ⇒ *'a qbs-measure quasi-borel* **where**
monadM-qbs $X \equiv$ *Abs-quasi-borel* $(\{s.$ *qbs-space-of* $s = X\}$, $\{\lambda r.\ [\![X,\ \alpha,\ k\ r]\!]_{sfin}\ |\alpha\ k.\ \alpha \in$ *qbs-Mx* $X \land$ *s-finite-kernel borel borel* $k\})$

lemma
shows *monadM-qbs-space*: *qbs-space* (*monadM-qbs* X) = $\{s.$ *qbs-space-of* $s = X\}$
and *monadM-qbs-Mx*: *qbs-Mx* (*monadM-qbs* X) = $\{\lambda r.\ [\![X,\ \alpha,\ k\ r]\!]_{sfin}\ |\alpha\ k.\ \alpha \in$ *qbs-Mx* $X \land$ *s-finite-kernel borel borel* $k\}$
⟨*proof*⟩

lemma *monadM-qbs-empty-iff*: *qbs-space* $X = \{\} \longleftrightarrow$ *qbs-space* (*monadM-qbs* X) $= \{\}$
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *in-space-monadM*[*qbs*]: $[\![X, \alpha, \mu]\!]_{sfin} \in$ *qbs-space* (*monadM-qbs* X)
⟨*proof*⟩

lemma *rep-qbs-space-monadM*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X)
obtains α μ **where** $s = [\![X, \alpha, \mu]\!]_{sfin}$ *qbs-s-finite* X α μ
⟨*proof*⟩

lemma *rep-qbs-space-monadM-sigma-finite*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X)
obtains α μ **where** $s = [\![X, \alpha, \mu]\!]_{sfin}$ *qbs-s-finite* X α μ *sigma-finite-measure* μ
⟨*proof*⟩

lemma *qbs-space-of-in*: $s \in$ *qbs-space* (*monadM-qbs* X) $\Longrightarrow$ *qbs-space-of* $s = X$
⟨*proof*⟩

lemma *in-qbs-space-of*: $s \in$ *qbs-space* (*monadM-qbs* (*qbs-space-of* s))
⟨*proof*⟩

4.1.3 *l*

lift-definition *qbs-l* :: *'a qbs-measure* $\Rightarrow$ *'a measure*
is λp. *distr* (*snd* (*snd* p)) (*qbs-to-measure* (*fst* p)) (*fst* (*snd* p))
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-l*: *qbs-l* $[\![X, \alpha, \mu]\!]_{sfin}$ = *distr* μ (*qbs-to-measure* X) α
⟨*proof*⟩

interpretation *qbs-l-s-finite*: *s-finite-measure qbs-l* (s :: *'a qbs-measure*)
⟨*proof*⟩

lemma *space-qbs-l*: *qbs-space* (*qbs-space-of* s) = *space* (*qbs-l* s)
⟨*proof*⟩

lemma *space-qbs-l-ne*: *space* (*qbs-l* s) $\neq \{\}$
⟨*proof*⟩

lemma *qbs-l-sets*: *sets* (*qbs-to-measure* (*qbs-space-of* s)) = *sets* (*qbs-l* s)
⟨*proof*⟩

lemma *qbs-null-measure-in-Mx*: *qbs-space* $X \neq \{\} \Longrightarrow$ *qbs-null-measure* $X \in$ *qbs-space* (*monadM-qbs* X)
⟨*proof*⟩

lemma *qbs-null-measure-null-measure*:*qbs-space* $X \neq \{\} \Longrightarrow$ *qbs-l* (*qbs-null-measure* X) = *null-measure* (*qbs-to-measure* X)
⟨*proof*⟩

lemma *space-qbs-l-in*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* X)
  shows *space* (*qbs-l* s) = *qbs-space* X
  ⟨*proof*⟩

lemma *sets-qbs-l*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* X)
  shows *sets* (*qbs-l* s) = *sets* (*qbs-to-measure* X)
  ⟨*proof*⟩

lemma *measurable-qbs-l*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* X)
  shows *qbs-l* $s \rightarrow_M M = X \rightarrow_Q$ *measure-to-qbs* M
  ⟨*proof*⟩

lemma *measurable-qbs-l'*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* X)
  shows *qbs-l* $s \rightarrow_M M =$ *qbs-to-measure* $X \rightarrow_M M$
  ⟨*proof*⟩

lemma *rep-qbs-Mx-monadM*:
  assumes $\gamma \in$ *qbs-Mx* (*monadM-qbs* X)
  obtains α k **where** $\gamma = (\lambda r.\ [\![X, \alpha, k\ r]\!]_{sfin})$ $\alpha \in$ *qbs-Mx* X *s-finite-kernel borel borel* k $\bigwedge r.$ *qbs-s-finite* X α (k r)
⟨*proof*⟩

lemma *qbs-l-measurable*[*measurable*]:*qbs-l* $\in$ *qbs-to-measure* (*monadM-qbs* X) $\rightarrow_M$ *s-finite-measure-algebra* (*qbs-to-measure* X)
⟨*proof*⟩

lemma *qbs-l-measure-kernel*: *measure-kernel* (*qbs-to-measure* (*monadM-qbs* X)) (*qbs-to-measure* X) *qbs-l*
⟨*proof*⟩

lemma *qbs-l-inj*: *inj-on* *qbs-l* (*qbs-space* (*monadM-qbs* X))
  ⟨*proof*⟩

lemma *qbs-l-morphism*:
  assumes [*measurable*]:$A \in$ *sets* (*qbs-to-measure* X)
  shows ($\lambda s.$ *qbs-l* s A) $\in$ *monadM-qbs* $X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-l-finite-pred*: *qbs-pred* (*monadM-qbs* X) ($\lambda s.$ *finite-measure* (*qbs-l* s))
⟨*proof*⟩

lemma *qbs-l-subprob-pred*: *qbs-pred* (*monadM-qbs* X) ($\lambda s.$ *subprob-space* (*qbs-l* s))
⟨*proof*⟩

lemma *qbs-l-prob-pred*: $qbs\text{-}pred\ (monadM\text{-}qbs\ X)\ (\lambda s.\ prob\text{-}space\ (qbs\text{-}l\ s))$
$\langle proof \rangle$

4.1.4 Return

definition *return-qbs* :: $'a\ quasi\text{-}borel \Rightarrow\ 'a \Rightarrow\ 'a\ qbs\text{-}measure$ **where**
$return\text{-}qbs\ X\ x \equiv [\![X,\ \lambda r.\ x,\ SOME\ \mu.\ real\text{-}distribution\ \mu]\!]_{sfin}$

lemma(**in** *real-distribution*)
  assumes $x \in qbs\text{-}space\ X$
  shows *return-qbs*:$return\text{-}qbs\ X\ x = [\![X,\ \lambda r.\ x,\ M]\!]_{sfin}$
    and *return-qbs-prob*:$qbs\text{-}prob\ X\ (\lambda r.\ x)\ M$
    and *return-qbs-s-finite*:$qbs\text{-}s\text{-}finite\ X\ (\lambda r.\ x)\ M$
$\langle proof \rangle$

lemma *return-qbs-comp*:
  assumes $\alpha \in qbs\text{-}Mx\ X$
  shows $(return\text{-}qbs\ X \circ \alpha) = (\lambda r.\ [\![X,\ \alpha,\ return\ borel\ r]\!]_{sfin})$
$\langle proof \rangle$

corollary *return-qbs-morphism*[*qbs*]: $return\text{-}qbs\ X \in X \rightarrow_Q monadM\text{-}qbs\ X$
$\langle proof \rangle$

4.1.5 Bind

definition *bind-qbs* :: $['a\ qbs\text{-}measure,\ 'a \Rightarrow\ 'b\ qbs\text{-}measure] \Rightarrow\ 'b\ qbs\text{-}measure$ **where**
$bind\text{-}qbs\ s\ f \equiv (let\ (X,\ \alpha,\ \mu) = rep\text{-}qbs\text{-}measure\ s;$
$\quad Y = qbs\text{-}space\text{-}of\ (f\ (\alpha\ undefined));$
$\quad (\beta,\ k) = (SOME\ (\beta,\ k).\ f \circ \alpha = (\lambda r.\ [\![Y,\ \beta,\ k\ r]\!]_{sfin}) \wedge \beta \in qbs\text{-}Mx\ Y \wedge s\text{-}finite\text{-}kernel\ borel\ borel\ k)\ in$
$\quad [\![Y,\ \beta,\ \mu \gg\!=_k k]\!]_{sfin})$

adhoc-overloading *Monad-Syntax.bind bind-qbs*

lemma(**in** *qbs-s-finite*)
  assumes $s = [\![X,\ \alpha,\ \mu]\!]_{sfin}$
    $f \in X \rightarrow_Q monadM\text{-}qbs\ Y$
    $\beta \in qbs\text{-}Mx\ Y$
    $s\text{-}finite\text{-}kernel\ borel\ borel\ k$
    and $(f \circ \alpha) = (\lambda r.\ [\![Y,\ \beta,\ k\ r]\!]_{sfin})$
  shows *bind-qbs-s-finite*:$qbs\text{-}s\text{-}finite\ Y\ \beta\ (\mu \gg\!=_k k)$
    and *bind-qbs*: $s \gg\!= f = [\![Y,\ \beta,\ \mu \gg\!=_k k]\!]_{sfin}$
$\langle proof \rangle$

lemma *bind-qbs-morphism'*:
  assumes $f \in X \rightarrow_Q monadM\text{-}qbs\ Y$
  shows $(\lambda x.\ x \gg\!= f) \in monadM\text{-}qbs\ X \rightarrow_Q monadM\text{-}qbs\ Y$
$\langle proof \rangle$

lemma *bind-qbs-return'*:
  assumes $x \in$ *qbs-space* (*monadM-qbs* X)
  shows $x \gg\!= $ *return-qbs* $X = x$
⟨*proof*⟩

lemma *bind-qbs-return*:
  assumes $f \in X \to_Q$ *monadM-qbs* Y
    and $x \in$ *qbs-space* X
   shows *return-qbs* $X\ x \gg\!= f = f\ x$
⟨*proof*⟩

lemma *bind-qbs-assoc*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* X)
      $f \in X \to_Q$ *monadM-qbs* Y
    and $g \in Y \to_Q$ *monadM-qbs* Z
  shows $s \gg\!= (\lambda x.\ f\ x \gg\!= g) = (s \gg\!= f) \gg\!= g$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-qbs-cong*:
  assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X)
      $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
    and [*qbs*]:$f \in X \to_Q$ *monadM-qbs* Y
   shows $s \gg\!= f = s \gg\!= g$
⟨*proof*⟩

4.1.6 The Functorial Action

definition *distr-qbs* :: [*'a quasi-borel*, *'b quasi-borel*,*'a* ⇒ *'b*,*'a qbs-measure*] ⇒ *'b qbs-measure* **where**
distr-qbs - $Y\ f\ sx \equiv sx \gg\!=$ *return-qbs* $Y \circ f$

lemma *distr-qbs-morphism'*:
  assumes $f \in X \to_Q Y$
  shows *distr-qbs* $X\ Y\ f \in$ *monadM-qbs* $X \to_Q$ *monadM-qbs* Y
  ⟨*proof*⟩

lemma(**in** *qbs-s-finite*)
  assumes $s = [\![X, \alpha, \mu]\!]_{sfin}$
    and $f \in X \to_Q Y$
   shows *distr-qbs-s-finite*:*qbs-s-finite* Y $(f \circ \alpha)$ μ
    and *distr-qbs*: *distr-qbs* $X\ Y\ f\ s = [\![Y, f \circ \alpha, \mu]\!]_{sfin}$
  ⟨*proof*⟩

lemma(**in** *qbs-prob*)
  assumes $s = [\![X, \alpha, \mu]\!]_{sfin}$
    and $f \in X \to_Q Y$
   shows *distr-qbs-prob*:*qbs-prob* Y $(f \circ \alpha)$ μ
  ⟨*proof*⟩

We show that M is a functor i.e. M preserve identity and composition.

lemma *distr-qbs-id*:
  assumes $s \in$ *qbs-space* (*monadM-qbs X*)
  shows *distr-qbs X X id s = s*
  ⟨*proof*⟩

lemma *distr-qbs-comp*:
  assumes $s \in$ *qbs-space* (*monadM-qbs X*)
      $f \in X \rightarrow_Q Y$
    and $g \in Y \rightarrow_Q Z$
   shows ((*distr-qbs Y Z g*) $\circ$ (*distr-qbs X Y f*)) *s* = *distr-qbs X Z* ($g \circ f$) *s*
⟨*proof*⟩

4.1.7 Join

definition *join-qbs* :: *'a qbs-measure qbs-measure* $\Rightarrow$ *'a qbs-measure* **where**
join-qbs $\equiv$ (λ*sst. sst* $\gg=$ *id*)

lemma *join-qbs-morphism*[*qbs*]: *join-qbs* $\in$ *monadM-qbs* (*monadM-qbs X*) $\rightarrow_Q$ *monadM-qbs X*
  ⟨*proof*⟩

lemma
  assumes *qbs-s-finite* (*monadM-qbs X*) β μ
      ssx = $[\![$*monadM-qbs X*, β, $\mu]\!]_{sfin}$
      $\alpha \in$ *qbs-Mx X*
      s-finite-kernel borel borel k
    and β =(λr. $[\![X, \alpha, k\ r]\!]_{sfin}$)
   shows *qbs-s-finite-join-qbs-s-finite*: *qbs-s-finite X* α ($\mu \gg=_k k$)
    and *qbs-s-finite-join-qbs*: *join-qbs ssx* = $[\![X, \alpha, \mu \gg=_k k]\!]_{sfin}$
  ⟨*proof*⟩

4.1.8 Strength

definition *strength-qbs* :: [*'a quasi-borel*,*'b quasi-borel*,*'a* $\times$ *'b qbs-measure*] $\Rightarrow$ (*'a* $\times$ *'b*) *qbs-measure* **where**
strength-qbs W X = (λ(*w*,*sx*). *let* (-,α,μ) = *rep-qbs-measure sx*
              in $[\![W \bigotimes_Q X, \lambda r. (w,\alpha\ r), \mu]\!]_{sfin}$)

lemma(**in** *qbs-s-finite*)
  assumes $w \in$ *qbs-space W*
    and *sx* = $[\![X, \alpha, \mu]\!]_{sfin}$
   shows *strength-qbs-s-finite*: *qbs-s-finite* ($W \bigotimes_Q X$) (λr. (w,$\alpha\ r$)) μ
    and *strength-qbs*: *strength-qbs W X* (*w*,*sx*) = $[\![W \bigotimes_Q X, \lambda r. (w,\alpha\ r), \mu]\!]_{sfin}$
⟨*proof*⟩

lemma(**in** *qbs-prob*)
  assumes $w \in$ *qbs-space W*
    and *sx* = $[\![X, \alpha, \mu]\!]_{sfin}$

shows *strength-qbs-prob*: $qbs\text{-}prob\ (W \bigotimes_Q X)\ (\lambda r.\ (w,\alpha\ r))\ \mu$
⟨*proof*⟩

lemma *strength-qbs-natural*:
assumes $f \in X \to_Q X'$
$g \in Y \to_Q Y'$
$x \in qbs\text{-}space\ X$
and $sy \in qbs\text{-}space\ (monadM\text{-}qbs\ Y)$
shows $(distr\text{-}qbs\ (X \bigotimes_Q Y)\ (X' \bigotimes_Q Y')\ (map\text{-}prod\ f\ g) \circ strength\text{-}qbs\ X\ Y)\ (x,sy) = (strength\text{-}qbs\ X'\ Y' \circ map\text{-}prod\ f\ (distr\text{-}qbs\ Y\ Y'\ g))\ (x,sy)$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

context
begin

interpretation rr : $standard\text{-}borel\text{-}ne\ borel \bigotimes_M borel :: (real \times real)\ measure$
⟨*proof*⟩

declare *rr.from-real-to-real*[*simplified space-pair-measure*,*simplified*,*simp*]

lemma *rr-from-real-to-real-id*[*simp*]: $rr.from\text{-}real \circ rr.to\text{-}real = id$
⟨*proof*⟩

lemma
assumes $\alpha \in qbs\text{-}Mx\ X$
$\beta \in qbs\text{-}Mx\ (monadM\text{-}qbs\ Y)$
$\gamma \in qbs\text{-}Mx\ Y$
s-finite-kernel borel borel k
and $\beta = (\lambda r.\ [\![Y,\ \gamma,\ k\ r]\!]_{sfin})$
shows *strength-qbs-ab-r-s-finite*: $qbs\text{-}s\text{-}finite\ (X \bigotimes_Q Y)\ (map\text{-}prod\ \alpha\ \gamma \circ rr.from\text{-}real)\ (distr\ (return\ borel\ r \bigotimes_M k\ r)\ borel\ rr.to\text{-}real)$
and *strength-qbs-ab-r*: $strength\text{-}qbs\ X\ Y\ (\alpha\ r,\ \beta\ r) = [\![X \bigotimes_Q Y,\ map\text{-}prod\ \alpha\ \gamma \circ rr.from\text{-}real,\ distr\ (return\ borel\ r \bigotimes_M k\ r)\ borel\ rr.to\text{-}real]\!]_{sfin}$ (**is** *?goal2*)
⟨*proof*⟩

lemma *strength-qbs-morphism*[*qbs*]: $strength\text{-}qbs\ X\ Y \in X \bigotimes_Q monadM\text{-}qbs\ Y \to_Q monadM\text{-}qbs\ (X \bigotimes_Q Y)$
⟨*proof*⟩

lemma *bind-qbs-morphism*[*qbs*]: $(\gg=) \in monadM\text{-}qbs\ X \to_Q (X \Rightarrow_Q monadM\text{-}qbs\ Y) \Rightarrow_Q monadM\text{-}qbs\ Y$
⟨*proof*⟩

lemma *strength-qbs-law1*:
assumes $x \in qbs\text{-}space\ (unit\text{-}quasi\text{-}borel \bigotimes_Q monadM\text{-}qbs\ X)$
shows $snd\ x = (distr\text{-}qbs\ (unit\text{-}quasi\text{-}borel \bigotimes_Q X)\ X\ snd \circ strength\text{-}qbs\ unit\text{-}quasi\text{-}borel\ X)\ x$
⟨*proof*⟩

lemma *strength-qbs-law2*:
assumes $x \in$ *qbs-space* $((X \bigotimes_Q Y) \bigotimes_Q$ *monadM-qbs* $Z)$
shows (*strength-qbs* X $(Y \bigotimes_Q Z) \circ$ (*map-prod id* (*strength-qbs* Y Z)) $\circ$ $(\lambda((x,y),z).\ (x,(y,z))))\ x =$
(*distr-qbs* $((X \bigotimes_Q Y) \bigotimes_Q Z)$ $(X \bigotimes_Q (Y \bigotimes_Q Z))$ $(\lambda((x,y),z).\ (x,(y,z)))$ $\circ$ *strength-qbs* $(X \bigotimes_Q Y)$ Z) x
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *strength-qbs-law3*:
assumes $x \in$ *qbs-space* $(X \bigotimes_Q Y)$
shows *return-qbs* $(X \bigotimes_Q Y)$ x = (*strength-qbs* X Y $\circ$ (*map-prod id* (*return-qbs* Y))) x
⟨*proof*⟩

lemma *strength-qbs-law4*:
assumes $x \in$ *qbs-space* $(X \bigotimes_Q$ *monadM-qbs* (*monadM-qbs* Y))
shows (*strength-qbs* X Y $\circ$ *map-prod id join-qbs*) x = (*join-qbs* $\circ$ *distr-qbs* $(X \bigotimes_Q$ *monadM-qbs* Y) (*monadM-qbs* $(X \bigotimes_Q Y)$) (*strength-qbs* X Y) $\circ$ *strength-qbs* X (*monadM-qbs* Y)) x
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *distr-qbs-morphism*[*qbs*]: *distr-qbs* X $Y \in (X \Rightarrow_Q Y) \rightarrow_Q$ (*monadM-qbs* X $\Rightarrow_Q$ *monadM-qbs* Y)
⟨*proof*⟩

lemma
assumes $\alpha \in$ *qbs-Mx* X $\beta \in$ *qbs-Mx* Y
shows *return-qbs-pair-Mx*: *return-qbs* $(X \bigotimes_Q Y)$ $(\alpha\ r, \beta\ k) = [\![X \bigotimes_Q Y$,*map-prod* α β $\circ$ *rr.from-real*, *distr* (*return borel* $r \bigotimes_M$ *return borel* k) *borel rr.to-real*$]\!]_{sfin}$
and *return-qbs-pair-Mx-prob*: *qbs-prob* $(X \bigotimes_Q Y)$ (*map-prod* α β $\circ$ *rr.from-real*) (*distr* (*return borel* $r \bigotimes_M$ *return borel* k) *borel rr.to-real*)
⟨*proof*⟩

lemma *bind-bind-return-distr*:
assumes *s-finite-measure* μ
and *s-finite-measure* ν
and [*measurable-cong*]: *sets* μ = *sets borel sets* ν = *sets borel*
shows $\mu \gg\!=_k (\lambda r.\ \nu \gg\!=_k (\lambda l.$ *distr* (*return borel* $r \bigotimes_M$ *return borel* l) *borel rr.to-real*))
= *distr* $(\mu \bigotimes_M \nu)$ *borel rr.to-real*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

context

begin
interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

lemma *from-real-rr-qbs-morphism*[*qbs*]: *rr.from-real* ∈ *qbs-borel* $\to_Q$ *qbs-borel* $\bigotimes_Q$ *qbs-borel*
⟨*proof*⟩

end

context *pair-qbs-s-finites*
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

sublocale *qbs-s-finite* $X \bigotimes_Q Y$ *map-prod* α β ∘ *rr.from-real distr* ($\mu \bigotimes_M \nu$) *borel rr.to-real*
⟨*proof*⟩

lemma *qbs-bind-bind-return-qp*:
$[\![Y,\beta,\nu]\!]_{sfin} \gg\!= (\lambda y.\ [\![X,\alpha,\mu]\!]_{sfin} \gg\!= (\lambda x.\ \textit{return-qbs}\ (X \bigotimes_Q Y)\ (x,y))) = [\![X \bigotimes_Q Y,\ \textit{map-prod}\ \alpha\ \beta \circ \textit{rr.from-real},\ \textit{distr}\ (\mu \bigotimes_M \nu)\ \textit{borel}\ \textit{rr.to-real}]\!]_{sfin}$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-bind-bind-return-pq*:
$[\![X,\alpha,\mu]\!]_{sfin} \gg\!= (\lambda x.\ [\![Y,\beta,\nu]\!]_{sfin} \gg\!= (\lambda y.\ \textit{return-qbs}\ (X \bigotimes_Q Y)\ (x,y))) = [\![X \bigotimes_Q Y,\ \textit{map-prod}\ \alpha\ \beta \circ \textit{rr.from-real},\ \textit{distr}\ (\mu \bigotimes_M \nu)\ \textit{borel}\ \textit{rr.to-real}]\!]_{sfin}$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

lemma *bind-qbs-return-rotate*:
assumes $p \in$ *qbs-space* (*monadM-qbs* X)
and $q \in$ *qbs-space* (*monadM-qbs* Y)
shows $q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.\ \textit{return-qbs}\ (X \bigotimes_Q Y)\ (x,y))) = p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.\ \textit{return-qbs}\ (X \bigotimes_Q Y)\ (x,y)))$
⟨*proof*⟩

lemma *qbs-bind-bind-return1*:
assumes [*qbs*]: $f \in X \bigotimes_Q Y \to_Q$ *monadM-qbs* Z
$p \in$ *qbs-space* (*monadM-qbs* X)
$q \in$ *qbs-space* (*monadM-qbs* Y)
shows $q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.\ f\ (x,y))) = (q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.\ \textit{return-qbs}\ (X \bigotimes_Q Y)\ (x,y)))) \gg\!= f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-bind-bind-return2*:
assumes $[qbs]{:}f \in X \bigotimes_Q Y \to_Q$ *monadM-qbs* Z
$p \in$ *qbs-space* (*monadM-qbs* X) $q \in$ *qbs-space* (*monadM-qbs* Y)
shows $p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.\ f\ (x,y))) = (p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.\ \mathit{return\text{-}qbs}\ (X \bigotimes_Q Y)\ (x,y)))) \gg\!= f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

corollary *bind-qbs-rotate*:
assumes $f \in X \bigotimes_Q Y \to_Q$ *monadM-qbs* Z
$p \in$ *qbs-space* (*monadM-qbs* X)
and $q \in$ *qbs-space* (*monadM-qbs* Y)
shows $q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.\ f\ (x,y))) = p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.\ f\ (x,y)))$
⟨*proof*⟩

context *pair-qbs-s-finites*
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

lemma
assumes $[qbs]{:}f \in X \bigotimes_Q Y \to_Q Z$
shows *qbs-bind-bind-return*: $[\![X,\alpha,\mu]\!]_{sfin} \gg\!= (\lambda x.\ [\![Y,\beta,\nu]\!]_{sfin} \gg\!= (\lambda y.\ \mathit{return\text{-}qbs}\ Z\ (f\ (x,y)))) = [\![Z,\ f \circ (\mathit{map\text{-}prod}\ \alpha\ \beta \circ \mathit{rr.from\text{-}real}),\ \mathit{distr}\ (\mu \bigotimes_M \nu)\ \mathit{borel}\ \mathit{rr.to\text{-}real}]\!]_{sfin}$ (**is** *?lhs* = *?rhs*)
and *qbs-bind-bind-return-s-finite*: *qbs-s-finite* Z ($f \circ$ (*map-prod* α $\beta \circ$ *rr.from-real*)) (*distr* ($\mu \bigotimes_M \nu$) *borel rr.to-real*)
⟨*proof*⟩

end

4.1.9 The Probability Monad

definition *monadP-qbs* $X \equiv$ *sub-qbs* (*monadM-qbs* X) {*s. prob-space* (*qbs-l s*)}

lemma
shows *qbs-space-monadPM*: $s \in$ *qbs-space* (*monadP-qbs* X) $\Longrightarrow s \in$ *qbs-space* (*monadM-qbs* X)
and *qbs-Mx-monadPM*: $f \in$ *qbs-Mx* (*monadP-qbs* X) $\Longrightarrow f \in$ *qbs-Mx* (*monadM-qbs* X)
⟨*proof*⟩

lemma *monadP-qbs-space*: *qbs-space* (*monadP-qbs* X) = {*s. qbs-space-of s* = X ∧ *prob-space* (*qbs-l s*)}
⟨*proof*⟩

lemma *rep-qbs-space-monadP*:

assumes $s \in$ *qbs-space* (*monadP-qbs* X)
obtains α μ **where** $s = [\![X, \alpha, \mu]\!]_{sfin}$ *qbs-prob* X α μ
⟨*proof*⟩

lemma *qbs-l-prob-space*:
$s \in$ *qbs-space* (*monadP-qbs* X) $\Longrightarrow$ *prob-space* (*qbs-l* s)
⟨*proof*⟩

lemma *monadP-qbs-empty-iff*:
(*qbs-space* $X = \{\}$) $=$ (*qbs-space* (*monadP-qbs* X) $= \{\}$)
⟨*proof*⟩

lemma *in-space-monadP-qbs-pred*: *qbs-pred* (*monadM-qbs* X) ($\lambda s.\ s \in$ *monadP-qbs* X)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *in-space-monadP*[*qbs*]: $[\![X, \alpha, \mu]\!]_{sfin} \in$ *qbs-space* (*monadP-qbs* X)
⟨*proof*⟩

lemma *qbs-morphism-monadPD*: $f \in X \to_Q$ *monadP-qbs* $Y \Longrightarrow f \in X \to_Q$ *monadM-qbs* Y
⟨*proof*⟩

lemma *qbs-morphism-monadPD'*: $f \in$ *monadM-qbs* $X \to_Q Y \Longrightarrow f \in$ *monadP-qbs* $X \to_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-monadPI*:
assumes $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow$ *prob-space* (*qbs-l* ($f\,x$)) $f \in X \to_Q$ *monadM-qbs* Y
shows $f \in X \to_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *qbs-morphism-monadPI'*:
assumes $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\,x \in$ *qbs-space* (*monadP-qbs* Y) $f \in X \to_Q$ *monadM-qbs* Y
shows $f \in X \to_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *qbs-morphism-monadPI''*:
assumes $f \in$ *monadM-qbs* $X \to_Q$ *monadM-qbs* Y $\bigwedge s.\ s \in$ *qbs-space* (*monadP-qbs* X) $\Longrightarrow f\,s \in$ *qbs-space* (*monadP-qbs* Y)
shows $f \in$ *monadP-qbs* $X \to_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *monadP-qbs-Mx*: *qbs-Mx* (*monadP-qbs* X) $= \{\lambda r.\ [\![X, \alpha, k\ r]\!]_{sfin} \mid \alpha\ k.\ \alpha \in$ *qbs-Mx* $X \wedge k \in$ *borel* $\to_M$ *prob-algebra borel*$\}$
⟨*proof*⟩

lemma *rep-qbs-Mx-monadP*:
  assumes $\gamma \in$ *qbs-Mx* (*monadP-qbs X*)
  obtains α k **where** $\gamma = (\lambda r.\ [\![X,\ \alpha,\ k\ r]\!]_{sfin})$ $\alpha \in$ *qbs-Mx X* $k \in$ *borel* $\rightarrow_M$ *prob-algebra borel* $\bigwedge r.$ *qbs-prob X* α $(k\ r)$
⟨*proof*⟩

lemma *qbs-l-monadP-le1*:$s \in$ *qbs-space* (*monadP-qbs X*) $\Longrightarrow$ *qbs-l s A* $\leq$ *1*
  ⟨*proof*⟩

lemma *qbs-l-inj-P*: *inj-on qbs-l* (*qbs-space* (*monadP-qbs X*))
  ⟨*proof*⟩

lemma *qbs-l-measurable-prob*[*measurable*]:*qbs-l* $\in$ *qbs-to-measure* (*monadP-qbs X*) $\rightarrow_M$ *prob-algebra* (*qbs-to-measure X*)
⟨*proof*⟩

lemma *return-qbs-morphismP*: *return-qbs X* $\in X \rightarrow_Q$ *monadP-qbs X*
⟨*proof*⟩

lemma(**in** *qbs-prob*)
  assumes $s = [\![X,\ \alpha,\ \mu]\!]_{sfin}$
      $f \in X \rightarrow_Q$ *monadP-qbs Y*
      $\beta \in$ *qbs-Mx Y*
    and *g*[*measurable*]:$g \in$ *borel* $\rightarrow_M$ *prob-algebra borel*
    and $(f \circ \alpha) = (\lambda r.\ [\![Y,\ \beta,\ g\ r]\!]_{sfin})$
   shows *bind-qbs-prob*:*qbs-prob Y* β $(\mu \ggg= g)$
    and *bind-qbs'*: $s \ggg= f = [\![Y,\ \beta,\ \mu \ggg= g]\!]_{sfin}$
⟨*proof*⟩

lemma *bind-qbs-morphism'P*:
  assumes $f \in X \rightarrow_Q$ *monadP-qbs Y*
  shows $(\lambda x.\ x \ggg= f) \in$ *monadP-qbs X* $\rightarrow_Q$ *monadP-qbs Y*
⟨*proof*⟩

lemma *distr-qbs-morphismP'*:
  assumes $f \in X \rightarrow_Q Y$
  shows *distr-qbs X Y f* $\in$ *monadP-qbs X* $\rightarrow_Q$ *monadP-qbs Y*
  ⟨*proof*⟩

lemma *join-qbs-morphismP*: *join-qbs* $\in$ *monadP-qbs* (*monadP-qbs X*) $\rightarrow_Q$ *monadP-qbs X*
  ⟨*proof*⟩

lemma
  assumes *qbs-prob* (*monadP-qbs X*) β μ
      $ssx = [\![$*monadP-qbs X*$,\ \beta,\ \mu]\!]_{sfin}$
      $\alpha \in$ *qbs-Mx X*
      $g \in$ *borel* $\rightarrow_M$ *prob-algebra borel*

and β =(λr. $[\![X,\ \alpha,\ g\ r]\!]_{sfin}$)
shows *qbs-prob-join-qbs-s-finite*: *qbs-prob* X α ($\mu \gg= g$)
and *qbs-prob-join-qbs*: *join-qbs ssx* = $[\![X,\ \alpha,\ \mu \gg= g]\!]_{sfin}$
⟨*proof*⟩

context
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

lemma *strength-qbs-ab-r-prob*:
assumes $\alpha \in$ *qbs-Mx X*
$\beta \in$ *qbs-Mx* (*monadP-qbs Y*)
$\gamma \in$ *qbs-Mx Y*
and [*measurable*]:$g \in$ *borel* $\rightarrow_M$ *prob-algebra borel*
and β = (λr. $[\![Y,\ \gamma,\ g\ r]\!]_{sfin}$)
shows *qbs-prob* ($X \bigotimes_Q Y$) (*map-prod* α γ ∘ *rr.from-real*) (*distr* (*return borel* r $\bigotimes_M$ *g r*) *borel rr.to-real*)
⟨*proof*⟩

lemma *strength-qbs-morphismP*: *strength-qbs X Y* ∈ $X \bigotimes_Q$ *monadP-qbs Y* $\rightarrow_Q$ *monadP-qbs* ($X \bigotimes_Q Y$)
⟨*proof*⟩

end

lemma *bind-qbs-morphismP*: ($\gg=$) ∈ *monadP-qbs X* $\rightarrow_Q$ ($X \Rightarrow_Q$ *monadP-qbs Y*) $\Rightarrow_Q$ *monadP-qbs Y*
⟨*proof*⟩

corollary *strength-qbs-law1P*:
assumes $x \in$ *qbs-space* (*unit-quasi-borel* $\bigotimes_Q$ *monadP-qbs X*)
shows *snd x* = (*distr-qbs* (*unit-quasi-borel* $\bigotimes_Q$ *X*) *X snd* ∘ *strength-qbs unit-quasi-borel X*) *x*
⟨*proof*⟩

corollary *strength-qbs-law2P*:
assumes $x \in$ *qbs-space* (($X \bigotimes_Q Y$) $\bigotimes_Q$ *monadP-qbs Z*)
shows (*strength-qbs X* ($Y \bigotimes_Q Z$) ∘ (*map-prod id* (*strength-qbs Y Z*)) ∘ (λ((*x*,*y*),*z*). (*x*,(*y*,*z*)))) *x* =
(*distr-qbs* (($X \bigotimes_Q Y$) $\bigotimes_Q Z$) ($X \bigotimes_Q$ ($Y \bigotimes_Q Z$)) (λ((*x*,*y*),*z*). (*x*,(*y*,*z*))) ∘ *strength-qbs* ($X \bigotimes_Q Y$) *Z*) *x*
⟨*proof*⟩

lemma *strength-qbs-law4P*:
assumes $x \in$ *qbs-space* ($X \bigotimes_Q$ *monadP-qbs* (*monadP-qbs Y*))
shows (*strength-qbs X Y* ∘ *map-prod id join-qbs*) *x* = (*join-qbs* ∘ *distr-qbs* (X $\bigotimes_Q$ *monadP-qbs Y*) (*monadP-qbs* ($X \bigotimes_Q Y$)) (*strength-qbs X Y*) ∘ *strength-qbs*

X ($monadP$-qbs Y)) x
(**is** $?lhs = ?rhs$)
⟨*proof*⟩

lemma *distr-qbs-morphismP*: $distr\text{-}qbs\ X\ Y \in X \Rightarrow_Q\ Y \rightarrow_Q\ monadP\text{-}qbs\ X \Rightarrow_Q\ monadP\text{-}qbs\ Y$
⟨*proof*⟩

lemma *bind-qbs-return-rotateP*:
assumes $p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$
and $q \in qbs\text{-}space\ (monadP\text{-}qbs\ Y)$
shows $q \gg= (\lambda y.\ p \gg= (\lambda x.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y))) = p \gg= (\lambda x.\ q \gg= (\lambda y.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y)))$
⟨*proof*⟩

lemma *qbs-bind-bind-return1P*:
assumes $f \in\ X \bigotimes_Q\ Y \rightarrow_Q\ monadP\text{-}qbs\ Z$
$p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$
$q \in qbs\text{-}space\ (monadP\text{-}qbs\ Y)$
shows $q \gg= (\lambda y.\ p \gg= (\lambda x.\ f\ (x,y))) = (q \gg= (\lambda y.\ p \gg= (\lambda x.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y)))) \gg= f$
⟨*proof*⟩

corollary *qbs-bind-bind-return1P′*:
assumes $[qbs]{:}f \in qbs\text{-}space\ (X \Rightarrow_Q\ Y \Rightarrow_Q\ monadP\text{-}qbs\ Z)$
$p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$
$q \in qbs\text{-}space\ (monadP\text{-}qbs\ Y)$
shows $q \gg= (\lambda y.\ p \gg= (\lambda x.\ f\ x\ y)) = (q \gg= (\lambda y.\ p \gg= (\lambda x.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y)))) \gg= (case\text{-}prod\ f)$
⟨*proof*⟩

lemma *qbs-bind-bind-return2P*:
assumes $f \in\ X \bigotimes_Q\ Y \rightarrow_Q\ monadP\text{-}qbs\ Z$
$p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$ $q \in qbs\text{-}space\ (monadP\text{-}qbs\ Y)$
shows $p \gg= (\lambda x.\ q \gg= (\lambda y.\ f\ (x,y))) = (p \gg= (\lambda x.\ q \gg= (\lambda y.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y)))) \gg= f$
⟨*proof*⟩

corollary *qbs-bind-bind-return2P′*:
assumes $[qbs]{:}f \in qbs\text{-}space\ (X \Rightarrow_Q\ Y \Rightarrow_Q\ monadP\text{-}qbs\ Z)$
$p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$
$q \in qbs\text{-}space\ (monadP\text{-}qbs\ Y)$
shows $p \gg= (\lambda x.\ q \gg= (\lambda y.\ f\ x\ y)) = (p \gg= (\lambda x.\ q \gg= (\lambda y.\ return\text{-}qbs\ (X \bigotimes_Q\ Y)\ (x,y)))) \gg= (case\text{-}prod\ f)$
⟨*proof*⟩

corollary *bind-qbs-rotateP*:
assumes $f \in\ X \bigotimes_Q\ Y \rightarrow_Q\ monadP\text{-}qbs\ Z$
$p \in qbs\text{-}space\ (monadP\text{-}qbs\ X)$

and *q* ∈ *qbs-space* (*monadP-qbs Y*)
shows *q* ≫= (λ*y*. *p* ≫= (λ*x*. *f* (*x*,*y*))) = *p* ≫= (λ*x*. *q* ≫= (λ*y*. *f* (*x*,*y*)))
⟨*proof*⟩

context *pair-qbs-probs*
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

sublocale *qbs-prob X* $\bigotimes_Q$ *Y map-prod* α β ∘ *rr.from-real distr* (μ $\bigotimes_M$ ν) *borel rr.to-real*
⟨*proof*⟩

lemma *qbs-bind-bind-return-prob*:
assumes [*qbs*]:*f* ∈ *X* $\bigotimes_Q$ *Y* $\to_Q$ *Z*
shows *qbs-prob Z* (*f* ∘ (*map-prod* α β ∘ *rr.from-real*)) (*distr* (μ $\bigotimes_M$ ν) *borel rr.to-real*)
⟨*proof*⟩

end

4.1.10 Almost Everywhere

lift-definition *qbs-almost-everywhere* :: [*'a qbs-measure*, *'a* ⇒ *bool*] ⇒ *bool*
is λ(*X*,α,μ). *almost-everywhere* (*distr* μ (*qbs-to-measure X*) α)
⟨*proof*⟩

syntax
-qbs-almost-everywhere :: *pttrn* ⇒ *'a* ⇒ *bool* ⇒ *bool* (AE_Q - *in* -. - [*0*,*0*,*10*] *10*)

translations
AE_Q *x in p*. *P* ⇌ *CONST qbs-almost-everywhere p* (λ*x*. *P*)

lemma *AEq-qbs-l*: (AE_Q *x in p*. *P x*) = (*AE x in qbs-l p*. *P x*)
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *AEq-def*:
(AE_Q *x in* $[\![X, \alpha, \mu]\!]_{sfin}$. *P x*) = (*AE x in* (*distr* μ (*qbs-to-measure X*) α). *P x*)
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *AEq-AE*: (AE_Q *x in* $[\![X, \alpha, \mu]\!]_{sfin}$. *P x*) ⟹ (*AE x in* μ. *P* (α *x*))
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *AEq-AE-iff*:
assumes [*qbs*]:*qbs-pred X P*
shows (AE_Q *x in* $[\![X, \alpha, \mu]\!]_{sfin}$. *P x*) ⟷ (*AE x in* μ. *P* (α *x*))
⟨*proof*⟩

lemma *AEq-qbs-pred*[*qbs*]: *qbs-almost-everywhere* $\in$ *monadM-qbs* $X \rightarrow_Q (X \Rightarrow_Q$ *qbs-count-space UNIV*) $\Rightarrow_Q$ *qbs-count-space UNIV*
⟨*proof*⟩

lemma *AEq-I2*[*simp*]:
assumes $p \in$ *qbs-space* (*monadM-qbs* X) $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow P\ x$
shows AE_Q *x in p*. $P\ x$
⟨*proof*⟩

lemma *AEq-mp*[*elim*!]:
assumes AE_Q *x in s*. $P\ x$ AE_Q *x in s*. $P\ x \longrightarrow Q\ x$
shows AE_Q *x in s*. $Q\ x$
⟨*proof*⟩

lemma
shows *AEq-iffI*: AE_Q *x in s*. $P\ x \Longrightarrow AE_Q$ *x in s*. $P\ x \longleftrightarrow Q\ x \Longrightarrow AE_Q$ *x in s*. $Q\ x$
and *AEq-disjI1*: AE_Q *x in s*. $P\ x \Longrightarrow AE_Q$ *x in s*. $P\ x \vee Q\ x$
and *AEq-disjI2*: AE_Q *x in s*. $Q\ x \Longrightarrow AE_Q$ *x in s*. $P\ x \vee Q\ x$
and *AEq-conjI*: AE_Q *x in s*. $P\ x \Longrightarrow AE_Q$ *x in s*. $Q\ x \Longrightarrow AE_Q$ *x in s*. $P\ x \wedge Q\ x$
and *AEq-conj-iff*[*simp*]: (AE_Q *x in s*. $P\ x \wedge Q\ x$) $\longleftrightarrow$ (AE_Q *x in s*. $P\ x$) $\wedge$ (AE_Q *x in s*. $Q\ x$)
⟨*proof*⟩

lemma *AEq-symmetric*:
assumes AE_Q *x in s*. $P\ x = Q\ x$
shows AE_Q *x in s*. $Q\ x = P\ x$
⟨*proof*⟩

lemma *AEq-impI*: ($P \Longrightarrow AE_Q$ *x in M*. $Q\ x$) $\Longrightarrow AE_Q$ *x in M*. $P \longrightarrow Q\ x$
⟨*proof*⟩

lemma *AEq-Ball-mp*:
$s \in$ *qbs-space* (*monadM-qbs* X) $\Longrightarrow$ ($\bigwedge x.\ x \in$*qbs-space* $X \Longrightarrow P\ x$) $\Longrightarrow AE_Q$ *x in s*. $P\ x \longrightarrow Q\ x \Longrightarrow AE_Q$ *x in s*. $Q\ x$
⟨*proof*⟩

lemma *AEq-cong*:
$s \in$ *qbs-space* (*monadM-qbs* X) $\Longrightarrow$ ($\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow P\ x \longleftrightarrow Q\ x$) $\Longrightarrow$ (AE_Q *x in s*. $P\ x$) $\longleftrightarrow$ (AE_Q *x in s*. $Q\ x$)
⟨*proof*⟩

lemma *AEq-cong-simp*: $s \in$ *qbs-space* (*monadM-qbs* X) $\Longrightarrow$ ($\bigwedge x.\ x \in$ *qbs-space* X =*simp*=> $P\ x = Q\ x$) $\Longrightarrow$ (AE_Q *x in s*. $P\ x$) $\longleftrightarrow$ (AE_Q *x in s*. $Q\ x$)
⟨*proof*⟩

lemma *AEq-all-countable*: (AE_Q *x in s*. $\forall i.\ P\ i\ x$) $\longleftrightarrow$ ($\forall i$::*'i*::*countable*. AE_Q *x*

in s. P i x)
⟨*proof*⟩

lemma *AEq-ball-countable*: *countable X* $\Longrightarrow$ (AE_Q *x in s.* ∀ *y*∈*X. P x y*) $\longleftrightarrow$ (∀ *y*∈*X.* AE_Q *x in s. P x y*)
⟨*proof*⟩

lemma *AEq-ball-countable'*: ($\bigwedge$*N. N* ∈ *I* $\Longrightarrow$ AE_Q *x in s. P N x*) $\Longrightarrow$ *countable I* $\Longrightarrow$ AE_Q *x in s.* ∀ *N* ∈ *I. P N x*
⟨*proof*⟩

lemma *AEq-pairwise*: *countable F* $\Longrightarrow$ *pairwise* (λ*A B.* AE_Q *x in s. R x A B*) *F* $\longleftrightarrow$ (AE_Q *x in s. pairwise* (*R x*) *F*)
⟨*proof*⟩

lemma *AEq-finite-all*: *finite S* $\Longrightarrow$ (AE_Q *x in s.* ∀ *i*∈*S. P i x*) $\longleftrightarrow$ (∀ *i*∈*S.* AE_Q *x in s. P i x*)
⟨*proof*⟩

lemma *AE-finite-allI*:*finite S* $\Longrightarrow$ ($\bigwedge$*s. s* ∈ *S* $\Longrightarrow$ AE_Q *x in M. Q s x*) $\Longrightarrow$ AE_Q *x in M.* ∀ *s*∈*S. Q s x*
⟨*proof*⟩

4.1.11 Integral

lift-definition *qbs-nn-integral* :: [*'a qbs-measure*, *'a* ⇒ *ennreal*] ⇒ *ennreal*
is λ(*X*,α,μ) *f*.(∫ ⁺*x. f x* ∂*distr* μ (*qbs-to-measure X*) α)
⟨*proof*⟩

lift-definition *qbs-integral* :: [*'a qbs-measure*, *'a* ⇒ (*'b* :: {*banach*,*second-countable-topology*})] ⇒ *'b*
is λ*p f. if f* ∈ (*fst p*) $\rightarrow_Q$ *qbs-borel then* (∫ *x. f* (*fst* (*snd p*) *x*) ∂ (*snd* (*snd p*))) *else 0*
⟨*proof*⟩

syntax
-qbs-nn-integral :: *pttrn* ⇒ *ennreal* ⇒ *'a qbs-measure* ⇒ *ennreal* ($∫^+{}_Q$((*2* -./ -)/ ∂-) [*60*,*61*] *110*)

translations
$∫^+{}_Q$ *x. f* ∂*p* ⇌ *CONST qbs-nn-integral p* (λ*x. f*)

syntax
-qbs-integral :: *pttrn* ⇒ - ⇒ *'a qbs-measure* ⇒ - ($∫_Q$((*2* -./ -)/ ∂-) [*60*,*61*] *110*)

translations
$∫_Q$ *x. f* ∂*p* ⇌ *CONST qbs-integral p* (λ*x. f*)

lemma(**in** *qbs-s-finite*)

shows *qbs-nn-integral-def*: $f \in X \rightarrow_Q$ *qbs-borel* $\Longrightarrow (\int^+{}_Q\ x.\ f\ x\ \partial[\![X,\ \alpha,\ \mu]\!]_{sfin}) = (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu)$
and *qbs-nn-integral-def2*:$(\int^+{}_Q\ x.\ f\ x\ \partial[\![X,\ \alpha,\ \mu]\!]_{sfin}) = (\int^+ x.\ f\ x\ \partial(distr\ \mu\ (qbs\text{-}to\text{-}measure\ X)\ \alpha))$
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-integral-def*:
$f \in X \rightarrow_Q$ *qbs-borel* $\Longrightarrow (\int_Q\ x.\ f\ x\ \partial[\![X,\ \alpha,\ \mu]\!]_{sfin}) = (\int x.\ f\ (\alpha\ x)\ \partial\ \mu)$
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-integral-def2*: $(\int_Q\ x.\ f\ x\ \partial[\![X,\ \alpha,\ \mu]\!]_{sfin}) = (\int x.\ f\ x\ \partial(distr\ \mu\ (qbs\text{-}to\text{-}measure\ X)\ \alpha))$
⟨*proof*⟩

lemma *qbs-measure-eqI*:
assumes [*qbs*]:$p \in$ *qbs-space* (*monadM-qbs* X) $q \in$ *qbs-space* (*monadM-qbs* X)
and $\bigwedge f.\ f \in X \rightarrow_Q$ *qbs-borel* $\Longrightarrow (\int^+{}_Q\ x.\ f\ x\ \partial p) = (\int^+{}_Q\ x.\ f\ x\ \partial q)$
shows $p = q$
⟨*proof*⟩

lemma *qbs-nn-integral-def2-l*: *qbs-nn-integral* $s\ f$ = $integral^N$ (*qbs-l* s) f
⟨*proof*⟩

lemma *qbs-integral-def2-l*: *qbs-integral* $s\ f$ = $integral^L$ (*qbs-l* s) f
⟨*proof*⟩

lift-definition *qbs-integrable* :: *'a qbs-measure* ⇒ (*'a* ⇒ *'b*::{*second-countable-topology*,*banach*}) ⇒ *bool*
is $\lambda p\ f.\ f \in fst\ p \rightarrow_Q$ *qbs-borel* $\wedge$ *integrable* (*snd* (*snd* p)) ($f \circ$ (*fst* (*snd* p)))
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-integrable-def*:
qbs-integrable $[\![X,\ \alpha,\ \mu]\!]_{sfin}\ f \longleftrightarrow f \in X \rightarrow_Q$ *qbs-borel* $\wedge$ *integrable* $\mu\ (\lambda x.\ f\ (\alpha\ x))$
⟨*proof*⟩

lemma *qbs-integrable-morphism-dest*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X)
and *qbs-integrable* $s\ f$
shows $f \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-integrable-morphismP*:
assumes $s \in$ *qbs-space* (*monadP-qbs* X)
and *qbs-integrable* $s\ f$
shows $f \in X \rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-integrable-measurable*[*simp*]:

assumes *qbs-integrable* $[\![X,\alpha,\mu]\!]_{sfin}$ *f*
shows $f \in$ *qbs-to-measure* $X \rightarrow_M$ *borel*
⟨*proof*⟩

lemma *qbs-integrable-iff-integrable*:
(*qbs-integrable* (*s*::′*a qbs-measure*) (f :: ′$a \Rightarrow$ ′b::{*second-countable-topology*,*banach*}))
= (*integrable* (*qbs-l s*) *f*)
⟨*proof*⟩

corollary(**in** *qbs-s-finite*) *qbs-integrable-distr*: *qbs-integrable* $[\![X, \alpha, \mu]\!]_{sfin}$ f = *integrable* (*distr* μ (*qbs-to-measure X*) α) *f*
⟨*proof*⟩

lemma *qbs-integrable-morphism*[*qbs*]: *qbs-integrable* $\in$ *monadM-qbs* $X \rightarrow_Q (X \Rightarrow_Q$ (*qbs-borel* :: (′a :: {*banach*, *second-countable-topology*}) *quasi-borel*)) $\Rightarrow_Q$ *qbs-count-space UNIV*
⟨*proof*⟩

lemma(**in** *qbs-s-finite*) *qbs-integrable-iff-integrable*:
assumes $f \in$ *qbs-to-measure* $X \rightarrow_M$ *borel*
shows *qbs-integrable* $[\![X, \alpha, \mu]\!]_{sfin}$ f = *integrable* μ ($\lambda x.\ f\ (\alpha\ x)$)
⟨*proof*⟩

lemma *qbs-integrable-iff-bounded*:
assumes $s \in$ *qbs-space* (*monadM-qbs X*)
shows *qbs-integrable s f* $\longleftrightarrow f \in X \rightarrow_Q$ *qbs-borel* $\wedge$ ($\int^{+}_Q x.$ *ennreal* (*norm* (f x)) ∂s) $< \infty$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *not-qbs-integrable-qbs-integral*: $\neg$ *qbs-integrable s f* $\Longrightarrow$ *qbs-integral s f* = *0*
⟨*proof*⟩

lemma *qbs-integrable-cong-AE*:
assumes $s \in$ *qbs-space* (*monadM-qbs X*)
AE_Q *x in s. f x = g x*
and *qbs-integrable s f* $g \in X \rightarrow_Q$ *qbs-borel*
shows *qbs-integrable s g*
⟨*proof*⟩

lemma *qbs-integrable-cong*:
assumes $s \in$ *qbs-space* (*monadM-qbs X*)
$\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
and *qbs-integrable s f*
shows *qbs-integrable s g*
⟨*proof*⟩

lemma *qbs-integrable-zero*[*simp*, *intro*]: *qbs-integrable s* ($\lambda x.$ *0*)

$\langle proof \rangle$

lemma *qbs-integrable-const*:
assumes $s \in$ *qbs-space* (*monadP-qbs* X)
shows *qbs-integrable* s ($\lambda x.\ c$)
$\langle proof \rangle$

lemma *qbs-integrable-add*[*simp*,*intro*!]:
assumes *qbs-integrable* $s\ f$
and *qbs-integrable* $s\ g$
shows *qbs-integrable* s ($\lambda x.\ f\ x + g\ x$)
$\langle proof \rangle$

lemma *qbs-integrable-diff*[*simp*,*intro*!]:
assumes *qbs-integrable* $s\ f$
and *qbs-integrable* $s\ g$
shows *qbs-integrable* s ($\lambda x.\ f\ x - g\ x$)
$\langle proof \rangle$

lemma *qbs-integrable-sum*[*simp*, *intro*!]: ($\bigwedge i.\ i \in I \Longrightarrow$ *qbs-integrable* s ($f\ i$)) $\Longrightarrow$ *qbs-integrable* s ($\lambda x.\ \sum i{\in}I.\ f\ i\ x$)
$\langle proof \rangle$

lemma *qbs-integrable-scaleR-left*[*simp*, *intro*!]: *qbs-integrable* $s\ f \Longrightarrow$ *qbs-integrable* s ($\lambda x.\ f\ x *_R$ (c :: $'a$ :: {*second-countable-topology*,*banach*}))
$\langle proof \rangle$

lemma *qbs-integrable-scaleR-right*[*simp*, *intro*!]: *qbs-integrable* $s\ f \Longrightarrow$ *qbs-integrable* s ($\lambda x.\ c *_R$ ($f\ x$ :: $'a$ :: {*second-countable-topology*,*banach*}))
$\langle proof \rangle$

lemma *qbs-integrable-mult-iff*:
fixes f :: $'a \Rightarrow$ *real*
shows (*qbs-integrable* s ($\lambda x.\ c * f\ x$)) = ($c = 0 \vee$ *qbs-integrable* $s\ f$)
$\langle proof \rangle$

lemma
fixes c :: -::{*real-normed-algebra*,*second-countable-topology*}
assumes *qbs-integrable* $s\ f$
shows *qbs-integrable-mult-right*:*qbs-integrable* s ($\lambda x.\ c * f\ x$)
and *qbs-integrable-mult-left*: *qbs-integrable* s ($\lambda x.\ f\ x * c$)
$\langle proof \rangle$

lemma *qbs-integrable-divide-zero*[*simp*, *intro*!]:
fixes c :: -::{*real-normed-field*, *field*, *second-countable-topology*}
shows *qbs-integrable* $s\ f \Longrightarrow$ *qbs-integrable* s ($\lambda x.\ f\ x\ /\ c$)
$\langle proof \rangle$

lemma *qbs-integrable-inner-left*[*simp*, *intro*!]:

$qbs\text{-}integrable\ s\ f \Longrightarrow qbs\text{-}integrable\ s\ (\lambda x.\ f\ x \cdot c)$
⟨*proof*⟩

lemma *qbs-integrable-inner-right*[*simp*, *intro*!]:
$qbs\text{-}integrable\ s\ f \Longrightarrow qbs\text{-}integrable\ s\ (\lambda x.\ c \cdot f\ x)$
⟨*proof*⟩

lemma *qbs-integrable-minus*[*simp*, *intro*!]:
$qbs\text{-}integrable\ s\ f \Longrightarrow qbs\text{-}integrable\ s\ (\lambda x.\ -\ f\ x)$
⟨*proof*⟩

lemma [*simp*, *intro*]:
assumes *qbs-integrable s f*
shows *qbs-integrable-Re*: $qbs\text{-}integrable\ s\ (\lambda x.\ Re\ (f\ x))$
and *qbs-integrable-Im*: $qbs\text{-}integrable\ s\ (\lambda x.\ Im\ (f\ x))$
and *qbs-integrable-cnj*: $qbs\text{-}integrable\ s\ (\lambda x.\ cnj\ (f\ x))$
⟨*proof*⟩

lemma *qbs-integrable-of-real*[*simp*, *intro*!]:
$qbs\text{-}integrable\ s\ f \Longrightarrow qbs\text{-}integrable\ s\ (\lambda x.\ of\text{-}real\ (f\ x))$
⟨*proof*⟩

lemma [*simp*, *intro*]:
assumes *qbs-integrable s f*
shows *qbs-integrable-fst*: $qbs\text{-}integrable\ s\ (\lambda x.\ fst\ (f\ x))$
and *qbs-integrable-snd*: $qbs\text{-}integrable\ s\ (\lambda x.\ snd\ (f\ x))$
⟨*proof*⟩

lemma *qbs-integrable-norm*:
assumes *qbs-integrable s f*
shows $qbs\text{-}integrable\ s\ (\lambda x.\ norm\ (f\ x))$
⟨*proof*⟩

lemma *qbs-integrable-abs*:
fixes f :: - ⇒ *real*
assumes *qbs-integrable s f*
shows $qbs\text{-}integrable\ s\ (\lambda x.\ |f\ x|)$
⟨*proof*⟩

lemma *qbs-integrable-sq*:
fixes c :: -::{*real-normed-field*,*second-countable-topology*}
assumes $qbs\text{-}integrable\ s\ (\lambda x.\ c)$ *qbs-integrable s f*
and $qbs\text{-}integrable\ s\ (\lambda x.\ (f\ x)^2)$
shows $qbs\text{-}integrable\ s\ (\lambda x.\ (f\ x - c)^2)$
⟨*proof*⟩

lemma *qbs-nn-integral-eq-integral-AEq*:
assumes *qbs-integrable s f* $AE_Q\ x\ in\ s.\ 0 \leq f\ x$
shows $(\int^{+}_{Q}\ x.\ ennreal\ (f\ x)\ \partial s) = ennreal\ (\int_{Q}\ x.\ f\ x\ \partial s)$

$\langle proof \rangle$

lemma *qbs-nn-integral-eq-integral*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) *qbs-integrable* s f
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow 0 \leq f\ x$
shows $(\int^{+}{}_{Q}\ x.\ ennreal\ (f\ x)\ \partial s) = ennreal\ (\int_{Q}\ x.\ f\ x\ \partial s)$
$\langle proof \rangle$

lemma *qbs-nn-integral-cong-AEq*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) AE_Q x *in* s. $f\ x = g\ x$
shows *qbs-nn-integral* s f = *qbs-nn-integral* s g
$\langle proof \rangle$

lemma *qbs-nn-integral-cong*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
shows *qbs-nn-integral* s f = *qbs-nn-integral* s g
$\langle proof \rangle$

lemma *qbs-nn-integral-const*:
$(\int^{+}{}_{Q}\ x.\ c\ \partial s) = c * qbs\text{-}l\ s\ (qbs\text{-}space\ (qbs\text{-}space\text{-}of\ s))$
$\langle proof \rangle$

lemma *qbs-nn-integral-const-prob*:
assumes $s \in$ *qbs-space* (*monadP-qbs* X)
shows $(\int^{+}{}_{Q}\ x.\ c\ \partial s) = c$
$\langle proof \rangle$

lemma *qbs-nn-integral-add*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X)
and [*qbs*]:$f \in X \rightarrow_Q$ *qbs-borel* $g \in X \rightarrow_Q$ *qbs-borel*
shows $(\int^{+}{}_{Q}\ x.\ f\ x + g\ x\ \partial s) = (\int^{+}{}_{Q}\ x.\ f\ x\ \partial s) + (\int^{+}{}_{Q}\ x.\ g\ x\ \partial s)$
$\langle proof \rangle$

lemma *qbs-nn-integral-cmult*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) **and** [*qbs*]:$f \in X \rightarrow_Q$ *qbs-borel*
shows $(\int^{+}{}_{Q}\ x.\ c * f\ x\ \partial s) = c * (\int^{+}{}_{Q}\ x.\ f\ x\ \partial s)$
$\langle proof \rangle$

lemma *qbs-integral-cong-AEq*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \rightarrow_Q$ *qbs-borel* $g \in X \rightarrow_Q$ *qbs-borel*
and AE_Q x *in* s. $f\ x = g\ x$
shows *qbs-integral* s f = *qbs-integral* s g
$\langle proof \rangle$

lemma *qbs-integral-cong*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
shows *qbs-integral* s f = *qbs-integral* s g
$\langle proof \rangle$

lemma *qbs-integral-nonneg-AEq*:
  fixes f :: - $\Rightarrow$ *real*
  shows AE_Q *x in s.* $0 \leq f\ x \Longrightarrow 0 \leq$ *qbs-integral s f*
  ⟨*proof*⟩

lemma *qbs-integral-nonneg*:
  fixes f :: - $\Rightarrow$ *real*
  assumes $s \in$ *qbs-space* (*monadM-qbs X*) $\bigwedge x.\ x \in$ *qbs-space X* $\Longrightarrow 0 \leq f\ x$
  shows $0 \leq$ *qbs-integral s f*
  ⟨*proof*⟩

lemma *qbs-integral-mono-AEq*:
  fixes f :: - $\Rightarrow$ *real*
  assumes *qbs-integrable s f qbs-integrable s g* AE_Q *x in s.* $f\ x \leq g\ x$
    shows *qbs-integral s f* $\leq$ *qbs-integral s g*
  ⟨*proof*⟩

lemma *qbs-integral-mono*:
  fixes f :: - $\Rightarrow$ *real*
  assumes $s \in$ *qbs-space* (*monadM-qbs X*)
      and *qbs-integrable s f qbs-integrable s g* $\bigwedge x.\ x \in$ *qbs-space X* $\Longrightarrow f\ x \leq g\ x$
    shows *qbs-integral s f* $\leq$ *qbs-integral s g*
  ⟨*proof*⟩

lemma *qbs-integral-const-prob*:
  assumes $s \in$ *qbs-space* (*monadP-qbs X*)
  shows $(\int_Q x.\ c\ \partial s) = c$
  ⟨*proof*⟩

lemma
  assumes *qbs-integrable s f qbs-integrable s g*
  shows *qbs-integral-add*:$(\int_Q x.\ f\ x + g\ x\ \partial s) = (\int_Q x.\ f\ x\ \partial s) + (\int_Q x.\ g\ x\ \partial s)$
    and *qbs-integral-diff*: $(\int_Q x.\ f\ x - g\ x\ \partial s) = (\int_Q x.\ f\ x\ \partial s) - (\int_Q x.\ g\ x\ \partial s)$
  ⟨*proof*⟩

lemma [*simp*]:
  fixes c :: -::{*real-normed-field*,*second-countable-topology*}
  shows *qbs-integral-mult-right-zero*:$(\int_Q x.\ c * f\ x\ \partial s) = c * (\int_Q x.\ f\ x\ \partial s)$
    and *qbs-integral-mult-left-zero*:$(\int_Q x.\ f\ x * c\ \partial s) = (\int_Q x.\ f\ x\ \partial s) * c$
    and *qbs-integral-divide-zero*: $(\int_Q x.\ f\ x\ /\ c\ \partial s) = (\int_Q x.\ f\ x\ \partial s)\ /\ c$
   ⟨*proof*⟩

lemma *qbs-integral-minus*[*simp*]: $(\int_Q x.\ - f\ x\ \partial s) = - (\int_Q x.\ f\ x\ \partial s)$
   ⟨*proof*⟩

lemma [*simp*]:
  shows *qbs-integral-scaleR-right*:$(\int_Q x.\ c *_R f\ x\ \partial s) = c *_R (\int_Q x.\ f\ x\ \partial s)$
    and *qbs-integral-scaleR-left*: $(\int_Q x.\ f\ x *_R c\ \partial s) = (\int_Q x.\ f\ x\ \partial s) *_R c$

⟨*proof*⟩

lemma [*simp*]:
shows *qbs-integral-inner-left*: *qbs-integrable s f* $\Longrightarrow$ ($\int_Q$ *x*. *f x* $\cdot$ *c* ∂s) = ($\int_Q$ *x*. *f x* ∂s) $\cdot$ *c*
and *qbs-integral-inner-right*: *qbs-integrable s f* $\Longrightarrow$ ($\int_Q$ *x*. *c* $\cdot$ *f x* ∂s) = *c* $\cdot$ ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma *integral-complex-of-real*[*simp*]: ($\int_Q$ *x*. *complex-of-real* (*f x*) ∂s)= *of-real* ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma *integral-cnj*[*simp*]: ($\int_Q$ *x*. *cnj* (*f x*) ∂s) = *cnj* ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma [*simp*]:
assumes *qbs-integrable s f*
shows *qbs-integral-Im*: ($\int_Q$ *x*. *Im* (*f x*) ∂s) = *Im* ($\int_Q$ *x*. *f x* ∂s)
and *qbs-integral-Re*: ($\int_Q$ *x*. *Re* (*f x*) ∂s) = *Re* ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma *qbs-integral-of-real*[*simp*]:*qbs-integrable s f* $\Longrightarrow$ ($\int_Q$ *x*. *of-real* (*f x*) ∂s) = *of-real* ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma [*simp*]:
assumes *qbs-integrable s f*
shows *qbs-integral-fst*: ($\int_Q$ *x*. *fst* (*f x*) ∂s) = *fst* ($\int_Q$ *x*. *f x* ∂s)
and *qbs-integral-snd*: ($\int_Q$ *x*. *snd* (*f x*) ∂s) = *snd* ($\int_Q$ *x*. *f x* ∂s)
⟨*proof*⟩

lemma *real-qbs-integral-def*:
assumes *qbs-integrable s f*
shows *qbs-integral s f* = *enn2real* ($\int^+{}_Q$ *x*. *ennreal* (*f x*) ∂s) $-$ *enn2real* ($\int^+{}_Q$ *x*. *ennreal* ($-$ *f x*) ∂s)
⟨*proof*⟩

lemma *Markov-inequality-qbs-prob*:
qbs-integrable s f $\Longrightarrow$ AE_Q *x in s*. $0 \le f\,x \Longrightarrow 0 < c \Longrightarrow \mathcal{P}$(*x in qbs-l s*. $c \le f\,x$) $\le$ ($\int_Q$ *x*. *f x* ∂s) / *c*
⟨*proof*⟩

lemma *Chebyshev-inequality-qbs-prob*:
assumes $s \in$ *qbs-space* (*monadP-qbs X*)
and $f \in X \to_Q$ *qbs-borel qbs-integrable s* (λx. $(f\,x)^2$)
and $0 < e$
shows $\mathcal{P}$(*x in qbs-l s*. $e \le |f\,x -$ ($\int_Q$ *x*. *f x* ∂s)$|$) $\le$ ($\int_Q$ *x*. (*f x* $-$ ($\int_Q$ *x*. *f x* ∂s))^2 ∂s) / e^2

⟨*proof*⟩

lemma *qbs-l-return-qbs*:
assumes $x \in$ *qbs-space X*
shows *qbs-l* (*return-qbs X x*) = *return* (*qbs-to-measure X*) *x*
⟨*proof*⟩

lemma *qbs-l-bind-qbs*:
assumes [*qbs*]: $s \in$ *qbs-space* (*monadM-qbs X*) $f \in X \rightarrow_Q$ *monadM-qbs Y*
shows *qbs-l* ($s \gg= f$) = *qbs-l s* $\gg=_k$ *qbs-l* $\circ$ *f* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-l-bind-qbsP*:
assumes [*qbs*]: $s \in$ *qbs-space* (*monadP-qbs X*) $f \in X \rightarrow_Q$ *monadP-qbs Y*
shows *qbs-l* ($s \gg= f$) = *qbs-l s* $\gg=$ *qbs-l* $\circ$ *f*
⟨*proof*⟩

lemma *qbs-integrable-return*[*simp*, *intro*]:
assumes $x \in$ *qbs-space X* $f \in X \rightarrow_Q$ *qbs-borel*
shows *qbs-integrable* (*return-qbs X x*) *f*
⟨*proof*⟩

lemma *qbs-integrable-bind-return*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs X*) $f \in Y \rightarrow_Q$ *qbs-borel* $g \in X \rightarrow_Q Y$
shows *qbs-integrable* ($s \gg=$ ($\lambda x.$ *return-qbs Y* (*g x*))) *f* = *qbs-integrable s* ($f \circ$ *g*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-nn-integral-morphism*[*qbs*]: *qbs-nn-integral* $\in$ *monadM-qbs X* $\rightarrow_Q$ (*X* $\Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ *qbs-borel*
⟨*proof*⟩

lemma *qbs-nn-integral-return*:
assumes $f \in X \rightarrow_Q$ *qbs-borel*
and $x \in$ *qbs-space X*
shows *qbs-nn-integral* (*return-qbs X x*) *f* = *f x*
⟨*proof*⟩

lemma *qbs-nn-integral-bind*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs X*)
$f \in X \rightarrow_Q$ *monadM-qbs Y* $g \in Y \rightarrow_Q$ *qbs-borel*
shows *qbs-nn-integral* ($s \gg= f$) *g* = *qbs-nn-integral s* ($\lambda y.$ (*qbs-nn-integral* (*f y*) *g*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-nn-integral-bind-return*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs Y*) $f \in Z \rightarrow_Q$ *qbs-borel* $g \in Y \rightarrow_Q Z$
shows *qbs-nn-integral* ($s \gg=$ ($\lambda y.$ *return-qbs Z* (*g y*))) *f* = *qbs-nn-integral s* ($f \circ$ *g*)

⟨*proof*⟩

lemma *qbs-integral-morphism*[*qbs*]:
qbs-integral $\in$ *monadM-qbs* $X \to_Q$ ($X \Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ (*qbs-borel* :: ('*b* :: {*second-countable-topology*,*banach*}) *quasi-borel*)
⟨*proof*⟩

lemma *qbs-integral-return*:
assumes [*qbs*]:$f \in X \to_Q$ *qbs-borel* $x \in$ *qbs-space* X
shows *qbs-integral* (*return-qbs* X x) $f = f\ x$
⟨*proof*⟩

lemma
assumes [*qbs*]: $s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *monadM-qbs* Y $g \in Y \to_Q$ *qbs-borel*
and *qbs-integrable* s ($\lambda x.$ $\int_Q y.$ *norm* ($g\ y$) $\partial f\ x$) AE_Q x *in* s. *qbs-integrable* ($f\ x$) g
shows *qbs-integrable-bind*: *qbs-integrable* ($s \gg= f$) g (**is** *?goal1*)
and *qbs-integral-bind*:($\int_Q y.$ $g\ y$ $\partial(s \gg= f)$) = ($\int_Q x.$ $\int_Q y.$ $g\ y$ $\partial f\ x$ ∂s) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-integral-bind-return*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* Y) $f \in Z \to_Q$ *qbs-borel* $g \in Y \to_Q Z$
shows *qbs-integral* ($s \gg=$ ($\lambda y.$ *return-qbs* Z ($g\ y$))) f = *qbs-integral* s ($f \circ g$)
⟨*proof*⟩

4.1.12 Binary Product Measures

definition *qbs-pair-measure* :: ['*a qbs-measure*, '*b qbs-measure*] $\Rightarrow$ ('*a* $\times$ '*b*) *qbs-measure* (**infix** $\bigotimes_{Qmes}$ *80*) **where**
qbs-pair-measure-def':*qbs-pair-measure* p q $\equiv$ ($p \gg=$ ($\lambda x.$ $q \gg=$ ($\lambda y.$ *return-qbs* (*qbs-space-of* p $\bigotimes_Q$ *qbs-space-of* q) (x, y))))

context *pair-qbs-s-finites*
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* $\times$ *real*) *measure*
⟨*proof*⟩

lemma
shows *qbs-pair-measure*: $[\![X,\ \alpha,\ \mu]\!]_{sfin}$ $\bigotimes_{Qmes}$ $[\![Y,\ \beta,\ \nu]\!]_{sfin}$ = $[\![X \bigotimes_Q Y$, *map-prod* α β $\circ$ *rr.from-real*, *distr* ($\mu \bigotimes_M \nu$) *borel rr.to-real*$]\!]_{sfin}$
and *qbs-pair-measure-s-finite*: *qbs-s-finite* ($X \bigotimes_Q Y$) (*map-prod* α β $\circ$ *rr.from-real*) (*distr* ($\mu \bigotimes_M \nu$) *borel rr.to-real*)
⟨*proof*⟩

lemma *qbs-l-qbs-pair-measure*:

qbs-l $(\llbracket X,\ \alpha,\ \mu\rrbracket_{sfin} \bigotimes_{Qmes} \llbracket Y,\ \beta,\ \nu\rrbracket_{sfin})$ = *distr* $(\mu \bigotimes_M \nu)$ (*qbs-to-measure* $(X \bigotimes_Q\ Y)$) (*map-prod* $\alpha\ \beta$)
⟨*proof*⟩

lemma *qbs-nn-integral-pair-measure*:
assumes [*qbs*]:$f \in X \bigotimes_Q\ Y \to_Q$ *qbs-borel*
shows $(\int^+{}_Q\ z.\ f\ z\ \partial(\llbracket X,\ \alpha,\ \mu\rrbracket_{sfin} \bigotimes_{Qmes} \llbracket Y,\ \beta,\ \nu\rrbracket_{sfin})) = (\int^+\ z.\ (f \circ$ *map-prod* $\alpha\ \beta)\ z\ \partial(\mu \bigotimes_M \nu))$
⟨*proof*⟩

lemma *qbs-integral-pair-measure*:
assumes [*qbs*]:$f \in X \bigotimes_Q\ Y \to_Q$ *qbs-borel*
shows $(\int_Q z.\ f\ z\ \partial(\llbracket X,\ \alpha,\ \mu\rrbracket_{sfin} \bigotimes_{Qmes} \llbracket Y,\ \beta,\ \nu\rrbracket_{sfin})) = (\int\ z.\ (f \circ$ *map-prod* $\alpha\ \beta)\ z\ \partial(\mu \bigotimes_M \nu))$
⟨*proof*⟩

lemma *qbs-pair-measure-integrable-eq*:
qbs-integrable $(\llbracket X,\ \alpha,\ \mu\rrbracket_{sfin} \bigotimes_{Qmes} \llbracket Y,\ \beta,\ \nu\rrbracket_{sfin})\ f \longleftrightarrow f \in X \bigotimes_Q\ Y \to_Q$ *qbs-borel* $\wedge$ *integrable* $(\mu \bigotimes_M \nu)\ (f \circ ($*map-prod* $\alpha\ \beta))$ (**is** *?h* $\longleftrightarrow$ *?h1* $\wedge$ *?h2*)
⟨*proof*⟩

end

lemmas(**in** *pair-qbs-probs*) *qbs-pair-measure-prob* = *qbs-prob-axioms*

context
fixes $X\ Y\ p\ q$
assumes p[*qbs*]:$p \in$ *qbs-space* (*monadM-qbs* X) **and** q[*qbs*]:$q \in$ *qbs-space* (*monadM-qbs* Y)
begin

lemma *qbs-pair-measure-def*: $p \bigotimes_{Qmes} q = p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.$ *return-qbs* $(X \bigotimes_Q\ Y)\ (x,y)))$
⟨*proof*⟩

lemma *qbs-pair-measure-def2*: $p \bigotimes_{Qmes} q = q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.$ *return-qbs* $(X \bigotimes_Q\ Y)\ (x,y)))$
⟨*proof*⟩

lemma
assumes $f \in\ X \bigotimes_Q\ Y \to_Q$ *monadM-qbs* Z
shows *qbs-pair-bind-bind-return1'*:$q \gg\!= (\lambda y.\ p \gg\!= (\lambda x.\ f\ (x,y))) = p \bigotimes_{Qmes} q \gg\!= f$
and *qbs-pair-bind-bind-return2'*:$p \gg\!= (\lambda x.\ q \gg\!= (\lambda y.\ f\ (x,y))) = p \bigotimes_{Qmes} q \gg\!= f$
⟨*proof*⟩

lemma
assumes [*qbs*]:$f \in\ X \to_Q$ *exp-qbs* Y (*monadM-qbs* Z)

shows *qbs-pair-bind-bind-return1''*: $q \gg= (\lambda y.\ p \gg= (\lambda x.\ f\ x\ y)) = p \bigotimes_{Qmes} q \gg= (\lambda x.\ f\ (fst\ x)\ (snd\ x))$
and *qbs-pair-bind-bind-return2''*: $p \gg= (\lambda x.\ q \gg= (\lambda y.\ f\ x\ y)) = p \bigotimes_{Qmes} q \gg= (\lambda x.\ f\ (fst\ x)\ (snd\ x))$
⟨*proof*⟩

lemma *qbs-nn-integral-Fubini-fst*:
assumes [*qbs*]:$f \in X \bigotimes_Q Y \to_Q$ *qbs-borel*
shows $(\int^+_Q x.\ \int^+_Q y.\ f\ (x,y)\ \partial q\ \partial p) = (\int^+_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-nn-integral-Fubini-snd*:
assumes [*qbs*]:$f \in X \bigotimes_Q Y \to_Q$ *qbs-borel*
shows $(\int^+_Q y.\ \int^+_Q x.\ f\ (x,y)\ \partial p\ \partial q) = (\int^+_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q))$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-ennintegral-indep-mult*:
assumes [*qbs*]: $f \in X \to_Q$ *qbs-borel* $g \in Y \to_Q$ *qbs-borel*
shows $(\int^+_Q z.\ f\ (fst\ z) * g\ (snd\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int^+_Q x.\ f\ x\ \partial p) * (\int^+_Q y.\ g\ y\ \partial q)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

lemma *qbs-l-qbs-pair-measure*:
assumes *standard-borel M standard-borel N*
defines $X \equiv$ *measure-to-qbs M* **and** $Y \equiv$ *measure-to-qbs N*
assumes [*qbs*]: $p \in$ *qbs-space* (*monadM-qbs X*) $q \in$ *qbs-space* (*monadM-qbs Y*)
shows *qbs-l* $(p \bigotimes_{Qmes} q)$ = *qbs-l* $p \bigotimes_M$ *qbs-l* q
⟨*proof*⟩

lemma *qbs-pair-measure-morphism*[*qbs*]: *qbs-pair-measure* $\in$ *monadM-qbs* $X \to_Q$ *monadM-qbs* $Y \Rightarrow_Q$ *monadM-qbs* $(X \bigotimes_Q Y)$
⟨*proof*⟩

lemma *qbs-pair-measure-morphismP*:
qbs-pair-measure $\in$ *monadP-qbs* $X \to_Q$ *monadP-qbs* $Y \Rightarrow_Q$ *monadP-qbs* $(X \bigotimes_Q Y)$
⟨*proof*⟩

lemma *qbs-nn-integral-indep1*:
assumes [*qbs*]:$p \in$ *qbs-space* (*monadM-qbs X*) $q \in$ *qbs-space* (*monadP-qbs X*) $f \in X \to_Q$ *qbs-borel*
shows $(\int^+_Q z.\ f\ (fst\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int^+_Q x.\ f\ x\ \partial p)$
⟨*proof*⟩

lemma *qbs-nn-integral-indep2*:

assumes [*qbs*]:$q \in$ *qbs-space* (*monadM-qbs* Y) $p \in$ *qbs-space* (*monadP-qbs* X) $f \in Y \to_Q$ *qbs-borel*
shows ($\int^{+}{}_Q$ z. f (*snd* z) $\partial(p \bigotimes_{Qmes} q)$) = ($\int^{+}{}_Q$ y. f y ∂q)
⟨*proof*⟩

context
begin

interpretation *rr* : *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* × *real*) *measure*
⟨*proof*⟩

lemma *qbs-integrable-pair-swap*:
assumes *qbs-integrable* ($p \bigotimes_{Qmes} q$) f
shows *qbs-integrable* ($q \bigotimes_{Qmes} p$) ($\lambda(x,y)$. f (y,x))
⟨*proof*⟩

lemma *qbs-integrable-pair1′*:
assumes [*qbs*]:$p \in$ *qbs-space* (*monadM-qbs* X)
$q \in$ *qbs-space* (*monadM-qbs* Y)
$f \in X \bigotimes_Q Y \to_Q$ *qbs-borel*
qbs-integrable p (λx. $\int_Q$ y. *norm* (f (x,y)) ∂q)
and AE_Q x *in* p. *qbs-integrable* q (λy. f (x,y))
shows *qbs-integrable* ($p \bigotimes_{Qmes} q$) f
⟨*proof*⟩

lemma
assumes *qbs-integrable* ($p \bigotimes_{Qmes} q$) f
shows *qbs-integrable-pair1D1′*: *qbs-integrable* p (λx. $\int_Q$ y. f (x,y) ∂q) (**is** *?g1*)
and *qbs-integrable-pair1D1-norm′*: *qbs-integrable* p (λx. $\int_Q$ y. *norm* (f (x,y)) ∂q) (**is** *?g2*)
and *qbs-integrable-pair1D2′*: AE_Q x *in* p. *qbs-integrable* q (λy. f (x,y)) (**is** *?g3*)
and *qbs-integrable-pair2D1′*: *qbs-integrable* q (λy. $\int_Q$ x. f (x,y) ∂p) (**is** *?g4*)
and *qbs-integrable-pair2D1-norm′*: *qbs-integrable* q (λy. $\int_Q$ x. *norm* (f (x,y)) ∂p) (**is** *?g5*)
and *qbs-integrable-pair2D2′*: AE_Q y *in* q. *qbs-integrable* p (λx. f (x,y)) (**is** *?g6*)
and *qbs-integral-Fubini-fst′*: ($\int_Q$ x. $\int_Q$ y. f (x,y) ∂q ∂p) = ($\int_Q$ z. f z $\partial(p \bigotimes_{Qmes} q)$) (**is** *?g7*)
and *qbs-integral-Fubini-snd′*: ($\int_Q$ y. $\int_Q$ x. f (x,y) ∂p ∂q) = ($\int_Q$ z. f z $\partial(p \bigotimes_{Qmes} q)$) (**is** *?g8*)
⟨*proof*⟩

end

lemma
assumes *h*:*qbs-integrable* ($p \bigotimes_{Qmes} q$) (*case-prod* f)

shows *qbs-integrable-pair1D1*: *qbs-integrable p* ($\lambda x.$ $\int_Q$ *y*. *f x y* ∂q)
and *qbs-integrable-pair1D1-norm*: *qbs-integrable p* ($\lambda x.$ $\int_Q$ *y*. *norm* (*f x y*) ∂q)
and *qbs-integrable-pair1D2*: AE_Q *x in p*. *qbs-integrable q* ($\lambda y.$ *f x y*)
and *qbs-integrable-pair2D1*: *qbs-integrable q* ($\lambda y.$ $\int_Q$ *x*. *f x y* ∂p)
and *qbs-integrable-pair2D1-norm*: *qbs-integrable q* ($\lambda y.$ $\int_Q$ *x*. *norm* (*f x y*) ∂p)
and *qbs-integrable-pair2D2*: AE_Q *y in q*. *qbs-integrable p* ($\lambda x.$ *f x y*)
and *qbs-integral-Fubini-fst*: ($\int_Q$ *x*. $\int_Q$ *y*. *f x y* ∂q ∂p) = ($\int_Q$ (*x*,*y*). *f x y* ∂(*p* $\bigotimes_{Qmes}$ *q*)) (**is** *?g7*)
and *qbs-integral-Fubini-snd*: ($\int_Q$ *y*. $\int_Q$ *x*. *f x y* ∂p ∂q) = ($\int_Q$ (*x*,*y*). *f x y* ∂(*p* $\bigotimes_{Qmes}$ *q*)) (**is** *?g8*)
⟨*proof*⟩

lemma *qbs-integrable-pair2'*:
assumes $p \in$ *qbs-space* (*monadM-qbs X*)
$q \in$ *qbs-space* (*monadM-qbs Y*)
$f \in X \bigotimes_Q Y \to_Q$ *qbs-borel*
qbs-integrable q ($\lambda y.$ $\int_Q$ *x*. *norm* (*f* (*x*,*y*)) ∂p)
and AE_Q *y in q*. *qbs-integrable p* ($\lambda x.$ *f* (*x*,*y*))
shows *qbs-integrable* (*p* $\bigotimes_{Qmes}$ *q*) *f*
⟨*proof*⟩

lemma *qbs-integrable-indep-mult*:
fixes *f* :: - ⇒ -::{*real-normed-div-algebra*,*second-countable-topology*}
assumes *qbs-integrable p f qbs-integrable q g*
shows *qbs-integrable* (*p* $\bigotimes_{Qmes}$ *q*) ($\lambda x.$ *f* (*fst x*) $*$ *g* (*snd x*))
⟨*proof*⟩

lemma *qbs-integrable-indep1*:
fixes *f* :: - ⇒ -::{*real-normed-div-algebra*,*second-countable-topology*}
assumes *qbs-integrable p f* $q \in$ *qbs-space* (*monadP-qbs Y*)
shows *qbs-integrable* (*p* $\bigotimes_{Qmes}$ *q*) ($\lambda x.$ *f* (*fst x*))
⟨*proof*⟩

lemma *qbs-integral-indep1*:
fixes *f* :: - ⇒ -::{*real-normed-div-algebra*,*second-countable-topology*}
assumes *qbs-integrable p f* $q \in$ *qbs-space* (*monadP-qbs Y*)
shows ($\int_Q$ *z*. *f* (*fst z*) ∂(*p* $\bigotimes_{Qmes}$ *q*)) = ($\int_Q$ *x*. *f x* ∂p)
⟨*proof*⟩

lemma *qbs-integrable-indep2*:
fixes *g* :: - ⇒ -::{*real-normed-div-algebra*,*second-countable-topology*}
assumes *qbs-integrable q g* $p \in$ *qbs-space* (*monadP-qbs X*)
shows *qbs-integrable* (*p* $\bigotimes_{Qmes}$ *q*) ($\lambda x.$ *g* (*snd x*))
⟨*proof*⟩

lemma *qbs-integral-indep2*:
fixes *g* :: - ⇒ -::{*real-normed-div-algebra*,*second-countable-topology*}
assumes *qbs-integrable q g* $p \in$ *qbs-space* (*monadP-qbs X*)
shows ($\int_Q$ *z*. *g* (*snd z*) ∂(*p* $\bigotimes_{Qmes}$ *q*)) = ($\int_Q$ *y*. *g y* ∂q)

⟨*proof*⟩

lemma *qbs-integral-indep-mult1*:
fixes *f* **and** *g*:: - ⇒ -::{*real-normed-field*,*second-countable-topology*}
assumes $p \in$ *qbs-space* (*monadP-qbs X*) $q \in$ *qbs-space* (*monadP-qbs Y*)
and *qbs-integrable p f qbs-integrable q g*
shows $(\int_Q z.\ f\ (fst\ z) * g\ (snd\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q x.\ f\ x\ \partial p) * (\int_Q y.\ g\ y\ \partial q)$
⟨*proof*⟩

lemma *qbs-integral-indep-mult2*:
fixes *f* **and** *g*:: - ⇒ -::{*real-normed-field*,*second-countable-topology*}
assumes $p \in$ *qbs-space* (*monadP-qbs X*) $q \in$ *qbs-space* (*monadP-qbs Y*)
and *qbs-integrable p f qbs-integrable q g*
shows $(\int_Q z.\ g\ (snd\ z) * f\ (fst\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q y.\ g\ y\ \partial q) * (\int_Q x.\ f\ x\ \partial p)$
⟨*proof*⟩

4.1.13 The Inverse Function of *l*

definition *qbs-l-inverse* :: *'a measure* ⇒ *'a qbs-measure* **where**
qbs-l-inverse M ≡ $[\![$*measure-to-qbs M*, *from-real-into M*, *distr M borel* (*to-real-on M*)$]\!]_{sfin}$

context *standard-borel-ne*
begin

lemma *qbs-l-inverse-def2*:
assumes [*measurable-cong*]: *sets* μ = *sets M*
and *s-finite-measure* μ
shows *qbs-l-inverse* μ = $[\![$*measure-to-qbs M*, *from-real*, *distr* μ *borel to-real*$]\!]_{sfin}$
⟨*proof*⟩

lemma
assumes [*measurable-cong*]:*sets* μ = *sets M*
shows *qbs-l-inverse-s-finite*: *s-finite-measure* $\mu \Longrightarrow$ *qbs-s-finite* (*measure-to-qbs M*) *from-real* (*distr* μ *borel to-real*)
and *qbs-l-inverse-qbs-prob*: *prob-space* $\mu \Longrightarrow$ *qbs-prob* (*measure-to-qbs M*) *from-real* (*distr* μ *borel to-real*)
⟨*proof*⟩

corollary
assumes [*measurable-cong*]:*sets* μ = *sets M*
shows *qbs-l-inverse-in-space-monadM*: *s-finite-measure* $\mu \Longrightarrow$ *qbs-l-inverse* $\mu \in$ *qbs-space* (*monadM-qbs M*)
and *qbs-l-inverse-in-space-monadP*: *prob-space* $\mu \Longrightarrow$ *qbs-l-inverse* $\mu \in$ *qbs-space* (*monadP-qbs M*)
⟨*proof*⟩

lemma *qbs-l-qbs-l-inverse*:
  assumes [*measurable-cong*]: *sets* μ = *sets* *M* *s-finite-measure* μ
  shows *qbs-l* (*qbs-l-inverse* μ) = μ
⟨*proof*⟩

corollary *qbs-l-qbs-l-inverse-prob*:
  sets μ = *sets* *M* $\Longrightarrow$ *prob-space* μ $\Longrightarrow$ *qbs-l* (*qbs-l-inverse* μ) = μ
  ⟨*proof*⟩

lemma *qbs-l-inverse-qbs-l*:
  assumes $s \in$ *qbs-space* (*monadM-qbs* (*measure-to-qbs* *M*))
  shows *qbs-l-inverse* (*qbs-l* *s*) = *s*
⟨*proof*⟩

corollary *qbs-l-inverse-qbs-l-prob*:
  assumes $s \in$ *qbs-space* (*monadP-qbs* (*measure-to-qbs* *M*))
  shows *qbs-l-inverse* (*qbs-l* *s*) = *s*
  ⟨*proof*⟩

lemma *s-finite-kernel-qbs-morphism*:
  assumes *s-finite-kernel* *N* *M* *k*
  shows (λx. *qbs-l-inverse* (*k* *x*)) $\in$ *measure-to-qbs* *N* $\rightarrow_Q$ *monadM-qbs* (*measure-to-qbs* *M*)
⟨*proof*⟩

lemma *prob-kernel-qbs-morphism*:
  assumes [*measurable*]:$k \in N \rightarrow_M$ *prob-algebra* *M*
  shows (λx. *qbs-l-inverse* (*k* *x*)) $\in$ *measure-to-qbs* *N* $\rightarrow_Q$ *monadP-qbs* (*measure-to-qbs* *M*)
⟨*proof*⟩

lemma *qbs-l-inverse-return*:
  assumes $x \in$ *space* *M*
  shows *qbs-l-inverse* (*return* *M* *x*) = *return-qbs* (*measure-to-qbs* *M*) *x*
⟨*proof*⟩

lemma *qbs-l-inverse-bind-kernel*:
  assumes *standard-borel-ne* *N* *s-finite-measure* *M* *s-finite-kernel* *M* *N* *k*
    shows *qbs-l-inverse* ($M \gg\!=_k k$) = *qbs-l-inverse* *M* $\gg\!=$ (λx. *qbs-l-inverse* (*k* *x*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-l-inverse-bind*:
  assumes *standard-borel-ne* *N* *s-finite-measure* *M* $k \in M \rightarrow_M$ *prob-algebra* *N*
  shows *qbs-l-inverse* ($M \gg\!= k$) = *qbs-l-inverse* *M* $\gg\!=$ (λx. *qbs-l-inverse* (*k* *x*))
  ⟨*proof*⟩

end

4.1.14 PMF and SPMF

definition *qbs-pmf* $\equiv$ (λp. *qbs-l-inverse* (*measure-pmf p*))
definition *qbs-spmf* $\equiv$ (λp. *qbs-l-inverse* (*measure-spmf p*))

declare [[*coercion qbs-pmf*]]

lemma *qbs-pmf-qbsP*:
  fixes *p* :: (- :: *countable*) *pmf*
  shows *qbs-pmf p* $\in$ *qbs-space* (*monadP-qbs* (*count-space*$_Q$ *UNIV*))
  ⟨*proof*⟩

lemma *qbs-pmf-qbs*[*qbs*]:
  fixes *p* :: (- :: *countable*) *pmf*
  shows *qbs-pmf p* $\in$ *qbs-space* (*monadM-qbs* (*count-space*$_Q$ *UNIV*))
  ⟨*proof*⟩

lemma *qbs-spmf-qbs*[*qbs*]:
  fixes *q* :: (- :: *countable*) *spmf*
  shows *qbs-spmf q* $\in$ *qbs-space* (*monadM-qbs* (*count-space*$_Q$ *UNIV*))
  ⟨*proof*⟩

lemma [*simp*]:
  fixes *p* :: (- :: *countable*) *pmf* **and** *q* :: (- :: *countable*) *spmf*
  shows *qbs-l-qbs-pmf*: *qbs-l* (*qbs-pmf p*) = *measure-pmf p*
    and *qbs-l-qbs-spmf*: *qbs-l* (*qbs-spmf q*) = *measure-spmf q*
  ⟨*proof*⟩

lemma *qbs-pmf-return-pmf*:
  fixes *x* :: - :: *countable*
  shows *qbs-pmf* (*return-pmf x*) = *return-qbs* (*count-space*$_Q$ *UNIV*) *x*
⟨*proof*⟩

lemma *qbs-pmf-bind-pmf*:
  fixes *p* :: (*'a* :: *countable*) *pmf* **and** *f* :: *'a* $\Rightarrow$ (*'b* :: *countable*) *pmf*
  shows *qbs-pmf* (*p* $\gg=$ *f*) = *qbs-pmf p* $\gg=$ (λx. *qbs-pmf* (*f x*))
  ⟨*proof*⟩

lemma *qbs-pair-pmf*:
  fixes *p* :: (*'a* :: *countable*) *pmf* **and** *q* :: (*'b* :: *countable*) *pmf*
  shows *qbs-pmf p* $\bigotimes_{Qmes}$ *qbs-pmf q* = *qbs-pmf* (*pair-pmf p q*)
⟨*proof*⟩

4.1.15 Density

lift-definition *density-qbs* :: [*'a qbs-measure*, *'a* $\Rightarrow$ *ennreal*] $\Rightarrow$ *'a qbs-measure*
is λ(X,α,μ) *f*. *if f* $\in$ *X* $\to_Q$ *qbs-borel then* (X, α, *density* μ (*f* $\circ$ α)) *else* (X, *SOME a*. *a* $\in$ *qbs-Mx X*, *null-measure borel*)
⟨*proof*⟩

lemma(**in** *qbs-s-finite*)
assumes $f \in X \to_Q$ *qbs-borel*
shows *density-qbs*:*density-qbs* $[\![X,\alpha,\ \mu]\!]_{sfin}\ f = [\![X,\ \alpha,\ density\ \mu\ (f \circ \alpha)]\!]_{sfin}$
and *density-qbs-s-finite*: *qbs-s-finite* $X\ \alpha\ (density\ \mu\ (f \circ \alpha))$
⟨*proof*⟩

lemma *density-qbs-density-qbs-eq*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *qbs-borel* $g \in X \to_Q$ *qbs-borel*
shows *density-qbs* (*density-qbs* $s\ f$) g = *density-qbs* s ($\lambda x.\ f\ x * g\ x$)
⟨*proof*⟩

lemma *qbs-l-density-qbs*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *qbs-borel*
shows *qbs-l* (*density-qbs* $s\ f$) = *density* (*qbs-l* s) f
⟨*proof*⟩

corollary *qbs-l-density-qbs-indicator*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X) *qbs-pred* $X\ P$
shows *qbs-l* (*density-qbs* s (*indicator* $\{x \in$ *qbs-space* $X.\ P\ x\}$)) (*qbs-space* X) = *qbs-l* s $\{x \in$ *qbs-space* $X.\ P\ x\}$
⟨*proof*⟩

lemma *qbs-nn-integral-density-qbs*:
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *qbs-borel* $g \in X \to_Q$ *qbs-borel*
shows ($\int^{+}{}_Q\ x.\ g\ x\ \partial$(*density-qbs* $s\ f$)) = ($\int^{+}{}_Q\ x.\ f\ x * g\ x\ \partial s$)
⟨*proof*⟩

lemma *qbs-integral-density-qbs*:
fixes g :: $'a \Rightarrow\ 'b$::{*banach*, *second-countable-topology*} **and** f :: $'a \Rightarrow$ *real*
assumes [*qbs*]:$s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *qbs-borel* $g \in X \to_Q$ *qbs-borel*
and $AE_Q\ x\ in\ s.\ f\ x \geq 0$
shows ($\int_Q\ x.\ g\ x\ \partial$(*density-qbs* $s\ f$)) = ($\int_Q\ x.\ f\ x *_R g\ x\ \partial s$)
⟨*proof*⟩

lemma *density-qbs-morphism*[*qbs*]: *density-qbs* $\in$ *monadM-qbs* $X \to_Q (X \Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ *monadM-qbs* X
⟨*proof*⟩

lemma *density-qbs-cong-AE*:
assumes [*qbs*]: $s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \to_Q$ *qbs-borel* $g \in X \to_Q$ *qbs-borel*
and $AE_Q\ x\ in\ s.\ f\ x = g\ x$
shows *density-qbs* $s\ f$ = *density-qbs* $s\ g$
⟨*proof*⟩

corollary *density-qbs-cong*:

assumes [*qbs*]: $s \in$ *qbs-space* (*monadM-qbs* X) $f \in X \rightarrow_Q$ *qbs-borel* $g \in X \rightarrow_Q$ *qbs-borel*
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
shows *density-qbs* $s\ f$ = *density-qbs* $s\ g$
$\langle proof \rangle$

lemma *density-qbs-1*[*simp*]: *density-qbs* s ($\lambda x.\ 1$) = s
$\langle proof \rangle$

lemma *pair-density-qbs*:
assumes [*qbs*]: $p \in$ *qbs-space* (*monadM-qbs* X) $q \in$ *qbs-space* (*monadM-qbs* Y)
and [*qbs*]: $f \in X \rightarrow_Q$ *qbs-borel* $g \in Y \rightarrow_Q$ *qbs-borel*
shows *density-qbs* $p\ f \bigotimes_{Qmes}$ *density-qbs* $q\ g$ = *density-qbs* ($p \bigotimes_{Qmes} q$) ($\lambda(x,y).\ f\ x * g\ y$)
$\langle proof \rangle$

4.1.16 Normalization

definition *normalize-qbs* :: $'a$ *qbs-measure* $\Rightarrow$ $'a$ *qbs-measure* **where**
normalize-qbs $s \equiv$ (*let* X = *qbs-space-of* s;
r = *qbs-l* s (*qbs-space* X) *in*
if $r \neq 0 \wedge r \neq \infty$ *then density-qbs* s ($\lambda x.\ 1\ /\ r$)
else qbs-null-measure X)

lemma
assumes $s \in$ *qbs-space* (*monadM-qbs* X)
shows *normalize-qbs*: *qbs-l* s (*qbs-space* X) $\neq 0 \Longrightarrow$ *qbs-l* s (*qbs-space* X) $\neq \infty \Longrightarrow$ *normalize-qbs* s = *density-qbs* s ($\lambda x.\ 1\ /$ *emeasure* (*qbs-l* s) (*qbs-space* X))
and *normalize-qbs0*: *qbs-l* s (*qbs-space* X) = $0 \Longrightarrow$ *normalize-qbs* s = *qbs-null-measure* X
and *normalize-qbsinfty*: *qbs-l* s (*qbs-space* X) = $\infty \Longrightarrow$ *normalize-qbs* s = *qbs-null-measure* X
$\langle proof \rangle$

lemma *normalize-qbs-prob*:
assumes $s \in$ *qbs-space* (*monadM-qbs* X) *qbs-l* s (*qbs-space* X) $\neq 0$ *qbs-l* s (*qbs-space* X) $\neq \infty$
shows *normalize-qbs* $s \in$ *qbs-space* (*monadP-qbs* X)
$\langle proof \rangle$

lemma *normalize-qbs-morphism*[*qbs*]: *normalize-qbs* $\in$ *monadM-qbs* $X \rightarrow_Q$ *monadM-qbs* X
$\langle proof \rangle$

lemma *normalize-qbs-morphismP*:
assumes [*qbs*]:$s \in X \rightarrow_Q$ *monadM-qbs* Y
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow$ *qbs-l* ($s\ x$) (*qbs-space* Y) $\neq 0$ $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow$ *qbs-l* ($s\ x$) (*qbs-space* Y) $\neq \infty$
shows ($\lambda x.$ *normalize-qbs* ($s\ x$)) $\in X \rightarrow_Q$ *monadP-qbs* Y

$\langle proof \rangle$

lemma *normalize-qbs-monadP-ident*:
assumes $s \in$ *qbs-space* (*monadP-qbs* X)
shows *normalize-qbs* $s = s$
$\langle proof \rangle$

corollary *normalize-qbs-idenpotent*: *normalize-qbs* (*normalize-qbs* s) = *normalize-qbs* s
$\langle proof \rangle$

4.1.17 Product Measures

definition *PiQ-measure* :: $['a\ set, 'a \Rightarrow 'b\ qbs\text{-}measure] \Rightarrow ('a \Rightarrow 'b)\ qbs\text{-}measure$
where
PiQ-measure $\equiv$ (λI *si*. *if* ($\forall i{\in}I$. $\exists Mi$. *standard-borel-ne Mi* $\wedge$ *si i* $\in$ *qbs-space* (*monadM-qbs* (*measure-to-qbs Mi*)))
then if countable I $\wedge$ ($\forall i{\in}I$. *prob-space* (*qbs-l* (*si i*))) *then qbs-l-inverse* (Π_M $i{\in}I$. *qbs-l* (*si i*))
else if finite I $\wedge$ ($\forall i{\in}I$. *sigma-finite-measure* (*qbs-l* (*si i*))) *then qbs-l-inverse* (Π_M $i{\in}I$. *qbs-l* (*si i*))
else qbs-null-measure (Π_Q $i{\in}I$. *qbs-space-of* (*si i*))
else qbs-null-measure (Π_Q $i{\in}I$. *qbs-space-of* (*si i*)))

syntax
-PiQ-measure :: *pttrn* $\Rightarrow$ *'i set* $\Rightarrow$ *'a qbs-measure* $\Rightarrow$ (*'i* => *'a*) *qbs-measure* ((*3*Π_{Qmeas} -∈-./ -) *10*)
translations
Π_{Qmeas} $x{\in}I$. X == *CONST PiQ-measure I* (λx. X)

context
fixes I **and** *Mi*
assumes *standard-borel-ne*: $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel-ne* (*Mi i*)
begin

context
assumes *countableI*: *countable I*
begin

interpretation *sb*:*standard-borel-ne* Π_M $i{\in}I$. (*borel* :: *real measure*)
$\langle proof \rangle$

interpretation *sbM*: *standard-borel-ne* Π_M $i{\in}I$. *Mi i*
$\langle proof \rangle$

lemma
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *si i* $\in$ *qbs-space* (*monadP-qbs* (*measure-to-qbs* (*Mi i*)))
and $\bigwedge i.\ i \in I \Longrightarrow$ *si i* = $[\![$*measure-to-qbs* (*Mi i*), α *i*, μ *i*$]\!]_{sfin}$ $\bigwedge i.\ i \in I \Longrightarrow$ *qbs-prob* (*measure-to-qbs* (*Mi i*)) (α *i*) (μ *i*)

shows *PiQ-measure-prob-eq*: $(\Pi_{Q\,meas}\ i{\in}I.\ si\ i) = [\![measure\text{-}to\text{-}qbs\ (\Pi_M\ i{\in}I.\ Mi\ i),\ sbM.from\text{-}real,\ distr\ (\Pi_M\ i{\in}I.\ qbs\text{-}l\ (si\ i))\ borel\ sbM.to\text{-}real]\!]_{sfin}$ (**is** - = *?rhs*)
and *PiQ-measure-qbs-prob*: $qbs\text{-}prob\ (measure\text{-}to\text{-}qbs\ (\Pi_M\ i{\in}I.\ Mi\ i))\ sbM.from\text{-}real\ (distr\ (\Pi_M\ i{\in}I.\ qbs\text{-}l\ (si\ i))\ borel\ sbM.to\text{-}real)$ (**is** *?qbsprob*)
⟨*proof*⟩

lemma *qbs-l-PiQ-measure-prob*:
assumes $\bigwedge i.\ i \in I \Longrightarrow si\ i \in qbs\text{-}space\ (monadP\text{-}qbs\ (measure\text{-}to\text{-}qbs\ (Mi\ i)))$
shows $qbs\text{-}l\ (\Pi_{Q\,meas}\ i{\in}I.\ si\ i) = (\Pi_M\ i{\in}I.\ qbs\text{-}l\ (si\ i))$
⟨*proof*⟩

end

context
assumes *finI*: *finite I*
begin

interpretation *sb*:*standard-borel-ne* $\Pi_M\ i{\in}I.\ (borel :: real\ measure)$
⟨*proof*⟩

interpretation *sbM*: *standard-borel-ne* $\Pi_M\ i{\in}I.\ Mi\ i$
⟨*proof*⟩

lemma *qbs-l-PiQ-measure*:
assumes $\bigwedge i.\ i \in I \Longrightarrow si\ i \in qbs\text{-}space\ (monadM\text{-}qbs\ (measure\text{-}to\text{-}qbs\ (Mi\ i)))$
and $\bigwedge i.\ i \in I \Longrightarrow sigma\text{-}finite\text{-}measure\ (qbs\text{-}l\ (si\ i))$
shows $qbs\text{-}l\ (\Pi_{Q\,meas}\ i{\in}I.\ si\ i) = (\Pi_M\ i{\in}I.\ qbs\text{-}l\ (si\ i))$
⟨*proof*⟩

end

end

4.2 Measures

4.2.1 The Lebesgue Measure

definition *lborel-qbs* ($lborel_Q$) **where** *lborel-qbs* ≡ *qbs-l-inverse lborel*

lemma *lborel-qbs-qbs*[*qbs*]: $lborel\text{-}qbs \in qbs\text{-}space\ (monadM\text{-}qbs\ qbs\text{-}borel)$
⟨*proof*⟩

lemma *qbs-l-lborel-qbs*[*simp*]: $qbs\text{-}l\ lborel_Q = lborel$
⟨*proof*⟩

corollary
shows *qbs-integral-lborel*: $(\int_Q x.\ f\ x\ \partial lborel\text{-}qbs) = (\int x.\ f\ x\ \partial lborel)$
and *qbs-nn-integral-lborel*: $(\int^+{}_Q\ x.\ f\ x\ \partial lborel\text{-}qbs) = (\int^+ x.\ f\ x\ \partial lborel)$

$\langle proof \rangle$

lemma(**in** *standard-borel-ne*) *measure-with-args-morphism*:
assumes *s-finite-kernel X M k*
shows $qbs\text{-}l\text{-}inverse \circ k \in measure\text{-}to\text{-}qbs\ X \rightarrow_Q monadM\text{-}qbs\ (measure\text{-}to\text{-}qbs\ M)$
$\langle proof \rangle$

lemma(**in** *standard-borel-ne*) *measure-with-args-morphismP*:
assumes [*measurable*]:$\mu \in X \rightarrow_M prob\text{-}algebra\ M$
shows $qbs\text{-}l\text{-}inverse \circ \mu \in measure\text{-}to\text{-}qbs\ X \rightarrow_Q monadP\text{-}qbs\ (measure\text{-}to\text{-}qbs\ M)$
$\langle proof \rangle$

4.2.2 Counting Measure

abbreviation $counting\text{-}measure\text{-}qbs\ A \equiv qbs\text{-}l\text{-}inverse\ (count\text{-}space\ A)$

lemma *qbs-nn-integral-count-space-nat*:
fixes $f :: nat \Rightarrow ennreal$
shows $(\int^{+}{}_{Q}\ i.\ f\ i\ \partial counting\text{-}measure\text{-}qbs\ UNIV) = (\sum i.\ f\ i)$
$\langle proof \rangle$

4.2.3 Normal Distribution

lemma *qbs-normal-distribution-qbs*: $(\lambda\mu\ \sigma.\ density\text{-}qbs\ lborel_Q\ (normal\text{-}density\ \mu\ \sigma)) \in qbs\text{-}borel \Rightarrow_Q qbs\text{-}borel \Rightarrow_Q monadM\text{-}qbs\ qbs\text{-}borel$
$\langle proof \rangle$

lemma *qbs-l-qbs-normal-distribution*[*simp*]: $qbs\text{-}l\ (density\text{-}qbs\ lborel_Q\ (normal\text{-}density\ \mu\ \sigma)) = density\ lborel\ (normal\text{-}density\ \mu\ \sigma)$
$\langle proof \rangle$

lemma *qbs-normal-distribution-P*: $\sigma > 0 \Longrightarrow density\text{-}qbs\ lborel_Q\ (normal\text{-}density\ \mu\ \sigma) \in qbs\text{-}space\ (monadP\text{-}qbs\ qbs\text{-}borel)$
$\langle proof \rangle$

lemma *qbs-normal-distribution-integral*:
$(\int_Q x.\ f\ x\ \partial\ (density\text{-}qbs\ lborel_Q\ (normal\text{-}density\ \mu\ \sigma))) = (\int x.\ f\ x\ \partial\ (density\ lborel\ (\lambda x.\ ennreal\ (normal\text{-}density\ \mu\ \sigma\ x))))$
$\langle proof \rangle$

lemma *qbs-normal-distribution-expectation*:
assumes [*measurable*]:$f \in borel\text{-}measurable\ borel$ **and** [*arith*]: $\sigma > 0$
shows $(\int_Q x.\ f\ x\ \partial\ (density\text{-}qbs\ lborel_Q\ (normal\text{-}density\ \mu\ \sigma))) = (\int x.\ normal\text{-}density\ \mu\ \sigma\ x * f\ x\ \partial\ lborel)$
$\langle proof \rangle$

lemma *qbs-normal-posterior*:

assumes [*arith*]: $\sigma > 0$ $\sigma' > 0$
shows *normalize-qbs* (*density-qbs* (*density-qbs lborel*$_Q$ (*normal-density* μ σ)) (*normal-density* μ' σ')) = *density-qbs lborel*$_Q$ (*normal-density* (($\mu * \sigma'^2 + \mu' * \sigma^2$) / ($\sigma^2 + \sigma'^2$)) ($\sigma * \sigma'$ / *sqrt* ($\sigma^2 + \sigma'^2$))) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

4.2.4 Uniform Distribution

definition *uniform-qbs* :: *'a qbs-measure* ⇒ *'a set* ⇒ *'a qbs-measure* **where**
uniform-qbs ≡ (λ*s A*. *qbs-l-inverse* (*uniform-measure* (*qbs-l s*) *A*))

lemma(**in** *standard-borel-ne*) *qbs-l-uniform-qbs'*:
assumes *sets* μ = *sets M* *s-finite-measure* μ μ *A* ≠ *0*
shows *qbs-l* (*uniform-qbs* (*qbs-l-inverse* μ) *A*) = *uniform-measure* μ *A* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

corollary(**in** *standard-borel-ne*) *qbs-l-uniform-qbs*:
assumes *s* ∈ *qbs-space* (*monadM-qbs* (*measure-to-qbs M*)) *qbs-l s A* ≠ *0*
shows *qbs-l* (*uniform-qbs s A*) = *uniform-measure* (*qbs-l s*) *A*
⟨*proof*⟩

lemma *interval-uniform-qbs*: (λ*a b*. *uniform-qbs lborel*$_Q$ {*a*<..<*b*::*real*}) ∈ *borel*$_Q$ ⇒$_Q$ *borel*$_Q$ ⇒$_Q$ *monadM-qbs borel*$_Q$
⟨*proof*⟩

context
fixes *a b* :: *real*
assumes [*arith*]:*a* < *b*
begin

lemma *qbs-uniform-distribution-expectation*:
assumes *f* ∈ *qbs-borel* →$_Q$ *qbs-borel*
shows ($\int^+_Q$ *x*. *f x* ∂*uniform-qbs lborel*$_Q$ {*a*<..<*b*}) = ($\int^+$*x* ∈ {*a*<..<*b*}. *f x* ∂*lborel*) / (*b* − *a*)
⟨*proof*⟩

end

4.2.5 Bernoulli Distribution

abbreviation *qbs-bernoulli* :: *real* ⇒ *bool qbs-measure* **where**
qbs-bernoulli ≡ (λ*x*. *qbs-pmf* (*bernoulli-pmf x*))

lemma *bernoulli-measurable*:
(λ*x*. *measure-pmf* (*bernoulli-pmf x*)) ∈ *borel* →$_M$ *prob-algebra* (*count-space UNIV*)
⟨*proof*⟩

lemma *qbs-bernoulli-morphism*: *qbs-bernoulli* ∈ *qbs-borel* →$_Q$ *monadP-qbs* (*qbs-count-space UNIV*)

⟨*proof*⟩

lemma *qbs-bernoulli-expectation*:
assumes [*simp*]: $0 \leq p$ $p \leq 1$
shows $(\int_Q x.\ f\ x\ \partial qbs\text{-}bernoulli\ p) = f\ True * p + f\ False * (1 - p)$
⟨*proof*⟩

end

5 Examples

5.1 Montecarlo Approximation

theory *Montecarlo*
imports *Monad-QuasiBorel*
begin

declare [[*coercion qbs-l*]]

abbreviation *real-quasi-borel* :: *real quasi-borel* ($\mathbb{R}_Q$) **where**
real-quasi-borel ≡ *qbs-borel*
abbreviation *nat-quasi-borel* :: *nat quasi-borel* ($\mathbb{N}_Q$) **where**
nat-quasi-borel ≡ *qbs-count-space UNIV*

primrec *montecarlo* :: *'a qbs-measure* ⇒ (*'a* ⇒ *real*) ⇒ *nat* ⇒ *real qbs-measure*
where
montecarlo - - 0 = *return-qbs* $\mathbb{R}_Q$ *0* |
montecarlo d h (*Suc n*) = *do* { *m* ← *montecarlo d h n*;
x ← *d*;
return-qbs $\mathbb{R}_Q$ ((*h x* + *m* ∗ (*real n*)) / (*real* (*Suc n*)))}

declare
bind-qbs-morphismP[*qbs*]
return-qbs-morphismP[*qbs*]
qbs-pair-measure-morphismP[*qbs*]

lemma *montecarlo-qbs-morphism*[*qbs*]: *montecarlo* ∈ *qbs-space* (*monadP-qbs X* $\Rightarrow_Q$ (*X* $\Rightarrow_Q$ $\mathbb{R}_Q$) $\Rightarrow_Q$ $\mathbb{N}_Q$ $\Rightarrow_Q$ *monadP-qbs* $\mathbb{R}_Q$)
⟨*proof*⟩

lemma *qbs-integrable-indep-mult2*[*simp*, *intro*!]:
fixes *f* :: - ⇒ *real*
assumes *qbs-integrable p f*
and *qbs-integrable q g*
shows *qbs-integrable* ($p \bigotimes_{Qmes} q$) ($\lambda x.\ g\ (snd\ x) * f\ (fst\ x)$)
⟨*proof*⟩

lemma *montecarlo-integrable*:
  assumes [*qbs*]:$p \in$ *qbs-space* (*monadP-qbs* X) $h \in X \to_Q \mathbb{R}_Q$ *qbs-integrable* p h *qbs-integrable* p ($\lambda x.\ h\ x * h\ x$)
  shows *qbs-integrable* (*montecarlo* p h n) ($\lambda x.\ x$) *qbs-integrable* (*montecarlo* p h n) ($\lambda x.\ x * x$)
⟨*proof*⟩

lemma
  fixes n :: *nat*
  assumes [*qbs*]:$p \in$ *qbs-space* (*monadP-qbs* X) $h \in X \to_Q \mathbb{R}_Q$ *qbs-integrable* p h *qbs-integrable* p ($\lambda x.\ h\ x * h\ x$)
    and e:$e > 0$
    and $(\int_Q x.\ h\ x\ \partial p) = \mu$ $(\int_Q x.\ (h\ x - \mu)^2\ \partial p) = \sigma^2$
    and n:$n > 0$
  shows $\mathcal{P}(y$ *in montecarlo* p h n. $|y - \mu| \geq e) \leq \sigma^2$ / (*real* $n * e^2$) (**is** *?P* $\leq$ -)
⟨*proof*⟩

end

5.2 Query

theory *Query*
  imports *Monad-QuasiBorel*
begin

declare [[*coercion qbs-l*]]
abbreviation *qbs-real* :: *real quasi-borel* ($\mathbb{R}_Q$) **where** $\mathbb{R}_Q \equiv$ *qbs-borel*
abbreviation *qbs-ennreal* :: *ennreal quasi-borel* ($\mathbb{R}_{Q\geq 0}$) **where** $\mathbb{R}_{Q\geq 0} \equiv$ *qbs-borel*
abbreviation *qbs-nat* :: *nat quasi-borel* ($\mathbb{N}_Q$) **where** $\mathbb{N}_Q \equiv$ *qbs-count-space UNIV*
abbreviation *qbs-bool* :: *bool quasi-borel* ($\mathbb{B}_Q$) **where** $\mathbb{B}_Q \equiv$ *count-space*$_Q$ *UNIV*

definition *query* :: [$'a$ *qbs-measure*, $'a \Rightarrow$ *ennreal*] $\Rightarrow$ $'a$ *qbs-measure* **where**
query $\equiv$ ($\lambda s\ f.$ *normalize-qbs* (*density-qbs* s f))

lemma *query-qbs-morphism*[*qbs*]: *query* $\in$ *monadM-qbs* $X \to_Q (X \Rightarrow_Q$ *qbs-borel*) $\Rightarrow_Q$ *monadM-qbs* X
  ⟨*proof*⟩

definition *condition* $\equiv$ ($\lambda s\ P.$ *query* s ($\lambda x.$ *if* $P\ x$ *then 1 else 0*))

lemma *condition-qbs-morphism*[*qbs*]: *condition* $\in$ *monadM-qbs* $X \Rightarrow_Q (X \Rightarrow_Q \mathbb{B}_Q)$ $\Rightarrow_Q$ *monadM-qbs* X
  ⟨*proof*⟩

lemma *condition-morphismP*:

assumes $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow \mathcal{P}(y$ *in qbs-l* $(s\ x).\ P\ x\ y) \neq 0$
and [*qbs*]: $s \in X \to_Q$ *monadP-qbs* Y $P \in X \to_Q Y \Rightarrow_Q$ *qbs-count-space UNIV*
shows $(\lambda x.$ *condition* $(s\ x)\ (P\ x)) \in X \to_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *query-Bayes*:
assumes [*qbs*]: $s \in$ *qbs-space* (*monadP-qbs* X) *qbs-pred* X P *qbs-pred* X Q
shows $\mathcal{P}(x$ *in condition* $s\ P.\ Q\ x) = \mathcal{P}(x$ *in* $s.\ Q\ x \mid P\ x)$ (**is** *?lhs* = *?pq*)
⟨*proof*⟩

lemma *qbs-pmf-cond-pmf*:
fixes p :: $'a$ :: *countable pmf*
assumes *set-pmf* $p \cap \{x.\ P\ x\} \neq \{\}$
shows *condition* (*qbs-pmf* p) P = *qbs-pmf* (*cond-pmf* p $\{x.\ P\ x\}$)
⟨*proof*⟩

5.2.1 `twoUs`

Example from Section 2 in [3].

definition *Uniform* $\equiv$ ($\lambda a\ b$::*real*. *uniform-qbs lborel-qbs* $\{a<..<b\}$)

lemma *Uniform-qbs*[*qbs*]: *Uniform* $\in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q$ *monadM-qbs* $\mathbb{R}_Q$
⟨*proof*⟩

definition *twoUs* :: (*real* $\times$ *real*) *qbs-measure* **where**
twoUs $\equiv$ *do* {
let u1 = Uniform 0 1;
let u2 = Uniform 0 1;
let $y = u1 \bigotimes_{Qmes} u2$;
condition y $(\lambda(x,y).\ x < 0.5 \vee y > 0.5)$
}

lemma *twoUs-qbs*: *twoUs* $\in$ *monadM-qbs* $(\mathbb{R}_Q \bigotimes_Q \mathbb{R}_Q)$
⟨*proof*⟩

interpretation *rr*: *standard-borel-ne borel* $\bigotimes_M$ *borel* :: (*real* $\times$ *real*) *measure*
⟨*proof*⟩

lemma *qbs-l-Uniform*[*simp*]: $a < b \Longrightarrow$ *qbs-l* (*Uniform* a b) = *uniform-measure lborel* $\{a<..<b\}$
⟨*proof*⟩

lemma *Uniform-qbsP*:
assumes [*arith*]: $a < b$
shows *Uniform* a $b \in$ *monadP-qbs* $\mathbb{R}_Q$
⟨*proof*⟩

interpretation *UniformP-pair*: *pair-prob-space uniform-measure lborel* $\{0<..<1$::*real*$\}$ *uniform-measure lborel* $\{0<..<1$::*real*$\}$

⟨*proof*⟩

lemma *qbs-l-Uniform-pair*: $a < b \Longrightarrow$ *qbs-l* (*Uniform a b* $\bigotimes_{Qmes}$ *Uniform a b*) = *uniform-measure lborel* $\{a<..<b\}$ $\bigotimes_M$ *uniform-measure lborel* $\{a<..<b\}$
⟨*proof*⟩

lemma *Uniform-pair-qbs*[*qbs*]:
assumes $a < b$
shows *Uniform a b* $\bigotimes_{Qmes}$ *Uniform a b* $\in$ *qbs-space* (*monadP-qbs* ($\mathbb{R}_Q$ $\bigotimes_Q$ $\mathbb{R}_Q$))
⟨*proof*⟩

lemma *twoUs-prob1*: $\mathcal{P}$(*z in Uniform 0 1* $\bigotimes_{Qmes}$ *Uniform 0 1*. *fst z* $< 0.5 \vee$ *snd z* > 0.5) $= 3 / 4$
⟨*proof*⟩

lemma *twoUs-prob2*:$\mathcal{P}$(*z in Uniform 0 1* $\bigotimes_{Qmes}$ *Uniform 0 1*. $1/2 <$ *fst z* $\wedge$ (*fst z* $< 1/2 \vee$ *snd z* $> 1/2$)) $= 1 / 4$
⟨*proof*⟩

lemma *twoUs-qbs-prob*: *twoUs* $\in$ *qbs-space* (*monadP-qbs* ($\mathbb{R}_Q$ $\bigotimes_Q$ $\mathbb{R}_Q$))
⟨*proof*⟩

lemma $\mathcal{P}$((*x*,*y*) *in twoUs*. $1/2 < x$) $= 1 / 3$
⟨*proof*⟩

5.2.2 Two Dice

Example from Adrian [2, Sect. 2.3].

abbreviation *die* $\equiv$ *qbs-pmf* (*pmf-of-set* $\{$*Suc 0..6*$\}$)

lemma *die-qbs*[*qbs*]: *die* $\in$ *monadM-qbs* $\mathbb{N}_Q$
⟨*proof*⟩

definition *two-dice* :: *nat qbs-measure* **where**
two-dice $\equiv$ *do* {
let die1 = *die*;
let die2 = *die*;
let twodice = *die1* $\bigotimes_{Qmes}$ *die2*;
(*x*,*y*) $\leftarrow$ *condition twodice*
(λ(*x*,*y*). $x = 4 \vee y = 4$);
return-qbs $\mathbb{N}_Q$ ($x + y$)
}

lemma *two-dice-qbs*: *two-dice* $\in$ *monadM-qbs* $\mathbb{N}_Q$
⟨*proof*⟩

lemma *prob-die2*: $\mathcal{P}$(*x in qbs-l* (*die* $\bigotimes_{Qmes}$ *die*). *P x*) = *real* (*card* ($\{x.\ P\ x\} \cap$ ($\{1..6\} \times \{1..6\}$))) / *36* (**is** *?P* = *?rhs*)

⟨*proof*⟩

lemma *dice-prob1*: $\mathcal{P}(z \textit{ in qbs-l } (\textit{die} \bigotimes_{Qmes} \textit{die}).\ \textit{fst } z = 4 \vee \textit{snd } z = 4) = 11 / 36$
⟨*proof*⟩

lemma *dice-program-prob*: $\mathcal{P}(x \textit{ in two-dice}.\ P\ x) = 2 * (\sum n{\in}\{5,6,7,9,10\}.\ \textit{of-bool}\ (P\ n) / 11) + \textit{of-bool}\ (P\ 8) / 11$ (**is** *?P = ?rp*)
⟨*proof*⟩

corollary
$\mathcal{P}(x \textit{ in two-dice}.\ x = 5) = 2 / 11$
$\mathcal{P}(x \textit{ in two-dice}.\ x = 6) = 2 / 11$
$\mathcal{P}(x \textit{ in two-dice}.\ x = 7) = 2 / 11$
$\mathcal{P}(x \textit{ in two-dice}.\ x = 8) = 1 / 11$
$\mathcal{P}(x \textit{ in two-dice}.\ x = 9) = 2 / 11$
$\mathcal{P}(x \textit{ in two-dice}.\ x = 10) = 2 / 11$

⟨*proof*⟩

5.2.3 Gaussian Mean Learning

Example from Sato et al. Section 8. 2 in [3].

definition $\textit{Gauss} \equiv (\lambda\mu\ \sigma.\ \textit{density-qbs lborel}_Q\ (\textit{normal-density}\ \mu\ \sigma))$

lemma *Gauss-qbs*[*qbs*]: $\textit{Gauss} \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \textit{monadM-qbs}\ \mathbb{R}_Q$
⟨*proof*⟩

primrec $\textit{GaussLearn}' :: [\textit{real}, \textit{real qbs-measure}, \textit{real list}] \Rightarrow \textit{real qbs-measure}$ **where**
$\textit{GaussLearn}'\ \text{-}\ p\ [] = p$
$|\ \textit{GaussLearn}'\ \sigma\ p\ (y\#ls) = \textit{query}\ (\textit{GaussLearn}'\ \sigma\ p\ ls)\ (\textit{normal-density}\ y\ \sigma)$

lemma *GaussLearn′-qbs*[*qbs*]: $\textit{GaussLearn}' \in \mathbb{R}_Q \Rightarrow_Q \textit{monadM-qbs}\ \mathbb{R}_Q \Rightarrow_Q \textit{list-qbs}\ \mathbb{R}_Q \Rightarrow_Q \textit{monadM-qbs}\ \mathbb{R}_Q$
⟨*proof*⟩

context
fixes $\sigma :: \textit{real}$
assumes [*arith*]: $\sigma > 0$
begin

abbreviation $\textit{GaussLearn} \equiv \textit{GaussLearn}'\ \sigma$

lemma *GaussLearn-qbs*[*qbs*]: $\textit{GaussLearn} \in \textit{qbs-space}\ (\textit{monadM-qbs}\ \mathbb{R}_Q \Rightarrow_Q \textit{list-qbs}\ \mathbb{R}_Q \Rightarrow_Q \textit{monadM-qbs}\ \mathbb{R}_Q)$
⟨*proof*⟩

definition *Total* :: *real list* ⇒ *real* **where** *Total* = ($\lambda l.$ *foldr* (+) *l 0*)

lemma *Total-simp*: *Total* [] = *0* *Total* (*y#ls*) = *y* + *Total ls*
⟨*proof*⟩

lemma *Total-qbs*[*qbs*]: *Total* ∈ *list-qbs* $\mathbb{R}_Q \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *GaussLearn-Total*:
assumes [*arith*]: $\xi > 0$ $n = length\ L$
shows *GaussLearn* (*Gauss* δ ξ) *L* = *Gauss* ((*Total L*$*\xi^2+\delta*\sigma^2$)/($n*\xi^2+\sigma^2$)) (*sqrt* (($\xi^2*\sigma^2$)/($n*\xi^2+\sigma^2$)))
⟨*proof*⟩

lemma *GaussLearn-KL-divergence-lem1*:
fixes *a* :: *real*
assumes [*arith*]: $a > 0$ $b > 0$ $c > 0$ $d > 0$
shows ($\lambda n.$ *ln* (($b * (n * d + c)$) / ($d * (n * b + a)$))) ⟶ *0*
⟨*proof*⟩

lemma *GaussLearn-KL-divergence-lem1'*:
fixes *b* :: *real*
assumes [*arith*]: $b > 0$ $d > 0$ $s > 0$
shows ($\lambda n.$ *ln* (*sqrt* ($b^2 * s^2$ / (*real n* $* b^2 + s^2$)) / *sqrt* ($d^2 * s^2$ / (*real n* $* d^2 + s^2$)))) ⟶ *0* (**is** *?f* ⟶ *0*)
⟨*proof*⟩

lemma *GaussLearn-KL-divergence-lem2*:
fixes *s* :: *real*
assumes [*arith*]: $s > 0$ $b > 0$ $d > 0$
shows ($\lambda n.$ (($d * s$) / ($n * d + s$)) / ($2 * ((b * s) / (n * b + s))$)) ⟶ *1 / 2*
⟨*proof*⟩

lemma *GaussLearn-KL-divergence-lem2'*:
fixes *s* :: *real*
assumes [*arith*]: $s > 0$ $b > 0$ $d > 0$
shows ($\lambda n.$ ((d^2 ∗ s^2) / (n ∗ d^2 + s^2)) / (2 ∗ ((b^2 ∗ s^2) / (n ∗ b^2 + s^2))) − 1 / 2) ⟶ *0*
⟨*proof*⟩

lemma *GaussLearn-KL-divergence-lem3*:
fixes *a b c d s K L* :: *real*
assumes [*arith*]: $b > 0$ $d > 0$ $s > 0$
shows ((K ∗ d + c ∗ s) / (n ∗ d + s) − (L ∗ b + a ∗ s) / (n ∗ b + s))^2 / (2 ∗ ((b ∗ s) / (n ∗ b + s))) = (((((((K − L) ∗ d ∗ b ∗ *real n* + c ∗ s ∗ b ∗ *real n* + K ∗ d ∗ s + c ∗ s ∗ s) − a ∗ s ∗ d ∗ *real n* − L ∗ b ∗ s − a ∗ s ∗ s))^2 / (d ∗ d ∗ b ∗ (*real n* ∗ *real n* ∗ *real n*) + s ∗ s ∗ b ∗ *real n* + 2 ∗ d ∗ s ∗ b ∗ (*real n* ∗ *real n*) + d ∗ d ∗ (*real n* ∗ *real n*) ∗ s + s ∗ s ∗ s + 2 ∗ d ∗ s ∗ s ∗ *real n*))) / (2 ∗ (b

$*\ s))$ (**is** $?lhs = ?rhs$)
$\langle proof \rangle$

lemma *GaussLearn-KL-divergence-lem4*:
fixes $a\ b\ c\ d\ s\ K\ L :: real$
assumes $[arith]$: $b > 0\ d > 0\ s > 0$
shows $(\lambda n.\ (|c * s * b * real\ n| + |K * (real\ n) * d * s| + |c * s * s| + |a * s * d * real\ n| + |L * (real\ n) * b * s| + |a * s * s|)^2 / (d * d * b * (real\ n * real\ n * real\ n) + s * s * b * real\ n + 2 * d * s * b * (real\ n * real\ n) + d * d * (real\ n * real\ n) * s + s * s * s + 2 * d * s * s * real\ n) / (2 * (b * s))) \longrightarrow 0$ (**is** $(\lambda n.\ ?f\ n) \longrightarrow 0$)
$\langle proof \rangle$

lemma *GaussLearn-KL-divergence-lem5*:
fixes $a\ b\ c\ d\ K :: real$
assumes $[arith]$: $b > 0\ d > 0\ s > 0\ K > 0\ |f\ l| < K * length\ l$
shows $|(c * s * b * real\ (length\ l) + f\ l * d * s + c * s * s - a * s * d * real\ (length\ l) - f\ l * b * s - a * s * s)^2 / (d * d * b * (real\ (length\ l) * real\ (length\ l) * real\ (length\ l)) + s * s * b * real\ (length\ l) + 2 * d * s * b * (real\ (length\ l) * real\ (length\ l)) + d * d * (real\ (length\ l) * real\ (length\ l)) * s + s * s * s + 2 * d * s * s * real\ (length\ l)) / (2 * (b * s))| \le |(|c * s * b * real\ (length\ l)| + |K * real\ (length\ l) * d * s| + |c * s * s| + |a * s * d * real\ (length\ l)| + |- K * real\ (length\ l) * b * s| + |a * s * s|)^2 / (d * d * b * (real\ (length\ l) * real\ (length\ l) * real\ (length\ l)) + s * s * b * real\ (length\ l) + 2 * d * s * b * (real\ (length\ l) * real\ (length\ l)) + d * d * (real\ (length\ l) * real\ (length\ l)) * s + s * s * s + 2 * d * s * s * real\ (length\ l)) / (2 * (b * s))|$ (**is** $|(?l)\hat{}2 / ?c1 / ?c2| \le |(?r)\hat{}2 / - / -|$)
$\langle proof \rangle$

lemma *GaussLearn-KL-divergence-lem6*:
fixes $a\ e\ b\ c\ d\ K :: real$ **and** $f :: 'a\ list \Rightarrow real$
assumes $[arith]$:$e > 0\ b > 0\ d > 0\ s > 0$
shows $\exists N.\ \forall l.\ length\ l \ge N \longrightarrow |f\ l| < K * length\ l \longrightarrow |((f\ l * d + c * s) / (length\ l * d + s) - (f\ l * b + a * s) / (length\ l * b + s))\hat{}2 / (2 * ((b * s) / (length\ l * b + s)))\ | < e$
$\langle proof \rangle$

lemma *GaussLearn-KL-divergence*:
fixes $a\ b\ c\ d\ e\ K :: real$
assumes $[arith]$:$e > 0\ b > 0\ d > 0$
shows $\exists N.\ \forall L.\ length\ L > N \longrightarrow |Total\ L / length\ L| < K$
$\longrightarrow$ *KL-divergence* $(exp\ 1)$ $(GaussLearn\ (Gauss\ a\ b)\ L)$ $(GaussLearn\ (Gauss\ c\ d)\ L) < e$
$\langle proof \rangle$

end

5.2.4 Continuous Distributions

The following (highr-order) program receives a non-negative function f and returns the distribution whose density function is (noramlized) f if f is integrable w.r.t. the Lebesgue measure.

definition *dens-to-dist* :: $['a$ :: *euclidean-space* $\Rightarrow$ *real*$] \Rightarrow$ $'a$ *qbs-measure* **where**
dens-to-dist $\equiv$ (λf. *do* {
query $lborel_Q$ *f*
})

lemma *dens-to-dist-qbs*[*qbs*]: *dens-to-dist* $\in$ ($borel_Q \Rightarrow_Q \mathbb{R}_Q$) $\rightarrow_Q$ *monadM-qbs* $borel_Q$
⟨*proof*⟩

context
fixes *f* :: $'a$ :: *euclidean-space* $\Rightarrow$ *real*
assumes *f-qbs*[*qbs*]: $f \in$ *qbs-borel* $\rightarrow_Q \mathbb{R}_Q$
and *f-le0*: $\bigwedge x.\ f\ x \geq 0$
and *f-int-ne0*: *qbs-l* (*density-qbs lborel-qbs f*) *UNIV* $\neq$ *0*
and *f-integrable*: *qbs-integrable lborel-qbs f*
begin

lemma *f-integrable'*[*measurable*]: *integrable lborel f*
⟨*proof*⟩

lemma *f-int-neinfty*:
qbs-l (*density-qbs lborel-qbs f*) *UNIV* $\neq \infty$
⟨*proof*⟩

lemma *dens-to-dist*: *dens-to-dist f* = *density-qbs lborel-qbs* (λx. *ennreal* (*1* / *measure* (*qbs-l* (*density-qbs lborel-qbs f*)) *UNIV* $*$ *f x*))
⟨*proof*⟩

corollary *qbs-l-dens-to-dist*: *qbs-l* (*dens-to-dist f*) = *density lborel* (λx. *ennreal* (*1* / *measure* (*qbs-l* (*density-qbs lborel-qbs f*)) *UNIV* $*$ *f x*))
⟨*proof*⟩

corollary *qbs-integral-dens-to-dist*:
assumes [*qbs*]: $g \in$ *qbs-borel* $\rightarrow_Q \mathbb{R}_Q$
shows ($\int_Q$ *x. g x* ∂*dens-to-dist f*) = ($\int_Q$ *x. 1* / *measure* (*qbs-l* (*density-qbs lborel-qbs f*)) *UNIV* $*$ *f x* $*$ *g x* $\partial lborel_Q$)
⟨*proof*⟩

lemma *dens-to-dist-prob*[*qbs*]:*dens-to-dist f* $\in$ *qbs-space* (*monadP-qbs* $borel_Q$)
⟨*proof*⟩

end

5.2.5 Normal Distribution

context
  fixes μ σ :: *real*
  assumes *sigma-pos*[*arith*]: $\sigma > 0$
begin

We use an unnormalized density function.

definition *normal-f* $\equiv$ ($\lambda x.$ *exp* $(-(x - \mu)^2 / (2 * \sigma^2)))$

lemma *nc-normal-f*: *qbs-l* (*density-qbs lborel-qbs normal-f*) *UNIV* = *ennreal* (*sqrt* $(2 * pi * \sigma^2)$)
⟨*proof*⟩

corollary *measure-qbs-l-dens-to-dist-normal-f*: *measure* (*qbs-l* (*density-qbs lborel-qbs normal-f*)) *UNIV* = *sqrt* $(2 * pi * \sigma^2)$
  ⟨*proof*⟩

lemma *normal-f*:
  shows *normal-f* $\in$ *qbs-borel* $\to_Q$ $\mathbb{R}_Q$
    and $\bigwedge x.$ *normal-f* $x \geq 0$
    and *qbs-l* (*density-qbs lborel-qbs normal-f*) *UNIV* $\neq 0$
    and *qbs-integrable lborel-qbs normal-f*
  ⟨*proof*⟩

lemma *qbs-l-densto-dist-normal-f*: *qbs-l* (*dens-to-dist normal-f*) = *density lborel* (*normal-density* μ σ)
  ⟨*proof*⟩

end

5.2.6 Half Normal Distribution

context
  fixes μ σ :: *real*
  assumes *sigma-pos*[*arith*]:$\sigma > 0$
begin

definition *hnormal-f* $\equiv$ ($\lambda x.$ *if* $x \leq \mu$ *then 0 else normal-density* μ σ x)

lemma *nc-hnormal-f*: *qbs-l* (*density-qbs lborel-qbs hnormal-f*) *UNIV* = *ennreal* $(1 / 2)$
⟨*proof*⟩

corollary *measure-qbs-l-dens-to-dist-hnormal-f*: *measure* (*qbs-l* (*density-qbs lborel-qbs hnormal-f*)) *UNIV* = $1 / 2$
  ⟨*proof*⟩

lemma *hnormal-f*:

shows *hnormal-f* $\in$ *qbs-borel* $\to_Q$ $\mathbb{R}_Q$
and $\bigwedge x$. *hnormal-f x* $\geq$ *0*
and *qbs-l* (*density-qbs lborel-qbs hnormal-f*) *UNIV* $\neq$ *0*
and *qbs-integrable lborel-qbs hnormal-f*
⟨*proof*⟩

lemma *qbs-l* (*dens-to-dist local.hnormal-f*) = *density lborel* (λx. *ennreal* (*2* $*$ (*if x* $\leq$ μ *then 0 else normal-density* μ σ *x*)))
⟨*proof*⟩

end

5.2.7 Erlang Distribution

context
fixes *k* :: *nat* **and** *l* :: *real*
assumes *l-pos*[*arith*]: *l* > *0*
begin

definition *erlang-f* $\equiv$ (λx. *if x* < *0 then 0 else x*^*k* $*$ *exp* (− *l* $*$ *x*))

lemma *nc-erlang-f*: *qbs-l* (*density-qbs lborel-qbs erlang-f*) *UNIV* = *ennreal* (*fact k* / *l*^(*Suc k*))
⟨*proof*⟩

corollary *measure-qbs-l-dens-to-dist-erlang-f*: *measure* (*qbs-l* (*density-qbs lborel-qbs erlang-f*)) *UNIV* = *fact k* / *l*^(*Suc k*)
⟨*proof*⟩

lemma *erlang-f*:
shows *erlang-f* $\in$ *qbs-borel* $\to_Q$ $\mathbb{R}_Q$
and $\bigwedge x$. *erlang-f x* $\geq$ *0*
and *qbs-l* (*density-qbs lborel-qbs erlang-f*) *UNIV* $\neq$ *0*
and *qbs-integrable lborel-qbs erlang-f*
⟨*proof*⟩

lemma *qbs-l* (*dens-to-dist erlang-f*) = *density lborel* (*erlang-density k l*)
⟨*proof*⟩

end

5.2.8 Uniform Distribution on $(0,1) \times (0,1)$.

definition *uniform-f* $\equiv$ *indicat-real* ({*0*<..<*1*::*real*}×{*0*<..<*1*::*real*})

lemma
shows *uniform-f-qbs'*[*qbs*]: *uniform-f* $\in$ *qbs-borel* $\to_Q$ $\mathbb{R}_Q$
and *uniform-f-qbs*[*qbs*]: *uniform-f* $\in$ $\mathbb{R}_Q \bigotimes_Q \mathbb{R}_Q \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *uniform-f-measurable*[*measurable*]: *uniform-f* $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *nc-uniform-f*: *qbs-l* (*density-qbs lborel-qbs uniform-f*) *UNIV* = *1*
⟨*proof*⟩

corollary *measure-qbs-l-dens-to-dist-uniform-f*: *measure* (*qbs-l* (*density-qbs lborel-qbs uniform-f*)) *UNIV* = *1*
⟨*proof*⟩

lemma *uniform-f*:
shows *uniform-f* $\in$ *qbs-borel* $\rightarrow_Q \mathbb{R}_Q$
and $\bigwedge x$. *uniform-f x* $\geq$ *0*
and *qbs-l* (*density-qbs lborel-qbs uniform-f*) *UNIV* $\neq$ *0*
and *qbs-integrable lborel-qbs uniform-f*
⟨*proof*⟩

lemma *qbs-l-dens-to-dist-uniform-f*:*qbs-l* (*dens-to-dist uniform-f*) = *density lborel* (λx. *ennreal* (*uniform-f x*))
⟨*proof*⟩

lemma *dens-to-dist uniform-f* = *Uniform 0 1* $\bigotimes_{Qmes}$ *Uniform 0 1*
⟨*proof*⟩

5.2.9 If then else

definition *gt* :: (*real* $\Rightarrow$ *real*) $\Rightarrow$ *real* $\Rightarrow$ *bool qbs-measure* **where**
gt $\equiv$ (λf *r*. *do* {
x $\leftarrow$ *dens-to-dist* (*normal-f 0 1*);
if f x > *r*
then return-qbs $\mathbb{B}_Q$ *True*
else return-qbs $\mathbb{B}_Q$ *False*
})

declare *normal-f*(*1*)[*of 1 0*,*simplified*]

lemma *gt-qbs*[*qbs*]: *gt* $\in$ *qbs-space* (($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) $\Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q$ *monadP-qbs* $\mathbb{B}_Q$)
⟨*proof*⟩

lemma
assumes [*qbs*]: *f* $\in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$
shows $\mathcal{P}$(*b in gt f r*. *b* = *True*) = $\mathcal{P}$(*x in std-normal-distribution*. *f x* > *r*) (**is** *?P1* = *?P2*)
⟨*proof*⟩

Examples from Staton [5, Sect. 2.2].

5.2.10 Weekend

Example from Staton [5, Sect. 2.2.1].

This example is formalized in Coq by Affeldt et al. [1].

definition *weekend* :: *bool qbs-measure* **where**
weekend ≡ *do* {
    let x = *qbs-bernoulli* (*2* / *7*);
        f = (λx. *let* r = *if* x *then 3 else 10 in pmf* (*poisson-pmf* r) *4*)
    in query x f
    }

lemma *weekend-qbs*[*qbs*]:*weekend* ∈ *qbs-space* (*monadM-qbs* $\mathbb{B}_Q$)
⟨*proof*⟩

lemma *weekend-nc*:
defines N ≡ *2* / *7* ∗ *pmf* (*poisson-pmf 3*) *4* + *5* / *7* ∗ *pmf* (*poisson-pmf 10*) *4*
shows *qbs-l* (*density-qbs* (*bernoulli-pmf* (*2*/*7*)) (λx. (*pmf* (*poisson-pmf* (*if* x *then 3 else 10*)) *4*))) *UNIV* = N
⟨*proof*⟩

lemma *qbs-l-weekend*:
defines N ≡ *2* / *7* ∗ *pmf* (*poisson-pmf 3*) *4* + *5* / *7* ∗ *pmf* (*poisson-pmf 10*) *4*
shows *qbs-l weekend* = *qbs-l* (*density-qbs* (*qbs-bernoulli* (*2* / *7*)) (λx. *ennreal* (*let* r = *if* x *then 3 else 10 in* r ^ *4* ∗ *exp* (− r) / (*fact 4* ∗ N)))) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma
defines N ≡ *2* / *7* ∗ *pmf* (*poisson-pmf 3*) *4* + *5* / *7* ∗ *pmf* (*poisson-pmf 10*) *4*
shows $\mathcal{P}$(*b in weekend*. *b* = *True*) = *2* / *7* ∗ (*3*^*4* ∗ *exp* (− *3*)) / *fact 4* ∗ *1* / N
⟨*proof*⟩

5.2.11 Whattime

Example from Staton [5, Sect. 2.2.3]

f is given as a parameter.

definition *whattime* :: (*real* ⇒ *real*) ⇒ *real qbs-measure* **where**
whattime ≡ (λf. *do* {
    let T = *Uniform 0 24 in*
    query T (λt. *let* r = f t *in*
        exponential-density r (*1* / *60*))
    })

lemma *whattime-qbs*[*qbs*]: *whattime* ∈ ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) $\Rightarrow_Q$ *monadM-qbs* $\mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-l-whattime-sub*:
assumes [*qbs*]: $f \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$
shows *qbs-l* (*density-qbs* (*Uniform 0 24*) ($\lambda x.$ *exponential-density* ($f\,x$) (*1 / 60*))) = *density lborel* ($\lambda x.$ *indicator* {*0<..<24*} *x / 24* $*$ *exponential-density* ($f\,x$) (*1 / 60*))
⟨*proof*⟩

lemma
assumes [*qbs*]: $f \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ **and** [*measurable*]: $U \in$ *sets borel*
and $\bigwedge r.\ f\ r \geq 0$
defines $N \equiv (\int t{\in}\{0{<}..{<}24\}.\ (f\ t * exp\ (-\ 1/\ 60 * f\ t))\ \partial lborel)$
defines $N' \equiv (\int^{+} t{\in}\{0{<}..{<}24\}.\ (f\ t * exp\ (-\ 1/\ 60 * f\ t))\ \partial lborel)$
assumes $N' \neq 0$ **and** $N' \neq \infty$
shows $\mathcal{P}(t\ in\ whattime\ f.\ t \in U) = (\int t{\in}\{0{<}..{<}24\}{\cap}U.\ (f\ t * exp\ (-\ 1/\ 60 * f\ t))\ \partial lborel)\ /\ N$
⟨*proof*⟩

5.2.12 Distributions on Functions

definition *a-times-x* :: (*real* ⇒ *real*) *qbs-measure* **where**
a-times-x ≡ *do* {
  a ← *Uniform* (−*2*) *2*;
  return-qbs ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) ($\lambda x.\ a * x$)
}

lemma *a-times-x-qbs*[*qbs*]: *a-times-x* ∈ *monadM-qbs* ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$)
⟨*proof*⟩

lemma *a-times-x-qbsP*: *a-times-x* ∈ *monadP-qbs* ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$)
⟨*proof*⟩

definition *a-times-x′* :: (*real* ⇒ *real*) *qbs-measure* **where**
a-times-x′ ≡ *do* {
  condition a-times-x ($\lambda f.\ f\ 1 \geq 0$)
}

lemma *a-times-x′-qbs*[*qbs*]: *a-times-x′* ∈ *monadM-qbs* ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$)
⟨*proof*⟩

lemma *prob-a-times-x*:
assumes [*measurable*]: *Measurable.pred borel P*
shows $\mathcal{P}(f\ in\ a\text{-}times\text{-}x.\ P\ (f\ r)) = \mathcal{P}(a\ in\ Uniform\ (-2)\ 2.\ P\ (a * r))$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma $\mathcal{P}(f\ in\ a\text{-}times\text{-}x'.\ f\ 1 \geq 1) = 1\ /\ 2$ (**is** *?P* = -)
⟨*proof*⟩

Almost everywhere, integrable, and integrations are also interpreted as programs.

lemma $(\lambda g\ f\ x.\ \mathit{if}\ (AE_Q\ y\ \mathit{in}\ g\ x.\ f\ x\ y \neq \infty)\ \mathit{then}\ (\int^{+}{}_Q\ y.\ f\ x\ y\ \partial(g\ x))\ \mathit{else}\ 0)$
$\in (\mathbb{R}_Q \Rightarrow_Q \mathit{monadM\text{-}qbs}\ \mathbb{R}_Q) \Rightarrow_Q (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_{Q\geq 0}) \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_{Q\geq 0}$
$\langle \mathit{proof} \rangle$

lemma $(\lambda g\ f\ x.\ \mathit{if}\ \mathit{qbs\text{-}integrable}\ (g\ x)\ (f\ x)\ \mathit{then}\ \mathit{Some}\ (\int{}_Q\ y.\ f\ x\ y\ \partial(g\ x))\ \mathit{else}\ \mathit{None})$
$\in (\mathbb{R}_Q \Rightarrow_Q \mathit{monadM\text{-}qbs}\ \mathbb{R}_Q) \Rightarrow_Q (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \Rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathit{option\text{-}qbs}\ \mathbb{R}_Q$
$\langle \mathit{proof} \rangle$

end

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