

The Rogers–Ramanujan Identities

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Abstract

This entry formalises the Rogers–Ramanujan Identities:

$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2}}{\prod_{j=1}^k (1 - q^j)} = \left(\prod_{n=0}^{\infty} (1 - q^{1+5n})(1 - q^{4+5n}) \right)^{-1}$$
$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2+k}}{\prod_{j=1}^k (1 - q^j)} = \left(\prod_{n=0}^{\infty} (1 - q^{2+5n})(1 - q^{3+5n}) \right)^{-1}$$

The formalisation follows the elegant proof given in Andrews and Eriksson *Integer Partitions*, using the Jacobi triple product.

1 The Rogers–Ramanujan identities

```
theory Rogers_Ramanujan
  imports "Theta_Functions.Jacobi_Triple_Product"
begin
```

Acknowledgement: I would like to thank George Andrews for giving me a crucial hint about a uniform convergence issue that I struggled with.

```
unbundle qpochhammer_inf_notation
```

First of all, we show two auxiliary results concerned with the (absolute) convergence of two infinite sums that will appear in our proof of the identities.

```
lemma summable_rogers_ramanujan_aux1:
  fixes q :: "'a :: {real_normed_field, banach}" and M :: int
  assumes q: "q ≠ 0" "norm q < 1"
  shows "(λj. norm q powi (j*(5*j+M) div 2)) summable_on UNIV"
⟨proof⟩
```

```
lemma summable_rogers_ramanujan_aux2:
  fixes q :: "'a :: {real_normed_field, banach}" and M :: int
  assumes q: "q ≠ 0" "norm q < 1"
  shows "summable (λj. norm (q ^ (j^2 + c * j) / qpochhammer (int j) q
q))"
⟨proof⟩
```

Next, we apply the Jacobi triple product to show that for $N \in \{0, 1\}$ we have

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+2N+1)/2} = \frac{(q; q)_{\infty}}{\prod_{i \in I} (q^i; q^5)_{\infty}}$$

where $I = \{1, \dots, 4\} \setminus \{2 - N, 3 + N\}$.

```
lemma rogers_ramanujan_aux:
  fixes q :: complex and N :: nat
  assumes q: "norm q < 1" and N: "N < 2"
  shows "((λn. (-1) powi n * q powi (n*(5*n+2*N+1) div 2)) has_sum
(q; q)_{\infty} / ((q^{1+N}; q^5)_{\infty} * (q^{4-N}; q^5)_{\infty})) UNIV"
⟨proof⟩
```

```
theorem rogers_ramanujan_complex:
  fixes q :: complex
  assumes "norm q < 1"
  shows "((λj. q ^ (j^2) / qpochhammer j q q) has_sum (1 / ((q; q^5)_{\infty}
* (q^4; q^5)_{\infty}))) UNIV"
  and "((λj. q ^ (j^2 + j) / qpochhammer j q q) has_sum (1 / ((q^2; q^5)_{\infty}
* (q^3; q^5)_{\infty}))) UNIV"
⟨proof⟩
```

```
lemma rogers_ramanujan_real:
```

```

    fixes q :: real
    assumes "|q| < 1"
    shows "( $(\lambda j. q^{j^2}) / \text{qpochhammer } j \text{ } q \text{ } q$ ) has\_sum (1 /  $((q; q^5)_\infty$ 
*  $(q^4; q^5)_\infty$ ))) UNIV"
    and "( $(\lambda j. q^{j^2 + j}) / \text{qpochhammer } j \text{ } q \text{ } q$ ) has\_sum (1 /  $((q^2; q^5)_\infty$ 
*  $(q^3; q^5)_\infty$ ))) UNIV"
  <proof>

unbundle no_qpochhammer_inf_notation

end

```

References

- [1] G. Andrews and K. Eriksson. *Integer Partitions*. Cambridge University Press, 2004.