

# Reducing Rewrite Properties to Properties on Ground Terms\*

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## Abstract

This AFP entry relates important rewriting properties between the set of terms and the set of ground terms induced by a given signature. The properties considered are confluence, strong/local confluence, the normal form property, unique normal forms with respect to reduction and conversion, commutation, conversion equivalence, and normalization equivalence.

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# 1 Introduction

Rewriting is an abstract model of computation. Among other things, it studies important properties including the following:

CR:	$\forall s \forall t \forall u (s \rightarrow^* t \wedge s \rightarrow^* u \implies t \downarrow u)$	confluence
SCR:	$\forall s \forall t \forall u (s \rightarrow t \wedge s \rightarrow u \implies \exists v (t \rightarrow^= v \wedge u \rightarrow^* v))$	strong confluence
WCR:	$\forall s \forall t \forall u (s \rightarrow t \wedge s \rightarrow u \implies t \downarrow u)$	local confluence
NFP:	$\forall s \forall t \forall u (s \rightarrow^* t \wedge s \rightarrow^! u \implies t \rightarrow^! u)$	normal form property
UNR:	$\forall s \forall t \forall u (s \rightarrow^! t \wedge s \rightarrow^! u \implies t = u)$	unique normal forms with respect to reduction
UNC:	$\forall t \forall u (t \leftrightarrow^* u \wedge \text{NF}(t) \wedge \text{NF}(u) \implies t = u)$	unique normal forms with respect to conversion

We also consider the following properties involving two TRSs  $\mathcal{R}$  and  $\mathcal{S}$ :

COM:	$\forall s \forall t \forall u (s \rightarrow_{\mathcal{R}}^* t \wedge s \rightarrow_{\mathcal{S}}^* u \implies \exists v (t \rightarrow_{\mathcal{S}}^* v \wedge u \rightarrow_{\mathcal{R}}^* v))$	commutation
CE:	$\forall s \forall t (s \leftrightarrow_{\mathcal{R}}^* t \iff s \leftrightarrow_{\mathcal{S}}^* t)$	conversion equivalence
NE:	$\forall s \forall t (s \rightarrow_{\mathcal{R}}^! t \iff s \rightarrow_{\mathcal{S}}^! t)$	normalization equivalence

An interesting observation is that for each of these properties there exists a rewrite system that satisfies the property when restricted to ground terms but not when arbitrary terms are allowed. Consider the left-linear right-ground TRS  $\mathcal{R}$  consisting of the rules

$$a \rightarrow b \qquad f(a, x) \rightarrow b \qquad f(b, b) \rightarrow b$$

over the signature  $\mathcal{F} = \{a, b, f\}$ . It is ground-confluent because every ground term in  $\mathcal{T}(\mathcal{F})$  rewrites to  $b$ . Confluence does not hold; the term  $f(a, x)$  rewrites to the different normal forms  $b$  and  $f(b, x)$ .

In this AFP entry, properties on arbitrary terms are reduced to the corresponding properties on ground terms, for left-linear right-ground rewrite systems and for linear variable-separated systems. To do this, I formalized fundamental term rewriting operations that include the root step and the one step rewriting relations. Also, I added definitions for conversion equivalence, normalization equivalence, strong confluence and the normal form property extending the list of important rewriting properties of the AFP entry ‘‘Abstract Rewriting’’ [1].

Rewrite sequences that contain a root step play an important role in the formalization. The table of contents should give the reader a good overview of the content of this entry.

## 2 Preliminaries

**theory** *Terms-Positions*  
**imports** *Regular-Tree-Relations.Ground-Terms*  
**begin**

### 2.1 Additional operations on terms and positions

#### 2.1.1 Linearity

**fun** *linear-term* :: ('f, 'v) term  $\Rightarrow$  bool **where**  
*linear-term* (Var -) = True |  
*linear-term* (Fun f ts) = (is-partition (map vars-term ts)  $\wedge$  ( $\forall t \in \text{set } ts. \text{linear-term } t$ ))  
**abbreviation** *linear-sys*  $\mathcal{R} \equiv \forall (l, r) \in \mathcal{R}. \text{linear-term } l \wedge \text{linear-term } r$

#### 2.1.2 Positions induced by contexts, by variables and by given subterms

**definition** *possc*  $C = \{p \mid p \ t. p \in \text{poss } C \langle t \rangle\}$   
**definition** *varposs*  $s = \{p \mid p. p \in \text{poss } s \wedge \text{is-Var } (s \mid - p)\}$   
**definition** *poss-of-term*  $u \ t = \{p. p \in \text{poss } t \wedge t \mid - p = u\}$

#### 2.1.3 Replacing functions symbols that aren't specified in the signature by variables

**definition** *funas-rel*  $\mathcal{R} = (\bigcup (l, r) \in \mathcal{R}. \text{funas-term } l \cup \text{funas-term } r)$

**fun** *term-to-sig* **where**  
*term-to-sig*  $\mathcal{F} \ v \ (\text{Var } x) = \text{Var } x$   
| *term-to-sig*  $\mathcal{F} \ v \ (\text{Fun } f \ ts) =$   
    (if ( $f, \text{length } ts \in \mathcal{F}$ ) then  $\text{Fun } f \ (\text{map } (\text{term-to-sig } \mathcal{F} \ v) \ ts)$  else  $\text{Var } v$ )

**fun** *ctxt-well-def-hole-path* **where**  
*ctxt-well-def-hole-path*  $\mathcal{F} \ \text{Hole} \ \longleftrightarrow \ \text{True}$   
| *ctxt-well-def-hole-path*  $\mathcal{F} \ (\text{More } f \ ss \ C \ ts) \ \longleftrightarrow \ (f, \text{Suc } (\text{length } ss + \text{length } ts)) \in \mathcal{F} \wedge \text{ctxt-well-def-hole-path } \mathcal{F} \ C$

**fun** *inv-const-ctxt* **where**  
*inv-const-ctxt*  $\mathcal{F} \ v \ \text{Hole} = \text{Hole}$   
| *inv-const-ctxt*  $\mathcal{F} \ v \ ((\text{More } f \ ss \ C \ ts))$   
    = ( $\text{More } f \ (\text{map } (\text{term-to-sig } \mathcal{F} \ v) \ ss) \ (\text{inv-const-ctxt } \mathcal{F} \ v \ C) \ (\text{map } (\text{term-to-sig } \mathcal{F} \ v) \ ts)$ )

**fun** *inv-const-ctxt'* **where**  
*inv-const-ctxt'*  $\mathcal{F} \ v \ \text{Hole} = \text{Var } v$   
| *inv-const-ctxt'*  $\mathcal{F} \ v \ ((\text{More } f \ ss \ C \ ts))$   
    = (if ( $f, \text{Suc } (\text{length } ss + \text{length } ts) \in \mathcal{F}$ ) then  $\text{Fun } f \ (\text{map } (\text{term-to-sig } \mathcal{F} \ v) \ ss \ @ \ \text{inv-const-ctxt}' \ \mathcal{F} \ v \ C \ \# \ \text{map } (\text{term-to-sig } \mathcal{F} \ v) \ ts)$  else  $\text{Var } v$ )

### 2.1.4 Replace term at a given position in contexts

**fun** *replace-term-context-at* :: ('f, 'v) *ctxt* ⇒ *pos* ⇒ ('f, 'v) *term* ⇒ ('f, 'v) *ctxt*  
 (-[- ← -]<sub>C</sub> [1000, 0] 1000) **where**  
*replace-term-context-at* □ *p u* = □  
 | *replace-term-context-at* (More *f ss C ts*) (*i # ps*) *u* =  
   (if *i* < length *ss* then More *f (ss[i := (ss ! i)[ps ← u]]) C ts*  
   else if *i* = length *ss* then More *f ss (replace-term-context-at C ps u) ts*  
   else More *f ss C (ts[(i - Suc (length ss)) := (ts ! (i - Suc (length ss)))[ps ← u]])*)

**abbreviation** *constT c* ≡ *Fun c* []

### 2.1.5 Multihole context closure of a term relation as inductive set

**definition** *all-ctxt-closed* **where**

*all-ctxt-closed F r* ⇔ (∀ *f ts ss. (f, length ss) ∈ F* → *length ts* = *length ss* →  
 (∀ *i. i* < *length ts* → (*ts ! i, ss ! i*) ∈ *r*) →  
 (∀ *i. i* < *length ts* → *funas-term (ts ! i) ∪ funas-term (ss ! i) ⊆ F*) → (*Fun f ts, Fun f ss*) ∈ *r*) ∧  
 (∀ *x. (Var x, Var x) ∈ r*)

## 2.2 Destruction and introduction of *all-ctxt-closed*

**lemma** *all-ctxt-closedD*: *all-ctxt-closed F r* ⇒ (*f, length ss*) ∈ *F* ⇒ *length ts* = *length ss*

⇒ [∧ *i. i* < *length ts* ⇒ (*ts ! i, ss ! i*) ∈ *r* ]  
 ⇒ [∧ *i. i* < *length ts* ⇒ *funas-term (ts ! i) ⊆ F* ]  
 ⇒ [∧ *i. i* < *length ts* ⇒ *funas-term (ss ! i) ⊆ F* ]  
 ⇒ (*Fun f ts, Fun f ss*) ∈ *r*  
 ⟨*proof*⟩

**lemma** *trans-ctxt-sig-imp-all-ctxt-closed*: **assumes** *tran*: *trans r*

**and** *refl*: ∩ *t. funas-term t ⊆ F* ⇒ (*t, t*) ∈ *r*

**and** *ctxt*: ∩ *C s t. funas-ctxt C ⊆ F* ⇒ *funas-term s ⊆ F* ⇒ *funas-term t ⊆ F* ⇒ (*s, t*) ∈ *r* ⇒ (*C < s*), *C < t*) ∈ *r*

**shows** *all-ctxt-closed F r* ⟨*proof*⟩

## 2.3 Lemmas for *poss* and ordering of positions

**lemma** *subst-poss-mono*: *poss s ⊆ poss (s · σ)*

⟨*proof*⟩

**lemma** *par-pos-prefix* [*simp*]:

(*i # p*) ⊥ (*i # q*) ⇒ *p* ⊥ *q*

⟨*proof*⟩

**lemma** *pos-diff-itself* [*simp*]: *p* −<sub>*p*</sub> *p* = []

⟨*proof*⟩

**lemma** *pos-les-eq-append-diff* [simp]:

$$p \leq_p q \implies p @ (q -_p p) = q$$

⟨proof⟩

**lemma** *pos-diff-append-itself* [simp]:  $(p @ q) -_p p = q$

⟨proof⟩

**lemma** *poss-pos-diffI*:

$$p \leq_p q \implies q \in \text{poss } s \implies q -_p p \in \text{poss } (s \mid\!-\! p)$$

⟨proof⟩

**lemma** *less-eq-poss-append-itself* [simp]:  $p \leq_p (p @ q)$

⟨proof⟩

**lemma** *poss-ctxt-apply* [simp]:

$$\text{hole-pos } C @ p \in \text{poss } C \langle s \rangle \longleftrightarrow p \in \text{poss } s$$

⟨proof⟩

**lemma** *pos-replace-at-pres*:

$$p \in \text{poss } s \implies p \in \text{poss } s[p \leftarrow t]$$

⟨proof⟩

**lemma** *par-pos-replace-pres*:

$$p \in \text{poss } s \implies p \perp q \implies p \in \text{poss } s[q \leftarrow t]$$

⟨proof⟩

**lemma** *poss-of-termE* [elim]:

**assumes**  $p \in \text{poss-of-term } u \ s$

**and**  $p \in \text{poss } s \implies s \mid\!-\! p = u \implies P$

**shows**  $P$  ⟨proof⟩

**lemma** *poss-of-term-Cons*:

$$i \# p \in \text{poss-of-term } u \ (Fun \ f \ ts) \implies p \in \text{poss-of-term } u \ (ts ! i)$$

⟨proof⟩

**lemma** *poss-of-term-const-ctxt-apply*:

**assumes**  $p \in \text{poss-of-term } (constT \ c) \ C \langle s \rangle$

**shows**  $p \perp (\text{hole-pos } C) \vee (\text{hole-pos } C) \leq_p p$  ⟨proof⟩

## 2.4 Lemmas for $(\mid\!-\!)$ and *replace-term-at*

**lemma** *subt-at-append-dist*:

$$p @ q \in \text{poss } s \implies s \mid\!-\! (p @ q) = (s \mid\!-\! p) \mid\!-\! q$$

⟨proof⟩

**lemma** *ctxt-apply-term-subt-at-hole-pos* [simp]:

$$C \langle s \rangle \mid\!-\! (\text{hole-pos } C @ q) = s \mid\!-\! q$$

⟨proof⟩

**lemma** *subst-subt-at-dist*:

$$p \in \text{poss } s \implies s \cdot \sigma \mid\!-\! p = s \mid\!-\! p \cdot \sigma$$

*<proof>*

**lemma** *replace-term-at-subt-at-id* [simp]:  $s[p \leftarrow (s \mid\!-\! p)] = s$

*<proof>*

**lemma** *replace-term-at-same-pos* [simp]:

$$s[p \leftarrow u][p \leftarrow t] = s[p \leftarrow t]$$

*<proof>*

**lemma** *subt-at-vars-term*:

$$p \in \text{poss } s \implies s \mid\!-\! p = \text{Var } x \implies x \in \text{vars-term } s$$

*<proof>*

**lemma** *linear-term-varposs-subst-replace-term*:

$$\begin{aligned} & \text{linear-term } s \implies p \in \text{varposs } s \implies p \leq_p q \implies \\ & (s \cdot \sigma)[q \leftarrow u] = s \cdot (\lambda x. \text{if } \text{Var } x = s \mid\!-\! p \text{ then } (\sigma x)[q \!-\!_p p \leftarrow u] \text{ else } (\sigma x)) \end{aligned}$$

*<proof>*

**lemma** *par-hole-pos-replace-term-context-at*:

$$p \perp \text{hole-pos } C \implies C\langle s \rangle[p \leftarrow u] = (C[p \leftarrow u]_C)\langle s \rangle$$

*<proof>*

**lemma** *par-pos-replace-term-at*:

$$p \in \text{poss } s \implies p \perp q \implies s[q \leftarrow t] \mid\!-\! p = s \mid\!-\! p$$

*<proof>*

**lemma** *less-eq-subt-at-replace*:

$$p \in \text{poss } s \implies p \leq_p q \implies s[q \leftarrow t] \mid\!-\! p = (s \mid\!-\! p)[q \!-\!_p p \leftarrow t]$$

*<proof>*

**lemma** *greater-eq-subt-at-replace*:

$$p \in \text{poss } s \implies q \leq_p p \implies s[q \leftarrow t] \mid\!-\! p = t \mid\!-\! (p \!-\!_p q)$$

*<proof>*

**lemma** *replace-subterm-at-itself* [simp]:

$$s[p \leftarrow (s \mid\!-\! p)[q \leftarrow t]] = s[p \text{ @ } q \leftarrow t]$$

*<proof>*

**lemma** *hole-pos-replace-term-at* [simp]:

$$\text{hole-pos } C \leq_p p \implies C\langle s \rangle[p \leftarrow u] = C\langle s[p \!-\!_p \text{hole-pos } C \leftarrow u] \rangle$$

*<proof>*

**lemma** *ctxt-of-pos-term-apply-replace-at-ident*:

$$\text{assumes } p \in \text{poss } s$$

**shows**  $(\text{ctxt-at-pos } s \ p)\langle t \rangle = s[p \leftarrow t]$   
 $\langle \text{proof} \rangle$

**lemma** *ctxt-apply-term-replace-term-hole-pos* [simp]:  
 $C\langle s \rangle[\text{hole-pos } C \ @ \ q \leftarrow u] = C\langle s[q \leftarrow u] \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *ctxt-apply-subst-at-hole-pos* [simp]:  $C\langle s \rangle \mid\text{- hole-pos } C = s$   
 $\langle \text{proof} \rangle$

**lemma** *subst-at-imp-supteq'*:  
**assumes**  $p \in \text{poss } s$  **and**  $s \mid\text{-} p = t$  **shows**  $s \supseteq t$   $\langle \text{proof} \rangle$

**lemma** *subst-at-imp-supteq*:  
**assumes**  $p \in \text{poss } s$  **shows**  $s \supseteq s \mid\text{-} p$   
 $\langle \text{proof} \rangle$

## 2.5 term-to-sig invariants and distributions

**lemma** *funas-term-term-to-sig* [simp]: *funas-term* (*term-to-sig*  $\mathcal{F} \ v \ t$ )  $\subseteq \mathcal{F}$   
 $\langle \text{proof} \rangle$

**lemma** *term-to-sig-id* [simp]:  
*funas-term*  $t \subseteq \mathcal{F} \implies \text{term-to-sig } \mathcal{F} \ v \ t = t$   
 $\langle \text{proof} \rangle$

**lemma** *term-to-sig-subst-sig* [simp]:  
*funas-term*  $t \subseteq \mathcal{F} \implies \text{term-to-sig } \mathcal{F} \ v \ (t \cdot \sigma) = t \cdot (\lambda x. \text{term-to-sig } \mathcal{F} \ v \ (\sigma \ x))$   
 $\langle \text{proof} \rangle$

**lemma** *funas-ctxt-ctxt-inv-const-ctxt-ind* [simp]:  
*funas-ctxt*  $C \subseteq \mathcal{F} \implies \text{inv-const-ctxt } \mathcal{F} \ v \ C = C$   
 $\langle \text{proof} \rangle$

**lemma** *term-to-sig-ctxt-apply* [simp]:  
 $\text{ctxt-well-def-hole-path } \mathcal{F} \ C \implies \text{term-to-sig } \mathcal{F} \ v \ C\langle s \rangle = (\text{inv-const-ctxt } \mathcal{F} \ v \ C)\langle \text{term-to-sig } \mathcal{F} \ v \ s \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *term-to-sig-ctxt-apply'* [simp]:  
 $\neg \text{ctxt-well-def-hole-path } \mathcal{F} \ C \implies \text{term-to-sig } \mathcal{F} \ v \ C\langle s \rangle = \text{inv-const-ctxt}' \mathcal{F} \ v \ C$   
 $\langle \text{proof} \rangle$

**lemma** *funas-ctxt-ctxt-well-def-hole-path*:  
*funas-ctxt*  $C \subseteq \mathcal{F} \implies \text{ctxt-well-def-hole-path } \mathcal{F} \ C$   
 $\langle \text{proof} \rangle$

## 2.6 Misc

**lemma** *funas-term-subst-at*:

$(f, n) \in \text{funas-term } t \implies (\exists p \text{ ts. } p \in \text{poss } t \wedge t \mid\text{- } p = \text{Fun } f \text{ ts} \wedge \text{length } \text{ts} = n)$   
 $\langle \text{proof} \rangle$

**lemma** *finite-poss*: *finite* (*poss* *s*)  
 $\langle \text{proof} \rangle$

**lemma** *finite-varposs*: *finite* (*varposs* *s*)  
 $\langle \text{proof} \rangle$

**lemma** *ground-linear* [*simp*]: *ground* *t*  $\implies$  *linear-term* *t*  
 $\langle \text{proof} \rangle$

**declare** *ground-substI* [*intro*, *simp*]

**lemma** *ground-ctxt-substI*:  
 $(\bigwedge x. x \in \text{vars-ctxt } C \implies \text{ground } (\sigma x)) \implies \text{ground-ctxt } (C \cdot_c \sigma)$   
 $\langle \text{proof} \rangle$

**lemma** *funas-ctxt-subst-apply-ctxt*:  
 $\text{funas-ctxt } (C \cdot_c \sigma) = \text{funas-ctxt } C \cup (\bigcup (\text{funas-term } ' \sigma ' \text{vars-ctxt } C))$   
 $\langle \text{proof} \rangle$

**lemma** *varposs-Var* [*simp*]:  
 $\text{varposs } (\text{Var } x) = \{\}\}$   
 $\langle \text{proof} \rangle$

**lemma** *varposs-Fun* [*simp*]:  
 $\text{varposs } (\text{Fun } f \text{ ts}) = \{ i \# p \mid i p. i < \text{length } \text{ts} \wedge p \in \text{varposs } (\text{ts } ! i) \}$   
 $\langle \text{proof} \rangle$

**lemma** *vars-term-varposs-iff*:  
 $x \in \text{vars-term } s \iff (\exists p \in \text{varposs } s. s \mid\text{- } p = \text{Var } x)$   
 $\langle \text{proof} \rangle$

**lemma** *vars-term-empty-ground*:  
 $\text{vars-term } s = \{\} \implies \text{ground } s$   
 $\langle \text{proof} \rangle$

**lemma** *ground-subst-apply*: *ground* *t*  $\implies t \cdot \sigma = t$   
 $\langle \text{proof} \rangle$

**lemma** *varposs-imp-poss*:  
 $p \in \text{varposs } s \implies p \in \text{poss } s$   $\langle \text{proof} \rangle$

**lemma** *varposs-empty-ground*:  
 $\text{varposs } s = \{\} \iff \text{ground } s$   
 $\langle \text{proof} \rangle$

**lemma** *funas-term-subterm-atI* [*intro*]:

$p \in \text{poss } s \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } (s \mid\!-\! p) \subseteq \mathcal{F}$   
 ⟨proof⟩

**lemma** *varposs-ground-replace-at*:

$p \in \text{varposs } s \implies \text{ground } u \implies \text{varposs } s[p \leftarrow u] = \text{varposs } s - \{p\}$   
 ⟨proof⟩

**lemma** *funas-term-replace-at-upper*:

$\text{funas-term } s[p \leftarrow t] \subseteq \text{funas-term } s \cup \text{funas-term } t$   
 ⟨proof⟩

**lemma** *funas-term-replace-at-lower*:

$p \in \text{poss } s \implies \text{funas-term } t \subseteq \text{funas-term } (s[p \leftarrow t])$   
 ⟨proof⟩

**lemma** *poss-of-term-possI* [intro!]:

$p \in \text{poss } s \implies s \mid\!-\! p = u \implies p \in \text{poss-of-term } u \ s$   
 ⟨proof⟩

**lemma** *poss-of-term-replace-term-at*:

$p \in \text{poss } s \implies p \in \text{poss-of-term } u \ s[p \leftarrow u]$   
 ⟨proof⟩

**lemma** *constT-nfunas-term-poss-of-term-empty*:

$(c, 0) \notin \text{funas-term } t \iff \text{poss-of-term } (\text{constT } c) \ t = \{\}$   
 ⟨proof⟩

**lemma** *poss-of-term-poss-emptyD*:

**assumes**  $\text{poss-of-term } u \ s = \{\}$   
**shows**  $p \in \text{poss } s \implies s \mid\!-\! p \neq u$  ⟨proof⟩

**lemma** *possc-subt-at-ctxt-apply*:

$p \in \text{possc } C \implies p \perp \text{hole-pos } C \implies C\langle s \rangle \mid\!-\! p = C\langle t \rangle \mid\!-\! p$   
 ⟨proof⟩

end

## 3 Rewriting

**theory** *Rewriting*

**imports** *Terms-Positions*

**begin**

### 3.1 Basic rewrite definitions

#### 3.1.1 Rewrite steps with implicit signature declaration (encoded in the type)

**inductive-set** *rrstep* ::  $(\iota, \iota) \text{ term rel} \Rightarrow (\iota, \iota) \text{ term rel}$  for  $\mathcal{R}$  where

[intro]:  $(l, r) \in \mathcal{R} \implies (l \cdot \sigma, r \cdot \sigma) \in rstep \mathcal{R}$

**inductive-set**  $rstep :: ('f, 'v) \text{ term rel} \Rightarrow ('f, 'v) \text{ term rel}$  for  $\mathcal{R}$  where  
 $(s, t) \in rstep \mathcal{R} \implies (C\langle s \rangle, C\langle t \rangle) \in rstep \mathcal{R}$

### 3.1.2 Restrict relations to terms induced by a given signature

**definition**  $sig\text{-step } \mathcal{F} \mathcal{R} = Restr \mathcal{R} (Collect (\lambda s. funas\text{-term } s \subseteq \mathcal{F}))$

### 3.1.3 Rewriting under a given signature/restricted to ground terms

**abbreviation**  $srrstep \mathcal{F} \mathcal{R} \equiv sig\text{-step } \mathcal{F} (rstep \mathcal{R})$

**abbreviation**  $srstep \mathcal{F} \mathcal{R} \equiv sig\text{-step } \mathcal{F} (rstep \mathcal{R})$

**abbreviation**  $gsrstep \mathcal{F} \mathcal{R} \equiv Restr (sig\text{-step } \mathcal{F} (rstep \mathcal{R})) (Collect \text{ground})$

### 3.1.4 Rewriting sequences involving a root step

**abbreviation** (input)  $relto :: 'a \text{ rel} \Rightarrow 'a \text{ rel} \Rightarrow 'a \text{ rel}$  where  
 $relto R S \equiv S^* \circ R \circ S^*$

**definition**  $srsteps\text{-with-root-step } \mathcal{F} \mathcal{R} \equiv relto (sig\text{-step } \mathcal{F} (rstep \mathcal{R})) (srstep \mathcal{F} \mathcal{R})$

## 3.2 Monotonicity laws

**lemma**  $Restr\text{-mono}: Restr r A \subseteq r \langle \text{proof} \rangle$

**lemma**  $Restr\text{-trancl-mono-set}: (Restr r A)^+ \subseteq A \times A$   
 $\langle \text{proof} \rangle$

**lemma**  $rstep\text{-rstep-mono}: rstep \mathcal{R} \subseteq rstep \mathcal{R}$   
 $\langle \text{proof} \rangle$

**lemma**  $sig\text{-step-mono}$ :  
 $\mathcal{F} \subseteq \mathcal{G} \implies sig\text{-step } \mathcal{F} \mathcal{R} \subseteq sig\text{-step } \mathcal{G} \mathcal{R}$   
 $\langle \text{proof} \rangle$

**lemma**  $sig\text{-step-mono2}$ :  
 $\mathcal{R} \subseteq \mathcal{L} \implies sig\text{-step } \mathcal{F} \mathcal{R} \subseteq sig\text{-step } \mathcal{F} \mathcal{L}$   
 $\langle \text{proof} \rangle$

**lemma**  $srrstep\text{-monp}$ :  
 $\mathcal{F} \subseteq \mathcal{G} \implies srrstep \mathcal{F} \mathcal{R} \subseteq srrstep \mathcal{G} \mathcal{R}$   
 $\langle \text{proof} \rangle$

**lemma**  $srstep\text{-monp}$ :  
 $\mathcal{F} \subseteq \mathcal{G} \implies srstep \mathcal{F} \mathcal{R} \subseteq srstep \mathcal{G} \mathcal{R}$   
 $\langle \text{proof} \rangle$

**lemma**  $srsteps\text{-monp}$ :

$\mathcal{F} \subseteq \mathcal{G} \implies (\text{srstep } \mathcal{F} \mathcal{R})^+ \subseteq (\text{srstep } \mathcal{G} \mathcal{R})^+$   
 ⟨proof⟩

**lemma** *srsteps-eq-monp*:

$\mathcal{F} \subseteq \mathcal{G} \implies (\text{srstep } \mathcal{F} \mathcal{R})^* \subseteq (\text{srstep } \mathcal{G} \mathcal{R})^*$   
 ⟨proof⟩

**lemma** *srsteps-with-root-step-sig-mono*:

$\mathcal{F} \subseteq \mathcal{G} \implies \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \subseteq \text{srsteps-with-root-step } \mathcal{G} \mathcal{R}$   
 ⟨proof⟩

### 3.3 Introduction, elimination, and destruction rules for *sig-step*, *rstep*, *rrstep*, *srrstep*, and *srstep*

**lemma** *sig-stepE* [*elim, consumes 1*]:

$(s, t) \in \text{sig-step } \mathcal{F} \mathcal{R} \implies [(s, t) \in \mathcal{R} \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } t \subseteq \mathcal{F} \implies P] \implies P$   
 ⟨proof⟩

**lemma** *sig-stepI* [*intro*]:

$\text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } t \subseteq \mathcal{F} \implies (s, t) \in \mathcal{R} \implies (s, t) \in \text{sig-step } \mathcal{F} \mathcal{R}$   
 ⟨proof⟩

**lemma** *rrstep-subst* [*elim, consumes 1*]:

**assumes**  $(s, t) \in \text{rrstep } \mathcal{R}$   
**obtains**  $l \ r \ \sigma$  **where**  $(l, r) \in \mathcal{R} \ s = l \cdot \sigma \ t = r \cdot \sigma$  ⟨proof⟩

**lemma** *rstep-imp-C-s-r*:

**assumes**  $(s, t) \in \text{rstep } \mathcal{R}$   
**shows**  $\exists C \ \sigma \ l \ r. (l, r) \in \mathcal{R} \wedge s = C \langle l \cdot \sigma \rangle \wedge t = C \langle r \cdot \sigma \rangle$  ⟨proof⟩

**lemma** *rstep-imp-C-s-r'* [*elim, consumes 1*]:

**assumes**  $(s, t) \in \text{rstep } \mathcal{R}$   
**obtains**  $C \ l \ r \ \sigma$  **where**  $(l, r) \in \mathcal{R} \ s = C \langle l \cdot \sigma \rangle \ t = C \langle r \cdot \sigma \rangle$  ⟨proof⟩

**lemma** *rrstep-basicI* [*intro*]:

$(l, r) \in \mathcal{R} \implies (l, r) \in \text{rrstep } \mathcal{R}$   
 ⟨proof⟩

**lemma** *rstep-ruleI* [*intro*]:

$(l, r) \in \mathcal{R} \implies (l, r) \in \text{rstep } \mathcal{R}$   
 ⟨proof⟩

**lemma** *rstepI* [*intro*]:

$(l, r) \in \mathcal{R} \implies s = C \langle l \cdot \sigma \rangle \implies t = C \langle r \cdot \sigma \rangle \implies (s, t) \in \text{rstep } \mathcal{R}$   
 ⟨proof⟩

**lemma** *rstep-substI* [intro]:

$$(s, t) \in \text{rstep } \mathcal{R} \implies (s \cdot \sigma, t \cdot \sigma) \in \text{rstep } \mathcal{R}$$

*<proof>*

**lemma** *rstep-ctxtI* [intro]:

$$(s, t) \in \text{rstep } \mathcal{R} \implies (C\langle s \rangle, C\langle t \rangle) \in \text{rstep } \mathcal{R}$$

*<proof>*

**lemma** *srrstepD*:

$$(s, t) \in \text{srrstep } \mathcal{F} \mathcal{R} \implies (s, t) \in \text{rrstep } \mathcal{R} \wedge \text{funas-term } s \subseteq \mathcal{F} \wedge \text{funas-term } t \subseteq \mathcal{F}$$

*<proof>*

**lemma** *srstepD*:

$$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R}) \implies (s, t) \in \text{rstep } \mathcal{R} \wedge \text{funas-term } s \subseteq \mathcal{F} \wedge \text{funas-term } t \subseteq \mathcal{F}$$

*<proof>*

**lemma** *srstepsD*:

$$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+ \implies (s, t) \in (\text{rstep } \mathcal{R})^+ \wedge \text{funas-term } s \subseteq \mathcal{F} \wedge \text{funas-term } t \subseteq \mathcal{F}$$

*<proof>*

### 3.3.1 Transitive and reflexive closure distribution over *sig-step*

**lemma** *funas-rel-converse*:

$$\text{funas-rel } \mathcal{R} \subseteq \mathcal{F} \implies \text{funas-rel } (\mathcal{R}^{-1}) \subseteq \mathcal{F} \text{ <proof>}$$

**lemma** *rstep-term-to-sig-r*:

**assumes**  $(s, t) \in \text{rstep } \mathcal{R}$  **and**  $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$  **and**  $\text{funas-term } s \subseteq \mathcal{F}$   
**shows**  $(s, \text{term-to-sig } \mathcal{F} v t) \in \text{rstep } \mathcal{R}$

*<proof>*

**lemma** *rstep-term-to-sig-l*:

**assumes**  $(s, t) \in \text{rstep } \mathcal{R}$  **and**  $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$  **and**  $\text{funas-term } t \subseteq \mathcal{F}$   
**shows**  $(\text{term-to-sig } \mathcal{F} v s, t) \in \text{rstep } \mathcal{R}$

*<proof>*

**lemma** *rstep-trancl-sig-step-r*:

**assumes**  $(s, t) \in (\text{rstep } \mathcal{R})^+$  **and**  $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$  **and**  $\text{funas-term } s \subseteq \mathcal{F}$   
**shows**  $(s, \text{term-to-sig } \mathcal{F} v t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$  *<proof>*

**lemma** *rstep-trancl-sig-step-l*:

**assumes**  $(s, t) \in (\text{rstep } \mathcal{R})^+$  **and**  $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$  **and**  $\text{funas-term } t \subseteq \mathcal{F}$   
**shows**  $(\text{term-to-sig } \mathcal{F} v s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$  *<proof>*

**lemma** *rstep-srstepI* [intro]:

$$\text{funas-rel } \mathcal{R} \subseteq \mathcal{F} \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } t \subseteq \mathcal{F} \implies (s, t) \in \text{rstep } \mathcal{R} \implies (s, t) \in \text{srstep } \mathcal{F} \mathcal{R}$$

*<proof>*

**lemma** *rsteps-srstepsI* [*intro*]:

*funas-rel*  $\mathcal{R} \subseteq \mathcal{F} \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } t \subseteq \mathcal{F} \implies (s, t) \in (\text{rstep } \mathcal{R})^+ \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$

*<proof>*

**lemma** *rsteps-eq-srsteps-eqI* [*intro*]:

*funas-rel*  $\mathcal{R} \subseteq \mathcal{F} \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{funas-term } t \subseteq \mathcal{F} \implies (s, t) \in (\text{rstep } \mathcal{R})^* \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$

*<proof>*

**lemma** *rsteps-eq-relcomp-srsteps-eq-relcompI* [*intro*]:

**assumes** *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  *funas-rel*  $\mathcal{S} \subseteq \mathcal{F}$

**and** *funas*: *funas-term*  $s \subseteq \mathcal{F}$  *funas-term*  $t \subseteq \mathcal{F}$

**and** *steps*:  $(s, t) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$

**shows**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^* \ O (\text{srstep } \mathcal{F} \mathcal{S})^*$

*<proof>*

### 3.3.2 Distributivity laws

**lemma** *rstep-smycl-dist*:

$(\text{rstep } \mathcal{R})^{\leftrightarrow} = \text{rstep } (\mathcal{R}^{\leftrightarrow})$

*<proof>*

**lemma** *sig-step-symcl-dist*:

$(\text{sig-step } \mathcal{F} \mathcal{R})^{\leftrightarrow} = \text{sig-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow})$

*<proof>*

**lemma** *srstep-symcl-dist*:

$(\text{srstep } \mathcal{F} \mathcal{R})^{\leftrightarrow} = \text{srstep } \mathcal{F} (\mathcal{R}^{\leftrightarrow})$

*<proof>*

**lemma** *Restr-smycl-dist*:

$(\text{Restr } \mathcal{R} \ \mathcal{A})^{\leftrightarrow} = \text{Restr } (\mathcal{R}^{\leftrightarrow}) \ \mathcal{A}$

*<proof>*

**lemmas** *rew-symcl-inwards* = *rstep-smycl-dist* *sig-step-symcl-dist* *srstep-symcl-dist* *Restr-smycl-dist*

**lemmas** *rew-symcl-outwards* = *rew-symcl-inwards*[*symmetric*]

**lemma** *rstep-converse-dist*:

$(\text{rstep } \mathcal{R})^{-1} = \text{rstep } (\mathcal{R}^{-1})$

*<proof>*

**lemma** *srrstep-converse-dist*:

$(\text{srrstep } \mathcal{F} \mathcal{R})^{-1} = \text{srrstep } \mathcal{F} (\mathcal{R}^{-1})$

*<proof>*

**lemma** *sig-step-converse-rstep*:  
 $(srstep \mathcal{F} \mathcal{R})^{-1} = sig\text{-step } \mathcal{F} ((rstep \mathcal{R})^{-1})$   
 ⟨proof⟩

**lemma** *srstep-converse-dist*:  
 $(srstep \mathcal{F} \mathcal{R})^{-1} = srstep \mathcal{F} (\mathcal{R}^{-1})$   
 ⟨proof⟩

**lemma** *Restr-converse*:  $(Restr \mathcal{R} A)^{-1} = Restr (\mathcal{R}^{-1}) A$   
 ⟨proof⟩

**lemmas** *rew-converse-inwards* = *rstep-converse-dist* *srrstep-converse-dist* *sig-step-converse-rstep*  
*srstep-converse-dist* *Restr-converse* *tranc1-converse*[*symmetric*] *rtranc1-converse*[*symmetric*]

**lemmas** *rew-converse-outwards* = *rew-converse-inwards*[*symmetric*]

**lemma** *sig-step-rsteps-dist*:  
 $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \implies sig\text{-step } \mathcal{F} ((rstep \mathcal{R})^+) = (srstep \mathcal{F} \mathcal{R})^+$   
 ⟨proof⟩

**lemma** *sig-step-rsteps-eq-dist*:  
 $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \implies sig\text{-step } \mathcal{F} ((rstep \mathcal{R})^+) \cup Id = (srstep \mathcal{F} \mathcal{R})^*$   
 ⟨proof⟩

**lemma** *sig-step-conversion-dist*:  
 $(srstep \mathcal{F} \mathcal{R})^{\leftrightarrow*} = (srstep \mathcal{F} (\mathcal{R}^{\leftrightarrow}))^*$   
 ⟨proof⟩

**lemma** *gsrstep-conversion-dist*:  
 $(gsrstep \mathcal{F} \mathcal{R})^{\leftrightarrow*} = (gsrstep \mathcal{F} (\mathcal{R}^{\leftrightarrow}))^*$   
 ⟨proof⟩

**lemma** *sig-step-grstep-dist*:  
 $gsrstep \mathcal{F} \mathcal{R} = sig\text{-step } \mathcal{F} (Restr (rstep \mathcal{R}) (Collect\ ground))$   
 ⟨proof⟩

### 3.4 Substitution closure of *srstep*

**lemma** *srstep-subst-closed*:  
 assumes  $(s, t) \in srstep \mathcal{F} \mathcal{R} \wedge x. funas\text{-term } (\sigma x) \subseteq \mathcal{F}$   
 shows  $(s \cdot \sigma, t \cdot \sigma) \in srstep \mathcal{F} \mathcal{R}$  ⟨proof⟩

**lemma** *srsteps-subst-closed*:  
 assumes  $(s, t) \in (srstep \mathcal{F} \mathcal{R})^+ \wedge x. funas\text{-term } (\sigma x) \subseteq \mathcal{F}$   
 shows  $(s \cdot \sigma, t \cdot \sigma) \in (srstep \mathcal{F} \mathcal{R})^+$  ⟨proof⟩

**lemma** *srsteps-eq-subst-closed*:  
 assumes  $(s, t) \in (srstep \mathcal{F} \mathcal{R})^* \wedge x. funas\text{-term } (\sigma x) \subseteq \mathcal{F}$   
 shows  $(s \cdot \sigma, t \cdot \sigma) \in (srstep \mathcal{F} \mathcal{R})^*$  ⟨proof⟩

**lemma** *srsteps-eq-subst-relcomp-closed*:

**assumes**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^* \circ (\text{srstep } \mathcal{F} \mathcal{S})^* \wedge x. \text{funas-term } (\sigma x) \subseteq \mathcal{F}$   
**shows**  $(s \cdot \sigma, t \cdot \sigma) \in (\text{srstep } \mathcal{F} \mathcal{R})^* \circ (\text{srstep } \mathcal{F} \mathcal{S})^*$   
 $\langle \text{proof} \rangle$

### 3.5 Context closure of *srstep*

**lemma** *srstep-ctxt-closed*:

**assumes** *funas-ctxt*  $C \subseteq \mathcal{F}$  **and**  $(s, t) \in \text{srstep } \mathcal{F} \mathcal{R}$   
**shows**  $(C\langle s \rangle, C\langle t \rangle) \in \text{srstep } \mathcal{F} \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-ctxt-closed*:

**assumes** *funas-ctxt*  $C \subseteq \mathcal{F}$  **and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$   
**shows**  $(C\langle s \rangle, C\langle t \rangle) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$   $\langle \text{proof} \rangle$

**lemma** *srsteps-eq-ctxt-closed*:

**assumes** *funas-ctxt*  $C \subseteq \mathcal{F}$  **and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$   
**shows**  $(C\langle s \rangle, C\langle t \rangle) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$   $\langle \text{proof} \rangle$

**lemma** *sig-steps-join-ctxt-closed*:

**assumes** *funas-ctxt*  $C \subseteq \mathcal{F}$  **and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^\downarrow$   
**shows**  $(C\langle s \rangle, C\langle t \rangle) \in (\text{srstep } \mathcal{F} \mathcal{R})^\downarrow$   $\langle \text{proof} \rangle$

The following lemma shows that every rewrite sequence either contains a root step or is root stable

**lemma** *nsrsteps-with-root-step-step-on-args*:

**assumes**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+ (s, t) \notin \text{srsteps-with-root-step } \mathcal{F} \mathcal{R}$   
**shows**  $\exists f \text{ ss ts. } s = \text{Fun } f \text{ ss} \wedge t = \text{Fun } f \text{ ts} \wedge \text{length ss} = \text{length ts} \wedge$   
 $(\forall i < \text{length ts. } (\text{ss} ! i, \text{ts} ! i) \in (\text{srstep } \mathcal{F} \mathcal{R})^*)$   $\langle \text{proof} \rangle$

**lemma** *rstep-to-pos-replace*:

**assumes**  $(s, t) \in \text{rstep } \mathcal{R}$   
**shows**  $\exists p \text{ l } r \sigma. p \in \text{poss } s \wedge (l, r) \in \mathcal{R} \wedge s \mid\text{-} p = l \cdot \sigma \wedge t = s[p \leftarrow r \cdot \sigma]$   
 $\langle \text{proof} \rangle$

**lemma** *pos-replace-to-rstep*:

**assumes**  $p \in \text{poss } s (l, r) \in \mathcal{R}$   
**and**  $s \mid\text{-} p = l \cdot \sigma t = s[p \leftarrow r \cdot \sigma]$   
**shows**  $(s, t) \in \text{rstep } \mathcal{R}$   
 $\langle \text{proof} \rangle$

**end**

**theory** *Replace-Constant*

**imports** *Rewriting*

**begin**

### 3.6 Removing/Replacing constants in a rewrite sequence that do not appear in the rewrite system

**lemma** *funas-term-const-subst-conv*:

$(c, 0) \notin \text{funas-term } l \iff \neg (l \supseteq \text{constT } c)$   
 $\langle \text{proof} \rangle$

**lemma** *fresh-const-single-step-replace*:

**assumes** *lin*: linear-sys  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *occ*:  $p \in \text{poss-of-term } (\text{constT } c) s$  **and** *step*:  $(s, t) \in \text{rstep } \mathcal{R}$   
**shows**  $(s[p \leftarrow u], t) \in \text{rstep } \mathcal{R} \vee$   
 $(\exists q. q \in \text{poss-of-term } (\text{constT } c) t \wedge (s[p \leftarrow u], t[q \leftarrow u]) \in \text{rstep } \mathcal{R})$   
 $\langle \text{proof} \rangle$

**lemma** *fresh-const-steps-replace*:

**assumes** *lin*: linear-sys  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *occ*:  $p \in \text{poss-of-term } (\text{constT } c) s$  **and** *steps*:  $(s, t) \in (\text{rstep } \mathcal{R})^+$   
**shows**  $(s[p \leftarrow u], t) \in (\text{rstep } \mathcal{R})^+ \vee$   
 $(\exists q. q \in \text{poss-of-term } (\text{constT } c) t \wedge (s[p \leftarrow u], t[q \leftarrow u]) \in (\text{rstep } \mathcal{R})^+)$   
 $\langle \text{proof} \rangle$

**lemma** *remove-const-lhs-steps*:

**assumes** *lin*: linear-sys  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } t$   
**and** *pos*:  $p \in \text{poss-of-term } (\text{constT } c) s$   
**and** *steps*:  $(s, t) \in (\text{rstep } \mathcal{R})^+$   
**shows**  $(s[p \leftarrow u], t) \in (\text{rstep } \mathcal{R})^+ \langle \text{proof} \rangle$

Now we can show that we may remove a constant substitution

**definition** *const-replace-closed where*

*const-replace-closed*  $c U = (\forall s t u p.$   
 $p \in \text{poss-of-term } (\text{constT } c) s \implies (s, t) \in U \implies$   
 $(\exists q. q \in \text{poss-of-term } (\text{constT } c) t \wedge (s[p \leftarrow u], t[q \leftarrow u]) \in U) \vee (s[p \leftarrow u],$   
 $t) \in U)$

**lemma** *const-replace-closedD*:

**assumes** *const-replace-closed*  $c U$   $p \in \text{poss-of-term } (\text{constT } c) s$   $(s, t) \in U$   
**shows**  $(s[p \leftarrow u], t) \in U \vee (\exists q. q \in \text{poss-of-term } (\text{constT } c) t \wedge (s[p \leftarrow u], t[q \leftarrow u]) \in U) \langle \text{proof} \rangle$

**lemma** *const-replace-closedI*:

**assumes**  $\bigwedge s t u p. p \in \text{poss-of-term } (\text{constT } c) s \implies (s, t) \in U \implies$   
 $(\exists q. q \in \text{poss-of-term } (\text{constT } c) t \wedge (s[p \leftarrow u], t[q \leftarrow u]) \in U) \vee (s[p \leftarrow u],$   
 $t) \in U$   
**shows** *const-replace-closed*  $c U \langle \text{proof} \rangle$

**abbreviation** *const-subst*  $:: 'f \Rightarrow 'v \Rightarrow ('f, 'v) \text{ Term.term where}$

*const-subst*  $c \equiv (\lambda x. \text{Fun } c \ [])$

**lemma** *lin-fresh-rstep-const-replace-closed*:

*linear-sys*  $\mathcal{R} \implies (c, 0) \notin \text{funas-rel } \mathcal{R} \implies \text{const-replace-closed } c \text{ (rstep } \mathcal{R})$   
 ⟨proof⟩

**lemma** *const-replace-closed-symcl*:  
 $\text{const-replace-closed } c \ U \implies \text{const-replace-closed } c \ (U^=)$   
 ⟨proof⟩

**lemma** *const-replace-closed-trancl*:  
 $\text{const-replace-closed } c \ U \implies \text{const-replace-closed } c \ (U^+)$   
 ⟨proof⟩

**lemma** *const-replace-closed-rtrancl*:  
 $\text{const-replace-closed } c \ U \implies \text{const-replace-closed } c \ (U^*)$   
 ⟨proof⟩

**lemma** *const-replace-closed-relcomp*:  
 $\text{const-replace-closed } c \ U \implies \text{const-replace-closed } c \ V \implies \text{const-replace-closed } c \ (U \ O \ V)$   
 ⟨proof⟩

*const-replace-closed* allow the removal of a fresh constant substitution

**lemma** *const-replace-closed-remove-subst-lhs*:  
**assumes** *repl*:  $\text{const-replace-closed } c \ U$   
**and** *const*:  $(c, 0) \notin \text{funas-term } t$   
**and** *steps*:  $(s \cdot \text{const-subst } c, t) \in U$   
**shows**  $(s, t) \in U$  ⟨proof⟩

### 3.6.1 Removal lemma applied to various rewrite relations

**lemma** *remove-const-subst-step-lhs*:  
**assumes** *lin*: *linear-sys*  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } t$   
**and** *step*:  $(s \cdot \text{const-subst } c, t) \in (\text{rstep } \mathcal{R})$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})$   
 ⟨proof⟩

**lemma** *remove-const-subst-steps-lhs*:  
**assumes** *lin*: *linear-sys*  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } t$   
**and** *steps*:  $(s \cdot \text{const-subst } c, t) \in (\text{rstep } \mathcal{R})^+$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^+$   
 ⟨proof⟩

**lemma** *remove-const-subst-steps-eq-lhs*:  
**assumes** *lin*: *linear-sys*  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } t$   
**and** *steps*:  $(s \cdot \text{const-subst } c, t) \in (\text{rstep } \mathcal{R})^*$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^*$  ⟨proof⟩

**lemma** *remove-const-subst-steps-rhs*:

**assumes** *lin*: linear-sys  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } s$   
**and** *steps*:  $(s, t \cdot \text{const-subst } c) \in (\text{rstep } \mathcal{R})^+$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^+$   
 $\langle \text{proof} \rangle$

**lemma** *remove-const-subst-steps-eq-rhs*:  
**assumes** *lin*: linear-sys  $\mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *const*:  $(c, 0) \notin \text{funas-term } s$   
**and** *steps*:  $(s, t \cdot \text{const-subst } c) \in (\text{rstep } \mathcal{R})^*$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^*$   
 $\langle \text{proof} \rangle$

Main lemmas

**lemma** *const-subst-eq-ground-eq*:  
**assumes**  $s \cdot \text{const-subst } c = t \cdot \text{const-subst } d$   $c \neq d$   
**and**  $(c, 0) \notin \text{funas-term } t$   $(d, 0) \notin \text{funas-term } s$   
**shows**  $s = t$   $\langle \text{proof} \rangle$

**lemma** *remove-const-subst-steps*:  
**assumes** linear-sys  $\mathcal{R}$  **and**  $(c, 0) \notin \text{funas-rel } \mathcal{R}$  **and**  $(d, 0) \notin \text{funas-rel } \mathcal{R}$   
**and**  $c \neq d$   $(c, 0) \notin \text{funas-term } t$   $(d, 0) \notin \text{funas-term } s$   
**and**  $(s \cdot \text{const-subst } c, t \cdot \text{const-subst } d) \in (\text{rstep } \mathcal{R})^*$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^*$   
 $\langle \text{proof} \rangle$

**lemma** *remove-const-subst-relcomp-lhs*:  
**assumes** *sys*: linear-sys  $\mathcal{R}$  linear-sys  $\mathcal{S}$   
**and** *fr*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$  **and** *fs*:  $(c, 0) \notin \text{funas-rel } \mathcal{S}$   
**and** *funas*:  $(c, 0) \notin \text{funas-term } t$   
**and** *seq*:  $(s \cdot \text{const-subst } c, t) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$   $\langle \text{proof} \rangle$

**lemma** *remove-const-subst-relcomp-rhs*:  
**assumes** *sys*: linear-sys  $\mathcal{R}$  linear-sys  $\mathcal{S}$   
**and** *fr*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$  **and** *fs*:  $(c, 0) \notin \text{funas-rel } \mathcal{S}$   
**and** *funas*:  $(c, 0) \notin \text{funas-term } s$   
**and** *seq*:  $(s, t \cdot \text{const-subst } c) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$   
**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$   
 $\langle \text{proof} \rangle$

**lemma** *remove-const-subst-relcomp*:  
**assumes** *sys*: linear-sys  $\mathcal{R}$  linear-sys  $\mathcal{S}$   
**and** *fr*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   $(d, 0) \notin \text{funas-rel } \mathcal{R}$   
**and** *fs*:  $(c, 0) \notin \text{funas-rel } \mathcal{S}$   $(d, 0) \notin \text{funas-rel } \mathcal{S}$   
**and** *diff*:  $c \neq d$  **and** *funas*:  $(c, 0) \notin \text{funas-term } t$   $(d, 0) \notin \text{funas-term } s$   
**and** *seq*:  $(s \cdot \text{const-subst } c, t \cdot \text{const-subst } d) \in (\text{rstep } \mathcal{R})^* \ O (\text{rstep } \mathcal{S})^*$

**shows**  $(s, t) \in (\text{rstep } \mathcal{R})^* \circ (\text{rstep } \mathcal{S})^*$   
 ⟨proof⟩

**end**

## 4 Confluence related rewriting properties

**theory** *Rewriting-Properties*

**imports** *Rewriting*

*Abstract-Rewriting.Abstract-Rewriting*

**begin**

### 4.1 Confluence related ARS properties

**definition** *SCR-on*  $r A \equiv (\forall a \in A. \forall b c. (a, b) \in r \wedge (a, c) \in r \longrightarrow (\exists d. (b, d) \in r^= \wedge (c, d) \in r^*))$

**abbreviation** *SCR*  $:: 'a \text{ rel} \Rightarrow \text{bool}$  **where** *SCR*  $r \equiv \text{SCR-on } r \text{ UNIV}$

**definition** *NFP-on*  $:: 'a \text{ rel} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$  **where**

*NFP-on*  $r A \longleftrightarrow (\forall a \in A. \forall b c. (a, b) \in r^* \wedge (a, c) \in r^! \longrightarrow (b, c) \in r^*)$

**abbreviation** *NFP*  $:: 'a \text{ rel} \Rightarrow \text{bool}$  **where** *NFP*  $r \equiv \text{NFP-on } r \text{ UNIV}$

**definition** *CE-on*  $:: 'a \text{ rel} \Rightarrow 'a \text{ rel} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$  **where**

*CE-on*  $r s A \longleftrightarrow (\forall a \in A. \forall b. (a, b) \in r^{\leftrightarrow*} \longleftrightarrow (a, b) \in s^{\leftrightarrow*})$

**abbreviation** *CE*  $:: 'a \text{ rel} \Rightarrow 'a \text{ rel} \Rightarrow \text{bool}$  **where** *CE*  $r s \equiv \text{CE-on } r s \text{ UNIV}$

**definition** *NE-on*  $:: 'a \text{ rel} \Rightarrow 'a \text{ rel} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$  **where**

*NE-on*  $r s A \longleftrightarrow (\forall a \in A. \forall b. (a, b) \in r^! \longleftrightarrow (a, b) \in s^!)$

**abbreviation** *NE*  $:: 'a \text{ rel} \Rightarrow 'a \text{ rel} \Rightarrow \text{bool}$  **where** *NE*  $r s \equiv \text{NE-on } r s \text{ UNIV}$

### 4.2 Signature closure of relation to model multihole context closure

**lemma** *all-ctxt-closed-sig-rsteps* [intro]:

**fixes**  $\mathcal{R} :: ('f, 'v) \text{ term rel}$

**shows** *all-ctxt-closed*  $\mathcal{F} ((\text{srstep } \mathcal{F} \mathcal{R})^*)$  (**is** *all-ctxt-closed* - ( $?R^*$ ))

⟨proof⟩

**lemma** *sigstep-trancl-funass*:

$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{S})^* \Longrightarrow s \neq t \Longrightarrow \text{funas-term } s \subseteq \mathcal{F}$

$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{S})^* \Longrightarrow s \neq t \Longrightarrow \text{funas-term } t \subseteq \mathcal{F}$

⟨proof⟩

**lemma** *srrstep-to-srstep*:

$(s, t) \in \text{srrstep } \mathcal{F} \mathcal{R} \Longrightarrow (s, t) \in \text{srstep } \mathcal{F} \mathcal{R}$

$\langle proof \rangle$

**lemma** *srsteps-with-root-step-srstepsD*:

$(s, t) \in srsteps-with-root-step \mathcal{F} \mathcal{R} \implies (s, t) \in (srstep \mathcal{F} \mathcal{R})^+$   
 $\langle proof \rangle$

**lemma** *srsteps-with-root-step-srsteps-eqD*:

$(s, t) \in srsteps-with-root-step \mathcal{F} \mathcal{R} \implies (s, t) \in (srstep \mathcal{F} \mathcal{R})^*$   
 $\langle proof \rangle$

**lemma** *symcl-srstep-conversion*:

$(s, t) \in srstep \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies (s, t) \in (srstep \mathcal{F} \mathcal{R})^{\leftrightarrow*}$   
 $\langle proof \rangle$

**lemma** *symcl-srsteps-conversion*:

$(s, t) \in (srstep \mathcal{F} (\mathcal{R}^{\leftrightarrow}))^* \implies (s, t) \in (srstep \mathcal{F} \mathcal{R})^{\leftrightarrow*}$   
 $\langle proof \rangle$

**lemma** *NF-srstep-args*:

**assumes**  $Fun f ss \in NF (srstep \mathcal{F} \mathcal{R})$  *funas-term*  $(Fun f ss) \subseteq \mathcal{F} \ i < length \ ss$   
**shows**  $ss \ ! \ i \in NF (srstep \mathcal{F} \mathcal{R})$   
 $\langle proof \rangle$

**lemma** *all-ctxt-closed-srstep-conversions [simp]*:

*all-ctxt-closed*  $\mathcal{F} ((srstep \mathcal{F} \mathcal{R})^{\leftrightarrow*})$   
 $\langle proof \rangle$

**lemma** *NFP-stepD*:

$NFP \ r \implies (a, b) \in r^* \implies (a, c) \in r^* \implies c \in NF \ r \implies (b, c) \in r^*$   
 $\langle proof \rangle$

**lemma** *NE-symmetric*:  $NE \ r \ s \implies NE \ s \ r$

$\langle proof \rangle$

**lemma** *CE-symmetric*:  $CE \ r \ s \implies CE \ s \ r$

$\langle proof \rangle$

Reducing the quantification over rewrite sequences for properties *CR* ...  
to rewrite sequences containing at least one root step

**lemma** *all-ctxt-closed-sig-reflE*:

*all-ctxt-closed*  $\mathcal{F} \mathcal{R} \implies funas-term \ t \subseteq \mathcal{F} \implies (t, t) \in \mathcal{R}$   
 $\langle proof \rangle$

**lemma** *all-ctxt-closed-relcomp [intro]*:

$(\bigwedge s t. (s, t) \in \mathcal{R} \implies s \neq t \implies \text{funas-term } s \subseteq \mathcal{F} \wedge \text{funas-term } t \subseteq \mathcal{F}) \implies$   
 $(\bigwedge s t. (s, t) \in \mathcal{S} \implies s \neq t \implies \text{funas-term } s \subseteq \mathcal{F} \wedge \text{funas-term } t \subseteq \mathcal{F}) \implies$   
 $\text{all-ctxt-closed } \mathcal{F} \mathcal{R} \implies \text{all-ctxt-closed } \mathcal{F} \mathcal{S} \implies \text{all-ctxt-closed } \mathcal{F} (\mathcal{R} \circ \mathcal{S})$   
 <proof>

**abbreviation**  $\text{prop-to-rel } P \equiv \{(s, t) \mid s t. P s t\}$

**abbreviation**  $\text{prop-mctxt-cl } \mathcal{F} P \equiv \text{all-ctxt-closed } \mathcal{F} (\text{prop-to-rel } P)$

**lemma**  $\text{prop-mctxt-cl-Var}$ :  
 $\text{prop-mctxt-cl } \mathcal{F} P \implies P (\text{Var } x) (\text{Var } x)$   
 <proof>

**lemma**  $\text{prop-mctxt-cl-refl-on}$ :  
 $\text{prop-mctxt-cl } \mathcal{F} P \implies \text{funas-term } t \subseteq \mathcal{F} \implies P t t$   
 <proof>

**lemma**  $\text{prop-mctxt-cl-reflcl-on}$ :  
 $\text{prop-mctxt-cl } \mathcal{F} P \implies \text{funas-term } s \subseteq \mathcal{F} \implies P s s$   
 <proof>

**lemma**  $\text{reduction-relations-to-root-step}$ :  
**assumes**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \implies P s t$   
**and cl:**  $\text{prop-mctxt-cl } \mathcal{F} P$   
**and well:**  $\text{funas-term } s \subseteq \mathcal{F} \text{funas-term } t \subseteq \mathcal{F}$   
**and steps:**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$   
**shows**  $P s t$  <proof>

**abbreviation**  $\text{comp-rrstep-rel } \mathcal{F} \mathcal{R} \mathcal{S} \equiv \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \circ ((\text{srstep } \mathcal{F} \mathcal{S})^* \cup (\text{srstep } \mathcal{F} \mathcal{R})^* \circ \text{srsteps-with-root-step } \mathcal{F} \mathcal{S})$

**abbreviation**  $\text{comp-rrstep-rel}' \mathcal{F} \mathcal{R} \mathcal{S} \equiv \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \circ ((\text{srstep } \mathcal{F} \mathcal{S})^+ \cup (\text{srstep } \mathcal{F} \mathcal{R})^+ \circ \text{srsteps-with-root-step } \mathcal{F} \mathcal{S})$

**lemma**  $\text{reduction-join-relations-to-root-step}$ :  
**assumes**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel } \mathcal{F} \mathcal{R} \mathcal{S} \implies P s t$   
**and cl:**  $\text{prop-mctxt-cl } \mathcal{F} P$   
**and well:**  $\text{funas-term } s \subseteq \mathcal{F} \text{funas-term } t \subseteq \mathcal{F}$   
**and steps:**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^* \circ ((\text{srstep } \mathcal{F} \mathcal{S})^* \circ \text{srsteps-with-root-step } \mathcal{F} \mathcal{S})$   
**shows**  $P s t$  <proof>

**definition**  $\text{commute-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t \iff (s, t) \in ((\text{srstep } \mathcal{F} \mathcal{S})^* \circ ((\text{srstep } \mathcal{F} \mathcal{R})^{-1})^*)$

**declare** *subsetI*[rule del]

**lemma** *commute-redp-mctxt-cl*:

*prop-mctxt-cl*  $\mathcal{F}$  (*commute-redp*  $\mathcal{F}$   $\mathcal{R}$   $\mathcal{S}$ )

$\langle$ *proof* $\rangle$

**declare** *subsetI*[intro!]

**lemma** *commute-rrstep-intro*:

**assumes**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel}' \mathcal{F} (\mathcal{R}^{-1}) \mathcal{S} \implies \text{commute-redp } \mathcal{F} \mathcal{R} \mathcal{S}$   
 $s t$

**shows** *commute* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

$\langle$ *proof* $\rangle$

**lemma** *commute-to-rrstep*:

**assumes** *commute* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel } \mathcal{F} (\mathcal{R}^{-1}) \mathcal{S} \implies \text{commute-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t$   
 $\langle$ *proof* $\rangle$

**lemma** *CR-Aux*:

**assumes**  $\bigwedge s t. (s, t) \in (\text{srstep } \mathcal{F} (\mathcal{R}^{-1}))^* \text{ } O \text{ srsteps-with-root-step } \mathcal{F} \mathcal{R} \implies$   
*commute-redp*  $\mathcal{F} \mathcal{R} \mathcal{R} s t$

**shows**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel } \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R} \implies \text{commute-redp } \mathcal{F} \mathcal{R} \mathcal{R} s t$   
 $\langle$ *proof* $\rangle$

**lemma** *CR-rrstep-intro*:

**assumes**  $\bigwedge s t. (s, t) \in (\text{srstep } \mathcal{F} (\mathcal{R}^{-1}))^+ \text{ } O \text{ srsteps-with-root-step } \mathcal{F} \mathcal{R} \implies$   
*commute-redp*  $\mathcal{F} \mathcal{R} \mathcal{R} s t$

**shows** *CR* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle$ *proof* $\rangle$

**lemma** *CR-to-rrstep*:

**assumes** *CR* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel } \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R} \implies \text{commute-redp } \mathcal{F} \mathcal{R} \mathcal{R} s t$   
 $\langle$ *proof* $\rangle$

**definition** *NFP-redp* **where**

*NFP-redp*  $\mathcal{F} \mathcal{R} s t \iff t \in \text{NF } (\text{srstep } \mathcal{F} \mathcal{R}) \longrightarrow (s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$

**lemma** *prop-mctxt-cl-NFP-redp*:

*prop-mctxt-cl*  $\mathcal{F}$  (*NFP-redp*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle$ *proof* $\rangle$

**lemma** *NFP-rrstep-intro*:

**assumes**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel}' \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R} \implies \text{NFP-redp } \mathcal{F} \mathcal{R} s t$

**shows** *NFP* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle$ *proof* $\rangle$

**lemma** *NFP-lift-to-conversion*:

**assumes** *NFP*  $r$  ( $s, t) \in (r^{\leftrightarrow})^*$  **and**  $t \in \text{NF } r$

**shows**  $(s, t) \in r^*$   $\langle$ *proof* $\rangle$

**lemma** *NFP-to-rrstep*:

**assumes** *NFP* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies \text{NFP-redp } \mathcal{F} \mathcal{R} s t$   
 $\langle \text{proof} \rangle$

**definition** *UN-redp*  $\mathcal{F} \mathcal{R} s t \iff s \in \text{NF} (\text{srstep } \mathcal{F} \mathcal{R}) \wedge t \in \text{NF} (\text{srstep } \mathcal{F} \mathcal{R}) \longrightarrow s = t$

**lemma** *prop-mctxt-cl-UN-redp*:

*prop-mctxt-cl*  $\mathcal{F}$  (*UN-redp*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle \text{proof} \rangle$

**lemma** *UNC-rrstep-intro*:

**assumes**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies \text{UN-redp } \mathcal{F} \mathcal{R} s t$

**shows** *UNC* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle \text{proof} \rangle$

**lemma** *UNC-to-rrstep*:

**assumes** *UNC* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies \text{UN-redp } \mathcal{F} \mathcal{R} s t$

$\langle \text{proof} \rangle$

**lemma** *UNF-rrstep-intro*:

**assumes**  $\bigwedge t u. (t, u) \in \text{comp-rrstep-rel}' \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R} \implies \text{UN-redp } \mathcal{F} \mathcal{R} t u$

**shows** *UNF* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

$\langle \text{proof} \rangle$

**lemma** *UNF-to-rrstep*:

**assumes** *UNF* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{comp-rrstep-rel } \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R} \implies \text{UN-redp } \mathcal{F} \mathcal{R} s t$

$\langle \text{proof} \rangle$

**lemma** *CE-rrstep-intro*:

**assumes**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{S})^{\leftrightarrow*}$

**and**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{S}^{\leftrightarrow}) \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^{\leftrightarrow*}$

**shows** *CE* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

$\langle \text{proof} \rangle$

**lemma** *CE-to-rrstep*:

**assumes** *CE* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

**shows**  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow}) \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{S})^{\leftrightarrow*}$

$\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} (\mathcal{S}^{\leftrightarrow}) \implies (s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^{\leftrightarrow*}$

$\langle \text{proof} \rangle$

**definition** *NE-redp* **where**

*NE-redp*  $\mathcal{F} \mathcal{R} \mathcal{S} s t \iff t \in \text{NF} (\text{srstep } \mathcal{F} \mathcal{R}) \longrightarrow t \in \text{NF} (\text{srstep } \mathcal{F} \mathcal{R}) \longrightarrow (s, t) \in (\text{srstep } \mathcal{F} \mathcal{S})^*$

**lemma** *prop-mctxt-cl-NE-redp*:  
*prop-mctxt-cl*  $\mathcal{F}$  (*NE-redp*  $\mathcal{F}$   $\mathcal{R}$   $\mathcal{S}$ )  
 ⟨*proof*⟩

**lemma** *NE-rrstep-intro*:  
 assumes  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \implies \text{NE-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t$   
 and  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{S} \implies \text{NE-redp } \mathcal{F} \mathcal{S} \mathcal{R} s t$   
 and  $\text{NF } (\text{srstep } \mathcal{F} \mathcal{R}) = \text{NF } (\text{srstep } \mathcal{F} \mathcal{S})$   
 shows  $\text{NE } (\text{srstep } \mathcal{F} \mathcal{R}) (\text{srstep } \mathcal{F} \mathcal{S})$   
 ⟨*proof*⟩

**lemma** *NE-to-rrstep*:  
 assumes  $\text{NE } (\text{srstep } \mathcal{F} \mathcal{R}) (\text{srstep } \mathcal{F} \mathcal{S})$   
 shows  $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{R} \implies \text{NE-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t$   
 $\bigwedge s t. (s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{S} \implies \text{NE-redp } \mathcal{F} \mathcal{S} \mathcal{R} s t$   
 ⟨*proof*⟩

**lemma** *NE-NF-eq*:  
 $\text{NE } \mathcal{R} \mathcal{S} \implies \text{NF } \mathcal{R} = \text{NF } \mathcal{S}$   
 ⟨*proof*⟩

**abbreviation**  $\text{SCRp } \mathcal{F} \mathcal{R} t u \equiv \exists v. (t, v) \in (\text{srstep } \mathcal{F} \mathcal{R})^\# \wedge (u, v) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$

**lemma** *SCR-rrstep-intro*:  
 assumes  $\bigwedge s t u. (s, t) \in \text{sig-step } \mathcal{F} (\text{rrstep } \mathcal{R}) \implies (s, u) \in \text{srstep } \mathcal{F} \mathcal{R} \implies \text{SCRp } \mathcal{F} \mathcal{R} t u$   
 and  $\bigwedge s t u. (s, t) \in \text{srstep } \mathcal{F} \mathcal{R} \implies (s, u) \in \text{sig-step } \mathcal{F} (\text{rrstep } \mathcal{R}) \implies \text{SCRp } \mathcal{F} \mathcal{R} t u$   
 shows  $\text{SCR } (\text{srstep } \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**lemma** *SCE-to-rrstep*:  
 assumes  $\text{SCR } (\text{srstep } \mathcal{F} \mathcal{R})$   
 shows  $\bigwedge s t u. (s, t) \in \text{sig-step } \mathcal{F} (\text{rrstep } \mathcal{R}) \implies (s, u) \in \text{srstep } \mathcal{F} \mathcal{R} \implies \text{SCRp } \mathcal{F} \mathcal{R} t u$   
 $\bigwedge s t u. (s, t) \in \text{srstep } \mathcal{F} \mathcal{R} \implies (s, u) \in \text{sig-step } \mathcal{F} (\text{rrstep } \mathcal{R}) \implies \text{SCRp } \mathcal{F} \mathcal{R} t u$   
 ⟨*proof*⟩

**lemma** *WCR-rrstep-intro*:  
 assumes  $\bigwedge s t u. (s, t) \in \text{sig-step } \mathcal{F} (\text{rrstep } \mathcal{R}) \implies (s, u) \in \text{srstep } \mathcal{F} \mathcal{R} \implies (t, u) \in (\text{srstep } \mathcal{F} \mathcal{R})^\downarrow$   
 shows  $\text{WCR } (\text{srstep } \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**end**

**theory** *Rewriting-LLRG-LV-Mondaic*  
**imports** *Rewriting*  
*Replace-Constant*  
**begin**

### 4.3 Specific results about rewriting under a linear variable-separated system

**lemma** *card-varposs-ground*:

$\text{card } (\text{varposs } s) = 0 \iff \text{ground } s$   
 $\langle \text{proof} \rangle$

**lemma** *poss-of-term-subst-apply-varposs*:

**assumes**  $p \in \text{poss-of-term } (\text{constT } c) (s \cdot \sigma) (c, 0) \notin \text{funas-term } s$   
**shows**  $\exists q. q \in \text{varposs } s \wedge q \leq_p p \langle \text{proof} \rangle$

**lemma** *poss-of-term-hole-poss*:

**assumes**  $p \in \text{poss-of-term } t C \langle s \rangle$  **and**  $\text{hole-pos } C \leq_p p$   
**shows**  $p \dashv_p \text{hole-pos } C \in \text{poss-of-term } t s \langle \text{proof} \rangle$

**lemma** *remove-const-subst-from-match*:

**assumes**  $s \cdot \text{const-subst } c = C \langle l \cdot \sigma \rangle (c, 0) \notin \text{funas-term } l \text{ linear-term } l$   
**shows**  $\exists D \tau. s = D \langle l \cdot \tau \rangle \langle \text{proof} \rangle$

**definition** *llrg*  $\mathcal{R} \iff (\forall (l, r) \in \mathcal{R}. \text{linear-term } l \wedge \text{ground } r)$

**definition** *lv*  $\mathcal{R} \iff (\forall (l, r) \in \mathcal{R}. \text{linear-term } l \wedge \text{linear-term } r \wedge \text{vars-term } l \cap \text{vars-term } r = \{\})$

**definition** *monadic*  $\mathcal{F} \iff (\forall (f, n) \in \mathcal{F}. n \leq \text{Suc } 0)$

— NF of ground terms

**lemma** *ground-NF-srstep-gsrstep*:

$\text{ground } s \implies s \in \text{NF } (\text{srstep } \mathcal{F} \mathcal{R}) \implies s \in \text{NF } (\text{gsrstep } \mathcal{F} \mathcal{R})$   
 $\langle \text{proof} \rangle$

**lemma** *NF-to-fresh-const-subst-NF*:

**assumes** *lin*:  $\text{linear-sys } \mathcal{R}$  **and** *fresh-const*:  $(c, 0) \notin \text{funas-rel } \mathcal{R} \text{ funas-rel } \mathcal{R} \subseteq \mathcal{F}$

**and** *nf-f*:  $\text{funas-term } s \subseteq \mathcal{F} \ s \in \text{NF } (\text{srstep } \mathcal{F} \mathcal{R})$

**shows**  $s \cdot \text{const-subst } c \in \text{NF } (\text{gsrstep } \mathcal{H} \mathcal{R})$

$\langle \text{proof} \rangle$

**lemma** *fresh-const-subst-NF-pres*:

**assumes** *fresh-const*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$

**and** *nf-f*:  $\text{funas-term } s \subseteq \mathcal{F}$   $\mathcal{F} \subseteq \mathcal{H}$   $(c, 0) \in \mathcal{H}$   $s \cdot \text{const-subst } c \in \text{NF } (\text{gsrstep } \mathcal{H} \mathcal{R})$

**shows**  $s \in \text{NF } (\text{srrstep } \mathcal{F} \mathcal{R})$

*<proof>*

**lemma** *linear-sys-gNF-eq-NF-eq*:

**assumes** *lin*:  $\text{linear-sys } \mathcal{R}$   $\text{linear-sys } \mathcal{S}$

**and** *well*:  $\text{funas-rel } \mathcal{R} \subseteq \mathcal{F}$   $\text{funas-rel } \mathcal{S} \subseteq \mathcal{F}$

**and** *fresh*:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$   $(c, 0) \notin \text{funas-rel } \mathcal{S}$

**and** *lift*:  $\mathcal{F} \subseteq \mathcal{H}$   $(c, 0) \in \mathcal{H}$

**and** *nf*:  $\text{NF } (\text{gsrstep } \mathcal{H} \mathcal{R}) = \text{NF } (\text{gsrstep } \mathcal{H} \mathcal{S})$

**shows**  $\text{NF } (\text{srrstep } \mathcal{F} \mathcal{R}) = \text{NF } (\text{srrstep } \mathcal{F} \mathcal{S})$

*<proof>*

**lemma** *gsrsteps-to-srsteps*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^+ \implies (s, t) \in (\text{srrstep } \mathcal{F} \mathcal{R})^+$

*<proof>*

**lemma** *gsrsteps-eq-to-srsteps-eq*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^* \implies (s, t) \in (\text{srrstep } \mathcal{F} \mathcal{R})^*$

*<proof>*

**lemma** *gsrsteps-to-rsteps*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^+ \implies (s, t) \in (\text{rstep } \mathcal{R})^+$

*<proof>*

**lemma** *gsrsteps-eq-to-rsteps-eq*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^* \implies (s, t) \in (\text{rstep } \mathcal{R})^*$

*<proof>*

**lemma** *gsrsteps-eq-relcomp-srsteps-relcompD*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^* \text{ } O \text{ } (\text{gsrstep } \mathcal{F} \mathcal{S})^* \implies (s, t) \in (\text{srrstep } \mathcal{F} \mathcal{R})^* \text{ } O \text{ } (\text{srrstep } \mathcal{F} \mathcal{S})^*$

*<proof>*

**lemma** *gsrsteps-eq-relcomp-to-rsteps-relcomp*:

$(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^* \text{ } O \text{ } (\text{gsrstep } \mathcal{F} \mathcal{S})^* \implies (s, t) \in (\text{rstep } \mathcal{R})^* \text{ } O \text{ } (\text{rstep } \mathcal{S})^*$

*<proof>*

**lemma** *ground-srsteps-gsrsteps*:

**assumes** *ground*  $s$  *ground*  $t$

**and**  $(s, t) \in (\text{srrstep } \mathcal{F} \mathcal{R})^+$

**shows**  $(s, t) \in (\text{gsrstep } \mathcal{F} \mathcal{R})^+$

*<proof>*

**lemma** *ground-srsteps-eq-gsrsteps-eq*:

**assumes** *ground s ground t*  
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^*$   
**shows**  $(s, t) \in (\text{gsrstep } \mathcal{F} \ \mathcal{R})^*$   
*<proof>*

**lemma** *srsteps-eq-relcomp-gsrsteps-relcomp*:

**assumes**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^* \ O \ (\text{srstep } \mathcal{F} \ \mathcal{S})^*$   
**and** *ground s ground t*  
**shows**  $(s, t) \in (\text{gsrstep } \mathcal{F} \ \mathcal{R})^* \ O \ (\text{gsrstep } \mathcal{F} \ \mathcal{S})^*$   
*<proof>*

**lemma** *llrg-ground-rhs*:

$\text{llrg } \mathcal{R} \implies (l, r) \in \mathcal{R} \implies \text{ground } r$   
*<proof>*

**lemma** *llrg-rrsteps-groundness*:

**assumes**  $\text{llrg } \mathcal{R}$  **and**  $(s, t) \in (\text{srrstep } \mathcal{F} \ \mathcal{R})$   
**shows** *ground t* *<proof>*

**lemma** *llrg-rsteps-pres-groundness*:

**assumes**  $\text{llrg } \mathcal{R}$  *ground s*  
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^*$   
**shows** *ground t* *<proof>*

**lemma** *llrg-srsteps-with-root-step-ground*:

**assumes**  $\text{llrg } \mathcal{R}$  **and**  $(s, t) \in \text{srsteps-with-root-step } \mathcal{F} \ \mathcal{R}$   
**shows** *ground t* *<proof>*

**lemma** *llrg-srsteps-with-root-step-inv-ground*:

**assumes**  $\text{llrg } \mathcal{R}$  **and**  $(s, t) \in \text{srsteps-with-root-step } \mathcal{F} \ (\mathcal{R}^{-1})$   
**shows** *ground s* *<proof>*

**lemma** *llrg-funas-term-step-pres*:

**assumes**  $\text{llrg } \mathcal{R}$  **and**  $(s, t) \in (\text{rstep } \mathcal{R})$   
**shows**  $\text{funas-term } t \subseteq \text{funas-rel } \mathcal{R} \cup \text{funas-term } s$   
*<proof>*

**lemma** *llrg-funas-term-steps-pres*:

**assumes**  $\text{llrg } \mathcal{R}$  **and**  $(s, t) \in (\text{rstep } \mathcal{R})^*$   
**shows**  $\text{funas-term } t \subseteq \text{funas-rel } \mathcal{R} \cup \text{funas-term } s$   
*<proof>*

**lemma** *monadic-ground-ctxt-apply*:

$\text{monadic } \mathcal{F} \implies \text{funas-ctxt } C \subseteq \mathcal{F} \implies \text{ground } r \implies \text{ground } C\langle r \rangle$   
*<proof>*

**lemma** *llrg-monadic-rstep-pres-groundness*:

**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and**  $(s, t) \in \text{srstep } \mathcal{F} \ \mathcal{R}$   
**shows** *ground*  $t$   $\langle \text{proof} \rangle$

**lemma** *llrg-monadic-rsteps-groundness*:

**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^+$   
**shows** *ground*  $t$   $\langle \text{proof} \rangle$

**fun** *monadic-term* **where**

*monadic-term*  $(\text{Var } x) = \text{True}$   
 $|$  *monadic-term*  $(\text{Fun } f \ \square) = \text{True}$   
 $|$  *monadic-term*  $(\text{Fun } f \ ts) = (\text{length } ts = \text{Suc } 0 \wedge \text{monadic-term } (\text{hd } ts))$

**fun** *monadic-get-leave* **where**

*monadic-get-leave*  $(\text{Var } x) = (\text{Var } x)$   
 $|$  *monadic-get-leave*  $(\text{Fun } f \ \square) = \text{Fun } f \ \square$   
 $|$  *monadic-get-leave*  $(\text{Fun } f \ ts) = \text{monadic-get-leave } (\text{hd } ts)$

**fun** *monadic-replace-leave* **where**

*monadic-replace-leave*  $t$   $(\text{Var } x) = t$   
 $|$  *monadic-replace-leave*  $t$   $(\text{Fun } f \ \square) = t$   
 $|$  *monadic-replace-leave*  $t$   $(\text{Fun } f \ ts) = \text{Fun } f \ [\text{monadic-replace-leave } t \ (\text{hd } ts)]$

**lemma** *monadic-replace-leave-undo-const-subst*:

**assumes** *monadic-term*  $s$   
**shows** *monadic-replace-leave*  $(\text{monadic-get-leave } s) (s \cdot \text{const-subst } c) = s$   $\langle \text{proof} \rangle$

**lemma** *monadic-replace-leave-context*:

**assumes** *monadic-term*  $C \langle s \rangle$   
**shows** *monadic-replace-leave*  $t$   $C \langle s \rangle = C \langle \text{monadic-replace-leave } t \ s \rangle$   $\langle \text{proof} \rangle$

**lemma** *monadic-replace-leave-subst*:

**assumes** *monadic-term*  $(s \cdot \sigma) \neg \text{ground } s$   
**shows** *monadic-replace-leave*  $t$   $(s \cdot \sigma) = s \cdot (\lambda x. \text{monadic-replace-leave } t \ (\sigma x))$   
 $\langle \text{proof} \rangle$

**lemma** *monadic-sig*:

*monadic*  $\mathcal{F} \implies (f, \text{length } ts) \in \mathcal{F} \implies \text{length } ts \leq \text{Suc } 0$   
 $\langle \text{proof} \rangle$

**lemma** *monadic-sig-funas-term-mt*:

*monadic*  $\mathcal{F} \implies \text{funas-term } s \subseteq \mathcal{F} \implies \text{monadic-term } s$   
 $\langle \text{proof} \rangle$

**lemma** *monadic-term-const-pres* [intro]:

*monadic-term*  $s \implies \text{monadic-term } (s \cdot \text{const-subst } c)$

*<proof>*

**lemma** *remove-const-lv-mondaic-step-lhs:*

**assumes** *lv: lv  $\mathcal{R}$  and fresh:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$*

**and** *mon: monadic  $\mathcal{F}$*

**and** *step:  $(s \cdot \text{const-subst } c, t) \in (\text{srstep } \mathcal{F} \mathcal{R})$*

**shows**  *$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})$*

*<proof>*

**lemma** *remove-const-lv-mondaic-step-rhs:*

**assumes** *lv: lv  $\mathcal{R}$  and fresh:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$*

**and** *mon: monadic  $\mathcal{F}$*

**and** *step:  $(s, t \cdot \text{const-subst } c) \in (\text{srstep } \mathcal{F} \mathcal{R})$*

**shows**  *$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})$*

*<proof>*

**lemma** *remove-const-lv-mondaic-steps-lhs:*

**assumes** *lv: lv  $\mathcal{R}$  and fresh:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$*

**and** *mon: monadic  $\mathcal{F}$*

**and** *steps:  $(s \cdot \text{const-subst } c, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

**shows**  *$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

*<proof>*

**lemma** *remove-const-lv-mondaic-steps-rhs:*

**assumes** *lv: lv  $\mathcal{R}$  and fresh:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$*

**and** *mon: monadic  $\mathcal{F}$*

**and** *steps:  $(s, t \cdot \text{const-subst } c) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

**shows**  *$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

*<proof>*

**lemma** *remove-const-lv-mondaic-steps:*

**assumes** *lv: lv  $\mathcal{R}$  and fresh:  $(c, 0) \notin \text{funas-rel } \mathcal{R}$*

**and** *mon: monadic  $\mathcal{F}$*

**and** *steps:  $(s \cdot \text{const-subst } c, t \cdot \text{const-subst } c) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

**shows**  *$(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$*

*<proof>*

**lemma** *lv-root-step-idep-subst:*

**assumes** *lv  $\mathcal{R}$*

**and**  *$(s, t) \in \text{srrstep } \mathcal{F} \mathcal{R}$*

**and** *well:  $\bigwedge x. \text{funas-term } (\sigma x) \subseteq \mathcal{F} \bigwedge x. \text{funas-term } (\tau x) \subseteq \mathcal{F}$*

**shows**  *$(s \cdot \sigma, t \cdot \tau) \in \text{srrstep } \mathcal{F} \mathcal{R}$*

*<proof>*

**lemma** *lv-srsteps-with-root-step-idep-subst:*

**assumes** *lv  $\mathcal{R}$*

**and**  *$(s, t) \in \text{srsteps-with-root-step } \mathcal{F} \mathcal{R}$*

**and well:**  $\bigwedge x. \text{funas-term } (\sigma x) \subseteq \mathcal{F} \bigwedge x. \text{funas-term } (\tau x) \subseteq \mathcal{F}$   
**shows**  $(s \cdot \sigma, t \cdot \tau) \in \text{srrsteps-with-root-step } \mathcal{F} \mathcal{R} \langle \text{proof} \rangle$

**end**  
**theory** *Rewriting-GTRS*  
**imports** *Rewriting*  
*Replace-Constant*  
**begin**

#### 4.4 Specific results about rewriting under a ground system

**abbreviation** *ground-sys*  $\mathcal{R} \equiv (\forall (s, t) \in \mathcal{R}. \text{ground } s \wedge \text{ground } t)$

**lemma** *srrstep-ground:*  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srrstep } \mathcal{F} \mathcal{R}$   
**shows**  $\text{ground } s \text{ ground } t \langle \text{proof} \rangle$

**lemma** *srstep-pres-ground-l:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $s$   
**and**  $(s, t) \in \text{srstep } \mathcal{F} \mathcal{R}$   
**shows**  $\text{ground } t \langle \text{proof} \rangle$

**lemma** *srstep-pres-ground-r:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $t$   
**and**  $(s, t) \in \text{srstep } \mathcal{F} \mathcal{R}$   
**shows**  $\text{ground } s \langle \text{proof} \rangle$

**lemma** *srsteps-pres-ground-l:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $s$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$   
**shows**  $\text{ground } t \langle \text{proof} \rangle$

**lemma** *srsteps-pres-ground-r:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $t$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^+$   
**shows**  $\text{ground } s \langle \text{proof} \rangle$

**lemma** *srsteps-eq-pres-ground-l:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $s$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$   
**shows**  $\text{ground } t \langle \text{proof} \rangle$

**lemma** *srsteps-eq-pres-ground-r:*  
**assumes** *ground-sys*  $\mathcal{R}$  *ground*  $t$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \mathcal{R})^*$   
**shows**  $\text{ground } s \langle \text{proof} \rangle$

**lemma** *srsteps-with-root-step-ground*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srsteps-with-root-step } \mathcal{F} \ \mathcal{R}$   
**shows** *ground*  $s$  *ground*  $t$   $\langle \text{proof} \rangle$

## 4.5 funas

**lemma** *srrstep-funas*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srrstep } \mathcal{F} \ \mathcal{R}$   
**shows** *funas-term*  $s \subseteq \text{funas-rel } \mathcal{R}$  *funas-term*  $t \subseteq \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srstep-funas-l*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srstep } \mathcal{F} \ \mathcal{R}$   
**shows** *funas-term*  $t \subseteq \text{funas-term } s \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srstep-funas-r*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srstep } \mathcal{F} \ \mathcal{R}$   
**shows** *funas-term*  $s \subseteq \text{funas-term } t \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-funas-l*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^+$   
**shows** *funas-term*  $t \subseteq \text{funas-term } s \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-funas-r*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^+$   
**shows** *funas-term*  $s \subseteq \text{funas-term } t \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-eq-funas-l*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^*$   
**shows** *funas-term*  $t \subseteq \text{funas-term } s \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-eq-funas-r*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in (\text{srstep } \mathcal{F} \ \mathcal{R})^*$   
**shows** *funas-term*  $s \subseteq \text{funas-term } t \cup \text{funas-rel } \mathcal{R}$   $\langle \text{proof} \rangle$

**lemma** *srsteps-with-root-step-funas*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and**  $(s, t) \in \text{srsteps-with-root-step } \mathcal{F} \ \mathcal{R}$   
**shows** *funas-term*  $s \subseteq \text{funas-rel } \mathcal{R}$  *funas-term*  $t \subseteq \text{funas-rel } \mathcal{R}$   
 $\langle \text{proof} \rangle$

**end**

## 5 Reducing Rewrite Properties to Properties on Ground Terms over Left-Linear Right-Ground Systems

```

theory Ground-Reduction-on-LLRG
  imports
    Rewriting-Properties
    Rewriting-LLRG-LV-Mondaic
begin

```

```

lemma llrg-linear-sys:
  llrg  $\mathcal{R} \implies$  linear-sys  $\mathcal{R}$ 
  <proof>

```

## 6 LLRG results

```

lemma llrg-commute:
  assumes sig: funas-rel  $\mathcal{R} \subseteq \mathcal{F}$  funas-rel  $\mathcal{S} \subseteq \mathcal{F}$ 
  and fresh:  $(c, 0) \notin \mathcal{F}$ 
  and llrg: llrg  $\mathcal{R}$  llrg  $\mathcal{S}$ 
  and com: commute (gsrstep (insert  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ ) (gsrstep (insert  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{S}$ )
  shows commute (srsstep  $\mathcal{F}$   $\mathcal{R}$ ) (srsstep  $\mathcal{F}$   $\mathcal{S}$ )
  <proof>

```

```

lemma llrg-CR:
  assumes sig: funas-rel  $\mathcal{R} \subseteq \mathcal{F}$ 
  and fresh:  $(c, 0) \notin \mathcal{F}$ 
  and llrg: llrg  $\mathcal{R}$ 
  and com: CR (gsrstep (insert  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ )
  shows CR (srsstep  $\mathcal{F}$   $\mathcal{R}$ )
  <proof>

```

```

lemma llrg-SCR:
  assumes sig: funas-rel  $\mathcal{R} \subseteq \mathcal{F}$  and fresh:  $(c, 0) \notin \mathcal{F}$ 
  and llrg: llrg  $\mathcal{R}$ 
  and scr: SCR (gsrstep (insert  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ )
  shows SCR (srsstep  $\mathcal{F}$   $\mathcal{R}$ )
  <proof>

```

```

lemma llrg-WCR:
  assumes sig: funas-rel  $\mathcal{R} \subseteq \mathcal{F}$  and fresh:  $(c, 0) \notin \mathcal{F}$ 
  and llrg: llrg  $\mathcal{R}$ 
  and wcr: WCR (gsrstep (insert  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ )
  shows WCR (srsstep  $\mathcal{F}$   $\mathcal{R}$ )
  <proof>

```

**lemma** *llrg-UNF*:

**assumes** *sig*: *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *llrg*: *llrg*  $\mathcal{R}$   
**and** *unf*: *UNF* (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ )  
**shows** *UNF* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

*<proof>*

**lemma** *llrg-NFP*:

**assumes** *sig*: *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *llrg*: *llrg*  $\mathcal{R}$   
**and** *nfp*: *NFP* (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ )  
**shows** *NFP* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ )

*<proof>*

**lemma** *llrg-NE-aux*:

**assumes**  $(s, t) \in$  *srsteps-with-root-step*  $\mathcal{F}$   $\mathcal{R}$   
**and** *sig*: *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  *funas-rel*  $\mathcal{S} \subseteq \mathcal{F}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *llrg*: *llrg*  $\mathcal{R}$  *llrg*  $\mathcal{S}$   
**and** *ne*: *NE* (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ ) (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{S}$ )  
**shows** *NE-redp*  $\mathcal{F}$   $\mathcal{R}$   $\mathcal{S}$   $s$   $t$

*<proof>*

**lemma** *llrg-NE*:

**assumes** *sig*: *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  *funas-rel*  $\mathcal{S} \subseteq \mathcal{F}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *llrg*: *llrg*  $\mathcal{R}$  *llrg*  $\mathcal{S}$   
**and** *ne*: *NE* (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{R}$ ) (*gsrstep* (*insert*  $(c, 0)$   $\mathcal{F}$ )  $\mathcal{S}$ )  
**shows** *NE* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

*<proof>*

## 6.1 Specialized for monadic signature

**lemma** *monadic-commute*:

**assumes** *llrg*  $\mathcal{R}$  *llrg*  $\mathcal{S}$  *monadic*  $\mathcal{F}$   
**and** *com*: *commute* (*gsrstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*gsrstep*  $\mathcal{F}$   $\mathcal{S}$ )  
**shows** *commute* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) (*srstep*  $\mathcal{F}$   $\mathcal{S}$ )

*<proof>*

**lemma** *monadic-CR*:

**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and** *CR* (*gsrstep*  $\mathcal{F}$   $\mathcal{R}$ )  
**shows** *CR* (*srstep*  $\mathcal{F}$   $\mathcal{R}$ ) *<proof>*

**lemma** *monadic-SCR*:

**assumes** *sig*: *funas-rel*  $\mathcal{R} \subseteq \mathcal{F}$  *monadic*  $\mathcal{F}$   
**and** *llrg*: *llrg*  $\mathcal{R}$

**and** *scr*:  $SCR (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $SCR (srstep \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**lemma** *monadic-WCR*:  
**assumes** *sig*:  $funas-rel \mathcal{R} \subseteq \mathcal{F}$  *monadic*  $\mathcal{F}$   
**and** *llrg*:  $llrg \mathcal{R}$   
**and** *wcr*:  $WCR (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $WCR (srstep \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**lemma** *monadic-UNF*:  
**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and** *unf*:  $UNF (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $UNF (srstep \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**lemma** *monadic-UNC*:  
**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and** *well*:  $funas-rel \mathcal{R} \subseteq \mathcal{F}$   
**and** *unc*:  $UNC (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $UNC (srstep \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

**lemma** *monadic-NFP*:  
**assumes** *llrg*  $\mathcal{R}$  *monadic*  $\mathcal{F}$   
**and** *nfp*:  $NFP (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $NFP (srstep \mathcal{F} \mathcal{R})$   
 ⟨*proof*⟩

end

## 7 Reducing Rewrite Properties to Properties on Ground Terms over Ground Systems

**theory** *Ground-Reduction-on-GTRS*  
**imports**  
   *Rewriting-Properties*  
   *Rewriting-GTRS*  
   *Rewriting-LLRG-LV-Mondaic*  
**begin**

**lemma** *ground-sys-nf-eq-lift*:  
**fixes**  $\mathcal{R} :: ('f, 'v)$  *term rel*  
**assumes** *gtrs*:  $ground-sys \mathcal{R}$   $ground-sys \mathcal{S}$

**and**  $nf: NF (gsrstep \mathcal{F} \mathcal{R}) = NF (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $NF (srstep \mathcal{F} \mathcal{R}) = NF (srstep \mathcal{F} \mathcal{S})$   
 $\langle proof \rangle$

**lemma** *ground-sys-inv*:  
 $ground\text{-}sys \mathcal{R} \implies ground\text{-}sys (\mathcal{R}^{-1}) \langle proof \rangle$

**lemma** *ground-sys-symcl*:  
 $ground\text{-}sys \mathcal{R} \implies ground\text{-}sys (\mathcal{R}^{\leftrightarrow}) \langle proof \rangle$

**lemma** *ground-sys-comp-rrstep-rel'-ground*:  
**assumes**  $ground\text{-}sys \mathcal{R} \ ground\text{-}sys \mathcal{S}$   
**and**  $(s, t) \in comp\text{-}rrstep\text{-}rel' \mathcal{F} \mathcal{R} \mathcal{S}$   
**shows**  $ground\ s \ ground\ t$   
 $\langle proof \rangle$

**lemma** *GTRS-commute*:  
**assumes**  $ground\text{-}sys \mathcal{R} \ ground\text{-}sys \mathcal{S}$   
**and**  $com: commute (gsrstep \mathcal{F} \mathcal{R}) (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $commute (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$   
 $\langle proof \rangle$

**lemma** *GTRS-CR*:  
**assumes**  $ground\text{-}sys \mathcal{R}$   
**and**  $CR (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $CR (srstep \mathcal{F} \mathcal{R}) \langle proof \rangle$

**lemma** *GTRS-SCR*:  
**assumes**  $gtrs: ground\text{-}sys \mathcal{R}$   
**and**  $scr: SCR (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $SCR (srstep \mathcal{F} \mathcal{R})$   
 $\langle proof \rangle$

**lemma** *GTRS-WCR*:  
**assumes**  $gtrs: ground\text{-}sys \mathcal{R}$   
**and**  $wcr: WCR (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $WCR (srstep \mathcal{F} \mathcal{R})$   
 $\langle proof \rangle$

**lemma** *GTRS-UNF*:  
**assumes**  $gtrs: ground\text{-}sys \mathcal{R}$   
**and**  $unf: UNF (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $UNF (srstep \mathcal{F} \mathcal{R})$   
 $\langle proof \rangle$

**lemma** *GTRS-UNC*:  
**assumes**  $gtrs: ground\text{-}sys \mathcal{R}$

**and** *unc*:  $UNC (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $UNC (srstep \mathcal{F} \mathcal{R})$   
 <proof>

**lemma** *GTRS-NFP*:  
**assumes** *ground-sys*  $\mathcal{R}$   
**and** *nfp*:  $NFP (gsrstep \mathcal{F} \mathcal{R})$   
**shows**  $NFP (srstep \mathcal{F} \mathcal{R})$   
 <proof>

**lemma** *GTRS-NE-aux*:  
**assumes**  $(s, t) \in srsteps\text{-with-root-step } \mathcal{F} \mathcal{R}$   
**and** *gtrs*: *ground-sys*  $\mathcal{R}$  *ground-sys*  $\mathcal{S}$   
**and** *ne*:  $NE (gsrstep \mathcal{F} \mathcal{R}) (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $NE\text{-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t$   
 <proof>

**lemma** *GTRS-NE*:  
**assumes** *gtrs*: *ground-sys*  $\mathcal{R}$  *ground-sys*  $\mathcal{S}$   
**and** *ne*:  $NE (gsrstep \mathcal{F} \mathcal{R}) (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $NE (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$   
 <proof>

**lemma** *gtrs-CE-aux*:  
**assumes** *step*:  $(s, t) \in srsteps\text{-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow})$   
**and** *gtrs*: *ground-sys*  $\mathcal{R}$  *ground-sys*  $\mathcal{S}$   
**and** *ce*:  $CE (gsrstep \mathcal{F} \mathcal{R}) (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $(s, t) \in (srstep \mathcal{F} \mathcal{S})^{\leftrightarrow*}$   
 <proof>

**lemma** *gtrs-CE*:  
**assumes** *gtrs*: *ground-sys*  $\mathcal{R}$  *ground-sys*  $\mathcal{S}$   
**and** *ce*:  $CE (gsrstep \mathcal{F} \mathcal{R}) (gsrstep \mathcal{F} \mathcal{S})$   
**shows**  $CE (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$   
 <proof>

end

## 8 Reducing Rewrite Properties to Properties on Ground Terms over Linear Variable-Separated Systems

**theory** *Ground-Reduction-on-LV*  
**imports**

*Rewriting-Properties*  
*Rewriting-LLRG-LV-Mondaic*

**begin**

**lemma** *lv-linear-sys*:  $lv \mathcal{R} \implies linear\text{-}sys \mathcal{R}$   
 $\langle proof \rangle$

**lemma** *comp-rrstep-rel'-sig-mono*:  
 $\mathcal{F} \subseteq \mathcal{G} \implies comp\text{-}rrstep\text{-}rel' \mathcal{F} \mathcal{R} \mathcal{S} \subseteq comp\text{-}rrstep\text{-}rel' \mathcal{G} \mathcal{R} \mathcal{S}$   
 $\langle proof \rangle$

**lemma** *srsteps-eqD*:  $(s, t) \in (srstep \mathcal{F} \mathcal{R})^* \implies (s, t) \in (rstep \mathcal{R})^*$   
 $\langle proof \rangle$

## 9 Linear-variable separated results

**locale** *open-terms-two-const-lv* =  
**fixes**  $\mathcal{R} :: ('f, 'v) \text{ term rel}$  **and**  $\mathcal{F} \ c \ d$   
**assumes** *lv*:  $lv \mathcal{R}$  **and** *sig*:  $funas\text{-}rel \ \mathcal{R} \subseteq \mathcal{F}$   
**and** *fresh*:  $(c, 0) \notin \mathcal{F} \ (d, 0) \notin \mathcal{F}$   
**and** *diff*:  $c \neq d$   
**begin**

**abbreviation**  $\mathcal{H} \equiv insert \ (c, 0) \ (insert \ (d, 0) \ \mathcal{F})$

**abbreviation**  $\sigma_c \equiv const\text{-}subst \ c$

**abbreviation**  $\sigma_d \equiv const\text{-}subst \ d$

**lemma** *sig-mono*:  $\mathcal{F} \subseteq \mathcal{H} \ \langle proof \rangle$

**lemma** *fresh-sym-c*:  $(c, 0) \notin funas\text{-}rel \ \mathcal{R} \ \langle proof \rangle$

**lemma** *fresh-sym-d*:  $(d, 0) \notin funas\text{-}rel \ \mathcal{R} \ \langle proof \rangle$

**lemma** *fresh-sym-c-inv*:  $(c, 0) \notin funas\text{-}rel \ (\mathcal{R}^{-1}) \ \langle proof \rangle$

**lemma** *fresh-sym-d-inv*:  $(d, 0) \notin funas\text{-}rel \ (\mathcal{R}^{-1}) \ \langle proof \rangle$

**lemmas** *all-fresh* = *fresh-sym-c fresh-sym-d fresh-sym-c-inv fresh-sym-d-inv*

**lemma** *sig-inv*:  $funas\text{-}rel \ (\mathcal{R}^{-1}) \subseteq \mathcal{F} \ \langle proof \rangle$

**lemma** *lv-inv*:  $lv \ (\mathcal{R}^{-1}) \ \langle proof \rangle$

**lemma** *well-subst*:

$\bigwedge x. funas\text{-}term \ ((const\text{-}subst \ c) \ x) \subseteq \mathcal{H}$

$\bigwedge x. funas\text{-}term \ ((const\text{-}subst \ d) \ x) \subseteq \mathcal{H}$

$\langle proof \rangle$

**lemma** *srsteps-with-root-step-to-grsteps*:

**assumes**  $(s, t) \in srsteps\text{-}with\text{-}root\text{-}step \ \mathcal{F} \ \mathcal{R}$

**shows**  $(s \cdot \sigma_c, t \cdot \sigma_d) \in (gsrstep \ \mathcal{H} \ \mathcal{R})^*$

$\langle proof \rangle$

**lemma** *comp-rrstep-rel'-to-grsteps*:

**assumes**  $(s, t) \in \text{comp-rrstep-rel}' \mathcal{F} (\mathcal{R}^{-1}) \mathcal{R}$   
**shows**  $(s \cdot \sigma_c, t \cdot \sigma_d) \in (\text{gsrstep } \mathcal{H} (\mathcal{R}^{-1}))^* O (\text{gsrstep } \mathcal{H} \mathcal{R})^*$   
 ⟨proof⟩

**lemma** *gsrsteps-eq-to-srsteps*:  
**assumes**  $(s \cdot \sigma_c, t \cdot \sigma_d) \in (\text{gsrstep } \mathcal{H} \mathcal{R})^*$   
**and** *funas*: *funas-term*  $s \subseteq \mathcal{F}$  *funas-term*  $t \subseteq \mathcal{F}$   
**shows**  $(s, t) \in (\text{srrstep } \mathcal{F} \mathcal{R})^*$   
 ⟨proof⟩

**lemma** *convert-NF-to-GNF*:  
*funas-term*  $t \subseteq \mathcal{F} \implies t \in \text{NF} (\text{srrstep } \mathcal{F} \mathcal{R}) \implies t \cdot \sigma_c \in \text{NF} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
*funas-term*  $t \subseteq \mathcal{F} \implies t \in \text{NF} (\text{srrstep } \mathcal{F} \mathcal{R}) \implies t \cdot \sigma_d \in \text{NF} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
 ⟨proof⟩

**lemma** *convert-GNF-to-NF*:  
*funas-term*  $t \subseteq \mathcal{F} \implies t \cdot \sigma_c \in \text{NF} (\text{gsrstep } \mathcal{H} \mathcal{R}) \implies t \in \text{NF} (\text{srrstep } \mathcal{F} \mathcal{R})$   
*funas-term*  $t \subseteq \mathcal{F} \implies t \cdot \sigma_d \in \text{NF} (\text{gsrstep } \mathcal{H} \mathcal{R}) \implies t \in \text{NF} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-CR*:  
**assumes** *cr*:  $\text{CR} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
**shows**  $\text{CR} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-WCR*:  
**assumes** *wcr*:  $\text{WCR} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
**shows**  $\text{WCR} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-NFP*:  
**assumes** *nfp*:  $\text{NFP} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
**shows**  $\text{NFP} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-UNF*:  
**assumes** *unf*:  $\text{UNF} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
**shows**  $\text{UNF} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-UNC*:  
**assumes** *unc*:  $\text{UNC} (\text{gsrstep } \mathcal{H} \mathcal{R})$   
**shows**  $\text{UNC} (\text{srrstep } \mathcal{F} \mathcal{R})$   
 ⟨proof⟩

**lemma** *lv-SCR*:

**assumes** *scr*:  $SCR (gsrstep \mathcal{H} \mathcal{R})$   
**shows**  $SCR (srstep \mathcal{F} \mathcal{R})$   
 $\langle proof \rangle$

**end**

**locale** *open-terms-two-const-lv-two-sys* =  
*open-terms-two-const-lv*  $\mathcal{R}$   
**for**  $\mathcal{R} :: ('f, 'v) \text{ term rel} +$   
**fixes**  $\mathcal{S} :: ('f, 'v) \text{ term rel}$   
**assumes** *lv-S*:  $lv \mathcal{S}$  **and** *sig-S*:  $funas\text{-rel } \mathcal{S} \subseteq \mathcal{F}$   
**begin**

**lemma** *fresh-sym-c-S*:  $(c, 0) \notin funas\text{-rel } \mathcal{S} \langle proof \rangle$

**lemma** *fresh-sym-d-S*:  $(d, 0) \notin funas\text{-rel } \mathcal{S} \langle proof \rangle$

**lemma** *lv-commute*:

**assumes** *com*:  $commute (gsrstep \mathcal{H} \mathcal{R}) (gsrstep \mathcal{H} \mathcal{S})$

**shows**  $commute (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$

$\langle proof \rangle$

**lemma** *lv-NE*:

**assumes** *ne*:  $NE (gsrstep \mathcal{H} \mathcal{R}) (gsrstep \mathcal{H} \mathcal{S})$

**shows**  $NE (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$

$\langle proof \rangle$

**end**

— CE is special as it only needs one additional constant therefore not included in the locale

**lemma** *lv-CE-aux*:

**assumes**  $(s, t) \in srsteps\text{-with-root-step } \mathcal{F} (\mathcal{R}^{\leftrightarrow})$

**and** *sig*:  $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \text{ funas-rel } \mathcal{S} \subseteq \mathcal{F}$

**and** *fresh*:  $(c, 0) \notin \mathcal{F}$  **and** *const*:  $(a, 0) \in \mathcal{F}$

**and** *lv*:  $lv \mathcal{R} \text{ lv } \mathcal{S}$

**and** *ce*:  $CE (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{R}) (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{S})$

**shows**  $(s, t) \in (srstep \mathcal{F} \mathcal{S})^{\leftrightarrow*}$

$\langle proof \rangle$

**lemma** *lv-CE*:

**assumes** *sig*:  $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \text{ funas-rel } \mathcal{S} \subseteq \mathcal{F}$

**and** *fresh*:  $(c, 0) \notin \mathcal{F}$  **and** *const*:  $(a, 0) \in \mathcal{F}$

**and** *lv*:  $lv \mathcal{R} \text{ lv } \mathcal{S}$

**and** *ce*:  $CE (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{R}) (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{S})$

**shows**  $CE (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$   
*<proof>*

## 9.1 Specialized for monadic signature

**lemma** *lv-NE-aux*:

**assumes**  $(s, t) \in srsteps\text{-with-root-step } \mathcal{F} \mathcal{R}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *sig*:  $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \text{ funas-rel } \mathcal{S} \subseteq \mathcal{F}$   
**and** *lv*:  $lv \mathcal{R} \text{ lv } \mathcal{S}$   
**and** *mon*: *monadic*  $\mathcal{F}$   
**and** *ne*:  $NE (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{R}) (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{S})$   
**shows**  $NE\text{-redp } \mathcal{F} \mathcal{R} \mathcal{S} s t$   
*<proof>*

**lemma** *lv-NE*:

**assumes** *sig*:  $funas\text{-rel } \mathcal{R} \subseteq \mathcal{F} \text{ funas-rel } \mathcal{S} \subseteq \mathcal{F}$   
**and** *mon*: *monadic*  $\mathcal{F}$  **and** *fresh*:  $(c, 0) \notin \mathcal{F}$   
**and** *lv*:  $lv \mathcal{R} \text{ lv } \mathcal{S}$   
**and** *ne*:  $NE (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{R}) (gsrstep (insert (c, 0) \mathcal{F}) \mathcal{S})$   
**shows**  $NE (srstep \mathcal{F} \mathcal{R}) (srstep \mathcal{F} \mathcal{S})$   
*<proof>*

**end**

## References

- [1] C. Sternagel and R. Thiemann. Abstract rewriting. *Archive of Formal Proofs*, June 2010. <https://isa-afp.org/entries/Abstract-Rewriting.html>, Formal proof development.