

Relational Characterisations of Paths

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March 24, 2023

Abstract

Binary relations are one of the standard ways to encode, characterise and reason about graphs. Relation algebras provide equational axioms for a large fragment of the calculus of binary relations. Although relations are standard tools in many areas of mathematics and computing, researchers usually fall back to point-wise reasoning when it comes to arguments about paths in a graph. We present a purely algebraic way to specify different kinds of paths in Kleene relation algebras, which are relation algebras equipped with an operation for reflexive transitive closure. We study the relationship between paths with a designated root vertex and paths without such a vertex. Since we stay in first-order logic this development helps with mechanising proofs. To demonstrate the applicability of the algebraic framework we verify the correctness of three basic graph algorithms.

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Overview

A path in a graph can be defined as a connected subgraph of edges where each vertex has at most one incoming edge and at most one outgoing edge [3, 12]. We develop a theory of paths based on this representation and use it for algorithm verification. All reasoning is done in variants of relation algebras and Kleene algebras [8, 9, 11].

Section 1 presents fundamental results that hold in relation algebras. Relation-algebraic characterisations of various kinds of paths are introduced and compared in Section 2. We extend this to paths with a designated root in Section 3. Section 4 verifies the correctness of a few basic graph algorithms.

These Isabelle/HOL theories formally verify results in [2]. See this paper for further details and related work.

1 (More) Relation Algebra

This theory presents fundamental properties of relation algebras, which are not present in the AFP entry on relation algebras but could be integrated there [1]. Many theorems concern vectors and points.

theory *More-Relation-Algebra*

imports *Relation-Algebra.Relation-Algebra-RTC*
Relation-Algebra.Relation-Algebra-Functions

begin

no-notation
trancl ((⁺) [1000] 999)

context *relation-algebra*
begin

notation
converse ((^T) [102] 101)

abbreviation *bijjective*
where *bijjective* $x \equiv is-inj\ x \wedge is-sur\ x$

abbreviation *reflexive*
where *reflexive* $R \equiv 1' \leq R$

abbreviation *symmetric*
where *symmetric* $R \equiv R = R^T$

abbreviation *transitive*

where *transitive* $R \equiv R;R \leq R$

General theorems

lemma *x-leq-triple-x*:

$x \leq x;x^T;x$

proof –

have $x = x;1' \cdot 1$

by *simp*

also have $\dots \leq (x \cdot 1;1'^T);(1' \cdot x^T;1)$

by (*rule dedekind*)

also have $\dots = x;(x^T;1 \cdot 1')$

by (*simp add: inf commute*)

also have $\dots \leq x;(x^T \cdot 1';1'^T);(1 \cdot (x^T)^T;1')$

by (*metis comp-assoc dedekind mult-isol*)

also have $\dots \leq x;x^T;x$

by *simp*

finally show *?thesis* .

qed

lemma *inj-triple*:

assumes *is-inj* x

shows $x = x;x^T;x$

by (*metis assms order.eq-iff inf-absorb2 is-inj-def mult-1-left mult-subdistr x-leq-triple-x*)

lemma *p-fun-triple*:

assumes *is-p-fun* x

shows $x = x;x^T;x$

by (*metis assms comp-assoc order.eq-iff is-p-fun-def mult-isol mult-oner x-leq-triple-x*)

lemma *loop-backward-forward*:

$x^T \leq -(1') + x$

by (*metis conv-e conv-times inf.cobounded2 test-dom test-domain test-eq-conv galois-2 inf commute sup commute*)

lemma *inj-sur-semi-swap*:

assumes *is-sur* z

and *is-inj* x

shows $z \leq y;x \implies x \leq y^T;z$

proof –

assume $z \leq y;x$

hence $z;x^T \leq y;(x;x^T)$

by (*metis mult-isol mult-assoc*)

hence $z;x^T \leq y$

using $\langle is-inj\ x \rangle$ **unfolding** *is-inj-def*

by (*metis mult-isol order.trans mult-1-right*)

hence $(z^T;z);x^T \leq z^T;y$

by (*metis mult-isol mult-assoc*)
 hence $x^T \leq z^T; y$
 using $\langle is-sur\ z \rangle$ **unfolding** *is-sur-def*
 by (*metis mult-isol order.trans mult-1-left*)
 thus *?thesis*
 using *conv-iso* by *fastforce*
qed

lemma *inj-sur-semi-swap-short*:

assumes *is-sur* z
 and *is-inj* x
 shows $z \leq y^T; x \implies x \leq y; z$

proof –

assume *as*: $z \leq y^T; x$
 hence $z; x^T \leq y^T$
 using $\langle z \leq y^T; x \rangle$ $\langle is-inj\ x \rangle$ **unfolding** *is-inj-def*
 by (*metis assms(2) conv-invol inf.orderI inf-absorb1 inj-p-fun ss-422iii*)
 hence $x^T \leq z^T; y^T$
 using $\langle is-sur\ z \rangle$ **unfolding** *is-sur-def*
 by (*metis as assms inj-sur-semi-swap conv-contrav conv-invol conv-iso*)
 thus $x \leq y; z$
 using *conv-iso* by *fastforce*
qed

lemma *bij-swap*:

assumes *bijective* z
 and *bijective* x
 shows $z \leq y^T; x \longleftrightarrow x \leq y; z$
 by (*metis assms inj-sur-semi-swap conv-invol*)

The following result is [10, Proposition 4.2.2(iv)].

lemma *ss422iv*:

assumes *is-p-fun* y
 and $x \leq y$
 and $y; 1 \leq x; 1$
 shows $x = y$
proof –
 have $y \leq (x; 1) \cdot y$
 using *assms(3) le-infI maddux-20 order-trans* by *blast*
 also have $\dots \leq x; x^T; y$
 by (*metis inf-top-left modular-1-var comp-assoc*)
 also have $\dots \leq x; y^T; y$
 using *assms(2) conv-iso mult-double-iso* by *blast*
 also have $\dots \leq x$
 using *assms(1) comp-assoc is-p-fun-def mult-isol mult-1-right*
 by *fastforce*
finally show *?thesis*
 by (*simp add: assms(2) order.antisym*)
qed

The following results are variants of [10, Proposition 4.2.3].

lemma *ss423conv*:
assumes *bijjective x*
shows $x ; y \leq z \longleftrightarrow y \leq x^T ; z$
by (*metis assms conv-contrav conv-iso inj-p-fun is-map-def ss423 sur-total*)

lemma *ss423bij*:
assumes *bijjective x*
shows $y ; x^T \leq z \longleftrightarrow y \leq z ; x$
by (*simp add: assms is-map-def p-fun-inj ss423 total-sur*)

lemma *inj-distr*:
assumes *is-inj z*
shows $(x \cdot y);z = (x;z) \cdot (y;z)$
apply (*rule order.antisym*)
using *mult-subdistr-var* **apply** *blast*
using *assms conv-iso inj-p-fun p-fun-distl* **by** *fastforce*

lemma *test-converse*:
 $x \cdot 1' = x^T \cdot 1'$
by (*metis conv-e conv-times inf-le2 is-test-def test-eq-conv*)

lemma *injective-down-closed*:
assumes *is-inj x*
and $y \leq x$
shows *is-inj y*
by (*meson assms conv-iso dual-order.trans is-inj-def mult-isol-var*)

lemma *injective-sup*:
assumes *is-inj t*
and $e; t^T \leq 1'$
and *is-inj e*
shows *is-inj (t + e)*
proof –
have $1: t; e^T \leq 1'$
using *assms(2) conv-contrav conv-e conv-invol conv-iso* **by** *fastforce*
have $(t + e); (t + e)^T = t; t^T + t; e^T + e; t^T + e; e^T$
by (*metis conv-add distrib-left distrib-right' sup-assoc*)
also have $\dots \leq 1'$
using 1 *assms* **by** (*simp add: is-inj-def le-supI*)
finally show *?thesis*
unfolding *is-inj-def* .
qed

Some (more) results about vectors

lemma *vector-meet-comp*:
assumes *is-vector v*
and *is-vector w*
shows $v; w^T = v \cdot w^T$

by (metis assms conv-contrav conv-one inf-top-right is-vector-def vector-1)

lemma *vector-meet-comp'*:

assumes *is-vector v*

shows $v;v^T = v \cdot v^T$

using *assms vector-meet-comp* by blast

lemma *vector-meet-comp-x*:

$x;1;x^T = x;1 \cdot 1;x^T$

by (metis *comp-assoc inf-top.right-neutral is-vector-def one-idem-mult vector-1*)

lemma *vector-meet-comp-x'*:

$x;1;x = x;1 \cdot 1;x$

by (metis *inf-commute inf-top.right-neutral ra-1*)

lemma *vector-prop1*:

assumes *is-vector v*

shows $-v^T;v = 0$

by (metis *assms compl-inf-bot inf-top.right-neutral one-compl one-idem-mult vector-2*)

The following results and a number of others in this theory are from [5].

lemma *ee*:

assumes *is-vector v*

and $e \leq v; -v^T$

shows $e;e = 0$

proof –

have $e;v \leq 0$

by (metis *assms annir mult-isor vector-prop1 comp-assoc*)

thus *?thesis*

by (metis *assms(2) annil order.antisym bot-least comp-assoc mult-isol*)

qed

lemma *et*:

assumes *is-vector v*

and $e \leq v; -v^T$

and $t \leq v; v^T$

shows $e;t = 0$

and $e;t^T = 0$

proof –

have $e;t \leq v; -v^T; v; v^T$

by (metis *assms(2-3) mult-isol-var comp-assoc*)

thus $e;t = 0$

by (*simp add: assms(1) comp-assoc le-bot vector-prop1*)

next

have $t^T \leq v; v^T$

using *assms(3) conv-iso* by *fastforce*

hence $e;t^T \leq v; -v^T; v; v^T$

by (metis *assms(2) mult-isol-var comp-assoc*)

thus $e; t^T = 0$
by (*simp add: assms(1) comp-assoc le-bot vector-prop1*)
qed

Some (more) results about points

definition *point*
where *point* $x \equiv \text{is-vector } x \wedge \text{bijective } x$

lemma *point-swap*:
assumes *point* p
and *point* q
shows $p \leq x; q \longleftrightarrow q \leq x^T; p$
by (*metis assms conv-invol inj-sur-semi-swap point-def*)

Some (more) results about singletons

abbreviation *singleton*
where *singleton* $x \equiv \text{bijective } (x; 1) \wedge \text{bijective } (x^T; 1)$

lemma *singleton-injective*:
assumes *singleton* x
shows *is-inj* x
using *assms injective-down-closed maddux-20* **by** *blast*

lemma *injective-inv*:
assumes *is-vector* v
and *singleton* e
and $e \leq v; -v^T$
and $t \leq v; v^T$
and *is-inj* t
shows *is-inj* $(t + e)$
by (*metis assms singleton-injective injective-sup bot-least et(2)*)

lemma *singleton-is-point*:
assumes *singleton* p
shows *point* $(p; 1)$
by (*simp add: assms comp-assoc is-vector-def point-def*)

lemma *singleton-transp*:
assumes *singleton* p
shows *singleton* (p^T)
by (*simp add: assms*)

lemma *point-to-singleton*:
assumes *singleton* p
shows *singleton* $(1'. p; p^T)$
using *assms dom-def-aux-var dom-one is-vector-def point-def* **by** *fastforce*

lemma *singleton-singletonT*:
assumes *singleton* p

shows $p; p^T \leq 1'$
using *assms singleton-injective is-inj-def* **by** *blast*

Minimality

abbreviation *minimum*
where *minimum* $x \ v \equiv v \cdot -(x^T; v)$

Regressively finite

abbreviation *regressively-finite*
where *regressively-finite* $x \equiv \forall v . \text{is-vector } v \wedge v \leq x^T; v \longrightarrow v = 0$

lemma *regressively-finite-minimum*:
regressively-finite $R \implies \text{is-vector } v \implies v \neq 0 \implies \text{minimum } R \ v \neq 0$
using *galois-aux2* **by** *blast*

lemma *regressively-finite-irreflexive*:

assumes *regressively-finite* x
shows $x \leq -1'$

proof –

have $1: \text{is-vector } ((x^T \cdot 1'); 1)$
by (*simp add: is-vector-def mult-assoc*)
have $(x^T \cdot 1'); 1 = (x^T \cdot 1'); (x^T \cdot 1'); 1$
by (*simp add: is-test-def test-comp-eq-mult*)
with 1 **have** $(x^T \cdot 1'); 1 = 0$
by (*metis assms comp-assoc mult-subdistr*)
thus *?thesis*

by (*metis conv-e conv-invol conv-times conv-zero galois-aux ss-p18*)

qed

end

1.1 Relation algebras satisfying the Tarski rule

class *relation-algebra-tarski* = *relation-algebra* +
assumes *tarski*: $x \neq 0 \longleftrightarrow 1; x; 1 = 1$
begin

Some (more) results about points

lemma *point-equations*:

assumes *is-point* p

shows $p; 1 = p$

and $1; p = 1$

and $p^T; 1 = 1$

and $1; p^T = p^T$

apply (*metis assms is-point-def is-vector-def*)

using *assms is-point-def is-vector-def tarski vector-comp* **apply** *fastforce*

apply (*metis assms conv-contrav conv-one conv-zero is-point-def is-vector-def tarski*)

by (*metis assms conv-contrav conv-one is-point-def is-vector-def*)

The following result is [10, Proposition 2.4.5(i)].

lemma *point-singleton*:
assumes *is-point* p
and *is-vector* v
and $v \neq 0$
and $v \leq p$
shows $v = p$
proof –
have $1;v = 1$
using *assms(2,3) comp-assoc is-vector-def tarski* **by** *fastforce*
hence $p = 1;v \cdot p$
by *simp*
also have $\dots \leq (1 \cdot p;v^T);(v \cdot 1^T;p)$
using *dedekind* **by** *blast*
also have $\dots \leq p;v^T;v$
by (*simp add: mult-subdistl*)
also have $\dots \leq p;p^T;v$
using *assms(4) conv-iso mult-double-iso* **by** *blast*
also have $\dots \leq v$
by (*metis assms(1) is-inj-def is-point-def mult-isor mult-onel*)
finally show *?thesis*
using *assms(4)* **by** *simp*
qed

lemma *point-not-equal-aux*:
assumes *is-point* p
and *is-point* q
shows $p \neq q \iff p \cdot -q \neq 0$
proof
show $p \neq q \implies p \cdot -q \neq 0$
proof (*rule contrapos-nn*)
assume $p \cdot -q = 0$
thus $p = q$
using *assms galois-aux2 is-point-def point-singleton* **by** *fastforce*
qed
next
show $p \cdot -q \neq 0 \implies p \neq q$
using *inf-compl-bot* **by** *blast*
qed

The following result is part of [10, Proposition 2.4.5(ii)].

lemma *point-not-equal*:
assumes *is-point* p
and *is-point* q
shows $p \neq q \iff p \leq -q$
and $p \leq -q \iff p;q^T \leq -1'$
and $p;q^T \leq -1' \iff p^T;q \leq 0$
proof –
have $p \neq q \implies p \leq -q$

by (*metis* *assms* *point-not-equal-aux* *is-point-def* *vector-compl* *vector-mult* *point-singleton*
inf.orderI *inf.cobounded1*)
thus $p \neq q \longleftrightarrow p \leq -q$
by (*metis* *assms*(1) *galois-aux* *inf.orderE* *is-point-def* *order.refl*)
next
show $(p \leq -q) = (p ; q^T \leq -1')$
by (*simp* *add*: *conv-galois-2*)
next
show $(p ; q^T \leq -1') = (p^T ; q \leq 0)$
by (*metis* *assms*(2) *compl-bot-eq* *conv-galois-2* *galois-aux* *maddux-141* *mult-1-right*
point-equations(4))
qed

lemma *point-is-point*:
point $x \longleftrightarrow$ *is-point* x
apply (*rule* *iffI*)
apply (*simp* *add*: *is-point-def* *point-def* *surj-one* *tarski*)
using *is-point-def* *is-vector-def* *mult-assoc* *point-def* *sur-def-var1* *tarski* **by**
fastforce

lemma *point-in-vector-or-complement*:
assumes *point* p
and *is-vector* v
shows $p \leq v \vee p \leq -v$
proof (*cases* $p \leq -v$)
assume $p \leq -v$
thus *?thesis*
by *simp*
next
assume $\neg(p \leq -v)$
hence $p \cdot v \neq 0$
by (*simp* *add*: *galois-aux*)
hence $1 ; (p \cdot v) = 1$
using *assms* *comp-assoc* *is-vector-def* *point-def* *tarski* *vector-mult* **by** *fastforce*
hence $p \leq p ; (p \cdot v)^T ; (p \cdot v)$
by (*metis* *inf-top*.*left-neutral* *modular-2-var*)
also have $\dots \leq p ; p^T ; v$
by (*simp* *add*: *mult-isol-var*)
also have $\dots \leq v$
using *assms*(1) *comp-assoc* *point-def* *ss423conv* **by** *fastforce*
finally show *?thesis* ..
qed

lemma *point-in-vector-or-complement-iff*:
assumes *point* p
and *is-vector* v
shows $p \leq v \longleftrightarrow \neg(p \leq -v)$

by (*metis assms annir compl-top-eq galois-aux inf.orderE one-compl point-def ss423conv tarski top-greatest point-in-vector-or-complement*)

lemma *different-points-consequences*:

assumes *point p*
 and *point q*
 and $p \neq q$
 shows $p^T; -q = 1$
 and $-q^T; p = 1$
 and $-(p^T; -q) = 0$
 and $-(-q^T; p) = 0$
proof –
 have $p \leq -q$
 by (*metis assms compl-le-swap1 inf.absorb1 inf.absorb2 point-def point-in-vector-or-complement*)
 thus 1: $p^T; -q = 1$
 using *assms(1)* by (*metis is-vector-def point-def ss423conv top-le*)
 thus 2: $-q^T; p = 1$
 using *conv-compl conv-one* by *force*
 from 1 show $-(p^T; -q) = 0$
 by *simp*
 from 2 show $-(-q^T; p) = 0$
 by *simp*
qed

Some (more) results about singletons

lemma *singleton-pq*:

assumes *point p*
 and *point q*
 shows *singleton (p; q^T)*
 using *assms comp-assoc point-def point-equations(1,3) point-is-point* by *fastforce*

lemma *singleton-equal-aux*:

assumes *singleton p*
 and *singleton q*
 and $q \leq p$
 shows $p \leq q; 1$
proof –
 have $pLp: p; 1; p^T \leq 1'$
 by (*simp add: assms(1) maddux-21 ss423conv*)

 have $p = 1; (q^T; q; 1) \cdot p$
 using *tarski*
 by (*metis assms(2) annir singleton-injective inf commute inf-top.right-neutral inj-triple mult-assoc surj-one*)
 also have $\dots \leq (1 \cdot p; (q^T; q; 1)^T); (q^T; q; 1 \cdot 1; p)$
 using *dedekind* by (*metis conv-one*)

also have $\dots \leq p; 1; q^T; q; q^T; q; 1$
by (*simp add: comp-assoc mult-isol*)
also have $\dots \leq p; 1; p^T; q; q^T; q; 1$
using *assms(3)* **by** (*metis comp-assoc conv-iso mult-double-iso*)
also have $\dots \leq 1'; q; q^T; q; 1$
using *pLp* **using** *mult-isor* **by** *blast*
also have $\dots \leq q; 1$
using *assms(2)* *singleton-singletonT* **by** (*simp add: comp-assoc mult-isol*)
finally show *?thesis* .
qed

lemma *singleton-equal*:

assumes *singleton p*
and *singleton q*
and $q \leq p$
shows $q = p$

proof –

have $p1: p \leq q; 1$
using *assms* **by** (*rule singleton-equal-aux*)
have $p^T \leq q^T; 1$
using *assms singleton-equal-aux singleton-transp conv-iso* **by** *fastforce*
hence $p2: p \leq 1; q$
using *conv-iso* **by** *force*

have $p \leq q; 1 \cdot 1; q$
using *p1 p2 inf.boundedI* **by** *blast*
also have $\dots \leq (q \cdot 1; q; 1); (1 \cdot q^T; 1; q)$
using *dedekind* **by** (*metis comp-assoc conv-one*)
also have $\dots \leq q; q^T; 1; q$
by (*simp add: mult-isor comp-assoc*)
also have $\dots \leq q; 1'$
by (*metis assms(2) conv-contrav conv-invol conv-one is-inj-def mult-assoc mult-isol*)

one-idem-mult)

also have $\dots \leq q$
by *simp*
finally have $p \leq q$.
thus $q = p$
using *assms(3)* **by** *simp*

qed

lemma *singleton-nonsplit*:

assumes *singleton p*
and $x \leq p$
shows $x = 0 \vee x = p$

proof (*cases x=0*)

assume $x = 0$
thus *?thesis* ..

next

```

assume 1:  $x \neq 0$ 
have singleton  $x$ 
proof (safe)
  show is-inj ( $x; 1$ )
    using assms injective-down-closed mult-isor by blast
  show is-inj ( $x^T; 1$ )
    using assms conv-iso injective-down-closed mult-isol-var by blast
  show is-sur ( $x; 1$ )
    using 1 comp-assoc sur-def-var1 tarski by fastforce
  thus is-sur ( $x^T; 1$ )
    by (metis conv-contrav conv-one mult.semigroup-axioms sur-def-var1
semigroup.assoc)
  qed
thus ?thesis
  using assms singleton-equal by blast
qed

```

```

lemma singleton-nonzero:
  assumes singleton  $p$ 
  shows  $p \neq 0$ 
proof
  assume  $p = 0$ 
  hence point 0
  using assms singleton-is-point by fastforce
  thus False
  by (simp add: is-point-def point-is-point)
qed

```

```

lemma singleton-sum:
  assumes singleton  $p$ 
  shows  $p \leq x + y \iff (p \leq x \vee p \leq y)$ 
proof
  show  $p \leq x + y \implies p \leq x \vee p \leq y$ 
  proof –
    assume as:  $p \leq x + y$ 
    show  $p \leq x \vee p \leq y$ 
    proof (cases  $p \leq x$ )
      assume  $p \leq x$ 
      thus ?thesis ..
    next
      assume a:  $\neg(p \leq x)$ 
      hence  $p \cdot x \neq p$ 
      using a inf.orderI by fastforce
      hence  $p \leq -x$ 
      using assms singleton-nonsplit galois-aux inf-le1 by blast
      hence  $p \leq y$ 
      using as by (metis galois-1 inf.orderE)
    thus ?thesis
    by simp

```

```

qed
qed
next
show  $p \leq x \vee p \leq y \implies p \leq x + y$ 
  using sup.coboundedI1 sup.coboundedI2 by blast
qed

```

```

lemma singleton-iff:
  singleton  $x \longleftrightarrow x \neq 0 \wedge x^T;1;x + x;1;x^T \leq 1'$ 
by (smt comp-assoc conv-contrav conv-invol conv-one is-inj-def le-sup-iff
one-idem-mult
  sur-def-var1 tarski)

```

```

lemma singleton-not-atom-in-relation-algebra-tarski:
  assumes  $p \neq 0$ 
  and  $\forall x . x \leq p \longrightarrow x = 0 \vee x = p$ 
  shows singleton  $p$ 
nitpick [expect=genuine] oops

end

```

1.2 Relation algebras satisfying the point axiom

```

class relation-algebra-point = relation-algebra +
  assumes point-axiom:  $x \neq 0 \longrightarrow (\exists y z . \text{point } y \wedge \text{point } z \wedge y; z^T \leq x)$ 
begin

```

Some (more) results about points

```

lemma point-exists:
   $\exists x . \text{point } x$ 
by (metis (full-types) order.eq-iff is-inj-def is-sur-def is-vector-def point-axiom
point-def)

```

```

lemma point-below-vector:
  assumes is-vector  $v$ 
  and  $v \neq 0$ 
  shows  $\exists x . \text{point } x \wedge x \leq v$ 
proof -
  from assms(2) obtain  $y$  and  $z$  where  $1: \text{point } y \wedge \text{point } z \wedge y; z^T \leq v$ 
  using point-axiom by blast
  have  $z^T;1 = (1;z)^T$ 
  using conv-contrav conv-one by simp
  hence  $y;(1;z)^T \leq v$ 
  using  $1$  by (metis assms(1) comp-assoc is-vector-def mult-isor)
  thus ?thesis
  using  $1$  by (metis conv-one is-vector-def point-def sur-def-var1)
qed

```

```

end

```

class *relation-algebra-tarski-point* = *relation-algebra-tarski* +
relation-algebra-point
begin

lemma *atom-is-singleton*:

assumes $p \neq 0$
and $\forall x . x \leq p \longrightarrow x = 0 \vee x = p$
shows *singleton* p
by (*metis* *assms* *singleton-nonzero* *singleton-pq* *point-axiom*)

lemma *singleton-iff-atom*:

singleton $p \iff p \neq 0 \wedge (\forall x . x \leq p \longrightarrow x = 0 \vee x = p)$
using *singleton-nonsplit* *singleton-nonzero* *atom-is-singleton* **by** *blast*

lemma *maddux-tarski*:

assumes $x \neq 0$
shows $\exists y . y \neq 0 \wedge y \leq x \wedge \text{is-}p\text{-fun } y$
proof –
obtain p q **where** $1: \text{point } p \wedge \text{point } q \wedge p; q^T \leq x$
using *assms* *point-axiom* **by** *blast*
hence $2: p; q^T \neq 0$
by (*simp* *add: singleton-nonzero* *singleton-pq*)
have *is-}p-fun* $(p; q^T)$
using 1 **by** (*meson* *singleton-singletonT* *singleton-pq* *singleton-transp*
is-inj-def *p-fun-inj*)
thus *?thesis*
using 1 2 **by** *force*
qed

Intermediate Point Theorem [10, Proposition 2.4.8]

lemma *intermediate-point-theorem*:

assumes *point* p
and *point* r
shows $p \leq x; y; r \iff (\exists q . \text{point } q \wedge p \leq x; q \wedge q \leq y; r)$
proof
assume $1: p \leq x; y; r$
let $?v = x^T; p \cdot y; r$
have $2: \text{is-vector } ?v$
using *assms* *comp-assoc* *is-vector-def* *point-def* *vector-mult* **by** *fastforce*
have $?v \neq 0$
using 1 **by** (*metis* *assms*(1) *inf.absorb2* *is-point-def* *maddux-141*
point-is-point *mult.assoc*)
hence $\exists q . \text{point } q \wedge q \leq ?v$
using 2 *point-below-vector* **by** *blast*
thus $\exists q . \text{point } q \wedge p \leq x; q \wedge q \leq y; r$
using *assms*(1) *point-swap* **by** *auto*
next
assume $\exists q . \text{point } q \wedge p \leq x; q \wedge q \leq y; r$
thus $p \leq x; y; r$

using *comp-assoc mult-isol order-trans* by *fastforce*
qed

end

context *relation-algebra*
begin

lemma *unfoldl-inductl-implies-unfoldr*:

assumes $\bigwedge x. 1' + x; (rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z. x+y; z \leq z \implies rtc(y); x \leq z$
shows $1' + rtc(x); x \leq rtc\ x$

by (*metis assms le-sup-iff mult-oner order.trans subdistl-eq sup-absorb2 sup-ge1*)

lemma *star-transpose-swap*:

assumes $\bigwedge x. 1' + x; (rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z. x+y; z \leq z \implies rtc(y); x \leq z$
shows $rtc(x^T) = (rtc\ x)^T$

apply(*simp only: order.eq-iff; rule conjI*)

apply (*metis assms conv-add conv-contrav conv-e conv-iso mult-1-right*
unfoldl-inductl-implies-unfoldr)

by (*metis assms conv-add conv-contrav conv-e conv-invol conv-iso mult-1-right*
unfoldl-inductl-implies-unfoldr)

lemma *unfoldl-inductl-implies-inductr*:

assumes $\bigwedge x. 1' + x; (rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z. x+y; z \leq z \implies rtc(y); x \leq z$
shows $x+z; y \leq z \implies x; rtc(y) \leq z$

by (*metis assms conv-add conv-contrav conv-iso star-transpose-swap*)

end

context *relation-algebra-rtc*
begin

abbreviation *tc* $((-^+) [101] 100)$ where $tc\ x \equiv x; x^*$

abbreviation *is-acyclic*

where *is-acyclic* $x \equiv x^+ \leq -1'$

General theorems

lemma *star-denest-10*:

assumes $x; y=0$
shows $(x+y)^* = y; y^*; x^* + x^*$

using *assms bubble-sort sup commute* by *auto*

lemma *star-star-plus*:

$x^* + y^* = x^+ + y^*$
by (*metis (full-types) sup.left-commute star-plus-one star-unfoldl-eq sup commute*)

The following two lemmas are from [6].

lemma *cancel-separate*:

assumes $x ; y \leq 1'$

shows $x^* ; y^* \leq x^* + y^*$

proof –

have $x ; y^* = x + x ; y ; y^*$

by (*metis comp-assoc conway.dagger-unfoldl-distr distrib-left mult-oner*)

also have $\dots \leq x + y^*$

by (*metis assms join-isol star-invol star-plus-one star-subdist-var-2 sup.absorb2 sup.assoc*)

also have $\dots \leq x^* + y^*$

using *join-iso* **by** *fastforce*

finally have $x ; (x^* + y^*) \leq x^* + y^*$

by (*simp add: distrib-left le-supI1*)

thus *?thesis*

by (*simp add: rtc-inductl*)

qed

lemma *cancel-separate-inj-converse*:

assumes *is-inj* x

shows $x^* ; x^{T^*} = x^* + x^{T^*}$

apply (*rule order.antisym*)

using *assms cancel-separate is-inj-def* **apply** *blast*

by (*metis conway.dagger-unfoldl-distr le-supI mult-1-right mult-isol sup.cobounded1*)

lemma *cancel-separate-p-fun-converse*:

assumes *is-p-fun* x

shows $x^{T^*} ; x^* = x^* + x^{T^*}$

using *sup-commute assms cancel-separate-inj-converse p-fun-inj* **by** *fastforce*

lemma *cancel-separate-converse-idempotent*:

assumes *is-inj* x

and *is-p-fun* x

shows $(x^* + x^{T^*});(x^* + x^{T^*}) = x^* + x^{T^*}$

by (*metis assms cancel-separate cancel-separate-p-fun-converse church-rosser-equiv is-inj-def star-denest-var-6*)

lemma *triple-star*:

assumes *is-inj* x

and *is-p-fun* x

shows $x^*;x^{T^*};x^* = x^* + x^{T^*}$

by (*simp add: assms cancel-separate-inj-converse cancel-separate-p-fun-converse*)

lemma *inj-xcts*:

assumes *is-inj* x
shows $x; x^{T^*} \leq x^* + x^{T^*}$
by (*metis* *assms* *cancel-separate-inj-converse* *distrib-right* *less-eq-def* *star-ext*)

lemma *plus-top*:
 $x^+; 1 = x; 1$
by (*metis* *comp-assoc* *conway.dagger-unfoldr-distr* *sup-top-left*)

lemma *top-plus*:
 $1; x^+ = 1; x$
by (*metis* *comp-assoc* *conway.dagger-unfoldr-distr* *star-denest-var-2* *star-ext* *star-slide-var* *sup-top-left* *top-unique*)

lemma *plus-conv*:
 $(x^+)^T = x^{T^+}$
by (*simp* *add*: *star-conv* *star-slide-var*)

lemma *inj-implies-step-forwards-backwards*:
assumes *is-inj* x
shows $x^*; (x^+ \cdot 1^'); 1 \leq x^T; 1$
proof –
have $(x^+ \cdot 1^'); 1 \leq (x^* \cdot x^T); (x \cdot (x^*)^T); 1$
by (*metis* *conv-contrav* *conv-e* *dedekind* *mult-1-right* *mult-isor* *star-slide-var*)
also have $\dots \leq (x^* \cdot x^T); 1$
by (*simp* *add*: *comp-assoc* *mult-isol*)
finally have $1; (x^+ \cdot 1^'); 1 \leq (x^* \cdot x^T); 1$.

have $x; (x^* \cdot x^T); 1 \leq (x^+ \cdot x; x^T); 1$
by (*metis* *inf-idem* *meet-interchange* *mult-isor*)
also have $\dots \leq (x^+ \cdot 1^'); 1$
using *assms* *is-inj-def* *meet-isor* *mult-isor* **by** *fastforce*
finally have $x; (x^* \cdot x^T); 1 \leq (x^+ \cdot 1^'); 1$
using 1 **by** *fastforce*
hence $x^*; (x^+ \cdot 1^'); 1 \leq (x^* \cdot x^T); 1$
using 1 **by** (*simp* *add*: *comp-assoc* *rtc-inductl*)
thus $x^*; (x^+ \cdot 1^'); 1 \leq x^T; 1$
using *inf.cobounded2* *mult-isor* *order-trans* **by** *blast*
qed

Acyclic relations

The following result is from [4].

lemma *acyclic-inv*:
assumes *is-acyclic* t
and *is-vector* v
and $e \leq v; -v^T$
and $t \leq v; v^T$
shows *is-acyclic* $(t + e)$
proof –

have $t^+;e \leq t^+;v;-v^T$
 by (*simp add: assms(3) mult-assoc mult-isol*)
also have $\dots \leq v;v^T;t^*;v;-v^T$
 by (*simp add: assms(4) mult-isol*)
also have $\dots \leq v;-v^T$
 by (*metis assms(2) mult-double-iso top-greatest is-vector-def mult-assoc*)
also have $\dots \leq -1'$
 by (*simp add: conv-galois-1*)
finally have $1: t^+;e \leq -1'$.
have $e \leq v;-v^T$
 using *assms(3)* by *simp*
also have $\dots \leq -1'$
 by (*simp add: conv-galois-1*)
finally have $2: t^+;e + e \leq -1'$
 using 1 by *simp*
have $3: e;t^* = e$
 by (*metis assms(2-4) et(1) independence2*)
have $4: e^* = 1' + e$
 using *assms(2-3) ee boffa-var bot-least* by *blast*
have $(t + e)^+ = (t + e);t^*;(e;t^*)^*$
 by (*simp add: comp-assoc*)
also have $\dots = (t + e);t^*;(1' + e)$
 using $3\ 4$ by *simp*
also have $\dots = t^*;(1' + e) + e;t^*;(1' + e)$
 by *simp*
also have $\dots = t^*;(1' + e) + e;(1' + e)$
 using 3 by *simp*
also have $\dots = t^*;(1' + e) + e$
 using 4 *assms(2-3) ee independence2* by *fastforce*
also have $\dots = t^+ + t^+;e + e$
 by (*simp add: distrib-left*)
also have $\dots \leq -1'$
 using *assms(1) 2* by *simp*
finally show *?thesis* .

qed

lemma *acyclic-single-step*:

assumes *is-acyclic x*

shows $x \leq -1'$

by (*metis assms dual-order.trans mult-isol mult-oner star-ref*)

lemma *acyclic-reachable-points*:

assumes *is-point p*

and *is-point q*

and $p \leq x;q$

and *is-acyclic x*

shows $p \neq q$

proof

assume $p=q$

hence $p \leq x; q \cdot q$
by (*simp add: assms(3) order.eq-iff inf.absorb2*)
also have $\dots = (x \cdot 1')$; q
using *assms(2) inj-distr is-point-def* **by** *simp*
also have $\dots \leq (-1' \cdot 1')$; q
using *acyclic-single-step assms(4)* **by** (*metis abel-semigroup commute*
inf.abel-semigroup-axioms
meet-isor mult-isor)
also have $\dots = 0$
by *simp*
finally have $p \leq 0$.
thus *False*
using *assms(1) bot-unique is-point-def* **by** *blast*
qed

lemma *acyclic-trans*:
assumes *is-acyclic x*
shows $x \leq -(x^{T+})$
proof –
have $\exists c \geq x. c \leq -(x^+)^T$
by (*metis assms compl-mono conv-galois-2 conv-iso double-compl mult-onel*
star-1l)
thus *?thesis*
by (*metis dual-order.trans plus-conv*)
qed

lemma *acyclic-trans'*:
assumes *is-acyclic x*
shows $x^* \leq -(x^{T+})$
proof –
have $x^* \leq -(-(-x^T; -(-1'))); (x^*)^T$
by (*metis assms conv-galois-1 conv-galois-2 order-trans star-trans*)
then show *?thesis*
by (*simp add: star-conv*)
qed

Regressively finite

lemma *regressively-finite-acyclic*:
assumes *regressively-finite x*
shows *is-acyclic x*
proof –
have $1: \text{is-vector } ((x^+ \cdot 1'); 1)$
by (*simp add: is-vector-def mult-assoc*)
have $(x^+ \cdot 1'); 1 = (x^{T+} \cdot 1'); 1$
by (*metis plus-conv test-converse*)
also have $\dots \leq x^T; (1'; x^{T*} \cdot x); 1$
by (*metis conv-invol modular-1-var mult-isor mult-oner mult-onel*)
also have $\dots \leq x^T; (1' \cdot x^+); x^{T*}; 1$
by (*metis comp-assoc conv-invol modular-2-var mult-isol mult-isor star-conv*)

```

also have ... =  $x^T;(x^+ \cdot 1')$ ;1
  by (metis comp-assoc conway.dagger-unfoldr-distr inf.commute
sup.cobounded1 top-le)
finally have  $(x^+ \cdot 1')$ ;1 = 0
  using 1 assms by (simp add: comp-assoc)
thus ?thesis
  by (simp add: galois-aux ss-p18)
qed

notation power (infixr  $\uparrow$  80)

lemma power-suc-below-plus:
   $x \uparrow \text{Suc } n \leq x^+$ 
  apply (induct n)
  using mult-isol star-ref apply fastforce
by (simp add: mult-isol-var order-trans)

end

class relation-algebra-rtc-tarski = relation-algebra-rtc + relation-algebra-tarski
begin

lemma point-loop-not-acyclic:
  assumes is-point p
    and  $p \leq x \uparrow \text{Suc } n ; p$ 
  shows  $\neg$  is-acyclic x
proof –
  have  $p \leq x^+ ; p$ 
    by (meson assms dual-order.trans point-def point-is-point ss423bij
power-suc-below-plus)
  hence  $p ; p^T \leq x^+$ 
    using assms(1) point-def point-is-point ss423bij by blast
  thus ?thesis
    using assms(1) order.trans point-not-equal(1) point-not-equal(2) by blast
qed

end

class relation-algebra-rtc-point = relation-algebra-rtc + relation-algebra-point

class relation-algebra-rtc-tarski-point = relation-algebra-rtc-tarski +
  relation-algebra-rtc-point +
  relation-algebra-tarski-point

  Finite graphs: the axiom says the algebra has finitely many elements.
  This means the relations have a finite base set.

class relation-algebra-rtc-tarski-point-finite = relation-algebra-rtc-tarski-point +
  finite
begin

```

For a finite acyclic relation, the powers eventually vanish.

lemma *acyclic-power-vanishes*:

assumes *is-acyclic* x

shows $\exists n . x \uparrow \text{Suc } n = 0$

proof –

let $?n = \text{card } \{ p . \text{is-point } p \}$

let $?p = x \uparrow ?n$

have $?p = 0$

proof (*rule ccontr*)

assume $?p \neq 0$

from this obtain $p \ q$ **where** $1: \text{point } p \wedge \text{point } q \wedge p; q^T \leq ?p$

using *point-axiom* **by** *blast*

hence $2: p \leq ?p; q$

using *point-def ss423bij* **by** *blast*

have $\forall n \leq ?n . (\exists f . \forall i \leq n . \text{is-point } (f \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; f \ i \wedge f \ i \leq x \uparrow (i - j) ; f \ j))$

proof

fix n

show $n \leq ?n \longrightarrow (\exists f . \forall i \leq n . \text{is-point } (f \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; f \ i \wedge f \ i \leq x \uparrow (i - j) ; f \ j))$

proof (*induct* n)

case 0

thus *?case*

using $1 \ 2$ *point-is-point* **by** *fastforce*

next

case (*Suc* n)

fix n

assume $3: n \leq ?n \longrightarrow (\exists f . \forall i \leq n . \text{is-point } (f \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; f \ i \wedge f \ i \leq x \uparrow (i - j) ; f \ j))$

show $\text{Suc } n \leq ?n \longrightarrow (\exists f . \forall i \leq \text{Suc } n . \text{is-point } (f \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; f \ i \wedge f \ i \leq x \uparrow (i - j) ; f \ j))$

proof

assume $4: \text{Suc } n \leq ?n$

from this obtain f **where** $5: \forall i \leq n . \text{is-point } (f \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; f \ i \wedge f \ i \leq x \uparrow (i - j) ; f \ j)$

using 3 **by** *auto*

have $p \leq x \uparrow (?n - n) ; f \ n$

using 5 **by** *blast*

also have $\dots = x \uparrow (?n - n - \text{one-class.one}) ; x ; f \ n$

using 4 **by** (*metis* (*no-types*) *Suc-diff-le* *diff-Suc-1* *diff-Suc-Suc* *power-Suc2*)

finally obtain r **where** $6: \text{point } r \wedge p \leq x \uparrow (?n - \text{Suc } n) ; r \wedge r \leq x ; f$

n

using $1 \ 5$ *intermediate-point-theorem* *point-is-point* **by** *fastforce*

let $?g = \lambda m . \text{if } m = \text{Suc } n \text{ then } r \text{ else } f \ m$

have $\forall i \leq \text{Suc } n . \text{is-point } (?g \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n - i) ; ?g \ i \wedge ?g \ i \leq x \uparrow (i - j) ; ?g \ j)$

proof

fix i

```

show  $i \leq \text{Suc } n \longrightarrow \text{is-point } (?g \ i) \wedge (\forall j \leq i . p \leq x \uparrow (?n-i) ; ?g \ i \wedge ?g$ 
 $i \leq x \uparrow (i-j) ; ?g \ j)$ 
proof (cases  $i \leq n$ )
  case True
    thus ?thesis
    using 5 by simp
  next
    case False
      have  $\text{is-point } (?g \ (\text{Suc } n)) \wedge (\forall j \leq \text{Suc } n . p \leq x \uparrow (?n-\text{Suc } n) ; ?g$ 
 $(\text{Suc } n) \wedge ?g \ (\text{Suc } n) \leq x \uparrow (\text{Suc } n-j) ; ?g \ j)$ 
      proof
        show  $\text{is-point } (?g \ (\text{Suc } n))$ 
        using 6 point-is-point by fastforce
      next
        show  $\forall j \leq \text{Suc } n . p \leq x \uparrow (?n-\text{Suc } n) ; ?g \ (\text{Suc } n) \wedge ?g \ (\text{Suc } n) \leq$ 
 $x \uparrow (\text{Suc } n-j) ; ?g \ j$ 
        proof
          fix  $j$ 
          show  $j \leq \text{Suc } n \longrightarrow p \leq x \uparrow (?n-\text{Suc } n) ; ?g \ (\text{Suc } n) \wedge ?g \ (\text{Suc } n)$ 
 $\leq x \uparrow (\text{Suc } n-j) ; ?g \ j$ 
          proof
            assume 7:  $j \leq \text{Suc } n$ 
            show  $p \leq x \uparrow (?n-\text{Suc } n) ; ?g \ (\text{Suc } n) \wedge ?g \ (\text{Suc } n) \leq x \uparrow (\text{Suc}$ 
 $n-j) ; ?g \ j$ 
            proof
              show  $p \leq x \uparrow (?n-\text{Suc } n) ; ?g \ (\text{Suc } n)$ 
              using 6 by simp
            next
              show  $?g \ (\text{Suc } n) \leq x \uparrow (\text{Suc } n-j) ; ?g \ j$ 
              proof (cases  $j = \text{Suc } n$ )
                case True
                  thus ?thesis
                  by simp
                next
                  case False
                    hence  $f \ n \leq x \uparrow (n-j) ; f \ j$ 
                    using 5 7 by fastforce
                    hence  $x ; f \ n \leq x \uparrow (\text{Suc } n-j) ; f \ j$ 
                    using 7 False Suc-diff-le comp-assoc mult-isol by fastforce
                    thus ?thesis
                    using 6 False by fastforce
                  qed
                qed
              qed
            qed
          thus ?thesis
          by (simp add: False le-Suc-eq)
        qed
      qed

```

```

      qed
      thus  $\exists f . \forall i \leq \text{Suc } n . \text{is-point } (f i) \wedge (\forall j \leq i . p \leq x \uparrow (?n-i) ; f i \wedge f i$ 
 $\leq x \uparrow (i-j) ; f j)$ 
      by auto
      qed
      qed
      from this obtain f where 8:  $\forall i \leq ?n . \text{is-point } (f i) \wedge (\forall j \leq i . p \leq x \uparrow$ 
 $(?n-i) ; f i \wedge f i \leq x \uparrow (i-j) ; f j)$ 
      by fastforce
      let ?A = { k . k ≤ ?n }
      have f ' ?A ⊆ { p . is-point p }
      using 8 by blast
      hence card (f ' ?A) ≤ ?n
      by (simp add: card-mono)
      hence ¬ inj-on f ?A
      by (simp add: pigeonhole)
      from this obtain i j where 9:  $i \leq ?n \wedge j \leq ?n \wedge i \neq j \wedge f i = f j$ 
      by (metis (no-types, lifting) inj-on-def mem-Collect-eq)
      show False
      apply (cases i < j)
      using 8 9 apply (metis Suc-diff-le Suc-leI assms diff-Suc-Suc
order-less-imp-le
point-loop-not-acyclic)
      using 8 9 by (metis assms neqE point-loop-not-acyclic Suc-diff-le Suc-leI
assms diff-Suc-Suc
order-less-imp-le)
      qed
      thus ?thesis
      by (metis annir power.simps(2))
    qed

```

Hence finite acyclic relations are regressively finite.

lemma *acyclic-regressively-finite*:

assumes *is-acyclic* x
shows *regressively-finite* x

proof

have *is-acyclic* (x^T)
using *assms acyclic-trans'* *compl-le-swap1* *order-trans* *star-ref* by blast
from this obtain n where 1: $x^T \uparrow \text{Suc } n = 0$
using *acyclic-power-vanishes* by fastforce

fix v
show *is-vector* $v \wedge v \leq x^T ; v \longrightarrow v = 0$

proof

assume 2: *is-vector* $v \wedge v \leq x^T ; v$
have $v \leq x^T \uparrow \text{Suc } n ; v$
proof (*induct* n)
case 0
thus ?case


```

    using 2 by simp
  next
  case (Suc n)
  hence  $x^T ; v \leq x^T \uparrow \text{Suc} (\text{Suc } n) ; v$ 
    by (simp add: comp-assoc mult-isol)
  thus ?case
    using 2 dual-order.trans by blast
  qed
  thus  $v = 0$ 
    using 1 by (simp add: le-bot)
  qed
  qed
end

lemma acyclic-is-regressively-finite:
  is-acyclic  $x \longleftrightarrow$  regressively-finite  $x$ 
using acyclic-regressively-finite regressively-finite-acyclic by blast

end

end

```

2 Relational Characterisation of Paths

This theory provides the relation-algebraic characterisations of paths, as defined in Sections 3–5 of [2].

theory *Paths*

imports *More-Relation-Algebra*

begin

context *relation-algebra-tarski*

begin

lemma *path-concat-aux-0*:

assumes *is-vector* v

and $v \neq 0$

and $w;v^T \leq x$

and $v;z \leq y$

shows $w;1;z \leq x;y$

proof –

from *tarski assms(1,2)* **have** $1 = 1;v^T;v;1$

by (*metis conv-contrav conv-one eq-refl inf-absorb1 inf-top-left is-vector-def ra-2*)

hence $w;1;z = w;1;v^T;v;1;z$

by (*simp add: mult-isor mult-isol mult-assoc*)

also from *assms(1)* **have** $\dots = w;v^T;v;z$

by (*metis is-vector-def comp-assoc conv-contrav conv-one*)

also from *assms(3)* **have** $\dots \leq x;v;z$
 by (*simp add: mult-isol*)
also from *assms(4)* **have** $\dots \leq x;y$
 by (*simp add: mult-isol mult-assoc*)
finally show *?thesis* .
qed

end

2.1 Consequences without the Tarski rule

context *relation-algebra-rtc*
begin

Definitions for path classifications

abbreviation *connected*
where *connected* $x \equiv x;1;x \leq x^* + x^{T^*}$

abbreviation *many-strongly-connected*
where *many-strongly-connected* $x \equiv x^* = x^{T^*}$

abbreviation *one-strongly-connected*
where *one-strongly-connected* $x \equiv x^T;1;x^T \leq x^*$

definition *path*
where *path* $x \equiv \text{connected } x \wedge \text{is-p-fun } x \wedge \text{is-inj } x$

abbreviation *cycle*
where *cycle* $x \equiv \text{path } x \wedge \text{many-strongly-connected } x$

abbreviation *start-points*
where *start-points* $x \equiv x;1 \cdot -(x^T;1)$

abbreviation *end-points*
where *end-points* $x \equiv x^T;1 \cdot -(x;1)$

abbreviation *no-start-points*
where *no-start-points* $x \equiv x;1 \leq x^T;1$

abbreviation *no-end-points*
where *no-end-points* $x \equiv x^T;1 \leq x;1$

abbreviation *no-start-end-points*
where *no-start-end-points* $x \equiv x;1 = x^T;1$

abbreviation *has-start-points*
where *has-start-points* $x \equiv 1 = -(1;x);x;1$

abbreviation *has-end-points*
where *has-end-points* $x \equiv 1 = 1;x;-(x;1)$

abbreviation *has-start-end-points*
where *has-start-end-points* $x \equiv 1 = -(1;x);x;1 \cdot 1;x;-(x;1)$

abbreviation *backward-terminating*
where *backward-terminating* $x \equiv x \leq -(1;x);x;1$

abbreviation *forward-terminating*
where *forward-terminating* $x \equiv x \leq 1;x;-(x;1)$

abbreviation *terminating*
where *terminating* $x \equiv x \leq -(1;x);x;1 \cdot 1;x;-(x;1)$

abbreviation *backward-finite*
where *backward-finite* $x \equiv x \leq x^{T^*} + -(1;x);x;1$

abbreviation *forward-finite*
where *forward-finite* $x \equiv x \leq x^{T^*} + 1;x;-(x;1)$

abbreviation *finite*
where *finite* $x \equiv x \leq x^{T^*} + -(1;x);x;1 \cdot 1;x;-(x;1)$

abbreviation *no-start-points-path*
where *no-start-points-path* $x \equiv \text{path } x \wedge \text{no-start-points } x$

abbreviation *no-end-points-path*
where *no-end-points-path* $x \equiv \text{path } x \wedge \text{no-end-points } x$

abbreviation *no-start-end-points-path*
where *no-start-end-points-path* $x \equiv \text{path } x \wedge \text{no-start-end-points } x$

abbreviation *has-start-points-path*
where *has-start-points-path* $x \equiv \text{path } x \wedge \text{has-start-points } x$

abbreviation *has-end-points-path*
where *has-end-points-path* $x \equiv \text{path } x \wedge \text{has-end-points } x$

abbreviation *has-start-end-points-path*
where *has-start-end-points-path* $x \equiv \text{path } x \wedge \text{has-start-end-points } x$

abbreviation *backward-terminating-path*
where *backward-terminating-path* $x \equiv \text{path } x \wedge \text{backward-terminating } x$

abbreviation *forward-terminating-path*
where *forward-terminating-path* $x \equiv \text{path } x \wedge \text{forward-terminating } x$

abbreviation *terminating-path*
where *terminating-path* $x \equiv \text{path } x \wedge \text{terminating } x$

abbreviation *backward-finite-path*
where *backward-finite-path* $x \equiv \text{path } x \wedge \text{backward-finite } x$

abbreviation *forward-finite-path*
where *forward-finite-path* $x \equiv \text{path } x \wedge \text{forward-finite } x$

abbreviation *finite-path*
where *finite-path* $x \equiv \text{path } x \wedge \text{finite } x$

General properties

lemma *reachability-from-z-in-y*:
assumes $x \leq y^*; z$
and $x \cdot z = 0$
shows $x \leq y^+; z$
by (*metis* *assms* *conway.dagger-unfoldl-distr* *galois-1* *galois-aux* *inf.orderE*)

lemma *reachable-imp*:
assumes *point* p
and *point* q
and $p^*; q \leq p^{T^*}; p$
shows $p \leq p^*; q$
by (*metis* *assms* *conway.dagger-unfoldr-distr* *le-supE* *point-swap* *star-conv*)

Basic equivalences

lemma *no-start-end-points-iff*:
no-start-end-points $x \longleftrightarrow \text{no-start-points } x \wedge \text{no-end-points } x$
by *fastforce*

lemma *has-start-end-points-iff*:
has-start-end-points $x \longleftrightarrow \text{has-start-points } x \wedge \text{has-end-points } x$
by (*metis* *inf-eq-top-iff*)

lemma *terminating-iff*:
terminating $x \longleftrightarrow \text{backward-terminating } x \wedge \text{forward-terminating } x$
by *simp*

lemma *finite-iff*:
finite $x \longleftrightarrow \text{backward-finite } x \wedge \text{forward-finite } x$
by (*simp* *add: sup-inf-distrib1* *inf.boundedI*)

lemma *no-start-end-points-path-iff*:
no-start-end-points-path $x \longleftrightarrow \text{no-start-points-path } x \wedge \text{no-end-points-path } x$
by *fastforce*

lemma *has-start-end-points-path-iff*:
has-start-end-points-path $x \longleftrightarrow \text{has-start-points-path } x \wedge \text{has-end-points-path } x$
using *has-start-end-points-iff* **by** *blast*

lemma *terminating-path-iff*:

terminating-path $x \iff \text{backward-terminating-path } x \wedge \text{forward-terminating-path } x$
by *fastforce*

lemma *finite-path-iff*:
finite-path $x \iff \text{backward-finite-path } x \wedge \text{forward-finite-path } x$
using *finite-iff* **by** *fastforce*

Closure under converse

lemma *connected-conv*:
connected $x \iff \text{connected } (x^T)$
by (*metis comp-assoc conv-add conv-contrav conv-iso conv-one star-conv*)

lemma *conv-many-strongly-connected*:
many-strongly-connected $x \iff \text{many-strongly-connected } (x^T)$
by *fastforce*

lemma *conv-one-strongly-connected*:
one-strongly-connected $x \iff \text{one-strongly-connected } (x^T)$
by (*metis comp-assoc conv-contrav conv-iso conv-one star-conv*)

lemma *conv-path*:
path $x \iff \text{path } (x^T)$
using *connected-conv inj-p-fun path-def* **by** *fastforce*

lemma *conv-cycle*:
cycle $x \iff \text{cycle } (x^T)$
using *conv-path* **by** *fastforce*

lemma *conv-no-start-points*:
no-start-points $x \iff \text{no-end-points } (x^T)$
by *simp*

lemma *conv-no-start-end-points*:
no-start-end-points $x \iff \text{no-start-end-points } (x^T)$
by *fastforce*

lemma *conv-has-start-points*:
has-start-points $x \iff \text{has-end-points } (x^T)$
by (*metis comp-assoc conv-compl conv-contrav conv-invol conv-one*)

lemma *conv-has-start-end-points*:
has-start-end-points $x \iff \text{has-start-end-points } (x^T)$
by (*metis comp-assoc conv-compl conv-contrav conv-invol conv-one inf-eq-top-iff*)

lemma *conv-backward-terminating*:
backward-terminating $x \iff \text{forward-terminating } (x^T)$
by (*metis comp-assoc conv-compl conv-contrav conv-iso conv-one*)

lemma *conv-terminating*:
terminating $x \longleftrightarrow$ *terminating* (x^T)
apply (*rule iffI*)
apply (*metis conv-compl conv-contrav conv-one conv-times inf.commute le-iff-inf mult-assoc*)
by (*metis conv-compl conv-contrav conv-invol conv-one conv-times inf.commute le-iff-inf mult-assoc*)

lemma *conv-backward-finite*:
backward-finite $x \longleftrightarrow$ *forward-finite* (x^T)
by (*metis comp-assoc conv-add conv-compl conv-contrav conv-iso conv-one star-conv*)

lemma *conv-finite*:
finite $x \longleftrightarrow$ *finite* (x^T)
by (*metis finite-iff conv-backward-finite conv-invol*)

lemma *conv-no-start-points-path*:
no-start-points-path $x \longleftrightarrow$ *no-end-points-path* (x^T)
using *conv-path* **by** *fastforce*

lemma *conv-no-start-end-points-path*:
no-start-end-points-path $x \longleftrightarrow$ *no-start-end-points-path* (x^T)
using *conv-path* **by** *fastforce*

lemma *conv-has-start-points-path*:
has-start-points-path $x \longleftrightarrow$ *has-end-points-path* (x^T)
using *conv-has-start-points conv-path* **by** *fastforce*

lemma *conv-has-start-end-points-path*:
has-start-end-points-path $x \longleftrightarrow$ *has-start-end-points-path* (x^T)
using *conv-has-start-end-points conv-path* **by** *fastforce*

lemma *conv-backward-terminating-path*:
backward-terminating-path $x \longleftrightarrow$ *forward-terminating-path* (x^T)
using *conv-backward-terminating conv-path* **by** *fastforce*

lemma *conv-terminating-path*:
terminating-path $x \longleftrightarrow$ *terminating-path* (x^T)
using *conv-path conv-terminating* **by** *fastforce*

lemma *conv-backward-finite-path*:
backward-finite-path $x \longleftrightarrow$ *forward-finite-path* (x^T)
using *conv-backward-finite conv-path* **by** *fastforce*

lemma *conv-finite-path*:
finite-path $x \longleftrightarrow$ *finite-path* (x^T)
using *conv-finite conv-path* **by** *blast*

Equivalences for *connected*

lemma *connected-iff2*:

assumes *is-inj* x

and *is-p-fun* x

shows $\text{connected } x \iff x;1;x^T \leq x^* + x^{T^*}$

proof

assume 1 : *connected* x

have $x;1;x^T \leq x;1;x;x^T$

by (*metis conv-invol modular-var-3 vector-meet-comp-x'*)

also have $\dots \leq (x^+ + x^{T^*});x^T$

using 1 *mult-isol star-star-plus* **by** *fastforce*

also have $\dots \leq x^*;x;x^T + x^{T^*}$

using *join-isol star-slide-var* **by** *simp*

also from *assms(1)* **have** $\dots \leq x^* + x^{T^*}$

by (*metis is-inj-def comp-assoc join-iso mult-1-right mult-isol*)

finally show $x;1;x^T \leq x^* + x^{T^*}$.

next

assume 2 : $x;1;x^T \leq x^* + x^{T^*}$

have $x;1;x \leq x;1;x^T;x$

by (*simp add: modular-var-3 vector-meet-comp-x*)

also have $\dots \leq (x^* + x^{T^+});x$

using 2 **by** (*metis mult-isol star-star-plus sup-commute*)

also have $\dots \leq x^* + x^{T^*};x^T;x$

using *join-iso star-slide-var* **by** *simp*

also from *assms(2)* **have** $\dots \leq x^* + x^{T^*}$

by (*metis comp-assoc is-p-fun-def join-isol mult-1-right mult-isol*)

finally show *connected* x .

qed

lemma *connected-iff3*:

assumes *is-inj* x

and *is-p-fun* x

shows $\text{connected } x \iff x^T;1;x \leq x^* + x^{T^*}$

by (*metis assms connected-conv connected-iff2 inj-p-fun p-fun-inj conv-invol add-commute*)

lemma *connected-iff4*:

$\text{connected } x \iff x^T;1;x^T \leq x^* + x^{T^*}$

by (*metis connected-conv conv-invol add-commute*)

lemma *connected-iff5*:

$\text{connected } x \iff x^+;1;x^+ \leq x^* + x^{T^*}$

using *comp-assoc plus-top top-plus* **by** *fastforce*

lemma *connected-iff6*:

assumes *is-inj* x

and *is-p-fun* x

shows $\text{connected } x \iff x^+;1;(x^+)^T \leq x^* + x^{T^*}$

using *assms connected-iff2 comp-assoc plus-conv plus-top top-plus* **by** *fastforce*

lemma *connected-iff7*:
assumes *is-inj x*
and *is-p-fun x*
shows *connected x $\longleftrightarrow (x^+)^T; 1; x^+ \leq x^* + x^{T*}$*
by (*metis assms connected-iff3 conv-contrav conv-invol conv-one top-plus vector-meet-comp-x*)

lemma *connected-iff8*:
connected x $\longleftrightarrow (x^+)^T; 1; (x^+)^T \leq x^ + x^{T*}$*
by (*metis connected-iff4 comp-assoc conv-contrav conv-invol conv-one plus-conv star-conv top-plus*)

Equivalences and implications for *many-strongly-connected*

lemma *many-strongly-connected-iff-1*:
many-strongly-connected x $\longleftrightarrow x^T \leq x^$*
apply (*rule iffI, simp*)
by (*metis conv-invol conv-iso order.eq-iff star-conv star-invol star-iso*)

lemma *many-strongly-connected-iff-2*:
many-strongly-connected x $\longleftrightarrow x^T \leq x^+$
proof
assume *as: many-strongly-connected x*
hence *$x^T \leq x^* \cdot -(1')$*
by (*metis many-strongly-connected-iff-1 loop-backward-forward inf-greatest*)
also have *$\dots \leq (x^* \cdot -(1')) + (x^* \cdot x)$*
by (*simp add: inf-sup-distrib1*)
also have *$\dots \leq x^+$*
by (*metis as order.eq-iff mult-1-right mult-isol star-ref sup.absorb1 conv-invol eq-refl galois-1 inf.absorb-iff1 inf commute star-unfoldl-eq sup-mono many-strongly-connected-iff-1*)
finally show *$x^T \leq x^+$* .
next
show *$x^T \leq x^+ \implies many-strongly-connected x$*
using *order-trans star-1l many-strongly-connected-iff-1* **by** *blast*
qed

lemma *many-strongly-connected-iff-3*:
many-strongly-connected x $\longleftrightarrow x \leq x^{T}$*
by (*metis conv-invol many-strongly-connected-iff-1*)

lemma *many-strongly-connected-iff-4*:
many-strongly-connected x $\longleftrightarrow x \leq x^{T+}$
by (*metis conv-invol many-strongly-connected-iff-2*)

lemma *many-strongly-connected-iff-5*:
many-strongly-connected x $\longleftrightarrow x^; x^T \leq x^+$*
by (*metis comp-assoc conv-contrav conway.dagger-unfoldr-distr star-conv star-denest-var-2*)

star-invol star-trans-eq star-unfoldl-eq sup.boundedE
many-strongly-connected-iff-2)

lemma *many-strongly-connected-iff-6:*

many-strongly-connected $x \longleftrightarrow x^T; x^* \leq x^+$

by (*metis dual-order.trans star-1l star-conv star-inductl-star star-invol*
star-slide-var

many-strongly-connected-iff-1 many-strongly-connected-iff-5)

lemma *many-strongly-connected-iff-7:*

many-strongly-connected $x \longleftrightarrow x^{T+} = x^+$

by (*metis order.antisym conv-invol star-slide-var star-unfoldl-eq*
many-strongly-connected-iff-5)

lemma *many-strongly-connected-iff-5-eq:*

many-strongly-connected $x \longleftrightarrow x^*; x^T = x^+$

by (*metis order.refl star-slide-var many-strongly-connected-iff-5*
many-strongly-connected-iff-7)

lemma *many-strongly-connected-iff-6-eq:*

many-strongly-connected $x \longleftrightarrow x^T; x^* = x^+$

using *many-strongly-connected-iff-6 many-strongly-connected-iff-7* **by** *force*

lemma *many-strongly-connected-implies-no-start-end-points:*

assumes *many-strongly-connected* x

shows *no-start-end-points* x

by (*metis assms conway.dagger-unfoldl-distr mult-assoc sup-top-left conv-invol*
many-strongly-connected-iff-7)

lemma *many-strongly-connected-implies-8:*

assumes *many-strongly-connected* x

shows $x; x^T \leq x^+$

by (*simp add: assms mult-isol*)

lemma *many-strongly-connected-implies-9:*

assumes *many-strongly-connected* x

shows $x^T; x \leq x^+$

by (*metis assms eq-refl phl-cons1 star-ext star-slide-var*)

lemma *many-strongly-connected-implies-10:*

assumes *many-strongly-connected* x

shows $x; x^T; x^* \leq x^+$

by (*simp add: assms comp-assoc mult-isol*)

lemma *many-strongly-connected-implies-10-eq:*

assumes *many-strongly-connected* x

shows $x; x^T; x^* = x^+$

proof (*rule order.antisym*)

show $x; x^T; x^* \leq x^+$

by (*simp add: assms comp-assoc mult-isol*)
next
 have $x^+ \leq x; x^T; x; x^*$
 using *mult-isol x-leq-triple-x* by *blast*
 thus $x^+ \leq x; x^T; x^*$
 by (*simp add: comp-assoc mult-isol order-trans*)
qed

lemma *many-strongly-connected-implies-11:*
 assumes *many-strongly-connected* x
 shows $x^*; x^T; x \leq x^+$
 by (*metis assms conv-contrav conv-iso mult-isol star-1l star-slide-var*)

lemma *many-strongly-connected-implies-11-eq:*
 assumes *many-strongly-connected* x
 shows $x^*; x^T; x = x^+$
 by (*metis assms comp-assoc conv-invol many-strongly-connected-iff-5-eq many-strongly-connected-implies-10-eq*)

lemma *many-strongly-connected-implies-12:*
 assumes *many-strongly-connected* x
 shows $x^*; x; x^T \leq x^+$
 by (*metis assms comp-assoc mult-isol star-1l star-slide-var*)

lemma *many-strongly-connected-implies-12-eq:*
 assumes *many-strongly-connected* x
 shows $x^*; x; x^T = x^+$
 by (*metis assms comp-assoc star-slide-var many-strongly-connected-implies-10-eq*)

lemma *many-strongly-connected-implies-13:*
 assumes *many-strongly-connected* x
 shows $x^T; x; x^* \leq x^+$
 by (*metis assms star-slide-var many-strongly-connected-implies-11 mult.assoc*)

lemma *many-strongly-connected-implies-13-eq:*
 assumes *many-strongly-connected* x
 shows $x^T; x; x^* = x^+$
 by (*metis assms conv-invol many-strongly-connected-iff-7 many-strongly-connected-implies-10-eq*)

lemma *many-strongly-connected-iff-8:*
 assumes *is-p-fun* x
 shows *many-strongly-connected* $x \iff x; x^T \leq x^+$
apply (*rule iffI*)
apply (*simp add: mult-isol*)
apply (*simp add: many-strongly-connected-iff-1*)
by (*metis comp-assoc conv-invol dual-order.trans mult-isol x-leq-triple-x assms comp-assoc dual-order.trans is-p-fun-def order.refl prod-star-closure star-ref*)

lemma *many-strongly-connected-iff-9*:
assumes *is-inj x*
shows *many-strongly-connected x $\longleftrightarrow x^T; x \leq x^+$*
by (*metis assms conv-contrav conv-iso inj-p-fun star-conv star-slide-var many-strongly-connected-iff-1 many-strongly-connected-iff-8*)

lemma *many-strongly-connected-iff-10*:
assumes *is-p-fun x*
shows *many-strongly-connected x $\longleftrightarrow x; x^T; x^* \leq x^+$*
apply (*rule iffI*)
apply (*simp add: comp-assoc mult-isol*)
by (*metis assms mult-isol mult-oner order-trans star-ref many-strongly-connected-iff-8*)

lemma *many-strongly-connected-iff-10-eq*:
assumes *is-p-fun x*
shows *many-strongly-connected x $\longleftrightarrow x; x^T; x^* = x^+$*
using *assms many-strongly-connected-iff-10*
many-strongly-connected-implies-10-eq **by** *fastforce*

lemma *many-strongly-connected-iff-11*:
assumes *is-inj x*
shows *many-strongly-connected x $\longleftrightarrow x^*; x^T; x \leq x^+$*
by (*metis assms comp-assoc conv-contrav conv-iso inj-p-fun plus-conv star-conv many-strongly-connected-iff-10 many-strongly-connected-iff-2*)

lemma *many-strongly-connected-iff-11-eq*:
assumes *is-inj x*
shows *many-strongly-connected x $\longleftrightarrow x^*; x^T; x = x^+$*
using *assms many-strongly-connected-iff-11*
many-strongly-connected-implies-11-eq **by** *fastforce*

lemma *many-strongly-connected-iff-12*:
assumes *is-p-fun x*
shows *many-strongly-connected x $\longleftrightarrow x^*; x; x^T \leq x^+$*
by (*metis assms dual-order.trans mult-double-iso mult-oner star-ref star-slide-var many-strongly-connected-iff-8 many-strongly-connected-implies-12*)

lemma *many-strongly-connected-iff-12-eq*:
assumes *is-p-fun x*
shows *many-strongly-connected x $\longleftrightarrow x^*; x; x^T = x^+$*
using *assms many-strongly-connected-iff-12*
many-strongly-connected-implies-12-eq **by** *fastforce*

lemma *many-strongly-connected-iff-13*:
assumes *is-inj x*
shows *many-strongly-connected x $\longleftrightarrow x^T; x; x^* \leq x^+$*
by (*metis assms comp-assoc conv-contrav conv-iso inj-p-fun star-conv*)

star-slide-var
many-strongly-connected-iff-1 many-strongly-connected-iff-12)

lemma *many-strongly-connected-iff-13-eq*:
assumes *is-inj x*
shows *many-strongly-connected x $\longleftrightarrow x^T;x;x^* = x^+$*
using *assms many-strongly-connected-iff-13*
many-strongly-connected-implies-13-eq **by** *fastforce*

Equivalences and implications for *one-strongly-connected*

lemma *one-strongly-connected-iff*:
one-strongly-connected x \longleftrightarrow connected x \wedge many-strongly-connected x
apply (*rule iffI*)
apply (*metis top-greatest x-leq-triple-x mult-double-iso top-greatest*
dual-order.trans
many-strongly-connected-iff-1 comp-assoc conv-contrav conv-invol
conv-iso le-supI2
star-conv)
by (*metis comp-assoc conv-contrav conv-iso conv-one conway.dagger-denest*
star-conv star-invol
star-sum-unfold star-trans-eq)

lemma *one-strongly-connected-iff-1*:
one-strongly-connected x $\longleftrightarrow x^T;1;x^T \leq x^+$
proof
assume *1: one-strongly-connected x*
have *$x^T;1;x^T \leq x^T;x;x^T;1;x^T$*
by (*metis conv-invol mult-isor x-leq-triple-x*)
also from 1 have *$\dots \leq x^T;x;x^*$*
by (*metis distrib-left mult-assoc sup.absorb-iff1*)
also from 1 have *$\dots \leq x^+$*
using *many-strongly-connected-implies-13 one-strongly-connected-iff* **by** *blast*
finally show *$x^T;1;x^T \leq x^+$*

next
assume *$x^T;1;x^T \leq x^+$*
thus *one-strongly-connected x*
using *dual-order.trans star-1l* **by** *blast*
qed

lemma *one-strongly-connected-iff-1-eq*:
one-strongly-connected x $\longleftrightarrow x^T;1;x^T = x^+$
apply (*rule iffI, simp-all*)
by (*metis comp-assoc conv-contrav conv-invol mult-double-iso plus-conv*
star-slide-var top-greatest
top-plus many-strongly-connected-implies-10-eq one-strongly-connected-iff
order.eq-iff
one-strongly-connected-iff-1)

lemma *one-strongly-connected-iff-2*:
 $one\text{-strongly}\text{-connected } x \longleftrightarrow x;1;x \leq x^{T\star}$
by (*metis conv-invol eq-refl less-eq-def one-strongly-connected-iff*)

lemma *one-strongly-connected-iff-3*:
 $one\text{-strongly}\text{-connected } x \longleftrightarrow x;1;x \leq x^{T+}$
by (*metis comp-assoc conv-contrav conv-invol conv-iso conv-one star-conv one-strongly-connected-iff-1*)

lemma *one-strongly-connected-iff-3-eq*:
 $one\text{-strongly}\text{-connected } x \longleftrightarrow x;1;x = x^{T+}$
by (*metis conv-invol one-strongly-connected-iff-1-eq one-strongly-connected-iff-2*)

lemma *one-strongly-connected-iff-4-eq*:
 $one\text{-strongly}\text{-connected } x \longleftrightarrow x^T;1;x = x^+$
apply (*rule iffI*)
apply (*metis comp-assoc top-plus many-strongly-connected-iff-7 one-strongly-connected-iff one-strongly-connected-iff-1-eq*)
by (*metis comp-assoc conv-contrav conv-invol conv-one plus-conv top-plus one-strongly-connected-iff-1-eq*)

lemma *one-strongly-connected-iff-5-eq*:
 $one\text{-strongly}\text{-connected } x \longleftrightarrow x;1;x^T = x^+$
using *comp-assoc conv-contrav conv-invol conv-one plus-conv top-plus many-strongly-connected-iff-7*
one-strongly-connected-iff one-strongly-connected-iff-3-eq **by** *metis*

lemma *one-strongly-connected-iff-6-aux*:
 $x;x^+ \leq x;1;x$
by (*metis comp-assoc maddux-21 mult-isol top-plus*)

lemma *one-strongly-connected-implies-6-eq*:
assumes *one-strongly-connected* x
shows $x;1;x = x;x^+$
by (*metis assms comp-assoc many-strongly-connected-iff-7 many-strongly-connected-implies-10-eq one-strongly-connected-iff one-strongly-connected-iff-3-eq*)

lemma *one-strongly-connected-iff-7-aux*:
 $x^+ \leq x;1;x$
by (*metis le-infI maddux-20 maddux-21 plus-top top-plus vector-meet-comp-x'*)

lemma *one-strongly-connected-implies-7-eq*:
assumes *one-strongly-connected* x
shows $x;1;x = x^+$
using *assms many-strongly-connected-iff-7 one-strongly-connected-iff one-strongly-connected-iff-3-eq*
by *force*

lemma *one-strongly-connected-implies-8*:

assumes *one-strongly-connected* x

shows $x;1;x \leq x^*$

using *assms one-strongly-connected-iff* **by** *fastforce*

lemma *one-strongly-connected-iff-4*:

assumes *is-inj* x

shows *one-strongly-connected* $x \longleftrightarrow x^T;1;x \leq x^+$

proof

assume *one-strongly-connected* x

thus $x^T;1;x \leq x^+$

by (*simp add: one-strongly-connected-iff-4-eq*)

next

assume $1: x^T;1;x \leq x^+$

hence $x^T;1;x^T \leq x^*;x;x^T$

by (*metis mult-isor star-slide-var comp-assoc conv-invol modular-var-3 vector-meet-comp-x*

order.trans)

also from *assms have* $\dots \leq x^*$

using *comp-assoc is-inj-def mult-isol mult-oner* **by** *fastforce*

finally show *one-strongly-connected* x

using *dual-order.trans star-1l* **by** *fastforce*

qed

lemma *one-strongly-connected-iff-5*:

assumes *is-p-fun* x

shows *one-strongly-connected* $x \longleftrightarrow x;1;x^T \leq x^+$

apply (*rule iffI*)

using *one-strongly-connected-iff-5-eq* **apply** *simp*

by (*metis assms comp-assoc mult-double-iso order.trans star-slide-var top-greatest top-plus*

many-strongly-connected-iff-12 many-strongly-connected-iff-7

one-strongly-connected-iff-3)

lemma *one-strongly-connected-iff-6*:

assumes *is-p-fun* x

and *is-inj* x

shows *one-strongly-connected* $x \longleftrightarrow x;1;x \leq x;x^+$

proof

assume *one-strongly-connected* x

thus $x;1;x \leq x;x^+$

by (*simp add: one-strongly-connected-implies-6-eq*)

next

assume $1: x;1;x \leq x;x^+$

have $x^T;1;x \leq x^T;x;x^T;1;x$

by (*metis conv-invol mult-isor x-leq-triple-x*)

also have $\dots \leq x^T;x;1;x$

by (*metis comp-assoc mult-double-iso top-greatest*)

also from 1 have $\dots \leq x^T; x; x^+$
by (*simp add: comp-assoc mult-isol*)
also from *assms(1)* **have** $\dots \leq x^+$
by (*metis comp-assoc is-p-fun-def mult-isor mult-onel*)
finally show *one-strongly-connected x*
using *assms(2) one-strongly-connected-iff-4* **by** *blast*
qed

lemma *one-strongly-connected-iff-6-eq*:
assumes *is-p-fun x*
and *is-inj x*
shows *one-strongly-connected x \longleftrightarrow x;1;x = x;x⁺*
apply (*rule iffI*)
using *one-strongly-connected-implies-6-eq* **apply** *blast*
by (*simp add: assms one-strongly-connected-iff-6*)

Start points and end points

lemma *start-end-implies-terminating*:
assumes *has-start-points x*
and *has-end-points x*
shows *terminating x*
using *assms* **by** *simp*

lemma *start-points-end-points-conv*:
start-points x = end-points (x^T)
by *simp*

lemma *start-point-at-most-one*:
assumes *path x*
shows *is-inj (start-points x)*
proof –
have *isvec: is-vector (x;1 · -(x^T;1))*
by (*simp add: comp-assoc is-vector-def one-compl vector-1*)

have $x;1 \cdot 1; x^T \leq x;1;x;x^T$
by (*metis comp-assoc conv-contrav conv-one inf.cobounded2 mult-1-right mult-isol one-conv ra-2*)
also have $\dots \leq (x^* + x^{T*}); x^T$
using $\langle \text{path } x \rangle$ **by** (*metis path-def mult-isor*)
also have $\dots = x^T + x^+; x^T + x^{T+}$
by (*simp add: star-slide-var*)
also have $\dots \leq x^{T+} + x^+; x^T + x^{T+}$
by (*metis add-iso mult-1-right star-unfoldl-eq subdistl*)
also have $\dots \leq x^*; x; x^T + x^{T+}$
by (*simp add: star-slide-var add-comm*)
also have $\dots \leq x^*; 1' + x^{T+}$
using $\langle \text{path } x \rangle$ **by** (*metis path-def is-inj-def comp-assoc distrib-left join-iso less-eq-def*)
also have $\dots = 1' + x^*; x + x^T; x^{T*}$

by *simp*
 also have $\dots \leq 1' + 1; x + x^T; 1$
 by (*metis join-isol mult-isol mult-isor sup-mono top-greatest*)
 finally have *aux*: $x; 1 \cdot 1; x^T \leq 1' + 1; x + x^T; 1$.

 from *aux* have $x; 1 \cdot 1; x^T \cdot -(x^T; 1) \cdot -(1; x) \leq 1'$
 by (*simp add: galois-1 sup-commute*)
 hence $(x; 1 \cdot -(x^T; 1)) \cdot (x; 1 \cdot -(x^T; 1))^T \leq 1'$
 by (*simp add: conv-compl inf.assoc inf.left-commute*)
 with *isvec* have $(x; 1 \cdot -(x^T; 1)) ; (x; 1 \cdot -(x^T; 1))^T \leq 1'$
 by (*metis vector-meet-comp'*)
 thus *is-inj* (*start-points x*)
 by (*simp add: conv-compl is-inj-def*)
 qed

lemma *start-point-zero-point*:
 assumes *path x*
 shows *start-points x = 0* \vee *is-point (start-points x)*
 using *assms start-point-at-most-one comp-assoc is-point-def is-vector-def*
vector-compl vector-mult
 by *simp*

lemma *start-point-iff1*:
 assumes *path x*
 shows *is-point (start-points x)* \longleftrightarrow $\neg(\text{no-start-points } x)$
 using *assms start-point-zero-point galois-aux2 is-point-def* by *blast*

lemma *end-point-at-most-one*:
 assumes *path x*
 shows *is-inj (end-points x)*
 by (*metis assms conv-path compl-bot-eq conv-invol inj-def-var1 is-point-def*
top-greatest
start-point-zero-point)

lemma *end-point-zero-point*:
 assumes *path x*
 shows *end-points x = 0* \vee *is-point (end-points x)*
 using *assms conv-path start-point-zero-point* by *fastforce*

lemma *end-point-iff1*:
 assumes *path x*
 shows *is-point (end-points x)* \longleftrightarrow $\neg(\text{no-end-points } x)$
 using *assms end-point-zero-point galois-aux2 is-point-def* by *blast*

lemma *predecessor-point'*:
 assumes *path x*
 and *point s*
 and *point e*
 and $e; s^T \leq x$

shows $x;s = e$
proof (*rule order.antisym*)
show $1: e \leq x ; s$
using *assms(2,4) point-def ss423bij* **by** *blast*
show $x ; s \leq e$
proof –
have $e^T ; (x ; s) = 1$
using 1 **by** (*metis assms(3) order.eq-iff is-vector-def point-def ss423conv top-greatest*)
thus *?thesis*
by (*metis assms(1-3) comp-assoc conv-contrav conv-invol order.eq-iff inj-compose is-vector-def mult-isol path-def point-def ss423conv sur-def-var1 top-greatest*)
qed
qed

lemma *predecessor-point*:
assumes *path x*
and *point s*
and *point e*
and $e;s^T \leq x$
shows *point(x;s)*
using *predecessor-point' assms* **by** *blast*

lemma *points-of-path-iff*:
shows $(x + x^T);1 = x^T;1 + \text{start-points}(x)$
and $(x + x^T);1 = x;1 + \text{end-points}(x)$
using *aux9 inf.commute sup.commute* **by** *auto*

Path concatenation preliminaries

lemma *path-concat-aux-1*:
assumes $x;1 \cdot y;1 \cdot y^T;1 = 0$
and *end-points x = start-points y*
shows $x;1 \cdot y;1 = 0$
proof –
have $x;1 \cdot y;1 = (x;1 \cdot y;1 \cdot y^T;1) + (x;1 \cdot y;1 \cdot -(y^T;1))$
by *simp*
also from *assms(1)* **have** $\dots = x;1 \cdot y;1 \cdot -(y^T;1)$
by (*metis aux6-var de-morgan-3 inf.left-commute inf-compl-bot inf-sup-absorb*)
also from *assms(2)* **have** $\dots = x;1 \cdot x^T;1 \cdot -(x;1)$
by (*simp add: inf.assoc*)
also have $\dots = 0$
by (*simp add: inf.commute inf.assoc*)
finally show *?thesis* .
qed

lemma *path-concat-aux-2*:
assumes $x;1 \cdot x^T;1 \cdot y^T;1 = 0$
and *end-points x = start-points y*

shows $x^T;1 \cdot y^T;1 = 0$
proof –
have $y^T;1 \cdot x^T;1 \cdot (x^T)^T;1 = 0$
using *assms(1) inf.assoc inf.commute* **by force**
thus *?thesis*
by (*metis assms(2) conv-invol inf.commute path-concat-aux-1*)
qed

lemma *path-concat-aux3-1*:
assumes *path x*
shows $x;1;x^T \leq x^* + x^{T*}$
proof –
have $x;1;x^T \leq x;1;x^T;x;x^T$
by (*metis comp-assoc conv-invol mult-isol x-leq-triple-x*)
also have $\dots \leq x;1;x;x^T$
by (*metis mult-isol mult-isol mult-assoc top-greatest*)
also from *assms* **have** $\dots \leq (x^* + x^{T*});x^T$
using *path-def comp-assoc mult-isol* **by blast**
also have $\dots = x^*;x;x^T + x^{T*};x^T$
by (*simp add: star-slide-var star-star-plus*)
also have $\dots \leq x^*;1' + x^{T*};x^T$
by (*metis assms path-def is-inj-def join-iso mult-isol mult-assoc*)
also have $\dots \leq x^* + x^{T*}$
using *join-iso* **by simp**
finally show *?thesis* .
qed

lemma *path-concat-aux3-2*:
assumes *path x*
shows $x^T;1;x \leq x^* + x^{T*}$
proof –
have $x^T;1;x \leq x^T;x;x^T;1;x$
by (*metis comp-assoc conv-invol mult-isol x-leq-triple-x*)
also have $\dots \leq x^T;x;1;x$
by (*metis mult-isol mult-isol mult-assoc top-greatest*)
also from *assms* **have** $\dots \leq x^T;(x^* + x^{T*})$
by (*simp add: comp-assoc mult-isol path-def*)
also have $\dots = x^T;x;x^* + x^T;x^{T*}$
by (*simp add: comp-assoc distrib-left star-star-plus*)
also have $\dots \leq 1';x^* + x^T;x^{T*}$
by (*metis assms path-def is-p-fun-def join-iso mult-isol mult-assoc*)
also have $\dots \leq x^* + x^{T*}$
using *join-iso* **by simp**
finally show *?thesis* .
qed

lemma *path-concat-aux3-3*:
assumes *path x*
shows $x^T;1;x^T \leq x^* + x^{T*}$

proof –
have $x^T;1;x^T \leq x^T;x;x^T;1;x^T$
by (*metis comp-assoc conv-invol mult-isor x-leq-triple-x*)
also have $\dots \leq x^T;x;1;x^T$
by (*metis mult-isol mult-isor mult-assoc top-greatest*)
also from *assms* **have** $\dots \leq x^T;(x^* + x^{T*})$
using *path-concat-aux3-1* **by** (*simp add: mult-assoc mult-isol*)
also have $\dots = x^T;x;x^* + x^T;x^{T*}$
by (*simp add: comp-assoc distrib-left star-star-plus*)
also have $\dots \leq 1';x^* + x^T;x^{T*}$
by (*metis assms path-def is-p-fun-def join-iso mult-isor mult-assoc*)
also have $\dots \leq x^* + x^{T*}$
using *join-isol* **by** *simp*
finally show *?thesis* .
qed

lemma *path-concat-aux-3*:

assumes *path x*
and $y \leq x^+ + x^{T+}$
and $z \leq x^+ + x^{T+}$
shows $y;1;z \leq x^* + x^{T*}$

proof –
from *assms(2,3)* **have** $y;1;z \leq (x^+ + x^{T+});1;(x^+ + x^{T+})$
using *mult-isol-var mult-isor* **by** *blast*
also have $\dots = x^+;1;x^+ + x^+;1;x^{T+} + x^{T+};1;x^+ + x^{T+};1;x^{T+}$
by (*simp add: distrib-left sup-commute sup-left-commute*)
also have $\dots = x;x^*;1;x^*;x + x;x^*;1;x^{T*};x^T + x^T;x^{T*};1;x^*;x + x^T;x^{T*};1;x^{T*};x^T$
by (*simp add: comp-assoc star-slide-var*)
also have $\dots \leq x;1;x + x;x^*;1;x^{T*};x^T + x^T;x^{T*};1;x^*;x + x^T;x^{T*};1;x^{T*};x^T$
by (*metis comp-assoc mult-double-iso top-greatest join-iso*)
also have $\dots \leq x;1;x + x;1;x^T + x^T;x^{T*};1;x^*;x + x^T;x^{T*};1;x^{T*};x^T$
by (*metis comp-assoc mult-double-iso top-greatest join-iso join-isol*)
also have $\dots \leq x;1;x + x;1;x^T + x^T;1;x + x^T;x^{T*};1;x^{T*};x^T$
by (*metis comp-assoc mult-double-iso top-greatest join-iso join-isol*)
also have $\dots \leq x;1;x + x;1;x^T + x^T;1;x + x^T;1;x^T$
by (*metis comp-assoc mult-double-iso top-greatest join-iso*)
also have $\dots \leq x^* + x^{T*}$
using *assms(1) path-def path-concat-aux3-1 path-concat-aux3-2 path-concat-aux3-3 join-iso join-isol*
by *simp*
finally show *?thesis* .
qed

lemma *path-concat-aux-4*:

$x^* + x^{T*} \leq x^* + x^T;1$

by (*metis star-star-plus add-comm join-isol mult-isol top-greatest*)

lemma *path-concat-aux-5*:

assumes *path* x
and $y \leq \text{start-points } x$
and $z \leq x + x^T$
shows $y;1;z \leq x^*$
proof –
from *assms(1)* **have** $x;1;x \leq x^* + x^T;1$
using *path-def path-concat-aux-4 dual-order.trans* **by** *blast*
hence *aux1*: $x;1;x \cdot -(x^T;1) \leq x^*$
by (*simp add: galois-1 sup-commute*)

from *assms(1)* **have** $x;1;x^T \leq x^* + x^T;1$
using *dual-order.trans path-concat-aux3-1 path-concat-aux-4* **by** *blast*
hence *aux2*: $x;1;x^T \cdot -(x^T;1) \leq x^*$
by (*simp add: galois-1 sup-commute*)

from *assms(2,3)* **have** $y;1;z \leq (x;1 \cdot -(x^T;1));1;(x + x^T)$
by (*simp add: mult-isol-var mult-isol*)
also have $\dots = (x;1 \cdot -(x^T;1));1;x + (x;1 \cdot -(x^T;1));1;x^T$
using *distrib-left* **by** *blast*
also have $\dots = (x;1 \cdot -(x^T;1) \cdot 1;x) + (x;1 \cdot -(x^T;1));1;x^T$
by (*metis comp-assoc inf-top-right is-vector-def one-idem-mult vector-1 vector-compl*)
also have $\dots = (x;1 \cdot -(x^T;1) \cdot 1;x) + (x;1 \cdot -(x^T;1) \cdot 1;x^T)$
by (*metis comp-assoc inf-top-right is-vector-def one-idem-mult vector-1 vector-compl*)
also have $\dots = (x;1;x \cdot -(x^T;1)) + (x;1;x^T \cdot -(x^T;1))$
using *vector-meet-comp-x vector-meet-comp-x' diff-eq inf.assoc inf.commute*
by *simp*
also from *aux1 aux2* **have** $\dots \leq x^*$
by (*simp add: diff-eq join-iso*)
finally show *?thesis* .
qed

lemma *path-conditions-disjoint-points-iff*:
 $x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0 \wedge \text{start-points } x \cdot \text{end-points } y = 0 \iff x;1 \cdot y^T;1 = 0$
proof
assume $1: x;1 \cdot y^T;1 = 0$
hence $g1: x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0$
by (*metis inf.left-commute inf-bot-right inf-commute*)
have $g2: \text{start-points } x \cdot \text{end-points } y = 0$
using 1 **by** (*metis compl-inf-bot inf.assoc inf.commute inf.left-idem*)
show $x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0 \wedge \text{start-points } x \cdot \text{end-points } y = 0$
using $g1$ **and** $g2$ **by** *simp*
next
assume $a: x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0 \wedge \text{start-points } x \cdot \text{end-points } y = 0$
from a **have** $a1: x;1 \cdot x^T;1 \cdot y^T;1 = 0$
by (*simp add: inf.commute inf-sup-distrib1*)
from a **have** $a2: x;1 \cdot y;1 \cdot y^T;1 = 0$

by (*simp add: inf.commute inf-sup-distrib1*)
from *a* **have** *a3*: *start-points x · end-points y = 0*
 by *blast*

have $x;1 \cdot y^T;1 = x;1 \cdot x^T;1 \cdot y^T;1 + x;1 \cdot -(x^T;1) \cdot y^T;1$
 by (*metis aux4 inf-sup-distrib2*)
also from *a1* **have** $\dots = x;1 \cdot -(x^T;1) \cdot y^T;1$
 using *sup-bot-left* **by** *blast*
also have $\dots = x;1 \cdot -(x^T;1) \cdot y;1 \cdot y^T;1 + x;1 \cdot -(x^T;1) \cdot -(y;1) \cdot y^T;1$
 by (*metis aux4 inf-sup-distrib2*)
also have $\dots \leq x;1 \cdot y;1 \cdot y^T;1 + x;1 \cdot -(x^T;1) \cdot -(y;1) \cdot y^T;1$
 using *join-iso meet-iso* **by** *simp*
also from *a2* **have** $\dots = \text{start-points } x \cdot \text{end-points } y$
 using *sup-bot-left inf.commute inf.left-commute* **by** *simp*
also from *a3* **have** $\dots = 0$
 by *blast*
finally show $x;1 \cdot y^T;1 = 0$
 using *le-bot* **by** *blast*

qed

end

2.2 Consequences with the Tarski rule

context *relation-algebra-rtc-tarski*
begin

General theorems

lemma *reachable-implies-predecessor*:

assumes $p \neq q$
 and *point p*
 and *point q*
 and $x^*;q \leq x^{T*};p$
shows $x;q \neq 0$

proof

assume *contra*: $x;q=0$
with *assms(4)* **have** $q \leq x^{T*};p$
 by (*simp add: independence1*)
hence $p \leq x^*;q$
 by (*metis assms(2,3) point-swap star-conv*)
with *contra assms(2,3)* **have** $p=q$
 by (*simp add: independence1 is-point-def point-singleton point-is-point*)
with *assms(1)* **show** *False*
 by *simp*

qed

lemma *acyclic-imp-one-step-different-points*:

assumes *is-acyclic x*
 and *point p*
 and *point q*

and $p \leq x;q$
shows $p \leq -q$ **and** $p \neq q$
using *acyclic-reachable-points* *assms* *point-is-point* *point-not-equal(1)* **by** *auto*

Start points and end points

lemma *start-point-iff2*:

assumes *path* x

shows *is-point* (*start-points* x) \longleftrightarrow *has-start-points* x

proof –

have *has-start-points* $x \longleftrightarrow 1 \leq -(1;x);x;1$

by (*simp* *add*: *order.eq-iff*)

also have $\dots \longleftrightarrow 1 \leq 1;x^T;-(x^T;1)$

by (*metis* *comp-assoc* *conv-compl* *conv-contrav* *conv-iso* *conv-one*)

also have $\dots \longleftrightarrow 1 \leq 1;(x;1 \cdot -(x^T;1))$

by (*metis* (*no-types*) *conv-contrav* *conv-one* *inf.commute* *is-vector-def* *one-idem-mult* *ra-2* *vector-1*

vector-meet-comp-x)

also have $\dots \longleftrightarrow 1 = 1;(x;1 \cdot -(x^T;1))$

by (*simp* *add*: *order.eq-iff*)

also have $\dots \longleftrightarrow x;1 \cdot -(x^T;1) \neq 0$

by (*metis* *tarski* *comp-assoc* *one-compl* *ra-1* *ss-p18*)

also have $\dots \longleftrightarrow$ *is-point* (*start-points* x)

using *assms* *is-point-def* *start-point-zero-point* **by** *blast*

finally show *?thesis* ..

qed

lemma *end-point-iff2*:

assumes *path* x

shows *is-point* (*end-points* x) \longleftrightarrow *has-end-points* x

by (*metis* *assms* *conv-invol* *conv-has-start-points* *conv-path* *start-point-iff2*)

lemma *edge-is-path*:

assumes *is-point* p

and *is-point* q

shows *path* ($p;q^T$)

apply (*unfold* *path-def*; *intro* *conjI*)

apply (*metis* *assms* *comp-assoc* *is-point-def* *le-supI1* *star-ext* *vector-rectangle* *point-equations(3)*)

apply (*metis* *is-p-fun-def* *assms* *comp-assoc* *conv-contrav* *conv-invol* *is-inj-def* *is-point-def*

vector-2-var *vector-meet-comp-x'* *point-equations*)

by (*metis* *is-inj-def* *assms* *conv-invol* *conv-times* *is-point-def* *p-fun-mult-var* *vector-meet-comp*)

lemma *edge-start*:

assumes *is-point* p

and *is-point* q

and $p \neq q$

shows *start-points* ($p;q^T$) = p

using *assms* **by** (*simp add: comp-assoc point-equations(1,3) point-not-equal inf.absorb1*)

lemma *edge-end*:
assumes *is-point p*
and *is-point q*
and $p \neq q$
shows $\text{end-points}(p; q^T) = q$
using *assms edge-start* **by** *simp*

lemma *loop-no-start*:
assumes *is-point p*
shows $\text{start-points}(p; p^T) = 0$
by *simp*

lemma *loop-no-end*:
assumes *is-point p*
shows $\text{end-points}(p; p^T) = 0$
by *simp*

lemma *start-point-no-predecessor*:
 $x; \text{start-points}(x) = 0$
by (*metis inf-top.right-neutral modular-1-aux'*)

lemma *end-point-no-successor*:
 $x^T; \text{end-points}(x) = 0$
by (*metis conv-invol start-point-no-predecessor*)

lemma *start-to-end*:
assumes *path x*
shows $\text{start-points}(x); \text{end-points}(x)^T \leq x^*$
proof (*cases end-points(x) = 0*)
assume $\text{end-points}(x) = 0$
thus *?thesis*
by *simp*

next
assume *ass: end-points(x) $\neq 0$*
hence *nz: x; end-points(x) $\neq 0$*
by (*metis comp-res-aux compl-bot-eq inf.left-idem*)
have *a: x; end-points(x); end-points(x)^T $\leq x + x^T$*
by (*metis end-point-at-most-one assms(1) is-inj-def comp-assoc mult-isol mult-oner le-supII*)

have $\text{start-points}(x); \text{end-points}(x)^T = \text{start-points}(x); 1; \text{end-points}(x)^T$
using *ass* **by** (*simp add: comp-assoc is-vector-def one-compl vector-1*)
also **have** $\dots = \text{start-points}(x); 1; x; \text{end-points}(x); 1; \text{end-points}(x)^T$
using *nz tarski* **by** (*simp add: comp-assoc*)
also **have** $\dots = \text{start-points}(x); 1; x; \text{end-points}(x); \text{end-points}(x)^T$
using *ass* **by** (*simp add: comp-assoc is-vector-def one-compl vector-1*)

also with a *assms*(1) **have** $\dots \leq x^*$
using *path-concat-aux-5 comp-assoc eq-refl* **by** *simp*
finally show *?thesis* .
qed

lemma *path-acyclic*:
assumes *has-start-points-path x*
shows *is-acyclic x*
proof –
let $?r = \text{start-points}(x)$
have $pt: \text{point}(?r)$
using *assms point-is-point start-point-iff2* **by** *blast*
have $x^+ \cdot 1' = (x^+)^T \cdot x^+ \cdot 1'$
by (*metis conv-e conv-times inf.assoc inf.left-idem inf-le2*
many-strongly-connected-iff-7
mult-oner star-subid)
also have $\dots \leq x^T; 1 \cdot x^+ \cdot 1'$
by (*metis conv-contrav inf commute maddux-20 meet-double-iso plus-top*
star-conv star-slide-var)
finally have $?r; (x^+ \cdot 1') \leq ?r; (x^T; 1 \cdot x^+ \cdot 1')$
using *mult-isol* **by** *blast*
also have $\dots = (?r \cdot 1; x); (x^+ \cdot 1')$
by (*metis (no-types, lifting) comp-assoc conv-contrav conv-invol conv-one*
inf.assoc
is-vector-def one-idem-mult vector-2)
also have $\dots = ?r; x; (x^+ \cdot 1')$
by (*metis comp-assoc inf-top.right-neutral is-vector-def one-compl*
one-idem-mult vector-1)
also have $\dots \leq (x^* + x^{T*}); (x^+ \cdot 1')$
using *assms(1) mult-isol*
by (*meson connected-iff4 dual-order.trans mult-subdistr path-concat-aux3-3*)
also have $\dots = x^*; (x^+ \cdot 1') + x^{T*}; (x^+ \cdot 1')$
by (*metis distrib-right star-star-plus sup commute*)
also have $\dots \leq x^*; (x^+ \cdot 1') + x^T; 1$
by (*metis join-isol mult-isol plus-top top-greatest*)
finally have $?r; (x^+ \cdot 1'); 1 \leq x^*; (x^+ \cdot 1'); 1 + x^T; 1$
by (*metis distrib-right inf-absorb2 mult-assoc mult-subdistr one-idem-mult*)
hence $1: ?r; (x^+ \cdot 1'); 1 \leq x^T; 1$
using *assms(1) path-def inj-implies-step-forwards-backwards sup-absorb2* **by**
simp
have $x^+ \cdot 1' \leq (x^+ \cdot 1'); 1$
by (*simp add: maddux-20*)
also have $\dots \leq ?r^T; ?r; (x^+ \cdot 1'); 1$
using *pt comp-assoc point-def ss423conv* **by** *fastforce*
also have $\dots \leq ?r^T; x^T; 1$
using 1 **by** (*simp add: comp-assoc mult-isol*)
also have $\dots = 0$
by (*metis start-point-no-predecessor annil conv-contrav conv-zero*)
finally show *?thesis*

using *galois-aux le-bot* by *blast*
qed

Equivalences for *terminating*

lemma *backward-terminating-iff1*:

assumes *path x*

shows *backward-terminating x* \longleftrightarrow *has-start-points x* \vee $x = 0$

proof

assume *backward-terminating x*

hence $1;x;1 \leq 1;-(1;x);x;1;1$

by (*metis mult-isor mult-isol comp-assoc*)

also have $\dots = -(1;x);x;1$

by (*metis conv-compl conv-contrav conv-invol conv-one mult-assoc one-compl one-idem-mult*)

finally have $1;x;1 \leq -(1;x);x;1$.

with *tarski* show *has-start-points x* \vee $x = 0$

by (*metis top-le*)

next

show *has-start-points x* \vee $x = 0 \implies$ *backward-terminating x*

by *fastforce*

qed

lemma *backward-terminating-iff2-aux*:

assumes *path x*

shows $x;1 \cdot 1;x^T \cdot -(1;x) \leq x^{T*}$

proof –

have $x;1 \cdot 1;x^T \leq x;1;x;x^T$

by (*metis conv-invol modular-var-3 vector-meet-comp-x vector-meet-comp-x'*)

also from *assms* have $\dots \leq (x^* + x^{T*});x^T$

using *path-def mult-isor* by *blast*

also have $\dots \leq x^*;x;x^T + x^{T*};x^T$

by (*simp add: star-star-plus star-slide-var add-comm*)

also from *assms* have $\dots \leq x^*;1' + x^{T*};x^T$

by (*metis path-def is-inj-def join-iso mult-assoc mult-isol*)

also have $\dots = x^+ + x^{T*}$

by (*metis mult-1-right star-slide-var star-star-plus sup commute*)

also have $\dots \leq x^{T*} + 1;x$

by (*metis join-iso mult-isor star-slide-var top-greatest add-comm*)

finally have $x;1 \cdot 1;x^T \leq x^{T*} + 1;x$.

thus *?thesis*

by (*simp add: galois-1 sup commute*)

qed

lemma *backward-terminating-iff2*:

assumes *path x*

shows *backward-terminating x* \longleftrightarrow $x \leq x^{T*};-(x^T;1)$

proof

assume *backward-terminating x*

with *assms* **have** *has-start-points* $x \vee x = 0$
by (*simp add: backward-terminating-iff1*)
thus $x \leq x^{T^*}; -(x^T; 1)$
proof
assume $x = 0$
thus *?thesis*
by *simp*
next
assume *has-start-points* x
hence *aux1*: $1 = 1; x^T; -(x^T; 1)$
by (*metis comp-assoc conv-compl conv-contrav conv-one*)
have $x = x \cdot 1$
by *simp*
also have $\dots \leq (x; -(1; x) \cdot 1; x^T); -(x^T; 1)$
by (*metis inf commute aux1 conv-compl conv-contrav conv-invol conv-one modular-2-var*)
also have $\dots = (x; 1 \cdot -(1; x) \cdot 1; x^T); -(x^T; 1)$
by (*metis comp-assoc conv-compl conv-contrav conv-invol conv-one inf commute inf-top-left one-compl ra-1*)
also from *assms* **have** $\dots \leq x^{T^*}; -(x^T; 1)$
using *backward-terminating-iff2-aux inf commute inf.assoc mult-isor* **by** *fastforce*
finally show $x \leq x^{T^*}; -(x^T; 1)$.
qed
next
assume $x \leq x^{T^*}; -(x^T; 1)$
hence $x \leq x^{T^*}; -(x^T; 1) \cdot x$
by *simp*
also have $\dots = (x^{T^*} \cdot -(1; x)); 1 \cdot x$
by (*metis one-compl conv-compl conv-contrav conv-invol conv-one inf-top-left ra-2*)
also have $\dots \leq (x^{T^*} \cdot -(1; x)); (1 \cdot (x^* \cdot -(1; x)^T); x)$
by (*metis (mono-tags) conv-compl conv-invol conv-times modular-1-var star-conv*)
also have $\dots \leq -(1; x); x^*; x$
by (*simp add: mult-assoc mult-isol-var*)
also have $\dots \leq -(1; x); x; 1$
by (*simp add: mult-assoc mult-isol star-slide-var*)
finally show *backward-terminating* x .
qed

lemma *backward-terminating-iff3-aux*:
assumes *path* x
shows $x^T; 1 \cdot 1; x^T \cdot -(1; x) \leq x^{T^*}$
proof –
have $x^T; 1 \cdot 1; x^T \leq x^T; 1; x; x^T$
by (*metis conv-invol modular-var-3 vector-meet-comp-x vector-meet-comp-x*)
also from *assms* **have** $\dots \leq (x^* + x^{T^*}); x^T$

using *mult-isor path-concat-aux3-2* by *blast*
 also have $\dots \leq x^*;x;x^T + x^{T*};x^T$
 by (*simp add: star-star-plus star-slide-var add-comm*)
 also from *assms* have $\dots \leq x^*;1' + x^{T*};x^T$
 by (*metis path-def is-inj-def join-iso mult-assoc mult-isol*)
 also have $\dots = x^+ + x^{T*}$
 by (*metis mult-1-right star-slide-var star-star-plus sup commute*)
 also have $\dots \leq x^{T*} + 1;x$
 by (*metis join-iso mult-isor star-slide-var top-greatest add-comm*)
 finally have $x^T;1 \cdot 1;x^T \leq x^{T*} + 1;x$.
 thus *?thesis*
 by (*simp add: galois-1 sup commute*)
qed

lemma *backward-terminating-iff3*:

assumes *path x*

shows *backward-terminating x* $\longleftrightarrow x^T \leq x^{T*};-(x^T;1)$

proof

assume *backward-terminating x*

with *assms* have *has-start-points x* $\vee x = 0$

by (*simp add: backward-terminating-iff1*)

thus $x^T \leq x^{T*};-(x^T;1)$

proof

assume $x = 0$

thus *?thesis*

by *simp*

next

assume *has-start-points x*

hence *aux1*: $1 = 1;x^T;-(x^T;1)$

by (*metis comp-assoc conv-compl conv-contrav conv-one*)

have $x^T = x^T \cdot 1$

by *simp*

also have $\dots \leq (x^T;-(1;x) \cdot 1;x^T);-(x^T;1)$

by (*metis inf commute aux1 conv-compl conv-contrav conv-invol conv-one modular-2-var*)

also have $\dots = (x^T;1 \cdot -(1;x) \cdot 1;x^T);-(x^T;1)$

by (*metis comp-assoc conv-compl conv-contrav conv-invol conv-one inf commute inf-top-left one-compl ra-1*)

also from *assms* have $\dots \leq x^{T*};-(x^T;1)$

using *backward-terminating-iff3-aux inf commute inf.assoc mult-isor* by *fastforce*

finally show $x^T \leq x^{T*};-(x^T;1)$.

qed

next

have *1*: $-(1;x) \cdot x = 0$

by (*simp add: galois-aux2 inf commute maddux-21*)

assume $x^T \leq x^{T*};-(x^T;1)$

hence $x = -(1;x);x^* \cdot x$

by (*metis (mono-tags, lifting) conv-compl conv-contrav conv-iso conv-one*)

inf.absorb2 star-conv
also have $\dots = -(1;x);x^+ + -(1;x);1' \cdot x$
by (*metis distrib-left star-unfoldl-eq sup-commute*)
also have $\dots = -(1;x);x^+ \cdot x + -(1;x) \cdot x$
by (*simp add: inf-sup-distrib2*)
also have $\dots \leq -(1;x);x^+$
using 1 by *simp*
also have $\dots \leq -(1;x);x;1$
by (*simp add: mult-assoc mult-isol star-slide-var*)
finally show *backward-terminating x* .
qed

lemma *backward-terminating-iff4*:
assumes *path x*
shows *backward-terminating x* $\longleftrightarrow x \leq -(1;x);x^*$
apply (*subst backward-terminating-iff3*)
apply (*rule assms*)
by (*metis (mono-tags, lifting) conv-compl conv-iso star-conv conv-contrav conv-one*)

lemma *forward-terminating-iff1*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow \text{has-end-points } x \vee x = 0$
by (*metis comp-assoc eq-refl le-bot one-compl tarski top-greatest*)

lemma *forward-terminating-iff2*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow x^T \leq x^*;-(x;1)$
by (*metis assms backward-terminating-iff1 backward-terminating-iff2 end-point-iff2 forward-terminating-iff1 compl-bot-eq conv-compl conv-invol conv-one conv-path double-compl start-point-iff2*)

lemma *forward-terminating-iff3*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow x \leq x^*;-(x;1)$
by (*metis assms backward-terminating-iff1 backward-terminating-iff3 end-point-iff2 forward-terminating-iff1 compl-bot-eq conv-compl conv-invol conv-one conv-path double-compl start-point-iff2*)

lemma *forward-terminating-iff4*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow x \leq -(1;x^T);x^{T*}$
using *forward-terminating-iff2 conv-contrav conv-iso star-conv assms conv-compl*
by *force*

lemma *terminating-iff1*:
assumes *path x*
shows *terminating x* \longleftrightarrow *has-start-end-points x* \vee $x = 0$
using *assms backward-terminating-iff1 forward-terminating-iff1* **by** *fastforce*

lemma *terminating-iff2*:
assumes *path x*
shows *terminating x* \longleftrightarrow $x \leq x^{T*}; -(x^T; 1) \cdot -(1; x^T); x^{T*}$
using *assms backward-terminating-iff2 forward-terminating-iff2 conv-compl conv-iso star-conv*
by *force*

lemma *terminating-iff3*:
assumes *path x*
shows *terminating x* \longleftrightarrow $x \leq x^*; -(x; 1) \cdot -(1; x); x^*$
using *assms backward-terminating-iff4 forward-terminating-iff3* **by** *fastforce*

lemma *backward-terminating-path-irreflexive*:
assumes *backward-terminating-path x*
shows $x \leq -1'$
proof –
have $1: x; x^T \leq 1'$
using *assms is-inj-def path-def* **by** *blast*
have $x; (x^T \cdot 1') \leq x; x^T \cdot x$
by (*metis inf.bounded-iff inf commute mult-1-right mult-subdistl*)
also have $\dots \leq 1' \cdot x$
using *1 meet-iso* **by** *blast*
also have $\dots = 1' \cdot x^T$
by (*metis conv-e conv-times inf.cobounded1 is-test-def test-eq-conv*)
finally have $2: x^T; -(x^T \cdot 1') \leq -(x^T \cdot 1')$
by (*metis compl-le-swap1 conv-galois-1 inf commute*)
have $x^T \cdot 1' \leq x^T; 1$
by (*simp add: le-infI1 maddux-20*)
hence $-(x^T; 1) \leq -(x^T \cdot 1')$
using *compl-mono* **by** *blast*
hence $x^T; -(x^T \cdot 1') + -(x^T; 1) \leq -(x^T \cdot 1')$
using *2* **by** (*simp add: le-supI*)
hence $x^{T*}; -(x^T; 1) \leq -(x^T \cdot 1')$
by (*simp add: rtc-inductl*)
hence $x^T \cdot 1' \cdot x^{T*}; -(x^T; 1) = 0$
by (*simp add: compl-le-swap1 galois-aux*)
hence $x^T \cdot 1' = 0$
using *assms backward-terminating-iff3 inf.order-iff le-infI1* **by** *blast*
hence $x \cdot 1' = 0$
by (*simp add: conv-self-conjugate*)
thus *?thesis*
by (*simp add: galois-aux*)
qed

lemma *forward-terminating-path-end-points-1*:
assumes *forward-terminating-path x*
shows $x \leq x^+$; *end-points x*
proof –
have $1: -(x;1) \cdot x = 0$
by (*simp add: galois-aux maddux-20*)
have $x = x^+; -(x;1) \cdot x$
using *assms forward-terminating-iff3 inf.absorb2* **by** *fastforce*
also have $\dots = (x^+; -(x;1) + 1'; -(x;1)) \cdot x$
by (*simp add: sup.commute*)
also have $\dots = x^+; -(x;1) \cdot x + -(x;1) \cdot x$
using *inf-sup-distrib2* **by** *fastforce*
also have $\dots = x^+; -(x;1) \cdot x$
using 1 **by** *simp*
also have $\dots \leq x^+; -(x;1) \cdot (x^+)^T; x$
using *modular-1-var* **by** *blast*
also have $\dots = x^+; -(x;1) \cdot x^{T+}; x$
using *plus-conv* **by** *fastforce*
also have $\dots \leq x^+$; *end-points x*
by (*metis inf-commute inf-top-right modular-1' mult-subdistl plus-conv plus-top*)
finally show *?thesis* .
qed

lemma *forward-terminating-path-end-points-2*:
assumes *forward-terminating-path x*
shows $x^T \leq x^*$; *end-points x*
proof –
have $x^T \leq x^T; x; x^T$
by (*metis conv-invol x-leq-triple-x*)
also have $\dots \leq x^T; x; 1$
using *mult-isol top-greatest* **by** *blast*
also have $\dots \leq x^T; x^+; \text{end-points } x; 1$
by (*metis assms forward-terminating-path-end-points-1 comp-assoc mult-isol mult-isol*)
also have $\dots = x^T; x^+; \text{end-points } x$
by (*metis inf-commute mult-assoc one-compl ra-1*)
also have $\dots \leq x^*$; *end-points x*
by (*metis assms comp-assoc compl-le-swap1 conv-galois-1 conv-invol p-fun-compl path-def*)
finally show *?thesis* .
qed

lemma *forward-terminating-path-end-points-3*:
assumes *forward-terminating-path x*
shows *start-points* $x \leq x^+$; *end-points x*
proof –
have *start-points* $x \leq x^+$; *end-points x; 1*
using *assms forward-terminating-path-end-points-1 comp-assoc mult-isol*

inf.coboundedI1
 by *blast*
 also have ... = x^+ ; *end-points* x
 by (*metis inf-commute mult-assoc one-compl ra-1*)
 finally show ?*thesis* .
 qed

lemma *backward-terminating-path-start-points-1*:
 assumes *backward-terminating-path* x
 shows $x^T \leq x^{T+}$; *start-points* x
 using *assms forward-terminating-path-end-points-1*
conv-backward-terminating-path by *fastforce*

lemma *backward-terminating-path-start-points-2*:
 assumes *backward-terminating-path* x
 shows $x \leq x^{T*}$; *start-points* x
 using *assms forward-terminating-path-end-points-2*
conv-backward-terminating-path by *fastforce*

lemma *backward-terminating-path-start-points-3*:
 assumes *backward-terminating-path* x
 shows *end-points* $x \leq x^{T+}$; *start-points* x
 using *assms forward-terminating-path-end-points-3*
conv-backward-terminating-path by *fastforce*

lemma *path-aux1a*:
 assumes *forward-terminating-path* x
 shows $x \neq 0 \longleftrightarrow \text{end-points } x \neq 0$
 using *assms end-point-iff2 forward-terminating-iff1 end-point-iff1 galois-aux2* by
force

lemma *path-aux1b*:
 assumes *backward-terminating-path* y
 shows $y \neq 0 \longleftrightarrow \text{start-points } y \neq 0$
 using *assms start-point-iff2 backward-terminating-iff1 start-point-iff1 galois-aux2*
 by *force*

lemma *path-aux1*:
 assumes *forward-terminating-path* x
 and *backward-terminating-path* y
 shows $x \neq 0 \vee y \neq 0 \longleftrightarrow \text{end-points } x \neq 0 \vee \text{start-points } y \neq 0$
 using *assms path-aux1a path-aux1b* by *blast*

Equivalences for *finite*

lemma *backward-finite-iff-msc*:
backward-finite $x \longleftrightarrow \text{many-strongly-connected } x \vee \text{backward-terminating } x$

proof
assume 1 : *backward-finite* x
thus *many-strongly-connected* $x \vee$ *backward-terminating* x
proof (*cases* $-(1;x);x;1 = 0$)
 assume $-(1;x);x;1 = 0$
 thus *many-strongly-connected* $x \vee$ *backward-terminating* x
 using 1 **by** (*metis conv-invol many-strongly-connected-iff-1 sup-bot-right*)
next
 assume $-(1;x);x;1 \neq 0$
 hence $1;-(1;x);x;1 = 1$
 by (*simp add: comp-assoc tarski*)
 hence $-(1;x);x;1 = 1$
 by (*metis comp-assoc conv-compl conv-contrav conv-invol conv-one one-compl*)
 thus *many-strongly-connected* $x \vee$ *backward-terminating* x
 using 1 **by** *simp*
qed
next
assume *many-strongly-connected* $x \vee$ *backward-terminating* x
thus *backward-finite* x
by (*metis star-ext sup.coboundedI1 sup.coboundedI2*)
qed

lemma *forward-finite-iff-msc*:
forward-finite $x \iff$ *many-strongly-connected* $x \vee$ *forward-terminating* x
by (*metis backward-finite-iff-msc conv-backward-finite conv-backward-terminating conv-invol*)

lemma *finite-iff-msc*:
finite $x \iff$ *many-strongly-connected* $x \vee$ *terminating* x
using *backward-finite-iff-msc forward-finite-iff-msc finite-iff* **by** *fastforce*

Path concatenation

lemma *path-concatenation*:
assumes *forward-terminating-path* x
 and *backward-terminating-path* y
 and *end-points* $x =$ *start-points* y
 and $x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0$
shows *path* $(x+y)$
proof (*cases* $y = 0$)
 assume $y = 0$
 thus *?thesis*
 using *assms(1)* **by** *fastforce*
next
assume *as*: $y \neq 0$
show *?thesis*
proof (*unfold path-def; intro conjI*)
 from *assms(4)* **have** a : $x;1 \cdot x^T;1 \cdot y^T;1 + x;1 \cdot y;1 \cdot y^T;1 = 0$
 by (*simp add: inf-sup-distrib1 inf-sup-distrib2*)

hence $aux1: x;1 \cdot x^T;1 \cdot y^T;1 = 0$
using *sup-eq-bot-iff* **by** *blast*
from a **have** $aux2: x;1 \cdot y;1 \cdot y^T;1 = 0$
using *sup-eq-bot-iff* **by** *blast*

show *is-inj* $(x + y)$
proof (*unfold is-inj-def; auto simp add: distrib-left*)
show $x;x^T \leq 1'$
using *assms(1) path-def is-inj-def* **by** *blast*
show $y;y^T \leq 1'$
using *assms(2) path-def is-inj-def* **by** *blast*
have $y;x^T = 0$
by (*metis assms(3) aux1 annir comp-assoc conv-one le-bot modular-var-2 one-idem-mult path-concat-aux-2 schroeder-2*)
thus $y;x^T \leq 1'$
using *bot-least le-bot* **by** *blast*
thus $x;y^T \leq 1'$
using *conv-iso* **by** *force*
qed

show *is-p-fun* $(x + y)$
proof (*unfold is-p-fun-def; auto simp add: distrib-left*)
show $x^T;x \leq 1'$
using *assms(1) path-def is-p-fun-def* **by** *blast*
show $y^T;y \leq 1'$
using *assms(2) path-def is-p-fun-def* **by** *blast*
have $y^T;x \leq y^T;(y;1 \cdot x;1)$
by (*metis conjugation-prop2 inf.commute inf-top.left-neutral maddux-20 mult-isol order-trans schroeder-1-var*)
also have $\dots = 0$
using *assms(3) aux2 annir inf-commute path-concat-aux-1* **by** *fastforce*
finally show $y^T;x \leq 1'$
using *bot-least le-bot* **by** *blast*
thus $x^T;y \leq 1'$
using *conv-iso* **by** *force*
qed

show *connected* $(x + y)$
proof (*auto simp add: distrib-left*)
have $x;1;x \leq x^* + x^{T*}$
using *assms(1) path-def* **by** *simp*
also have $\dots \leq (x^*;y^*)^* + (x^{T*};y^{T*})^*$
using *join-iso join-isol star-subdist* **by** *simp*
finally show $x;1;x \leq (x^*;y^*)^* + (x^{T*};y^{T*})^*$
have $y;1;y \leq y^* + y^{T*}$
using *assms(2) path-def* **by** *simp*
also have $\dots \leq (x^*;y^*)^* + (x^{T*};y^{T*})^*$

by (*metis star-denest star-subdist sup.mono sup-commute*)
finally show $y;1;y \leq (x^*;y^*)^* + (x^{T^*};y^{T^*})^*$.

show $y;1;x \leq (x^*;y^*)^* + (x^{T^*};y^{T^*})^*$
proof –
 have $(y;1);1;(1;x) \leq y^{T^*};x^{T^*}$
proof (*rule-tac v=start-points y in path-concat-aux-0*)
 show *is-vector* (*start-points y*)
 by (*metis is-vector-def comp-assoc one-compl one-idem-mult ra-1*)
 show *start-points y* $\neq 0$
 using *as*
 by (*metis assms(2) conv-compl conv-contrav conv-one inf.orderE*
inf-bot-right
inf-top.right-neutral maddux-141)
 have $(\text{start-points } y);1;y^T \leq y^*$
 by (*rule path-concat-aux-5*) (*simp-all add: assms(2)*)
thus $y;1;(\text{start-points } y)^T \leq y^{T^*}$
 by (*metis (mono-tags, lifting) conv-iso comp-assoc conv-contrav*
conv-invol conv-one
star-conv)
 have *end-points* $x;1;x \leq x^{T^*}$
apply (*rule path-concat-aux-5*)
using *assms(1) conv-path by simp-all*
thus *start-points* $y;(1;x) \leq x^{T^*}$
 by (*metis assms(3) mult-assoc*)
qed
thus *?thesis*
 by (*metis comp-assoc le-supI2 less-eq-def one-idem-mult star-denest*
star-subdist-var-1
sup commute)
qed

show $x;1;y \leq (x^*;y^*)^* + (x^{T^*};y^{T^*})^*$
proof –
 have $(x;1);1;(1;y) \leq x^*;y^*$
proof (*rule-tac v=start-points y in path-concat-aux-0*)
 show *is-vector* (*start-points y*)
 by (*simp add: comp-assoc is-vector-def one-compl vector-1-comm*)
 show *start-points y* $\neq 0$
 using *as assms(2,4) backward-terminating-iff1 galois-aux2*
start-point-iff1 start-point-iff2
 by *blast*
 have *end-points* $x;1;x^T \leq x^{T^*}$
apply (*rule path-concat-aux-5*)
using *assms(1) conv-path by simp-all*
hence (*end-points* $x;1;x^T$)^T $\leq (x^{T^*})^T$
using *conv-iso by blast*
thus $x;1;(\text{start-points } y)^T \leq x^*$
 by (*simp add: assms(3) comp-assoc star-conv*)

```

      have start-points  $y;1;y \leq y^*$ 
        by (rule path-concat-aux-5) (simp-all add: assms(2))
      thus start-points  $y;(1;y) \leq y^*$ 
        by (simp add: mult-assoc)
    qed
  thus ?thesis
    by (metis comp-assoc dual-order.trans le-supI1 one-idem-mult star-ext)
  qed
qed
qed
qed

```

lemma *path-concatenation-with-edge*:

```

  assumes  $x \neq 0$ 
    and forward-terminating-path  $x$ 
    and is-point  $q$ 
    and  $q \leq -(1;x)$ 
  shows path  $(x+(end-points\ x);q^T)$ 
proof (rule path-concatenation)
  from assms(1,2) have 1: is-point  $(end-points\ x)$ 
    using end-point-zero-point path-aux1a by blast
  show 2: backward-terminating-path  $((end-points\ x);q^T)$ 
    apply (intro conjI)
    apply (metis edge-is-path 1 assms(3))
    by (metis assms(2-4) 1 bot-least comp-assoc compl-le-swap1 conv-galois-2
      double-compl
        end-point-iff1 le-supE point-equations(1) tarski top-le)
  thus end-points  $x = start-points\ ((end-points\ x);q^T)$ 
    by (metis assms(3) 1 edge-start comp-assoc compl-top-eq double-compl
      inf.absorb-iff2 inf commute
        inf-top-right modular-2-aux' point-equations(2))
  show  $x;1 \cdot (x^T;1 + ((end-points\ x);q^T);1) \cdot ((end-points\ x);q^T)^T;1 = 0$ 
    using 2 by (metis assms(3,4) annir compl-le-swap1 compl-top-eq
      conv-galois-2 double-compl
        inf.absorb-iff2 inf commute modular-1' modular-2-aux'
          point-equations(2))
  show forward-terminating-path  $x$ 
    by (simp add: assms(2))
qed

```

lemma *path-concatenation-cycle-free*:

```

  assumes forward-terminating-path  $x$ 
    and backward-terminating-path  $y$ 
    and end-points  $x = start-points\ y$ 
    and  $x;1 \cdot y^T;1 = 0$ 
  shows path  $(x+y)$ 
apply (rule path-concatenation,simp-all add: assms)
by (metis assms(4) inf.left-commute inf-bot-right inf-commute)

```

lemma *path-concatenation-start-points-approx*:

assumes *end-points* $x = \text{start-points } y$

shows *start-points* $(x+y) \leq \text{start-points } x$

proof –

have *start-points* $(x+y) = x;1 \cdot -(x^T;1) \cdot -(y^T;1) + y;1 \cdot -(x^T;1) \cdot -(y^T;1)$

by (*simp add: inf.assoc inf-sup-distrib2*)

also with *assms(1)* **have** $\dots = x;1 \cdot -(x^T;1) \cdot -(y^T;1) + x^T;1 \cdot -(x^T;1) \cdot -(x;1)$

by (*metis inf.assoc inf.left-commute*)

also have $\dots = x;1 \cdot -(x^T;1) \cdot -(y^T;1)$

by *simp*

also have $\dots \leq \text{start-points } x$

using *inf-le1* **by** *blast*

finally show *?thesis* .

qed

lemma *path-concatenation-end-points-approx*:

assumes *end-points* $x = \text{start-points } y$

shows *end-points* $(x+y) \leq \text{end-points } y$

proof –

have *end-points* $(x+y) = x^T;1 \cdot -(x;1) \cdot -(y;1) + y^T;1 \cdot -(x;1) \cdot -(y;1)$

by (*simp add: inf.assoc inf-sup-distrib2*)

also from *assms(1)* **have** $\dots = y;1 \cdot -(y^T;1) \cdot -(y;1) + y^T;1 \cdot -(x;1) \cdot -(y;1)$

by *simp*

also have $\dots = y^T;1 \cdot -(x;1) \cdot -(y;1)$

by (*simp add: inf.commute*)

also have $\dots \leq \text{end-points } y$

using *inf-le1 meet-iso* **by** *blast*

finally show *?thesis* .

qed

lemma *path-concatenation-start-points*:

assumes *end-points* $x = \text{start-points } y$

and $x;1 \cdot y^T;1 = 0$

shows *start-points* $(x+y) = \text{start-points } x$

proof –

from *assms(2)* **have** *aux*: $x;1 \cdot -(y^T;1) = x;1$

by (*simp add: galois-aux inf.absorb1*)

have *start-points* $(x+y) = (x;1 \cdot -(x^T;1) \cdot -(y^T;1)) + (y;1 \cdot -(x^T;1) \cdot -(y^T;1))$

by (*simp add: inf-sup-distrib2 inf.assoc*)

also from *assms(1)* **have** $\dots = (x;1 \cdot -(x^T;1) \cdot -(y^T;1)) + (x^T;1 \cdot -(x;1) \cdot -(x^T;1))$

using *inf.assoc inf.commute* **by** *simp*

also have $\dots = (x;1 \cdot -(x^T;1) \cdot -(y^T;1))$

by (*simp add: inf.assoc*)

also from *aux* **have** $\dots = x;1 \cdot -(x^T;1)$

by (*metis inf.assoc inf.commute*)
finally show ?thesis .
qed

lemma *path-concatenation-end-points*:

assumes *end-points* $x = \text{start-points } y$
and $x;1 \cdot y^T;1 = 0$
shows *end-points* $(x+y) = \text{end-points } y$

proof –

from *assms*(2) **have** *aux*: $y^T;1 \cdot -(x;1) = y^T;1$
using *galois-aux inf.absorb1 inf.commute* **by** *blast*

have *end-points* $(x+y) = (x^T;1 + y^T;1) \cdot -(x;1) \cdot -(y;1)$
using *inf.assoc* **by** *simp*

also from *assms*(1) **have** ... = $(y;1 \cdot -(y^T;1) \cdot -(y;1)) + (y^T;1 \cdot -(x;1) \cdot -(y;1))$

by (*simp add: inf-sup-distrib2*)

also have ... = $y^T;1 \cdot -(x;1) \cdot -(y;1)$

by (*simp add: inf.assoc*)

also from *aux* **have** ... = $y^T;1 \cdot -(y;1)$

by (*metis inf.assoc inf.commute*)

finally show ?thesis .

qed

lemma *path-concatenation-cycle-free-complete*:

assumes *forward-terminating-path* x
and *backward-terminating-path* y
and *end-points* $x = \text{start-points } y$
and $x;1 \cdot y^T;1 = 0$

shows *path* $(x+y) \wedge \text{start-points } (x+y) = \text{start-points } x \wedge \text{end-points } (x+y)$
= *end-points* y

using *assms path-concatenation-cycle-free path-concatenation-end-points*

path-concatenation-start-points

by *blast*

[Path restriction \(path from a given point\)](#)

lemma *reachable-points-iff*:

assumes *point* p

shows $(x^{T^*};p \cdot x) = (x^{T^*};p \cdot 1^{\wedge});x$

proof (*rule order.antisym*)

show $(x^{T^*};p \cdot 1^{\wedge});x \leq x^{T^*};p \cdot x$

proof (*rule le-infI*)

show $(x^{T^*};p \cdot 1^{\wedge});x \leq x^{T^*};p$

proof –

have $(x^{T^*};p \cdot 1^{\wedge});x \leq x^{T^*};p;1$

by (*simp add: mult-isol-var*)

also have ... $\leq x^{T^*};p$

using *assms* **by** (*simp add: comp-assoc order.eq-iff point-equations(1)*)

point-is-point)

```

    finally show ?thesis .
  qed
  show  $(x^{T^*}; p \cdot 1 \wedge); x \leq x$ 
    by (metis inf-le2 mult-isor mult-onel)
  qed
  show  $x^{T^*}; p \cdot x \leq (x^{T^*}; p \cdot 1 \wedge); x$ 
  proof -
    have  $(x^{T^*}; p); x^T \leq x^{T^*}; p + -1'$ 
      by (metis assms comp-assoc is-vector-def mult-isol point-def sup.coboundedI1
top-greatest)
    hence aux:  $(-(x^{T^*}; p) \cdot 1 \wedge); x \leq -(x^{T^*}; p)$ 
      using compl-mono conv-galois-2 by fastforce
    have  $x = (x^{T^*}; p \cdot 1 \wedge); x + (-(x^{T^*}; p) \cdot 1 \wedge); x$ 
      by (metis aux4 distrib-right inf-commute mult-1-left)
    also with aux have  $\dots \leq (x^{T^*}; p \cdot 1 \wedge); x + -(x^{T^*}; p)$ 
      using join-isol by blast
    finally have  $x \leq (x^{T^*}; p \cdot 1 \wedge); x + -(x^{T^*}; p)$  .
    thus ?thesis
      using galois-2 inf.commute by fastforce
  qed
qed

lemma path-from-given-point:
  assumes path x
    and point p
  shows path  $(x^{T^*}; p \cdot x)$ 
    and start-points  $(x^{T^*}; p \cdot x) \leq p$ 
    and end-points  $(x^{T^*}; p \cdot x) \leq \text{end-points}(x)$ 
proof (unfold path-def; intro conjI)
  show uni: is-p-fun  $(x^{T^*}; p \cdot x)$ 
    by (metis assms(1) inf-commute is-p-fun-def p-fun-mult-var path-def)
  show inj: is-inj  $(x^{T^*}; p \cdot x)$ 
    by (metis abel-semigroup.commute assms(1) conv-times
inf.abel-semigroup-axioms inj-p-fun
is-p-fun-def p-fun-mult-var path-def)
  show connected  $(x^{T^*}; p \cdot x)$ 
  proof -
    let ?t =  $x^{T^*}; p \cdot 1'$ 
    let ?u =  $-(x^{T^*}; p) \cdot 1'$ 

    have t-plus-u:  $?t + ?u = 1'$ 
      by (simp add: inf.commute)
    have t-times-u:  $?t ; ?u \leq 0$ 
      by (simp add: inf.left-commute is-test-def test-comp-eq-mult)
    have t-conv:  $?t^T = ?t$ 
      using inf.cobounded2 is-test-def test-eq-conv by blast
    have txu-zero:  $?t; x; ?u \leq 0$ 
  proof -
    have  $x^T; ?t; 1 \leq -?u$ 

```

proof –
have $x^T; ?t; 1 \leq x^T; x^{T*}; p$
using *assms(2)*
by (*simp add: is-vector-def mult.semigroup-axioms mult-isol-var*
mult-subdistr order.refl
point-def semigroup.assoc)
also have $\dots \leq -?u$
by (*simp add: le-supI1 mult-isol*)
finally show *?thesis* .
qed
thus *?thesis*
by (*metis compl-bot-eq compl-le-swap1 conv-contrav conv-galois-1 t-conv*)
qed
hence *txx-zero: ?t;x; ?u;x ≤ 0*
using *annil le-bot by fastforce*

have *tx-leq: ?t;x* ≤ (?t;x)**
proof –
have $?t;x^* = ?t;(?t;x + ?u;x)^*$
using *t-plus-u by (metis distrib-right' mult-onel)*
also have $\dots = ?t;(?u;x; (?u;x)^*; (?t;x)^* + (?t;x)^*)$
using *txx-zero star-denest-10 by (simp add: comp-assoc le-bot)*
also have $\dots = ?t; ?u;x; (?u;x)^*; (?t;x)^* + ?t; (?t;x)^*$
by (*simp add: comp-assoc distrib-left*)
also have $\dots \leq 0;x; (?u;x)^*; (?t;x)^* + ?t; (?t;x)^*$
using *le-bot t-times-u by blast*
also have $\dots \leq (?t;x)^*$
by (*metis annil inf.commute inf-bot-right le-supI mult-onel mult-subdistr*)
finally show *?thesis* .
qed

hence *aux: ?t;x*; ?t ≤ (?t;x)**
using *inf.cobounded2 order.trans prod-star-closure star-ref by blast*
with *t-conv have aux-trans: ?t;x^{T*}; ?t ≤ (?t;x)^{T*}*
by (*metis comp-assoc conv-contrav conv-self-conjugate-var g-iso star-conv*)

from *aux aux-trans have ?t;(x*+x^{T*}); ?t ≤ (?t;x)* + (?t;x)^{T*}*
by (*metis sup-mono distrib-right' distrib-left*)
with *assms(1) path-concat-aux3-1 have ?t;(x;1;x^T); ?t ≤ (?t;x)* + (?t;x)^{T*}*
using *dual-order.trans mult-double-iso by blast*
with *t-conv have (?t;x); 1; (?t;x)^T ≤ (?t;x)* + (?t;x)^{T*}*
using *comp-assoc conv-contrav by fastforce*
with *connected-iff2 show ?thesis*
using *assms(2) inj reachable-points-iff uni by fastforce*

qed
next
show *start-points (x^{T*}; p · x) ≤ p*
proof –

```

have 1: is-vector  $(x^{T^*}; p)$ 
  using assms(2) by (simp add: is-vector-def mult-assoc point-def)
hence  $(x^{T^*}; p \cdot x); 1 \leq x^{T^*}; p$ 
  by (simp add: inf.commute vector-1-comm)
also have  $\dots = x^{T^+}; p + p$ 
  by (simp add: sup.commute)
finally have 2:  $(x^{T^*}; p \cdot x); 1 \cdot -(x^{T^+}; p) \leq p$ 
  using galois-1 by blast
have  $(x^{T^*}; p \cdot x)^T; 1 = (x^T \cdot (x^{T^*}; p)^T); 1$ 
  by (simp add: inf.commute)
also have  $\dots = x^T; (x^{T^*}; p \cdot 1)$ 
  using 1 vector-2 by blast
also have  $\dots = x^{T^+}; p$ 
  by (simp add: comp-assoc)
finally show start-points  $(x^{T^*}; p \cdot x) \leq p$ 
  using 2 by simp
qed
next
show end-points  $(x^{T^*}; p \cdot x) \leq \text{end-points}(x)$ 
proof -
  have 1: is-vector  $(x^{T^*}; p)$ 
    using assms(2) by (simp add: is-vector-def mult-assoc point-def)
  have  $(x^{T^*}; p \cdot x)^T; 1 = ((x^{T^*}; p)^T \cdot x^T); 1$ 
    by (simp add: star-conv)
  also have  $\dots = x^T; (x^{T^*}; p \cdot 1)$ 
    using 1 vector-2 inf.commute by fastforce
  also have  $\dots \leq x^{T^*}; p$ 
    using comp-assoc mult-isor by fastforce
  finally have 2:  $(x^{T^*}; p \cdot x)^T; 1 \cdot -(x^{T^*}; p) = 0$ 
    using galois-aux2 by blast
  have  $(x^{T^*}; p \cdot x)^T; 1 \cdot -((x^{T^*}; p \cdot x); 1) = (x^{T^*}; p \cdot x)^T; 1 \cdot -(x^{T^*}; p) +$ 
 $- (x; 1)$ 
    using 1 vector-1 by fastforce
  also have  $\dots = (x^{T^*}; p \cdot x)^T; 1 \cdot -(x^{T^*}; p) + (x^{T^*}; p \cdot x)^T; 1 \cdot -(x; 1)$ 
    using inf-sup-distrib1 by blast
  also have  $\dots = (x^{T^*}; p \cdot x)^T; 1 \cdot -(x; 1)$ 
    using 2 by simp
  also have  $\dots \leq x^T; 1 \cdot -(x; 1)$ 
    using meet-iso mult-subdistr-var by fastforce
  finally show ?thesis .
qed
qed

lemma path-from-given-point':
  assumes has-start-points-path  $x$ 
    and point  $p$ 
    and  $p \leq x; 1$ 
  shows path  $(x^{T^*}; p \cdot x)$ 
    and start-points  $(x^{T^*}; p \cdot x) = p$ 

```


and $end\text{-}points(x^{T^*}; p \cdot x) = end\text{-}points(x)$
proof –
show $path(x^{T^*}; p \cdot x)$
using $assms\ path\text{-}from\text{-}given\text{-}point(1)$ **by** $blast$
next
show $start\text{-}points(x^{T^*}; p \cdot x) = p$
proof ($simp\ only: order.eq\text{-}iff; rule\ conjI$)
show $start\text{-}points(x^{T^*}; p \cdot x) \leq p$
using $assms\ path\text{-}from\text{-}given\text{-}point(2)$ **by** $blast$
show $p \leq start\text{-}points(x^{T^*}; p \cdot x)$
proof –
have $1: is\text{-}vector(x^{T^*}; p)$
using $assms(2)\ comp\text{-}assoc\ is\text{-}vector\text{-}def\ point\text{-}equations(1)\ point\text{-}is\text{-}point$
by $fastforce$
hence $a: p \leq (x^{T^*}; p \cdot x); 1$
by ($metis\ vector\text{-}1\ assms(3)\ conway.dagger\text{-}unfoldl\text{-}distr\ inf.orderI$
 $inf\text{-}greatest$
 $inf\text{-}sup\text{-}absorb$)

have $x^{T^+}; p \cdot p \leq (x^{T^+} \cdot 1')$; p
using $assms(2)\ inj\text{-}distr\ point\text{-}def$ **by** $fastforce$
also **have** $\dots \leq (-1^{T^+} \cdot 1')$; p
using $assms(1)\ path\text{-}acyclic$
by ($metis\ conv\text{-}contrav\ conv\text{-}e\ meet\text{-}iso\ mult\text{-}isor\ star\text{-}conv\ star\text{-}slide\text{-}var$
 $test\text{-}converse$)
also **have** $\dots \leq 0$
by $simp$
finally **have** $2: x^{T^+}; p \cdot p \leq 0$.

have $b: p \leq -((x^{T^*}; p \cdot x)^T; 1)$
proof –
have $(x^{T^*}; p \cdot x)^T; 1 = ((x^{T^*}; p)^T \cdot x^T); 1$
by ($simp\ add: star\text{-}conv$)
also **have** $\dots = x^T; (x^{T^*}; p \cdot 1)$
using $1\ vector\text{-}2\ inf.commute$ **by** $fastforce$
also **have** $\dots = x^T; x^{T^*}; p$
by ($simp\ add: comp\text{-}assoc$)
also **have** $\dots \leq -p$
using $2\ galois\text{-}aux\ le\text{-}bot$ **by** $blast$
finally **show** $?thesis$
using $compl\text{-}le\text{-}swap1$ **by** $blast$
qed
with a **show** $?thesis$
by $simp$
qed
qed
next
show $end\text{-}points(x^{T^*}; p \cdot x) = end\text{-}points(x)$
proof ($simp\ only: order.eq\text{-}iff; rule\ conjI$)

show $\text{end-points}(x^{T^*}; p \cdot x) \leq \text{end-points}(x)$
using *assms path-from-given-point(3)* **by** *blast*
show $\text{end-points}(x) \leq \text{end-points}(x^{T^*}; p \cdot x)$
proof –
have 1: $\text{is-vector}(x^{T^*}; p)$
using *assms(2) comp-assoc is-vector-def point-equations(1) point-is-point*
by *fastforce*
have 2: $\text{is-vector}(\text{end-points}(x))$
by (*simp add: comp-assoc is-vector-def one-compl vector-1-comm*)
have a: $\text{end-points}(x) \leq (x^{T^*}; p \cdot x)^T; 1$
proof –
have $x^T; 1 \cdot 1; x^T = x^T; 1; x^T$
by (*simp add: vector-meet-comp-x'*)
also have $\dots \leq x^{T^*} + x^*$
using *assms(1) path-concat-ax3-3 sup commute* **by** *fastforce*
also have $\dots = x^{T^*} + x^+$
by (*simp add: star-star-plus sup commute*)
also have $\dots \leq x^{T^*} + x; 1$
using *join-isol mult-isol* **by** *fastforce*
finally have $\text{end-points}(x) \cdot 1; x^T \leq x^{T^*}$
by (*metis galois-1 inf.assoc inf commute sup commute*)
hence $\text{end-points}(x) \cdot p^T \leq x^{T^*}$
using *assms(3)*
by (*metis conv-contrav conv-iso conv-one dual-order.trans inf.cobounded1*
inf.right-idem
inf-mono)
hence $\text{end-points}(x) ; p^T \leq x^{T^*}$
using *assms(2) 2* **by** (*simp add: point-def vector-meet-comp*)
hence $\text{end-points}(x) \leq x^{T^*}; p$
using *assms(2) point-def ss423bij* **by** *blast*
hence $x^T; 1 \leq x^{T^*}; p + x; 1$
by (*simp add: galois-1 sup commute*)
hence $x^T; 1 \leq x^{T^+}; p + p + x; 1$
by (*metis conway.dagger-unfoldl-distr sup commute*)
hence $x^T; 1 \leq x^{T^+}; p + x; 1$
by (*simp add: assms(3) sup.absorb2 sup.assoc*)
hence $\text{end-points}(x) \leq x^{T^+}; p$
by (*simp add: galois-1 sup commute*)
also have $\dots = (x^{T^*}; p \cdot x)^T; 1$
using 1 *inf commute mult-assoc vector-2* **by** *fastforce*
finally show *?thesis* .
qed
have $x^T; 1 \cdot (x^{T^*}; p \cdot x); 1 \leq x; 1$
by (*simp add: le-infI2 mult-isol*)
hence b: $\text{end-points}(x) \leq -((x^{T^*}; p \cdot x); 1)$
using *galois-1 galois-2* **by** *blast*
with a **show** *?thesis*
by *simp*
qed

qed
qed

Cycles

lemma *selfloop-is-cycle*:
 assumes *is-point* x
 shows *cycle* $(x;x^T)$
 by (*simp add: assms edge-is-path*)

lemma *start-point-no-cycle*:
 assumes *has-start-points-path* x
 shows \neg *cycle* x
using *assms many-strongly-connected-implies-no-start-end-points*
no-start-end-points-iff
 start-point-iff1 start-point-iff2 **by** *blast*

lemma *end-point-no-cycle*:
 assumes *has-end-points-path* x
 shows \neg *cycle* x
using *assms end-point-iff2 end-point-iff1*
many-strongly-connected-implies-no-start-end-points
 no-start-end-points-iff **by** *blast*

lemma *cycle-no-points*:
 assumes *cycle* x
 shows *start-points* $x = 0$
 and *end-points* $x = 0$
 by (*metis assms inf-compl-bot*
many-strongly-connected-implies-no-start-end-points)**+**

Path concatenation to cycle

lemma *path-path-equals-cycle-aux*:
 assumes *has-start-end-points-path* x
 and *has-start-end-points-path* y
 and *start-points* $x =$ *end-points* y
 and *end-points* $x =$ *start-points* y
shows $x \leq (x+y)^{T^*}$
proof –
 let $?e =$ *end-points*(x)
 let $?s =$ *start-points*(x)
 have $sp:$ *is-point*($?s$)
 using *assms(1) start-point-iff2 has-start-end-points-path-iff* **by** *blast*
 have $ep:$ *is-point*($?e$)
 using *assms(1) end-point-iff2 has-start-end-points-path-iff* **by** *blast*

 have $x \leq x^{T^*}; ?s; 1 \cdot 1; ?e^T; x^{T^*}$
 by (*metis assms(1) backward-terminating-path-start-points-2 end-point-iff2 ep*
 forward-terminating-iff1 forward-terminating-path-end-points-2
comp-assoc)

$conv\text{-}contrav\ conv\text{-}invol\ conv\text{-}iso\ inf.\text{bounded}I\ point\text{-}equations(1)$
 $point\text{-}equations(4)$
 $star\text{-}conv\ sp\ start\text{-}point\text{-}iff2$)
also have $\dots = x^{T^*}; ?s; 1; ?e^T; x^{T^*}$
by (*metis inf-commute inf-top-right ra-1*)
also have $\dots = x^{T^*}; ?s; ?e^T; x^{T^*}$
by (*metis ep comp-assoc point-equations(4)*)
also have $\dots \leq x^{T^*}; y^{T^*}; x^{T^*}$
by (*metis (mono-tags, lifting) assms(2-4) start-to-end comp-assoc*
 $conv\text{-}contrav\ conv\text{-}invol$
 $conv\text{-}iso\ mult\text{-}double\text{-}iso\ star\text{-}conv$)
also have $\dots = (x^*; y^*; x^*)^T$
by (*simp add: comp-assoc star-conv*)
also have $\dots \leq ((x+y)^*; (x+y)^*; (x+y)^*)^T$
by (*metis conv-invol conv-iso prod-star-closure star-conv star-denest star-ext*
 $star\text{-}iso$
 $star\text{-}trans\text{-}eq\ sup\text{-}ge1$)
also have $\dots = (x+y)^{T^*}$
by (*metis star-conv star-trans-eq*)
finally show $x: x \leq (x+y)^{T^*}$.
qed

lemma *path-path-equals-cycle*:

assumes *has-start-end-points-path* x
and *has-start-end-points-path* y
and *start-points* $x = \text{end-points } y$
and *end-points* $x = \text{start-points } y$
and $x; 1 \cdot (x^T; 1 + y; 1) \cdot y^T; 1 = 0$
shows $cycle(x + y)$
proof (*intro conjI*)
show $path(x + y)$
apply (*rule path-concatenation*)
using *assms* **by** (*simp-all add:has-start-end-points-iff*)
show *many-strongly-connected* $(x + y)$
by (*metis path-path-equals-cycle-aux assms(1-4) sup.commute le-supI*
 $many\text{-}strongly\text{-}connected\text{-}iff\text{-}3$)
qed

lemma *path-edge-equals-cycle*:

assumes *has-start-end-points-path* x
shows $cycle(x + \text{end-points}(x); (\text{start-points } x)^T)$
proof (*rule path-path-equals-cycle*)
let $?s = \text{start-points } x$
let $?e = \text{end-points } x$
let $?y = (?e; ?s^T)$

have $sp: is\text{-}point(?s)$
using *start-point-iff2* *assms* *has-start-end-points-path-iff* **by** *blast*
have $ep: is\text{-}point(?e)$

```

using end-point-iff2 assms has-start-end-points-path-iff by blast

show has-start-end-points-path x
  using assms by blast
show has-start-end-points-path ?y
  using edge-is-path
  by (metis assms edge-end edge-start end-point-iff2 end-point-iff1 galois-ax2
      has-start-end-points-iff inf.left-idem inf-compl-bot-right start-point-iff2)
show ?s = end-points ?y
  by (metis sp ep edge-end annil conv-zero inf.left-idem inf-compl-bot-right)
thus ?e = start-points ?y
  by (metis edge-start ep conv-contrav conv-invol sp)
show  $x;1 \cdot (x^T;1 + ?e;?s^T;1) \cdot (?e;?s^T)^T;1 = 0$ 
proof -
  have  $x;1 \cdot (x^T;1 + ?e;?s^T;1) \cdot (?e;?s^T)^T;1 = x;1 \cdot (x^T;1 + ?e;1;?s^T;1) \cdot$ 
   $(?s;?e^T);1$ 
  using sp comp-assoc point-equations(3) by fastforce
  also have  $\dots = x;1 \cdot (x^T;1 + ?e;1) \cdot ?s;1$ 
  by (metis sp ep comp-assoc point-equations(1,3))
  also have  $\dots \leq 0$ 
  by (simp add: sp ep inf.assoc point-equations(1))
  finally show ?thesis
  using bot-unique by blast
qed
qed

```

Break cycles

```

lemma cycle-remove-edge:
  assumes cycle x
    and point s
    and point e
    and  $e;s^T \leq x$ 
  shows path( $x \cdot -(e;s^T)$ )
    and start-points ( $x \cdot -(e;s^T)$ )  $\leq s$ 
    and end-points ( $x \cdot -(e;s^T)$ )  $\leq e$ 
proof -
show path( $x \cdot -(e;s^T)$ )
proof (unfold path-def; intro conjI)
  show 1: is-p-fun( $x \cdot -(e;s^T)$ )
    using assms(1) path-def is-p-fun-def p-fun-mult-var by blast
  show 2: is-inj( $x \cdot -(e;s^T)$ )
    using assms(1) path-def inf.cobounded1 injective-down-closed by blast
show connected ( $x \cdot -(e;s^T)$ )
proof -
  have  $x^* = ((x \cdot -(e;s^T)) + e;s^T)^*$ 
    by (metis assms(4) aux4-comm inf.absorb2)
  also have  $\dots = (x \cdot -(e;s^T))^* ; (e;s^T ; (x \cdot -(e;s^T))^*)^*$ 
    by simp
  also have  $\dots = (x \cdot -(e;s^T))^* ; (1' + e;s^T ; (x \cdot -(e;s^T))^*)^* ; (e;s^T ; (x \cdot$ 

```

$-(e;s^T))^*)^*$
by *fastforce*
also have ... = $(x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;s^T ; (x \cdot -(e;s^T))^*; (e;s^T ;$
 $(x \cdot -(e;s^T))^*)^*$
by (*simp add: distrib-left mult-assoc*)
also have ... = $(x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;(s^T ; (x \cdot$
 $-(e;s^T))^*; e)^*; s^T ; (x \cdot -(e;s^T))^*$
by (*simp add: comp-assoc star-slide*)
also have ... $\leq (x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;1;s^T ; (x \cdot -(e;s^T))^*$
using *top-greatest join-isol mult-double-iso* **by** (*metis mult-assoc*)
also have ... = $(x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;s^T ; (x \cdot -(e;s^T))^*$
using *assms(3)* **by** (*simp add: comp-assoc is-vector-def point-def*)
finally have $\exists: x^* \leq (x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;s^T ; (x \cdot -(e;s^T))^* .$

from *assms(4)* **have** $e;s^T \leq e;e^T;x$
using *assms(3) comp-assoc mult-isol point-def ss423conv* **by** *fastforce*
also have ... $\leq e;e^T;(x^*)^T$
using *assms(1) many-strongly-connected-iff-3 mult-isol star-conv* **by**
fastforce
also have ... $\leq e;e^T;((x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;s^T ; (x \cdot$
 $-(e;s^T))^*)^T$
using *3 conv-iso mult-isol* **by** *blast*
also have ... $\leq e;e^T;((x \cdot -(e;s^T))^{T*} + (x \cdot -(e;s^T))^{T*} ; s;e^T ; (x \cdot$
 $-(e;s^T))^{T*})$
by (*simp add: star-conv comp-assoc*)
also have ... $\leq e;e^T;(x \cdot -(e;s^T))^{T*} + e;e^T;(x \cdot -(e;s^T))^{T*} ; s;e^T ; (x \cdot$
 $-(e;s^T))^{T*}$
by (*simp add: comp-assoc distrib-left*)
also have ... $\leq e;e^T;(x \cdot -(e;s^T))^{T*} + e;1;e^T ; (x \cdot -(e;s^T))^{T*}$
by (*metis comp-assoc join-isol mult-isol mult-isol top-greatest*)
also have ... $\leq e;e^T;(x \cdot -(e;s^T))^{T*} + e;e^T;(x \cdot -(e;s^T))^{T*}$
using *assms(3)* **by** (*simp add: point-equations(1) point-is-point*)
also have ... = $e;e^T;(x \cdot -(e;s^T))^{T*}$
by *simp*
also have ... $\leq 1';(x \cdot -(e;s^T))^{T*}$
using *assms(3) is-inj-def point-def join-iso mult-isol* **by** *blast*
finally have $4: e;s^T \leq (x \cdot -(e;s^T))^{T*}$
by *simp*

have $(x \cdot -(e;s^T));1;(x \cdot -(e;s^T)) \leq x;1;x$
by (*simp add: mult-isol-var*)
also have ... $\leq x^*$
using *assms(1) connected-iff4 one-strongly-connected-iff*
one-strongly-connected-implies-8
path-concat-aux3-3 **by** *blast*
also have ... $\leq (x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; e;s^T ; (x \cdot -(e;s^T))^*$
by (*rule 3*)
also have ... $\leq (x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^* ; (x \cdot -(e;s^T))^{T*} ; (x \cdot$
 $-(e;s^T))^*$

```

    using 4 by (metis comp-assoc join-isol mult-isol mult-isor)
    also have ...  $\leq (x \cdot -(e;s^T))^* + (x \cdot -(e;s^T))^{T*}$ 
    using 1 2 triple-star by force
    finally show ?thesis .
  qed
qed
next
show start-points  $(x \cdot -(e;s^T)) \leq s$ 
proof -
  have 1: is-vector(-s)
    using assms(2) by (simp add: point-def vector-compl)
  have  $(x \cdot -(e;s^T)); 1 \cdot -s \leq x; 1 \cdot -s$ 
    using meet-iso mult-subdistr by blast
  also have ...  $\leq x^T; 1 \cdot -s$ 
    using assms(1) many-strongly-connected-implies-no-start-end-points meet-iso
      no-start-end-points-path-iff by blast
  also have ...  $\leq (x^T \cdot -s); 1$ 
    using 1 by (simp add: vector-1-comm)
  also have ...  $\leq (x^T \cdot -(s;e^T)); 1$ 
    by (metis 1 galois-aux inf.boundedI inf.cobounded1 inf commute mult-isor
schroeder-2
      vector-1-comm)
  also have ...  $= (x \cdot -(e;s^T))^T; 1$ 
    by (simp add: conv-compl)
  finally show ?thesis
    by (simp add: galois-1 sup-commute)
qed
next
show end-points  $(x \cdot -(e;s^T)) \leq e$ 
proof -
  have 1: is-vector(-e)
    using assms(3) by (simp add: point-def vector-compl)
  have  $(x \cdot -(e;s^T))^T; 1 \cdot -e \leq x^T; 1 \cdot -e$ 
    using meet-iso mult-subdistr by simp
  also have ...  $\leq x; 1 \cdot -e$ 
    using assms(1) many-strongly-connected-implies-no-start-end-points meet-iso
      no-start-end-points-path-iff by blast
  also have ...  $\leq (x \cdot -e); 1$ 
    using 1 by (simp add: vector-1-comm)
  also have ...  $\leq (x \cdot -(e;s^T)); 1$ 
    by (metis 1 galois-aux inf.boundedI inf.cobounded1 inf commute mult-isor
schroeder-2
      vector-1-comm)
  finally show ?thesis
    by (simp add: galois-1 sup-commute)
qed
qed
lemma cycle-remove-edge':

```

```

assumes cycle x
  and point s
  and point e
  and  $s \neq e$ 
  and  $e; s^T \leq x$ 
shows  $\text{path}(x \cdot -(e; s^T))$ 
  and  $s = \text{start-points}(x \cdot -(e; s^T))$ 
  and  $e = \text{end-points}(x \cdot -(e; s^T))$ 
proof -
  show  $\text{path}(x \cdot -(e; s^T))$ 
    using assms(1,2,3,5) cycle-remove-edge(1) by blast
next
show  $s = \text{start-points}(x \cdot -(e; s^T))$ 
proof (simp only: order.eq-iff; rule conjI)
  show  $s \leq \text{start-points}(x \cdot -(e; s^T))$ 
proof -
  have  $a: s \leq (x \cdot -(e; s^T)); 1$ 
proof -
  have  $1: \text{is-vector}(-e)$ 
    using assms(3) point-def vector-compl by blast
  from assms(2-4) have  $s = s \cdot -e$ 
    using comp-assoc edge-end point-equations(1) point-equations(3)
point-is-point by fastforce
  also have  $\dots \leq x^T; e \cdot -e$ 
    using assms(3,5) conv-iso meet-iso point-def ss423conv by fastforce
  also have  $\dots \leq x; 1 \cdot -e$ 
    by (metis assms(1) many-strongly-connected-implies-no-start-end-points
meet-iso mult-isol
    top-greatest)
  also have  $\dots \leq (x \cdot -e); 1$ 
    using  $1$  by (simp add: vector-1-comm)
  also have  $\dots \leq (x \cdot -(e; s^T)); 1$ 
    by (metis assms(3) comp-anti is-vector-def meet-isor mult-isol mult-isor
point-def
    top-greatest)
  finally show ?thesis .
qed
have  $b: s \leq -((x \cdot -(e; s^T))^T; 1)$ 
proof -
  have  $1: x; s = e$ 
    using assms predecessor-point' by blast
  have  $s \cdot x^T = s; (e^T + -(e^T)) \cdot x^T$ 
    using assms(2) point-equations(1) point-is-point by fastforce
  also have  $\dots = s; e^T \cdot x^T$ 
    by (metis 1 conv-contrav inf commute inf-sup-absorb modular-1')
  also have  $\dots \leq e^T$ 
    by (metis assms(3) inf.coboundedI1 mult-isor point-equations(4)
point-is-point
    top-greatest)

```



```

finally have  $s \cdot x^T \leq s \cdot e^T$ 
  by simp
also have  $\dots \leq s ; e^T$ 
  using assms(2,3) by (simp add: point-def vector-meet-comp)
finally have  $2: s \cdot x^T \cdot -(s ; e^T) = 0$ 
  using galois-aux2 by blast
thus ?thesis
proof –
  have  $s ; e^T = e^T \cdot s$ 
    using assms(2,3) inf-commute point-def vector-meet-comp by force
  thus ?thesis
    using  $2$ 
    by (metis assms(2,3) conv-compl conv-invol conv-one conv-times
galois-aux
      inf.assoc point-def point-equations(1) point-is-point schroeder-2
      vector-meet-comp)
  qed
qed
with a show ?thesis
  by simp
qed
show start-points  $(x \cdot -(e ; s^T)) \leq s$ 
  using assms(1,2,3,5) cycle-remove-edge(2) by blast
qed
next
show  $e = \text{end-points } (x \cdot -(e ; s^T))$ 
proof (simp only: order.eq-iff; rule conjI)
  show  $e \leq \text{end-points } (x \cdot -(e ; s^T))$ 

proof –
  have  $a: e \leq (x \cdot -(e ; s^T))^T ; 1$ 
proof –
  have  $1: \text{is-vector } (-s)$ 
    using assms(2) point-def vector-compl by blast
  from assms(2-4) have  $e = e \cdot -s$ 
    using comp-assoc edge-end point-equations(1) point-equations(3)
point-is-point by fastforce
  also have  $\dots \leq x ; s \cdot -s$ 
    using assms(2,5) meet-iso point-def ss423bij by fastforce
  also have  $\dots \leq x^T ; 1 \cdot -s$ 
    by (metis assms(1) many-strongly-connected-implies-no-start-end-points
meet-iso mult-isol
      top-greatest)
  also have  $\dots \leq (x^T \cdot -s) ; 1$ 
    using  $1$  by (simp add: vector-1-comm)
  also have  $\dots \leq (x^T \cdot -(s ; e^T)) ; 1$ 
    by (metis assms(2) comp-anti is-vector-def meet-isor mult-isol mult-isor
point-def
      top-greatest)

```

```

    finally show ?thesis
      by (simp add: conv-compl)
  qed
  have b:  $e \leq -((x \cdot - (e ; s^T));1)$ 
  proof -
    have 1:  $x^T ; e = s$ 
      using assms predecessor-point' by (metis conv-contrav conv-invol conv-iso
conv-path)
    have  $e \cdot x = e ; (s^T + -(s^T)) \cdot x$ 
      using assms(3) point-equations(1) point-is-point by fastforce
    also have  $\dots = e ; s^T \cdot x$ 
      by (metis 1 conv-contrav conv-invol inf.commute inf-sup-absorb
modular-1')
    also have  $\dots \leq s^T$ 
      by (metis assms(2) inf.coboundedI1 mult-isor point-equations(4)
point-is-point top-greatest)
    finally have  $e \cdot x \leq e \cdot s^T$ 
      by simp
    also have  $\dots \leq e ; s^T$ 
      using assms(2,3) by (simp add: point-def vector-meet-comp)
    finally have 2:  $e \cdot x \cdot -(e ; s^T) = 0$ 
      using galois-aux2 by blast
    thus ?thesis
  proof -
    have  $e ; s^T = s^T \cdot e$ 
      using assms(2,3) inf-commute point-def vector-meet-comp by force
    thus ?thesis
      using 2
      by (metis assms(2,3) conv-one galois-aux inf.assoc point-def
point-equations(1)
point-is-point schroeder-2 vector-meet-comp)
  qed
  qed
  with a show ?thesis
    by simp
  qed
  show end-points  $(x \cdot - (e ; s^T)) \leq e$ 
    using assms(1,2,3,5) cycle-remove-edge(3) by blast
  qed
  qed
end
end

```

3 Relational Characterisation of Rooted Paths

We characterise paths together with a designated root. This is important as often algorithms start with a single vertex, and then build up a path, a tree or another structure. An example is Dijkstra's shortest path algorithm.

theory *Rooted-Paths*

imports *Paths*

begin

context *relation-algebra*

begin

General theorems

lemma *step-has-target*:

assumes $x;r \neq 0$

shows $x^T;1 \neq 0$

using *assms inf commute inf-bot-right schroeder-1* **by** *fastforce*

lemma *end-point-char*:

$x^T;p = 0 \iff p \leq -(x;1)$

using *order.antisym bot-least compl-bot-eq conv-galois-1* **by** *fastforce*

end

context *relation-algebra-tarski*

begin

General theorems concerning points

lemma *successor-point*:

assumes *is-inj* x

and *point* r

and $x;r \neq 0$

shows *point* $(x;r)$

using *assms*

by (*simp add: inj-compose is-point-def is-vector-def mult-assoc point-is-point*)

lemma *no-end-point-char*:

assumes *point* p

shows $x^T;p \neq 0 \iff p \leq x;1$

by (*simp add: assms comp-assoc end-point-char is-vector-def point-in-vector-or-complement-iff*)

lemma *no-end-point-char-converse*:

assumes *point* p

shows $x;p \neq 0 \iff p \leq x^T;1$

using *assms no-end-point-char* **by** *force*

end

3.1 Consequences without the Tarski rule

context *relation-algebra-rtc*

begin

Definitions for path classifications

definition *path-root*

where $\text{path-root } r \ x \equiv r; x \leq x^* + x^{T^*} \wedge \text{is-inj } x \wedge \text{is-p-fun } x \wedge \text{point } r$

abbreviation *connected-root*

where $\text{connected-root } r \ x \equiv r; x \leq x^+$

definition *backward-finite-path-root*

where $\text{backward-finite-path-root } r \ x \equiv \text{connected-root } r \ x \wedge \text{is-inj } x \wedge \text{is-p-fun } x \wedge \text{point } r$

abbreviation *backward-terminating-path-root*

where $\text{backward-terminating-path-root } r \ x \equiv \text{backward-finite-path-root } r \ x \wedge x; r = 0$

abbreviation *cycle-root*

where $\text{cycle-root } r \ x \equiv r; x \leq x^+ \cdot x^T; 1 \wedge \text{is-inj } x \wedge \text{is-p-fun } x \wedge \text{point } r$

abbreviation *non-empty-cycle-root*

where $\text{non-empty-cycle-root } r \ x \equiv \text{backward-finite-path-root } r \ x \wedge r \leq x^T; 1$

abbreviation *finite-path-root-end*

where $\text{finite-path-root-end } r \ x \ e \equiv \text{backward-finite-path-root } r \ x \wedge \text{point } e \wedge r \leq x^*; e$

abbreviation *terminating-path-root-end*

where $\text{terminating-path-root-end } r \ x \ e \equiv \text{finite-path-root-end } r \ x \ e \wedge x^T; e = 0$

Equivalent formulations of *connected-root*

lemma *connected-root-iff1:*

assumes *point* r

shows $\text{connected-root } r \ x \longleftrightarrow 1; x \leq r^T; x^+$

by (*metis assms comp-assoc is-vector-def point-def ss423conv*)

lemma *connected-root-iff2:*

assumes *point* r

shows $\text{connected-root } r \ x \longleftrightarrow x^T; 1 \leq x^{T+}; r$

by (*metis assms conv-contrav conv-invol conv-iso conv-one star-conv star-slide-var connected-root-iff1*)

lemma *connected-root-aux:*

$x^{T+}; r \leq x^T; 1$

by (*simp add: comp-assoc mult-isol*)

lemma *connected-root-iff3:*

assumes *point r*
shows *connected-root r x \longleftrightarrow $x^T;1 = x^{T+};r$*
using *assms order.antisym connected-root-aux connected-root-iff2* **by** *fastforce*

lemma *connected-root-iff4:*

assumes *point r*
shows *connected-root r x \longleftrightarrow $1;x = r^T;x^+$*
by (*metis assms conv-contrav conv-invol conv-one star-conv star-slide-var connected-root-iff3*)

Consequences of *connected-root*

lemma *has-root-contra:*

assumes *connected-root r x*
and *point r*
and *$x^T;r = 0$*
shows *$x = 0$*
using *assms comp-assoc independence1 conv-zero ss-p18 connected-root-iff3*
by *force*

lemma *has-root:*

assumes *connected-root r x*
and *point r*
and *$x \neq 0$*
shows *$x^T;r \neq 0$*
using *has-root-contra assms* **by** *blast*

lemma *connected-root-move-root:*

assumes *connected-root r x*
and *$q \leq x^*;r$*
shows *connected-root q x*
by (*metis assms comp-assoc mult-isol phl-cons1 star-slide-var star-trans-eq*)

lemma *root-cycle-converse:*

assumes *connected-root r x*
and *point r*
and *$x;r \neq 0$*
shows *$x^T;r \neq 0$*
using *assms conv-zero has-root* **by** *fastforce*

Rooted paths

lemma *path-iff-aux-1:*

assumes *bijective r*
shows *$r;x \leq x^* + x^{T*} \longleftrightarrow x \leq r^T;(x^* + x^{T*})$*
by (*simp add: assms ss423conv*)

lemma *path-iff-aux-2:*

assumes *bijective r*
shows *$r;x \leq x^* + x^{T*} \longleftrightarrow x^T \leq (x^* + x^{T*});r$*
proof –

have $((x^* + x^{T*});r)^T = r^T;(x^* + x^{T*})$
by (*metis conv-add conv-contrav conv-invol star-conv sup commute*)
thus *?thesis*
by (*metis assms conv-invol conv-iso path-iff-aux-1*)
qed

lemma *path-iff-backward*:

assumes *is-inj x*
and *is-p-fun x*
and *point r*
and $r;x \leq x^* + x^{T*}$
shows *connected x*
proof –
have $x^T;1;x^T \leq (x^* + x^{T*});r;1;x^T$
using *assms(3,4) path-iff-aux-2 mult-isor point-def* **by** *blast*
also have $\dots = (x^* + x^{T*});r;1;x^T;x;x^T$
using *assms(1) comp-assoc inj-p-fun p-fun-triple* **by** *fastforce*
also have $\dots \leq (x^* + x^{T*});r;x;x^T$
by (*metis assms(3) mult-double-iso top-greatest point-def is-vector-def*
comp-assoc)
also have $\dots \leq (x^* + x^{T*});(x^* + x^{T*});x^T$
by (*metis assms(4) comp-assoc mult-double-iso*)
also have $\dots \leq (x^* + x^{T*});(x^* + x^{T*});(x^* + x^{T*})$
using *le-supI2 mult-isol star-ext* **by** *blast*
also have $\dots = x^* + x^{T*}$
using *assms(1,2) cancel-separate-converse-idempotent* **by** *fastforce*
finally show *?thesis*
by (*metis conv-add conv-contrav conv-invol conv-one mult-assoc star-conv*
sup.orderE sup.orderI
sup-commute)
qed

lemma *empty-path-root-end*:

assumes *terminating-path-root-end r x e*
shows $e = r \longleftrightarrow x = 0$
apply (*standard*)
using *assms has-root backward-finite-path-root-def* **apply** *blast*
by (*metis assms order.antisym conv-e conv-zero independence1 is-inj-def*
mult-oner point-swap
backward-finite-path-root-def ss423conv sur-def-var1 x-leq-triple-x)

lemma *path-root-acyclic*:

assumes *path-root r x*
and $x;r = 0$
shows *is-acyclic x*
proof –
have $x^+.1' = (x^+)^T.x^+.1'$
by (*metis conv-e conv-times inf.assoc inf.left-idem inf-le2*
many-strongly-connected-iff-7 mult-oner star-subid)

also have $\dots \leq x^T; 1 \cdot x^+ \cdot 1'$
by (*metis conv-contrav inf commute maddux-20 meet-double-iso plus-top star-conv star-slide-var*)
finally have $r; (x^+ \cdot 1') \leq r; (x^T; 1 \cdot x^+ \cdot 1')$
using *mult-isol by blast*
also have $\dots = (r \cdot 1; x); (x^+ \cdot 1')$
by (*metis (no-types, lifting) comp-assoc conv-contrav conv-invol conv-one inf.assoc is-vector-def one-idem-mult vector-2*)
also have $\dots = r; x; (x^+ \cdot 1')$
by (*metis asms(1) path-root-def point-def inf-top-right vector-1*)
also have $\dots \leq (x^* + x^{T*}); (x^+ \cdot 1')$
using *asms(1) mult-isor path-root-def by blast*
also have $\dots = x^*; (x^+ \cdot 1') + x^{T+}; (x^+ \cdot 1')$
by (*metis distrib-right star-star-plus sup commute*)
also have $\dots \leq x^*; (x^+ \cdot 1') + x^T; 1$
by (*metis join-isol mult-isol plus-top top-greatest*)
finally have $r; (x^+ \cdot 1'); 1 \leq x^*; (x^+ \cdot 1'); 1 + x^T; 1$
by (*metis distrib-right inf-absorb2 mult-assoc mult-subdistr one-idem-mult*)
hence $1: r; (x^+ \cdot 1'); 1 \leq x^T; 1$
by (*metis asms(1) inj-implies-step-forwards-backwards sup-absorb2 path-root-def*)
have $x^+ \cdot 1' \leq (x^+ \cdot 1'); 1$
by (*simp add: maddux-20*)
also have $\dots \leq r^T; r; (x^+ \cdot 1'); 1$
by (*metis asms(1) comp-assoc order.refl point-def ss423conv path-root-def*)
also have $\dots \leq r^T; x^T; 1$
using *1 by (simp add: comp-assoc mult-isol)*
also have $\dots = 0$
using *asms(2) annil conv-contrav conv-zero by force*
finally show *?thesis*
using *galois-aux le-bot by blast*
qed

Start points and end points

lemma *start-points-in-root-aux*:
assumes *backward-finite-path-root r x*
shows $x; 1 \leq x^{T*}; r$
proof –
have $x; 1 \leq x; x^{T+}; r$
by (*metis asms inf-top.left-neutral modular-var-2 mult-assoc connected-root-iff3 backward-finite-path-root-def*)
also have $\dots \leq 1'; x^{T*}; r$
by (*metis asms is-inj-def mult-assoc mult-isor backward-finite-path-root-def*)
finally show *?thesis*
by *simp*
qed

lemma *start-points-in-root*:

assumes *backward-finite-path-root* $r\ x$
shows *start-points* $x \leq r$
using *assms galois-1 sup-commute connected-root-iff3*
backward-finite-path-root-def
start-points-in-root-aux **by** *fastforce*

lemma *start-points-not-zero-contra*:

assumes *connected-root* $r\ x$
and *point* r
and *start-points* $x = 0$
and $x;r = 0$
shows $x = 0$
proof –
have $x;1 \leq x^T;1$
using *assms(3) galois-aux* **by** *force*
also have $\dots \leq -r$
using *assms(4) comp-res compl-bot-eq* **by** *blast*
finally show *?thesis*
using *assms(1,2) has-root-contra galois-aux schroeder-1* **by** *force*
qed

lemma *start-points-not-zero*:

assumes *connected-root* $r\ x$
and *point* r
and $x \neq 0$
and $x;r = 0$
shows *start-points* $x \neq 0$
using *assms start-points-not-zero-contra* **by** *blast*

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-aux*:

assumes *backward-terminating-path-root* $r\ x$
shows $x \leq x^{T^*}; -(x^T; 1)$
proof –
have $x^{T^*}; r \leq x^{T^*}; -(x^T; 1)$
using *assms comp-res compl-bot-eq compl-le-swap1 mult-isol* **by** *blast*
thus *?thesis*
using *assms dual-order.trans maddux-20 start-points-in-root-aux* **by** *blast*
qed

lemma *backward-finite-path-connected-aux*:

assumes *backward-finite-path-root* $r\ x$
shows $x^T;r;x^T \leq x^* + x^{T^*}$
proof –
have $x^T;r;x^T \cdot r^T = x^T;r;(x^T \cdot r^T)$
by (*metis conv-invol conv-times vector-1-comm comp-assoc conv-contrav assms*
backward-finite-path-root-def point-def)
also have $\dots \leq x^T;r;r^T$
by (*simp add: mult-isol*)

also have $1: \dots \leq x^T$
by (*metis assms comp-assoc is-inj-def mult-1-right mult-isol point-def backward-finite-path-root-def*)
also have $\dots \leq x^{T^*}$
by *simp*
finally have $2: x^T; r; x^T \cdot r^T \leq x^{T^*}$.
let $?v = x; 1 \cdot -r$
have $?v \leq x^{T^+}; r$
by (*simp add: assms galois-1 start-points-in-root-aux*)
hence $r^T; x \cdot ?v \leq r^T; x \cdot x^{T^+}; r$
using *meet-isor* **by** *blast*
also have $3: \dots = x^{T^+}; r \cdot 1; r^T; x$
by (*metis assms conv-contrav conv-one inf-commute is-vector-def point-def backward-finite-path-root-def*)
also have $\dots = (x^{T^+}; r \cdot 1); r^T; x$
using 3 **by** (*metis comp-assoc inf-commute is-vector-def star-conv vector-1 assms backward-finite-path-root-def point-def*)
also have $\dots = x^{T^+}; r; r^T; x$
by *simp*
also have $\dots \leq x^{T^+}; x$
using 1 **by** (*metis mult-assoc mult-isol mult-isor star-slide-var*)
also have $\dots = x^{T^*}; x^T; x$
by (*simp add: star-slide-var*)
also have $\dots \leq x^{T^*}$
by (*metis assms backward-finite-path-root-def is-p-fun-def mult-1-right mult-assoc mult-isol-var star-1l star-inductl-star*)
finally have $4: x^T; r \cdot ?v^T \leq x^*$
using *conv-iso star-conv* **by** *force*
have $x^T; r; x^T \cdot -r^T = (x^T; r \cdot 1); x^T \cdot -r^T$
by *simp*
also have $\dots = x^T; r \cdot 1; x^T \cdot -r^T$
by (*metis inf-commute is-vector-def comp-assoc vector-1 assms backward-finite-path-root-def point-def*)
also have $\dots \leq x^*$
using 4 **by** (*simp add: conv-compl inf.assoc*)
finally have $(x^T; r; x^T \cdot -r^T) + (x^T; r; x^T \cdot r^T) \leq x^* + x^{T^*}$
using 2 *sup.mono* **by** *blast*
thus *?thesis*
by *fastforce*
qed

lemma *backward-finite-path-connected:*

assumes *backward-finite-path-root* r x

shows *connected* x

proof –

from *assms* **obtain** r **where** $1: \text{backward-finite-path-root } r \ x \ ..$

have $x^T; (x^* + x^{T*}) = x^T; (1' + x^+) + x^{T+}$
by (*simp add: distrib-left*)
also have $\dots = x^T; x^+ + x^{T+}$
using *calculation distrib-left star-star-plus* **by** *fastforce*
also have $\dots \leq 1'; x^* + x^{T+}$
using *1* **by** (*metis add-iso comp-assoc is-p-fun-def mult-isor*)
backward-finite-path-root-def
also have $\dots \leq x^* + x^{T*}$
using *join-isol* **by** *fastforce*
finally have $x^T; r; x^T + x^T; (x^* + x^{T*}) \leq x^* + x^{T*}$
using *1 backward-finite-path-connected-aux* **by** *simp*
hence $x^{T*}; x^T; r; x^T \leq x^* + x^{T*}$
using *star-inductl comp-assoc* **by** *simp*
hence $x^T; 1; x^T \leq x^* + x^{T*}$
using *1 backward-finite-path-root-def connected-root-iff3 star-slide-var* **by**
fastforce
thus *?thesis*
by (*metis (mono-tags, lifting) sup commute comp-assoc conv-add conv-contrav*
conv-invol conv-iso
conv-one star-conv)
qed

lemma *backward-finite-path-root-path:*
assumes *backward-finite-path-root r x*
shows *path x*
using *assms path-def backward-finite-path-connected backward-finite-path-root-def*
by *blast*

lemma *backward-finite-path-root-path-root:*
assumes *backward-finite-path-root r x*
shows *path-root r x*
using *assms backward-finite-path-root-def le-supI1 star-star-plus path-root-def* **by**
fastforce

lemma *zero-backward-terminating-path-root:*
assumes *point r*
shows *backward-terminating-path-root r 0*
by (*simp add: assms is-inj-def is-p-fun-def backward-finite-path-root-def*)

lemma *backward-finite-path-root-move-root:*
assumes *backward-finite-path-root r x*
and *point q*
and $q \leq x^*; r$
shows *backward-finite-path-root q x*
using *assms connected-root-move-root backward-finite-path-root-def* **by** *blast*

Cycle

lemma *non-empty-cycle-root-var-axioms-1:*
non-empty-cycle-root r x \longleftrightarrow $x^T; 1 \leq x^{T+}; r \wedge is-inj x \wedge is-p-fun x \wedge point r \wedge$

$r \leq x^T;1$
using *connected-root-iff2 backward-finite-path-root-def* **by** *blast*

lemma *non-empty-cycle-root-loop*:
assumes *non-empty-cycle-root* $r\ x$
shows $r \leq x^{T+};r$
using *assms connected-root-iff3 backward-finite-path-root-def* **by** *fastforce*

lemma *cycle-root-end-empty*:
assumes *terminating-path-root-end* $r\ x\ e$
and *many-strongly-connected* x
shows $x = 0$
by (*metis assms has-root-contra point-swap backward-finite-path-root-def backward-finite-path-root-move-root star-conv*)

lemma *cycle-root-end-empty-var*:
assumes *terminating-path-root-end* $r\ x\ e$
and $x \neq 0$
shows \neg *many-strongly-connected* x
using *assms cycle-root-end-empty* **by** *blast*

Terminating path

lemma *terminating-path-root-end-connected*:
assumes *terminating-path-root-end* $r\ x\ e$
shows $x;1 \leq x^+;e$
proof –
have $x;1 \leq x;x^T;1$
by (*metis comp-assoc inf-top.left-neutral modular-var-2*)
also have $\dots = x;x^{T+};r$
using *assms backward-finite-path-root-def connected-root-iff3 comp-assoc* **by** *fastforce*
also have $\dots \leq x;x^{T+};x^*;e$
by (*simp add: assms comp-assoc mult-isol*)
also have $\dots = x;x^T;(x^* + x^{T*});e$
using *assms cancel-separate-p-fun-converse comp-assoc backward-finite-path-root-def* **by** *fastforce*
also have $\dots = x;x^T;(x^+ + x^{T*});e$
by (*simp add: star-star-plus*)
also have $\dots = x;x^T;x^+;e + x;x^{T+};e$
by (*simp add: comp-assoc distrib-left*)
also have $\dots = x;x^T;x^+;e$
by (*simp add: assms comp-assoc independence1*)
also have $\dots \leq x^+;e$
by (*metis assms annil independence1 is-inj-def mult-isor mult-oner backward-finite-path-root-def*)
finally show *?thesis* .
qed

lemma *terminating-path-root-end-forward-finite*:

assumes *terminating-path-root-end* $r\ x\ e$
shows *backward-finite-path-root* $e\ (x^T)$
using *assms terminating-path-root-end-connected inj-p-fun connected-root-iff2*
backward-finite-path-root-def **by** *force*

end

3.2 Consequences with the Tarski rule

context *relation-algebra-rtc-tarski*

begin

Some (more) results about points

lemma *point-reachable-converse*:

assumes *is-vector* v

and $v \neq 0$

and *point* r

and $v \leq x^{T+};r$

shows $r \leq x^+;v$

proof –

have $v^T;v \neq 0$

by (*metis assms(2) inf.idem inf-bot-right mult-1-right schroeder-1*)

hence $1;v^T;v = 1$

using *assms(1) is-vector-def mult-assoc tarski* **by** *force*

hence $1: r = r;v^T;v$

by (*metis assms(3) is-vector-def mult-assoc point-def*)

have $v;r^T \leq x^{T+}$

using *assms(3,4) point-def ss423bij* **by** *simp*

hence $r;v^T \leq x^+$

by (*metis conv-contrav conv-invol conv-iso star-conv star-slide-var*)

thus *?thesis*

using 1 **by** (*metis mult-isor*)

qed

Roots

lemma *root-in-start-points*:

assumes *connected-root* $r\ x$

and *is-vector* r

and $x \neq 0$

and $x;r = 0$

shows $r \leq \text{start-points } x$

proof –

have $r = r;x;1$

by (*metis assms(2,3) comp-assoc is-vector-def tarski*)

also have $\dots \leq x;1$

by (*metis assms(1) comp-assoc one-idem-mult phl-seq top-greatest*)

finally show *?thesis*

using *assms(4) comp-res compl-bot-eq compl-le-swap1 inf.boundedI* **by** *blast*

qed

lemma *root-equals-start-points*:
assumes *backward-terminating-path-root* r x
and $x \neq 0$
shows $r = \text{start-points } x$
using *assms order.antisym point-def backward-finite-path-root-def*
start-points-in-root root-in-start-points
by *fastforce*

lemma *root-equals-end-points*:
assumes *backward-terminating-path-root* r (x^T)
and $x \neq 0$
shows $r = \text{end-points } x$
by (*metis assms conv-invol step-has-target ss-p18 root-equals-start-points*)

lemma *root-in-edge-sources*:
assumes *connected-root* r x
and $x \neq 0$
and *is-vector* r
shows $r \leq x;1$
proof –
have $r;1;x;1 \leq x^+;1$
using *assms(1,3) is-vector-def mult-isor* **by** *fastforce*
thus *?thesis*
by (*metis assms(2) comp-assoc conway.dagger-unfoldl-distr dual-order.trans*
maddux-20 sup commute
sup-absorb2 tarski top-greatest)
qed

Rooted Paths

lemma *non-empty-path-root-iff-aux*:
assumes *path-root* r x
and $x \neq 0$
shows $r \leq (x + x^T);1$
proof –
have $(r;x \cdot 1^');1 = (x^T;r^T \cdot 1^');1$
by (*metis conv-contrav conv-e conv-times inf.cobounded2 is-test-def*
test-eq-conv)
also have $\dots \leq x^T;r^T;1$
using *mult-subdistr* **by** *blast*
also have $\dots \leq x^T;1$
by (*metis mult-assoc mult-double-iso one-idem-mult top-greatest*)
finally have $1: (r;x \cdot 1^');1 \leq x^T;1$.
have $r \leq r;1;x;1$
using *assms(2) comp-assoc maddux-20 tarski* **by** *fastforce*
also have $\dots = r;x;1$
using *assms(1) path-root-def point-def is-vector-def* **by** *simp*
also have $\dots = (r;x \cdot (x^* + x^{T*}));1$
using *assms(1) path-root-def* **by** (*simp add: inf.absorb-iff1*)
also have $\dots = (r;x \cdot (x^+ + x^{T+} + 1^'));1$

by (*metis star-star-plus star-unfoldl-eq sup-commute sup-left-commute*)
 also have $\dots \leq (x^+ + x^{T^+} + (r;x \cdot 1^'));1$
 by (*metis inf-le2 inf-sup-distrib1 mult-isor order-reft sup-mono*)
 also have $\dots \leq x;1 + x^T;1 + (r;x \cdot 1^');1$
 by (*simp add: plus-top*)
 also have $\dots = x;1 + x^T;1$
 using *1 sup.coboundedI2 sup.order-iff* by *fastforce*
 finally show *?thesis*
 by *simp*
 qed

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-2*:
 assumes *backward-terminating-path-root r x*
 shows *backward-terminating x*
 using *assms backward-terminating-iff2 path-def*
backward-terminating-path-root-aux
backward-finite-path-connected backward-finite-path-root-def by *blast*

lemma *backward-terminating-path-root*:
 assumes *backward-terminating-path-root r x*
 shows *backward-terminating-path x*
 using *assms backward-finite-path-root-path backward-terminating-path-root-2* by
fastforce

(Non-empty) Cycle

lemma *cycle-iff*:
 assumes *point r*
 shows $x;r \neq 0 \iff r \leq x^T;1$
 by (*simp add: assms no-end-point-char-converse*)

lemma *non-empty-cycle-root-iff*:
 assumes *connected-root r x*
 and *point r*
 shows $x;r \neq 0 \iff r \leq x^{T^+};r$
 using *assms connected-root-iff3 cycle-iff* by *simp*

lemma *backward-finite-path-root-terminating-or-cycle*:
backward-finite-path-root r x \iff *backward-terminating-path-root r x* \vee
non-empty-cycle-root r x
 using *cycle-iff backward-finite-path-root-def* by *blast*

lemma *non-empty-cycle-root-msc*:
 assumes *non-empty-cycle-root r x*
 shows *many-strongly-connected x*
proof –
 let $?p = x^T;r$
 have *1: is-point ?p*
 unfolding *is-point-def*

using *conjI* *assms is-vector-def mult-assoc point-def inj-compose p-fun-inj cycle-iff backward-finite-path-root-def root-cycle-converse* **by** *fastforce*
have $?p \leq x^{T+}; ?p$
by (*metis assms comp-assoc mult-isol star-slide-var non-empty-cycle-root-loop*)
hence $?p \leq x^+; ?p$
using *1 bot-least point-def point-is-point point-reachable-converse* **by** *blast*
also have $\dots = x^*; (x; x^T); r$
by (*metis comp-assoc star-slide-var*)
also have $\dots \leq x^*; 1'; r$
using *assms is-inj-def mult-double-iso backward-finite-path-root-def* **by** *blast*
finally have $2: ?p \leq x^*; r$
by *simp*
have $x^T; x^*; r = ?p + x^T; x^*; r$
by (*metis conway.dagger-unfoldl-distr distrib-left mult-assoc*)
also have $\dots \leq ?p + 1'; x^*; r$
by (*metis assms is-p-fun-def join-isol mult-assoc mult-isol backward-finite-path-root-def*)
also have $\dots = x^*; r$
using *2* **by** (*simp add: sup-absorb2*)
finally have $3: x^{T*}; r \leq x^*; r$
by (*metis star-inductl comp-assoc conway.dagger-unfoldl-distr le-supI order-prop*)
have $x^T \leq x^{T+}; r$
by (*metis assms maddux-20 connected-root-iff3 backward-finite-path-root-def*)
also have $\dots \leq x^*; r$
using *3* **by** (*metis assms conway.dagger-unfoldl-distr sup-absorb2 non-empty-cycle-root-loop*)
finally have $4: x^T \leq x^*; r$.
have $x^T \leq x^T; x; x^T$
by (*metis conv-invol x-leg-triple-x*)
also have $\dots \leq 1; x; x^T$
by (*simp add: mult-isol*)
also have $\dots = r^T; x^+; x^T$
using *assms connected-root-iff4 backward-finite-path-root-def* **by** *fastforce*
also have $\dots \leq r^T; x^*$
by (*metis assms is-inj-def mult-1-right mult-assoc mult-isol backward-finite-path-root-def star-slide-var*)
finally have $x^T \leq x^*; r \cdot r^T; x^*$
using *4* **by** *simp*
also have $\dots = x^*; r \cdot 1; r^T; x^*$
by (*metis assms conv-contrav conv-one is-vector-def point-def backward-finite-path-root-def*)
also have $\dots = (x^*; r \cdot 1); r^T; x^*$
by (*metis (no-types, lifting) assms is-vector-def mult-assoc point-def backward-finite-path-root-def vector-1*)
also have $\dots = x^*; r; r^T; x^*$
by *simp*
also have $\dots \leq x^*; x^*$

by (*metis* *assms* *is-inj-def* *mult-1-right* *mult-assoc* *mult-isol* *mult-isor* *point-def*
 backward-finite-path-root-def)
 also have $\dots \leq x^*$
 by *simp*
 finally show *?thesis*
 by (*simp* *add: many-strongly-connected-iff-1*)
 qed

lemma *non-empty-cycle-root-msc-cycle*:
 assumes *non-empty-cycle-root* r x
 shows *cycle* x
 using *assms* *backward-finite-path-root-path* *non-empty-cycle-root-msc* by *fastforce*

lemma *non-empty-cycle-root-non-empty*:
 assumes *non-empty-cycle-root* r x
 shows $x \neq 0$
 using *assms* *cycle-iff* *annil* *backward-finite-path-root-def* by *blast*

lemma *non-empty-cycle-root-rtc-symmetric*:
 assumes *non-empty-cycle-root* r x
 shows $x^*;r = x^{T*};r$
 using *assms* *non-empty-cycle-root-msc* by *fastforce*

lemma *non-empty-cycle-root-point-exchange*:
 assumes *non-empty-cycle-root* r x
 and *point* p
 shows $r \leq x^*;p \iff p \leq x^*;r$
 by (*metis* *assms*(1,2) *inj-sur-semi-swap* *point-def* *non-empty-cycle-root-msc*
 backward-finite-path-root-def *star-conv*)

lemma *non-empty-cycle-root-rtc-tc*:
 assumes *non-empty-cycle-root* r x
 shows $x^*;r = x^+;r$
proof (*rule* *order.antisym*)
 have $r \leq x^+;r$
 using *assms* *many-strongly-connected-iff-7* *non-empty-cycle-root-loop*
non-empty-cycle-root-msc
 by *simp*
 thus $x^*;r \leq x^+;r$
 using *sup-absorb2* by *fastforce*
next
 show $x^+;r \leq x^*;r$
 by (*simp* *add: mult-isor*)
 qed

lemma *non-empty-cycle-root-no-start-end-points*:
 assumes *non-empty-cycle-root* r x
 shows $x;1 = x^T;1$
 using *assms* *many-strongly-connected-implies-no-start-end-points*

non-empty-cycle-root-msc **by** *blast*

lemma *non-empty-cycle-root-move-root*:

assumes *non-empty-cycle-root* $r\ x$

and *point* q

and $q \leq x^*;r$

shows *non-empty-cycle-root* $q\ x$

by (*metis* *assms* *cycle-iff* *dual-order.trans* *backward-finite-path-root-move-root* *start-points-in-root*

root-equals-start-points *non-empty-cycle-root-non-empty*)

lemma *non-empty-cycle-root-loop-converse*:

assumes *non-empty-cycle-root* $r\ x$

shows $r \leq x^+;r$

using *assms* *less-eq-def* *non-empty-cycle-root-rtc-tc* **by** *fastforce*

lemma *non-empty-cycle-root-move-root-same-reachable*:

assumes *non-empty-cycle-root* $r\ x$

and *point* q

and $q \leq x^*;r$

shows $x^*;r = x^*;q$

by (*metis* *assms* *many-strongly-connected-iff-7* *connected-root-iff3* *connected-root-move-root*

backward-finite-path-root-def *non-empty-cycle-root-msc*

non-empty-cycle-root-rtc-tc)

lemma *non-empty-cycle-root-move-root-same-reachable-2*:

assumes *non-empty-cycle-root* $r\ x$

and *point* q

and $q \leq x^*;r$

shows $x^*;r = x^{T^*};q$

using *assms* *non-empty-cycle-root-move-root-same-reachable* *non-empty-cycle-root-msc* **by** *simp*

lemma *non-empty-cycle-root-move-root-msc*:

assumes *non-empty-cycle-root* $r\ x$

shows $x^{T^*};q = x^*;q$

using *assms* *non-empty-cycle-root-msc* **by** *simp*

lemma *non-empty-cycle-root-move-root-rtc-tc*:

assumes *non-empty-cycle-root* $r\ x$

and *point* q

and $q \leq x^*;r$

shows $x^*;q = x^+;q$

using *assms* *non-empty-cycle-root-move-root* *non-empty-cycle-root-rtc-tc* **by** *blast*

lemma *non-empty-cycle-root-move-root-loop-converse*:

assumes *non-empty-cycle-root* $r\ x$

and *point* q

and $q \leq x^*; r$
shows $q \leq x^{T+}; q$
using *assms non-empty-cycle-root-loop non-empty-cycle-root-move-root* **by** *blast*

lemma *non-empty-cycle-root-move-root-loop*:
assumes *non-empty-cycle-root r x*
and *point q*
and $q \leq x^*; r$
shows $q \leq x^+; q$
using *assms non-empty-cycle-root-loop-converse non-empty-cycle-root-move-root*
by *blast*

lemma *non-empty-cycle-root-msc-plus*:
assumes *non-empty-cycle-root r x*
shows $x^+; r = x^{T+}; r$
using *assms many-strongly-connected-iff-7 non-empty-cycle-root-msc* **by** *fastforce*

lemma *non-empty-cycle-root-tc-start-points*:
assumes *non-empty-cycle-root r x*
shows $x^+; r = x; 1$
by (*metis assms connected-root-iff3 backward-finite-path-root-def*
non-empty-cycle-root-msc-plus
non-empty-cycle-root-no-start-end-points)

lemma *non-empty-cycle-root-rtc-start-points*:
assumes *non-empty-cycle-root r x*
shows $x^*; r = x; 1$
by (*simp add: assms non-empty-cycle-root-rtc-tc*
non-empty-cycle-root-tc-start-points)

lemma *non-empty-cycle-root-converse-start-end-points*:
assumes *non-empty-cycle-root r x*
shows $x^T \leq x; 1; x$
by (*metis assms conv-contrav conv-invol conv-one inf.boundedI maddux-20*
maddux-21 vector-meet-comp-x
non-empty-cycle-root-no-start-end-points)

lemma *non-empty-cycle-root-start-end-points-plus*:
assumes *non-empty-cycle-root r x*
shows $x; 1; x \leq x^+$
using *assms order.eq-iff one-strongly-connected-iff*
one-strongly-connected-implies-7-eq
backward-finite-path-connected non-empty-cycle-root-msc **by** *blast*

lemma *non-empty-cycle-root-converse-plus*:
assumes *non-empty-cycle-root r x*
shows $x^T \leq x^+$
using *assms many-strongly-connected-iff-2 non-empty-cycle-root-msc* **by** *blast*

lemma *non-empty-cycle-root-plus-converse*:
assumes *non-empty-cycle-root* r x
shows $x^+ = x^{T+}$
using *assms many-strongly-connected-iff-7 non-empty-cycle-root-msc* **by** *fastforce*

lemma *non-empty-cycle-root-converse*:
assumes *non-empty-cycle-root* r x
shows *non-empty-cycle-root* r (x^T)
by (*metis assms conv-invol inj-p-fun connected-root-iff3*
backward-finite-path-root-def
non-empty-cycle-root-msc-plus non-empty-cycle-root-tc-start-points)

lemma *non-empty-cycle-root-move-root-forward*:
assumes *non-empty-cycle-root* r x
and *point* q
and $r \leq x^*;q$
shows *non-empty-cycle-root* q x
by (*metis assms backward-finite-path-root-move-root*
non-empty-cycle-root-no-start-end-points
non-empty-cycle-root-point-exchange non-empty-cycle-root-rtc-start-points)

lemma *non-empty-cycle-root-move-root-forward-cycle*:
assumes *non-empty-cycle-root* r x
and *point* q
and $r \leq x^*;q$
shows $x;q \neq 0 \wedge x^T;q \neq 0$
by (*metis assms comp-assoc independence1 ss-p18*
non-empty-cycle-root-move-root-forward
non-empty-cycle-root-msc-plus non-empty-cycle-root-non-empty
non-empty-cycle-root-tc-start-points)

lemma *non-empty-cycle-root-equivalences*:
assumes *non-empty-cycle-root* r x
and *point* q
shows $(r \leq x^*;q \longleftrightarrow q \leq x^*;r)$
and $(r \leq x^*;q \longleftrightarrow x;q \neq 0)$
and $(r \leq x^*;q \longleftrightarrow x^T;q \neq 0)$
and $(r \leq x^*;q \longleftrightarrow q \leq x;1)$
and $(r \leq x^*;q \longleftrightarrow q \leq x^T;1)$
using *assms cycle-iff no-end-point-char non-empty-cycle-root-no-start-end-points*
non-empty-cycle-root-point-exchange non-empty-cycle-root-rtc-start-points
by *metis+*

lemma *non-empty-cycle-root-chord*:
assumes *non-empty-cycle-root* r x
and *point* p
and *point* q
and $r \leq x^*;p$
and $r \leq x^*;q$

shows $p \leq x^*;q$
using *assms non-empty-cycle-root-move-root-same-reachable*
non-empty-cycle-root-point-exchange
by *fastforce*

lemma *non-empty-cycle-root-var-axioms-2:*
 $non\text{-}empty\text{-}cycle\text{-}root\ r\ x \longleftrightarrow x;1 \leq x^*;r \wedge is\text{-}inj\ x \wedge is\text{-}p\text{-}fun\ x \wedge point\ r \wedge r \leq x;1$
apply (*rule iffI*)
apply (*metis order.eq-iff backward-finite-path-root-def non-empty-cycle-root-no-start-end-points non-empty-cycle-root-tc-start-points*)
by (*metis conv-invol p-fun-inj connected-root-iff2 connected-root-iff3 non-empty-cycle-root-var-axioms-1 non-empty-cycle-root-msc-plus non-empty-cycle-root-rtc-start-points non-empty-cycle-root-rtc-tc*)

lemma *non-empty-cycle-root-var-axioms-3:*
 $non\text{-}empty\text{-}cycle\text{-}root\ r\ x \longleftrightarrow x;1 \leq x^*;r \wedge is\text{-}inj\ x \wedge is\text{-}p\text{-}fun\ x \wedge point\ r \wedge r \leq x^*;x;1$
apply (*rule iffI*)
apply (*metis comp-assoc eq-refl backward-finite-path-root-def star-inductl-var-eq2 non-empty-cycle-root-no-start-end-points non-empty-cycle-root-rtc-start-points non-empty-cycle-root-tc-start-points*)
by (*metis annir comp-assoc conv-contrav no-end-point-char non-empty-cycle-root-var-axioms-2*)

lemma *non-empty-cycle-root-subset-equals:*
assumes *non-empty-cycle-root r x*
and *non-empty-cycle-root r y*
and $x \leq y$
shows $x = y$
proof –
have $y;x^{T^*};r = y;x^{T^+};r$
using *assms(1) comp-assoc non-empty-cycle-root-msc non-empty-cycle-root-msc-plus non-empty-cycle-root-rtc-tc* **by** *fastforce*
also have $\dots \leq y;y^T;x^{T^*};r$
using *assms(3) comp-assoc conv-iso mult-double-iso* **by** *fastforce*
also have $\dots \leq x^{T^*};r$
using *assms(2) backward-finite-path-root-def is-inj-def*
by (*meson dual-order.trans mult-isor order.refl prod-star-closure star-ref*)
finally have $r + y;x^{T^*};r \leq x^{T^*};r$
by (*metis conway.dagger-unfoldl-distr le-supI sup.cobounded1*)
hence $y^*;r \leq x^{T^*};r$
by (*simp add: comp-assoc rtc-inductl*)
hence $y;1 \leq x;1$
using *assms(1,2) non-empty-cycle-root-msc non-empty-cycle-root-rtc-start-points* **by** *fastforce*

thus *?thesis*
using *assms(2,3) backward-finite-path-root-def ss422iv* **by** *blast*
qed

lemma *non-empty-cycle-root-subset-equals-change-root:*

assumes *non-empty-cycle-root r x*
and *non-empty-cycle-root q y*
and $x \leq y$
shows $x = y$
proof –
have $r \leq y;1$
by (*metis assms(1,3) dual-order.trans mult-isor*
non-empty-cycle-root-no-start-end-points)
hence *non-empty-cycle-root r y*
by (*metis assms(1,2) connected-root-move-root backward-finite-path-root-def*
non-empty-cycle-root-no-start-end-points
non-empty-cycle-root-rtc-start-points)
thus *?thesis*
using *assms(1,3) non-empty-cycle-root-subset-equals* **by** *blast*
qed

lemma *non-empty-cycle-root-equivalences-2:*

assumes *non-empty-cycle-root r x*
shows $(v \leq x^*;r \longleftrightarrow v \leq x^T;1)$
and $(v \leq x^*;r \longleftrightarrow v \leq x;1)$
using *assms non-empty-cycle-root-no-start-end-points*
non-empty-cycle-root-rtc-start-points
by *metis+*

lemma *cycle-root-non-empty:*

assumes $x \neq 0$
shows $\text{cycle-root } r \ x \longleftrightarrow \text{non-empty-cycle-root } r \ x$
proof
assume *1: cycle-root r x*
have $r \leq r;1;x;1$
using *assms comp-assoc maddux-20 tarski* **by** *fastforce*
also have $\dots \leq (x^+ \cdot x^T;1);1$
using *1* **by** (*simp add: is-vector-def mult-isor point-def*)
also have $\dots \leq x^T;1$
by (*simp add: ra-1*)
finally show *non-empty-cycle-root r x*
using *1 backward-finite-path-root-def inf.boundedE* **by** *blast*
next
assume *non-empty-cycle-root r x*
thus *cycle-root r x*
by (*metis backward-finite-path-root-def inf.orderE maddux-20*
non-empty-cycle-root-plus-converse
ra-1)
qed

Start points and end points

lemma *start-points-path-aux*:

assumes *backward-finite-path-root* $r\ x$

and *start-points* $x \neq 0$

shows $x;r = 0$

by (*metis* *assms* *compl-inf-bot* *inf commute*
non-empty-cycle-root-no-start-end-points
backward-finite-path-root-terminating-or-cycle)

lemma *start-points-path*:

assumes *backward-finite-path-root* $r\ x$

and *start-points* $x \neq 0$

shows *backward-terminating-path-root* $r\ x$

by (*simp* *add: assms* *start-points-path-aux*)

lemma *root-in-start-points-2*:

assumes *backward-finite-path-root* $r\ x$

and *start-points* $x \neq 0$

shows $r \leq \text{start-points } x$

by (*metis* *assms* *conv-zero* *eq-refl* *galois-aux2* *root-equals-start-points*
start-points-path-aux)

lemma *root-equals-start-points-2*:

assumes *backward-finite-path-root* $r\ x$

and *start-points* $x \neq 0$

shows $r = \text{start-points } x$

by (*metis* *assms* *inf-bot-left* *ss-p18* *root-equals-start-points* *start-points-path*)

lemma *start-points-injective*:

assumes *backward-finite-path-root* $r\ x$

shows *is-inj* (*start-points* x)

by (*metis* *assms* *compl-bot-eq* *inj-def-var1* *point-def* *backward-finite-path-root-def*
top-greatest
root-equals-start-points-2)

lemma *backward-terminating-path-root-aux-2*:

assumes *backward-finite-path-root* $r\ x$

and *start-points* $x \neq 0 \vee x = 0$

shows $x \leq x^{T*}; -(x^T; 1)$

using *assms* *bot-least* *backward-terminating-path-root-aux* *start-points-path* **by**
blast

lemma *start-points-not-zero-iff*:

assumes *backward-finite-path-root* $r\ x$

shows $x;r = 0 \wedge x \neq 0 \iff \text{start-points } x \neq 0$

by (*metis* *assms* *conv-zero* *inf-compl-bot* *backward-finite-path-root-def*
start-points-not-zero-contra
start-points-path-aux)

Backwards terminating and backwards finite: Part II

lemma *backward-finite-path-root-acyclic-terminating-aux*:
assumes *backward-finite-path-root* $r\ x$
and *is-acyclic* x
shows $x;r = 0$
proof (*cases* $x = 0$)
assume $x = 0$
thus *?thesis*
by *simp*
next
assume $x \neq 0$
hence $1: r \leq x;1$
using *assms(1) has-root-contr no-end-point-char backward-finite-path-root-def*
by *blast*
have $r.(x^T;1) = r.(x^{T+};r)$
using *assms(1) connected-root-iff3 backward-finite-path-root-def* **by** *fastforce*
also have $\dots \leq r.(-1';r)$
by (*metis assms(2) conv-compl conv-contrav conv-e conv-iso meet-isor*
mult-isor star-conv
star-slide-var)
also have $\dots = 0$
by (*metis (no-types) assms(1) inj-distr annil inf-compl-bot mult-1-left point-def*
backward-finite-path-root-def)
finally have $r \leq \text{start-points } x$
using 1 *galois-aux inf.boundedI le-bot* **by** *blast*
thus *?thesis*
using *assms(1) annir le-bot start-points-path* **by** *blast*
qed

lemma *backward-finite-path-root-acyclic-terminating-iff*:
assumes *backward-finite-path-root* $r\ x$
shows *is-acyclic* $x \longleftrightarrow x;r = 0$
apply (*rule iffI*)
apply (*simp add: assms backward-finite-path-root-acyclic-terminating-aux*)
using *assms backward-finite-path-root-path-root path-root-acyclic* **by** *blast*

lemma *backward-finite-path-root-acyclic-terminating*:
assumes *backward-finite-path-root* $r\ x$
and *is-acyclic* x
shows *backward-terminating-path-root* $r\ x$
by (*simp add: assms backward-finite-path-root-acyclic-terminating-aux*)

lemma *non-empty-cycle-root-one-strongly-connected*:
assumes *non-empty-cycle-root* $r\ x$
shows *one-strongly-connected* x
by (*metis assms one-strongly-connected-iff order-trans star-1l star-star-plus*
sup.absorb2
non-empty-cycle-root-msc non-empty-cycle-root-start-end-points-plus)

lemma *backward-finite-path-root-nodes-reachable*:
assumes *backward-finite-path-root* $r\ x$
and $v \leq x; 1 + x^T; 1$
and *is-sur* v
shows $r \leq x^*; v$
proof –
have $v \leq x; 1 + x^{T+}; r$
using *assms connected-root-iff3 backward-finite-path-root-def* **by** *fastforce*
also have $\dots \leq x^{T*}; r + x^{T+}; r$
using *assms(1) join-iso start-points-in-root-aux* **by** *blast*
also have $\dots = x^{T*}; r$
using *mult-isor sup.absorb1* **by** *fastforce*
finally show *?thesis*
using *assms(1,3)*
by (*simp add: inj-sur-semi-swap point-def backward-finite-path-root-def*
star-conv *inj-sur-semi-swap-short*)
qed

lemma *terminating-path-root-end-backward-terminating*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *backward-terminating-path-root* $r\ x$
using *assms non-empty-cycle-root-move-root-forward-cycle*
backward-finite-path-root-terminating-or-cycle **by** *blast*

lemma *terminating-path-root-end-converse*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *terminating-path-root-end* $e\ (x^T)\ r$
by (*metis assms terminating-path-root-end-backward-terminating*
backward-finite-path-root-def
conv-invol terminating-path-root-end-forward-finite point-swap star-conv)

lemma *terminating-path-root-end-forward-terminating*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *backward-terminating-path-root* $e\ (x^T)$
using *assms terminating-path-root-end-converse* **by** *blast*

end

3.3 Consequences with the Tarski rule and the point axiom

context *relation-algebra-rtc-tarski-point*
begin

Rooted paths

lemma *path-root-iff*:
 $(\exists r . \text{path-root } r\ x) \longleftrightarrow \text{path } x$

proof
assume $\exists r . \text{path-root } r\ x$
thus *path* x

using *path-def path-iff-backward point-def path-root-def* **by blast**
next
 assume 1: *path x*
 show $\exists r . \text{path-root } r \ x$
proof (*cases x = 0*)
 assume $x = 0$
 thus *?thesis*
 by (*simp add: is-inj-def is-p-fun-def point-exists path-root-def*)
next
 assume $\neg(x = 0)$
 hence $x;1 \neq 0$
 by (*simp add: ss-p18*)
from this obtain r where 2: point r \wedge r \leq x;1
 using *comp-assoc is-vector-def one-idem-mult point-below-vector* **by fastforce**
 hence $r;x \leq x;1;x$
 by (*simp add: mult-isor*)
 also have $\dots \leq x^* + x^{T*}$
 using 1 *path-def* **by blast**
finally show ?thesis
 using 1 2 *path-def path-root-def* **by blast**
qed
qed

lemma non-empty-path-root-iff:
 $(\exists r . \text{path-root } r \ x \wedge r \leq (x + x^T);1) \longleftrightarrow \text{path } x \wedge x \neq 0$
apply (*rule iffI*)
 using *non-empty-cycle-root-non-empty path-root-def*
 zero-backward-terminating-path-root path-root-iff
 apply fastforce
using path-root-iff non-empty-path-root-iff-aux **by blast**

(Non-empty) Cycle

lemma non-empty-cycle-root-iff:
 $(\exists r . \text{non-empty-cycle-root } r \ x) \longleftrightarrow \text{cycle } x \wedge x \neq 0$
proof
 assume $\exists r . \text{non-empty-cycle-root } r \ x$
 thus $\text{cycle } x \wedge x \neq 0$
 using *non-empty-cycle-root-msc-cycle non-empty-cycle-root-non-empty* **by fastforce**
next
 assume 1: $\text{cycle } x \wedge x \neq 0$
 hence $x^T;1 \neq 0$
 using *many-strongly-connected-implies-no-start-end-points ss-p18* **by blast**
from this obtain r where 2: point r \wedge r \leq x^T;1
 using *comp-assoc is-vector-def one-idem-mult point-below-vector* **by fastforce**
 have 3: $x^T;1;x^T \leq x^*$
 using 1 *one-strongly-connected-iff path-def* **by blast**
 have $r;x \leq x^T;1;x$
 using 2 **by** (*simp add: is-vector-def mult-isor point-def*)

also have $\dots \leq x^T; 1; x; x^T; x$
using *comp-assoc mult-isol x-leq-triple-x* **by** *fastforce*
also have $\dots \leq x^T; 1; x^T; x$
by (*metis mult-assoc mult-double-iso top-greatest*)
also have $\dots \leq x^*; x$
using *3 mult-isol* **by** *blast*
finally have *connected-root r x*
by (*simp add: star-slide-var*)
hence *non-empty-cycle-root r x*
using *1 2 path-def backward-finite-path-root-def* **by** *fastforce*
thus $\exists r . \text{non-empty-cycle-root } r \ x \ ..$
qed

lemma *non-empty-cycle-subset-equals*:
assumes *cycle x*
and *cycle y*
and $x \leq y$
and $x \neq 0$
shows $x = y$
by (*metis assms le-bot non-empty-cycle-root-subset-equals-change-root non-empty-cycle-root-iff*)

lemma *cycle-root-iff*:
 $(\exists r . \text{cycle-root } r \ x) \longleftrightarrow \text{cycle } x$
proof (*cases x = 0*)
assume $x = 0$
thus *?thesis*
using *path-def point-exists* **by** *fastforce*
next
assume $x \neq 0$
thus *?thesis*
using *cycle-root-non-empty non-empty-cycle-root-iff* **by** *simp*
qed

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-iff*:
 $(\exists r . \text{backward-terminating-path-root } r \ x) \longleftrightarrow \text{backward-terminating-path } x$
proof
assume $\exists r . \text{backward-terminating-path-root } r \ x$
thus *backward-terminating-path x*
using *backward-terminating-path-root* **by** *fastforce*
next
assume *1: backward-terminating-path x*
show $\exists r . \text{backward-terminating-path-root } r \ x$
proof (*cases x = 0*)
assume $x = 0$
thus *?thesis*
using *point-exists zero-backward-terminating-path-root* **by** *blast*
next

let $?r = \text{start-points } x$
assume $x \neq 0$
hence $2: \text{is-point } ?r$
using $1 \text{ start-point-iff2 backward-terminating-iff1}$ **by** *fastforce*
have $3: x; ?r = 0$
by *(metis inf-top.right-neutral modular-1-aux')*
have $x; 1; x \leq x; 1; x; x^T; x$
using *comp-assoc mult-isol x-leq-triple-x* **by** *fastforce*
also have $\dots \leq (x^* + x^{T*}); x^T; x$
using $1 \text{ mult-isol path-def}$ **by** *blast*
also have $\dots = (1' + x^+ + x^{T+}); x^T; x$
by *(metis star-star-plus star-unfoldl-eq sup commute)*
also have $\dots = x^T; x + x^+; x^T; x + x^{T+}; x^T; x$
by *(metis distrib-right' mult-onel)*
also have $\dots = x^T; (x + x^{T*}; x^T; x) + x^+; x^T; x$
using *comp-assoc distrib-left sup commute sup.assoc* **by** *simp*
also have $\dots \leq x^T; 1 + x^+; x^T; x$
using *join-iso mult-isol* **by** *fastforce*
also have $\dots \leq x^T; 1 + x^+; 1'$
using 1 **by** *(metis comp-assoc join-isol mult-isol path-def is-p-fun-def)*
finally have $-(x^T; 1) \cdot x; 1; x \leq x^+$
by *(simp add: galois-1 inf commute)*
hence $?r; x \leq x^+$
by *(metis inf-commute one-compl ra-1)*
hence *backward-terminating-path-root ?r x*
using $1 \ 2 \ 3$ **by** *(simp add: point-is-point backward-finite-path-root-def path-def)*
thus *?thesis ..*
qed
qed

lemma *non-empty-backward-terminating-path-root-iff*:
 $\text{backward-terminating-path-root (start-points } x) \ x \longleftrightarrow$
 $\text{backward-terminating-path } x \wedge x \neq 0$
apply *(rule iffI)*
apply *(metis backward-finite-path-root-path backward-terminating-path-root-2 conv-zero inf.cobounded1 non-empty-cycle-root-non-empty)*
using *backward-terminating-path-root-iff root-equals-start-points* **by** *blast*

lemma *non-empty-backward-terminating-path-root-iff'*:
 $\text{backward-finite-path-root (start-points } x) \ x \longleftrightarrow \text{backward-terminating-path } x \wedge x$
 $\neq 0$
using *start-point-no-predecessor non-empty-backward-terminating-path-root-iff* **by** *simp*

lemma *backward-finite-path-root-iff*:
 $(\exists r \ . \ \text{backward-finite-path-root } r \ x) \longleftrightarrow \text{backward-finite-path } x$
proof

```

assume  $\exists r . \text{backward-finite-path-root } r \ x$ 
thus  $\text{backward-finite-path } x$ 
  by (meson backward-finite-iff-msc non-empty-cycle-root-msc
backward-finite-path-root-path
      backward-finite-path-root-terminating-or-cycle
backward-terminating-path-root)
next
  assume  $\text{backward-finite-path } x$ 
  thus  $\exists r . \text{backward-finite-path-root } r \ x$ 
    by (metis backward-finite-iff-msc point-exists non-empty-cycle-root-iff
        zero-backward-terminating-path-root backward-terminating-path-root-iff)
qed

```

```

lemma non-empty-backward-finite-path-root-iff:
   $(\exists r . \text{backward-finite-path-root } r \ x \wedge r \leq x;1) \longleftrightarrow \text{backward-finite-path } x \wedge x \neq 0$ 
apply (rule iffI)
apply (metis backward-finite-path-root-iff annir backward-finite-path-root-def
le-bot
        no-end-point-char ss-p18)
using backward-finite-path-root-iff backward-finite-path-root-def point-def
root-in-edge-sources by blast

```

Terminating

```

lemma terminating-path-root-end-aux:
  assumes  $\text{terminating-path } x$ 
  shows  $\exists r \ e . \text{terminating-path-root-end } r \ x \ e$ 
proof (cases  $x = 0$ )
  assume  $x = 0$ 
  thus ?thesis
    using point-exists zero-backward-terminating-path-root by fastforce
next
  assume  $1: \neg(x = 0)$ 
  have  $2: \text{backward-terminating-path } x$ 
    using assms by simp
  from this obtain  $r$  where  $3: \text{backward-terminating-path-root } r \ x$ 
    using backward-terminating-path-root-iff by blast
  have  $\text{backward-terminating-path } (x^T)$ 
    using  $2$  by (metis assms backward-terminating-iff1
conv-backward-terminating-path conv-invol
        conv-zero inf-top.left-neutral)
  from this obtain  $e$  where  $4: \text{backward-terminating-path-root } e \ (x^T)$ 
    using backward-terminating-path-root-iff by blast
  have  $r \leq x;1$ 
    using  $1 \ 3$  root-in-edge-sources backward-finite-path-root-def point-def by
fastforce
  also have  $\dots = x^+;e$ 
    using  $4$  connected-root-iff3 backward-finite-path-root-def by fastforce
  also have  $\dots \leq x^*;e$ 

```

by (*simp add: mult-isor*)
 finally show *?thesis*
 using 3 4 *backward-finite-path-root-def* by *blast*
 qed

lemma *terminating-path-root-end-iff:*

$(\exists r e . \text{terminating-path-root-end } r \ x \ e) \longleftrightarrow \text{terminating-path } x$

proof

assume 1: $\exists r e . \text{terminating-path-root-end } r \ x \ e$

show *terminating-path* x

proof (*cases* $x = 0$)

 assume $x = 0$

 thus *?thesis*

 by (*simp add: is-inj-def is-p-fun-def path-def*)

next

 assume $\neg(x = 0)$

 hence 2: \neg *many-strongly-connected* x

 using 1 *cycle-root-end-empty* by *blast*

 hence 3: *backward-terminating-path* x

 using 1 *backward-terminating-path-root*

terminating-path-root-end-backward-terminating by *blast*

 have $\exists e . \text{backward-finite-path-root } e \ (x^T)$

 using 1 *terminating-path-root-end-converse* by *blast*

 hence *backward-terminating-path* (x^T)

 using 1 *backward-terminating-path-root* *terminating-path-root-end-converse*

by *blast*

 hence *forward-terminating-path* x

 by (*simp add: conv-backward-terminating-path*)

 thus *?thesis*

 using 3 by (*simp add: inf.boundedI*)

qed

next

 assume *terminating-path* x

 thus $\exists r e . \text{terminating-path-root-end } r \ x \ e$

 using *terminating-path-root-end-aux* by *blast*

qed

lemma *non-empty-terminating-path-root-end-iff:*

$\text{terminating-path-root-end } (\text{start-points } x) \ x \ (\text{end-points } x) \longleftrightarrow \text{terminating-path } x \wedge x \neq 0$

apply (*rule iffI*)

apply (*metis conv-zero non-empty-backward-terminating-path-root-iff*

terminating-path-root-end-iff)

using *terminating-path-root-end-iff* *terminating-path-root-end-forward-terminating*

root-equals-end-points *terminating-path-root-end-backward-terminating*

root-equals-start-points

by *blast*

lemma *non-empty-finite-path-root-end-iff:*

```

    finite-path-root-end (start-points  $x$ )  $x$  (end-points  $x$ )  $\longleftrightarrow$  terminating-path  $x \wedge x$ 
 $\neq 0$ 
using non-empty-terminating-path-root-end-iff end-point-no-successor by simp

end

end

```

4 Correctness of Path Algorithms

To show that our theory of paths integrates with verification tasks, we verify the correctness of three basic path algorithms. Algorithms at the presented level are executable and can serve prototyping purposes. Data refinement can be carried out to move from such algorithms to more efficient programs. The total-correctness proofs use a library developed in [7].

```
theory Path-Algorithms
```

```
imports HOL-Hoare.Hoare-Logic Rooted-Paths
```

```
begin
```

```
no-notation
```

```
  trancl (( $-^+$ ) [1000] 999)
```

```
class choose-singleton-point-signature =
```

```
  fixes choose-singleton :: 'a  $\Rightarrow$  'a
```

```
  fixes choose-point :: 'a  $\Rightarrow$  'a
```

```
class relation-algebra-rtc-tarski-choose-point =
```

```
  relation-algebra-rtc-tarski + choose-singleton-point-signature +
```

```
  assumes choose-singleton-singleton:  $x \neq 0 \implies \text{singleton } (\text{choose-singleton } x)$ 
```

```
  assumes choose-singleton-decreasing:  $\text{choose-singleton } x \leq x$ 
```

```
  assumes choose-point-point:  $\text{is-vector } x \implies x \neq 0 \implies \text{point } (\text{choose-point } x)$ 
```

```
  assumes choose-point-decreasing:  $\text{choose-point } x \leq x$ 
```

```
begin
```

```
no-notation
```

```
  composition (infixl ; 75) and
```

```
  times (infixl * 70)
```

```
notation
```

```
  composition (infixl * 75)
```

4.1 Construction of a path

Our first example is a basic greedy algorithm that constructs a path from a vertex x to a different vertex y of a directed acyclic graph D .

abbreviation *construct-path-inv* $q\ x\ y\ D\ W \equiv$
 $is\text{-}acyclic\ D \wedge point\ x \wedge point\ y \wedge point\ q \wedge$
 $D^* * q \leq D^{T^*} * x \wedge W \leq D \wedge terminating\text{-}path\ W \wedge$
 $(W = 0 \longleftrightarrow q=y) \wedge (W \neq 0 \longleftrightarrow q = start\text{-}points\ W \wedge y = end\text{-}points\ W)$

abbreviation *construct-path-inv-simp* $q\ x\ y\ D\ W \equiv$
 $is\text{-}acyclic\ D \wedge point\ x \wedge point\ y \wedge point\ q \wedge$
 $D^* * q \leq D^{T^*} * x \wedge W \leq D \wedge terminating\text{-}path\ W \wedge$
 $q = start\text{-}points\ W \wedge y = end\text{-}points\ W$

lemma *construct-path-pre*:
assumes *is-acyclic* D
and *point* y
and *point* x
and $D^* * y \leq D^{T^*} * x$
shows *construct-path-inv* $y\ x\ y\ D\ 0$
apply (*intro conjI, simp-all add: assms is-inj-def is-p-fun-def path-def*)
using *assms(2) cycle-iff* **by** *fastforce*

The following three lemmas are auxiliary lemmas for *construct-path-inv*.
They are pulled out of the main proof to have more structure.

lemma *path-inv-points*:
assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$
shows *point* q
and *point* (*choose-point* $(D*q)$)
using *assms* **apply** *blast*
by (*metis assms choose-point-point comp-assoc is-vector-def point-def*
reachable-implies-predecessor)

lemma *path-inv-choose-point-decrease*:
assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$
shows $W \neq 0 \implies choose\text{-}point\ (D*q) \leq -((W + choose\text{-}point\ (D*q) * q^T)^T * 1)$
proof –
let $?q = choose\text{-}point\ (D*q)$
let $?W = W + ?q * q^T$
assume *as*: $W \neq 0$
hence $q * W \leq W^+$
by (*metis assms conv-contrav conv-invol conv-iso conv-terminating-path*
forward-terminating-path-end-points-1 plus-conv point-def ss423bij
terminating-path-iff)
hence $?q \cdot W^T * 1 \leq D * q \cdot W^{T+} * q$
using *choose-point-decreasing meet-iso meet-isor inf-mono assms*
connected-root-iff2 **by** *simp*
also have $\dots \leq (D \cdot D^{T+}) * q$
by (*metis assms inj-distr point-def conv-contrav conv-invol conv-iso meet-isor*
mult-isol-var mult-isor star-conv star-slide-var star-subdist
sup commute sup.orderE)
also have $\dots \leq 0$

by (*metis acyclic-trans assms conv-zero step-has-target order.eq-iff galois-aux ss-p18*)
finally have $a: ?q \leq -(W^T * 1)$
using *galois-aux le-bot* **by** *blast*

have *point ?q*
using *assms* **by** (*rule path-inv-points(2)*)
hence $?q \leq -(q * ?q^T * 1)$
by (*metis assms acyclic-imp-one-step-different-points(2) point-is-point choose-point-decreasing edge-end end-point-char end-point-no-successor*)
with *a* **show** *?thesis*
by (*simp add: inf.boundedI*)
qed

lemma *end-points*:
assumes *construct-path-inv q x y D W \wedge q \neq x*
shows *choose-point (D*q) = start-points (W + choose-point (D*q) * q^T)*
and *y = end-points (W + choose-point (D*q) * q^T)*
proof –
let $?q = \text{choose-point } (D * q)$
let $?W = W + ?q * q^T$
show *1: ?q = start-points ?W*
proof (*rule order.antisym*)
show *start-points ?W \leq ?q*
by (*metis assms(1) path-inv-points(2) acyclic-imp-one-step-different-points(2) choose-point-decreasing edge-end edge-start sup commute path-concatenation-start-points-approx point-is-point order.eq-iff sup-bot-left*)
show $?q \leq \text{start-points } ?W$
proof –
have $a: ?q = ?q * q^T * 1$
by (*metis assms(1) comp-assoc point-equations(1) point-is-point aux4 conv-zero choose-point-decreasing choose-point-point conv-contrav conv-one point-def inf.orderE inf-compl-bot inf-compl-bot-right is-vector-def maddux-142 sup-bot-left sur-def-var1*)
hence $?q = (q \cdot -q) + (?q \cdot -q \cdot -(?W^T * 1))$
by (*metis assms path-inv-points(2) path-inv-choose-point-decrease acyclic-imp-one-step-different-points(1) choose-point-decreasing inf.orderE inf-compl-bot sup-inf-absorb edge-start point-is-point sup-bot-left*)
also have $\dots \leq (W * 1 \cdot -(?W^T * 1) \cdot -q) + (?q \cdot -q \cdot -(?W^T * 1))$
by *simp*
also have $\dots = (W * 1 + ?q) \cdot -(q + ?W^T * 1)$
by (*metis compl-sup inf-sup-distrib2 meet-assoc sup commute*)
also have $\dots \leq ?W * 1 \cdot -(?W^T * 1)$
using *a* **by** (*metis inf.left-commute distrib-right' compl-sup inf.cobounded2*)


```

    finally show  $?q \leq \text{start-points } ?W$  .
  qed
qed
show  $y = \text{end-points } ?W$ 
proof -
  have point-nq: point ?q
    using assms by(rule path-inv-points(2))
  hence yp:  $y \leq -?q$ 
    using 1 assms
    by (metis acyclic-imp-one-step-different-points(2) choose-point-decreasing
cycle-no-points(1)
      finite-iff finite-iff-msc forward-finite-iff-msc path-aux1a
path-edge-equals-cycle
      point-is-point point-not-equal(1) terminating-iff1)
  have  $y = y + (W*1 \cdot -(W^T*1) \cdot -(W*1))$ 
    by (simp add: inf commute)
  also have  $\dots = y + (q \cdot -(W*1))$ 
    using assms by fastforce
  also have  $\dots = y + (q \cdot -(W*1) \cdot -?q)$ 
    by (metis calculation sup-assoc sup-inf-absorb)
  also have  $\dots = (y \cdot -?q) + (q \cdot -(W*1) \cdot -?q)$ 
    using yp by (simp add: inf.absorb1)
  also have  $\dots = (W^T*1 \cdot -(W*1) \cdot -?q) + (q \cdot -(W*1) \cdot -?q)$ 
    using assms by fastforce
  also have  $\dots = (W^T*1 + q) \cdot -(W*1) \cdot -?q$ 
    by (simp add: inf-sup-distrib2)
  also have  $\dots = (W^T*1 + q) \cdot -(W*1 + ?q)$ 
    by (simp add: inf.assoc)
  also have  $\dots = (W^T*1 + q*?q^T*1) \cdot -(W*1 + ?q*?q^T*1)$ 
    using point-nq
    by(metis assms(1) comp-assoc conv-contrav conv-one is-vector-def point-def
sur-def-var1)
  also have  $\dots = (?W^T)*1 \cdot -(?W*1)$ 
    by simp
  finally show ?thesis .
qed
qed

```

```

lemma construct-path-inv:
  assumes construct-path-inv  $q\ x\ y\ D\ W \wedge q \neq x$ 
  shows construct-path-inv (choose-point  $(D*q)$ )  $x\ y\ D\ (W + \text{choose-point}$ 
 $(D*q)*q^T)$ 
proof (intro conjI)
  let  $?q = \text{choose-point } (D*q)$ 
  let  $?W = W + ?q * q^T$ 
  show is-acyclic  $D$ 
    using assms by blast
  show point-y: point  $y$ 
    using assms by blast

```

```

show point x
  using assms by blast
show ?W ≤ D
  using assms choose-point-decreasing le-sup-iff point-def ss423bij inf.boundedE
by blast
show  $D^* * ?q \leq D^{T^*} * x$ 
proof -
  have  $D^+ * q \leq D^{T^*} * x$ 
  using assms conv-galois-2 order-trans star-1l by blast
  thus ?thesis
  by (metis choose-point-decreasing comp-assoc dual-order.trans mult-isol
star-slide-var)
qed
show point-nq: point ?q
  using assms by(rule path-inv-points(2))
show pathW: path ?W
proof(cases W=0)
  assume W=0
  thus ?thesis
  using assms edge-is-path point-is-point point-nq by simp
next
  assume a:  $W \neq 0$ 
  have b:  $?q * q^T \leq 1 * ?q * q^T * - (?q * q^T * 1)$ 
  proof -
    have  $?q * q^T \leq 1$  by simp
    thus ?thesis
    using assms point-nq
    by(metis different-points-consequences(1) point-def sur-def-var1
acyclic-imp-one-step-different-points(2) choose-point-decreasing
comp-assoc
is-vector-def point-def point-equations(3,4) point-is-point)
  qed
  have c:  $W \leq -(1 * W) * W * 1$ 
  using assms terminating-path-iff by blast
  have d:  $(?q * q^T)^T * 1 \cdot -((?q * q^T) * 1) = W * 1 \cdot -(W^T * 1)$ 
  using a
  by (metis assms path-inv-points(2) acyclic-reachable-points
choose-point-decreasing
edge-end point-is-point comp-assoc point-def sur-total total-one)
  have e:  $?q * q^T * 1 \cdot W^T * 1 = 0$ 
  proof -
    have  $?q * q^T * 1 \cdot W^T * 1 = ?q \cdot W^T * 1$ 
    using assms point-nq
    by (metis comp-assoc conv-contrav conv-one is-vector-def point-def
sur-def-var1)
    also have  $\dots \leq -(?W^T * 1) \cdot ?W^T * 1$ 
    using assms path-inv-choose-point-decrease
    by (smt a conv-contrav conv-iso conv-one inf-mono less-eq-def subdistl-eq)
    also have  $\dots \leq 0$ 

```

```

    using compl-inf-bot eq-refl by blast
    finally show ?thesis
    using bot-unique by blast
qed
show ?thesis
using b c d e by (metis assms comp-assoc edge-is-path
path-concatenation-cycle-free
point-is-point sup.commute point-nq)

qed
show ?W = 0  $\longleftrightarrow$  ?q = y
  apply (rule iffI)
  apply (metis assms conv-zero dist-alt edge-start inf-compl-bot-right
modular-1-aux' modular-2-aux'
point-is-point sup.left-idem sup-bot-left point-nq)
  by (smt assms end-points(1) conv-contrav conv-invol cycle-no-points(1)
end-point-iff2 has-start-end-points-iff path-aux1b path-edge-equals-cycle
point-is-point start-point-iff2 sup-bot-left top-greatest pathW)
show ?W  $\neq$  0  $\longleftrightarrow$  ?q = start-points ?W  $\wedge$  y = end-points ?W
  apply (rule iffI)
  using assms end-points apply blast
  using assms by force
show terminating ?W
  by (smt assms end-points end-point-iff2 has-start-end-points-iff point-is-point
start-point-iff2
terminating-iff1 pathW point-nq)
qed

theorem construct-path-partial: VARS p q W
{ is-acyclic D  $\wedge$  point y  $\wedge$  point x  $\wedge$  D**y  $\leq$  DT*x }
W := 0;
q := y;
WHILE q  $\neq$  x
  INV { construct-path-inv q x y D W }
  DO p := choose-point (D*q);
  W := W + p*qT;
  q := p
OD
{ W  $\leq$  D  $\wedge$  terminating-path W  $\wedge$  (W=0  $\longleftrightarrow$  x=y)  $\wedge$  (W $\neq$ 0  $\longleftrightarrow$  x =
start-points W  $\wedge$  y = end-points W) }
  apply vcg
  using construct-path-pre apply blast
  using construct-path-inv apply blast
  by fastforce
end


```

For termination, we additionally need finiteness.

```

context finite
begin

```

```

lemma decrease-set:
  assumes  $\forall x::'a . Q\ x \longrightarrow P\ x$ 
    and  $P\ w$ 
    and  $\neg Q\ w$ 
  shows  $\text{card } \{ x . Q\ x \} < \text{card } \{ x . P\ x \}$ 
by (metis Collect-mono assms card-seteq finite mem-Collect-eq not-le)

```

end

```

class relation-algebra-rtc-tarski-choose-point-finite =
  relation-algebra-rtc-tarski-choose-point +
  relation-algebra-rtc-tarski-point-finite
begin

```

```

lemma decrease-variant:
  assumes  $y \leq z$ 
    and  $w \leq z$ 
    and  $\neg w \leq y$ 
  shows  $\text{card } \{ x . x \leq y \} < \text{card } \{ x . x \leq z \}$ 
by (metis Collect-mono assms card-seteq linorder-not-le dual-order.trans
  finite-code mem-Collect-eq)

```

```

lemma construct-path-inv-termination:
  assumes  $\text{construct-path-inv } q\ x\ y\ D\ W \wedge q \neq x$ 
  shows  $\text{card } \{ z . z \leq -(W + \text{choose-point } (D*q)*q^T) \} < \text{card } \{ z . z \leq -W \}$ 
}
proof –
  let  $?q = \text{choose-point } (D*q)$ 
  let  $?W = W + ?q * q^T$ 
  show ?thesis
  proof (rule decrease-variant)
    show  $\neg ?W \leq -W$ 
    by simp
    show  $?q * q^T \leq -W$ 
    by (metis assms galois-aux inf-compl-bot-right maddux-142 mult-isor
  order-trans top-greatest)
    show  $\neg (?q * q^T \leq -?W)$ 
    using assms end-points(1)
    by (smt acyclic-imp-one-step-different-points(2) choose-point-decreasing
  compl-sup inf.absorb1
  inf-compl-bot-right sup commute sup-bot.left-neutral conv-zero
  end-points(2))
  qed
qed

```

```

theorem construct-path-total: VARs p q W
  [ is-acyclic D  $\wedge$  point y  $\wedge$  point x  $\wedge D^*y \leq D^T*x$  ]
   $W := 0$ ;

```

```

q := y;
WHILE q ≠ x
  INV { construct-path-inv q x y D W }
  VAR { card { z . z ≤ -W } }
  DO p := choose-point (D*q);
    W := W + p*qT;
    q := p
  OD
[ W ≤ D ∧ terminating-path W ∧ (W=0 ↔ x=y) ∧ (W≠0 ↔ x =
start-points W ∧ y = end-points W) ]
apply vcg-tc
using construct-path-pre apply blast
apply (rule CollectI, rule conjI)
using construct-path-inv apply blast
using construct-path-inv-termination apply clarsimp
by fastforce

```

end

4.2 Topological sorting

In our second example we look at topological sorting. Given a directed acyclic graph, the problem is to construct a linear order of its vertices that contains x before y for each edge (x, y) of the graph. If the input graph models dependencies between tasks, the output is a linear schedule of the tasks that respects all dependencies.

context *relation-algebra-rtc-tarski-choose-point*
begin

abbreviation *topological-sort-inv*

where *topological-sort-inv* $q\ v\ R\ W \equiv$
 $regressively-finite\ R \wedge R \cdot v * v^T \leq W^+ \wedge terminating-path\ W \wedge W * 1 =$
 $v \cdot -q \wedge$
 $(W = 0 \vee q = end-points\ W) \wedge point\ q \wedge R * v \leq v \wedge q \leq v \wedge is-vector\ v$

lemma *topological-sort-pre:*

assumes *regressively-finite* R
shows *topological-sort-inv* (*choose-point* (*minimum* $R\ 1$)) (*choose-point*
(*minimum* $R\ 1$)) $R\ 0$

proof (*intro conjI, simp-all add:assms*)

let $?q = choose-point\ (-\ (R^T * 1))$

show *point-q: point* $?q$

using *assms* **by** (*metis* (*full-types*) *annir choose-point-point galois-aux2*
is-inj-def is-sur-def

is-vector-def one-idem-mult point-def ss-p18 inf-top-left

one-compl)

show $R \cdot ?q * ?q^T \leq 0$

by (*metis choose-point-decreasing conv-invol end-point-char order.eq-iff*
inf-bot-left schroeder-2)
show *path 0*
by (*simp add: is-inj-def is-p-fun-def path-def*)
show $R * ?q \leq ?q$
by (*metis choose-point-decreasing compl-bot-eq conv-galois-1 inf-compl-bot-left2*
le-inf-iff)
show *is-vector ?q*
using *point-q point-def by blast*
qed

lemma *topological-sort-inv:*

assumes $v \neq 1$
and *topological-sort-inv q v R W*
shows *topological-sort-inv (choose-point (minimum R (- v))) (v +*
*choose-point (minimum R (- v))) R (W + q * choose-point*
(minimum R (- v))^T)
proof (*intro conjI*)
let $?p = \text{choose-point (minimum R (- v))}$
let $?W = W + q * ?p^T$
let $?v = v + ?p$
show *point-p: point ?p*
using *assms*
by (*metis choose-point-point compl-bot-eq double-compl galois-aux2 comp-assoc*
is-vector-def
vector-compl vector-mult)
hence *ep-np: end-points (q * ?p^T) = ?p*
using *assms(2)*
by (*metis aux4 choose-point-decreasing edge-end le-supI1*
point-in-vector-or-complement-iff
point-is-point)
hence *sp-q: start-points (q * ?p^T) = q*
using *assms(2) point-p*
by (*metis (no-types, lifting) conv-contrav conv-invol edge-start point-is-point*)
hence *ep-sp: W ≠ 0 ⇒ end-points W = start-points (q * ?p^T)*
using *assms(2) by force*
have $W * 1 \cdot (q * ?p^T)^T * 1 = v - q \cdot ?p$
using *assms(2) point-p is-vector-def mult-assoc point-def point-equations(3)*
point-is-point
by *auto*
hence $1: W * 1 \cdot (q * ?p^T)^T * 1 = 0$
by (*metis choose-point-decreasing dual-order.trans galois-aux inf.cobounded2*
inf commute)

show *regressively-finite R*
using *assms(2) by blast*
show $R \cdot ?v * ?v^T \leq ?W^+$
proof –
have $a: R \cdot v * v^T \leq ?W^+$

```

    using assms(2) by (meson mult-isol-var order.trans order-prop star-subdist)
  have b:  $R \cdot v * ?p^T \leq ?W^+$ 
  proof -
    have  $R \cdot v * ?p^T \leq W * 1 * ?p^T + q * ?p^T$ 
      by (metis inf-le2 assms(2) aux4 double-compl inf-absorb2 distrib-right)
    also have  $\dots = W * ?p^T + q * ?p^T$ 
      using point-p by (metis conv-contrav conv-one is-vector-def mult-assoc
point-def)
    also have  $\dots \leq W^+ * \text{end-points } W * ?p^T + q * ?p^T$ 
      using assms(2)
      by (meson forward-terminating-path-end-points-1 join-iso mult-isor
terminating-path-iff)
    also have  $\dots \leq W^+ * q * ?p^T + q * ?p^T$ 
      using assms(2) by (metis annil eq-refl)
    also have  $\dots = W^+ * q * ?p^T$ 
      using conway.dagger-unfoldl-distr mult-assoc sup-commute by fastforce
    also have  $\dots \leq ?W^+$ 
      by (metis mult-assoc mult-isol-var star-slide-var star-subdist sup-ge2)
    finally show ?thesis .
  qed
  have c:  $R \cdot ?p * v^T \leq ?W^+$ 
  proof -
    have  $v \leq -?p$ 
      using choose-point-decreasing compl-le-swap1 inf-le1 order-trans by blast
    hence  $R * v \leq -?p$ 
      using assms(2) order.trans by blast
    thus ?thesis
      by (metis galois-aux inf-le2 schroeder-2)
  qed
  have d:  $R \cdot ?p * ?p^T \leq ?W^+$ 
  proof -
    have  $R \cdot ?p * ?p^T \leq R \cdot 1'$ 
      using point-p is-inj-def meet-isor point-def by blast
    also have  $\dots = 0$ 
      using assms(2) regressively-finite-irreflexive galois-aux by blast
    finally show ?thesis
      using bot-least inf.absorb-iff2 by simp
  qed
  have  $R \cdot ?v * ?v^T = (R \cdot v * v^T) + (R \cdot v * ?p^T) + (R \cdot ?p * v^T) + (R \cdot ?p * ?p^T)$ 
    by (metis conv-add distrib-left distrib-right inf-sup-distrib1 sup.commute
sup.left-commute)
  also have  $\dots \leq ?W^+$ 
    using a b c d by (simp add: le-sup-iff)
  finally show ?thesis .
  qed
  show pathW: path ?W
  proof (cases  $W = 0$ )
    assume  $W = 0$ 
    thus ?thesis

```

```

    using assms(2) point-p edge-is-path point-is-point sup-bot-left by auto
next
assume a1: W ≠ 0
have fw-path: forward-terminating-path W
  using assms(2) terminating-iff by blast
have bw-path: backward-terminating-path (q*?pT)
  using assms point-p sp-q
  by (metis conv-backward-terminating conv-has-start-points conv-path
edge-is-path
      forward-terminating-iff1 point-is-point start-point-iff2)
show ?thesis
  using fw-path bw-path ep-sp 1 a1 path-concatenation-cycle-free by blast
qed
show terminating ?W
proof (rule start-end-implies-terminating)
  show has-start-points ?W
    apply (cases W = 0)
    using assms(2) sp-q pathW
    apply (metis (no-types, lifting) point-is-point start-point-iff2
sup-bot.left-neutral)
    using assms(2) ep-sp 1 pathW
    by (metis has-start-end-points-iff path-concatenation-start-points
start-point-iff2
      terminating-iff1)
  show has-end-points ?W
    apply (cases W = 0)
    using point-p ep-np ep-sp pathW end-point-iff2 point-is-point apply force
    using point-p ep-np ep-sp 1 pathW
    by (metis end-point-iff2 path-concatenation-end-points point-is-point)
qed
show ?W*1 = ?v.-?p
proof -
  have ?W*1 = v
    by (metis assms(2) point-p is-vector-def mult-assoc point-def
point-equations(3)
      point-is-point aux4 distrib-right' inf-absorb2 sup commute)
  also have ... = v.-?p
    by (metis choose-point-decreasing compl-le-swap1 inf.cobounded1 inf.orderE
order-trans)
  finally show ?thesis
    by (simp add: inf-sup-distrib2)
qed
show ?W = 0 ∨ ?p = end-points ?W
  using ep-np ep-sp 1 by (metis path-concatenation-end-points sup-bot-left)
show R*?v ≤ ?v
  using assms(2)
  by (meson choose-point-decreasing conv-galois-1 inf.cobounded2 order.trans
sup.coboundedI1
      sup-least)

```



```

show ?p ≤ ?v
  by simp
show is-vector ?v
  using assms(2) point-p point-def vector-add by blast
qed

```

lemma *topological-sort-post*:

```

assumes ¬ v ≠ 1
  and topological-sort-inv q v R W
shows R ≤ W+ ∧ terminating-path W ∧ (W + WT)*1 = -1'*1
proof (intro conjI, simp-all add: assms)
show R ≤ W+
  using assms by force
show backward-terminating W ∧ W ≤ 1 * W * (-v + q)
  using assms by force
show v · -q + WT * 1 = -1' * 1
  proof (cases W = 0)
    assume W = 0
    thus ?thesis
    using assms
    by (metis compl-bot-eq conv-one conv-zero double-compl inf-top.left-neutral
is-inj-def
le-bot mult-1-right one-idem-mult point-def ss-p18 star-zero
sup.absorb2 top-le)
  next
    assume a1: W ≠ 0
    hence -1' ≠ 0
    using assms backward-terminating-path-irreflexive le-bot by fastforce
    hence 1 = 1*-1'*1
    by (simp add: tarski)
    also have ... = -1'*1
    by (metis comp-assoc distrib-left mult-1-left sup-top-left distrib-right
sup-compl-top)
    finally have a: 1 = -1'*1 .
    have W*1 + WT*1 = 1
    using assms a1 by (metis double-compl galois-aux4 inf.absorb-iff2
inf-top.left-neutral)
    thus ?thesis
    using a by (simp add: assms(2))
  qed
qed

```

theorem *topological-sort-partial*: VARS p q v W

{ regressively-finite R }

W := 0;

q := choose-point (minimum R 1);

v := q;

WHILE v ≠ 1

INV { topological-sort-inv q v R W }

```

    DO p := choose-point (minimum R (-v));
      W := W + q*pT;
      q := p;
      v := v + p
    OD
  { R ≤ W+ ∧ terminating-path W ∧ (W + WT)*1 = -1'*1 }
  apply vcg
  using topological-sort-pre apply blast
  using topological-sort-inv apply blast
  using topological-sort-post by blast

end

context relation-algebra-rtc-tarski-choose-point-finite
begin

lemma topological-sort-inv-termination:
  assumes v ≠ 1
    and topological-sort-inv q v R W
  shows card { z . z ≤ -(v + choose-point (minimum R (-v))) } < card { z . z
≤ -v }
proof (rule decrease-variant)
  let ?p = choose-point (minimum R (-v))
  let ?v = v + ?p
  show -?v ≤ -v
    by simp
  show ?p ≤ -v
    using choose-point-decreasing inf.boundedE by blast
  have point ?p
    using assms
    by (metis choose-point-point compl-bot-eq double-compl galois-aux2 comp-assoc
is-vector-def
vector-compl vector-mult)
  thus ¬ (?p ≤ -?v)
    by (metis annir compl-sup inf.absorb1 inf-compl-bot-right maddux-20
no-end-point-char)
qed

```

Use precondition *is-acyclic* instead of *regressively-finite*. They are equivalent for finite graphs.

```

theorem topological-sort-total: VARS p q v W
  [ is-acyclic R ]
  W := 0;
  q := choose-point (minimum R 1);
  v := q;
  WHILE v ≠ 1
    INV { topological-sort-inv q v R W }
    VAR { card { z . z ≤ -v } }
    DO p := choose-point (minimum R (-v));

```

```

      W := W + q*pT;
      q := p;
      v := v + p
    OD
  [ R ≤ W+ ∧ terminating-path W ∧ (W + WT)*1 = -1'*1 ]
apply vcg-tc
apply (drule acyclic-regressively-finite)
using topological-sort-pre apply blast
apply (rule CollectI, rule conjI)
using topological-sort-inv apply blast
using topological-sort-inv-termination apply auto[1]
using topological-sort-post by blast

end

```

4.3 Construction of a tree

Our last application is a correctness proof of an algorithm that constructs a non-empty cycle for a given directed graph. This works in two steps. The first step is to construct a directed tree from a given root along the edges of the graph.

context *relation-algebra-rtc-tarski-choose-point*
begin

abbreviation *construct-tree-pre*

where *construct-tree-pre* $x\ y\ R \equiv y \leq R^{T*} * x \wedge \text{point } x$

abbreviation *construct-tree-inv*

where *construct-tree-inv* $v\ x\ y\ D\ R \equiv \text{construct-tree-pre } x\ y\ R \wedge \text{is-acyclic } D \wedge \text{is-inj } D \wedge$

$D^* \wedge D \leq v * v^T \wedge$

$D \leq R \wedge D * x = 0 \wedge v = x + D^T * 1 \wedge x * v^T \leq$

is-vector v

abbreviation *construct-tree-post*

where *construct-tree-post* $x\ y\ D\ R \equiv \text{is-acyclic } D \wedge \text{is-inj } D \wedge D \leq R \wedge D * x = 0 \wedge D^T * 1 \leq D^{T*} * x \wedge$

$D^* * y \leq D^{T*} * x$

lemma *construct-tree-pre*:

assumes *construct-tree-pre* $x\ y\ R$

shows *construct-tree-inv* $x\ x\ y\ 0\ R$

using *assms* **by** (*simp add: is-inj-def point-def*)

lemma *construct-tree-inv-aux*:

assumes $\neg y \leq v$

and *construct-tree-inv* $v\ x\ y\ D\ R$

shows *singleton* (*choose-singleton* ($v * - v^T \cdot R$))

proof (*rule choose-singleton-singleton, rule notI*)

assume $v * - v^T \cdot R = 0$

hence $R^{T*} * v \leq v$

by (*metis galois-aux conv-compl conv-galois-1 conv-galois-2 conv-invol*
double-compl
star-inductl-var)
 hence $y = 0$
 using *assms* by (*meson mult-isol order-trans sup.cobounded1*)
 thus *False*
 using *assms point-is-point* by *auto*
 qed

lemma *construct-tree-inv*:

assumes $\neg y \leq v$
 and *construct-tree-inv* $v x y D R$
 shows *construct-tree-inv* $(v + \text{choose-singleton } (v^* - v^T \cdot R)^T * 1) x y (D + \text{choose-singleton } (v^* - v^T \cdot R)) R$

proof (*intro conjI*)

let $?e = \text{choose-singleton } (v^* - v^T \cdot R)$
 let $?D = D + ?e$
 let $?v = v + ?e^T * 1$
 have 1: $?e \leq v^* - v^T$
 using *choose-singleton-decreasing inf.boundedE* by *blast*
 show *point* x
 by (*simp add: assms*)
 show $y \leq R^T * x$
 by (*simp add: assms*)
 show *is-acyclic* $?D$
 using 1 *assms acyclic-inv* by *fastforce*
 show *is-inj* $?D$
 using 1 *construct-tree-inv-aux assms injective-inv* by *blast*
 show $?D \leq R$
 apply (*rule sup.boundedI*)
 using *assms* apply *blast*
 using *choose-singleton-decreasing inf.boundedE* by *blast*
 show $?D * x = 0$
proof –
 have $?D * x = ?e * x$
 by (*simp add: assms*)
 also have $\dots \leq ?e * v$
 by (*simp add: assms mult-isol*)
 also have $\dots \leq v^* - v^T * v$
 using 1 *mult-isol* by *blast*
 also have $\dots = 0$
 by (*metis assms(2) annir comp-assoc vector-prop1*)
 finally show *thesis*
 using *le-bot* by *blast*
 qed
 show $?v = x + ?D^T * 1$
 by (*simp add: assms sup-assoc*)
 show $x * ?v^T \leq ?D^*$
proof –

```

have  $x * ?v^T = x * v^T + x * 1 * ?e$ 
  by (simp add: distrib-left mult-assoc)
also have  $\dots \leq D^* + x * 1 * (?e \cdot v * -v^T)$ 
  using 1 by (metis assms(2) inf.absorb1 join-iso)
also have  $\dots = D^* + x * 1 * (?e \cdot v \cdot -v^T)$ 
  by (metis assms(2) comp-assoc conv-compl inf.assoc vector-compl
vector-meet-comp)
also have  $\dots \leq D^* + x * 1 * (?e \cdot v)$ 
  using join-isol mult-subdistl by fastforce
also have  $\dots = D^* + x * (1 \cdot v^T) * ?e$ 
  by (metis assms(2) inf commute mult-assoc vector-2)
also have  $\dots = D^* + x * v^T * ?e$ 
  by simp
also have  $\dots \leq D^* + D^* * ?e$ 
  using assms join-isol mult-isor by blast
also have  $\dots \leq ?D^*$ 
  by (meson le-sup-iff prod-star-closure star-ext star-subdist)
finally show ?thesis .
qed
show  $?D \leq ?v * ?v^T$ 
proof (rule sup.boundedI)
  show  $D \leq ?v * ?v^T$ 
  using assms
  by (meson conv-add distrib-left le-supI1 conv-iso dual-order.trans
mult-isol-var order-prop)
  have  $?e \leq v * (-v^T \cdot v^T * ?e)$ 
  using 1 inf.absorb-iff2 modular-1' by fastforce
  also have  $\dots \leq v * 1 * ?e$ 
  by (simp add: comp-assoc le-infI2 mult-isol-var)
  also have  $\dots \leq ?v * ?v^T$ 
  by (metis conv-contrav conv-invol conv-iso conv-one mult-assoc mult-isol-var
sup.cobounded1
sup-ge2)
  finally show  $?e \leq ?v * ?v^T$ 
  by simp
qed
show is-vector ?v
  using assms comp-assoc is-vector-def by fastforce
qed

lemma construct-tree-post:
  assumes  $y \leq v$ 
  and construct-tree-inv  $v x y D R$ 
  shows construct-tree-post  $x y D R$ 
proof -
  have  $v * x^T \leq D^T *$ 
  by (metis (no-types, lifting) assms(2) conv-contrav conv-invol conv-iso
star-conv)
  hence 1:  $v \leq D^T * * x$ 

```

```

    using assms point-def ss423bij by blast
  hence 2:  $D^T * 1 \leq D^{T*} * x$ 
    using assms le-supE by blast
  have  $D^* * y \leq D^{T*} * x$ 
  proof (rule star-inductl, rule sup.boundedI)
    show  $y \leq D^{T*} * x$ 
      using 1 assms order.trans by blast
  next
    have  $D*(D^{T*} * x) = D*x + D*D^{T+} * x$ 
      by (metis Conway.dagger-unfoldl-distr distrib-left mult-assoc)
    also have  $\dots = D*D^{T+} * x$ 
      using assms by simp
    also have  $\dots \leq 1' * D^{T*} * x$ 
      by (metis assms(2) is-inj-def mult-assoc mult-isor)
    finally show  $D*(D^{T*} * x) \leq D^{T*} * x$ 
      by simp
  qed
  thus construct-tree-post x y D R
    using 2 assms by simp
qed

```

```

theorem construct-tree-partial:  $\text{VARs } e \ v \ D$ 
  { construct-tree-pre x y R }
   $D := 0;$ 
   $v := x;$ 
  WHILE  $\neg y \leq v$ 
    INV { construct-tree-inv v x y D R }
    DO  $e := \text{choose-singleton } (v * -v^T \cdot R);$ 
       $D := D + e;$ 
       $v := v + e^T * 1$ 
    OD
  { construct-tree-post x y D R }
apply vcg
using construct-tree-pre apply blast
using construct-tree-inv apply blast
using construct-tree-post by blast

```

end

```

context relation-algebra-rtc-tarski-choose-point-finite
begin

```

lemma construct-tree-inv-termination:

```

assumes  $\neg y \leq v$ 
and construct-tree-inv v x y D R
shows  $\text{card } \{ z \cdot z \leq -(v + \text{choose-singleton } (v * -v^T \cdot R)^T * 1) \} < \text{card } \{ z \cdot$ 
 $z \leq -v \}$ 
proof (rule decrease-variant)
  let ?e =  $\text{choose-singleton } (v * -v^T \cdot R)$ 

```

```

let ?v = v + ?eT*1
have 1: ?e ≤ v*-vT
  using choose-singleton-decreasing inf.boundedE by blast
have 2: singleton ?e
  using construct-tree-inv-aux assms by auto
show -?v ≤ -v
  by simp
have ?eT ≤ -v*vT
  using 1 conv-compl conv-iso by force
also have ... ≤ -v*1
  by (simp add: mult-isol)
finally show ?eT*1 ≤ -v
  using assms by (metis is-vector-def mult-isor one-compl)
thus ¬ (?eT*1 ≤ -?v)
  using 2 by (metis annir compl-sup inf.absorb1 inf-compl-bot-right surj-one
tarski)
qed

```

```

theorem construct-tree-total: VARS e v D
[ construct-tree-pre x y R ]
D := 0;
v := x;
WHILE ¬ y ≤ v
  INV { construct-tree-inv v x y D R }
  VAR { card { z . z ≤ -v } }
  DO e := choose-singleton (v*-vT · R);
    D := D + e;
    v := v + eT*1
  OD
[ construct-tree-post x y D R ]
apply vcg-tc
using construct-tree-pre apply blast
apply (rule CollectI, rule conjI)
using construct-tree-inv apply blast
using construct-tree-inv-termination apply force
using construct-tree-post by blast

```

end

4.4 Construction of a non-empty cycle

The second step is to construct a path from the root to a given vertex in the tree. Adding an edge back to the root gives the cycle.

```

context relation-algebra-rtc-tarski-choose-point
begin

```

```

abbreviation comment
  where comment - ≡ SKIP
abbreviation construct-cycle-inv

```

where $\text{construct-cycle-inv } v \ x \ y \ D \ R \equiv \text{construct-tree-inv } v \ x \ y \ D \ R \wedge \text{point } y \wedge y * x^T \leq R$

lemma *construct-cycle-pre*:

assumes $\neg \text{is-acyclic } R$
and $y = \text{choose-point } ((R^+ \cdot 1') * 1)$
and $x = \text{choose-point } (R^* * y \cdot R^T * y)$
shows $\text{construct-cycle-inv } x \ x \ y \ 0 \ R$
proof(*rule conjI, rule-tac [2] conjI*)
show *point-y: point y*
using *assms by (simp add: choose-point-point is-vector-def mult-assoc galois-aux ss-p18)*
have $R^* * y \cdot R^T * y \neq 0$
proof
have $R^+ \cdot 1' = (R^+)^T \cdot 1'$
by (*metis (mono-tags, opaque-lifting) conv-e conv-times inf.cobounded1 inf commute many-strongly-connected-iff-6-eq mult-oner star-subid*)
also have $\dots = R^{T+} \cdot 1'$
using *plus-conv by fastforce*
also have $\dots \leq (R^{T*} \cdot R) * R^T$
by (*metis conv-contrav conv-e conv-invol modular-2-var mult-oner star-slide-var*)
also have $\dots \leq (R^{T*} \cdot R) * 1$
by (*simp add: mult-isol*)
finally have $a: (R^+ \cdot 1') * 1 \leq (R^{T*} \cdot R) * 1$
by (*metis mult-assoc mult-isor one-idem-mult*)
assume $R^* * y \cdot R^T * y = 0$
hence $(R^* \cdot R^T) * y = 0$
using *point-y inj-distr point-def by blast*
hence $(R^* \cdot R^T)^T * 1 \leq -y$
by (*simp add: conv-galois-1*)
hence $y \leq -((R^* \cdot R^T)^T * 1)$
using *compl-le-swap1 by blast*
also have $\dots = -((R^{T*} \cdot R) * 1)$
by (*simp add: star-conv*)
also have $\dots \leq -((R^+ \cdot 1') * 1)$
using *a comp-anti by blast*
also have $\dots \leq -y$
by (*simp add: assms galois-aux ss-p18 choose-point-decreasing*)
finally have $y = 0$
using *inf.absorb2 by fastforce*
thus *False*
using *point-y annir point-equations(2) point-is-point tarski by force*
qed
hence *point-x: point x*
by (*metis point-y assms(3) inj-distr is-vector-def mult-assoc point-def choose-point-point*)
hence $y \leq R^{T*} * x$

by (*metis* *assms*(3) *point-y choose-point-decreasing inf-le1 order.trans point-swap star-conv*)
thus *tree-inv: construct-tree-inv x x y 0 R*
using *point-x construct-tree-pre* **by** *blast*
show $y * x^T \leq R$
proof –
have $x \leq R^* * y \cdot R^T * y$
using *assms*(3) *choose-point-decreasing* **by** *blast*
also have $\dots = (R^* \cdot R^T) * y$
using *point-y inj-distr point-def* **by** *fastforce*
finally have $x * y^T \leq R^* \cdot R^T$
using *point-y point-def ss423bij* **by** *blast*
also have $\dots \leq R^T$
by *simp*
finally show *?thesis*
using *conv-iso* **by** *force*
qed
qed

lemma *construct-cycle-pre2*:
assumes $y \leq v$
and *construct-cycle-inv v x y D R*
shows *construct-path-inv y x y D 0 $\wedge D \leq R \wedge D * x = 0 \wedge y * x^T \leq R$*
proof(*intro conjI, simp-all add: assms*)
show $D^* * y \leq D^{T*} * x$
using *assms construct-tree-post* **by** *blast*
show *path 0*
by (*simp add: is-inj-def is-p-fun-def path-def*)
show $y \neq 0$
using *assms*(2) *is-point-def point-is-point* **by** *blast*
qed

lemma *construct-cycle-post*:
assumes $\neg q \neq x$
and (*construct-path-inv q x y D W $\wedge D \leq R \wedge D * x = 0 \wedge y * x^T \leq R$*)
shows $W + y * x^T \neq 0 \wedge W + y * x^T \leq R \wedge \text{cycle } (W + y * x^T)$
proof(*intro conjI*)
let $?C = W + y * x^T$
show $?C \neq 0$
by (*metis* *assms acyclic-imp-one-step-different-points*(2) *no-trivial-inverse point-def ss423bij*
sup-bot.monoid-axioms monoid.left-neutral)
show $?C \leq R$
using *assms*(2) *order-trans sup.boundedI* **by** *blast*
show *path (W + y * x^T)*
by (*metis* *assms construct-tree-pre edge-is-path less-eq-def path-edge-equals-cycle*
point-is-point terminating-iff1)
show *many-strongly-connected (W + y * x^T)*

by (*metis assms construct-tree-pre bot-least conv-zero less-eq-def
path-edge-equals-cycle star-conv star-subid terminating-iff1*)

qed

theorem *construct-cycle-partial*: VARS $e\ p\ q\ v\ x\ y\ C\ D\ W$
 $\{ \neg \text{is-acyclic } R \}$
 $y := \text{choose-point } ((R^+ \cdot 1') * 1);$
 $x := \text{choose-point } (R^* * y \cdot R^T * y);$
 $D := 0;$
 $v := x;$
WHILE $\neg y \leq v$
 INV $\{ \text{construct-cycle-inv } v\ x\ y\ D\ R \}$
 DO $e := \text{choose-singleton } (v * -v^T \cdot R);$
 $D := D + e;$
 $v := v + e^T * 1$
OD;
comment $\{ \text{is-acyclic } D \wedge \text{point } y \wedge \text{point } x \wedge D^* * y \leq D^T * x \};$
 $W := 0;$
 $q := y;$
WHILE $q \neq x$
 INV $\{ \text{construct-path-inv } q\ x\ y\ D\ W \wedge D \leq R \wedge D * x = 0 \wedge y * x^T \leq R \}$
 DO $p := \text{choose-point } (D * q);$
 $W := W + p * q^T;$
 $q := p$
OD;
comment $\{ W \leq D \wedge \text{terminating-path } W \wedge (W = 0 \iff q = y) \wedge (W \neq 0 \iff q = \text{start-points } W \wedge y = \text{end-points } W) \};$
 $C := W + y * x^T$
 $\{ C \neq 0 \wedge C \leq R \wedge \text{cycle } C \}$
apply *vcg*
using *construct-cycle-pre* **apply** *blast*
using *construct-tree-inv* **apply** *blast*
using *construct-cycle-pre2* **apply** *blast*
using *construct-path-inv* **apply** *blast*
using *construct-cycle-post* **by** *blast*

end

context *relation-algebra-rtc-tarski-choose-point-finite*

begin

theorem *construct-cycle-total*: VARS $e\ p\ q\ v\ x\ y\ C\ D\ W$
 $[\neg \text{is-acyclic } R]$
 $y := \text{choose-point } ((R^+ \cdot 1') * 1);$
 $x := \text{choose-point } (R^* * y \cdot R^T * y);$
 $D := 0;$
 $v := x;$
WHILE $\neg y \leq v$
 INV $\{ \text{construct-cycle-inv } v\ x\ y\ D\ R \}$

```

VAR { card { z . z ≤ -v } }
DO e := choose-singleton (v*-vT · R);
  D := D + e;
  v := v + eT*1
OD;
comment { is-acyclic D ∧ point y ∧ point x ∧ D*y ≤ DT*x };
W := 0;
q := y;
WHILE q ≠ x
  INV { construct-path-inv q x y D W ∧ D ≤ R ∧ D*x = 0 ∧ y*xT ≤ R }
  VAR { card { z . z ≤ -W } }
  DO p := choose-point (D*q);
    W := W + p*qT;
    q := p
  OD;
  comment { W ≤ D ∧ terminating-path W ∧ (W = 0 ↔ q=y) ∧ (W ≠ 0
↔ q = start-points W ∧ y = end-points W) };
  C := W + y*xT
  [ C ≠ 0 ∧ C ≤ R ∧ cycle C ]
  apply vcg-tc
  using construct-cycle-pre apply blast
  apply (rule CollectI, rule conjI)
  using construct-tree-inv apply blast
  using construct-tree-inv-termination apply force
  using construct-cycle-pre2 apply blast
  apply (rule CollectI, rule conjI)
  using construct-path-inv apply blast
  using construct-path-inv-termination apply clarsimp
  using construct-cycle-post by blast
end
end

```

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