Relational Minimum Spanning Tree Algorithms

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Abstract

We verify the correctness of Prim's, Kruskal's and Borůvka's minimum spanning tree algorithms based on algebras for aggregation and minimisation.

Contents

1 Overview

The theories described in this document prove the correctness of Prim's, Kruskal's and Borůvka's minimum spanning tree algorithms. Specifications and algorithms work in Stone-Kleene relation algebras extended by operations for aggregation and minimisation. The algorithms are implemented in a simple imperative language and their proof uses Hoare logic. The correctness proofs are discussed in [3, 5, 6, 8].

1.1 Prim's and Kruskal's minimum spanning tree algorithms

A framework based on Stone relation algebras and Kleene algebras and extended by operations for aggregation and minimisation was presented by the first author in [3, 5] and used to formally verify the correctness of Prim's minimum spanning tree algorithm. It was extended in [6] and applied to prove the correctness of Kruskal's minimum spanning tree algorithm.

Two theories, one each for Prim's and Kruskal's algorithms, prove total correctness of these algorithms. As case studies for the algebraic framework, these two theories combined were originally part of another AFP entry [4].

1.2 Borůvka's minimum spanning tree algorithm

Otakar Borůvka formalised the minimum spanning tree problem and proposed a solution to it [1]. Borůvka's original paper is written in Czech; translations of varying completeness can be found in [2, 7].

The theory for Borůvka's minimum spanning tree algorithm proves partial correctness of this algorithm. This work is based on the same algebraic framework as the proof of Kruskal's algorithm; in particular it uses many theories from the hierarchy underlying [4].

The theory for Borůvka's algorithm formally verifies results from the second author's Master's thesis [8].

2 Kruskal's Minimum Spanning Tree Algorithm

In this theory we prove total correctness of Kruskal's minimum spanning tree algorithm. The proof uses the following steps [6]. We first establish that the algorithm terminates and constructs a spanning tree. This is a constructive proof of the existence of a spanning tree; any spanning tree algorithm could be used for this. We then conclude that a minimum spanning tree exists. This is necessary to establish the invariant for the actual correctness proof, which shows that Kruskal's algorithm produces a minimum spanning tree.

theory *Kruskal*

imports *HOL−Hoare.Hoare-Logic Aggregation-Algebras.Aggregation-Algebras*

begin

context *m-kleene-algebra*
begin

definition *spanning-forest* f $g \equiv$ *forest* $f \wedge f \leq --g \wedge$ *components* $g \leq$ *forest-components* $f \wedge$ *regular* f
definition *minimum-spanning-forest* f $g \equiv$ *spanning-forest* f $g \wedge (\forall u\ .$ *spanning-forest* u $g \longrightarrow$ *sum* $(f \sqcap g) \leq$ *sum* $(u \sqcap g))$

definition *kruskal-spanning-invariant f g h* $\equiv$ *symmetric g* $\wedge$ $h = h^T$ $\wedge$ $g \sqcap --h = h$ $\wedge$ *spanning-forest f* $(-h \sqcap g)$
definition *kruskal-invariant f g h* $\equiv$ *kruskal-spanning-invariant f g h* $\wedge$ $(\exists w$. *minimum-spanning-forest w g* $\wedge$ $f \leq w \sqcup w^T)$

We first show two verification conditions which are used in both correctness proofs.

lemma *kruskal-vc-1*:
  assumes *symmetric g*
    shows *kruskal-spanning-invariant bot g g*
proof (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
  show *symmetric g*
    using *assms* **by** *simp*
next
  show $g = g^T$
    using *assms* **by** *simp*
next
  show $g \sqcap --g = g$
    using *inf.sup-monoid.add-commute selection-closed-id* **by** *simp*
next
  show *spanning-forest bot* $(-g \sqcap g)$
    using *star.circ-transitive-equal spanning-forest-def* **by** *simp*
qed

lemma *kruskal-vc-2*:
  assumes *kruskal-spanning-invariant f g h*
      and $h \neq bot$
    shows (*minarc h* $\leq$ $-$*forest-components f* $\longrightarrow$ *kruskal-spanning-invariant* $((f \sqcap -(top * minarc\ h * f^{T\star})) \sqcup (f \sqcap top * minarc\ h * f^{T\star})^T \sqcup minarc\ h)\ g\ (h \sqcap -minarc\ h \sqcap -minarc\ h^T)$
$\wedge$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -minarc\ h \wedge x \leq -minarc\ h^T\ \} <$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h\ \}) \wedge$
(¬ *minarc h* $\leq$ $-$*forest-components f* $\longrightarrow$ *kruskal-spanning-invariant f g* $(h \sqcap -minarc\ h \sqcap -minarc\ h^T)$
$\wedge$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -minarc\ h \wedge x \leq -minarc\ h^T\ \} <$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h\ \})$
proof $-$
  let $?e = minarc\ h$
  let $?f = (f \sqcap -(top * ?e * f^{T\star})) \sqcup (f \sqcap top * ?e * f^{T\star})^T \sqcup ?e$
  let $?h = h \sqcap -?e \sqcap -?e^T$
  let $?F = forest\text{-}components\ f$
  let $?n1 = card\ \{\ x\ .\ regular\ x \wedge x \leq --h\ \}$
  let $?n2 = card\ \{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -?e \wedge x \leq -?e^T\ \}$
  have *1*: *regular f* $\wedge$ *regular ?e*
    by (*metis assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def minarc-regular*)
  hence *2*: *regular ?f* $\wedge$ *regular ?F* $\wedge$ *regular* $(?e^T)$
    using *regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
  have *3*: $\neg\ ?e \leq -?e$

using *assms*(*2*) *inf.orderE minarc-bot-iff* **by** *fastforce*
have *4*: *?n2* < *?n1*
apply (*rule psubset-card-mono*)
using *finite-regular* **apply** *simp*
using *1 3 kruskal-spanning-invariant-def minarc-below* **by** *auto*
show (*?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*) ∧ (¬ *?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant f g ?h* ∧ *?n2* < *?n1*)
proof (*rule conjI*)
have *5*: *injective ?f*
apply (*rule kruskal-injective-inv*)
using *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
apply (*simp add*: *covector-mult-closed*)
apply (*simp add*: *comp-associative comp-isotone star.right-plus-below-circ*)
apply (*meson mult-left-isotone order-lesseq-imp star-outer-increasing top.extremum*)
using *assms*(*1*,*2*) *kruskal-spanning-invariant-def kruskal-injective-inv-2 minarc-arc spanning-forest-def* **apply** *simp*
using *assms*(*2*) *arc-injective minarc-arc* **apply** *blast*
using *assms*(*1*,*2*) *kruskal-spanning-invariant-def kruskal-injective-inv-3 minarc-arc spanning-forest-def* **by** *simp*
show *?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*
proof
assume *6*: *?e* ≤ −*?F*
have *7*: *equivalence ?F*
using *assms*(*1*) *kruskal-spanning-invariant-def forest-components-equivalence spanning-forest-def* **by** *simp*
have $?e^T * top * ?e^T = ?e^T$
using *assms*(*2*) **by** (*simp add*: *arc-top-arc minarc-arc*)
hence $?e^T * top * ?e^T \leq -?F$
using *6 7 conv-complement conv-isotone* **by** *fastforce*
hence *8*: *?e* ∗ *?F* ∗ *?e* = *bot*
using *le-bot triple-schroeder-p* **by** *simp*
show *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*
proof (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
show *symmetric g*
using *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*
next
show $?h = ?h^T$
using *assms*(*1*) **by** (*simp add*: *conv-complement conv-dist-inf inf-commute inf-left-commute kruskal-spanning-invariant-def*)
next
show *g* ⊓ −−*?h* = *?h*
using *1 2* **by** (*metis* (*opaque-lifting*) *assms*(*1*) *kruskal-spanning-invariant-def inf-assoc pp-dist-inf*)
next
show *spanning-forest ?f* (−*?h* ⊓ *g*)
proof (*unfold spanning-forest-def*, *intro conjI*)
show *injective ?f*

using *5* **by** *simp*
next
show *acyclic ?f*
apply (*rule kruskal-acyclic-inv*)
using *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def*
apply *simp*
apply (*simp add*: *covector-mult-closed*)
using *8 assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def*
kruskal-acyclic-inv-1 **apply** *simp*
using *8* **apply** (*metis comp-associative mult-left-sub-dist-sup-left*
star.circ-loop-fixpoint sup-commute le-bot)
using *6* **by** (*simp add*: *p-antitone-iff*)
next
show $?f \le --(-?h \sqcap g)$
apply (*rule kruskal-subgraph-inv*)
using *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def*
apply *simp*
using *assms*(*1*) **apply** (*metis kruskal-spanning-invariant-def*
minarc-below order.trans pp-isotone-inf)
using *assms*(*1*) *kruskal-spanning-invariant-def* **apply** *simp*
using *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*
next
show *components* $(-?h \sqcap g) \le$ *forest-components ?f*
apply (*rule kruskal-spanning-inv*)
using *5* **apply** *simp*
using *1 regular-closed-star regular-conv-closed regular-mult-closed*
apply *simp*
using *1* **apply** *simp*
using *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **by**
simp
next
show *regular ?f*
using *2* **by** *simp*
qed
next
show $?n2 < ?n1$
using *4* **by** *simp*
qed
qed
next
show $\neg\ ?e \le -?F \longrightarrow$ *kruskal-spanning-invariant f g ?h* $\wedge\ ?n2 < ?n1$
proof
assume $\neg\ ?e \le -?F$
hence *9*: $?e \le ?F$
using *2 assms*(*2*) *arc-in-partition minarc-arc* **by** *fastforce*
show *kruskal-spanning-invariant f g ?h* $\wedge\ ?n2 < ?n1$
proof (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
show *symmetric g*
using *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*

```
      next
        show ?h = ?h^T
          using assms(1) by (simp add: conv-complement conv-dist-inf
inf-commute inf-left-commute kruskal-spanning-invariant-def)
      next
        show g ⊓ −−?h = ?h
          using 1 2 by (metis (opaque-lifting) assms(1)
kruskal-spanning-invariant-def inf-assoc pp-dist-inf)
      next
        show spanning-forest f (−?h ⊓ g)
        proof (unfold spanning-forest-def, intro conjI)
          show injective f
            using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
        next
          show acyclic f
            using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
        next
          have f ≤ −−(−h ⊓ g)
            using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
          also have ... ≤ −−(−?h ⊓ g)
            using comp-inf.mult-right-isotone inf.sup-monoid.add-commute
inf-left-commute p-antitone-inf pp-isotone by presburger
          finally show f ≤ −−(−?h ⊓ g)
            by simp
        next
          show components (−?h ⊓ g) ≤ ?F
            apply (rule kruskal-spanning-inv-1)
            using 9 apply simp
            using 1 apply simp
            using assms(1) kruskal-spanning-invariant-def spanning-forest-def
apply simp
            using assms(1) kruskal-spanning-invariant-def
forest-components-equivalence spanning-forest-def by simp
        next
          show regular f
            using 1 by simp
        qed
      next
        show ?n2 < ?n1
          using 4 by simp
      qed
    qed
  qed
qed
```

The following result shows that Kruskal's algorithm terminates and constructs a spanning tree. We cannot yet show that this is a minimum spanning tree.

theorem *kruskal-spanning*:
  VARS e f h
  [*symmetric g*]
  $f := bot;$
  $h := g;$
  WHILE $h \neq bot$
    INV { *kruskal-spanning-invariant f g h* }
    VAR { *card* { x . *regular* $x \wedge x \leq --h$ } }
    DO $e := minarc\ h;$
      IF $e \leq -$*forest-components f THEN*
        $f := (f \sqcap -(top * e * f^{T\star})) \sqcup (f \sqcap top * e * f^{T\star})^T \sqcup e$
      ELSE
        SKIP
      FI;
      $h := h \sqcap -e \sqcap -e^T$
    OD
  [*spanning-forest f g*]
  apply *vcg-tc-simp*
  using *kruskal-vc-1* **apply** *simp*
  using *kruskal-vc-2* **apply** *simp*
  using *kruskal-spanning-invariant-def* **by** *auto*

Because we have shown total correctness, we conclude that a spanning tree exists.

lemma *kruskal-exists-spanning*:
  symmetric g $\Longrightarrow \exists f$. *spanning-forest f g*
  using *tc-extract-function kruskal-spanning* **by** *blast*

This implies that a minimum spanning tree exists, which is used in the subsequent correctness proof.

lemma *kruskal-exists-minimal-spanning*:
  assumes *symmetric g*
    shows $\exists f$. *minimum-spanning-forest f g*
proof –
  let *?s* = { *f* . *spanning-forest f g* }
  have $\exists m \in ?s$. $\forall z \in ?s$. $sum\ (m \sqcap g) \leq sum\ (z \sqcap g)$
    apply (*rule finite-set-minimal*)
    using *finite-regular spanning-forest-def* **apply** *simp*
    using *assms kruskal-exists-spanning* **apply** *simp*
    using *sum-linear* **by** *simp*
  thus *?thesis*
    using *minimum-spanning-forest-def* **by** *simp*
qed

Kruskal's minimum spanning tree algorithm terminates and is correct. This is the same algorithm that is used in the previous correctness proof, with the same precondition and variant, but with a different invariant and postcondition.

```
theorem kruskal:
  VARS e f h
  [ symmetric g ]
  f := bot;
  h := g;
  WHILE h ≠ bot
    INV { kruskal-invariant f g h }
    VAR { card { x . regular x ∧ x ≤ −−h } }
     DO e := minarc h;
        IF e ≤ −forest-components f THEN
          f := (f ⊓ −(top ∗ e ∗ f^T⋆)) ⊔ (f ⊓ top ∗ e ∗ f^T⋆)^T ⊔ e
        ELSE
          SKIP
        FI;
        h := h ⊓ −e ⊓ −e^T
     OD
  [ minimum-spanning-forest f g ]
proof vcg-tc-simp
  assume symmetric g
  thus kruskal-invariant bot g g
    using kruskal-vc-1 kruskal-exists-minimal-spanning kruskal-invariant-def by
simp
next
  fix f h
  let ?e = minarc h
  let ?f = (f ⊓ −(top ∗ ?e ∗ f^T⋆)) ⊔ (f ⊓ top ∗ ?e ∗ f^T⋆)^T ⊔ ?e
  let ?h = h ⊓ −?e ⊓ −?e^T
  let ?F = forest-components f
  let ?n1 = card { x . regular x ∧ x ≤ −−h }
  let ?n2 = card { x . regular x ∧ x ≤ −−h ∧ x ≤ −?e ∧ x ≤ −?e^T }
  assume 1: kruskal-invariant f g h ∧ h ≠ bot
  from 1 obtain w where 2: minimum-spanning-forest w g ∧ f ≤ w ⊔ w^T
    using kruskal-invariant-def by auto
  hence 3: regular f ∧ regular w ∧ regular ?e
    using 1 by (metis kruskal-invariant-def kruskal-spanning-invariant-def
minimum-spanning-forest-def spanning-forest-def minarc-regular)
  show (?e ≤ −?F ⟶ kruskal-invariant ?f g ?h ∧ ?n2 < ?n1) ∧ (¬ ?e ≤ −?F
⟶ kruskal-invariant f g ?h ∧ ?n2 < ?n1)
  proof (rule conjI)
    show ?e ≤ −?F ⟶ kruskal-invariant ?f g ?h ∧ ?n2 < ?n1
    proof
      assume 4: ?e ≤ −?F
      have 5: equivalence ?F
        using 1 kruskal-invariant-def kruskal-spanning-invariant-def
forest-components-equivalence spanning-forest-def by simp
      have ?e^T ∗ top ∗ ?e^T = ?e^T
        using 1 by (simp add: arc-top-arc minarc-arc)
      hence ?e^T ∗ top ∗ ?e^T ≤ −?F
        using 4 5 conv-complement conv-isotone by fastforce
```

```
    hence 6: ?e * ?F * ?e = bot
      using le-bot triple-schroeder-p by simp
    show kruskal-invariant ?f g ?h ∧ ?n2 < ?n1
    proof (unfold kruskal-invariant-def, intro conjI)
      show kruskal-spanning-invariant ?f g ?h
        using 1 4 kruskal-vc-2 kruskal-invariant-def by simp
    next
      show ∃ w . minimum-spanning-forest w g ∧ ?f ≤ w ⊔ w^T
      proof
        let ?p = w ⊓ top * ?e * w^T⋆
        let ?v = (w ⊓ −(top * ?e * w^T⋆)) ⊔ ?p^T
        have 7: regular ?p
          using 3 regular-closed-star regular-conv-closed regular-mult-closed by
simp
        have 8: injective ?v
          apply (rule kruskal-exchange-injective-inv-1)
          using 2 minimum-spanning-forest-def spanning-forest-def apply simp
          apply (simp add: covector-mult-closed)
          apply (simp add: comp-associative comp-isotone
star.right-plus-below-circ)
          using 1 2 kruskal-injective-inv-3 minarc-arc
minimum-spanning-forest-def spanning-forest-def by simp
        have 9: components g ≤ forest-components ?v
          apply (rule kruskal-exchange-spanning-inv-1)
          using 8 apply simp
          using 7 apply simp
          using 2 minimum-spanning-forest-def spanning-forest-def by simp
        have 10: spanning-forest ?v g
        proof (unfold spanning-forest-def, intro conjI)
          show injective ?v
            using 8 by simp
        next
          show acyclic ?v
            apply (rule kruskal-exchange-acyclic-inv-1)
            using 2 minimum-spanning-forest-def spanning-forest-def apply simp
            by (simp add: covector-mult-closed)
        next
          show ?v ≤ −−g
            apply (rule sup-least)
            using 2 inf.coboundedI1 minimum-spanning-forest-def
spanning-forest-def apply simp
            using 1 2 by (metis kruskal-invariant-def
kruskal-spanning-invariant-def conv-complement conv-dist-inf order.trans
inf.absorb2 inf.cobounded1 minimum-spanning-forest-def spanning-forest-def)
        next
          show components g ≤ forest-components ?v
            using 9 by simp
        next
          show regular ?v
```

using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
qed
have *11*: *sum* $(?v \sqcap g) = sum\ (w \sqcap g)$
proof −
have *sum* $(?v \sqcap g) = sum\ (w \sqcap -(top * ?e * w^{T\star}) \sqcap g) + sum\ (?p^T \sqcap g)$
using *2* **by** (*metis conv-complement conv-top epm-8 inf-import-p inf-top-right regular-closed-top vector-top-closed minimum-spanning-forest-def spanning-forest-def sum-disjoint*)
also have ... $= sum\ (w \sqcap -(top * ?e * w^{T\star}) \sqcap g) + sum\ (?p \sqcap g)$
using *1 kruskal-invariant-def kruskal-spanning-invariant-def sum-symmetric* **by** *simp*
also have ... $= sum\ (((w \sqcap -(top * ?e * w^{T\star})) \sqcup ?p) \sqcap g)$
using *inf-commute inf-left-commute sum-disjoint* **by** *simp*
also have ... $= sum\ (w \sqcap g)$
using *3 7 maddux-3-11-pp* **by** *simp*
finally show *?thesis*
by *simp*
qed
have *12*: $?v \sqcup ?v^T = w \sqcup w^T$
proof −
have $?v \sqcup ?v^T = (w \sqcap -?p) \sqcup ?p^T \sqcup (w^T \sqcap -?p^T) \sqcup ?p$
using *conv-complement conv-dist-inf conv-dist-sup inf-import-p sup-assoc* **by** *simp*
also have ... $= w \sqcup w^T$
using *3 7 conv-complement conv-dist-inf inf-import-p maddux-3-11-pp sup-monoid.add-assoc sup-monoid.add-commute* **by** *simp*
finally show *?thesis*
by *simp*
qed
have *13*: $?v * ?e^T = bot$
apply (*rule kruskal-reroot-edge*)
using *1* **apply** (*simp add*: *minarc-arc*)
using *2 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
have $?v \sqcap ?e \leq ?v \sqcap top * ?e$
using *inf.sup-right-isotone top-left-mult-increasing* **by** *simp*
also have ... $\leq ?v * (top * ?e)^T$
using *covector-restrict-comp-conv covector-mult-closed vector-top-closed* **by** *simp*
finally have *14*: $?v \sqcap ?e = bot$
using *13* **by** (*metis conv-dist-comp mult-assoc le-bot mult-left-zero*)
let $?d = ?v \sqcap top * ?e^T * ?v^{T\star} \sqcap ?F * ?e^T * top \sqcap top * ?e * -?F$
let $?w = (?v \sqcap -?d) \sqcup ?e$
have *15*: *regular ?d*
using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
have *16*: $?F \leq -?d$
apply (*rule kruskal-edge-between-components-1*)

using *5* **apply** *simp*
using *1 conv-dist-comp minarc-arc mult-assoc* **by** *simp*
have *17*: $f \sqcup f^T \leq (?v \sqcap -?d \sqcap -?d^T) \sqcup (?v^T \sqcap -?d \sqcap -?d^T)$
apply (*rule kruskal-edge-between-components-2*)
using *16* **apply** *simp*
using *1 kruskal-invariant-def kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
using *2 12* **by** (*metis conv-dist-sup conv-involutive conv-isotone le-supI sup-commute*)
show *minimum-spanning-forest ?w g* $\wedge$ $?f \leq ?w \sqcup ?w^T$
proof (*intro conjI*)
have *18*: $?e^T \leq ?v^\star$
apply (*rule kruskal-edge-arc-1*[**where** *g*=*g* **and** *h*=*h*])
using *minarc-below* **apply** *simp*
using *1* **apply** (*metis kruskal-invariant-def kruskal-spanning-invariant-def inf-le1*)
using *1 kruskal-invariant-def kruskal-spanning-invariant-def* **apply** *simp*
using *9* **apply** *simp*
using *13* **by** *simp*
have *19*: *arc ?d*
apply (*rule kruskal-edge-arc*)
using *5* **apply** *simp*
using *10 spanning-forest-def* **apply** *blast*
using *1* **apply** (*simp add*: *minarc-arc*)
using *3* **apply** (*metis conv-complement pp-dist-star regular-mult-closed*)
using *2 8 12* **apply** (*simp add*: *kruskal-forest-components-inf*)
using *10 spanning-forest-def* **apply** *simp*
using *13* **apply** *simp*
using *6* **apply** *simp*
using *18* **by** *simp*
show *minimum-spanning-forest ?w g*
proof (*unfold minimum-spanning-forest-def*, *intro conjI*)
have $(?v \sqcap -?d) * ?e^T \leq ?v * ?e^T$
using *inf-le1 mult-left-isotone* **by** *simp*
hence $(?v \sqcap -?d) * ?e^T = bot$
using *13 le-bot* **by** *simp*
hence *20*: $?e * (?v \sqcap -?d)^T = bot$
using *conv-dist-comp conv-involutive conv-bot* **by** *force*
have *21*: *injective ?w*
apply (*rule injective-sup*)
using *8* **apply** (*simp add*: *injective-inf-closed*)
using *20* **apply** *simp*
using *1 arc-injective minarc-arc* **by** *blast*
show *spanning-forest ?w g*
proof (*unfold spanning-forest-def*, *intro conjI*)
show *injective ?w*
using *21* **by** *simp*

next
show *acyclic ?w*
apply (*rule kruskal-exchange-acyclic-inv-2*)
using *10 spanning-forest-def* **apply** *blast*
using *8* **apply** *simp*
using *inf.coboundedI1* **apply** *simp*
using *19* **apply** *simp*
using *1* **apply** (*simp add*: *minarc-arc*)
using *inf.cobounded2 inf.coboundedI1* **apply** *simp*
using *13* **by** *simp*
next
have $?w \leq ?v \sqcup ?e$
using *inf-le1 sup-left-isotone* **by** *simp*
also have ... $\leq --g \sqcup ?e$
using *10 sup-left-isotone spanning-forest-def* **by** *blast*
also have ... $\leq --g \sqcup --h$
by (*simp add*: *le-supI2 minarc-below*)
also have ... $= --g$
using *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def pp-isotone-inf sup.orderE*)
finally show $?w \leq --g$
by *simp*
next
have *22*: $?d \leq (?v \sqcap -?d)^{T\star} * ?e^T * top$
apply (*rule kruskal-exchange-spanning-inv-2*)
using *8* **apply** *simp*
using *13* **apply** (*metis semiring.mult-not-zero star-absorb star-simulation-right-equal*)
using *17* **apply** *simp*
by (*simp add*: *inf.coboundedI1*)
have *components g* $\leq$ *forest-components ?v*
using *10 spanning-forest-def* **by** *auto*
also have ... $\leq$ *forest-components ?w*
apply (*rule kruskal-exchange-forest-components-inv*)
using *21* **apply** *simp*
using *15* **apply** *simp*
using *1* **apply** (*simp add*: *arc-top-arc minarc-arc*)
apply (*simp add*: *inf.coboundedI1*)
using *13* **apply** *simp*
using *8* **apply** *simp*
apply (*simp add*: *le-infI1*)
using *22* **by** *simp*
finally show *components g* $\leq$ *forest-components ?w*
by *simp*
next
show *regular ?w*
using *3 7 regular-conv-closed* **by** *simp*
qed
next

have *23*: $?e \sqcap g \neq bot$
using *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def comp-inf.semiring.mult-zero-right inf.sup-monoid.add-assoc inf.sup-monoid.add-commute minarc-bot-iff minarc-meet-bot*)
have $g \sqcap -h \leq (g \sqcap -h)^\star$
using *star.circ-increasing* **by** *simp*
also have ... $\leq (--(g \sqcap -h))^\star$
using *pp-increasing star-isotone* **by** *blast*
also have ... $\leq ?F$
using *1 kruskal-invariant-def kruskal-spanning-invariant-def inf.sup-monoid.add-commute spanning-forest-def* **by** *simp*
finally have *24*: $g \sqcap -h \leq ?F$
by *simp*
have $?d \leq --g$
using *10 inf.coboundedI1 spanning-forest-def* **by** *blast*
hence $?d \leq --g \sqcap -?F$
using *16 inf.boundedI p-antitone-iff* **by** *simp*
also have ... $= --(g \sqcap -?F)$
by *simp*
also have ... $\leq --h$
using *24 p-shunting-swap pp-isotone* **by** *fastforce*
finally have *25*: $?d \leq --h$
by *simp*
have $?d = bot \longrightarrow top = bot$
using *19* **by** (*metis mult-left-zero mult-right-zero*)
hence $?d \neq bot$
using *1 le-bot* **by** *auto*
hence *26*: $?d \sqcap h \neq bot$
using *25* **by** (*metis inf.absorb-iff2 inf-commute pseudo-complement*)
have *sum* $(?e \sqcap g) =$ *sum* $(?e \sqcap --h \sqcap g)$
by (*simp add*: *inf.absorb1 minarc-below*)
also have ... $=$ *sum* $(?e \sqcap h)$
using *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def inf.left-commute inf.sup-monoid.add-commute*)
also have ... $\leq$ *sum* $(?d \sqcap h)$
using *19 26 minarc-min* **by** *simp*
also have ... $=$ *sum* $(?d \sqcap (--h \sqcap g))$
using *1 kruskal-invariant-def kruskal-spanning-invariant-def inf-commute* **by** *simp*
also have ... $=$ *sum* $(?d \sqcap g)$
using *25* **by** (*simp add*: *inf.absorb2 inf-assoc inf-commute*)
finally have *27*: *sum* $(?e \sqcap g) \leq$ *sum* $(?d \sqcap g)$
by *simp*
have $?v \sqcap ?e \sqcap -?d = bot$
using *14* **by** *simp*
hence *sum* $(?w \sqcap g) =$ *sum* $(?v \sqcap -?d \sqcap g) +$ *sum* $(?e \sqcap g)$
using *sum-disjoint inf-commute inf-assoc* **by** *simp*
also have ... $\leq$ *sum* $(?v \sqcap -?d \sqcap g) +$ *sum* $(?d \sqcap g)$

using *23 27 sum-plus-right-isotone* **by** *simp*
also have ... $=$ *sum* $(((?v \sqcap -?d) \sqcup ?d) \sqcap g)$
using *sum-disjoint inf-le2 pseudo-complement* **by** *simp*
also have ... $=$ *sum* $((?v \sqcup ?d) \sqcap (-?d \sqcup ?d) \sqcap g)$
by (*simp add*: *sup-inf-distrib2*)
also have ... $=$ *sum* $((?v \sqcup ?d) \sqcap g)$
using *15* **by** (*metis inf-top-right stone*)
also have ... $=$ *sum* $(?v \sqcap g)$
by (*simp add*: *inf.sup-monoid.add-assoc*)
finally have *sum* $(?w \sqcap g) \leq$ *sum* $(?v \sqcap g)$
by *simp*
thus $\forall u$. *spanning-forest u g* $\longrightarrow$ *sum* $(?w \sqcap g) \leq$ *sum* $(u \sqcap g)$
using *2 11 minimum-spanning-forest-def* **by** *auto*
qed
next
have $?f \leq f \sqcup f^T \sqcup ?e$
using *conv-dist-inf inf-le1 sup-left-isotone sup-mono* **by** *presburger*
also have ... $\leq (?v \sqcap -?d \sqcap -?d^T) \sqcup (?v^T \sqcap -?d \sqcap -?d^T) \sqcup ?e$
using *17 sup-left-isotone* **by** *simp*
also have ... $\leq (?v \sqcap -?d) \sqcup (?v^T \sqcap -?d \sqcap -?d^T) \sqcup ?e$
using *inf.cobounded1 sup-inf-distrib2* **by** *presburger*
also have ... $= ?w \sqcup (?v^T \sqcap -?d \sqcap -?d^T)$
by (*simp add*: *sup-assoc sup-commute*)
also have ... $\leq ?w \sqcup (?v^T \sqcap -?d^T)$
using *inf.sup-right-isotone inf-assoc sup-right-isotone* **by** *simp*
also have ... $\leq ?w \sqcup ?w^T$
using *conv-complement conv-dist-inf conv-dist-sup sup-right-isotone*
by *simp*
finally show $?f \leq ?w \sqcup ?w^T$
by *simp*
qed
qed
next
show $?n2 < ?n1$
using *1 kruskal-vc-2 kruskal-invariant-def* **by** *auto*
qed
qed
next
show $\neg\ ?e \leq -?F \longrightarrow$ *kruskal-invariant f g ?h* $\wedge\ ?n2 < ?n1$
using *1 kruskal-vc-2 kruskal-invariant-def* **by** *auto*
qed
next
fix *f*
assume *28*: *kruskal-invariant f g bot*
hence *29*: *spanning-forest f g*
using *kruskal-invariant-def kruskal-spanning-invariant-def* **by** *auto*
from *28* **obtain** *w* **where** *30*: *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$
using *kruskal-invariant-def* **by** *auto*
hence $w = w \sqcap --g$

by (*simp add*: *inf.absorb1 minimum-spanning-forest-def spanning-forest-def*)
also have ... $\leq w \sqcap$ *components g*
by (*metis inf.sup-right-isotone star.circ-increasing*)
also have ... $\leq w \sqcap f^{T\star} * f^\star$
using *29 spanning-forest-def inf.sup-right-isotone* **by** *simp*
also have ... $\leq f \sqcup f^T$
apply (*rule cancel-separate-6*[**where** z=w **and** y=w^T])
using *30 minimum-spanning-forest-def spanning-forest-def* **apply** *simp*
using *30* **apply** (*metis conv-dist-inf conv-dist-sup conv-involutive inf.cobounded2 inf.orderE*)
using *30* **apply** (*simp add*: *sup-commute*)
using *30 minimum-spanning-forest-def spanning-forest-def* **apply** *simp*
using *30* **by** (*metis acyclic-star-below-complement comp-inf.mult-right-isotone inf-p le-bot minimum-spanning-forest-def spanning-forest-def*)
finally have *31*: $w \leq f \sqcup f^T$
by *simp*
have *sum* $(f \sqcap g)$ = *sum* $((w \sqcup w^T) \sqcap (f \sqcap g))$
using *30* **by** (*metis inf-absorb2 inf.assoc*)
also have ... = *sum* $(w \sqcap (f \sqcap g))$ + *sum* $(w^T \sqcap (f \sqcap g))$
using *30 inf.commute acyclic-asymmetric sum-disjoint minimum-spanning-forest-def spanning-forest-def* **by** *simp*
also have ... = *sum* $(w \sqcap (f \sqcap g))$ + *sum* $(w \sqcap (f^T \sqcap g^T))$
by (*metis conv-dist-inf conv-involutive sum-conv*)
also have ... = *sum* $(f \sqcap (w \sqcap g))$ + *sum* $(f^T \sqcap (w \sqcap g))$
using *28 inf.commute inf.assoc kruskal-invariant-def kruskal-spanning-invariant-def* **by** *simp*
also have ... = *sum* $((f \sqcup f^T) \sqcap (w \sqcap g))$
using *29 acyclic-asymmetric inf.sup-monoid.add-commute sum-disjoint spanning-forest-def* **by** *simp*
also have ... = *sum* $(w \sqcap g)$
using *31* **by** (*metis inf-absorb2 inf.assoc*)
finally show *minimum-spanning-forest f g*
using *29 30 minimum-spanning-forest-def* **by** *simp*
qed

end

end

3 Prim's Minimum Spanning Tree Algorithm

In this theory we prove total correctness of Prim's minimum spanning tree algorithm. The proof has the same overall structure as the total-correctness proof of Kruskal's algorithm [6]. The partial-correctness proof of Prim's algorithm is discussed in [3, 5].

theory *Prim*

imports *HOL−Hoare.Hoare-Logic Aggregation-Algebras.Aggregation-Algebras*

begin

context *m-kleene-algebra*
begin

abbreviation *component* $g\ r \equiv r^T * (--g)^\star$
definition *spanning-tree* $t\ g\ r \equiv$ *forest* $t \wedge t \leq$ (*component* $g\ r)^T *$ (*component* g $r) \sqcap --g \wedge$ *component* $g\ r \leq r^T * t^\star \wedge$ *regular* t
definition *minimum-spanning-tree* $t\ g\ r \equiv$ *spanning-tree* $t\ g\ r \wedge (\forall u\ .$ *spanning-tree* $u\ g\ r \longrightarrow$ *sum* $(t \sqcap g) \leq$ *sum* $(u \sqcap g))$
definition *prim-precondition* $g\ r \equiv g = g^T \wedge$ *injective* $r \wedge$ *vector* $r \wedge$ *regular* r
definition *prim-spanning-invariant* $t\ v\ g\ r \equiv$ *prim-precondition* $g\ r \wedge v^T = r^T * t^\star \wedge$ *spanning-tree* $t\ (v * v^T \sqcap g)\ r$
definition *prim-invariant* $t\ v\ g\ r \equiv$ *prim-spanning-invariant* $t\ v\ g\ r \wedge (\exists w\ .$ *minimum-spanning-tree* $w\ g\ r \wedge t \leq w)$

lemma *span-tree-split*:
  assumes *vector* r
    shows $t \leq$ (*component* $g\ r)^T *$ (*component* $g\ r) \sqcap --g \longleftrightarrow (t \leq$ (*component* $g\ r)^T \wedge t \leq$ *component* $g\ r \wedge t \leq --g)$
proof –
  have (*component* $g\ r)^T *$ (*component* $g\ r) =$ (*component* $g\ r)^T \sqcap$ *component* $g\ r$
    by (*metis assms conv-involutive covector-mult-closed vector-conv-covector vector-covector*)
  thus *?thesis*
    by *simp*
qed

lemma *span-tree-component*:
  assumes *spanning-tree* $t\ g\ r$
    shows *component* $g\ r =$ *component* $t\ r$
  using *assms* **by** (*simp add*: *order.antisym mult-right-isotone star-isotone spanning-tree-def*)

We first show three verification conditions which are used in both correctness proofs.

lemma *prim-vc-1*:
  assumes *prim-precondition* $g\ r$
    shows *prim-spanning-invariant bot* $r\ g\ r$
proof (*unfold prim-spanning-invariant-def*, *intro conjI*)
  show *prim-precondition* $g\ r$
    using *assms* **by** *simp*
next
  show $r^T = r^T * bot^\star$
    by (*simp add*: *star-absorb*)
next
  let $?ss = r * r^T \sqcap g$
  show *spanning-tree bot* $?ss\ r$

proof (*unfold spanning-tree-def*, *intro conjI*)
show *injective bot*
by *simp*
next
show *acyclic bot*
by *simp*
next
show $bot \leq (component\ ?ss\ r)^T * (component\ ?ss\ r) \sqcap --?ss$
by *simp*
next
have $component\ ?ss\ r \leq component\ (r * r^T)\ r$
by (*simp add*: *mult-right-isotone star-isotone*)
also have $... \leq r^T * 1^\star$
using *assms* **by** (*metis order.eq-iff p-antitone regular-one-closed star-sub-one prim-precondition-def*)
also have $... = r^T * bot^\star$
by (*simp add*: *star.circ-zero star-one*)
finally show $component\ ?ss\ r \leq r^T * bot^\star$
.
next
show *regular bot*
by *simp*
qed
qed

lemma *prim-vc-2*:
assumes *prim-spanning-invariant t v g r*
and $v * -v^T \sqcap g \neq bot$
shows $prim\text{-}spanning\text{-}invariant\ (t \sqcup minarc\ (v * -v^T \sqcap g))\ (v \sqcup minarc\ (v * -v^T \sqcap g)^T * top)\ g\ r \wedge card\ \{\ x\ .\ regular\ x \wedge x \leq component\ g\ r \wedge x \leq -(v \sqcup minarc\ (v * -v^T \sqcap g)^T * top)^T\ \} < card\ \{\ x\ .\ regular\ x \wedge x \leq component\ g\ r \wedge x \leq -v^T\ \}$
proof −
let $?vcv = v * -v^T \sqcap g$
let $?e = minarc\ ?vcv$
let $?t = t \sqcup ?e$
let $?v = v \sqcup ?e^T * top$
let $?c = component\ g\ r$
let $?g = --g$
let $?n1 = card\ \{\ x\ .\ regular\ x \wedge x \leq ?c \wedge x \leq -v^T\ \}$
let $?n2 = card\ \{\ x\ .\ regular\ x \wedge x \leq ?c \wedge x \leq -?v^T\ \}$
have *1*: $regular\ v \wedge regular\ (v * v^T) \wedge regular\ (?v * ?v^T) \wedge regular\ (top * ?e)$
using *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive regular-closed-top regular-closed-sup minarc-regular*)
hence *2*: $t \leq v * v^T \sqcap ?g$
using *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE*)
hence *3*: $t \leq v * v^T$

by *simp*
have *4*: $t \leq ?g$
using *2* **by** *simp*
have *5*: $?e \leq v * -v^T \sqcap ?g$
using *1* **by** (*metis minarc-below pp-dist-inf regular-mult-closed regular-closed-p*)
hence *6*: $?e \leq v * -v^T$
by *simp*
have *7*: *vector v*
using *assms*(*1*) *prim-spanning-invariant-def prim-precondition-def* **by** (*simp add*: *covector-mult-closed vector-conv-covector*)
hence *8*: $?e \leq v$
using *6* **by** (*metis conv-complement inf.boundedE vector-complement-closed vector-covector*)
have *9*: $?e * t = bot$
using *3 6 7 et*(*1*) **by** *blast*
have *10*: $?e * t^T = bot$
using *3 6 7 et*(*2*) **by** *simp*
have *11*: *arc ?e*
using *assms*(*2*) *minarc-arc* **by** *simp*
have $r^T \leq r^T * t^\star$
by (*metis mult-right-isotone order-refl semiring.mult-not-zero star.circ-separate-mult-1 star-absorb*)
hence *12*: $r^T \leq v^T$
using *assms*(*1*) **by** (*simp add*: *prim-spanning-invariant-def*)
have *13*: *vector r* $\wedge$ *injective r* $\wedge$ $v^T = r^T * t^\star$
using *assms*(*1*) *prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def reachable-restrict* **by** *simp*
have $g = g^T$
using *assms*(*1*) *prim-invariant-def prim-spanning-invariant-def prim-precondition-def* **by** *simp*
hence *14*: $?g^T = ?g$
using *conv-complement* **by** *simp*
show *prim-spanning-invariant ?t ?v g r* $\wedge$ $?n2 < ?n1$
proof (*rule conjI*)
show *prim-spanning-invariant ?t ?v g r*
proof (*unfold prim-spanning-invariant-def*, *intro conjI*)
show *prim-precondition g r*
using *assms*(*1*) *prim-spanning-invariant-def* **by** *simp*
next
show $?v^T = r^T * ?t^\star$
using *assms*(*1*) *6 7 9* **by** (*simp add*: *reachable-inv prim-spanning-invariant-def prim-precondition-def spanning-tree-def*)
next
let $?G = ?v * ?v^T \sqcap g$
show *spanning-tree ?t ?G r*
proof (*unfold spanning-tree-def*, *intro conjI*)
show *injective ?t*
using *assms*(*1*) *10 11* **by** (*simp add*: *injective-inv*

prim-spanning-invariant-def spanning-tree-def)
  next
  show *acyclic ?t*
  using *assms*(*1*) *3 6 7 acyclic-inv prim-spanning-invariant-def spanning-tree-def* **by** *simp*
  next
  show $?t \leq (component\ ?G\ r)^T * (component\ ?G\ r) \sqcap --?G$
  using *1 2 5 7 13 prim-subgraph-inv inf-pp-commute mst-subgraph-inv-2* **by** *auto*
  next
  show $component\ (?v * ?v^T \sqcap g)\ r \leq r^T * ?t^\star$
  proof −
  have *15*: $r^T * (v * v^T \sqcap ?g)^\star \leq r^T * t^\star$
  using *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def spanning-tree-def inf-pp-commute*)
  have $component\ (?v * ?v^T \sqcap g)\ r = r^T * (?v * ?v^T \sqcap ?g)^\star$
  using *1* **by** *simp*
  also have ... $\leq r^T * ?t^\star$
  using *2 6 7 11 12 13 14 15* **by** (*metis span-inv*)
  finally show *?thesis*
  .
  qed
  next
  show *regular ?t*
  using *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def regular-closed-sup minarc-regular*)
  qed
  qed
  next
  have *16*: $top * ?e \leq ?c$
  proof −
  have $top * ?e = top * ?e^T * ?e$
  using *11* **by** (*metis arc-top-edge mult-assoc*)
  also have ... $\leq v^T * ?e$
  using *7 8* **by** (*metis conv-dist-comp conv-isotone mult-left-isotone symmetric-top-closed*)
  also have ... $\leq v^T * ?g$
  using *5 mult-right-isotone* **by** *auto*
  also have ... $= r^T * t^\star * ?g$
  using *13* **by** *simp*
  also have ... $\leq r^T * ?g^\star * ?g$
  using *4* **by** (*simp add*: *mult-left-isotone mult-right-isotone star-isotone*)
  also have ... $\leq ?c$
  by (*simp add*: *comp-associative mult-right-isotone star.right-plus-below-circ*)
  finally show *?thesis*
  by *simp*
  qed
  have *17*: $top * ?e \leq -v^T$
  using *6 7* **by** (*simp add*: *schroeder-4-p vTeT*)

have *18*: $\neg$ *top* $*$ *?e* $\leq -($*top* $*$ *?e*$)$
by (*metis assms*(*2*) *inf.orderE minarc-bot-iff conv-complement-sub-inf inf-p inf-top.left-neutral p-bot symmetric-top-closed vector-top-closed*)
have *19*: $-?v^T = -v^T \sqcap -($*top* $*$ *?e*$)$
by (*simp add*: *conv-dist-comp conv-dist-sup*)
hence *20*: $\neg$ *top* $*$ *?e* $\leq -?v^T$
using *18* **by** *simp*
show *?n2* $<$ *?n1*
apply (*rule psubset-card-mono*)
using *finite-regular* **apply** *simp*
using *1 16 17 19 20* **by** *auto*
qed
qed

lemma *prim-vc-3*:
assumes *prim-spanning-invariant t v g r*
and $v * -v^T \sqcap g = bot$
shows *spanning-tree t g r*
proof $-$
let $?g = --g$
have *1*: *regular v* $\wedge$ *regular* $(v * v^T)$
using *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
have *2*: $v * -v^T \sqcap ?g = bot$
using *assms*(*2*) *pp-inf-bot-iff pp-pp-inf-bot-iff* **by** *simp*
have *3*: $v^T = r^T * t^\star \wedge$ *vector v*
using *assms*(*1*) **by** (*simp add*: *covector-mult-closed prim-invariant-def prim-spanning-invariant-def vector-conv-covector prim-precondition-def*)
have *4*: $t \leq v * v^T \sqcap ?g$
using *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def inf-pp-commute spanning-tree-def inf.boundedE*)
have $r^T * (v * v^T \sqcap ?g)^\star \leq r^T * t^\star$
using *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def inf-pp-commute spanning-tree-def*)
hence *5*: *component g r* $= v^T$
using *1 2 3 4* **by** (*metis span-post*)
have *regular* $(v * v^T)$
using *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
hence *6*: $t \leq v * v^T \sqcap ?g$
by (*metis assms*(*1*) *prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE*)
show *spanning-tree t g r*
apply (*unfold spanning-tree-def*, *intro conjI*)
using *assms*(*1*) *prim-spanning-invariant-def spanning-tree-def* **apply** *simp*
using *assms*(*1*) *prim-spanning-invariant-def spanning-tree-def* **apply** *simp*
using *5 6* **apply** *simp*

using *assms*(*1*) *5* *prim-spanning-invariant-def* **apply** *simp*
using *assms*(*1*) *prim-spanning-invariant-def* *spanning-tree-def* **by** *simp*
qed

The following result shows that Prim's algorithm terminates and constructs a spanning tree. We cannot yet show that this is a minimum spanning tree.

theorem *prim-spanning*:
VARS $t\ v\ e$
[*prim-precondition* $g\ r$]
$t := bot;$
$v := r;$
WHILE $v * -v^T \sqcap g \neq bot$
INV { *prim-spanning-invariant* $t\ v\ g\ r$ }
VAR { *card* { x . *regular* $x \wedge x \leq$ *component* $g\ r \sqcap -v^T$ } }
DO $e :=$ *minarc* $(v * -v^T \sqcap g);$
$t := t \sqcup e;$
$v := v \sqcup e^T * top$
OD
[*spanning-tree* $t\ g\ r$]
apply *vcg-tc-simp*
apply (*simp add*: *prim-vc-1*)
using *prim-vc-2* **apply** *blast*
using *prim-vc-3* **by** *auto*

Because we have shown total correctness, we conclude that a spanning tree exists.

lemma *prim-exists-spanning*:
prim-precondition $g\ r \Longrightarrow \exists t$. *spanning-tree* $t\ g\ r$
using *tc-extract-function* *prim-spanning* **by** *blast*

This implies that a minimum spanning tree exists, which is used in the subsequent correctness proof.

lemma *prim-exists-minimal-spanning*:
assumes *prim-precondition* $g\ r$
shows $\exists t$. *minimum-spanning-tree* $t\ g\ r$
proof –
let *?s* = { t . *spanning-tree* $t\ g\ r$ }
have $\exists m \in ?s$. $\forall z \in ?s$. *sum* $(m \sqcap g) \leq$ *sum* $(z \sqcap g)$
apply (*rule finite-set-minimal*)
using *finite-regular* *spanning-tree-def* **apply** *simp*
using *assms* *prim-exists-spanning* **apply** *simp*
using *sum-linear* **by** *simp*
thus *?thesis*
using *minimum-spanning-tree-def* **by** *simp*
qed

Prim's minimum spanning tree algorithm terminates and is correct. This is the same algorithm that is used in the previous correctness proof, with the same precondition and variant, but with a different invariant and post-condition.

theorem *prim*:
  VARS t v e
  [*prim-precondition g r* $\wedge$ ($\exists$ *w* . *minimum-spanning-tree w g r*)]
  $t := bot$;
  $v := r$;
  WHILE $v * -v^T \sqcap g \neq bot$
    INV { *prim-invariant t v g r* }
    VAR { *card* { x . *regular* $x \wedge x \leq$ *component g r* $\sqcap -v^T$ } }
     DO $e :=$ *minarc* $(v * -v^T \sqcap g)$;
        $t := t \sqcup e$;
        $v := v \sqcup e^T * top$
     OD
  [*minimum-spanning-tree t g r*]
proof *vcg-tc-simp*
  assume *prim-precondition g r* $\wedge$ ($\exists$ *w* . *minimum-spanning-tree w g r*)
  thus *prim-invariant bot r g r*
    using *prim-invariant-def prim-vc-1* **by** *simp*
next
  fix *t v*
  let $?vcv = v * -v^T \sqcap g$
  let $?vv = v * v^T \sqcap g$
  let $?e =$ *minarc* $?vcv$
  let $?t = t \sqcup ?e$
  let $?v = v \sqcup ?e^T * top$
  let $?c =$ *component g r*
  let $?g = --g$
  let $?n1 =$ *card* { x . *regular* $x \wedge x \leq ?c \wedge x \leq -v^T$ }
  let $?n2 =$ *card* { x . *regular* $x \wedge x \leq ?c \wedge x \leq -?v^T$ }
  assume *1*: *prim-invariant t v g r* $\wedge$ $?vcv \neq bot$
  hence *2*: *regular* $v \wedge$ *regular* $(v * v^T)$
    by (*metis* (*no-types*, *opaque-lifting*) *prim-invariant-def*
prim-spanning-invariant-def spanning-tree-def prim-precondition-def
regular-conv-closed regular-closed-star regular-mult-closed conv-involutive)
  have *3*: $t \leq v * v^T \sqcap ?g$
    using *1 2* **by** (*metis* (*no-types*, *opaque-lifting*) *prim-invariant-def*
prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE)
  hence *4*: $t \leq v * v^T$
    by *simp*
  have *5*: $t \leq ?g$
    using *3* **by** *simp*
  have *6*: $?e \leq v * -v^T \sqcap ?g$
    using *2* **by** (*metis minarc-below pp-dist-inf regular-mult-closed*
regular-closed-p)
  hence *7*: $?e \leq v * -v^T$
    by *simp*
  have *8*: *vector v*
    using *1 prim-invariant-def prim-spanning-invariant-def prim-precondition-def*

by (*simp add*: *covector-mult-closed vector-conv-covector*)
have *9*: *arc ?e*
using *1 minarc-arc* **by** *simp*
from *1* **obtain** *w* **where** *10*: *minimum-spanning-tree w g r* $\wedge$ $t \leq w$
by (*metis prim-invariant-def*)
hence *11*: *vector r* $\wedge$ *injective r* $\wedge$ $v^T = r^T * t^\star$ $\wedge$ *forest w* $\wedge$ $t \leq w \wedge w \leq ?c^T * ?c \sqcap ?g \wedge r^T * (?c^T * ?c \sqcap ?g)^\star \leq r^T * w^\star$
using *1 2 prim-invariant-def prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def reachable-restrict* **by** *simp*
hence *12*: $w * v \leq v$
using *predecessors-reachable reachable-restrict* **by** *auto*
have *13*: $g = g^T$
using *1 prim-invariant-def prim-spanning-invariant-def prim-precondition-def* **by** *simp*
hence *14*: $?g^T = ?g$
using *conv-complement* **by** *simp*
show *prim-invariant ?t ?v g r* $\wedge$ $?n2 < ?n1$
proof (*unfold prim-invariant-def*, *intro conjI*)
show *prim-spanning-invariant ?t ?v g r*
using *1 prim-invariant-def prim-vc-2* **by** *blast*
next
show $\exists\, w$. *minimum-spanning-tree w g r* $\wedge$ $?t \leq w$
proof
let $?f = w \sqcap v * -v^T \sqcap top * ?e * w^{T\star}$
let $?p = w \sqcap -v * -v^T \sqcap top * ?e * w^{T\star}$
let $?fp = w \sqcap -v^T \sqcap top * ?e * w^{T\star}$
let $?w = (w \sqcap -?fp) \sqcup ?p^T \sqcup ?e$
have *15*: *regular ?f* $\wedge$ *regular ?fp* $\wedge$ *regular ?w*
using *2 10* **by** (*metis regular-conv-closed regular-closed-star regular-mult-closed regular-closed-top regular-closed-inf regular-closed-sup minarc-regular minimum-spanning-tree-def spanning-tree-def*)
show *minimum-spanning-tree ?w g r* $\wedge$ $?t \leq ?w$
proof (*intro conjI*)
show *minimum-spanning-tree ?w g r*
proof (*unfold minimum-spanning-tree-def*, *intro conjI*)
show *spanning-tree ?w g r*
proof (*unfold spanning-tree-def*, *intro conjI*)
show *injective ?w*
using *7 8 9 11 exchange-injective* **by** *blast*
next
show *acyclic ?w*
using *7 8 11 12 exchange-acyclic* **by** *blast*
next
show $?w \leq ?c^T * ?c \sqcap --g$
proof $-$
have *16*: $w \sqcap -?fp \leq ?c^T * ?c \sqcap --g$
using *10* **by** (*simp add*: *le-infI1 minimum-spanning-tree-def spanning-tree-def*)

have $?p^T \leq w^T$
by (*simp add*: *conv-isotone inf.sup-monoid.add-assoc*)
also have ... $\leq (?c^T * ?c \sqcap --g)^T$
using *11 conv-order* **by** *simp*
also have ... $= ?c^T * ?c \sqcap --g$
using *2 14 conv-dist-comp conv-dist-inf* **by** *simp*
finally have *17*: $?p^T \leq ?c^T * ?c \sqcap --g$
.
have $?e \leq ?c^T * ?c \sqcap ?g$
using *5 6 11 mst-subgraph-inv* **by** *auto*
thus *?thesis*
using *16 17* **by** *simp*
qed
next
show $?c \leq r^T * ?w^\star$
proof $-$
have $?c \leq r^T * w^\star$
using *10 minimum-spanning-tree-def spanning-tree-def* **by** *simp*
also have ... $\leq r^T * ?w^\star$
using *4 7 8 10 11 12 15* **by** (*metis mst-reachable-inv*)
finally show *?thesis*
.
qed
next
show *regular ?w*
using *15* **by** *simp*
qed
next
have *18*: $?f \sqcup ?p = ?fp$
using *2 8 epm-1* **by** *fastforce*
have *arc* $(w \sqcap --v * -v^T \sqcap top * ?e * w^{T\star})$
using *5 6 8 9 11 12 reachable-restrict arc-edge* **by** *auto*
hence *19*: *arc ?f*
using *2* **by** *simp*
hence $?f = bot \longrightarrow top = bot$
by (*metis mult-left-zero mult-right-zero*)
hence $?f \neq bot$
using *1 le-bot* **by** *auto*
hence $?f \sqcap v * -v^T \sqcap ?g \neq bot$
using *2 11* **by** (*simp add*: *inf.absorb1 le-infI1*)
hence $g \sqcap (?f \sqcap v * -v^T) \neq bot$
using *inf-commute pp-inf-bot-iff* **by** *simp*
hence *20*: $?f \sqcap ?vcv \neq bot$
by (*simp add*: *inf-assoc inf-commute*)
hence *21*: $?f \sqcap g = ?f \sqcap ?vcv$
using *2* **by** (*simp add*: *inf-assoc inf-commute inf-left-commute*)
have *22*: $?e \sqcap g = minarc\ ?vcv \sqcap ?vcv$
using *7* **by** (*simp add*: *inf.absorb2 inf.assoc inf.commute*)
hence *23*: $sum\ (?e \sqcap g) \leq sum\ (?f \sqcap g)$

using *15 19 20 21* **by** (*simp add*: *minarc-min*)
have *?e* $\neq$ *bot*
using *20 comp-inf.semiring.mult-not-zero semiring.mult-not-zero* **by** *blast*
hence *24*: *?e* $\sqcap$ *g* $\neq$ *bot*
using *22 minarc-meet-bot* **by** *auto*
have *sum* (*?w* $\sqcap$ *g*) = *sum* (*w* $\sqcap$ $-$*?fp* $\sqcap$ *g*) + *sum* (*?p*^{T} $\sqcap$ *g*) + *sum* (*?e* $\sqcap$ *g*)
using *7 8 10* **by** (*metis sum-disjoint-3 epm-8 epm-9 epm-10 minimum-spanning-tree-def spanning-tree-def*)
also have ... = *sum* (((*w* $\sqcap$ $-$*?fp*) $\sqcup$ *?p*^{T}) $\sqcap$ *g*) + *sum* (*?e* $\sqcap$ *g*)
using *11* **by** (*metis epm-8 sum-disjoint*)
also have ... $\leq$ *sum* (((*w* $\sqcap$ $-$*?fp*) $\sqcup$ *?p*^{T}) $\sqcap$ *g*) + *sum* (*?f* $\sqcap$ *g*)
using *23 24* **by** (*simp add*: *sum-plus-right-isotone*)
also have ... = *sum* (*w* $\sqcap$ $-$*?fp* $\sqcap$ *g*) + *sum* (*?p*^{T} $\sqcap$ *g*) + *sum* (*?f* $\sqcap$ *g*)
using *11* **by** (*metis epm-8 sum-disjoint*)
also have ... = *sum* (*w* $\sqcap$ $-$*?fp* $\sqcap$ *g*) + *sum* (*?p* $\sqcap$ *g*) + *sum* (*?f* $\sqcap$ *g*)
using *13 sum-symmetric* **by** *auto*
also have ... = *sum* (((*w* $\sqcap$ $-$*?fp*) $\sqcup$ *?p* $\sqcup$ *?f*) $\sqcap$ *g*)
using *2 8* **by** (*metis sum-disjoint-3 epm-11 epm-12 epm-13*)
also have ... = *sum* (*w* $\sqcap$ *g*)
using *2 8 15 18 epm-2* **by** *force*
finally have *sum* (*?w* $\sqcap$ *g*) $\leq$ *sum* (*w* $\sqcap$ *g*)
.
thus $\forall$ *u* . *spanning-tree u g r* $\longrightarrow$ *sum* (*?w* $\sqcap$ *g*) $\leq$ *sum* (*u* $\sqcap$ *g*)
using *10 order-lesseq-imp minimum-spanning-tree-def* **by** *auto*
qed
next
show *?t* $\leq$ *?w*
using *4 8 10 mst-extends-new-tree* **by** *simp*
qed
qed
next
show *?n2* < *?n1*
using *1 prim-invariant-def prim-vc-2* **by** *auto*
qed
next
fix *t v*
let *?g* = $--$*g*
assume *25*: *prim-invariant t v g r* $\wedge$ *v* $*$ $-$*v*^{T} $\sqcap$ *g* = *bot*
hence *26*: *regular v*
by (*metis prim-invariant-def prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
from *25* **obtain** *w* **where** *27*: *minimum-spanning-tree w g r* $\wedge$ *t* $\leq$ *w*
by (*metis prim-invariant-def*)
have *spanning-tree t g r*
using *25 prim-invariant-def prim-vc-3* **by** *blast*
hence *component g r* = *v*^{T}

by (*metis 25 prim-invariant-def span-tree-component prim-spanning-invariant-def spanning-tree-def*)
hence *28*: $w \le v * v^T$
using *26 27* **by** (*simp add*: *minimum-spanning-tree-def spanning-tree-def inf-pp-commute*)
have *vector* $r \wedge$ *injective* $r \wedge$ *forest* w
using *25 27* **by** (*simp add*: *prim-invariant-def prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def*)
hence $w = t$
using *25 27 28 prim-invariant-def prim-spanning-invariant-def mst-post* **by** *blast*
thus *minimum-spanning-tree* $t\ g\ r$
using *27* **by** *simp*
qed

end

end

4 Borůvka's Minimum Spanning Tree Algorithm

In this theory we prove partial correctness of Borůvka's minimum spanning tree algorithm.

theory *Boruvka*

imports
Aggregation-Algebras.M-Choose-Component
Relational-Disjoint-Set-Forests.Disjoint-Set-Forests
Kruskal

begin

4.1 General results

The proof is carried out in m-k-Stone-Kleene relation algebras. In this section we give results that hold more generally.

context *stone-kleene-relation-algebra*
begin

lemma *He-eq-He-THe-star*:
assumes *arc* e
and *equivalence* H
shows $H * e = H * e * (top * H * e)^\star$
proof −
let $?x = H * e$
have *1*: $H * e \le H * e * (top * H * e)^\star$
using *comp-isotone star.circ-reflexive* **by** *fastforce*
have $H * e * (top * H * e)^\star = H * e * (top * e)^\star$

by (*metis assms*(*2*) *preorder-idempotent surjective-var*)
also have ... $\leq H * e * (1 \sqcup top * (e * top)^\star * e)$
by (*metis eq-refl star.circ-mult-1*)
also have ... $\leq H * e * (1 \sqcup top * top * e)$
by (*simp add*: *star.circ-left-top*)
also have ... $= H * e \sqcup H * e * top * e$
by (*simp add*: *mult.semigroup-axioms semiring.distrib-left semigroup.assoc*)
finally have *2*: $H * e * (top * H * e)^\star \leq H * e$
using *assms arc-top-arc mult-assoc* **by** *auto*
thus *?thesis*
using *1 2* **by** *simp*
qed

lemma *path-through-components*:
assumes *equivalence H*
and *arc e*
shows $(H * (d \sqcup e))^\star = (H * d)^\star \sqcup (H * d)^\star * H * e * (H * d)^\star$
proof −
have $H * e * (H * d)^\star * H * e \leq H * e * top * H * e$
by (*simp add*: *comp-isotone*)
also have ... $= H * e * top * e$
by (*metis assms*(*1*) *preorder-idempotent surjective-var mult-assoc*)
also have ... $= H * e$
using *assms*(*2*) *arc-top-arc mult-assoc* **by** *auto*
finally have *1*: $H * e * (H * d)^\star * H * e \leq H * e$
by *simp*
have $(H * (d \sqcup e))^\star = (H * d \sqcup H * e)^\star$
by (*simp add*: *comp-left-dist-sup*)
also have ... $= (H * d)^\star \sqcup (H * d)^\star * H * e * (H * d)^\star$
using *1 star-separate-3* **by** (*simp add*: *mult-assoc*)
finally show *?thesis*
by *simp*
qed

lemma *simplify-f*:
assumes *regular p*
and *regular e*
shows $(f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup e^T \sqcup e = f \sqcup f^T \sqcup e \sqcup e^T$
proof −
have $(f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup e^T \sqcup e$
$= (f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f^T \sqcap - e \sqcap p^T) \sqcup (f^T \sqcap - e \sqcap - p^T) \sqcup e^T \sqcup e$
by (*simp add*: *conv-complement conv-dist-inf*)
also have ... =
$((f \sqcup (f \sqcap - e^T \sqcap p)) \sqcap (- e^T \sqcup (f \sqcap - e^T \sqcap p)) \sqcap (- p \sqcup (f \sqcap - e^T \sqcap p)))$
$\sqcup ((f^T \sqcup (f^T \sqcap - e \sqcap - p^T)) \sqcap (- e \sqcup (f^T \sqcap - e \sqcap - p^T)) \sqcap (p^T \sqcup (f^T \sqcap - e \sqcap - p^T)))$

$\sqcup e^T \sqcup e$
by (*metis sup-inf-distrib2 sup-assoc*)
also have ... =
$((f \sqcup f) \sqcap (f \sqcup - e^T) \sqcap (f \sqcup p) \sqcap (- e^T \sqcup f) \sqcap (- e^T \sqcup - e^T) \sqcap (- e^T \sqcup p) \sqcap (- p \sqcup f) \sqcap (- p \sqcup - e^T) \sqcap (- p \sqcup p))$
$\sqcup ((f^T \sqcup f^T) \sqcap (f^T \sqcup - e) \sqcap (f^T \sqcup - p^T) \sqcap (- e \sqcup f^T) \sqcap (- e \sqcup - e) \sqcap (- e \sqcup - p^T) \sqcap (p^T \sqcup f^T) \sqcap (p^T \sqcup - e) \sqcap (p^T \sqcup - p^T))$
$\sqcup e^T \sqcup e$
using *sup-inf-distrib1 sup-assoc inf-assoc sup-inf-distrib1* **by** *simp*
also have ... =
$((f \sqcup f) \sqcap (f \sqcup - e^T) \sqcap (f \sqcup p) \sqcap (f \sqcup - p) \sqcap (- e^T \sqcup f) \sqcap (- e^T \sqcup - e^T) \sqcap (- e^T \sqcup p) \sqcap (- e^T \sqcup - p) \sqcap (- p \sqcup p))$
$\sqcup ((f^T \sqcup f^T) \sqcap (f^T \sqcup - e) \sqcap (f^T \sqcup - p^T) \sqcap (- e \sqcup f^T) \sqcap (f^T \sqcup p^T) \sqcap (- e \sqcup - e) \sqcap (- e \sqcup - p^T) \sqcap (- e \sqcup p^T) \sqcap (p^T \sqcup - p^T))$
$\sqcup e^T \sqcup e$
by (*smt abel-semigroup.commute inf.abel-semigroup-axioms inf.left-commute sup.abel-semigroup-axioms*)
also have ... = $(f \sqcap - e^T \sqcap (- p \sqcup p)) \sqcup (f^T \sqcap - e \sqcap (p^T \sqcup - p^T)) \sqcup e^T \sqcup e$
by (*smt inf.sup-monoid.add-assoc inf.sup-monoid.add-commute inf-sup-absorb sup.idem*)
also have ... = $(f \sqcap - e^T) \sqcup (f^T \sqcap - e) \sqcup e^T \sqcup e$
by (*metis assms*(*1*) *conv-complement inf-top-right stone*)
also have ... = $(f \sqcup e^T) \sqcap (- e^T \sqcup e^T) \sqcup (f^T \sqcup e) \sqcap (- e \sqcup e)$
by (*metis sup.left-commute sup-assoc sup-inf-distrib2*)
finally show *?thesis*
by (*metis abel-semigroup.commute assms*(*2*) *conv-complement inf-top-right stone sup.abel-semigroup-axioms sup-assoc*)
qed

lemma *simplify-forest-components-f*:
assumes *regular p*
and *regular e*
and *injective* $(f \sqcap - e^T \sqcap - p \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
and *injective f*
shows *forest-components* $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) = (f \sqcup f^T \sqcup e \sqcup e^T)^\star$
proof −
have *forest-components* $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) =$ *wcc* $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
by (*simp add*: *assms*(*3*) *forest-components-wcc*)
also have ... = $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup (f \sqcap - e^T \sqcap p) \sqcup e^T)^\star$
using *conv-dist-sup sup-assoc* **by** *auto*
also have ... = $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup e^T \sqcup e)^\star$
using *sup-assoc sup-commute* **by** *auto*
also have ... = $(f \sqcup f^T \sqcup e \sqcup e^T)^\star$
using *assms*(*1*, *2*, *3*, *4*) *simplify-f* **by** *auto*
finally show *?thesis*

by *simp*
qed

lemma *components-disj-increasing*:
assumes *regular* p
and *regular* e
and *injective* $(f \sqcap - e^T \sqcap - p \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
and *injective* f
shows *forest-components* $f \leq$ *forest-components* $(f \sqcap - e^T \sqcap - p \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
proof $-$
have *1*: *forest-components* $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) = (f \sqcup f^T \sqcup e \sqcup e^T)^\star$
using *simplify-forest-components-f assms*(*1*, *2*, *3*, *4*) **by** *blast*
have *forest-components* $f =$ *wcc* f
by (*simp add*: *assms*(*4*) *forest-components-wcc*)
also have ... $\leq (f \sqcup f^T \sqcup e^T \sqcup e)^\star$
by (*simp add*: *le-supI2 star-isotone sup-commute*)
finally show *?thesis*
using *1 sup.left-commute sup-commute* **by** *simp*
qed

lemma *fch-equivalence*:
assumes *forest* h
shows *equivalence* (*forest-components* h)
by (*simp add*: *assms forest-components-equivalence*)

lemma *forest-modulo-equivalence-path-split-1*:
assumes *arc* a
and *equivalence* H
shows $(H * d)^\star * H * a * top = (H * (d \sqcap - a))^\star * H * a * top$
proof $-$
let $?H = H$
let $?x = ?H * (d \sqcap -a)$
let $?y = ?H * a$
let $?a = ?H * a * top$
let $?d = ?H * d$
have *1*: $?d^\star * ?a \leq ?x^\star * ?a$
proof $-$
have $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a * ?a$
by (*smt mult-left-isotone star.circ-right-top top-right-mult-increasing mult-assoc*)
also have ... $= ?x^\star * ?a * a * top$
by (*metis ex231e mult-assoc*)
also have ... $= ?x^\star * ?a$
by (*simp add*: *assms*(*1*) *mult-assoc*)
finally have *11*: $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a$
by *simp*
have $?d^\star * ?a = (?H * (d \sqcap a) \sqcup ?H * (d \sqcap -a))^\star * ?a$

proof −
have *12*: *regular a*
using *assms*(*1*) *arc-regular* **by** *simp*
have $?H * ((d \sqcap a) \sqcup (d \sqcap - a)) = ?H * (d \sqcap top)$
using *12* **by** (*metis inf-top-right maddux-3-11-pp*)
thus *?thesis*
using *mult-left-dist-sup* **by** *auto*
qed
also have ... $\leq (?y \sqcup ?x)^\star * ?a$
by (*metis comp-inf.coreflexive-idempotent comp-isotone inf.cobounded1 inf.sup-monoid.add-commute semiring.add-mono star-isotone top.extremum*)
also have ... $= (?x \sqcup ?y)^\star * ?a$
by (*simp add*: *sup-commute mult-assoc*)
also have ... $= ?x^\star * ?a \sqcup (?x^\star * ?y * (?x^\star * ?y)^\star * ?x^\star) * ?a$
by (*smt mult-right-dist-sup star.circ-sup-9 star.circ-unfold-sum mult-assoc*)
also have ... $\leq ?x^\star * ?a \sqcup (?x^\star * ?y * (top * ?y)^\star * ?x^\star) * ?a$
proof −
have $(?x^\star * ?y)^\star \leq (top * ?y)^\star$
by (*simp add*: *mult-left-isotone star-isotone*)
thus *?thesis*
by (*metis comp-inf.coreflexive-idempotent comp-inf.transitive-star eq-refl mult-left-dist-sup top.extremum mult-assoc*)
qed
also have ... $= ?x^\star * ?a \sqcup (?x^\star * ?y * ?x^\star) * ?a$
using *assms*(*1*, *2*) *He-eq-He-THe-star arc-regular mult-assoc* **by** *auto*
finally have *13*: $(?H * d)^\star * ?a \leq ?x^\star * ?a \sqcup ?x^\star * ?y * ?x^\star * ?a$
by (*simp add*: *mult-assoc*)
have *14*: $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a$
using *11 mult-assoc* **by** *auto*
thus *?thesis*
using *13 14 sup.absorb1* **by** *auto*
qed
have *2*: $?d^\star * ?a \geq ?x^\star * ?a$
by (*simp add*: *comp-isotone star-isotone*)
thus *?thesis*
using *1 2 order.antisym mult-assoc* **by** *simp*
qed

lemma *dTransHd-le-1*:
assumes *equivalence H*
and *univalent* $(H * d)$
shows $d^T * H * d \leq 1$
proof −
have $d^T * H^T * H * d \leq 1$
using *assms*(*2*) *conv-dist-comp mult-assoc* **by** *auto*
thus *?thesis*
using *assms*(*1*) *mult-assoc* **by** (*simp add*: *preorder-idempotent*)
qed

lemma *HcompaT-le-compHaT*:
  assumes *equivalence* H
    and *injective* $(a * top)$
  shows $-H * a * top \leq -(H * a * top)$
proof –
  have $a * top * a^T \leq 1$
    by (*metis assms*(*2*) *conv-dist-comp symmetric-top-closed vector-top-closed mult-assoc*)
  hence $a * top * a^T * H \leq H$
    using *assms*(*1*) *comp-isotone order-trans* **by** *blast*
  hence $a * top * top * a^T * H \leq H$
    by (*simp add*: *vector-mult-closed*)
  hence $a * top * (H * a * top)^T \leq H$
    by (*metis assms*(*1*) *conv-dist-comp symmetric-top-closed vector-top-closed mult-assoc*)
  thus *?thesis*
    using *assms*(*2*) *comp-injective-below-complement mult-assoc* **by** *auto*
qed

4.2 Forests modulo an equivalence

In the graphical interpretation, the arcs of d are directed towards the root(s) of the *forest-modulo-equivalence*.

definition *forest-modulo-equivalence* $x\ d \equiv$ *equivalence* $x \wedge$ *univalent* $(x * d) \wedge x \sqcap d * d^T \leq 1 \wedge (x * d)^+ \sqcap x \leq bot$

definition *forest-modulo-equivalence-path* $a\ b\ H\ d \equiv$ *arc* $a \wedge$ *arc* $b \wedge a^T * top \leq (H * d)^\star * H * b * top$

lemma *d-separates-forest-modulo-equivalence-1*:
  assumes *forest-modulo-equivalence* $x\ d$
  shows $x * d \leq -x$
proof –
  have $x * d \leq (x * d)^+$
    using *star.circ-mult-increasing* **by** *simp*
  also have ... $\leq -x$
    using *assms*(*1*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
  finally show *?thesis*
    by *simp*
qed

lemma *d-separates-forest-modulo-equivalence-2*:
  shows *forest-modulo-equivalence* $x\ d \Longrightarrow d * x \leq -x$
  using *forest-modulo-equivalence-def schroeder-6-p d-separates-forest-modulo-equivalence-1* **by** *metis*

lemma *d-separates-forest-modulo-equivalence-3*:
  assumes *forest-modulo-equivalence* $x\ d$

shows $d \leq -x$
proof $-$
have $1 \leq x$
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
then have $d \leq x * d$
using *mult-left-isotone* **by** *fastforce*
also have ... $\leq (x * d)^+$
using *star.circ-mult-increasing* **by** *simp*
also have ... $\leq -x$
using *assms*(*1*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
finally show *?thesis*
by *simp*
qed

lemma *d-separates-forest-modulo-equivalence-4*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow d^T \leq -x$
using *d-separates-forest-modulo-equivalence-3 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed* **by** *metis*

lemma *d-separates-forest-modulo-equivalence-5*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow d \sqcup d^T \leq -x$
using *d-separates-forest-modulo-equivalence-3 d-separates-forest-modulo-equivalence-4 sup-least* **by** *blast*

lemma *d-separates-forest-modulo-equivalence-6*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow d * x \sqcup x * d \leq -x$
using *d-separates-forest-modulo-equivalence-1 d-separates-forest-modulo-equivalence-2 sup-least* **by** *blast*

lemma *d-separates-forest-modulo-equivalence-7*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow x * (d \sqcup d^T) * x \leq -x$
using *d-separates-forest-modulo-equivalence-5 forest-modulo-equivalence-def inf.sup-monoid.add-commute preorder-idempotent pseudo-complement triple-schroeder-p* **by** *metis*

lemma *d-separates-forest-modulo-equivalence-8*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow (d * x)^T \leq -x$
using *d-separates-forest-modulo-equivalence-2 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed* **by** *metis*

lemma *d-separates-forest-modulo-equivalence-9*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow (x * d)^T \leq -x$
by (*metis d-separates-forest-modulo-equivalence-1 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed*)

lemma *d-separates-forest-modulo-equivalence-10*:
shows *forest-modulo-equivalence* $x\ d \Longrightarrow (d * x)^+ \leq -x$
using *forest-modulo-equivalence-def le-bot pseudo-complement schroeder-5-p*

star-slide mult-assoc **by** *metis*

lemma *d-separates-forest-modulo-equivalence-11*:
shows *forest-modulo-equivalence* x $d \Longrightarrow (x * d)^+ \leq -x$
using *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*

lemma *d-separates-forest-modulo-equivalence-12*:
shows *forest-modulo-equivalence* x $d \Longrightarrow (d * x)^{T+} \leq -x$
using *d-separates-forest-modulo-equivalence-10 forest-modulo-equivalence-def conv-isotone conv-plus-commute symmetric-complement-closed* **by** *metis*

lemma *d-separates-x-in-forest-13*:
shows *forest-modulo-equivalence* x $d \Longrightarrow (x * d)^{T+} \leq -x$
using *d-separates-forest-modulo-equivalence-11 forest-modulo-equivalence-def conv-isotone conv-plus-commute symmetric-complement-closed* **by** *metis*

lemma *irreflexive-d-in-forest-modulo-equivalence*:
shows *forest-modulo-equivalence* x $d \Longrightarrow$ *irreflexive* d
by (*metis d-separates-forest-modulo-equivalence-3 forest-modulo-equivalence-def inf.cobounded2 inf.left-commute inf.orderE pseudo-complement*)

lemma *univalent-d-in-forest-modulo-equivalence*:
assumes *forest-modulo-equivalence* x d
shows *univalent* d
proof –
have $d^T * d \leq d^T * x^T * x * d$
using *assms*(*1*) *forest-modulo-equivalence-def comp-isotone comp-right-one mult-sub-right-one* **by** *metis*
also have ... $\leq$ *1*
using *assms*(*1*) *forest-modulo-equivalence-def comp-associative conv-dist-comp* **by** *auto*
finally show *?thesis*
by *simp*
qed

lemma *acyclic-d-in-forest-modulo-equivalence*:
assumes *forest-modulo-equivalence* x d
shows *acyclic* d
proof –
have $d^\star \leq (x * d)^\star$
using *assms*(*1*) *forest-modulo-equivalence-def mult-left-isotone star.circ-circ-mult star.circ-circ-mult-1 star.circ-extra-circ star.left-plus-circ star-involutive star-isotone star-one star-slide mult-assoc* **by** *metis*
then have $d * d^\star \leq d * (x * d)^\star$
using *mult-right-isotone* **by** *blast*
also have ... $\leq x * d * (x * d)^\star$
using *assms*(*1*) *forest-modulo-equivalence-def eq-refl inf.order-trans mult-isotone star.circ-circ-mult-1 star-involutive star-one star-outer-increasing mult-assoc* **by** *metis*

also have ... $\leq -x$
using *assms d-separates-forest-modulo-equivalence-11* **by** *blast*
also have ... ≤ -1
using *assms*(*1*) *forest-modulo-equivalence-def p-antitone* **by** *blast*
finally show *?thesis*
by *simp*
qed

lemma *acyclic-dt-d-in-forest-modulo-equivalence*:
shows *forest-modulo-equivalence x d* $\Longrightarrow$ *acyclic* (d^T)
using *acyclic-d-in-forest-modulo-equivalence conv-plus-commute irreflexive-conv-closed* **by** *fastforce*

lemma *dt-forest-modulo-equivalence-forest*:
shows *forest-modulo-equivalence x d* $\Longrightarrow$ *forest* (d^T)
using *acyclic-dt-d-in-forest-modulo-equivalence univalent-d-in-forest-modulo-equivalence* **by** *simp*

lemma *var-forest-modulo-equivalence-axiom*:
shows *forest-modulo-equivalence x d* $\Longrightarrow d^T * x * d \leq 1$
using *forest-modulo-equivalence-def comp-associative conv-dist-comp preorder-idempotent* **by** *metis*

lemma *forest-modulo-equivalence-wcc*:
assumes *forest-modulo-equivalence x d*
shows $(x * d)^\star * (x * d)^{T\star} = ((x * d) \sqcup (x * d)^T)^\star$
using *assms*(*1*) *forest-modulo-equivalence-def fc-wcc* **by** *force*

lemma *forest-modulo-equivalence-direction-1*:
assumes *forest-modulo-equivalence x d*
shows $(x * d)^\star \sqcap (x * d)^T = bot$
using *assms*(*1*) *d-separates-forest-modulo-equivalence-11 forest-modulo-equivalence-def acyclic-star-below-complement-1 order-lesseq-imp p-antitone-iff* **by** *meson*

lemma *forest-modulo-equivalence-direction-2*:
assumes *forest-modulo-equivalence x d*
shows $(x * d)^{T\star} \sqcap (x * d) \leq bot$
using *assms*(*1*) *forest-modulo-equivalence-direction-1 comp-inf.idempotent-bot-closed conv-inf-bot-iff conv-star-commute inf.sup-left-divisibility* **by** *metis*

lemma *forest-modulo-equivalence-separate*:
assumes *forest-modulo-equivalence x d*
shows $(x * d)^\star * (x * d)^{T\star} \sqcap (x * d)^T * (x * d) \leq 1$
proof −
have $(x * d)^\star \sqcap (x * d)^T * (x * d) = (1 \sqcup (x * d)^+) \sqcap (x * d)^T * (x * d)$
using *star-left-unfold-equal* **by** *simp*
also have ... $= (1 \sqcap (x * d)^T * (x * d)) \sqcup ((x * d)^+ \sqcap (x * d)^T * (x * d))$

using *comp-inf.semiring.distrib-right* **by** *simp*
also have ... $\leq$ *1* $\sqcup ((x * d)^+ \sqcap (x * d)^T * (x * d))$
using *inf.coboundedl semiring.add-right-mono* **by** *blast*
also have ... $=$ *1* $\sqcup ((x * d)^\star \sqcap (x * d)^T) * (x * d)$
using *assms*(*1*) *forest-modulo-equivalence-def forest-modulo-equivalence-direction-1 comp-inf.semiring.mult-zero-right inf.sup-left-divisibility le-infI2 semiring.mult-not-zero sup.orderE* **by** *metis*
also have ... $\leq$ *1* $\sqcup$ *bot*
using *assms*(*1*) *forest-modulo-equivalence-direction-1* **by** *simp*
finally have *2*: $(x * d)^\star \sqcap (x * d)^T * (x * d) \leq$ *1*
by *simp*
then have *3*: $(x * d)^{T\star} \sqcap (x * d)^T * (x * d) \leq$ *1*
using *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-dist-inf conv-involutive conv-star-commute coreflexive-symmetric* **by** *metis*
have $((x * d)^\star \sqcup (x * d)^{T\star}) \sqcap ((x * d)^T * (x * d)) \leq$ *1*
using *2 3 inf-sup-distrib2* **by** *simp*
thus *?thesis*
using *assms*(*1*) *le-infI2 forest-modulo-equivalence-def* **by** *blast*
qed

lemma *forest-modulo-equivalence-path-trans-closure*:
assumes *forest-modulo-equivalence x d*
shows $(x * d^T)^+ * x * d * x * d^T \leq (x * d^T)^+$
proof −
have $(x * d^T)^+ * x * d * x * d^T = (x * d^T)^\star * x * d^T * x * d * x * d^T$
using *comp-associative star.circ-plus-same* **by** *metis*
also have ... $\leq (x * d^T)^\star * x *$ *1* $* x * d^T$
using *assms*(*1*) *forest-modulo-equivalence-def var-forest-modulo-equivalence-axiom comp-associative mult-left-isotone mult-right-isotone* **by** *metis*
also have ... $\leq (x * d^T)^\star * x * d^T$
using *assms*(*1*) *forest-modulo-equivalence-def* **by** (*simp add*: *preorder-idempotent mult-assoc*)
finally show *?thesis*
using *star.circ-plus-same mult-assoc* **by** *simp*
qed

The *forest-modulo-equivalence* structure is preserved if d is decreased.

lemma *forest-modulo-equivalence-decrease-d*:
assumes *forest-modulo-equivalence x d*
shows *forest-modulo-equivalence x* $(d \sqcap c)$
proof (*unfold forest-modulo-equivalence-def*, *intro conjI*)
show *reflexive x*
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *blast*
next
show *transitive x*
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *blast*
next
show *symmetric x*

using *assms*(*1*) *forest-modulo-equivalence-def* **by** *blast*
next
show *univalent* $(x * (d \sqcap c))$
proof −
have $(x * (d \sqcap c))^T * x * (d \sqcap c) \leq (x * d)^T * x * d$
using *conv-order mult-isotone* **by** *auto*
also have ... ≤ 1
using *assms*(*1*) *forest-modulo-equivalence-def mult-assoc* **by** *auto*
finally show *?thesis*
using *mult-assoc* **by** *auto*
qed
next
show *coreflexive* $(x \sqcap ((d \sqcap c) * (d \sqcap c)^T))$
proof −
have $x \sqcap (d \sqcap c) * (d \sqcap c)^T \leq x \sqcap d * d^T$
using *conv-dist-inf inf.sup-right-isotone mult-isotone* **by** *auto*
thus *?thesis*
using *assms*(*1*) *forest-modulo-equivalence-def order-lesseq-imp* **by** *blast*
qed
next
show $(x * (d \sqcap c))^+ \sqcap x \leq bot$
proof −
have $(x * (d \sqcap c))^+ \leq (x * d)^+$
using *comp-isotone star-isotone* **by** *simp*
thus *?thesis*
using *assms d-separates-forest-modulo-equivalence-11 dual-order.eq-iff dual-order.trans pseudo-complement* **by** *blast*
qed
qed

lemma *expand-forest-modulo-equivalence*:
assumes *forest-modulo-equivalence* H d
shows $(d^T * H)^\star * (H * d)^\star = (d^T * H)^\star \sqcup (H * d)^\star$
proof −
have $(H * d)^T * H * d \leq 1$
using *assms forest-modulo-equivalence-def mult-assoc* **by** *auto*
hence $d^T * H * H * d \leq 1$
using *assms forest-modulo-equivalence-def conv-dist-comp* **by** *auto*
thus *?thesis*
by (*simp add*: *cancel-separate-eq comp-associative*)
qed

lemma *forest-modulo-equivalence-path-bot*:
assumes *arc* a
and $a \leq d$
and *forest-modulo-equivalence* H d
shows $(d \sqcap - a)^T * (H * a * top) \leq bot$
proof −
have *1*: $d^T * H * d \leq 1$

using *assms*(*3*) *forest-modulo-equivalence-def dTransHd-le-1* **by** *blast*
hence $d * - 1 * d^T \leq - H$
using *triple-schroeder-p* **by** *force*
hence $d * - 1 * d^T \leq 1 \sqcup - H$
by (*simp add*: *le-supI2*)
hence $d * d^T \sqcup d * - 1 * d^T \leq 1 \sqcup - H$
by (*metis assms*(*3*) *forest-modulo-equivalence-def inf-commute regular-one-closed shunting-p le-supI*)
hence $d * 1 * d^T \sqcup d * - 1 * d^T \leq 1 \sqcup - H$
by *simp*
hence $d * (1 \sqcup - 1) * d^T \leq 1 \sqcup - H$
using *comp-associative mult-right-dist-sup* **by** (*simp add*: *mult-left-dist-sup*)
hence $d * top * d^T \leq 1 \sqcup - H$
using *regular-complement-top* **by** *auto*
hence $d * top * a^T \leq 1 \sqcup - H$
using *assms*(*2*) *conv-isotone dual-order.trans mult-right-isotone* **by** *blast*
hence $d * (a * top)^T \leq 1 \sqcup - H$
by (*simp add*: *comp-associative conv-dist-comp*)
hence $d \leq (1 \sqcup - H) * (a * top)$
by (*simp add*: *assms*(*1*) *shunt-bijective*)
hence $d \leq a * top \sqcup - H * a * top$
by (*simp add*: *comp-associative mult-right-dist-sup*)
also have ... $\leq a * top \sqcup - (H * a * top)$
using *assms*(*1*, *3*) *HcompaT-le-compHaT forest-modulo-equivalence-def sup-right-isotone* **by** *auto*
finally have $d \leq a * top \sqcup - (H * a * top)$
by *simp*
hence $d \sqcap --(H * a * top) \leq a * top$
using *shunting-var-p* **by** *auto*
hence *2*:$d \sqcap H * a * top \leq a * top$
using *inf.sup-right-isotone order.trans pp-increasing* **by** *blast*
have *3*:$d \sqcap H * a * top \leq top * a$
proof −
have $d \sqcap H * a * top \leq (H * a \sqcap d * top^T) * (top \sqcap (H * a)^T * d)$
by (*metis dedekind inf-commute*)
also have ... $= d * top \sqcap H * a * a^T * H^T * d$
by (*simp add*: *conv-dist-comp inf-vector-comp mult-assoc*)
also have ... $\leq d * top \sqcap H * a * d^T * H^T * d$
using *assms*(*2*) *mult-right-isotone mult-left-isotone conv-isotone inf.sup-right-isotone* **by** *auto*
also have ... $= d * top \sqcap H * a * d^T * H * d$
using *assms*(*3*) *forest-modulo-equivalence-def* **by** *auto*
also have ... $\leq d * top \sqcap H * a * 1$
using *1* **by** (*metis inf.sup-right-isotone mult-right-isotone mult-assoc*)
also have ... $\leq H * a$
by *simp*
also have ... $\leq top * a$
by (*simp add*: *mult-left-isotone*)
finally have $d \sqcap H * a * top \leq top * a$

by *simp*
thus *?thesis*
by *simp*
qed
have $d \sqcap H * a * top \le a * top \sqcap top * a$
using *2 3* **by** *simp*
also have ... $= a * top * top * a$
by (*metis comp-associative comp-inf.star.circ-decompose-9 comp-inf.star-star-absorb comp-inf-covector vector-inf-comp vector-top-closed*)
also have ... $= a * top * a$
by (*simp add*: *vector-mult-closed*)
finally have *4*:$d \sqcap H * a * top \le a$
by (*simp add*: *assms*(*1*) *arc-top-arc*)
hence $d \sqcap - a \le -(H * a * top)$
using *assms*(*1*) *arc-regular p-shunting-swap* **by** *fastforce*
hence $(d \sqcap - a) * top \le -(H * a * top)$
using *mult.semigroup-axioms p-antitone-iff schroeder-4-p semigroup.assoc* **by** *fastforce*
thus *?thesis*
by (*simp add*: *schroeder-3-p*)
qed

lemma *forest-modulo-equivalence-path-split-2*:
assumes *arc a*
and $a \le d$
and *forest-modulo-equivalence H d*
shows $(H * (d \sqcap - a))^\star * H * a * top = (H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top$
proof $-$
let *?lhs* $= (H * (d \sqcap - a))^\star * H * a * top$
have *1*: $d^T * H * d \le 1$
using *assms*(*3*) *forest-modulo-equivalence-def dTransHd-le-1* **by** *blast*
have *2*: $H * a * top \le$ *?lhs*
by (*metis le-iff-sup star.circ-loop-fixpoint star.circ-transitive-equal star-involutive sup-commute mult-assoc*)
have $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) = (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
proof $-$
have $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) = (d \sqcap - a)^T * (1 \sqcup H * (d \sqcap - a) * (H * (d \sqcap - a))^\star) * (H * a * top)$
by (*simp add*: *star-left-unfold-equal*)
also have ... $= (d \sqcap - a)^T * H * a * top \sqcup (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
by (*smt mult-left-dist-sup star.circ-loop-fixpoint star.circ-mult-1 star-slide sup-commute mult-assoc*)
also have ... $= bot \sqcup (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
by (*metis assms*(*1*, *2*, *3*) *forest-modulo-equivalence-path-bot mult-assoc le-bot*)

thus *?thesis*
by (*simp add*: *calculation*)
qed
also have ... $\leq d^T * H * d * (H * (d \sqcap - a))^\star * (H * a * top)$
using *conv-isotone inf.coboundedl mult-isotone* **by** *auto*
also have ... $\leq 1 * (H * (d \sqcap - a))^\star * (H * a * top)$
using *1* **by** (*metis le-iff-sup mult-right-dist-sup*)
finally have *3*: $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq$ *?lhs*
using *mult-assoc* **by** *auto*
hence *4*: $H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq$ *?lhs*
proof −
have $H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq H * (H * (d \sqcap - a))^\star * H * a * top$
using *3 mult-right-isotone mult-assoc* **by** *auto*
also have ... $= H * H * ((d \sqcap - a) * H)^\star * H * a * top$
by (*metis assms*(*3*) *forest-modulo-equivalence-def star-slide mult-assoc preorder-idempotent*)
also have ... $= H * ((d \sqcap - a) * H)^\star * H * a * top$
using *assms*(*3*) *forest-modulo-equivalence-def preorder-idempotent* **by** *fastforce*
finally show *?thesis*
by (*metis assms*(*3*) *forest-modulo-equivalence-def preorder-idempotent star-slide mult-assoc*)
qed
have *5*: $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T) * (H * (d \sqcap - a))^\star * H * a * top \leq$ *?lhs*
proof −
have *51*: $H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * H * a * top \leq (H * (d \sqcap - a))^\star * H * a * top$
using *star.left-plus-below-circ mult-left-isotone* **by** *simp*
have *52*: $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T) * (H * (d \sqcap - a))^\star * H * a * top = H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * H * a * top \sqcup H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * H * a * top$
using *mult-right-dist-sup* **by** *auto*
hence ... $\leq (H * (d \sqcap - a))^\star * H * a * top \sqcup H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * H * a * top$
using *star.left-plus-below-circ mult-left-isotone sup-left-isotone* **by** *auto*
thus *?thesis*
using *4 51 52 mult-assoc* **by** *auto*
qed
hence $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T)^\star * H * a * top \leq$ *?lhs*
proof −
have $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T)^\star * (H * (d \sqcap - a))^\star * H * a * top \leq$ *?lhs*
using *5 star-left-induct-mult-iff mult-assoc* **by** *auto*
thus *?thesis*
using *star.circ-decompose-11 star-decompose-1* **by** *auto*
qed
hence *6*: $(H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top \leq$ *?lhs*

using *mult-left-dist-sup* **by** *auto*
have *7*: $(H * (d \sqcap - a))^\star * H * a * top \leq (H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top$
by (*simp add*: *mult-left-isotone semiring.distrib-left star-isotone*)
thus *?thesis*
using *6 7* **by** (*simp add*: *mult-assoc*)
qed

end

4.3 An operation to select components

This section has been moved to theories *Stone-Relation-Algebras.Choose-Component* and *Aggregation-Algebras.M-Choose-Component*.

4.4 m-k-Stone-Kleene relation algebras

m-*k*-Stone-Kleene relation algebras are an extension of *m*-Kleene algebras where the *choose-component* operation has been added.

context *m-kleene-algebra-choose-component*
begin

A *selected-edge* is a minimum-weight edge whose source is in a component, with respect to h, j and g, and whose target is not in that component.

abbreviation *selected-edge h j g* ≡ *minarc* (*choose-component* (*forest-components h*) $j * -$ *choose-component* (*forest-components h*) $j^T \sqcap g$)

A *path* is any sequence of edges in the forest, f, of the graph, g, backwards from the target of the *selected-edge* to a root in f.

abbreviation *path f h j g* ≡ *top* ∗ *selected-edge h j g* ∗ ($f \sqcap -$ *selected-edge h j* $g^T)^{T\star}$

definition *boruvka-outer-invariant f g* ≡
symmetric g
∧ *forest f*
∧ $f \leq --g$
∧ *regular f*
∧ (∃ *w* . *minimum-spanning-forest w g* ∧ $f \leq w \sqcup w^T$)

definition *boruvka-inner-invariant j f h g d* ≡
boruvka-outer-invariant f g
∧ $g \neq bot$
∧ *regular d*
∧ *regular j* ∧ *vector j*
∧ *regular h* ∧ *forest h*
∧ *forest-components h* ∗ $j = j$
∧ *forest-modulo-equivalence* (*forest-components h*) *d*
∧ $d * top \leq - j$

$\wedge\ f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
$\wedge\ (\forall\ a\ b\ .\ \textit{forest-modulo-equivalence-path}\ a\ b\ (\textit{forest-components}\ h)\ d \wedge a \leq -(\textit{forest-components}\ h) \sqcap --\ g \wedge b \leq d \longrightarrow \textit{sum}(b \sqcap g) \leq \textit{sum}(a \sqcap g))$

lemma *F-is-H-and-d*:
assumes $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
and *injective f*
and *injective h*
shows *forest-components f* $=$ (*forest-components h* $*$ $(d \sqcup d^T))^\star *$ *forest-components h*
proof $-$
have *forest-components f* $= (f \sqcup f^T)^\star$
using *assms*(*2*) *cancel-separate-1* **by** *simp*
also have ... $= (h \sqcup h^T \sqcup d \sqcup d^T)^\star$
using *assms*(*1*) **by** *auto*
also have ... $= ((h \sqcup h^T)^\star * (d \sqcup d^T))^\star * (h \sqcup h^T)^\star$
using *star.circ-sup-9 sup-assoc* **by** *metis*
also have ... $=$ (*forest-components h* $* (d \sqcup d^T))^\star *$ *forest-components h*
using *assms*(*3*) *forest-components-wcc* **by** *simp*
finally show *?thesis*
by *simp*
qed

lemma *H-below-F*:
assumes $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
and *injective f*
and *injective h*
shows *forest-components h* $\leq$ *forest-components f*
using *assms*(*1*, *2*, *3*) *cancel-separate-1 dual-order.trans star.circ-sub-dist* **by** *metis*

lemma *H-below-regular-g*:
assumes $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
and $f \leq --g$
and *symmetric g*
shows $h \leq --g$
proof $-$
have $h \leq f \sqcup f^T$
using *assms*(*1*) *sup-assoc* **by** *simp*
also have ... $\leq --g$
using *assms*(*2*, *3*) *conv-complement conv-isotone* **by** *fastforce*
finally show *?thesis*
using *order-trans* **by** *blast*
qed

lemma *expression-equivalent-without-e-complement*:
assumes *selected-edge h j g* $\leq -$ *forest-components f*
shows $f \sqcap -$ (*selected-edge h j g*)$^T \sqcap -$ (*path f h j g*) $\sqcup$ ($f \sqcap -$ (*selected-edge h j g*)$^T \sqcap$ (*path f h j g*))$^T \sqcup$ (*selected-edge h j g*)

$= f \sqcap -\ (\textit{path } f\ h\ j\ g) \sqcup (f \sqcap (\textit{path } f\ h\ j\ g))^T \sqcup (\textit{selected-edge } h\ j\ g)$
proof $-$
  let $?p = \textit{path } f\ h\ j\ g$
  let $?e = \textit{selected-edge } h\ j\ g$
  let $?F = \textit{forest-components } f$
  have 1: $?e \leq -\ ?F$
    by (*simp add*: *assms*)
  have $f^T \leq\ ?F$
    by (*metis conv-dist-comp conv-involutive conv-order conv-star-commute forest-components-increasing*)
  hence $-\ ?F \leq -\ f^T$
    using *p-antitone* **by** *auto*
  hence $?e \leq -\ f^T$
    using *1 dual-order.trans* **by** *blast*
  hence $f^T \leq -\ ?e$
    by (*simp add*: *p-antitone-iff*)
  hence $f^{TT} \leq -\ ?e^T$
    by (*metis conv-complement conv-dist-inf inf.orderE inf.orderI*)
  hence $f \leq -\ ?e^T$
    by *auto*
  hence $f = f \sqcap -\ ?e^T$
    using *inf.orderE* **by** *blast*
  hence $f \sqcap -\ ?e^T \sqcap -\ ?p \sqcup (f \sqcap -\ ?e^T \sqcap\ ?p)^T \sqcup\ ?e = f \sqcap -\ ?p \sqcup (f \sqcap\ ?p)^T \sqcup\ ?e$
    by *auto*
  thus *?thesis* **by** *auto*
qed

The source of the *selected-edge* is contained in j, the vector describing those vertices still to be processed in the inner loop of Borůvka's algorithm.

lemma *et-below-j*:
  assumes *vector j*
    and *regular j*
    and $j \neq \textit{bot}$
  shows $\textit{selected-edge } h\ j\ g * \textit{top} \leq j$
proof $-$
  let $?e = \textit{selected-edge } h\ j\ g$
  let $?c = \textit{choose-component } (\textit{forest-components } h)\ j$
  have $?e * \textit{top} \leq --(?c * -?c^T \sqcap g) * \textit{top}$
    using *comp-isotone minarc-below* **by** *blast*
  also have ... $= (--(?c * -?c^T) \sqcap --g) * \textit{top}$
    by *simp*
  also have ... $= (?c * -?c^T \sqcap --g) * \textit{top}$
    using *component-is-regular regular-mult-closed* **by** *auto*
  also have ... $= (?c \sqcap -?c^T \sqcap --g) * \textit{top}$
    by (*metis component-is-vector conv-complement vector-complement-closed vector-covector*)
  also have ... $\leq\ ?c * \textit{top}$
    using *inf.cobounded1 mult-left-isotone order-trans* **by** *blast*

also have ... $\leq j * top$
  by (*metis assms*(*2*) *comp-inf*.*star*.*circ-sup-2* *comp-isotone* *component-in-v*)
also have ... $= j$
  by (*simp add*: *assms*(*1*))
finally show *?thesis*
  by *simp*
qed

4.4.1 Components of forests and forests modulo an equivalence

We prove a number of properties about *forest-modulo-equivalence* and *forest-components*.

lemma *fc-j-eq-j-inv*:
  assumes *forest h*
    and *forest-components h* $* j = j$
  shows *forest-components h* $* (j \sqcap -$ *choose-component* (*forest-components h*) $j)$ $= j \sqcap -$ *choose-component* (*forest-components h*) j
proof $-$
  let $?c =$ *choose-component* (*forest-components h*) j
  let $?H =$ *forest-components h*
  have *1*:*equivalence ?H*
    by (*simp add*: *assms*(*1*) *forest-components-equivalence*)
  have $?H * (j \sqcap - ?c) = ?H * j \sqcap ?H * - ?c$
    using *1* **by** (*metis assms*(*2*) *equivalence-comp-dist-inf* *inf*.*sup-monoid*.*add-commute*)
  hence *2*: $?H * (j \sqcap - ?c) = j \sqcap ?H * - ?c$
    by (*simp add*: *assms*(*2*))
  have *3*: $j \sqcap - ?c \leq ?H * - ?c$
    using *1* **by** (*metis assms*(*2*) *dedekind-1* *dual-order*.*trans* *equivalence-comp-dist-inf* *inf*.*cobounded2*)
  have $?H * ?c \leq ?c$
    using *component-single* **by** *auto*
  hence $?H^T * ?c \leq ?c$
    using *1* **by** *simp*
  hence $?H * - ?c \leq - ?c$
    using *component-is-regular schroeder-3-p* **by** *force*
  hence $j \sqcap ?H * - ?c \leq j \sqcap - ?c$
    using *inf*.*sup-right-isotone* **by** *auto*
  thus *?thesis*
    using *2 3 order*.*antisym* **by** *simp*
qed

There is a path in the *forest-modulo-equivalence* between edges a and b if and only if there is either a path in the *forest-modulo-equivalence* from a to b or one from a to c and one from c to b.

lemma *forest-modulo-equivalence-path-split-disj*:
  assumes *equivalence H*
    and *arc c*
    and *regular a* $\wedge$ *regular b* $\wedge$ *regular c* $\wedge$ *regular d* $\wedge$ *regular H*

shows *forest-modulo-equivalence-path a b H* $(d \sqcup c) \longleftrightarrow$ *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)

proof −

have *1*: *forest-modulo-equivalence-path a b H* $(d \sqcup c) \longrightarrow$ *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)

proof (*rule impI*)

assume *11*: *forest-modulo-equivalence-path a b H* $(d \sqcup c)$

hence $a^T * top \le (H * (d \sqcup c))^\star * H * b * top$

by (*simp add*: *forest-modulo-equivalence-path-def*)

also have ... $= ((H * d)^\star \sqcup (H * d)^\star * H * c * (H * d)^\star) * H * b * top$

using *assms*(*1*, *2*) *path-through-components* **by** *simp*

also have ... $= (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$

by (*simp add*: *mult-right-dist-sup*)

finally have *12*:$a^T * top \le (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$

by *simp*

have *13*: $a^T * top \le (H * d)^\star * H * b * top \vee a^T * top \le (H * d)^\star * H * c * (H * d)^\star * H * b * top$

proof (*rule point-in-vector-sup*)

show *point* $(a^T * top)$

using *11 forest-modulo-equivalence-path-def mult-assoc* **by** *auto*

next

show *vector* $((H * d)^\star * H * b * top)$

using *vector-mult-closed* **by** *simp*

next

show *regular* $((H * d)^\star * H * b * top)$

using *assms*(*3*) *pp-dist-comp pp-dist-star* **by** *auto*

next

show $a^T * top \le (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$

using *12* **by** *simp*

qed

thus *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)

proof (*cases* $a^T * top \le (H * d)^\star * H * b * top$)

case *True*

assume $a^T * top \le (H * d)^\star * H * b * top$

hence *forest-modulo-equivalence-path a b H d*

using *11 forest-modulo-equivalence-path-def* **by** *auto*

thus *?thesis*

by *simp*

next

case *False*

have *14*: $a^T * top \le (H * d)^\star * H * c * (H * d)^\star * H * b * top$

using *13* **by** (*simp add*: *False*)

hence *15*: $a^T * top \le (H * d)^\star * H * c * top$

by (*metis mult-right-isotone order-lesseq-imp top-greatest mult-assoc*)
have $c^T * top \leq (H * d)^\star * H * b * top$
proof (*rule ccontr*)
assume $\neg\ c^T * top \leq (H * d)^\star * H * b * top$
hence $c^T * top \leq -((H * d)^\star * H * b * top)$
by (*meson assms*(*2*, *3*) *point-in-vector-or-complement regular-closed-star regular-closed-top regular-mult-closed vector-mult-closed vector-top-closed*)
hence $c * (H * d)^\star * H * b * top \leq bot$
using *schroeder-3-p mult-assoc* **by** *auto*
thus *False*
using *13 False sup.absorb-iff1 mult-assoc* **by** *auto*
qed
hence *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*
using *11 15 assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
thus *?thesis*
by *simp*
qed
qed
have *2*: *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*) $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
proof −
have *21*: *forest-modulo-equivalence-path a b H d* $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
proof (*rule impI*)
assume *22*:*forest-modulo-equivalence-path a b H d*
hence $a^T * top \leq (H * d)^\star * H * b * top$
using *forest-modulo-equivalence-path-def* **by** *blast*
hence $a^T * top \leq (H * (d \sqcup c))^\star * H * b * top$
by (*simp add*: *mult-left-isotone mult-right-dist-sup mult-right-isotone order.trans star-isotone star-slide*)
thus *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
using *22 forest-modulo-equivalence-path-def* **by** *blast*
qed
have *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d* $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
proof (*rule impI*)
assume *23*: *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*
hence $a^T * top \leq (H * d)^\star * H * c * top$
using *forest-modulo-equivalence-path-def* **by** *blast*
also have ... $\leq (H * d)^\star * H * c * c^T * c * top$
by (*metis ex231c comp-inf.star.circ-sup-2 mult-isotone mult-right-isotone mult-assoc*)
also have ... $\leq (H * d)^\star * H * c * c^T * top$
by (*simp add*: *mult-right-isotone mult-assoc*)
also have ... $\leq (H * d)^\star * H * c * (H * d)^\star * H * b * top$
using *23 mult-right-isotone mult-assoc* **by** (*simp add*:

forest-modulo-equivalence-path-def)
also have ... $\leq (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$
by (*simp add*: *forest-modulo-equivalence-path-def*)
finally have $a^T * top \leq (H * (d \sqcup c))^\star * H * b * top$
using *assms*(*1*, *2*) *path-through-components mult-right-dist-sup* **by** *simp*
thus *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
using *23 forest-modulo-equivalence-path-def* **by** *blast*
qed
thus *?thesis*
using *21* **by** *auto*
qed
thus *?thesis*
using *1 2* **by** *blast*
qed

lemma *dT-He-eq-bot*:
assumes *vector j*
and *regular j*
and $d * top \leq - j$
and *forest-components* $h * j = j$
and $j \neq bot$
shows $d^T *$ *forest-components* $h *$ *selected-edge h j g* $\leq bot$
proof −
let *?e* = *selected-edge h j g*
let *?H* = *forest-components h*
have *1*: $?e * top \leq j$
using *assms*(*1*, *2*, *5*) *et-below-j* **by** *auto*
have $d^T * ?H * ?e \leq (d * top)^T * ?H * (?e * top)$
by (*simp add*: *comp-isotone conv-isotone top-right-mult-increasing*)
also have ... $\leq (d * top)^T * ?H * j$
using *1 mult-right-isotone* **by** *auto*
also have ... $\leq (- j)^T * ?H * j$
by (*simp add*: *assms*(*3*) *conv-isotone mult-left-isotone*)
also have ... $= (- j)^T * j$
using *assms*(*4*) *comp-associative* **by** *auto*
also have ... = *bot*
by (*simp add*: *assms*(*1*) *conv-complement covector-vector-comp*)
finally show *?thesis*
using *coreflexive-bot-closed le-bot* **by** *blast*
qed

lemma *forest-modulo-equivalence-d-U-e*:
assumes *forest f*
and *vector j*
and *regular j*
and *forest h*
and *forest-modulo-equivalence* (*forest-components h*) *d*
and $d * top \leq - j$

and *forest-components* $h * j = j$
and $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
and *selected-edge* $h\ j\ g \leq -$ *forest-components f*
and $j \neq bot$
shows *forest-modulo-equivalence* (*forest-components h*) ($d \sqcup$ *selected-edge h j g*)
proof (*cases selected-edge h j g = bot*)
let *?e = selected-edge h j g*
case *True*
assume *?e = bot*
thus *?thesis*
by (*simp add*: *True assms*(*5*))
next
let *?H = forest-components h*
let *?F = forest-components f*
let *?e = selected-edge h j g*
let $?d' = d \sqcup ?e$
case *False*
assume *e-not-bot*: $?e \neq bot$
have *forest-modulo-equivalence* (*forest-components h*) ($d \sqcup$ *selected-edge h j g*)
proof (*unfold forest-modulo-equivalence-def*, *intro conjI*)
show *01*: *reflexive ?H*
by (*simp add*: *assms*(*4*) *forest-components-equivalence*)
show *02*: *transitive ?H*
by (*simp add*: *assms*(*4*) *forest-components-equivalence*)
show *03*: *symmetric ?H*
by (*simp add*: *assms*(*4*) *forest-components-equivalence*)
have *04*: *equivalence ?H*
by (*simp add*: *01 02 03*)
show *univalent* $(?H * ?d')$
proof $-$
have $(?H * ?d')^T * (?H * ?d') = ?d'^T * ?H^T * ?H * ?d'$
using *conv-dist-comp mult-assoc* **by** *auto*
also have $... = ?d'^T * ?H * ?H * ?d'$
by (*simp add*: *conv-dist-comp conv-star-commute*)
also have $... = ?d'^T * ?H * ?d'$
using *01 02* **by** (*metis preorder-idempotent mult-assoc*)
finally have *21*: *univalent* $(?H * ?d') \longleftrightarrow ?e^T * ?H * d \sqcup d^T * ?H * ?e \sqcup ?e^T * ?H * ?e \sqcup d^T * ?H * d \leq 1$
using *conv-dist-sup semiring.distrib-left semiring.distrib-right* **by** *auto*
have *22*: $?e^T * ?H * ?e \leq 1$
proof $-$
have *221*: $?e^T * ?H * ?e \leq ?e^T * top * ?e$
by (*simp add*: *mult-left-isotone mult-right-isotone*)
have *arc ?e*
using *e-not-bot minarc-arc minarc-bot-iff* **by** *blast*
hence $?e^T * top * ?e \leq 1$
using *arc-expanded* **by** *blast*
thus *?thesis*
using *221 dual-order.trans* **by** *blast*

qed
have *24*: $d^T * ?H * ?e \leq 1$
by (*metis assms*(*2*, *3*, *6*, *7*, *10*) *dT-He-eq-bot coreflexive-bot-closed le-bot*)
hence $(d^T * ?H * ?e)^T \leq 1^T$
using *conv-isotone* **by** *blast*
hence $?e^T * ?H^T * d^{TT} \leq 1$
by (*simp add*: *conv-dist-comp mult-assoc*)
hence *25*: $?e^T * ?H * d \leq 1$
using *assms*(*4*) *fch-equivalence* **by** *auto*
have *8*: $d^T * ?H * d \leq 1$
using *04 assms*(*5*) *dTransHd-le-1 forest-modulo-equivalence-def* **by** *blast*
thus *?thesis*
using *21 22 24 25* **by** *simp*
qed
show *coreflexive* $(?H \sqcap ?d' * ?d'^T)$
proof $-$
have *coreflexive* $(?H \sqcap ?d' * ?d'^T) \longleftrightarrow ?H \sqcap (d \sqcup ?e) * (d^T \sqcup ?e^T) \leq 1$
by (*simp add*: *conv-dist-sup*)
also have ... $\longleftrightarrow ?H \sqcap (d * d^T \sqcup d * ?e^T \sqcup ?e * d^T \sqcup ?e * ?e^T) \leq 1$
by (*metis mult-left-dist-sup mult-right-dist-sup sup.left-commute sup-commute*)
finally have *1*: *coreflexive* $(?H \sqcap ?d' * ?d'^T) \longleftrightarrow ?H \sqcap d * d^T \sqcup ?H \sqcap d * ?e^T \sqcup ?H \sqcap ?e * d^T \sqcup ?H \sqcap ?e * ?e^T \leq 1$
by (*simp add*: *inf-sup-distrib1*)
have *31*: $?H \sqcap d * d^T \leq 1$
using *assms*(*5*) *forest-modulo-equivalence-def* **by** *blast*
have *32*: $?H \sqcap ?e * d^T \leq 1$
proof $-$
have $?e * d^T \leq ?e * top * (d * top)^T$
by (*simp add*: *conv-isotone mult-isotone top-right-mult-increasing*)
also have ... $\leq ?e * top * - j^T$
by (*metis assms*(*6*) *conv-complement conv-isotone mult-right-isotone*)
also have ... $\leq j * - j^T$
using *assms*(*2*, *3*, *10*) *et-below-j mult-left-isotone* **by** *auto*
also have ... $\leq - ?H$
using *03* **by** (*metis assms*(*2*, *3*, *7*) *conv-complement conv-dist-comp equivalence-top-closed mult-left-isotone schroeder-3-p vector-top-closed*)
finally have $?e * d^T \leq - ?H$
by *simp*
thus *?thesis*
by (*metis inf.coboundedI1 p-antitone-iff p-shunting-swap regular-one-closed*)
qed
have *33*: $?H \sqcap d * ?e^T \leq 1$
proof $-$
have *331*: *injective h*
by (*simp add*: *assms*(*4*))
have $(?H \sqcap ?e * d^T)^T \leq 1$
using *32 coreflexive-conv-closed* **by** *auto*

hence $?H \sqcap (?e * d^T)^T \leq 1$
using *331 conv-dist-inf forest-components-equivalence* **by** *auto*
thus *?thesis*
using *conv-dist-comp* **by** *auto*
qed
have *34*: $?H \sqcap ?e * ?e^T \leq 1$
proof $-$
have *341*:$arc\ ?e \wedge arc\ (?e^T)$
using *e-not-bot minarc-arc minarc-bot-iff* **by** *auto*
have $?H \sqcap ?e * ?e^T \leq ?e * ?e^T$
by *auto*
thus *?thesis*
using *341 arc-injective le-infI2* **by** *blast*
qed
thus *?thesis*
using *1 31 32 33 34* **by** *simp*
qed
show *4*:$(?H * (d \sqcup ?e))^+ \sqcap ?H \leq bot$
proof $-$
have *40*: $(?H * d)^+ \leq -?H$
using *assms*(*5*) *forest-modulo-equivalence-def bot-unique*
pseudo-complement **by** *blast*
have $?e \leq -\ ?F$
by (*simp add*: *assms*(*9*))
hence $?F \leq -\ ?e$
by (*simp add*: *p-antitone-iff*)
hence $?F^T * ?F \leq -\ ?e$
using *assms*(*1*) *fch-equivalence* **by** *fastforce*
hence $?F^T * ?F * ?F^T \leq -\ ?e$
by (*metis assms*(*1*) *fch-equivalence forest-components-star*
star.circ-decompose-9)
hence *41*: $?F * ?e * ?F \leq -\ ?F$
using *triple-schroeder-p* **by** *blast*
hence *42*:$(?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star \leq -\ ?F$
proof $-$
have *43*: $?F * ?F = ?F$
using *assms*(*1*) *forest-components-equivalence preorder-idempotent* **by**
auto
hence $?F * ?e * ?F = ?F * ?F * ?e * ?F$
by *simp*
also have ... $= (?F)^\star * ?F * ?e * (?F)^\star$
by (*simp add*: *assms*(*1*) *forest-components-star*)
also have ... $= (?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star$
using *43* **by** *simp*
finally show *?thesis*
using *41* **by** *simp*
qed
hence *44*: $(?H * d)^\star * ?H * ?e * (?H * d)^\star \leq -\ ?H$
proof $-$

have *45*: *?H* ≤ *?F*
using *assms*(*1*, *4*, *8*) *H-below-F* **by** *blast*
hence *46*:*?H* ∗ *?e* ≤ *?F* ∗ *?e*
by (*simp add*: *mult-left-isotone*)
have $d \leq f \sqcup f^T$
using *assms*(*8*) *sup.left-commute sup-commute* **by** *auto*
also have ... ≤ *?F*
by (*metis forest-components-increasing le-supI2 star.circ-back-loop-fixpoint star.circ-increasing sup.bounded-iff*)
finally have *d* ≤ *?F*
by *simp*
hence *?H* ∗ *d* ≤ *?F* ∗ *?F*
using *45 mult-isotone* **by** *auto*
hence *47*: $(?H * d)^\star \leq (?F * ?F)^\star$
by (*simp add*: *star-isotone*)
hence $(?H * d)^\star * ?H * ?e * (?H * d)^\star \leq (?H * d)^\star * ?F * ?e * (?H * d)^\star$
using *46* **by** (*metis mult-left-isotone mult-right-isotone mult-assoc*)
also have ... $\leq (?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star$
using *47 mult-left-isotone mult-right-isotone* **by** (*simp add*: *comp-isotone*)
also have ... ≤ − *?F*
using *42* **by** *simp*
also have ... ≤ − *?H*
using *45* **by** (*simp add*: *p-antitone*)
finally show *?thesis*
by *simp*
qed
have $(?H * (d \sqcup ?e))^+ = (?H * (d \sqcup ?e))^\star * (?H * (d \sqcup ?e))$
using *star.circ-plus-same* **by** *auto*
also have ... $= ((?H * d)^\star \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star) * (?H * (d \sqcup ?e))$
using *assms*(*4*) *e-not-bot forest-components-equivalence minarc-arc minarc-bot-iff path-through-components* **by** *auto*
also have ... $= (?H * d)^\star * (?H * (d \sqcup ?e)) \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * (d \sqcup ?e))$
using *mult-right-dist-sup* **by** *auto*
also have ... $= (?H * d)^\star * (?H * d \sqcup ?H * ?e) \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
by (*simp add*: *mult-left-dist-sup*)
also have ... $= (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
using *mult-left-dist-sup mult-assoc* **by** *auto*
also have ... $= (?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
by (*simp add*: *star.circ-plus-same mult-assoc*)
also have ... $= (?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * ?e$
by (*simp add*: *mult.semigroup-axioms semiring.distrib-left sup.semigroup-axioms semigroup.assoc*)

also have ... $\leq$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e$
proof –
have $?e * (?H * d)^\star * ?H * ?e \leq ?e * top * ?e$
by (*metis comp-associative comp-inf.coreflexive-idempotent comp-inf.coreflexive-transitive comp-isotone top.extremum*)
also have ... $\leq$ $?e$
using *e-not-bot arc-top-arc minarc-arc minarc-bot-iff* **by** *auto*
finally have $?e * (?H * d)^\star * ?H * ?e \leq ?e$
by *simp*
hence $(?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*simp add*: *comp-associative comp-isotone*)
thus *?thesis*
using *sup-right-isotone* **by** *blast*
qed
also have ... $=$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d$
by (*simp add*: *order.eq-iff ac-simps*)
also have ... $=$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^+$
using *star.circ-plus-same mult-assoc* **by** *auto*
also have ... $=$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e * (1 \sqcup (?H * d)^+)$
by (*simp add*: *mult-left-dist-sup sup-assoc*)
also have ... $=$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star$
by (*simp add*: *star-left-unfold-equal*)
also have ... $\leq$ $-\ ?H$
using *40 44 assms*(*5*) *sup.boundedI* **by** *blast*
finally show *?thesis*
using *pseudo-complement* **by** *force*
qed
qed
thus *?thesis*
by *blast*
qed

4.4.2 Identifying arcs

The expression $d \sqcap \top e^\top H \sqcap (Hd^\top)^* Ha^\top \top$ identifies the edge incoming to the component that the *selected-edge*, e, is outgoing from and which is on the path from edge a to e. Here, we prove this expression is an *arc*.

lemma *shows-arc-x*:
assumes *forest-modulo-equivalence H d*
and *forest-modulo-equivalence-path a e H d*
and $H * d * (H * d)^\star \leq -\ H$
and $\neg\ a^T * top \leq H * e * top$
and *regular a*
and *regular e*
and *regular H*
and *regular d*

shows *arc* $(d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
proof −
let *?x* $= d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top$
have *1*:*regular ?x*
using *assms*(*5*, *6*, *7*, *8*) *regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
have *2*: $a^T * top * a \leq 1$
using *arc-expanded assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
have *3*: $e * top * e^T \leq 1$
using *assms*(*2*) *forest-modulo-equivalence-path-def arc-expanded* **by** *blast*
have *4*: $top * ?x * top = top$
proof −
have $a^T * top \leq (H * d)^\star * H * e * top$
using *assms*(*2*) *forest-modulo-equivalence-path-def* **by** *blast*
also have ... $= H * e * top \sqcup (H * d)^\star * H * d * H * e * top$
by (*metis star.circ-loop-fixpoint star.circ-plus-same sup-commute mult-assoc*)
finally have $a^T * top \leq H * e * top \sqcup (H * d)^\star * H * d * H * e * top$
by *simp*
hence $a^T * top \leq H * e * top \vee a^T * top \leq (H * d)^\star * H * d * H * e * top$
using *assms*(*2*, *6*, *7*) *point-in-vector-sup forest-modulo-equivalence-path-def regular-mult-closed vector-mult-closed* **by** *auto*
hence $a^T * top \leq (H * d)^\star * H * d * H * e * top$
using *assms*(*4*) **by** *blast*
also have ... $= (H * d)^\star * H * d * (H * e * top \sqcap H * e * top)$
by (*simp add*: *mult-assoc*)
also have ... $= (H * d)^\star * H * (d \sqcap (H * e * top)^T) * H * e * top$
by (*metis comp-associative covector-inf-comp-3 star.circ-left-top star.circ-top*)
also have ... $= (H * d)^\star * H * (d \sqcap top^T * e^T * H^T) * H * e * top$
using *conv-dist-comp mult-assoc* **by** *auto*
also have ... $= (H * d)^\star * H * (d \sqcap top * e^T * H) * H * e * top$
using *assms*(*1*) **by** (*simp add*: *forest-modulo-equivalence-def*)
finally have *2*: $a^T * top \leq (H * d)^\star * H * (d \sqcap top * e^T * H) * H * e * top$
by *simp*
hence $e * top \leq ((H * d)^\star * H * (d \sqcap top * e^T * H) * H)^T * a^T * top$
proof −
have *bijective* $(e * top) \wedge$ *bijective* $(a^T * top)$
using *assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
thus *?thesis*
using *2* **by** (*metis bijective-reverse mult-assoc*)
qed
also have ... $= H^T * (d \sqcap top * e^T * H)^T * H^T * (H * d)^{\star T} * a^T * top$
by (*simp add*: *conv-dist-comp mult-assoc*)
also have ... $= H * (d \sqcap top * e^T * H)^T * H * (H * d)^{\star T} * a^T * top$
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
also have ... $= H * (d \sqcap top * e^T * H)^T * H * (d^T * H)^\star * a^T * top$
using *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-star-commute* **by** *auto*
also have ... $= H * (d^T \sqcap H * e * top) * H * (d^T * H)^\star * a^T * top$

using *assms*(*1*) *conv-dist-comp forest-modulo-equivalence-def comp-associative conv-dist-inf* **by** *auto*
also have ... $= H * (d^T \sqcap H * e * top) * (H * d^T)^\star * H * a^T * top$
by (*simp add*: *comp-associative star-slide*)
also have ... $= H * (d^T \sqcap H * e * top) * ((H * d^T)^\star * H * a^T * top \sqcap (H * d^T)^\star * H * a^T * top)$
using *mult-assoc* **by** *auto*
also have ... $= H * (d^T \sqcap H * e * top \sqcap ((H * d^T)^\star * H * a^T * top)^T) * (H * d^T)^\star * H * a^T * top$
by (*smt comp-inf-vector covector-comp-inf vector-conv-covector vector-top-closed mult-assoc*)
also have ... $= H * (d^T \sqcap (top * e^T * H)^T \sqcap ((H * d^T)^\star * H * a^T * top)^T) * (H * d^T)^\star * H * a^T * top$
using *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp mult-assoc* **by** *auto*
also have ... $= H * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)^T * (H * d^T)^\star * H * a^T * top$
by (*simp add*: *conv-dist-inf*)
finally have *3*: $e * top \leq H * ?x^T * (H * d^T)^\star * H * a^T * top$
by *auto*
have $?x \neq bot$
proof (*rule ccontr*)
assume $\neg\ ?x \neq bot$
hence $e * top = bot$
using *3 le-bot* **by** *auto*
thus *False*
using *assms*(*2*, *4*) *forest-modulo-equivalence-path-def mult-assoc semiring.mult-zero-right* **by** *auto*
qed
thus *?thesis*
using *1* **using** *tarski* **by** *blast*
qed
have *5*: $?x * top * ?x^T \leq 1$
proof −
have *51*: $H * (d * H)^\star \sqcap d * H * d^T \leq 1$
proof −
have *511*: $d * (H * d)^\star \leq - H$
using *assms*(*1*, *3*) *forest-modulo-equivalence-def preorder-idempotent schroeder-4-p triple-schroeder-p* **by** *fastforce*
hence $(d * H)^\star * d \leq - H$
using *star-slide* **by** *auto*
hence $H * (d^T * H)^\star \leq - d$
by (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-star-commute schroeder-4-p star-slide*)
hence $H * (d * H)^\star \leq - d^T$
using *511* **by** (*metis assms*(*1*) *forest-modulo-equivalence-def schroeder-5-p star-slide*)
hence $H * (d * H)^\star \leq - (H * d^T)$
by (*metis assms*(*3*) *p-antitone-iff schroeder-4-p star-slide mult-assoc*)

hence $H * (d * H)^\star \sqcap H * d^T \leq bot$
by (*simp add*: *bot-unique pseudo-complement*)
hence $H * d * (H * (d * H)^\star \sqcap H * d^T) \leq 1$
by (*simp add*: *bot-unique*)
hence *512*: $H * d * H * (d * H)^\star \sqcap H * d * H * d^T \leq 1$
using *univalent-comp-left-dist-inf assms*(*1*) *forest-modulo-equivalence-def mult-assoc* **by** *fastforce*
hence *513*: $H * d * H * (d * H)^\star \sqcap d * H * d^T \leq 1$
proof −
have $d * H * d^T \leq H * d * H * d^T$
by (*metis assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-involutive mult-1-right mult-left-isotone*)
thus *?thesis*
using *512* **by** (*smt dual-order.trans p-antitone p-shunting-swap regular-one-closed*)
qed
have $d^T * H * d \leq 1 \sqcup - H$
using *assms*(*1*) *forest-modulo-equivalence-def dTransHd-le-1 le-supI1* **by** *blast*
hence $(- 1 \sqcap H) * d^T * H \leq - d^T$
by (*metis assms*(*1*) *forest-modulo-equivalence-def dTransHd-le-1 inf.sup-monoid.add-commute le-infI2 p-antitone-iff regular-one-closed schroeder-4-p mult-assoc*)
hence $d * (- 1 \sqcap H) * d^T \leq - H$
by (*metis assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp schroeder-3-p triple-schroeder-p*)
hence $H \sqcap d * (- 1 \sqcap H) * d^T \leq 1$
by (*metis inf.coboundedI1 p-antitone-iff p-shunting-swap regular-one-closed*)
hence $H \sqcap d * d^T \sqcup H \sqcap d * (- 1 \sqcap H) * d^T \leq 1$
using *assms*(*1*) *forest-modulo-equivalence-def le-supI* **by** *blast*
hence $H \sqcap (d * 1 * d^T \sqcup d * (- 1 \sqcap H) * d^T) \leq 1$
using *comp-inf.semiring.distrib-left* **by** *auto*
hence $H \sqcap (d * (1 \sqcup (- 1 \sqcap H)) * d^T) \leq 1$
by (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
hence *514*: $H \sqcap d * H * d^T \leq 1$
by (*metis assms*(*1*) *forest-modulo-equivalence-def comp-inf.semiring.distrib-left inf.le-iff-sup inf.sup-monoid.add-commute inf-top-right regular-one-closed stone*)
thus *?thesis*
proof −
have $H \sqcap d * H * d^T \sqcup H * d * H * (d * H)^\star \sqcap d * H * d^T \leq 1$
using *513 514* **by** *simp*
hence $d * H * d^T \sqcap (H \sqcup H * d * H * (d * H)^\star) \leq 1$
by (*simp add*: *comp-inf.semiring.distrib-left inf.sup-monoid.add-commute*)
hence $d * H * d^T \sqcap H * (1 \sqcup d * H * (d * H)^\star) \leq 1$
by (*simp add*: *mult-left-dist-sup mult-assoc*)
thus *?thesis*
by (*simp add*: *inf.sup-monoid.add-commute star-left-unfold-equal*)

qed
qed
have $?x * top * ?x^T = (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top) * top * (d^T \sqcap H^T * e^{TT} * top^T \sqcap top^T * a^{TT} * H^T * (d^{TT} * H^T)^\star)$
by (*simp add*: *conv-dist-comp conv-dist-inf conv-star-commute mult-assoc*)
also have ... $= (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
also have ... $= (H * d^T)^\star * H * a^T * top \sqcap (d \sqcap top * e^T * H) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
by (*metis inf-vector-comp vector-export-comp*)
also have ... $= (H * d^T)^\star * H * a^T * top \sqcap (d \sqcap top * e^T * H) * top * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
by (*simp add*: *vector-mult-closed*)
also have ... $= (H * d^T)^\star * H * a^T * top \sqcap d * ((top * e^T * H)^T \sqcap top) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
by (*simp add*: *covector-comp-inf-1 covector-mult-closed*)
also have ... $= (H * d^T)^\star * H * a^T * top \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T \sqcap top * a * H * (d * H)^\star$
by (*smt comp-associative comp-inf.star-star-absorb comp-inf-vector conv-star-commute covector-comp-inf covector-conv-vector fc-top star.circ-top total-conv-surjective vector-conv-covector vector-inf-comp*)
also have ... $= (H * d^T)^\star * H * a^T * top \sqcap top * a * H * (d * H)^\star \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T$
using *inf.sup-monoid.add-assoc inf.sup-monoid.add-commute* **by** *auto*
also have ... $= (H * d^T)^\star * H * a^T * top * top * a * H * (d * H)^\star \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T$
by (*smt comp-inf.star.circ-decompose-9 comp-inf.star-star-absorb comp-inf-covector fc-top star.circ-decompose-11 star.circ-top vector-export-comp*)
also have ... $= (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \sqcap d * (H * e * top \sqcap top * e^T * H) * d^T$
using *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp mult-assoc* **by** *auto*
also have ... $= (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \sqcap d * H * e * top * e^T * H * d^T$
by (*metis comp-inf-covector inf-top.left-neutral mult-assoc*)
also have ... $\leq (H * d^T)^\star * (H * d)^\star * H \sqcap d * H * e * top * e^T * H * d^T$
proof −
have $(H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \leq (H * d^T)^\star * H * 1 * H * (d * H)^\star$
using *2* **by** (*metis comp-associative comp-isotone mult-left-isotone mult-semi-associative star.circ-transitive-equal*)
also have ... $= (H * d^T)^\star * H * (d * H)^\star$
using *assms*(*1*) *forest-modulo-equivalence-def mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*
also have ... $= (H * d^T)^\star * (H * d)^\star * H$
by (*metis star-slide mult-assoc*)
finally show *?thesis*
using *inf.sup-left-isotone* **by** *auto*

qed
also have ... $\leq (H * d^T)^\star * (H * d)^\star * H \sqcap d * H * d^T$
proof $-$
have $d * H * e * top * e^T * H * d^T \leq d * H * 1 * H * d^T$
using *3* **by** (*metis comp-isotone idempotent-one-closed mult-left-isotone mult-sub-right-one mult-assoc*)
also have ... $\leq d * H * d^T$
by (*metis assms*(*1*) *forest-modulo-equivalence-def mult-left-isotone mult-one-associative mult-semi-associative preorder-idempotent*)
finally show *?thesis*
using *inf.sup-right-isotone* **by** *auto*
qed
also have ... $= H * (d^T * H)^\star * (H * d)^\star * H \sqcap d * H * d^T$
by (*metis assms*(*1*) *forest-modulo-equivalence-def comp-associative preorder-idempotent star-slide*)
also have ... $= H * ((d^T * H)^\star \sqcup (H * d)^\star) * H \sqcap d * H * d^T$
by (*simp add*: *assms*(*1*) *expand-forest-modulo-equivalence mult.semigroup-axioms semigroup.assoc*)
also have ... $= (H * (d^T * H)^\star * H \sqcup H * (H * d)^\star * H) \sqcap d * H * d^T$
by (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
also have ... $= (H * d^T)^\star * H \sqcap d * H * d^T \sqcup H * (d * H)^\star \sqcap d * H * d^T$
by (*smt assms*(*1*) *forest-modulo-equivalence-def inf-sup-distrib2 mult.semigroup-axioms preorder-idempotent star-slide semigroup.assoc*)
also have ... $\leq (H * d^T)^\star * H \sqcap d * H * d^T \sqcup 1$
using *51 comp-inf.semiring.add-left-mono* **by** *blast*
finally have $?x * top * ?x^T \leq 1$
using *51* **by** (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-dist-inf conv-dist-sup conv-involutive conv-star-commute equivalence-one-closed mult.semigroup-axioms sup.absorb2 semigroup.assoc conv-isotone conv-order*)
thus *?thesis*
by *simp*
qed
have *6*: $?x^T * top * ?x \leq 1$
proof $-$
have $?x^T * top * ?x = (d^T \sqcap H^T * e^{TT} * top^T \sqcap top^T * a^{TT} * H^T * (d^{TT} * H^T)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
by (*simp add*: *conv-dist-comp conv-dist-inf conv-star-commute mult-assoc*)
also have ... $= (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
using *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
also have ... $= H * e * top \sqcap (d^T \sqcap top * a * H * (d * H)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
by (*smt comp-associative inf.sup-monoid.add-assoc inf.sup-monoid.add-commute star.circ-left-top star.circ-top vector-inf-comp*)
also have ... $= H * e * top \sqcap d^T * ((top * a * H * (d * H)^\star)^T \sqcap top) * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
by (*simp add*: *covector-comp-inf-1 covector-mult-closed*)
also have ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * top * (d \sqcap top *$

$e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$

using *assms(1) forest-modulo-equivalence-def comp-associative conv-dist-comp* **by** *auto*

also have ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * top * (d \sqcap (H * d^T)^\star * H * a^T * top) \sqcap top * e^T * H$

by (*smt comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf.sup-monoid.add-commute*)

also have ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * (top \sqcap ((H * d^T)^\star * H * a^T * top)^T) * d \sqcap top * e^T * H$

by (*metis comp-associative comp-inf-vector vector-conv-covector vector-top-closed*)

also have ... $= H * e * top \sqcap (H * e * top)^T \sqcap d^T * (d * H)^{\star T} * H * a^T * ((H * d^T)^\star * H * a^T * top)^T * d$

by (*smt assms(1) forest-modulo-equivalence-def conv-dist-comp inf.left-commute inf.sup-monoid.add-commute symmetric-top-closed mult-assoc inf-top.left-neutral*)

also have ... $= H * e * top * (H * e * top)^T \sqcap d^T * (d * H)^{\star T} * H * a^T * ((H * d^T)^\star * H * a^T * top)^T * d$

using *vector-covector vector-mult-closed* **by** *auto*

also have ... $= H * e * top * top^T * e^T * H^T \sqcap d^T * (d * H)^{\star T} * H * a^T * top^T * a^{TT} * H^T * (H * d^T)^{\star T} * d$

by (*smt conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)

also have ... $= H * e * top * top * e^T * H \sqcap d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d$

using *assms(1) forest-modulo-equivalence-def conv-dist-comp conv-star-commute* **by** *auto*

also have ... $= H * e * top * e^T * H \sqcap d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d$

using *vector-top-closed mult-assoc* **by** *auto*

also have ... $\leq H \sqcap d^T * (H * d^T)^\star * H * (d * H)^\star * d$

proof −

have $H * e * top * e^T * H \leq H * 1 * H$

using *3* **by** (*metis comp-associative mult-left-isotone mult-right-isotone*)

also have ... $= H$

using *assms(1) forest-modulo-equivalence-def preorder-idempotent* **by** *auto*

finally have *611*: $H * e * top * e^T * H \leq H$

by *simp*

have $d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d \leq d^T * (H * d^T)^\star * H * 1 * H * (d * H)^\star * d$

using *2* **by** (*metis comp-associative mult-left-isotone mult-right-isotone*)

also have ... $= d^T * (H * d^T)^\star * H * (d * H)^\star * d$

using *assms(1) forest-modulo-equivalence-def mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*

finally have $d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d \leq d^T * (H * d^T)^\star * H * (d * H)^\star * d$

by *simp*

thus *?thesis*

using *611 comp-inf.comp-isotone* **by** *blast*

qed
also have ... $= H \sqcap (d^T * H)^\star * d^T * H * d * (H * d)^\star$
using *star-slide mult-assoc* **by** *auto*
also have ... $\leq H \sqcap (d^T * H)^\star * (H * d)^\star$
proof −
have $(d^T * H)^\star * d^T * H * d * (H * d)^\star \leq (d^T * H)^\star * 1 * (H * d)^\star$
by (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp mult-left-isotone mult-right-isotone preorder-idempotent mult-assoc*)
also have ... $= (d^T * H)^\star * (H * d)^\star$
by *simp*
finally show *?thesis*
using *inf.sup-right-isotone* **by** *blast*
qed
also have ... $= H \sqcap ((d^T * H)^\star \sqcup (H * d)^\star)$
by (*simp add*: *assms*(*1*) *expand-forest-modulo-equivalence*)
also have ... $= H \sqcap (d^T * H)^\star \sqcup H \sqcap (H * d)^\star$
by (*simp add*: *comp-inf.semiring.distrib-left*)
also have ... $= 1 \sqcup H \sqcap (d^T * H)^+ \sqcup H \sqcap (H * d)^+$
proof −
have *612*: $H \sqcap (H * d)^\star = 1 \sqcup H \sqcap (H * d)^+$
using *assms*(*1*) *forest-modulo-equivalence-def reflexive-inf-star* **by** *blast*
have $H \sqcap (d^T * H)^\star = 1 \sqcup H \sqcap (d^T * H)^+$
using *assms*(*1*) *forest-modulo-equivalence-def reflexive-inf-star* **by** *auto*
thus *?thesis*
using *612 sup-assoc sup-commute* **by** *auto*
qed
also have ... ≤ 1
proof −
have *613*: $H \sqcap (H * d)^+ \leq 1$
by (*metis assms*(*3*) *inf.coboundedI1 p-antitone-iff p-shunting-swap regular-one-closed*)
hence $H \sqcap (d^T * H)^+ \leq 1$
by (*metis assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-dist-inf conv-plus-commute coreflexive-symmetric*)
thus *?thesis*
by (*simp add*: *613*)
qed
finally show *?thesis*
by *simp*
qed
have *7*:*bijective* (*?x* $*$ *top*)
using *4 5 6 arc-expanded* **by** *blast*
have *bijective* ($?x^T$ $*$ *top*)
using *4 5 6 arc-expanded* **by** *blast*
thus *?thesis*
using *7* **by** *simp*
qed

To maintain that f can be extended to a minimum spanning forest we identify an edge, $i = v \sqcap \overline{F}e\top \sqcap \top e^\top F$, that may be exchanged with the *selected-edge*, e. Here, we show that i is an *arc*.

lemma *boruvka-edge-arc*:
assumes *equivalence* F
and *forest* v
and *arc* e
and *regular* F
and $F \leq$ *forest-components* $(F \sqcap v)$
and *regular* v
and $v * e^T = bot$
and $e * F * e = bot$
and $e^T \leq v^\star$
and $e \neq bot$
shows *arc* $(v \sqcap -F * e * top \sqcap top * e^T * F)$
proof –
let $?i = v \sqcap -F * e * top \sqcap top * e^T * F$
have 1: $?i^T * top * ?i \leq 1$
proof –
have $?i^T * top * ?i = (v^T \sqcap top * e^T * -F \sqcap F * e * top) * top * (v \sqcap -F * e * top \sqcap top * e^T * F)$
using *assms*(*1*) *conv-complement conv-dist-comp conv-dist-inf mult.semigroup-axioms semigroup.assoc* **by** *fastforce*
also have ... $= F * e * top \sqcap (v^T \sqcap top * e^T * -F) * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
by (*smt covector-comp-inf covector-mult-closed inf-vector-comp vector-export-comp vector-top-closed*)
also have ... $= F * e * top \sqcap (v^T \sqcap top * e^T * -F) * top * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
by (*simp add*: *comp-associative*)
also have ... $= F * e * top \sqcap v^T * (top \sqcap (top * e^T * -F)^T) * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
using *comp-associative comp-inf-vector-1* **by** *auto*
also have ... $= F * e * top \sqcap v^T * (top \sqcap (top * e^T * -F)^T) * (top \sqcap (-F * e * top)^T) * v \sqcap top * e^T * F$
by (*smt comp-inf-vector conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)
also have ... $= F * e * top \sqcap v^T * (top * e^T * -F)^T * (-F * e * top)^T * v \sqcap top * e^T * F$
by *simp*
also have ... $= F * e * top \sqcap v^T * -F^T * e^{TT} * top^T * top^T * e^T * -F^T * v \sqcap top * e^T * F$
by (*metis comp-associative conv-complement conv-dist-comp*)
also have ... $= F * e * top \sqcap v^T * -F * e * top * top * e^T * -F * v \sqcap top * e^T * F$
by (*simp add*: *assms*(*1*))
also have ... $= F * e * top \sqcap v^T * -F * e * top \sqcap top * e^T * -F * v \sqcap top * e^T * F$
by (*metis comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf-top.left-neutral vector-top-closed*)
also have ... $= (F \sqcap v^T * -F) * e * top \sqcap top * e^T * -F * v \sqcap top * e^T * F$

using *assms*(*3*) *injective-comp-right-dist-inf mult-assoc* **by** *auto*
also have ... $= (F \sqcap v^T * -F) * e * top \sqcap top * e^T * (F \sqcap -F * v)$
using *assms*(*3*) *conv-dist-comp inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult.semigroup-axioms univalent-comp-left-dist-inf semigroup.assoc* **by** *fastforce*
also have ... $= (F \sqcap v^T * -F) * e * top * top * e^T * (F \sqcap -F * v)$
by (*metis comp-associative comp-inf-covector inf-top.left-neutral vector-top-closed*)
also have ... $= (F \sqcap v^T * -F) * e * top * e^T * (F \sqcap -F * v)$
by (*simp add*: *comp-associative*)
also have ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v)$
by (*smt assms*(*3*) *conv-dist-comp mult-left-isotone shunt-bijective symmetric-top-closed top-right-mult-increasing mult-assoc*)
also have ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v) \sqcap F$
by (*metis assms*(*1*) *inf.absorb1 inf.cobounded1 mult-isotone preorder-idempotent*)
also have ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v) \sqcap (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
using *assms*(*5*) *comp-inf.mult-right-isotone* **by** *auto*
also have ... $\leq (-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
proof −
have $F \sqcap v^T * -F \leq (v^T \sqcap F * -F^T) * -F$
by (*metis conv-complement dedekind-2 inf-commute*)
also have ... $= (v^T \sqcap -F^T) * -F$
using *assms*(*1*) *equivalence-comp-left-complement* **by** *simp*
finally have $F \sqcap v^T * -F \leq F \sqcap (v^T \sqcap -F) * -F$
using *assms*(*1*) **by** *auto*
hence *11*: $F \sqcap v^T * -F = F \sqcap (-F \sqcap v^T) * -F$
by (*metis inf.antisym-conv inf.sup-monoid.add-commute comp-left-subdist-inf inf.boundedE inf.sup-right-isotone*)
hence $F^T \sqcap -F^T * v^{TT} = F^T \sqcap -F^T * (-F^T \sqcap v^{TT})$
by (*metis* (*full-types*) *assms*(*1*) *conv-complement conv-dist-comp conv-dist-inf*)
hence *12*: $F \sqcap -F * v = F \sqcap -F * (-F \sqcap v)$
using *assms*(*1*) **by** (*simp add*: *abel-semigroup.commute inf.abel-semigroup-axioms*)
have $(F \sqcap v^T * -F) * (F \sqcap -F * v) = (F \sqcap (-F \sqcap v^T) * -F) * (F \sqcap -F * (-F \sqcap v))$
using *11 12* **by** *auto*
also have ... $\leq (-F \sqcap v^T) * -F * -F * (-F \sqcap v)$
by (*metis comp-associative comp-isotone inf.cobounded2*)
finally show *?thesis*
using *comp-inf.mult-left-isotone* **by** *blast*
qed
also have ... $= ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
by (*metis comp-associative inf-sup-distrib1 star.circ-loop-fixpoint*)
also have ... $= ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v^T) * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
using *assms*(*1*) *conv-dist-inf* **by** *auto*

also have ... $= bot \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
proof $-$
have $(-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v^T) * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq bot$
using *assms*(*1*, *2*) *forests-bot-2* **by** (*simp add*: *comp-associative*)
thus *?thesis*
using *le-bot* **by** *blast*
qed
also have ... $= (-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (1 \sqcup (F \sqcap v)^\star * (F \sqcap v))$
by (*simp add*: *star.circ-plus-same star-left-unfold-equal*)
also have ... $= ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap 1) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star * (F \sqcap v))$
by (*simp add*: *comp-inf.semiring.distrib-left*)
also have ... $\leq 1 \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star * (F \sqcap v))$
using *sup-left-isotone* **by** *auto*
also have ... $\leq 1 \sqcup bot$
using *assms*(*1*, *2*) *forests-bot-3 comp-inf.semiring.add-left-mono* **by** *simp*
finally show *?thesis*
by *simp*
qed
have *2*: $?i * top * ?i^T \leq 1$
proof $-$
have $?i * top * ?i^T = (v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap (-F * e * top)^T \sqcap (top * e^T * F)^T)$
by (*simp add*: *conv-dist-inf*)
also have ... $= (v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap top^T * e^T * -F^T \sqcap F^T * e^{TT} * top^T)$
by (*simp add*: *conv-complement conv-dist-comp mult-assoc*)
also have ... $= (v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap top * e^T * -F \sqcap F * e * top)$
by (*simp add*: *assms*(*1*))
also have ... $= -F * e * top \sqcap (v \sqcap top * e^T * F) * top * (v^T \sqcap top * e^T * -F \sqcap F * e * top)$
by (*smt inf.left-commute inf.sup-monoid.add-assoc vector-export-comp*)
also have ... $= -F * e * top \sqcap (v \sqcap top * e^T * F) * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
by (*smt comp-inf-covector inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult-assoc*)
also have ... $= -F * e * top \sqcap (v \sqcap top * e^T * F) * top * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
by (*simp add*: *mult-assoc*)
also have ... $= -F * e * top \sqcap v * ((top * e^T * F)^T \sqcap top) * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
by (*simp add*: *comp-inf-vector-1 mult.semigroup-axioms semigroup.assoc*)
also have ... $= -F * e * top \sqcap v * ((top * e^T * F)^T \sqcap top) * (top \sqcap (F * e * top)^T) * v^T \sqcap top * e^T * -F$
by (*smt comp-inf-vector covector-comp-inf vector-conv-covector*

vector-mult-closed vector-top-closed)
also have ... $= -F * e * top \sqcap v * (top * e^T * F)^T * (F * e * top)^T * v^T \sqcap top * e^T * -F$
by *simp*
also have ... $= -F * e * top \sqcap v * F^T * e^{TT} * top^T * top^T * e^T * F^T * v^T \sqcap top * e^T * -F$
by (*metis comp-associative conv-dist-comp*)
also have ... $= -F * e * top \sqcap v * F * e * top * top * e^T * F * v^T \sqcap top * e^T * -F$
using *assms*(*1*) **by** *auto*
also have ... $= -F * e * top \sqcap v * F * e * top \sqcap top * e^T * F * v^T \sqcap top * e^T * -F$
by (*smt comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf-top.left-neutral vector-top-closed*)
also have ... $= (-F \sqcap v * F) * e * top \sqcap top * e^T * F * v^T \sqcap top * e^T * -F$
using *injective-comp-right-dist-inf assms*(*3*) *mult.semigroup-axioms semigroup.assoc* **by** *fastforce*
also have ... $= (-F \sqcap v * F) * e * top \sqcap top * e^T * (F * v^T \sqcap -F)$
using *injective-comp-right-dist-inf assms*(*3*) *conv-dist-comp inf.sup-monoid.add-assoc mult.semigroup-axioms univalent-comp-left-dist-inf semigroup.assoc* **by** *fastforce*
also have ... $= (-F \sqcap v * F) * e * top * top * e^T * (F * v^T \sqcap -F)$
by (*metis inf-top-right vector-export-comp vector-top-closed*)
also have ... $= (-F \sqcap v * F) * e * top * e^T * (F * v^T \sqcap -F)$
by (*simp add*: *comp-associative*)
also have ... $\leq (-F \sqcap v * F) * (F * v^T \sqcap -F)$
by (*smt assms*(*3*) *conv-dist-comp mult.semigroup-axioms mult-left-isotone shunt-bijective symmetric-top-closed top-right-mult-increasing semigroup.assoc*)
also have ... $= (-F \sqcap v * F) * ((v * F)^T \sqcap -F)$
by (*simp add*: *assms*(*1*) *conv-dist-comp*)
also have ... $= (-F \sqcap v * F) * (-F \sqcap v * F)^T$
using *assms*(*1*) *conv-complement conv-dist-inf* **by** (*simp add*: *inf.sup-monoid.add-commute*)
also have ... $\leq (-F \sqcap v) * (F \sqcap v)^\star * (F \sqcap v)^{T\star} * (-F \sqcap v)^T$
proof −
let *?Fv* $= F \sqcap v$
have $-F \sqcap v * F \leq -F \sqcap v * (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
using *assms*(*5*) *inf.sup-right-isotone mult-right-isotone comp-associative* **by** *auto*
also have ... $\leq -F \sqcap v * (F \sqcap v)^\star$
proof −
have $v * v^T \leq 1$
by (*simp add*: *assms*(*2*))
hence $v * v^T * F \leq F$
using *assms*(*1*) *dual-order.trans mult-left-isotone* **by** *blast*
hence $v * v^T * F^{T\star} * F^\star \leq F$
by (*metis assms*(*1*) *mult-1-right preorder-idempotent star.circ-sup-one-right-unfold star.circ-transitive-equal star-one star-simulation-right-equal mult-assoc*)

hence $v * (F \sqcap v)^T * F^{T\star} * F^\star \leq F$
by (*meson conv-isotone dual-order.trans inf.cobounded2 inf.sup-monoid.add-commute mult-left-isotone mult-right-isotone*)
hence $v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq F$
by (*meson conv-isotone dual-order.trans inf.cobounded2 inf.sup-monoid.add-commute mult-left-isotone mult-right-isotone comp-isotone conv-dist-inf inf.cobounded1 star-isotone*)
hence $-F \sqcap v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq bot$
using *order.eq-iff p-antitone pseudo-complement* **by** *auto*
hence $(-F \sqcap v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup v * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
using *bot-least le-bot* **by** *fastforce*
hence $(-F \sqcup v * (v \sqcap F)^\star) \sqcap (v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcup v * (v \sqcap F)^\star) \leq v * (v \sqcap F)^\star$
by (*simp add*: *sup-inf-distrib2*)
hence $(-F \sqcup v * (v \sqcap F)^\star) \sqcap v * ((F \sqcap v)^T * (F \sqcap v)^{T\star} \sqcup 1) * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
by (*simp add*: *inf.sup-monoid.add-commute mult.semigroup-axioms mult-left-dist-sup mult-right-dist-sup semigroup.assoc*)
hence $(-F \sqcup v * (v \sqcap F)^\star) \sqcap v * (F \sqcap v)^{T\star} * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
by (*simp add*: *star-left-unfold-equal sup-commute*)
hence $-F \sqcap v * (F \sqcap v)^{T\star} * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
using *comp-inf.mult-right-sub-dist-sup-left inf.order-lesseq-imp* **by** *blast*
thus *?thesis*
by (*simp add*: *inf.sup-monoid.add-commute*)
qed
also have ... $\leq (v \sqcap -F * (F \sqcap v)^{T\star}) * (F \sqcap v)^\star$
by (*metis dedekind-2 conv-star-commute inf.sup-monoid.add-commute*)
also have ... $\leq (v \sqcap -F * F^{T\star}) * (F \sqcap v)^\star$
using *conv-isotone inf.sup-right-isotone mult-left-isotone mult-right-isotone star-isotone* **by** *auto*
also have ... $= (v \sqcap -F * F) * (F \sqcap v)^\star$
by (*metis assms*(*1*) *equivalence-comp-right-complement mult-left-one star-one star-simulation-right-equal*)
also have ... $= (-F \sqcap v) * (F \sqcap v)^\star$
using *assms*(*1*) *equivalence-comp-right-complement inf.sup-monoid.add-commute* **by** *auto*
finally have $-F \sqcap v * F \leq (-F \sqcap v) * (F \sqcap v)^\star$
by *simp*
hence $(-F \sqcap v * F) * (-F \sqcap v * F)^T \leq (-F \sqcap v) * (F \sqcap v)^\star * ((-F \sqcap v) * (F \sqcap v)^\star)^T$
by (*simp add*: *comp-isotone conv-isotone*)
also have ... $= (-F \sqcap v) * (F \sqcap v)^\star * (F \sqcap v)^{T\star} * (-F \sqcap v)^T$
by (*simp add*: *comp-associative conv-dist-comp conv-star-commute*)
finally show *?thesis*
by *simp*
qed
also have ... $\leq (-F \sqcap v) * ((F \sqcap v^\star) \sqcup (F \sqcap v^{T\star})) * (-F \sqcap v)^T$
proof −

have $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F^\star * F^{T\star}$
using *fc-isotone* **by** *auto*
also have ... $\leq F * F$
by (*metis assms*(*1*) *preorder-idempotent star.circ-sup-one-left-unfold star.circ-transitive-equal star-right-induct-mult*)
finally have *21*: $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F$
using *assms*(*1*) *dual-order.trans* **by** *blast*
have $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq v^\star * v^{T\star}$
by (*simp add*: *fc-isotone*)
hence $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F \sqcap v^\star * v^{T\star}$
using *21* **by** *simp*
also have ... $= F \sqcap (v^\star \sqcup v^{T\star})$
by (*simp add*: *assms*(*2*) *cancel-separate-eq*)
finally show *?thesis*
by (*metis assms*(*4*, *6*) *comp-associative comp-inf.semiring.distrib-left comp-isotone inf-pp-semi-commute mult-left-isotone regular-closed-inf*)
qed
also have ... $\leq (-F \sqcap v) * (F \sqcap v^{T\star}) * (-F \sqcap v)^T \sqcup (-F \sqcap v) * (F \sqcap v^\star) * (-F \sqcap v)^T$
by (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
also have ... $\leq (-F \sqcap v) * (-F \sqcap v)^T \sqcup (-F \sqcap v) * (-F \sqcap v)^T$
proof −
have $(-F \sqcap v) * (F \sqcap v^{T\star}) \leq (-F \sqcap v) * ((F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcap v^{T\star})$
by (*simp add*: *assms*(*5*) *inf.coboundedI1 mult-right-isotone*)
also have ... $= (-F \sqcap v) * ((F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcap v^{T\star}) \sqcup (-F \sqcap v) * ((F \sqcap v)^\star \sqcap v^{T\star})$
by (*metis comp-associative comp-inf.mult-right-dist-sup mult-left-dist-sup star.circ-loop-fixpoint*)
also have ... $\leq (-F \sqcap v) * (F \sqcap v)^T * top \sqcup (-F \sqcap v) * ((F \sqcap v)^\star \sqcap v^{T\star})$
by (*simp add*: *comp-associative comp-isotone inf.coboundedI2 inf.sup-monoid.add-commute le-supI1*)
also have ... $\leq (-F \sqcap v) * (F \sqcap v)^T * top \sqcup (-F \sqcap v) * (v^\star \sqcap v^{T\star})$
by (*smt comp-inf.mult-right-isotone comp-inf.semiring.add-mono order.eq-iff inf.cobounded2 inf.sup-monoid.add-commute mult-right-isotone star-isotone*)
also have ... $\leq bot \sqcup (-F \sqcap v) * (v^\star \sqcap v^{T\star})$
by (*metis assms*(*1*, *2*) *forests-bot-1 comp-associative comp-inf.semiring.add-right-mono mult-semi-associative vector-bot-closed*)
also have ... $\leq -F \sqcap v$
by (*simp add*: *assms*(*2*) *acyclic-star-inf-conv*)
finally have *22*: $(-F \sqcap v) * (F \sqcap v^{T\star}) \leq -F \sqcap v$
by *simp*
have $((-F \sqcap v) * (F \sqcap v^{T\star}))^T = (F \sqcap v^\star) * (-F \sqcap v)^T$
by (*simp add*: *assms*(*1*) *conv-dist-inf conv-star-commute conv-dist-comp*)
hence $(F \sqcap v^\star) * (-F \sqcap v)^T \leq (-F \sqcap v)^T$
using *22 conv-isotone* **by** *fastforce*
thus *?thesis*
using *22* **by** (*metis assms*(*4*, *6*) *comp-associative comp-inf.pp-comp-semi-commute comp-inf.semiring.add-mono comp-isotone*

inf-pp-commute mult-left-isotone)
qed
also have ... $= (-F \sqcap v) * (-F \sqcap v)^T$
by *simp*
also have ... $\leq v * v^T$
by (*simp add*: *comp-isotone conv-isotone*)
also have ... ≤ 1
by (*simp add*: *assms*(*2*))
thus *?thesis*
using *calculation dual-order.trans* **by** *blast*
qed
have *3*: $top * ?i * top = top$
proof −
have *31*: *regular* $(e^T * -F * v * F * e)$
using *assms*(*3*, *4*, *6*) *arc-regular regular-mult-closed* **by** *auto*
have *32*: *bijective* $((top * e^T)^T)$
using *assms*(*3*) **by** (*simp add*: *conv-dist-comp*)
have $top * ?i * top = top * (v \sqcap -F * e * top) * ((top * e^T * F)^T \sqcap top)$
by (*simp add*: *comp-associative comp-inf-vector-1*)
also have ... $= (top \sqcap (-F * e * top)^T) * v * ((top * e^T * F)^T \sqcap top)$
using *comp-inf-vector conv-dist-comp* **by** *auto*
also have ... $= (-F * e * top)^T * v * (top * e^T * F)^T$
by *simp*
also have ... $= top^T * e^T * -F^T * v * F^T * e^{TT} * top^T$
by (*simp add*: *comp-associative conv-complement conv-dist-comp*)
finally have *33*: $top * ?i * top = top * e^T * -F * v * F * e * top$
by (*simp add*: *assms*(*1*))
have $top * ?i * top \neq bot$
proof (*rule ccontr*)
assume $\neg\ top * (v \sqcap -\ F * e * top \sqcap top * e^T * F) * top \neq bot$
hence $top * e^T * -F * v * F * e * top = bot$
using *33* **by** *auto*
hence $e^T * -F * v * F * e = bot$
using *31 tarski comp-associative le-bot* **by** *fastforce*
hence $top * (-F * v * F * e)^T \leq -(e^T)$
by (*metis comp-associative conv-complement-sub-leq conv-involutive p-bot schroeder-5-p*)
hence $top * e^T * F^T * v^T * -F^T \leq -(e^T)$
by (*simp add*: *comp-associative conv-complement conv-dist-comp*)
hence $v * F * e * top * e^T \leq F$
by (*metis assms*(*1*, *4*) *comp-associative conv-dist-comp schroeder-3-p symmetric-top-closed*)
hence $v * F * e * top * top * e^T \leq F$
by (*simp add*: *comp-associative*)
hence $v * F * e * top \leq F * (top * e^T)^T$
using *32* **by** (*metis shunt-bijective comp-associative conv-involutive*)
hence $v * F * e * top \leq F * e * top$
using *comp-associative conv-dist-comp* **by** *auto*
hence $v^\star * F * e * top \leq F * e * top$

using *comp-associative star-left-induct-mult-iff* **by** *auto*
hence $e^T * F * e * top \leq F * e * top$
by (*meson assms*(*9*) *mult-left-isotone order-trans*)
hence $e^T * F * e * top * (e * top)^T \leq F$
using *32 shunt-bijective assms*(*3*) *mult-assoc* **by** *auto*
hence *34*: $e^T * F * e * top * top * e^T \leq F$
by (*metis conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)
hence $e^T \leq F$
proof −
have $e^T \leq e^T * e * e^T$
by (*metis conv-involutive ex231c*)
also have ... $\leq e^T * F * e * e^T$
using *assms*(*1*) *comp-associative mult-left-isotone mult-right-isotone* **by** *fastforce*
also have ... $\leq e^T * F * e * top * top * e^T$
by (*simp add*: *mult-left-isotone top-right-mult-increasing vector-mult-closed*)
finally show *?thesis*
using *34* **by** *simp*
qed
hence *35*: $e \leq F$
using *assms*(*1*) *conv-order* **by** *fastforce*
have $top * (F * e)^T \leq - e$
using *assms*(*8*) *comp-associative schroeder-4-p* **by** *auto*
hence $top * e^T * F \leq - e$
by (*simp add*: *assms*(*1*) *comp-associative conv-dist-comp*)
hence $(top * e^T)^T * e \leq - F$
using *schroeder-3-p* **by** *auto*
hence $e * top * e \leq - F$
by (*simp add*: *conv-dist-comp*)
hence $e \leq - F$
by (*simp add*: *assms*(*3*) *arc-top-arc*)
hence $e \leq F \sqcap - F$
using *35 inf.boundedI* **by** *blast*
hence $e = bot$
using *bot-unique* **by** *auto*
thus *False*
using *assms*(*10*) **by** *auto*
qed
thus *?thesis*
by (*metis assms*(*3*, *4*, *6*) *arc-regular regular-closed-inf regular-closed-top regular-conv-closed regular-mult-closed semiring.mult-not-zero tarski*)
qed
have *bijective* $(?i * top) \wedge$ *bijective* $(?i^T * top)$
using *1 2 3 arc-expanded* **by** *blast*
thus *?thesis*
by *blast*
qed

4.4.3 Comparison of edge weights

In this section we compare the weight of the *selected-edge* with other edges of interest. For example, Theorem *e-leq-c-c-complement-transpose-general* is used to show that the *selected-edge* has its source inside and its target outside the component it is chosen for.

lemma *e-leq-c-c-complement-transpose-general*:
  assumes $e = minarc\ (v * -(v)^T \sqcap g)$
    and *regular* v
  shows $e \le v * -(v)^T$
proof –
  have $e \le --\ (v * -\ v^T \sqcap g)$
    using *assms*(*1*) *minarc-below order-trans* **by** *blast*
  also have ... $\le --\ (v * -\ v^T)$
    using *order-lesseq-imp pp-isotone-inf* **by** *blast*
  also have ... $= v * -\ v^T$
    using *assms*(*2*) *regular-mult-closed* **by** *auto*
  finally show *?thesis*
    by *simp*
qed

lemma *x-leq-c-transpose-general*:
  assumes *vector-classes* $x\ v$
    and $a^T * top \le x * e * top$
    and $e \le v * -v^T$
  shows $a \le v^T$
proof –
  have *1*: *equivalence* x
    using *assms*(*1*) **using** *vector-classes-def* **by** *blast*
  have $a \le top * a$
    using *top-left-mult-increasing* **by** *blast*
  also have ... $\le (x * e * top)^T$
    using *assms*(*2*) *conv-dist-comp conv-isotone* **by** *fastforce*
  also have ... $= top * e^T * x$
    using *1* **by** (*simp add*: *conv-dist-comp mult-assoc*)
  also have ... $\le top * (v * -\ v^T)^T * x$
    by (*metis assms*(*3*) *conv-dist-comp conv-isotone mult-left-isotone*
symmetric-top-closed)
  also have ... $= top * (-\ v * v^T) * x$
    by (*simp add*: *conv-complement conv-dist-comp*)
  also have ... $\le top * v^T * x$
    by (*metis mult-left-isotone top.extremum mult-assoc*)
  also have ... $= v^T * x$
    using *assms*(*1*) *vector-classes-def vector-conv-covector* **by** *auto*
  also have ... $= v^T$
    by (*metis assms*(*1*) *order.antisym conv-dist-comp conv-order dual-order.trans*
mult-right-isotone mult-sub-right-one vector-classes-def)
  finally show *?thesis*
    by *simp*

qed

lemma *x-leq-c-complement-general*:
  assumes *vector v*
    and $v * v^T \leq x$
    and $a \leq v^T$
    and $a \leq -x$
  shows $a \leq -v$
proof –
  have $a \leq -x \sqcap v^T$
    using *assms*(*3*, *4*) **by** *auto*
  also have ... $\leq -\ v$
  proof –
    have $v \sqcap v^T \leq x$
      using *assms*(*1*, *2*) *vector-covector* **by** *auto*
    hence $-x \sqcap v \sqcap v^T \leq bot$
      using *inf.sup-monoid.add-assoc p-antitone pseudo-complement* **by** *fastforce*
    thus *?thesis*
      using *le-bot p-shunting-swap pseudo-complement* **by** *blast*
  qed
  finally show *?thesis*
    by *simp*
qed

lemma *sum-e-below-sum-a-when-outgoing-same-component-general*:
  assumes $e = minarc\ (v * -(v)^T \sqcap g)$
    and *symmetric g*
    and *arc a*
    and $a \leq -x \sqcap --\ g$
    and $a^T * top \leq x * e * top$
    and *unique-vector-class x v*
  shows $sum\ (e \sqcap g) \leq sum\ (a \sqcap g)$
proof –
  have *1*:$e \leq v * -\ v^T$
    using *assms*(*1*, *6*) *e-leq-c-c-complement-transpose-general*
unique-vector-class-def vector-classes-def **by** *auto*
  have *2*: $a \leq v^T$
    using *1 assms*(*5*) *assms*(*6*) *x-leq-c-transpose-general unique-vector-class-def*
by *blast*
  hence $a \leq -v$
    using *assms*(*4*, *6*) *inf.boundedE unique-vector-class-def vector-classes-def*
x-leq-c-complement-general **by** *meson*
  hence $a \leq -\ v \sqcap v^T$
    using *2* **by** *simp*
  hence $a \leq -\ v * v^T$
    using *assms*(*6*) *vector-complement-closed vector-covector*
unique-vector-class-def vector-classes-def **by** *metis*
  hence $a^T \leq v^{TT} * -\ v^T$
    using *conv-complement conv-dist-comp conv-isotone* **by** *metis*

hence *3*: $a^T \leq v * - v^T$
by *simp*
hence $a \leq - - g$
using *assms*(*4*) **by** *auto*
hence $a^T \leq - - g$
using *assms*(*2*) *conv-complement conv-isotone* **by** *fastforce*
hence $a^T \sqcap v * - v^T \sqcap - - g \neq bot$
using *3 assms*(*3*, *6*) *inf.orderE semiring.mult-not-zero unique-vector-class-def vector-classes-def* **by** *metis*
hence $a^T \sqcap v * - v^T \sqcap g \neq bot$
using *inf.sup-monoid.add-commute pp-inf-bot-iff* **by** *auto*
hence $sum\ (minarc\ (v * - v^T \sqcap g) \sqcap (v * - v^T \sqcap g)) \leq sum\ (a^T \sqcap v * - v^T \sqcap g)$
using *assms*(*3*) *minarc-min inf.sup-monoid.add-assoc* **by** *simp*
hence $sum\ (e \sqcap v * - v^T \sqcap g) \leq sum\ (a^T \sqcap v * - v^T \sqcap g)$
using *assms*(*1*, *6*) *inf.sup-monoid.add-assoc* **by** *simp*
hence $sum\ (e \sqcap g) \leq sum\ (a^T \sqcap g)$
using *1 3* **by** (*metis inf.orderE*)
hence $sum\ (e \sqcap g) \leq sum\ (a \sqcap g)$
by (*simp add*: *assms*(*2*) *sum-symmetric*)
thus *?thesis*
by *simp*
qed

lemma *sum-e-below-sum-x-when-outgoing-same-component*:
assumes *symmetric g*
and *vector j*
and *forest h*
and *regular h*
and $x \leq -\ forest\text{-}components\ h \sqcap - - g$
and $x^T * top \leq forest\text{-}components\ h * selected\text{-}edge\ h\ j\ g * top$
and $j \neq bot$
and *arc x*
shows $sum\ (selected\text{-}edge\ h\ j\ g \sqcap g) \leq sum\ (x \sqcap g)$
proof −
let *?e* = *selected-edge h j g*
let *?c* = *choose-component* (*forest-components h*) *j*
let *?H* = *forest-components h*
show *?thesis*
proof (*rule sum-e-below-sum-a-when-outgoing-same-component-general*)
next
show $?e = minarc\ (?c * - ?c^T \sqcap g)$
by *simp*
next
show *symmetric g*
by (*simp add*: *assms*(*1*))
next
show *arc x*
by (*simp add*: *assms*(*8*))

```
  next
    show x ≤ − ?H ⊓ −− g
      using assms(5) by auto
  next
    show x^T ∗ top ≤ ?H ∗ ?e ∗ top
      by (simp add: assms(6))
  next
    show unique-vector-class ?H ?c
    proof (unfold unique-vector-class-def, unfold vector-classes-def, intro conjI)
    next
      show regular ?H
        by (metis assms(4) conv-complement pp-dist-star regular-mult-closed)
    next
      show regular ?c
        using component-is-regular by auto
    next
      show reflexive ?H
        using assms(3) forest-components-equivalence by blast
    next
      show transitive ?H
        using assms(3) fch-equivalence by blast
    next
      show symmetric ?H
        by (simp add: assms(3) fch-equivalence)
    next
      show vector ?c
        by (simp add: assms(2, 6) component-is-vector)
    next
      show ?H ∗ ?c ≤ ?c
        using component-single by auto
    next
      show ?c ≠ bot
        using assms(2, 6 , 7, 8) inf-bot-left le-bot minarc-bot mult-left-zero
mult-right-zero by fastforce
    next
      show ?c ∗ ?c^T ≤ ?H
        by (simp add: component-is-connected)
    qed
  qed
qed
```

If there is a path in the *forest-modulo-equivalence* from an edge between components, a, to the *selected-edge*, e, then the weight of e is no greater than the weight of a. This is because either,

* the edges a and e are adjacent the same component so that we can use *sum-e-below-sum-x-when-outgoing-same-component*, or

* there is at least one edge between a and e, namely x, the edge incoming to the component that e is outgoing from. The path from a to e is split

on x using *forest-modulo-equivalence-path-split-disj*. We show that the weight of e is no greater than the weight of x by making use of lemma *sum-e-below-sum-x-when-outgoing-same-component*. We define x in a way that we can show that the weight of x is no greater than the weight of a using the invariant. Then, it follows that the weight of e is no greater than the weight of a owing to transitivity.

lemma *a-to-e-in-forest-modulo-equivalence*:
  assumes *symmetric g*
    and $f \leq --g$
    and *vector j*
    and *forest h*
    and *forest-modulo-equivalence* (*forest-components h*) *d*
    and $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
    and ($\forall$ *a b* . *forest-modulo-equivalence-path a b* (*forest-components h*) $d \wedge a \leq$ $-$(*forest-components h*) $\sqcap --\ g \wedge b \leq d \longrightarrow sum(b \sqcap g) \leq sum(a \sqcap g)$)
    and *regular d*
    and $j \neq bot$
    and $b =$ *selected-edge h j g*
    and *arc a*
    and *forest-modulo-equivalence-path a b* (*forest-components h*) ($d \sqcup$ *selected-edge h j g*)
    and $a \leq -$ *forest-components h* $\sqcap --\ g$
    and *regular h*
  shows *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
proof $-$
  let *?p* = *path f h j g*
  let *?e* = *selected-edge h j g*
  let *?F* = *forest-components f*
  let *?H* = *forest-components h*
  have *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
  proof (*cases* $a^T * top \leq ?H * ?e * top$)
    case *True*
    show $a^T * top \leq ?H * ?e * top \Longrightarrow$ *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
    proof$-$
      have *sum* $(?e \sqcap g) \leq$ *sum* $(a \sqcap g)$
      proof (*rule sum-e-below-sum-x-when-outgoing-same-component*)
        show *symmetric g*
          using *assms*(*1*) **by** *auto*
      next
        show *vector j*
          using *assms*(*3*) **by** *blast*
      next
        show *forest h*
          by (*simp add*: *assms*(*4*))
      next
        show $a \leq -\ ?H \sqcap --\ g$
          using *assms*(*13*) **by** *auto*
      next

show $a^T * top \leq ?H * ?e * top$
using *True* **by** *auto*
next
show $j \neq bot$
by (*simp add*: *assms*(*9*))
next
show *arc a*
by (*simp add*: *assms*(*11*))
next
show *regular h*
using *assms*(*14*) **by** *auto*
qed
thus *?thesis*
using *assms*(*10*) **by** *auto*
qed
next
case *False*
show $\neg\ a^T * top \leq ?H * ?e * top \Longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof −
let $?d' = d \sqcup ?e$
let $?x = d \sqcap top * ?e^T * ?H \sqcap (?H * d^T)^\star * ?H * a^T * top$
have *61*: *arc* (*?x*)
proof (*rule shows-arc-x*)
show *forest-modulo-equivalence ?H d*
by (*simp add*: *assms*(*5*))
next
show *forest-modulo-equivalence-path a ?e ?H d*
proof −
have *611*: *forest-modulo-equivalence-path a b ?H* ($d \sqcup b$)
using *assms*(*10*, *12*) **by** *auto*
have *616*: *regular h*
using *assms*(*14*) **by** *auto*
have *regular a*
using *611 forest-modulo-equivalence-path-def arc-regular* **by** *fastforce*
thus *?thesis*
using *616* **by** (*smt forest-modulo-equivalence-path-split-disj assms*(*4*, *8*, *10*, *12*) *forest-modulo-equivalence-path-def fch-equivalence minarc-regular regular-closed-star regular-conv-closed regular-mult-closed*)
qed
next
show $(?H * d)^+ \leq -\ ?H$
using *assms*(*5*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
next
show $\neg\ a^T * top \leq ?H * ?e * top$
by (*simp add*: *False*)
next
show *regular a*
using *assms*(*12*) *forest-modulo-equivalence-path-def arc-regular* **by** *auto*

next
  show *regular ?e*
    using *minarc-regular* **by** *auto*
next
  show *regular ?H*
    using *assms*(*14*) *pp-dist-star regular-conv-closed regular-mult-closed* **by** *auto*
next
  show *regular d*
    using *assms*(*8*) **by** *auto*
qed
have *62*: *bijective* $(a^T * top)$
  by (*simp add*: *assms*(*11*))
have *63*: *bijective* $(?x * top)$
  using *61* **by** *simp*
have *64*: $?x \leq (?H * d^T)^\star * ?H * a^T * top$
  by *simp*
hence $?x * top \leq (?H * d^T)^\star * ?H * a^T * top$
  using *mult-left-isotone inf-vector-comp* **by** *auto*
hence $a^T * top \leq ((?H * d^T)^\star * ?H)^T * ?x * top$
  using *62 63 64* **by** (*smt bijective-reverse mult-assoc*)
also have ... $= ?H * (d * ?H)^\star * ?x * top$
  using *conv-dist-comp conv-star-commute* **by** *auto*
also have ... $= (?H * d)^\star * ?H * ?x * top$
  by (*simp add*: *star-slide*)
finally have $a^T * top \leq (?H * d)^\star * ?H * ?x * top$
  by *simp*
hence *65*: *forest-modulo-equivalence-path a ?x ?H d*
  using *61 assms*(*12*) *forest-modulo-equivalence-path-def* **by** *blast*
have *66*: $?x \leq d$
  by (*simp add*: *inf.sup-monoid.add-assoc*)
hence *x-below-a*: *sum* $(?x \sqcap g) \leq$ *sum* $(a \sqcap g)$
  using *65 forest-modulo-equivalence-path-def assms*(*7*, *13*) **by** *blast*
have *sum* $(?e \sqcap g) \leq$ *sum* $(?x \sqcap g)$
proof (*rule sum-e-below-sum-x-when-outgoing-same-component*)
  show *symmetric g*
    using *assms*(*1*) **by** *auto*
next
  show *vector j*
    using *assms*(*3*) **by** *blast*
next
  show *forest h*
    by (*simp add*: *assms*(*4*))
next
  show $?x \leq -\ ?H \sqcap --\ g$
  proof −
    have *67*: $?x \leq -\ ?H$
    proof −
      have $?x \leq d$

```
          using 66 by blast
        also have ... ≤ ?H ∗ d
          using dual-order.trans star.circ-loop-fixpoint sup.cobounded2
mult-assoc by metis
        also have ... ≤ (?H ∗ d)⁺
          using star.circ-mult-increasing by blast
        also have ... ≤ − ?H
          using assms(5) bot-unique pseudo-complement
forest-modulo-equivalence-def by blast
        thus ?thesis
          using calculation inf.order-trans by blast
      qed
      have ?x ≤ d
        by (simp add: conv-isotone inf.sup-monoid.add-assoc)
      also have ... ≤ f ⊔ f^T
      proof −
        have h ⊔ h^T ⊔ d ⊔ d^T = f ⊔ f^T
          by (simp add: assms(6))
        thus ?thesis
          by (metis (no-types) le-supE sup.absorb-iff2 sup.idem)
      qed
      also have ... ≤ −− g
        using assms(1, 2) conv-complement conv-isotone by fastforce
      finally have ?x ≤ −− g
        by simp
      thus ?thesis
        by (simp add: 67)
    qed
  next
    show ?x^T ∗ top ≤ ?H ∗ ?e ∗ top
    proof −
      have ?x ≤ top ∗ ?e^T ∗ ?H
        using inf.coboundedI1 by auto
      hence ?x^T ≤ ?H ∗ ?e ∗ top
        using conv-dist-comp conv-dist-inf conv-star-commute inf.orderI
inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult-assoc by auto
      hence ?x^T ∗ top ≤ ?H ∗ ?e ∗ top ∗ top
        by (simp add: mult-left-isotone)
      thus ?thesis
        by (simp add: mult-assoc)
    qed
  next
    show j ≠ bot
      by (simp add: assms(9))
  next
    show arc (?x)
      using 61 by blast
  next
    show regular h
```

```
      using assms(14) by auto
    qed
    hence sum (?e ⊓ g) ≤ sum (a ⊓ g)
      using x-below-a order.trans by blast
    thus ?thesis
      by (simp add: assms(10))
  qed
qed
thus ?thesis
  by simp
qed
```

4.4.4 Maintenance of algorithm invariants

In this section, most of the work is done to maintain the invariants of the inner and outer loops of the algorithm. In particular, we use *exists-a-w* to maintain that f can be extended to a minimum spanning forest.

```
lemma boruvka-exchange-spanning-inv:
  assumes forest v
    and v⋆ ∗ e^T = e^T
    and i ≤ v ⊓ top ∗ e^T ∗ w^T⋆
    and arc i
    and arc e
    and v ≤ −−g
    and w ≤ −−g
    and e ≤ −−g
    and components g ≤ forest-components v
  shows i ≤ (v ⊓ −i)^T⋆ ∗ e^T ∗ top
proof −
  have 1: (v ⊓ −i ⊓ −i^T) ∗ (v^T ⊓ −i ⊓ −i^T) ≤ 1
    using assms(1) comp-isotone order.trans inf.cobounded1 by blast
  have 2: bijective (i ∗ top) ∧ bijective (e^T ∗ top)
    using assms(4, 5) mult-assoc by auto
  have i ≤ v ∗ (top ∗ e^T ∗ w^T⋆)^T
    using assms(3) covector-mult-closed covector-restrict-comp-conv
order-lesseq-imp vector-top-closed by blast
  also have ... ≤ v ∗ w^T⋆T ∗ e^TT ∗ top^T
    by (simp add: comp-associative conv-dist-comp)
  also have ... ≤ v ∗ w⋆ ∗ e ∗ top
    by (simp add: conv-star-commute)
  also have ... = v ∗ w⋆ ∗ e ∗ e^T ∗ e ∗ top
    using assms(5) arc-eq-1 by (simp add: comp-associative)
  also have ... ≤ v ∗ w⋆ ∗ e ∗ e^T ∗ top
    by (simp add: comp-associative mult-right-isotone)
  also have ... ≤ (−−g) ∗ (−−g)⋆ ∗ (−−g) ∗ e^T ∗ top
    using assms(6, 7, 8) by (simp add: comp-isotone star-isotone)
  also have ... ≤ (−−g)⋆ ∗ e^T ∗ top
    by (metis comp-isotone mult-left-isotone star.circ-increasing
star.circ-transitive-equal)
```

also have ... $\leq v^{T\star} * v^\star * e^T * top$
by (*simp add*: *assms*(*9*) *mult-left-isotone*)
also have ... $\leq v^{T\star} * e^T * top$
by (*simp add*: *assms*(*2*) *comp-associative*)
finally have $i \leq v^{T\star} * e^T * top$
by *simp*
hence $i * top \leq v^{T\star} * e^T * top$
by (*metis comp-associative mult-left-isotone vector-top-closed*)
hence $e^T * top \leq v^{T\star T} * i * top$
using *2* **by** (*metis bijective-reverse mult-assoc*)
also have ... $= v^\star * i * top$
by (*simp add*: *conv-star-commute*)
also have ... $\leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
proof $-$
have *3*: $i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *star.circ-loop-fixpoint sup-right-divisibility mult-assoc* **by** *auto*
have $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq i * top * i * top$
by (*metis comp-isotone inf.coboundedl inf.sup-monoid.add-commute mult-left-isotone top.extremum*)
also have ... $\leq i * top$
by *simp*
finally have *4*: $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *3 dual-order.trans* **by** *blast*
have *5*: $(v \sqcap -i \sqcap -i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
by (*metis mult-left-isotone star.circ-increasing star.left-plus-circ*)
have $v^+ \leq -1$
by (*simp add*: *assms*(*1*))
hence $v * v \leq -1$
by (*metis mult-left-isotone order-trans star.circ-increasing star.circ-plus-same*)
hence $v * 1 \leq -v^T$
by (*simp add*: *schroeder-5-p*)
hence $v \leq -v^T$
by *simp*
hence $v \sqcap v^T \leq bot$
by (*simp add*: *bot-unique pseudo-complement*)
hence *7*: $v \sqcap i^T \leq bot$
by (*metis assms*(*3*) *comp-inf.mult-right-isotone conv-dist-inf inf.boundedE inf.le-iff-sup le-bot*)
hence $(v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq bot$
using *le-bot semiring.mult-zero-left* **by** *fastforce*
hence *6*: $(v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *bot-least le-bot* **by** *blast*
have *8*: $v = (v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T)$
proof $-$
have *81*: *regular i*
by (*simp add*: *assms*(*4*) *arc-regular*)

have $(v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T) = (v \sqcap -i)$
using *7* **by** (*metis comp-inf.coreflexive-comp-inf-complement inf-import-p inf-p le-bot maddux-3-11-pp top.extremum*)
hence $(v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T) = (v \sqcap i) \sqcup (v \sqcap -i)$
by (*simp add*: *sup.semigroup-axioms semigroup.assoc*)
also have ... $= v$
using *81* **by** (*metis maddux-3-11-pp*)
finally show *?thesis*
by *simp*
qed
have $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \sqcup (v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \sqcup (v \sqcap -i \sqcap -i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *4 5 6* **by** *simp*
hence $((v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T)) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
by (*simp add*: *mult-right-dist-sup*)
hence $v * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *8* **by** *auto*
hence $i * top \sqcup v * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *3* **by** *auto*
hence *9*:$v^\star * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
by (*simp add*: *star-left-induct-mult mult-assoc*)
have $v^\star * i * top \leq v^\star * (v \sqcap -i \sqcap -i^T)^\star * i * top$
using *3 mult-right-isotone mult-assoc* **by** *auto*
thus *?thesis*
using *9 order.trans* **by** *blast*
qed
finally have $e^T * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
by *simp*
hence $i * top \leq (v \sqcap -i \sqcap -i^T)^{\star T} * e^T * top$
using *2* **by** (*metis bijective-reverse mult-assoc*)
also have ... $= (v^T \sqcap -i \sqcap -i^T)^\star * e^T * top$
using *comp-inf.inf-vector-comp conv-complement conv-dist-inf conv-star-commute inf.sup-monoid.add-commute* **by** *auto*
also have ... $\leq ((v \sqcap -i \sqcap -i^T) \sqcup (v^T \sqcap -i \sqcap -i^T))^\star * e^T * top$
by (*simp add*: *mult-left-isotone star-isotone*)
finally have $i \leq ((v^T \sqcap -i \sqcap -i^T) \sqcup (v \sqcap -i \sqcap -i^T))^\star * e^T * top$
using *dual-order.trans top-right-mult-increasing sup-commute* **by** *auto*
also have ... $= (v^T \sqcap -i \sqcap -i^T)^\star * (v \sqcap -i \sqcap -i^T)^\star * e^T * top$
using *1 cancel-separate-1* **by** (*simp add*: *sup-commute*)
also have ... $\leq (v^T \sqcap -i \sqcap -i^T)^\star * v^\star * e^T * top$
by (*simp add*: *inf-assoc mult-left-isotone mult-right-isotone star-isotone*)
also have ... $= (v^T \sqcap -i \sqcap -i^T)^\star * e^T * top$
using *assms*(*2*) *mult-assoc* **by** *simp*
also have ... $\leq (v^T \sqcap -i^T)^\star * e^T * top$
by (*metis mult-left-isotone star-isotone inf.cobounded2 inf.left-commute inf.sup-monoid.add-commute*)
also have ... $= (v \sqcap -i)^{T\star} * e^T * top$
using *conv-complement conv-dist-inf* **by** *auto*

finally show *?thesis*
by *simp*
qed

lemma *exists-a-w*:
assumes *symmetric g*
and *forest f*
and $f \leq --g$
and *regular f*
and $(\exists\, w\ .\ \textit{minimum-spanning-forest}\ w\ g \wedge f \leq w \sqcup w^T)$
and *vector j*
and *regular j*
and *forest h*
and *forest-modulo-equivalence* (*forest-components h*) *d*
and $d * top \leq -\ j$
and $\textit{forest-components}\ h * j = j$
and $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
and $(\forall\ a\ b\ .\ \textit{forest-modulo-equivalence-path}\ a\ b\ (\textit{forest-components}\ h)\ d \wedge a \leq -(\textit{forest-components}\ h) \sqcap --\ g \wedge b \leq d \longrightarrow sum(b \sqcap g) \leq sum(a \sqcap g))$
and *regular d*
and $\textit{selected-edge}\ h\ j\ g \leq -\ \textit{forest-components}\ f$
and $\textit{selected-edge}\ h\ j\ g \neq bot$
and $j \neq bot$
and *regular h*
and $h \leq --g$
shows $\exists\, w.\ \textit{minimum-spanning-forest}\ w\ g \wedge$
$f \sqcap -\ (\textit{selected-edge}\ h\ j\ g)^T \sqcap -\ (\textit{path}\ f\ h\ j\ g) \sqcup (f \sqcap -\ (\textit{selected-edge}\ h\ j\ g)^T \sqcap (\textit{path}\ f\ h\ j\ g))^T \sqcup (\textit{selected-edge}\ h\ j\ g) \leq w \sqcup w^T$
proof −
let $?p = \textit{path}\ f\ h\ j\ g$
let $?e = \textit{selected-edge}\ h\ j\ g$
let $?f = (f \sqcap -?e^T \sqcap -?p) \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
let $?F = \textit{forest-components}\ f$
let $?H = \textit{forest-components}\ h$
let $?ec = \textit{choose-component}\ (\textit{forest-components}\ h)\ j * -\ \textit{choose-component}\ (\textit{forest-components}\ h)\ j^T \sqcap g$
from *assms*(*4*) **obtain** *w* **where** *2*: $\textit{minimum-spanning-forest}\ w\ g \wedge f \leq w \sqcup w^T$
using *assms*(*5*) **by** *blast*
hence *3*: $\textit{regular}\ w \wedge \textit{regular}\ f \wedge \textit{regular}\ ?e$
by (*metis assms*(*4*) *minarc-regular minimum-spanning-forest-def spanning-forest-def*)
have *5*: *equivalence ?F*
using *assms*(*2*) *forest-components-equivalence* **by** *auto*
have $?e^T * top * ?e^T = ?e^T$
by (*metis arc-conv-closed arc-top-arc coreflexive-bot-closed coreflexive-symmetric minarc-arc minarc-bot-iff semiring.mult-not-zero*)
hence $?e^T * top * ?e^T \leq -?F$
using *5 assms*(*15*) *conv-complement conv-isotone* **by** *fastforce*

hence *6*: $?e * ?F * ?e = bot$
  using *assms*(*2*) *le-bot triple-schroeder-p* **by** *simp*
let $?q = w \sqcap top * ?e * w^{T\star}$
let $?v = (w \sqcap -(top * ?e * w^{T\star})) \sqcup ?q^T$
have *7*: *regular ?q*
  using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
have *8*: *injective ?v*
proof (*rule kruskal-exchange-injective-inv-1*)
  show *injective w*
    using *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
next
  show *covector* $(top * ?e * w^{T\star})$
    by (*simp add*: *covector-mult-closed*)
next
  show $top * ?e * w^{T\star} * w^T \leq top * ?e * w^{T\star}$
    by (*simp add*: *mult-right-isotone star.right-plus-below-circ mult-assoc*)
next
  show *coreflexive* $((top * ?e * w^{T\star})^T * (top * ?e * w^{T\star}) \sqcap w^T * w)$
    using *2* **by** (*metis comp-inf.semiring.mult-not-zero forest-bot kruskal-injective-inv-3 minarc-arc minarc-bot-iff minimum-spanning-forest-def semiring.mult-not-zero spanning-forest-def*)
qed
have *9*: *components* $g \leq$ *forest-components ?v*
proof (*rule kruskal-exchange-spanning-inv-1*)
  show *injective* $(w \sqcap - (top *?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)$
    using *8* **by** *simp*
next
  show *regular* $(w \sqcap top * ?e * w^{T\star})$
    using *7* **by** *simp*
next
  show *components* $g \leq$ *forest-components w*
    using *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
qed
have *10*: *spanning-forest ?v g*
proof (*unfold spanning-forest-def*, *intro conjI*)
  show *injective ?v*
    using *8* **by** *auto*
next
  show *acyclic ?v*
  proof (*rule kruskal-exchange-acyclic-inv-1*)
    show *pd-kleene-allegory-class.acyclic w*
      using *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
  next
    show *covector* $(top * ?e * w^{T\star})$
      by (*simp add*: *covector-mult-closed*)
  qed
next
  show $?v \leq --g$
  proof (*rule sup-least*)

show $w \sqcap -(top * ?e * w^{T\star}) \leq - - g$
using *7 inf.coboundedI1 minimum-spanning-forest-def spanning-forest-def 2*
by *blast*
next
show $(w \sqcap top * ?e * w^{T\star})^T \leq - - g$
using *2* **by** (*metis assms*(*1*) *conv-complement conv-isotone*
inf.coboundedI1 minimum-spanning-forest-def spanning-forest-def)
qed
next
show *components g* $\leq$ *forest-components ?v*
using *9* **by** *simp*
next
show *regular ?v*
using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
qed
have *11*: *sum* $(?v \sqcap g) =$ *sum* $(w \sqcap g)$
proof $-$
have *sum* $(?v \sqcap g) =$ *sum* $(w \sqcap -(top * ?e * w^{T\star}) \sqcap g) +$ *sum* $(?q^T \sqcap g)$
using *2* **by** (*smt conv-complement conv-top epm-8 inf-import-p inf-top-right*
regular-closed-top vector-top-closed minimum-spanning-forest-def
spanning-forest-def sum-disjoint)
also have ... $=$ *sum* $(w \sqcap -(top * ?e * w^{T\star}) \sqcap g) +$ *sum* $(?q \sqcap g)$
by (*simp add*: *assms*(*1*) *sum-symmetric*)
also have ... $=$ *sum* $(((w \sqcap -(top * ?e * w^{T\star})) \sqcup ?q) \sqcap g)$
using *inf-commute inf-left-commute sum-disjoint* **by** *simp*
also have ... $=$ *sum* $(w \sqcap g)$
using *3 7 8 maddux-3-11-pp* **by** *auto*
finally show *?thesis*
by *simp*
qed
have *12*: $?v \sqcup ?v^T = w \sqcup w^T$
proof $-$
have $?v \sqcup ?v^T = (w \sqcap -?q) \sqcup ?q^T \sqcup (w^T \sqcap -?q^T) \sqcup ?q$
using *conv-complement conv-dist-inf conv-dist-sup inf-import-p sup-assoc* **by**
simp
also have ... $= w \sqcup w^T$
using *3 7 conv-complement conv-dist-inf inf-import-p maddux-3-11-pp*
sup-monoid.add-assoc sup-monoid.add-commute **by** *auto*
finally show *?thesis*
by *simp*
qed
have *13*: $?v * ?e^T = bot$
proof (*rule kruskal-reroot-edge*)
show *injective* $(?e^T * top)$
using *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *blast*
next
show *pd-kleene-allegory-class.acyclic w*
using *2 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
qed

have $?v \sqcap ?e \leq ?v \sqcap top * ?e$
using *inf.sup-right-isotone top-left-mult-increasing* **by** *simp*
also have ... $\leq ?v * (top * ?e)^T$
using *covector-restrict-comp-conv covector-mult-closed vector-top-closed* **by** *simp*
finally have *14*: $?v \sqcap ?e = bot$
using *13* **by** (*metis conv-dist-comp mult-assoc le-bot mult-left-zero*)
let $?i = ?v \sqcap (- ?F) * ?e * top \sqcap top * ?e^T * ?F$
let $?w = (?v \sqcap -?i) \sqcup ?e$
have *15*: *regular* $?i$
using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
have *16*: $?F \leq -?i$
proof $-$
have *161*: *bijective* $(?e * top)$
using *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *auto*
have $?i \leq - ?F * ?e * top$
using *inf.cobounded2 inf.coboundedI1* **by** *blast*
also have ... $= - (?F * ?e * top)$
using *161 comp-bijective-complement* **by** (*simp add*: *mult-assoc*)
finally have $?i \leq - (?F * ?e * top)$
by *blast*
hence *162*: $?i \sqcap ?F \leq - (?F * ?e * top)$
using *inf.coboundedI1* **by** *blast*
have $?i \sqcap ?F \leq ?F \sqcap (top * ?e^T * ?F)$
by (*meson inf-le1 inf-le2 le-infI order-trans*)
also have ... $\leq ?F * (top * ?e^T * ?F)^T$
by (*simp add*: *covector-mult-closed covector-restrict-comp-conv*)
also have ... $= ?F * ?F^T * ?e^{TT} * top^T$
by (*simp add*: *conv-dist-comp mult-assoc*)
also have ... $= ?F * ?F * ?e * top$
by (*simp add*: *conv-dist-comp conv-star-commute*)
also have ... $= ?F * ?e * top$
by (*simp add*: *5 preorder-idempotent*)
finally have $?i \sqcap ?F \leq ?F * ?e * top$
by *simp*
hence $?i \sqcap ?F \leq ?F * ?e * top \sqcap - (?F * ?e * top)$
using *162 inf.bounded-iff* **by** *blast*
also have ... $= bot$
by *simp*
finally show *?thesis*
using *le-bot p-antitone-iff pseudo-complement* **by** *blast*
qed
have *17*: $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
proof $-$
have $?i \leq ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star$
using *2 8 12* **by** (*smt inf.sup-right-isotone kruskal-forest-components-inf mult-right-isotone mult-assoc*)
also have ... $= ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (1 \sqcup (?F \sqcap ?v)^\star * (?F \sqcap ?v))$

using *star-left-unfold-equal star.circ-right-unfold-1* **by** *auto*
also have ... $= ?v \sqcap - ?F * ?e * top \sqcap (top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
by (*simp add*: *mult-left-dist-sup mult-assoc*)
also have ... $= (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star}) \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
using *comp-inf.semiring.distrib-left* **by** *blast*
also have ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
using *comp-inf.semiring.add-right-mono inf-le2* **by** *blast*
also have ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
by (*simp add*: *conv-dist-inf*)
also have ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v))$
proof −
have $top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v) \le top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v)$
using *star-isotone* **by** (*simp add*: *comp-isotone*)
hence $?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v) \le ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v)$
using *inf.sup-right-isotone* **by** *blast*
thus *?thesis*
using *sup-right-isotone* **by** *blast*
qed
also have ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^\star * ?F^\star * (?F \sqcap ?v))$
using *5* **by** *auto*
also have ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F * ?F * (?F \sqcap ?v))$
by (*simp add*: *assms*(*2*) *forest-components-star*)
also have ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v))$
using *5 mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*
also have ... $= top * ?e^T * (?F \sqcap ?v)^{T\star}$
proof −
have $?e * top * ?e^T \le 1$
using *assms*(*16*) *arc-expanded minarc-arc minarc-bot-iff* **by** *auto*
hence $?F * ?e * top * ?e^T \le ?F * 1$
by (*metis comp-associative comp-isotone mult-semi-associative star.circ-transitive-equal*)
hence $?v * ?v^T * ?F * ?e * top * ?e^T \le 1 * ?F * 1$
using *8* **by** (*smt comp-isotone mult-assoc*)
hence *171*: $?v * ?v^T * ?F * ?e * top * ?e^T \le ?F$
by *simp*
hence $?v * (?F \sqcap ?v)^T * ?F * ?e * top * ?e^T \le ?F$
proof −
have $?v * (?F \sqcap ?v)^T * ?F * ?e * top * ?e^T \le ?v * ?v^T * ?F * ?e * top *$

$?e^T$
by ($simp\ add$: $conv$-$dist$-inf $mult$-$left$-$isotone$ $mult$-$right$-$isotone$)
thus $?thesis$
using 171 $order$-$trans$ **by** $blast$
qed
hence 172: $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T \leq -?v$
by (smt $schroeder$-4-p $comp$-$associative$ $order$-$lesseq$-imp pp-$increasing$)
have $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T = -?F * ?e^{TT} * top^T * ?e^T * ?F^T * (?F \sqcap ?v)^{TT}$
by ($simp\ add$: $comp$-$associative$ $conv$-$dist$-$comp$)
also have ... $= -?F * ?e * top * ?e^T * ?F * (?F \sqcap ?v)$
using 5 **by** $auto$
also have ... $= -?F * ?e * top * top * ?e^T * ?F * (?F \sqcap ?v)$
using $comp$-$associative$ **by** $auto$
also have ... $= -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v)$
by (smt $comp$-$associative$ $comp$-$inf.star.circ$-$decompose$-9 $comp$-$inf.star$-$star$-$absorb$ $comp$-inf-$covector$ inf-$vector$-$comp$ $vector$-top-$closed$)
finally have $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T = -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v)$
by $simp$
hence $-?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v) \leq -?v$
using 172 **by** $auto$
hence $?v \sqcap -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v) \leq bot$
by (smt bot-$unique$ $inf.sup$-$monoid.add$-$commute$ p-$shunting$-$swap$ $pseudo$-$complement$)
thus $?thesis$
using le-bot sup-$monoid.add$-0-$right$ **by** $blast$
qed
also have ... $= top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
using 16 **by** (smt $comp$-$inf.coreflexive$-$comp$-inf-$complement$ inf-top-$right$ p-bot $pseudo$-$complement$ $top.extremum$)
finally show $?thesis$
by $blast$
qed
have 18: $?i \leq top * ?e^T * ?w^{T\star}$
proof $-$
have $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
using 17 **by** $simp$
also have ... $\leq top * ?e^T * (?v \sqcap -?i)^{T\star}$
using $mult$-$right$-$isotone$ $conv$-$isotone$ $star$-$isotone$ $inf.cobounded2$ $inf.sup$-$monoid.add$-$assoc$ **by** ($simp\ add$: $inf.sup$-$monoid.add$-$assoc$ $order.eq$-iff $inf.sup$-$monoid.add$-$commute$)
also have ... $\leq top * ?e^T * ((?v \sqcap -?i) \sqcup ?e)^{T\star}$
using $mult$-$right$-$isotone$ $conv$-$isotone$ $star$-$isotone$ sup-$ge1$ **by** $simp$
finally show $?thesis$
by $blast$
qed
have 19: $?i \leq top * ?e^T * ?v^{T\star}$
proof $-$

have $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
using *17* **by** *simp*
also have ... $\leq top * ?e^T * (?v \sqcap -?i)^{T\star}$
using *mult-right-isotone conv-isotone star-isotone inf.cobounded2 inf.sup-monoid.add-assoc* **by** (*simp add*: *inf.sup-monoid.add-assoc order.eq-iff inf.sup-monoid.add-commute*)
also have ... $\leq top * ?e^T * (?v)^{T\star}$
using *mult-right-isotone conv-isotone star-isotone* **by** *auto*
finally show *?thesis*
by *blast*
qed
have *20*: $f \sqcup f^T \leq (?v \sqcap -?i \sqcap -?i^T) \sqcup (?v^T \sqcap -?i \sqcap -?i^T)$
proof (*rule kruskal-edge-between-components-2*)
show $?F \leq - ?i$
using *16* **by** *simp*
next
show *injective f*
by (*simp add*: *assms*(*2*))
next
show $f \sqcup f^T \leq w \sqcap - (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T \sqcup (w \sqcap - (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)^T$
using *2 12* **by** (*metis conv-dist-sup conv-involutive conv-isotone le-supI sup-commute*)
qed
have *minimum-spanning-forest* $?w\ g \wedge ?f \leq ?w \sqcup ?w^T$
proof (*intro conjI*)
have *211*: $?e^T \leq ?v^\star$
proof (*rule kruskal-edge-arc-1*[**where** *g*=*g* **and** *h*=*?ec*])
show $?e \leq -- ?ec$
using *minarc-below* **by** *blast*
next
show $?ec \leq g$
using *assms*(*4*) *inf.cobounded2* **by** (*simp add*: *boruvka-inner-invariant-def boruvka-outer-invariant-def conv-dist-inf*)
next
show *symmetric g*
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
next
show *components g* $\leq$ *forest-components* $(w \sqcap - (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)$
using *9* **by** *simp*
next
show $(w \sqcap - (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T) * ?e^T = bot$
using *13* **by** *blast*
qed
have *212*: *arc ?i*
proof (*rule boruvka-edge-arc*)
show *equivalence ?F*

by (*simp add*: *5*)
next
show *forest ?v*
using *10 spanning-forest-def* **by** *blast*
next
show *arc ?e*
using *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *blast*
next
show *regular ?F*
using *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
next
show $?F \leq$ *forest-components* $(?F \sqcap ?v)$
by (*simp add*: *12 2 8 kruskal-forest-components-inf*)
next
show *regular ?v*
using *10 spanning-forest-def* **by** *blast*
next
show $?v * ?e^T = bot$
using *13* **by** *auto*
next
show $?e * ?F * ?e = bot$
by (*simp add*: *6*)
next
show $?e^T \leq ?v^\star$
using *211* **by** *auto*

next
show $?e \neq bot$
by (*simp add*: *assms*(*16*))
qed
show *minimum-spanning-forest ?w g*
proof (*unfold minimum-spanning-forest-def*, *intro conjI*)
have $(?v \sqcap -?i) * ?e^T \leq ?v * ?e^T$
using *inf-le1 mult-left-isotone* **by** *simp*
hence $(?v \sqcap -?i) * ?e^T = bot$
using *13 le-bot* **by** *simp*
hence *221*: $?e * (?v \sqcap -?i)^T = bot$
using *conv-dist-comp conv-involutive conv-bot* **by** *force*
have *222*: *injective ?w*
proof (*rule injective-sup*)
show *injective* $(?v \sqcap -?i)$
using *8* **by** (*simp add*: *injective-inf-closed*)
next
show *coreflexive* $(?e * (?v \sqcap -?i)^T)$
using *221* **by** *simp*
next
show *injective ?e*
by (*metis arc-injective minarc-arc coreflexive-bot-closed*
coreflexive-injective minarc-bot-iff)

```
    qed
    show spanning-forest ?w g
    proof (unfold spanning-forest-def, intro conjI)
      show injective ?w
        using 222 by simp
    next
      show acyclic ?w
      proof (rule kruskal-exchange-acyclic-inv-2)
        show acyclic ?v
          using 10 spanning-forest-def by blast
      next
        show injective ?v
          using 8 by simp
      next
        show ?i ≤?v
          using inf.coboundedI1 by simp
      next
        show bijective (?i^T * top)
          using 212 by simp
      next
        show bijective (?e * top)
          using 14 212 by (smt assms(4) comp-inf.idempotent-bot-closed
conv-complement minarc-arc minarc-bot-iff p-bot regular-closed-bot
semiring.mult-not-zero symmetric-top-closed)
      next
        show ?i ≤ top * ?e^T *?v^T⋆
          using 19 by simp
      next
        show ?v * ?e^T * top = bot
          using 13 by simp
      qed
    next
      have ?w ≤ ?v ⊔ ?e
        using inf-le1 sup-left-isotone by simp
      also have ... ≤ --g ⊔ ?e
        using 10 sup-left-isotone spanning-forest-def by blast
      also have ... ≤ --g ⊔ --h
      proof -
        have 1: --g ≤ --g ⊔ --h
          by simp
        have 2: ?e ≤ --g ⊔ --h
          by (metis inf.coboundedI1 inf.sup-monoid.add-commute minarc-below
order.trans p-dist-inf p-dist-sup sup.cobounded1)
        thus ?thesis
          using 1 2 by simp
      qed
      also have ... ≤ --g
          using assms(18, 19) by auto
      finally show ?w ≤ --g
```

by *simp*
next
have *223*: $?i \leq (?v \sqcap -?i)^{T\star} * ?e^T * top$
proof (*rule boruvka-exchange-spanning-inv*)
show *forest ?v*
using *10 spanning-forest-def* **by** *blast*
next
show $?v^\star * ?e^T = ?e^T$
using *13* **by** (*smt conv-complement conv-dist-comp conv-involutive conv-star-commute dense-pp fc-top regular-closed-top star-absorb*)
next
show $?i \leq ?v \sqcap top * ?e^T * ?w^{T\star}$
using *18 inf.sup-monoid.add-assoc* **by** *auto*
next
show *arc ?i*
using *212* **by** *blast*
next
show *arc ?e*
using *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *auto*
next
show $?v \leq --g$
using *10 spanning-forest-def* **by** *blast*
next
show $?w \leq --g$
proof −
have *2231*: $?e \leq --g$
by (*metis inf.boundedE minarc-below pp-dist-inf*)
have $?w \leq ?v \sqcup ?e$
using *inf-le1 sup-left-isotone* **by** *simp*
also have ... $\leq --g$
using *2231 10 spanning-forest-def sup-least* **by** *blast*
finally show *?thesis*
by *blast*
qed
next
show $?e \leq --g$
by (*metis inf.boundedE minarc-below pp-dist-inf*)
next
show *components g* $\leq$ *forest-components ?v*
by (*simp add*: *9*)
qed
have *components g* $\leq$ *forest-components ?v*
using *10 spanning-forest-def* **by** *auto*
also have ... $\leq$ *forest-components ?w*
proof (*rule kruskal-exchange-forest-components-inv*)
next
show *injective* $((?v \sqcap -?i) \sqcup ?e)$
using *222* **by** *simp*
next

show *regular* $?i$
using *15* **by** *simp*
next
show $?e * top * ?e = ?e$
by (*metis arc-top-arc minarc-arc minarc-bot-iff semiring.mult-not-zero*)
next
show $?i \leq top * ?e^T * ?v^{T\star}$
using *19* **by** *blast*
next
show $?v * ?e^T * top = bot$
using *13* **by** *simp*
next
show *injective* $?v$
using *8* **by** *simp*
next
show $?i \leq ?v$
by (*simp add*: *le-infI1*)
next
show $?i \leq (?v \sqcap -?i)^{T\star} * ?e^T * top$
using *223* **by** *blast*
qed
finally show *components* $g \leq$ *forest-components* $?w$
by *simp*
next
show *regular* $?w$
using *3 7 regular-conv-closed* **by** *simp*
qed
next
have *224*: $?e \sqcap g \neq bot$
using *assms*(*16*) *inf.left-commute inf-bot-right minarc-meet-bot* **by** *fastforce*
have *225*: $sum\ (?e \sqcap g) \leq sum\ (?i \sqcap g)$
proof (*rule a-to-e-in-forest-modulo-equivalence*)
show *symmetric* g
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*
by *auto*
next
show $j \neq bot$
by (*simp add*: *assms*(*17*))
next
show $f \leq --\ g$
by (*simp add*: *assms*(*3*))
next
show *vector* j
using *assms*(*6*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest* h
by (*simp add*: *assms*(*8*))
next
show *forest-modulo-equivalence* (*forest-components* h) d

by (*simp add*: *assms*(*9*))
next
show $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
by (*simp add*: *assms*(*12*))
next
show $\forall a\ b.$ *forest-modulo-equivalence-path* $a\ b\ (?H)\ d \wedge a \leq -\ ?H \sqcap --\ g \wedge b \leq d \longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
by (*simp add*: *assms*(*13*))
next
show *regular d*
using *assms*(*14*) **by** *auto*
next
show $?e = ?e$
by *simp*
next
show *arc ?i*
using *212* **by** *blast*
next
show *forest-modulo-equivalence-path* $?i\ ?e\ ?H\ (d \sqcup ?e)$
proof −
have $d^T * ?H * ?e = bot$
using *assms*(*6*, *7*, *10* ,*11*, *17*) *dT-He-eq-bot le-bot* **by** *blast*
hence *251*: $d^T * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by *simp*
hence $d^T * ?H * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*metis assms*(*8*) *forest-components-star star.circ-decompose-9 mult-assoc*)
hence $d^T * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
proof −
have $d^T * ?H * d \leq 1$
using *assms*(*9*) *forest-modulo-equivalence-def dTransHd-le-1* **by** *blast*
hence $d^T * ?H * d * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*metis mult-left-isotone star.circ-circ-mult star-involutive star-one*)
hence $d^T * ?H * ?e \sqcup d^T * ?H * d * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
using *251* **by** *simp*
hence $d^T * (1 \sqcup ?H * d * (?H * d)^\star) * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*simp add*: *comp-associative comp-left-dist-sup semiring.distrib-right*)
thus *?thesis*
by (*simp add*: *star-left-unfold-equal*)
qed
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * (?H * d)^\star * ?H * ?e$
by (*simp add*: *mult-right-isotone mult-assoc*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * ?H * (d * ?H)^\star * ?e$
by (*smt star-slide mult-assoc*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * (d * ?H)^\star * ?e$
by (*metis assms*(*8*) *forest-components-star star.circ-decompose-9*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$

using *star-slide* **by** *auto*
hence $?H * d * (?H * d)^\star * ?H * ?e \sqcup ?H * d^T * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*smt le-supI star.circ-loop-fixpoint sup.cobounded2 sup-commute mult-assoc*)
hence $(?H * (d \sqcup d^T)) * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*simp add*: *semiring.distrib-left semiring.distrib-right*)
hence $(?H * (d \sqcup d^T))^\star * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*simp add*: *star-left-induct-mult mult-assoc*)
hence *252*: $(?H * (d \sqcup d^T))^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*smt mult-left-dist-sup star.circ-transitive-equal star-slide star-sup-1 mult-assoc*)
have $?i \leq top * ?e^T * ?F$
by *auto*
hence $?i^T \leq ?F^T * ?e^{TT} * top^T$
by (*simp add*: *conv-dist-comp conv-dist-inf mult-assoc*)
hence $?i^T * top \leq ?F^T * ?e^{TT} * top^T * top$
using *comp-isotone* **by** *blast*
also have ... $= ?F^T * ?e^{TT} * top^T$
by (*simp add*: *vector-mult-closed*)
also have ... $= ?F * ?e^{TT} * top^T$
by (*simp add*: *conv-dist-comp conv-star-commute*)
also have ... $= ?F * ?e * top$
by *simp*
also have ... $= (?H * (d \sqcup d^T))^\star * ?H * ?e * top$
using *assms*(*2*, *8*, *12*) *F-is-H-and-d* **by** *simp*
also have ... $\leq (?H * d)^\star * ?H * ?e * top$
by (*simp add*: *252 comp-isotone*)
also have ... $\leq (?H * (d \sqcup ?e))^\star * ?H * ?e * top$
by (*simp add*: *comp-isotone star-isotone*)
finally have $?i^T * top \leq (?H * (d \sqcup ?e))^\star * ?H * ?e * top$
by *blast*
thus *?thesis*
using *212 assms*(*16*) *forest-modulo-equivalence-path-def minarc-arc minarc-bot-iff* **by** *blast*
qed
next
show $?i \leq -?H \sqcap --g$
proof $-$
have *forest-components h* $\leq$ *forest-components f*
using *assms*(*2*, *8*, *12*) *H-below-F* **by** *blast*
then have *241*: $?i \leq -?H$
using *16 assms*(*9*) *inf.order-lesseq-imp p-antitone-iff* **by** *blast*
have $?i \leq --g$
using *10 inf.coboundedI1 spanning-forest-def* **by** *blast*
thus *?thesis*
using *241 inf-greatest* **by** *blast*
qed
next

show *regular h*
using *assms(18)* **by** *auto*
qed
have $?v \sqcap ?e \sqcap -?i = bot$
using *14* **by** *simp*
hence $sum\ (?w \sqcap g) = sum\ (?v \sqcap -?i \sqcap g) + sum\ (?e \sqcap g)$
using *sum-disjoint inf-commute inf-assoc* **by** *simp*
also have $... \leq sum\ (?v \sqcap -?i \sqcap g) + sum\ (?i \sqcap g)$
using *224 225 sum-plus-right-isotone* **by** *simp*
also have $... = sum\ (((?v \sqcap -?i) \sqcup ?i) \sqcap g)$
using *sum-disjoint inf-le2 pseudo-complement* **by** *simp*
also have $... = sum\ ((?v \sqcup ?i) \sqcap (-?i \sqcup ?i) \sqcap g)$
by (*simp add*: *sup-inf-distrib2*)
also have $... = sum\ ((?v \sqcup ?i) \sqcap g)$
using *15* **by** (*metis inf-top-right stone*)
also have $... = sum\ (?v \sqcap g)$
by (*simp add*: *inf.sup-monoid.add-assoc*)
finally have $sum\ (?w \sqcap g) \leq sum\ (?v \sqcap g)$
by *simp*
thus $\forall\, u\ .\ spanning\text{-}forest\ u\ g \longrightarrow sum\ (?w \sqcap g) \leq sum\ (u \sqcap g)$
using *2 11 minimum-spanning-forest-def* **by** *auto*
qed
next
have $?f \leq f \sqcup f^T \sqcup ?e$
by (*smt conv-dist-inf inf-le1 sup-left-isotone sup-mono inf.order-lesseq-imp*)
also have $... \leq (?v \sqcap -?i \sqcap -?i^T) \sqcup (?v^T \sqcap -?i \sqcap -?i^T) \sqcup ?e$
using *20 sup-left-isotone* **by** *simp*
also have $... \leq (?v \sqcap -?i) \sqcup (?v^T \sqcap -?i \sqcap -?i^T) \sqcup ?e$
by (*metis inf.cobounded1 sup-inf-distrib2*)
also have $... = ?w \sqcup (?v^T \sqcap -?i \sqcap -?i^T)$
by (*simp add*: *sup-assoc sup-commute*)
also have $... \leq ?w \sqcup (?v^T \sqcap -?i^T)$
using *inf.sup-right-isotone inf-assoc sup-right-isotone* **by** *simp*
also have $... \leq ?w \sqcup ?w^T$
using *conv-complement conv-dist-inf conv-dist-sup sup-right-isotone* **by** *simp*
finally show $?f \leq ?w \sqcup ?w^T$
by *simp*
qed
thus *?thesis* **by** *auto*
qed

lemma *boruvka-outer-invariant-when-e-not-bot*:
assumes *boruvka-inner-invariant j f h g d*
and $j \neq bot$
and $selected\text{-}edge\ h\ j\ g \leq -\ forest\text{-}components\ f$
and $selected\text{-}edge\ h\ j\ g \neq bot$
shows $boruvka\text{-}outer\text{-}invariant\ (f \sqcap -\ selected\text{-}edge\ h\ j\ g^T \sqcap -\ path\ f\ h\ j\ g \sqcup (f \sqcap -\ selected\text{-}edge\ h\ j\ g^T \sqcap path\ f\ h\ j\ g)^T \sqcup selected\text{-}edge\ h\ j\ g)\ g$
proof −

let *?c* = *choose-component* (*forest-components h*) *j*
let *?p* = *path f h j g*
let *?F* = *forest-components f*
let *?H* = *forest-components h*
let *?e* = *selected-edge h j g*
let $?f' = f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
let $?d' = d \sqcup ?e$
let $?j' = j \sqcap -?c$
show *boruvka-outer-invariant ?f′ g*
proof (*unfold boruvka-outer-invariant-def*, *intro conjI*)
show *symmetric g*
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
next
show *injective ?f′*
proof (*rule kruskal-injective-inv*)
show *injective* $(f \sqcap -\ ?e^T)$
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def injective-inf-closed*)
show *covector* (*?p*)
using *covector-mult-closed* **by** *simp*
show $?p * (f \sqcap -\ ?e^T)^T \leq\ ?p$
by (*simp add*: *mult-right-isotone star.left-plus-below-circ star-plus mult-assoc*)
show $?e \leq\ ?p$
by (*meson mult-left-isotone order.trans star-outer-increasing top.extremum*)
show $?p * (f \sqcap -\ ?e^T)^T \leq -\ ?e$
proof −
have $?p * (f \sqcap -\ ?e^T)^T \leq\ ?p * f^T$
by (*simp add*: *conv-dist-inf mult-right-isotone*)
also have ... $\leq top *\ ?e * (f)^{T\star} * f^T$
using *conv-dist-inf star-isotone comp-isotone* **by** *simp*
also have ... $\leq -\ ?e$
using *assms*(*1*, *4*) *boruvka-inner-invariant-def boruvka-outer-invariant-def kruskal-injective-inv-2 minarc-arc minarc-bot-iff* **by** *auto*
finally show *?thesis* **.**
qed
show *injective* (*?e*)
by (*metis arc-injective coreflexive-bot-closed minarc-arc minarc-bot-iff semiring.mult-not-zero*)
show *coreflexive* $(?p^T *\ ?p \sqcap (f \sqcap -\ ?e^T)^T * (f \sqcap -\ ?e^T))$
proof −
have $(?p^T *\ ?p \sqcap (f \sqcap -\ ?e^T)^T * (f \sqcap -\ ?e^T)) \leq\ ?p^T *\ ?p \sqcap f^T * f$
using *conv-dist-inf inf.sup-right-isotone mult-isotone* **by** *simp*
also have ... $\leq (top *\ ?e * f^{T\star})^T * (top *\ ?e * f^{T\star}) \sqcap f^T * f$
by (*metis comp-associative comp-inf.coreflexive-transitive comp-inf.mult-right-isotone comp-isotone conv-isotone inf.coboundedl inf.idem inf.sup-monoid.add-commute star-isotone top.extremum*)
also have ... ≤ 1
using *assms*(*1*, *4*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*

kruskal-injective-inv-3 minarc-arc minarc-bot-iff **by** *auto*
finally show *?thesis*
by *simp*
qed
qed
next
show *acyclic ?f'*
proof (*rule kruskal-acyclic-inv*)
show *acyclic* $(f \sqcap - ?e^T)$
proof $-$
have *f-intersect-below*: $(f \sqcap - ?e^T) \leq f$ **by** *simp*
have *acyclic f*
by (*meson assms*(*1*) *boruvka-inner-invariant-def*
boruvka-outer-invariant-def)
thus *?thesis*
using *comp-isotone dual-order.trans star-isotone f-intersect-below* **by** *blast*
qed
next
show *covector ?p*
by (*metis comp-associative vector-top-closed*)
next
show $(f \sqcap - ?e^T \sqcap ?p)^T * (f \sqcap - ?e^T)^\star * ?e = bot$
proof $-$
have $?e \leq - (f^{T\star} * f^\star)$
by (*simp add*: *assms*(*3*))
hence $?e * top * ?e \leq - (f^{T\star} * f^\star)$
by (*metis arc-top-arc minarc-arc minarc-bot-iff semiring.mult-not-zero*)
hence $?e^T * top * ?e^T \leq - (f^{T\star} * f^\star)^T$
by (*metis comp-associative conv-complement conv-dist-comp conv-isotone*
symmetric-top-closed)
hence $?e^T * top * ?e^T \leq - (f^{T\star} * f^\star)$
by (*simp add*: *conv-dist-comp conv-star-commute*)
hence $?e * (f^{T\star} * f^\star) * ?e \leq bot$
using *triple-schroeder-p* **by** *auto*
hence *1*: $?e * f^{T\star} * f^\star * ?e \leq bot$
using *mult-assoc* **by** *auto*
have *2*: $(f \sqcap - ?e^T)^{T\star} \leq f^{T\star}$
by (*simp add*: *conv-dist-inf star-isotone*)
have $(f \sqcap - ?e^T \sqcap ?p)^T * (f \sqcap - ?e^T)^\star * ?e \leq (f \sqcap ?p)^T * (f \sqcap - ?e^T)^\star$
$* ?e$
by (*simp add*: *comp-isotone conv-dist-inf inf.orderI*
inf.sup-monoid.add-assoc)
also have ... $\leq (f \sqcap ?p)^T * f^\star * ?e$
by (*simp add*: *comp-isotone star-isotone*)
also have ... $\leq (f \sqcap top * ?e * (f)^{T\star})^T * f^\star * ?e$
using *2* **by** (*metis comp-inf.comp-isotone comp-inf.coreflexive-transitive*
comp-isotone conv-isotone inf.idem top.extremum)
also have ... $= (f^T \sqcap (top * ?e * f^{T\star})^T) * f^\star * ?e$
by (*simp add*: *conv-dist-inf*)

also have ... $\leq$ *top* $*$ ($f^T \sqcap$ (*top* $*$ *?e* $*$ $f^{T\star}$)^T) $*$ $f^\star$ $*$ *?e*
using *top-left-mult-increasing mult-assoc* **by** *auto*
also have ... = (*top* $\sqcap$ *top* $*$ *?e* $*$ $f^{T\star}$) $*$ f^T $*$ $f^\star$ $*$ *?e*
by (*smt covector-comp-inf-1 covector-mult-closed order.eq-iff inf.sup-monoid.add-commute vector-top-closed*)
also have ... = *top* $*$ *?e* $*$ $f^{T\star}$ $*$ f^T $*$ $f^\star$ $*$ *?e*
by *simp*
also have ... $\leq$ *top* $*$ *?e* $*$ $f^{T\star}$ $*$ $f^\star$ $*$ *?e*
by (*smt conv-dist-comp conv-isotone conv-star-commute mult-left-isotone mult-right-isotone star.left-plus-below-circ mult-assoc*)
also have ... $\leq$ *bot*
using *1 covector-bot-closed le-bot mult-assoc* **by** *fastforce*
finally show *?thesis*
using *le-bot* **by** *auto*
qed
next
show *?e* $*$ ($f \sqcap -$ *?e*^{T})* $*$ *?e* = *bot*
proof $-$
have *1*: *?e* $\leq -$ *?F*
by (*simp add*: *assms*(*3*))
have *2*: *injective f*
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
have *3*: *equivalence ?F*
using *2 forest-components-equivalence* **by** *simp*
hence *4*: *?e*^{T} = *?e*^{T} $*$ *top* $*$ *?e*^{T}
by (*smt arc-conv-closed arc-top-arc covector-complement-closed covector-conv-vector ex231e minarc-arc minarc-bot-iff pp-surjective regular-closed-top vector-mult-closed vector-top-closed*)
also have ... $\leq -$ *?F* **using** *1 3 conv-isotone conv-complement calculation* **by** *fastforce*
finally have *5*: *?e* $*$ *?F* $*$ *?e* = *bot*
using *4* **by** (*smt triple-schroeder-p le-bot pp-total regular-closed-top vector-top-closed*)
have ($f \sqcap -$ *?e*^{T})* $\leq f^\star$
by (*simp add*: *star-isotone*)
hence *?e* $*$ ($f \sqcap -$ *?e*^{T})* $*$ *?e* $\leq$ *?e* $*$ $f^\star$ $*$ *?e*
using *mult-left-isotone mult-right-isotone* **by** *blast*
also have ... $\leq$ *?e* $*$ *?F* $*$ *?e*
by (*metis conv-star-commute forest-components-increasing mult-left-isotone mult-right-isotone star-involutive*)
also have *6*: ... = *bot*
using *5* **by** *simp*
finally show *?thesis* **using** *6 le-bot* **by** *blast*
qed
next
show *forest-components* ($f \sqcap -$*?e*^{T}) $\leq -$ *?e*
proof $-$
have *1*: *?e* $\leq -$ *?F*

by (*simp add*: *assms*(*3*))
have $f \sqcap - ?e^T \leq f$
by *simp*
hence *forest-components* $(f \sqcap - ?e^T) \leq ?F$
using *forest-components-isotone* **by** *blast*
thus *?thesis*
using *1 order-lesseq-imp p-antitone-iff* **by** *blast*
qed
qed
next
show $?f' \leq --g$
proof −
have *1*: $(f \sqcap - ?e^T \sqcap - ?p) \leq --g$
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def inf.coboundedI1*)
have *2*: $(f \sqcap - ?e^T \sqcap ?p)^T \leq --g$
proof −
have $(f \sqcap - ?e^T \sqcap ?p)^T \leq f^T$
by (*simp add*: *conv-isotone inf.sup-monoid.add-assoc*)
also have ... $\leq --g$
by (*metis assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def conv-complement conv-isotone*)
finally show *?thesis*
by *simp*
qed
have *3*: $?e \leq --g$
by (*metis inf.boundedE minarc-below pp-dist-inf*)
show *?thesis* **using** *1 2 3*
by *simp*
qed
next
show *regular* $?f'$
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def minarc-regular regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
next
show $\exists w.$ *minimum-spanning-forest* $w\ g \wedge ?f' \leq w \sqcup w^T$
proof (*rule exists-a-w*)
show *symmetric g*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show *forest f*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show $f \leq --g$
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next

show *regular f*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show ($\exists w$. *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$)
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show *vector j*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *regular j*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest h*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest-modulo-equivalence* (*forest-components h*) *d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $d * top \leq - j$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest-components* $h * j = j$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show ($\forall$ *a b* . *forest-modulo-equivalence-path a b* (*forest-components h*) *d* $\wedge$ $a \leq -$(*forest-components h*) $\sqcap --\ g \wedge b \leq d \longrightarrow$ *sum*($b \sqcap g$) $\leq$ *sum*($a \sqcap g$))
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *regular d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *selected-edge h j g* $\leq -$ *forest-components f*
by (*simp add*: *assms*(*3*))
next
show *selected-edge h j g* $\neq$ *bot*
by (*simp add*: *assms*(*4*))
next
show $j \neq$ *bot*
by (*simp add*: *assms*(*2*))
next
show *regular h*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next

```
      show h ≤ −−g
        using H-below-regular-g assms(1) boruvka-inner-invariant-def
boruvka-outer-invariant-def by auto
    qed
  qed
qed

lemma second-inner-invariant-when-e-not-bot:
  assumes boruvka-inner-invariant j f h g d
    and j ≠ bot
    and selected-edge h j g ≤ − forest-components f
    and selected-edge h j g ≠ bot
  shows boruvka-inner-invariant
        (j ⊓ − choose-component (forest-components h) j)
        (f ⊓ − selected-edge h j g^T ⊓ − path f h j g ⊔
         (f ⊓ − selected-edge h j g^T ⊓ path f h j g)^T ⊔
         selected-edge h j g)
        h g (d ⊔ selected-edge h j g)
proof −
  let ?c = choose-component (forest-components h) j
  let ?p = path f h j g
  let ?F = forest-components f
  let ?H = forest-components h
  let ?e = selected-edge h j g
  let ?f' = f ⊓ −?e^T ⊓ −?p ⊔ (f ⊓ −?e^T ⊓ ?p)^T ⊔ ?e
  let ?d' = d ⊔ ?e
  let ?j' = j ⊓ −?c
  show boruvka-inner-invariant ?j' ?f' h g ?d'
  proof (unfold boruvka-inner-invariant-def, intro conjI)
    show 1: boruvka-outer-invariant ?f' g
      using assms(1, 2, 3, 4) boruvka-outer-invariant-when-e-not-bot by blast
  next
    show g ≠ bot
      using assms(1) boruvka-inner-invariant-def by force
  next
    show regular ?d'
      using assms(1) boruvka-inner-invariant-def minarc-regular by auto
  next
    show regular ?j'
      using assms(1) boruvka-inner-invariant-def by auto
  next
    show vector ?j'
      using assms(1, 2) boruvka-inner-invariant-def component-is-vector
vector-complement-closed vector-inf-closed by simp
  next
    show regular h
      by (meson assms(1) boruvka-inner-invariant-def)
  next
    show injective h
```

by (*meson assms*(*1*) *boruvka-inner-invariant-def*)
next
show *pd-kleene-allegory-class.acyclic h*
by (*meson assms*(*1*) *boruvka-inner-invariant-def*)
next
show $?H * ?j' = ?j'$
using *fc-j-eq-j-inv assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest-modulo-equivalence* $?H$ $?d'$
using *assms*(*1*, *2*, *3*) *forest-modulo-equivalence-d-U-e*
boruvka-inner-invariant-def boruvka-outer-invariant-def **by** *auto*
next
show $?d' * top \le -?j'$
proof $-$
have *31*: $?d' * top = d * top \sqcup ?e * top$
by (*simp add*: *mult-right-dist-sup*)
have *32*: $d * top \le -?j'$
by (*meson assms*(*1*) *boruvka-inner-invariant-def inf.coboundedI1*
p-antitone-iff)
have *regular* $(?c * - ?c^T)$
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*
component-is-regular regular-conv-closed regular-mult-closed **by** *presburger*
then have $minarc(?c * - ?c^T \sqcap g) = minarc(?c \sqcap - ?c^T \sqcap g)$
by (*metis component-is-vector covector-comp-inf inf-top.left-neutral*
vector-conv-compl)
also have ... $\le --(?c \sqcap - ?c^T \sqcap g)$
using *minarc-below* **by** *blast*
also have ... $\le --?c$
by (*simp add*: *inf.sup-monoid.add-assoc*)
also have ... $= ?c$
using *component-is-regular* **by** *auto*
finally have $?e \le ?c$
by *simp*
then have $?e * top \le ?c$
by (*metis component-is-vector mult-left-isotone*)
also have ... $\le -j \sqcup ?c$
by *simp*
also have ... $= -(j \sqcap - ?c)$
using *component-is-regular* **by** *auto*
finally have *33*: $?e * top \le -(j \sqcap - ?c)$
by *simp*
show *?thesis*
using *31 32 33* **by** *auto*
qed
next
show $?f' \sqcup ?f'^T = h \sqcup h^T \sqcup ?d' \sqcup ?d'^T$
proof $-$
have $?f' \sqcup ?f'^T = f \sqcap - ?e^T \sqcap - ?p \sqcup (f \sqcap - ?e^T \sqcap ?p)^T \sqcup ?e \sqcup (f \sqcap -$
$?e^T \sqcap - ?p)^T \sqcup (f \sqcap - ?e^T \sqcap ?p) \sqcup ?e^T$

by (*simp add*: *conv-dist-sup sup-monoid.add-assoc*)
also have ... $= (f \sqcap - ?e^T \sqcap - ?p) \sqcup (f \sqcap - ?e^T \sqcap ?p) \sqcup (f \sqcap - ?e^T \sqcap ?p)^T \sqcup (f \sqcap - ?e^T \sqcap - ?p)^T \sqcup ?e^T \sqcup ?e$
by (*simp add*: *sup.left-commute sup-commute*)
also have ... $= f \sqcup f^T \sqcup ?e \sqcup ?e^T$
proof (*rule simplify-f*)
show *regular ?p*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def minarc-regular regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
next
show *regular ?e*
using *minarc-regular* **by** *blast*
qed
also have ... $= h \sqcup h^T \sqcup d \sqcup d^T \sqcup ?e \sqcup ?e^T$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *auto*
finally show *?thesis*
by (*smt conv-dist-sup sup.left-commute sup-commute*)
qed
next
show $\forall$ *a b* . *forest-modulo-equivalence-path a b ?H ?d'* $\wedge a \leq - ?H \sqcap -- g \wedge b \leq ?d' \longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*intro allI*, *rule impI*, *unfold forest-modulo-equivalence-path-def*)
fix *a b*
assume *1*: $(arc\ a \wedge arc\ b \wedge a^T * top \leq (?H * ?d')^\star * ?H * b * top) \wedge a \leq - ?H \sqcap -- g \wedge b \leq ?d'$
thus $sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*cases* $b = ?e$)
case *b-equals-e*: *True*
thus *?thesis*
proof (*cases* $a = ?e$)
case *True*
thus *?thesis*
using *b-equals-e* **by** *auto*
next
case *a-ne-e*: *False*
have $sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*rule a-to-e-in-forest-modulo-equivalence*)
show *symmetric g*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show $j \neq bot$
by (*simp add*: *assms*(*2*))
next
show $f \leq -- g$
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
next
show *vector j*

using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest h*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest-modulo-equivalence* (*forest-components h*) *d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $\forall$ *a b*. *forest-modulo-equivalence-path a b* (*?H*) $d \wedge a \leq -$ *?H* $\sqcap -$
$-\ g \wedge b \leq d \longrightarrow$ *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *regular d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $b =$ *?e*
using *b-equals-e* **by** *simp*
next
show *arc a*
using *1* **by** *simp*
next
show *forest-modulo-equivalence-path a b ?H ?d′*
using *1 forest-modulo-equivalence-path-def* **by** *simp*
next
show $a \leq -$ *?H* $\sqcap --\ g$
using *1* **by** *simp*
next
show *regular h*
using *assms*(*1*) *boruvka-inner-invariant-def*
boruvka-outer-invariant-def **by** *auto*
qed
thus *?thesis*
by *simp*
qed
next
case *b-not-equal-e*: *False*
then have *b-below-d*: $b \leq d$
using *1 assms*(*4*) *different-arc-in-sup-arc minarc-arc minarc-bot-iff* **by**
metis
thus *?thesis*
proof (*cases ?e* $\leq$ *d*)
case *True*
then have *forest-modulo-equivalence-path a b ?H d* $\wedge\ b \leq d$
using *1 forest-modulo-equivalence-path-def sup.absorb1* **by** *auto*
thus *?thesis*
using *1 assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*

next
case *e-not-less-than-d*: *False*
have *71*:*equivalence ?H*
using *assms*(*1*) *fch-equivalence boruvka-inner-invariant-def* **by** *auto*
then have *72*: *forest-modulo-equivalence-path a b ?H ?d'* $\longleftrightarrow$ *forest-modulo-equivalence-path a b ?H d* $\vee$ (*forest-modulo-equivalence-path a ?e ?H d* $\wedge$ *forest-modulo-equivalence-path ?e b ?H d*)
proof (*rule forest-modulo-equivalence-path-split-disj*)
show *arc ?e*
using *assms*(*4*) *minarc-arc minarc-bot-iff* **by** *blast*
next
show *regular a* $\wedge$ *regular b* $\wedge$ *regular ?e* $\wedge$ *regular d* $\wedge$ *regular ?H*
using *assms*(*1*) *1 boruvka-inner-invariant-def boruvka-outer-invariant-def arc-regular minarc-regular regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
qed
thus *?thesis*
proof (*cases forest-modulo-equivalence-path a b ?H d*)
case *True*
have *forest-modulo-equivalence-path a b ?H d* $\wedge$ $b \leq d$
using *1 True forest-modulo-equivalence-path-def sup.absorb1* **by** (*metis assms*(*4*) *b-not-equal-e minarc-arc minarc-bot-iff different-arc-in-sup-arc*)
thus *?thesis*
using *1 assms*(*1*) *b-below-d boruvka-inner-invariant-def* **by** *auto*
next
case *False*
have *73*:*forest-modulo-equivalence-path a ?e ?H d* $\wedge$ *forest-modulo-equivalence-path ?e b ?H d*
using *1 72 False forest-modulo-equivalence-path-def* **by** *blast*
have *74*: *?e* $\leq --g$
by (*metis inf.boundedE minarc-below pp-dist-inf*)
have *?H* $\leq$ *?F*
using *assms*(*1*) *H-below-F boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *blast*
then have *?e* $\leq -$ *?H*
using *assms*(*3*) *order-trans p-antitone* **by** *blast*
then have *?e* $\leq -$ *?H* $\sqcap --g$
using *74* **by** *simp*
then have *75*: *sum* ($b \sqcap g$) $\leq$ *sum* (*?e* $\sqcap g$)
using *assms*(*1*) *b-below-d 73 boruvka-inner-invariant-def* **by** *blast*
have *76*: *forest-modulo-equivalence-path a ?e ?H ?d'*
by (*meson 73 forest-modulo-equivalence-path-split-disj assms*(*1*) *forest-modulo-equivalence-path-def boruvka-inner-invariant-def boruvka-outer-invariant-def fch-equivalence arc-regular regular-closed-star regular-conv-closed regular-mult-closed*)
have *77*: *sum* (*?e* $\sqcap g$) $\leq$ *sum* ($a \sqcap g$)
proof (*rule a-to-e-in-forest-modulo-equivalence*)
show *symmetric g*
using *assms*(*1*) *boruvka-inner-invariant-def*

boruvka-outer-invariant-def **by** *auto*
next
show $j \neq bot$
by (*simp add*: *assms*(*2*))
next
show $f \leq --\ g$
using *assms*(*1*) *boruvka-inner-invariant-def*
boruvka-outer-invariant-def **by** *auto*
next
show *vector j*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest h*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *forest-modulo-equivalence* (*forest-components h*) *d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show $\forall$ *a b*. *forest-modulo-equivalence-path a b* (*?H*) $d \wedge a \leq -$ *?H* $\sqcap$
$- -\ g \wedge b \leq d \longrightarrow$ *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *regular d*
using *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
next
show *?e* = *?e*
by *simp*
next
show *arc a*
using *1* **by** *simp*
next
show *forest-modulo-equivalence-path a ?e ?H ?d′*
by (*simp add*: *76*)
next
show $a \leq -$ *?H* $\sqcap --g$
using *1* **by** *simp*
next
show *regular h*
using *assms*(*1*) *boruvka-inner-invariant-def*
boruvka-outer-invariant-def **by** *auto*
qed
thus *?thesis*
using *75 order.trans* **by** *blast*
qed
qed
qed

    qed
  qed
qed

lemma *second-inner-invariant-when-e-bot*:
  assumes *selected-edge h j g* = *bot*
    and *selected-edge h j g* $\leq$ − *forest-components f*
    and *boruvka-inner-invariant j f h g d*
  shows *boruvka-inner-invariant*
    ($j \sqcap -$ *choose-component* (*forest-components h*) *j*)
    ($f \sqcap -$ *selected-edge h j* $g^T \sqcap -$ *path f h j g* $\sqcup$
     ($f \sqcap -$ *selected-edge h j* $g^T \sqcap$ *path f h j g*)^T $\sqcup$
     selected-edge h j g)
    h g (*d* $\sqcup$ *selected-edge h j g*)
proof −
  let *?c* = *choose-component* (*forest-components h*) *j*
  let *?p* = *path f h j g*
  let *?F* = *forest-components f*
  let *?H* = *forest-components h*
  let *?e* = *selected-edge h j g*
  let $?f' = f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
  let $?d' = d \sqcup ?e$
  let $?j' = j \sqcap -?c$
  show *boruvka-inner-invariant ?j' ?f' h g ?d'*
  proof (*unfold boruvka-inner-invariant-def*, *intro conjI*)
  next
    show *boruvka-outer-invariant ?f' g*
      using *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
  next
    show $g \neq$ *bot*
      using *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
  next
    show *regular ?d'*
      using *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
  next
    show *regular ?j'*
      using *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
  next
    show *vector ?j'*
      by (*metis assms*(*3*) *boruvka-inner-invariant-def component-is-vector*
vector-complement-closed vector-inf-closed)
  next
    show *regular h*
      using *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
  next
    show *injective h*
      using *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
  next
    show *pd-kleene-allegory-class.acyclic h*

using *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
next
show $?H * ?j' = ?j'$
using *assms*(*3*) *fc-j-eq-j-inv boruvka-inner-invariant-def* **by** *blast*
next
show *forest-modulo-equivalence* $?H$ $?d'$
using *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
next
show $?d' * top \leq -?j'$
using *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** (*metis order.trans p-antitone-inf sup-monoid.add-0-right*)
next
show $?f' \sqcup ?f'^T = h \sqcup h^T \sqcup ?d' \sqcup ?d'^T$
using *assms*(*1*, *3*) *boruvka-inner-invariant-def* **by** *auto*
next
show $\forall a\ b.$ *forest-modulo-equivalence-path* $a\ b\ ?H\ ?d' \wedge a \leq -?H \sqcap --g \wedge b \leq ?d' \longrightarrow sum(b \sqcap g) \leq sum(a \sqcap g)$
using *assms*(*1*, *3*) *boruvka-inner-invariant-def* **by** *auto*
qed
qed

4.5 Formalization and correctness proof

The following result shows that Borůvka's algorithm constructs a minimum spanning forest. We have the same postcondition as the proof of Kruskal's minimum spanning tree algorithm. We show only partial correctness.

theorem *boruvka-mst*:
VARS f j h c e d
$\{$ *symmetric g* $\}$
$f := bot$;
WHILE $-(forest\text{-}components\ f) \sqcap g \neq bot$
INV $\{$ *boruvka-outer-invariant f g* $\}$
DO
$j := top$;
$h := f$;
$d := bot$;
WHILE $j \neq bot$
INV $\{$ *boruvka-inner-invariant j f h g d* $\}$
DO
$c :=$ *choose-component* (*forest-components* h) j;
$e := minarc(c * -c^T \sqcap g)$;
IF $e \leq -(forest\text{-}components\ f)$ *THEN*
$f := f \sqcap -e^T$;
$f := (f \sqcap -(top * e * f^{T\star})) \sqcup (f \sqcap top * e * f^{T\star})^T \sqcup e$;
$d := d \sqcup e$
ELSE
SKIP
FI;
$j := j \sqcap -c$

```
      OD
    OD
  { minimum-spanning-forest f g }
proof vcg-simp
  assume 1: symmetric g
  show boruvka-outer-invariant bot g
    using 1 boruvka-outer-invariant-def kruskal-exists-minimal-spanning by auto
next
  fix f
  let ?F = forest-components f
  assume 1: boruvka-outer-invariant f g ∧ − ?F ⊓ g ≠ bot
  have 2: equivalence ?F
    using 1 boruvka-outer-invariant-def forest-components-equivalence by auto
  show boruvka-inner-invariant top f f g bot
  proof (unfold boruvka-inner-invariant-def, intro conjI)
    show boruvka-outer-invariant f g
      by (simp add: 1)
  next
    show g ≠ bot
      using 1 by auto
  next
    show surjective top
      by simp
  next
    show regular top
      by simp
  next
    show regular bot
      by auto
  next
    show regular f
      using 1 boruvka-outer-invariant-def by blast
  next
    show injective f
      using 1 boruvka-outer-invariant-def by blast
  next
    show pd-kleene-allegory-class.acyclic f
      using 1 boruvka-outer-invariant-def by blast
  next
    show forest-modulo-equivalence ?F bot
      by (simp add: 2 forest-modulo-equivalence-def)
  next
    show bot * top ≤ − top
      by simp
  next
    show times-top-class.total (?F)
      by (simp add: star.circ-right-top mult-assoc)
  next
    show f ⊔ f^T = f ⊔ f^T ⊔ bot ⊔ bot^T
```

by *simp*
next
show $\forall\ a\ b.$ *forest-modulo-equivalence-path a b ?F bot* $\wedge$ $a \leq -\ ?F \sqcap --\ g \wedge$ $b \leq bot \longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
by (*metis* (*full-types*) *forest-modulo-equivalence-path-def bot-unique mult-left-zero mult-right-zero top.extremum*)
qed
next
fix *f j h d*
let $?c$ = *choose-component* (*forest-components h*) *j*
let $?p$ = *path f h j g*
let $?F$ = *forest-components f*
let $?H$ = *forest-components h*
let $?e$ = *selected-edge h j g*
let $?f' = f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
let $?d' = d \sqcup ?e$
let $?j' = j \sqcap -?c$
assume *1*: *boruvka-inner-invariant j f h g d* $\wedge\ j \neq bot$
show ($?e \leq -?F \longrightarrow$ *boruvka-inner-invariant* $?j'\ ?f'\ h\ g\ ?d'$) $\wedge$ ($\neg\ ?e \leq -?F \longrightarrow$ *boruvka-inner-invariant* $?j'$ *f h g d*)
proof (*intro conjI*)
show $?e \leq -?F \longrightarrow$ *boruvka-inner-invariant* $?j'\ ?f'\ h\ g\ ?d'$
proof (*cases* $?e = bot$)
case *True*
thus *?thesis*
using *1 second-inner-invariant-when-e-bot* **by** *simp*
next
case *False*
thus *?thesis*
using *1 second-inner-invariant-when-e-not-bot* **by** *simp*
qed
next
show $\neg\ ?e \leq -?F \longrightarrow$ *boruvka-inner-invariant* $?j'$ *f h g d*
proof (*rule impI*, *unfold boruvka-inner-invariant-def*, *intro conjI*)
show *boruvka-outer-invariant f g*
using *1 boruvka-inner-invariant-def* **by** *blast*
next
show $g \neq bot$
using *1 boruvka-inner-invariant-def* **by** *blast*
next
show *vector* $?j'$
using *1 boruvka-inner-invariant-def component-is-vector vector-complement-closed vector-inf-closed* **by** *auto*
next
show *regular* $?j'$
using *1 boruvka-inner-invariant-def* **by** *auto*
next
show *regular d*
using *1 boruvka-inner-invariant-def* **by** *blast*

```
    next
      show regular h
        using 1 boruvka-inner-invariant-def by blast
    next
      show injective h
        using 1 boruvka-inner-invariant-def by blast
    next
      show pd-kleene-allegory-class.acyclic h
        using 1 boruvka-inner-invariant-def by blast
    next
      show forest-modulo-equivalence ?H d
        using 1 boruvka-inner-invariant-def by blast
    next
      show d * top ≤ −?j′
        using 1 by (meson boruvka-inner-invariant-def dual-order.trans
p-antitone-inf)
    next
      show ?H * ?j′ = ?j′
        using 1 fc-j-eq-j-inv boruvka-inner-invariant-def by blast
    next
      show f ⊔ f^T = h ⊔ h^T ⊔ d ⊔ d^T
        using 1 boruvka-inner-invariant-def by blast
    next
      show ¬ ?e ≤ −?F ⟹ ∀ a b. forest-modulo-equivalence-path a b ?H d ∧ a ≤
−?H ⊓ −−g ∧ b ≤ d ⟶ sum(b ⊓ g) ≤ sum(a ⊓ g)
        using 1 boruvka-inner-invariant-def by blast
    qed
  qed
next
  fix f h d
  assume boruvka-inner-invariant bot f h g d
  thus boruvka-outer-invariant f g
    by (meson boruvka-inner-invariant-def)
next
  fix f
  assume 1: boruvka-outer-invariant f g ∧ − forest-components f ⊓ g = bot
  hence 2:spanning-forest f g
  proof (unfold spanning-forest-def, intro conjI)
    show injective f
      using 1 boruvka-outer-invariant-def by blast
  next
    show acyclic f
      using 1 boruvka-outer-invariant-def by blast
  next
    show f ≤ −−g
      using 1 boruvka-outer-invariant-def by blast
  next
    show components g ≤ forest-components f
    proof −
```

let *?F* = *forest-components f*
have $-?F \sqcap g \leq bot$
by (*simp add*: *1*)
hence $--g \leq bot \sqcup --?F$
using *1 shunting-p p-antitone pseudo-complement* **by** *auto*
hence $--g \leq ?F$
using *1 boruvka-outer-invariant-def pp-dist-comp pp-dist-star regular-conv-closed* **by** *auto*
hence $(--g)^\star \leq ?F^\star$
by (*simp add*: *star-isotone*)
thus *?thesis*
using *1 boruvka-outer-invariant-def forest-components-star* **by** *auto*
qed
next
show *regular f*
using *1 boruvka-outer-invariant-def* **by** *auto*
qed
from *1* **obtain** *w* **where** *3*: *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$
using *boruvka-outer-invariant-def* **by** *blast*
hence $w = w \sqcap --g$
by (*simp add*: *inf.absorb1 minimum-spanning-forest-def spanning-forest-def*)
also have ... $\leq w \sqcap$ *components g*
by (*metis inf.sup-right-isotone star.circ-increasing*)
also have ... $\leq w \sqcap f^{T\star} * f^\star$
using *2 spanning-forest-def inf.sup-right-isotone* **by** *simp*
also have ... $\leq f \sqcup f^T$
proof (*rule cancel-separate-6*[**where** z=w **and** y=w^T])
show *injective w*
using *3 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
next
show $f^T \leq w^T \sqcup w$
using *3* **by** (*metis conv-dist-inf conv-dist-sup conv-involutive inf.cobounded2 inf.orderE*)
next
show $f \leq w^T \sqcup w$
using *3* **by** (*simp add*: *sup-commute*)
next
show *injective w*
using *3 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
next
show $w \sqcap w^{T\star} = bot$
using *3* **by** (*metis acyclic-star-below-complement comp-inf.mult-right-isotone inf-p le-bot minimum-spanning-forest-def spanning-forest-def*)
qed
finally have *4*: $w \leq f \sqcup f^T$
by *simp*
have *sum* $(f \sqcap g)$ = *sum* $((w \sqcup w^T) \sqcap (f \sqcap g))$
using *3* **by** (*metis inf-absorb2 inf.assoc*)
also have ... = *sum* $(w \sqcap (f \sqcap g))$ + *sum* $(w^T \sqcap (f \sqcap g))$

using *3 inf.commute acyclic-asymmetric sum-disjoint minimum-spanning-forest-def spanning-forest-def* **by** *simp*
also have ... $= sum\ (w \sqcap (f \sqcap g)) + sum\ (w \sqcap (f^T \sqcap g^T))$
by (*metis conv-dist-inf conv-involutive sum-conv*)
also have ... $= sum\ (f \sqcap (w \sqcap g)) + sum\ (f^T \sqcap (w \sqcap g))$
proof −
have *51*:$f^T \sqcap (w \sqcap g) = f^T \sqcap (w \sqcap g^T)$
using *1 boruvka-outer-invariant-def* **by** *auto*
have *52*:$f \sqcap (w \sqcap g) = w \sqcap (f \sqcap g)$
by (*simp add*: *inf.left-commute*)
thus *?thesis*
using *51 52 abel-semigroup.left-commute inf.abel-semigroup-axioms* **by** *fastforce*
qed
also have ... $= sum\ ((f \sqcup f^T) \sqcap (w \sqcap g))$
using *2 acyclic-asymmetric inf.sup-monoid.add-commute sum-disjoint spanning-forest-def* **by** *simp*
also have ... $= sum\ (w \sqcap g)$
using *4* **by** (*metis inf-absorb2 inf.assoc*)
finally show *minimum-spanning-forest f g*
using *2 3 minimum-spanning-forest-def* **by** *simp*
qed

end

end

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