

# Cardinality and Representation of Stone Relation Algebras

Walter Guttmann

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## Abstract

In relation algebras, which model unweighted graphs, the cardinality operation counts the number of edges of a graph. We generalise the cardinality axioms to Stone relation algebras, which model weighted graphs, and study the relationships between various axioms for cardinality. We also give a representation theorem for Stone relation algebras.

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The theories formally verify results in [1]. See this papers for further details and related work. Stone relation algebras have been introduced in [2] and formalised in [3].

**theory** *Representation*

**imports** *Stone-Relation-Algebras.Matrix-Relation-Algebras*

**begin**

## 1 Representation of Stone Relation Algebras

We show that Stone relation algebras can be represented by matrices if we assume a point axiom. The matrix indices and entries and the point axiom are based on the concepts of ideals and ideal-points. We start with general results about sets and finite suprema.

**lemma** *finite-ne-subset-induct'* [*consumes 3, case-names singleton insert*]:

**assumes** *finite F*  
**and**  $F \neq \{\}$   
**and**  $F \subseteq S$   
**and** *singleton*:  $\bigwedge x . x \in S \implies P \{x\}$   
**and** *insert*:  $\bigwedge x F . \text{finite } F \implies F \neq \{\} \implies F \subseteq S \implies x \in S \implies x \notin F$   
 $\implies P F \implies P (\text{insert } x F)$   
**shows**  $P F$   
*<proof>*

**context** *order-bot*

**begin**

**abbreviation** *atom* :: '*a*  $\Rightarrow$  bool

**where**  $\text{atom } x \equiv x \neq \text{bot} \wedge (\forall y . y \neq \text{bot} \wedge y \leq x \longrightarrow y = x)$

**end**

**context** *semilattice-sup*

**begin**

```

lemma nested-sup-fin:
  assumes finite X
    and  $X \neq \{\}$ 
    and finite Y
    and  $Y \neq \{\}$ 
  shows  $\text{Sup-fin } \{ \text{Sup-fin } \{ f\ x\ y \mid x . x \in X \} \mid y . y \in Y \} = \text{Sup-fin } \{ f\ x\ y \mid$ 
 $x\ y . x \in X \wedge y \in Y \}$ 
   $\langle \text{proof} \rangle$ 

```

**end**

```

context bounded-semilattice-sup-bot
begin

```

```

lemma one-point-sup-fin:
  assumes finite X
    and  $y \in X$ 
  shows  $\text{Sup-fin } \{ (if\ x = y\ then\ f\ x\ else\ bot) \mid x . x \in X \} = f\ y$ 
   $\langle \text{proof} \rangle$ 

```

**end**

## 1.1 Ideals and Ideal-Points

We study ideals in Stone relation algebras, which are elements that are both a vector and a covector. We include general results about Stone relation algebras.

```

context times-top
begin

```

```

abbreviation ideal :: 'a  $\Rightarrow$  bool where ideal  $x \equiv \text{vector } x \wedge \text{covector } x$ 

```

**end**

```

context bounded-non-associative-left-semiring
begin

```

```

lemma ideal-fixpoint:
  ideal  $x \longleftrightarrow \text{top} * x * \text{top} = x$ 
   $\langle \text{proof} \rangle$ 

```

```

lemma ideal-top-closed:
  ideal top
   $\langle \text{proof} \rangle$ 

```

**end**

```

context bounded-idempotent-left-semiring
begin

```

```

lemma ideal-mult-closed:
  ideal  $x \implies \textit{ideal } y \implies \textit{ideal } (x * y)$ 
   $\langle \textit{proof} \rangle$ 

end

context bounded-idempotent-left-zero-semiring
begin

lemma ideal-sup-closed:
  ideal  $x \implies \textit{ideal } y \implies \textit{ideal } (x \sqcup y)$ 
   $\langle \textit{proof} \rangle$ 

end

context idempotent-semiring
begin

lemma sup-fin-sum:
  fixes  $f :: 'b :: \textit{finite} \Rightarrow 'a$ 
  shows  $\textit{Sup-fin } \{ f\ x \mid x . x \in \textit{UNIV} \} = (\bigsqcup_x f\ x)$ 
   $\langle \textit{proof} \rangle$ 

end

context stone-relation-algebra
begin

lemma dedekind-univalent:
  assumes univalent  $y$ 
  shows  $x * y \sqcap z = (x \sqcap z * y^T) * y$ 
   $\langle \textit{proof} \rangle$ 

lemma dedekind-injective:
  assumes injective  $x$ 
  shows  $x * y \sqcap z = x * (y \sqcap x^T * z)$ 
   $\langle \textit{proof} \rangle$ 

lemma domain-vector-conv:
   $1 \sqcap x * \textit{top} = 1 \sqcap x * x^T$ 
   $\langle \textit{proof} \rangle$ 

lemma domain-vector-covector:
   $1 \sqcap x * \textit{top} = 1 \sqcap \textit{top} * x^T$ 
   $\langle \textit{proof} \rangle$ 

lemma domain-covector-conv:
   $1 \sqcap \textit{top} * x^T = 1 \sqcap x * x^T$ 

```

$\langle \text{proof} \rangle$

**lemma** *ideal-bot-closed*:

*ideal bot*

$\langle \text{proof} \rangle$

**lemma** *ideal-inf-closed*:

*ideal x  $\implies$  ideal y  $\implies$  ideal (x  $\sqcap$  y)*

$\langle \text{proof} \rangle$

**lemma** *ideal-conv-closed*:

*ideal x  $\implies$  ideal (x<sup>T</sup>)*

$\langle \text{proof} \rangle$

**lemma** *ideal-complement-closed*:

*ideal x  $\implies$  ideal (−x)*

$\langle \text{proof} \rangle$

**lemma** *ideal-conv-id*:

*ideal x  $\implies$  x = x<sup>T</sup>*

$\langle \text{proof} \rangle$

**lemma** *ideal-mult-inf*:

*ideal x  $\implies$  ideal y  $\implies$  x \* y = x  $\sqcap$  y*

$\langle \text{proof} \rangle$

**lemma** *ideal-mult-import*:

*ideal x  $\implies$  y \* z  $\sqcap$  x = (y  $\sqcap$  x) \* (z  $\sqcap$  x)*

$\langle \text{proof} \rangle$

**lemma** *point-meet-one*:

*point x  $\implies$  x \* x<sup>T</sup> = x  $\sqcap$  1*

$\langle \text{proof} \rangle$

**lemma** *below-point-eq-domain*:

*point x  $\implies$  y  $\leq$  x  $\implies$  y = x \* x<sup>T</sup> \* y*

$\langle \text{proof} \rangle$

**lemma** *covector-mult-vector-ideal*:

*vector x  $\implies$  vector z  $\implies$  ideal (x<sup>T</sup> \* y \* z)*

$\langle \text{proof} \rangle$

**abbreviation** *ideal-point* :: 'a  $\Rightarrow$  bool **where** *ideal-point* x  $\equiv$  point x  $\wedge$  ( $\forall$  y z . point y  $\wedge$  ideal z  $\wedge$  z  $\neq$  bot  $\wedge$  y \* z  $\leq$  x  $\longrightarrow$  y  $\leq$  x)

**lemma** *different-ideal-points-disjoint*:

**assumes** *ideal-point* p

**and** *ideal-point* q

**and** p  $\neq$  q

**shows**  $p \sqcap q = \text{bot}$   
 $\langle \text{proof} \rangle$

**lemma** *points-disjoint-iff*:  
**assumes** *vector*  $x$   
**shows**  $x \sqcap y = \text{bot} \longleftrightarrow x^T * y = \text{bot}$   
 $\langle \text{proof} \rangle$

**lemma** *different-ideal-points-disjoint-2*:  
**assumes** *ideal-point*  $p$   
**and** *ideal-point*  $q$   
**and**  $p \neq q$   
**shows**  $p^T * q = \text{bot}$   
 $\langle \text{proof} \rangle$

**lemma** *mult-right-dist-sup-fin*:  
**assumes** *finite*  $X$   
**and**  $X \neq \{\}$   
**shows**  $\text{Sup-fin } \{ f\ x \mid x::'b . x \in X \} * y = \text{Sup-fin } \{ f\ x * y \mid x . x \in X \}$   
 $\langle \text{proof} \rangle$

**lemma** *mult-left-dist-sup-fin*:  
**assumes** *finite*  $X$   
**and**  $X \neq \{\}$   
**shows**  $y * \text{Sup-fin } \{ f\ x \mid x::'b . x \in X \} = \text{Sup-fin } \{ y * f\ x \mid x . x \in X \}$   
 $\langle \text{proof} \rangle$

**lemma** *inf-left-dist-sup-fin*:  
**assumes** *finite*  $X$   
**and**  $X \neq \{\}$   
**shows**  $y \sqcap \text{Sup-fin } \{ f\ x \mid x::'b . x \in X \} = \text{Sup-fin } \{ y \sqcap f\ x \mid x . x \in X \}$   
 $\langle \text{proof} \rangle$

**lemma** *top-one-sup-fin-iff*:  
**assumes** *finite*  $P$   
**and**  $P \neq \{\}$   
**and**  $\forall p \in P . \text{point } p$   
**shows**  $\text{top} = \text{Sup-fin } P \longleftrightarrow 1 = \text{Sup-fin } \{ p * p^T \mid p . p \in P \}$   
 $\langle \text{proof} \rangle$

**abbreviation** *ideals* :: 'a set **where** *ideals*  $\equiv \{ x . \text{ideal } x \}$   
**abbreviation** *ideal-points* :: 'a set **where** *ideal-points*  $\equiv \{ x . \text{ideal-point } x \}$

**lemma** *surjective-vector-top*:  
*surjective*  $x \implies \text{vector } x \implies x^T * x = \text{top}$   
 $\langle \text{proof} \rangle$

**lemma** *point-mult-top*:  
*point*  $x \implies x^T * x = \text{top}$

$\langle \text{proof} \rangle$

**lemma** *point-below-equal*:

$\text{point } p \implies \text{point } q \implies p \leq q \implies p = q$

$\langle \text{proof} \rangle$

**lemma** *ideal-point-without-ideal*:

$\text{ideal-point } p \iff (\text{point } p \wedge (\forall q . \text{point } q \longrightarrow q \leq p \vee q \leq -p))$

$\langle \text{proof} \rangle$

**lemma** *ideal-point-without-ideal-2*:

$\text{ideal-point } p \iff (\text{point } p \wedge (\forall q . \text{point } q \longrightarrow q = p \vee q \leq -p))$

$\langle \text{proof} \rangle$

**lemma** *ideal-point-without-ideal-3*:

$\text{ideal-point } p \iff (\text{point } p \wedge (\forall q . \text{point } q \wedge q \neq p \longrightarrow q \leq -p))$

$\langle \text{proof} \rangle$

**end**

## 1.2 Point Axiom

The following class captures the point axiom for Stone relation algebras.

**class** *stone-relation-algebra-pa* = *stone-relation-algebra* +

**assumes** *finite-ideal-points*: *finite ideal-points*

**assumes** *ne-ideal-points*: *ideal-points*  $\neq \{\}$

**assumes** *top-sup-ideal-points*: *top* = *Sup-fin ideal-points*

**begin**

**lemma** *one-sup-ideal-points*:

$1 = \text{Sup-fin } \{ p * p^T \mid p . \text{ideal-point } p \}$

$\langle \text{proof} \rangle$

**lemma** *ideal-point-rep-1*:

$x = \text{Sup-fin } \{ p * p^T * x * q * q^T \mid p q . \text{ideal-point } p \wedge \text{ideal-point } q \}$

$\langle \text{proof} \rangle$

**lemma** *atom-below-ideal-point*:

**assumes** *atom* *a*

**shows**  $\exists p . \text{ideal-point } p \wedge a \leq p$

$\langle \text{proof} \rangle$

**lemma** *point-ideal-point-1*:

**assumes** *point* *a*

**shows** *ideal-point* *a*

$\langle \text{proof} \rangle$

**lemma** *point-ideal-point*:

$\text{point } x \iff \text{ideal-point } x$

*<proof>*

**end**

### 1.3 Ideals, Ideal-Points and Matrices as Types

Stone relation algebras will be represented by matrices with ideal-points as entries and ideals as indices. To define the type of such matrices, we first derive types for the set of ideals and ideal-points.

**typedef** (overloaded) 'a ideal = ideals::'a::stone-relation-algebra-pa set  
*<proof>*

**setup-lifting** type-definition-ideal

**instantiation** ideal :: (stone-relation-algebra-pa) stone-algebra  
**begin**

**lift-definition** uminus-ideal :: 'a ideal  $\Rightarrow$  'a ideal **is** uminus  
*<proof>*

**lift-definition** inf-ideal :: 'a ideal  $\Rightarrow$  'a ideal  $\Rightarrow$  'a ideal **is** inf  
*<proof>*

**lift-definition** sup-ideal :: 'a ideal  $\Rightarrow$  'a ideal  $\Rightarrow$  'a ideal **is** sup  
*<proof>*

**lift-definition** bot-ideal :: 'a ideal **is** bot  
*<proof>*

**lift-definition** top-ideal :: 'a ideal **is** top  
*<proof>*

**lift-definition** less-eq-ideal :: 'a ideal  $\Rightarrow$  'a ideal  $\Rightarrow$  bool **is** less-eq *<proof>*

**lift-definition** less-ideal :: 'a ideal  $\Rightarrow$  'a ideal  $\Rightarrow$  bool **is** less *<proof>*

**instance**  
*<proof>*

**end**

**instantiation** ideal :: (stone-relation-algebra-pa) stone-relation-algebra  
**begin**

**lift-definition** conv-ideal :: 'a ideal  $\Rightarrow$  'a ideal **is** id  
*<proof>*

**lift-definition** times-ideal :: 'a ideal  $\Rightarrow$  'a ideal  $\Rightarrow$  'a ideal **is** inf  
*<proof>*



```

lift-definition one-ideal :: 'a ideal is top
  ⟨proof⟩

instance
  ⟨proof⟩

end

typedef (overloaded) 'a ideal-point = ideal-points::'a::stone-relation-algebra-pa
  set
  ⟨proof⟩

instantiation ideal-point :: (stone-relation-algebra-pa) finite
begin

instance
  ⟨proof⟩

end

type-synonym 'a ideal-matrix = ('a ideal-point, 'a ideal) square

interpretation ideal-matrix-stone-relation-algebra: stone-relation-algebra where
  sup = sup-matrix and inf = inf-matrix and less-eq = less-eq-matrix and less =
  less-matrix and bot = bot-matrix::'a::stone-relation-algebra-pa ideal-matrix and
  top = top-matrix and uminus = uminus-matrix and one = one-matrix and
  times = times-matrix and conv = conv-matrix
  ⟨proof⟩

lemma ideal-point-rep-2:
  assumes x = Sup-fin { Rep-ideal-point p * Rep-ideal (f p q) * (Rep-ideal-point
  q)T | p q . True }
  shows f r s = Abs-ideal ((Rep-ideal-point r)T * x * (Rep-ideal-point s))
  ⟨proof⟩

```

## 1.4 Isomorphism

The following two functions comprise the isomorphism between Stone relation algebras and matrices. We prove that they are inverses of each other and that the first one is a homomorphism.

**definition** sra-to-mat :: 'a::stone-relation-algebra-pa  $\Rightarrow$  'a ideal-matrix  
**where** sra-to-mat x  $\equiv$   $\lambda(p,q)$  . Abs-ideal ((Rep-ideal-point p)<sup>T</sup> \* x \* Rep-ideal-point q)

**definition** mat-to-sra :: 'a::stone-relation-algebra-pa ideal-matrix  $\Rightarrow$  'a  
**where** mat-to-sra f  $\equiv$  Sup-fin { Rep-ideal-point p \* Rep-ideal (f (p,q)) \* (Rep-ideal-point q)<sup>T</sup> | p q . True }

**lemma** *sra-mat-sra*:

$mat\text{-}to\text{-}sra\ (sra\text{-}to\text{-}mat\ x) = x$   
 $\langle proof \rangle$

**lemma** *mat-sra-mat*:

$sra\text{-}to\text{-}mat\ (mat\text{-}to\text{-}sra\ f) = f$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-sup-homomorphism*:

$sra\text{-}to\text{-}mat\ (x \sqcup y) = sra\text{-}to\text{-}mat\ x \sqcup sra\text{-}to\text{-}mat\ y$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-inf-homomorphism*:

$sra\text{-}to\text{-}mat\ (x \sqcap y) = sra\text{-}to\text{-}mat\ x \sqcap sra\text{-}to\text{-}mat\ y$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-conv-homomorphism*:

$sra\text{-}to\text{-}mat\ (x^T) = (sra\text{-}to\text{-}mat\ x)^t$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-complement-homomorphism*:

$sra\text{-}to\text{-}mat\ (-x) = -(sra\text{-}to\text{-}mat\ x)$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-bot-homomorphism*:

$sra\text{-}to\text{-}mat\ bot = bot$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-top-homomorphism*:

$sra\text{-}to\text{-}mat\ top = top$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-one-homomorphism*:

$sra\text{-}to\text{-}mat\ 1 = one\text{-}matrix$   
 $\langle proof \rangle$

**lemma** *Abs-ideal-dist-sup-fin*:

**assumes** *finite X*  
**and**  $X \neq \{\}$   
**and**  $\forall x \in X . ideal\ (f\ x)$   
**shows**  $Abs\text{-}ideal\ (Sup\text{-}fin\ \{ f\ x \mid x . x \in X \}) = Sup\text{-}fin\ \{ Abs\text{-}ideal\ (f\ x) \mid x . x \in X \}$   
 $\langle proof \rangle$

**lemma** *sra-to-mat-mult-homomorphism*:

$sra\text{-}to\text{-}mat\ (x * y) = sra\text{-}to\text{-}mat\ x \odot sra\text{-}to\text{-}mat\ y$   
 $\langle proof \rangle$

**end**

```

theory Cardinality

imports List-Infinite.InfiniteSet2 Representation

begin

context uminus
begin

no-notation uminus ( $-$  - [81] 80)

end

```

## 2 Atoms Below an Element in Partial Orders

We define the set and the number of atoms below an element in a partial order. To handle infinitely many atoms we use *enat*, which are natural numbers with infinity, and *icard*, which modifies *card* by giving a separate option of being infinite. We include general results about *enat*, *icard*, sets functions and atoms.

**lemma** *enat-mult-strict-mono*:  
**assumes**  $a < b$   $c < d$   $(0::\text{enat}) < b$   $0 \leq c$   
**shows**  $a * c < b * d$   
 $\langle \text{proof} \rangle$

**lemma** *enat-mult-strict-mono'*:  
**assumes**  $a < b$  **and**  $c < d$  **and**  $(0::\text{enat}) \leq a$  **and**  $0 \leq c$   
**shows**  $a * c < b * d$   
 $\langle \text{proof} \rangle$

**lemma** *finite-icard-card*:  
 $\text{finite } A \implies \text{icard } A = \text{icard } B \implies \text{card } A = \text{card } B$   
 $\langle \text{proof} \rangle$

**lemma** *icard-eq-sum*:  
 $\text{finite } A \implies \text{icard } A = \text{sum } (\lambda x. 1) A$   
 $\langle \text{proof} \rangle$

**lemma** *icard-sum-constant-function*:  
**assumes**  $\forall x \in A. f x = c$   
**and**  $\text{finite } A$   
**shows**  $\text{sum } f A = (\text{icard } A) * c$   
 $\langle \text{proof} \rangle$

**lemma** *icard-le-finite*:  
**assumes**  $\text{icard } A \leq \text{icard } B$   
**and**  $\text{finite } B$   
**shows**  $\text{finite } A$

*<proof>*

**lemma** *bij-betw-same-icard:*

*bij-betw*  $f$   $A$   $B \implies \text{icard } A = \text{icard } B$

*<proof>*

**lemma** *surj-icard-le:*  $B \subseteq f \text{ `` } A \implies \text{icard } B \leq \text{icard } A$

*<proof>*

**lemma** *icard-image-part-le:*

**assumes**  $\forall x \in A . f\ x \subseteq B$

**and**  $\forall x \in A . f\ x \neq \{\}$

**and**  $\forall x \in A . \forall y \in A . x \neq y \longrightarrow f\ x \cap f\ y = \{\}$

**shows**  $\text{icard } A \leq \text{icard } B$

*<proof>*

**lemma** *finite-image-part-le:*

**assumes**  $\forall x \in A . f\ x \subseteq B$

**and**  $\forall x \in A . f\ x \neq \{\}$

**and**  $\forall x \in A . \forall y \in A . x \neq y \longrightarrow f\ x \cap f\ y = \{\}$

**and** *finite*  $B$

**shows** *finite*  $A$

*<proof>*

**context** *semiring-1*

**begin**

**lemma** *sum-constant-function:*

**assumes**  $\forall x \in A . f\ x = c$

**shows**  $\text{sum } f\ A = \text{of-nat } (\text{card } A) * c$

*<proof>*

**end**

**context** *order*

**begin**

**lemma** *ne-finite-has-minimal:*

**assumes** *finite*  $S$

**and**  $S \neq \{\}$

**shows**  $\exists m \in S . \forall x \in S . x \leq m \longrightarrow x = m$

*<proof>*

**end**

**context** *order-bot*

**begin**

**abbreviation** *atoms-below*  $:: 'a \Rightarrow 'a \text{ set } (AB)$

**where**  $atoms\text{-}below\ x \equiv \{ a . atom\ a \wedge a \leq x \}$

**definition**  $num\text{-}atoms\text{-}below :: 'a \Rightarrow enat\ (nAB)$   
**where**  $num\text{-}atoms\text{-}below\ x \equiv icard\ (atoms\text{-}below\ x)$

**lemma**  $AB\text{-}iso$ :  
 $x \leq y \implies AB\ x \subseteq AB\ y$   
 $\langle proof \rangle$

**lemma**  $AB\text{-}bot$ :  
 $AB\ bot = \{\}$   
 $\langle proof \rangle$

**lemma**  $nAB\text{-}bot$ :  
 $nAB\ bot = 0$   
 $\langle proof \rangle$

**lemma**  $AB\text{-}atom$ :  
 $atom\ a \longleftrightarrow AB\ a = \{a\}$   
 $\langle proof \rangle$

**lemma**  $nAB\text{-}atom$ :  
 $atom\ a \implies nAB\ a = 1$   
 $\langle proof \rangle$

**lemma**  $nAB\text{-}iso$ :  
 $x \leq y \implies nAB\ x \leq nAB\ y$   
 $\langle proof \rangle$

**end**

**context**  $bounded\text{-}semilattice\text{-}sup\text{-}bot$   
**begin**

**lemma**  $nAB\text{-}iso\text{-}sup$ :  
 $nAB\ x \leq nAB\ (x \sqcup y)$   
 $\langle proof \rangle$

**end**

**context**  $bounded\text{-}lattice$   
**begin**

**lemma**  $different\text{-}atoms\text{-}disjoint$ :  
 $atom\ x \implies atom\ y \implies x \neq y \implies x \sqcap y = bot$   
 $\langle proof \rangle$

**lemma**  $AB\text{-}dist\text{-}inf$ :  
 $AB\ (x \sqcap y) = AB\ x \cap AB\ y$

$\langle proof \rangle$

**lemma** *AB-iso-inf*:  
 $AB (x \sqcap y) \subseteq AB x$   
 $\langle proof \rangle$

**lemma** *AB-iso-sup*:  
 $AB x \subseteq AB (x \sqcup y)$   
 $\langle proof \rangle$

**lemma** *AB-disjoint*:  
**assumes**  $x \sqcap y = bot$   
**shows**  $AB x \cap AB y = \{\}$   
 $\langle proof \rangle$

**lemma** *nAB-iso-inf*:  
 $nAB (x \sqcap y) \leq nAB x$   
 $\langle proof \rangle$

**end**

**context** *distrib-lattice-bot*  
**begin**

**lemma** *atom-in-sup*:  
**assumes** *atom*  $a$   
**and**  $a \leq x \sqcup y$   
**shows**  $a \leq x \vee a \leq y$   
 $\langle proof \rangle$

**lemma** *atom-in-sup-iff*:  
**assumes** *atom*  $a$   
**shows**  $a \leq x \sqcup y \longleftrightarrow a \leq x \vee a \leq y$   
 $\langle proof \rangle$

**lemma** *atom-in-sup-xor*:  
 $atom\ a \implies a \leq x \sqcup y \implies x \sqcap y = bot \implies (a \leq x \wedge \neg a \leq y) \vee (\neg a \leq x \wedge a \leq y)$   
 $\langle proof \rangle$

**lemma** *atom-in-sup-xor-iff*:  
**assumes** *atom*  $a$   
**and**  $x \sqcap y = bot$   
**shows**  $a \leq x \sqcup y \longleftrightarrow (a \leq x \wedge \neg a \leq y) \vee (\neg a \leq x \wedge a \leq y)$   
 $\langle proof \rangle$

**lemma** *AB-dist-sup*:  
 $AB (x \sqcup y) = AB x \cup AB y$   
 $\langle proof \rangle$

**end**

**context** *bounded-distrib-lattice*  
**begin**

**lemma** *nAB-add*:

$nAB\ x + nAB\ y = nAB\ (x \sqcup y) + nAB\ (x \sqcap y)$   
 $\langle proof \rangle$

**lemma** *nAB-split-disjoint*:

**assumes**  $x \sqcap y = bot$   
**shows**  $nAB\ (x \sqcup y) = nAB\ x + nAB\ y$   
 $\langle proof \rangle$

**end**

**context** *p-algebra*  
**begin**

**lemma** *atom-in-p*:

$atom\ a \implies a \leq x \vee a \leq -x$   
 $\langle proof \rangle$

**lemma** *atom-in-p-xor*:

$atom\ a \implies (a \leq x \wedge \neg a \leq -x) \vee (\neg a \leq x \wedge a \leq -x)$   
 $\langle proof \rangle$

The following two lemmas also hold in distributive lattices with a least element (see above). However, p-algebras are not necessarily distributive, so the following results are independent.

**lemma** *atom-in-sup'*:

$atom\ a \implies a \leq x \sqcup y \implies a \leq x \vee a \leq y$   
 $\langle proof \rangle$

**lemma** *AB-dist-sup'*:

$AB\ (x \sqcup y) = AB\ x \cup AB\ y$   
 $\langle proof \rangle$

**lemma** *AB-split-1*:

$AB\ x = AB\ ((x \sqcap y) \sqcup (x \sqcap -y))$   
 $\langle proof \rangle$

**lemma** *AB-split-2*:

$AB\ x = AB\ (x \sqcap y) \cup AB\ (x \sqcap -y)$   
 $\langle proof \rangle$

**lemma** *AB-split-2-disjoint*:

$AB\ (x \sqcap y) \cap AB\ (x \sqcap -y) = \{\}$

$\langle \text{proof} \rangle$

**lemma** *AB-pp*:

$$AB \ (--x) = AB \ x$$

$\langle \text{proof} \rangle$

**lemma** *nAB-pp*:

$$nAB \ (--x) = nAB \ x$$

$\langle \text{proof} \rangle$

**lemma** *nAB-split-1*:

$$nAB \ x = nAB \ ((x \sqcap y) \sqcup (x \sqcap -y))$$

$\langle \text{proof} \rangle$

**lemma** *nAB-split-2*:

$$nAB \ x = nAB \ (x \sqcap y) + nAB \ (x \sqcap -y)$$

$\langle \text{proof} \rangle$

**end**

### 3 Atoms Below an Element in Stone Relation Algebras

We extend our study of atoms below an element to Stone relation algebras. We consider combinations of the following five assumptions: the Stone relation algebra is atomic, atom-rectangular, atom-simple, a relation algebra, or has finitely many atoms. We include general properties of atoms, rectangles and simple elements.

**context** *stone-relation-algebra*

**begin**

**abbreviation** *rectangle*  $:: 'a \Rightarrow \text{bool}$  **where** *rectangle*  $x \equiv x * \text{top} * x \leq x$

**abbreviation** *simple*  $:: 'a \Rightarrow \text{bool}$  **where** *simple*  $x \equiv \text{top} * x * \text{top} = \text{top}$

**lemma** *rectangle-eq*:

$$\text{rectangle } x \iff x * \text{top} * x = x$$

$\langle \text{proof} \rangle$

**lemma** *arc-univalent-injective-rectangle-simple*:

$$\text{arc } a \iff \text{univalent } a \wedge \text{injective } a \wedge \text{rectangle } a \wedge \text{simple } a$$

$\langle \text{proof} \rangle$

**lemma** *conv-atom*:

$$\text{atom } x \implies \text{atom } (x^T)$$

$\langle \text{proof} \rangle$

**lemma** *conv-atom-iff*:



$atom\ x \longleftrightarrow atom\ (x^T)$   
 $\langle proof \rangle$

**lemma** *counterexample-different-atoms-top-disjoint:*  
 $atom\ x \implies atom\ y \implies x \neq y \implies x * top \sqcap y = bot$   
**nitpick**[*expect=genuine,card=4*]  
 $\langle proof \rangle$

**lemma** *counterexample-different-univalent-atoms-top-disjoint:*  
 $atom\ x \implies univalent\ x \implies atom\ y \implies univalent\ y \implies x \neq y \implies x * top \sqcap y = bot$   
**nitpick**[*expect=genuine,card=4*]  
 $\langle proof \rangle$

**lemma** *AB-card-4-1:*  
 $a \leq x \wedge a \leq y \longleftrightarrow a \leq x \sqcup y \wedge a \leq x \sqcap y$   
 $\langle proof \rangle$

**lemma** *AB-card-4-2:*  
**assumes** *atom a*  
**shows**  $(a \leq x \wedge \neg a \leq y) \vee (\neg a \leq x \wedge a \leq y) \longleftrightarrow a \leq x \sqcup y \wedge \neg a \leq x \sqcap y$   
 $\langle proof \rangle$

**lemma** *AB-card-4-3:*  
**assumes** *atom a*  
**shows**  $\neg a \leq x \wedge \neg a \leq y \longleftrightarrow \neg a \leq x \sqcup y \wedge \neg a \leq x \sqcap y$   
 $\langle proof \rangle$

**lemma** *AB-card-5-1:*  
**assumes** *atom a*  
**and**  $a \leq x^T * y \sqcap z$   
**shows**  $x * a \sqcap y \leq x * z \sqcap y$   
**and**  $x * a \sqcap y \neq bot$   
 $\langle proof \rangle$

**lemma** *AB-card-5-2:*  
**assumes** *univalent x*  
**and** *atom a*  
**and** *atom b*  
**and**  $b \leq x^T * y \sqcap z$   
**and**  $a \neq b$   
**shows**  $(x * a \sqcap y) \sqcap (x * b \sqcap y) = bot$   
**and**  $x * a \sqcap y \neq x * b \sqcap y$   
 $\langle proof \rangle$

**lemma** *AB-card-6-0:*  
**assumes** *univalent x*  
**and** *atom a*  
**and**  $a \leq x$

**and**  $\text{atom } b$   
**and**  $b \leq x$   
**and**  $a \neq b$   
**shows**  $a * \text{top} \sqcap b * \text{top} = \text{bot}$   
 $\langle \text{proof} \rangle$

**lemma** *AB-card-6-1*:  
**assumes**  $\text{atom } a$   
**and**  $a \leq x \sqcap y * z^T$   
**shows**  $a * z \sqcap y \leq x * z \sqcap y$   
**and**  $a * z \sqcap y \neq \text{bot}$   
 $\langle \text{proof} \rangle$

**lemma** *AB-card-6-2*:  
**assumes**  $\text{univalent } x$   
**and**  $\text{atom } a$   
**and**  $a \leq x \sqcap y * z^T$   
**and**  $\text{atom } b$   
**and**  $b \leq x \sqcap y * z^T$   
**and**  $a \neq b$   
**shows**  $(a * z \sqcap y) \sqcap (b * z \sqcap y) = \text{bot}$   
**and**  $a * z \sqcap y \neq b * z \sqcap y$   
 $\langle \text{proof} \rangle$

**lemma** *nAB-conv*:  
 $nAB \ x = nAB \ (x^T)$   
 $\langle \text{proof} \rangle$

**lemma** *domain-atom*:  
**assumes**  $\text{atom } a$   
**shows**  $\text{atom } (a * \text{top} \sqcap 1)$   
 $\langle \text{proof} \rangle$

**lemma** *codomain-atom*:  
**assumes**  $\text{atom } a$   
**shows**  $\text{atom } (\text{top} * a \sqcap 1)$   
 $\langle \text{proof} \rangle$

**lemma** *atom-rectangle-atom-one-rep*:  
 $(\forall a . \text{atom } a \longrightarrow a * \text{top} * a \leq a) \longleftrightarrow (\forall a . \text{atom } a \wedge a \leq 1 \longrightarrow a * \text{top} * a \leq 1)$   
 $\langle \text{proof} \rangle$

**lemma** *AB-card-2-1*:  
**assumes**  $a * \text{top} * a \leq a$   
**shows**  $(a * \text{top} \sqcap 1) * \text{top} * (\text{top} * a \sqcap 1) = a$   
 $\langle \text{proof} \rangle$

**lemma** *atomsimple-atom1simple*:

$(\forall a . \text{atom } a \longrightarrow \text{top} * a * \text{top} = \text{top}) \longleftrightarrow (\forall a . \text{atom } a \wedge a \leq 1 \longrightarrow \text{top} * a * \text{top} = \text{top})$   
 $\langle \text{proof} \rangle$

**lemma** *AB-card-2-2*:

**assumes** *atom a*  
**and**  $a \leq 1$   
**and** *atom b*  
**and**  $b \leq 1$   
**and**  $\forall a . \text{atom } a \longrightarrow \text{top} * a * \text{top} = \text{top}$   
**shows**  $a * \text{top} * b * \text{top} \sqcap 1 = a$  **and**  $\text{top} * a * \text{top} * b \sqcap 1 = b$   
 $\langle \text{proof} \rangle$

**abbreviation** *dom-cod* ::  $'a \Rightarrow 'a \times 'a$

**where**  $\text{dom-cod } a \equiv (a * \text{top} \sqcap 1, \text{top} * a \sqcap 1)$

**lemma** *dom-cod-atoms-1*:

$\text{dom-cod } 'AB \text{ top} \subseteq AB \ 1 \times AB \ 1$   
 $\langle \text{proof} \rangle$

**end**

**class** *stone-relation-algebra-simple* = *stone-relation-algebra* +

**assumes** *simple*:  $x \neq \text{bot} \longrightarrow \text{simple } x$

**begin**

**lemma** *point-ideal-point*:

$\text{point } x \longleftrightarrow \text{ideal-point } x$   
 $\langle \text{proof} \rangle$

**end**

### 3.1 Atomic

**class** *stone-relation-algebra-atomic* = *stone-relation-algebra* +

**assumes** *atomic*:  $x \neq \text{bot} \longrightarrow (\exists a . \text{atom } a \wedge a \leq x)$

**begin**

**lemma** *AB-nonempty*:

$x \neq \text{bot} \implies AB \ x \neq \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *AB-nonempty-iff*:

$x \neq \text{bot} \longleftrightarrow AB \ x \neq \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *atomsimple-simple*:

$(\forall a . a \neq \text{bot} \longrightarrow \text{top} * a * \text{top} = \text{top}) \longleftrightarrow (\forall a . \text{atom } a \longrightarrow \text{top} * a * \text{top} = \text{top})$

$\langle proof \rangle$

**lemma** *AB-card-2-3*:

**assumes**  $a \neq bot$

**and**  $a \leq 1$

**and**  $b \neq bot$

**and**  $b \leq 1$

**and**  $\forall a . a \neq bot \longrightarrow top * a * top = top$

**shows**  $a * top * b * top \sqcap 1 = a$  **and**  $top * a * top * b \sqcap 1 = b$

$\langle proof \rangle$

**lemma** *injective-down-closed*:

$x \leq y \implies injective\ y \implies injective\ x$

$\langle proof \rangle$

**lemma** *univalent-down-closed*:

$x \leq y \implies univalent\ y \implies univalent\ x$

$\langle proof \rangle$

**lemma** *nAB-bot-iff*:

$x = bot \longleftrightarrow nAB\ x = 0$

$\langle proof \rangle$

It is unclear if *atomic* is necessary for the following two results, but it seems likely.

**lemma** *nAB-univ-comp-meet*:

**assumes** *univalent*  $x$

**shows**  $nAB\ (x^T * y \sqcap z) \leq nAB\ (x * z \sqcap y)$

$\langle proof \rangle$

**lemma** *nAB-univ-meet-comp*:

**assumes** *univalent*  $x$

**shows**  $nAB\ (x \sqcap y * z^T) \leq nAB\ (x * z \sqcap y)$

$\langle proof \rangle$

**end**

### 3.2 Atom-rectangular

**class** *stone-relation-algebra-atomrect* = *stone-relation-algebra* +

**assumes** *atomrect*:  $atom\ a \longrightarrow rectangle\ a$

**begin**

**lemma** *atomrect-eq*:

$atom\ a \implies a * top * a = a$

$\langle proof \rangle$

**lemma** *AB-card-2-4*:

**assumes** *atom*  $a$

**shows**  $(a * \text{top} \sqcap 1) * \text{top} * (\text{top} * a \sqcap 1) = a$   
 $\langle \text{proof} \rangle$

**lemma** *simple-atom-2*:  
**assumes** *atom a*  
**and**  $a \leq 1$   
**and** *atom b*  
**and**  $b \leq 1$   
**and**  $x \neq \text{bot}$   
**and**  $x \leq a * \text{top} * b$   
**shows**  $x = a * \text{top} * b$   
 $\langle \text{proof} \rangle$

**lemma** *dom-cod-inj-atoms*:  
*inj-on dom-cod*  $(AB \text{ top})$   
 $\langle \text{proof} \rangle$

**lemma** *finite-AB-iff*:  
*finite*  $(AB \text{ top}) \longleftrightarrow \text{finite } (AB \ 1)$   
 $\langle \text{proof} \rangle$

**lemma** *nAB-top-1*:  
 $nAB \text{ top} \leq nAB \ 1 * nAB \ 1$   
 $\langle \text{proof} \rangle$

**lemma** *atom-vector-injective*:  
**assumes** *atom x*  
**shows** *injective*  $(x * \text{top})$   
 $\langle \text{proof} \rangle$

**lemma** *atom-injective*:  
 $\text{atom } x \implies \text{injective } x$   
 $\langle \text{proof} \rangle$

**lemma** *atom-covector-univalent*:  
 $\text{atom } x \implies \text{univalent } (\text{top} * x)$   
 $\langle \text{proof} \rangle$

**lemma** *atom-univalent*:  
 $\text{atom } x \implies \text{univalent } x$   
 $\langle \text{proof} \rangle$

**lemma** *counterexample-atom-simple*:  
 $\text{atom } x \implies \text{simple } x$   
**nitpick** $[\text{expect}=\text{genuine}, \text{card}=3]$   
 $\langle \text{proof} \rangle$

**lemma** *symmetric-atom-below-1*:  
**assumes** *atom x*

```

    and  $x = x^T$ 
    shows  $x \leq 1$ 
  <proof>

```

```

end

```

### 3.3 Atomic and Atom-Rectangular

```

class stone-relation-algebra-atomic-atomrect = stone-relation-algebra-atomic +
  stone-relation-algebra-atomrect
begin

```

```

lemma point-dense:
  assumes  $x \neq \text{bot}$ 
    and  $x \leq 1$ 
  shows  $\exists a . a \neq \text{bot} \wedge a * \text{top} * a \leq 1 \wedge a \leq x$ 
  <proof>

```

```

end

```

### 3.4 Atom-simple

```

class stone-relation-algebra-atomsimple = stone-relation-algebra +
  assumes atomsimple:  $\text{atom } a \longrightarrow \text{simple } a$ 
begin

```

```

lemma AB-card-2-5:
  assumes atom  $a$ 
    and  $a \leq 1$ 
    and atom  $b$ 
    and  $b \leq 1$ 
  shows  $a * \text{top} * b * \text{top} \sqcap 1 = a$  and  $\text{top} * a * \text{top} * b \sqcap 1 = b$ 
  <proof>

```

```

lemma simple-atom-1:
  atom  $a \implies \text{atom } b \implies a * \text{top} * b \neq \text{bot}$ 
  <proof>

```

```

end

```

### 3.5 Atomic and Atom-simple

```

class stone-relation-algebra-atomic-atomsimple = stone-relation-algebra-atomic +
  stone-relation-algebra-atomsimple
begin

```

```

subclass stone-relation-algebra-simple
  <proof>

```

```

lemma AB-card-2-6:

```

```

assumes  $a \neq \text{bot}$ 
  and  $a \leq 1$ 
  and  $b \neq \text{bot}$ 
  and  $b \leq 1$ 
shows  $a * \text{top} * b * \text{top} \sqcap 1 = a$  and  $\text{top} * a * \text{top} * b \sqcap 1 = b$ 
<proof>

lemma dom-cod-atoms-2:
   $AB\ 1 \times AB\ 1 \subseteq \text{dom-cod} \text{ ` } AB\ \text{top}$ 
<proof>

lemma dom-cod-atoms:
   $AB\ 1 \times AB\ 1 = \text{dom-cod} \text{ ` } AB\ \text{top}$ 
<proof>

end

```

### 3.6 Atom-rectangular and Atom-simple

```

class stone-relation-algebra-atomrect-atomsimple =
  stone-relation-algebra-atomrect + stone-relation-algebra-atomsimple
begin

```

```

lemma simple-atom:
  assumes atom  $a$ 
    and  $a \leq 1$ 
    and atom  $b$ 
    and  $b \leq 1$ 
  shows atom  $(a * \text{top} * b)$ 
<proof>

lemma nAB-top-2:
   $nAB\ 1 * nAB\ 1 \leq nAB\ \text{top}$ 
<proof>

lemma nAB-top:
   $nAB\ 1 * nAB\ 1 = nAB\ \text{top}$ 
<proof>

lemma atom-covector-mapping:
  atom  $a \implies \text{mapping} (\text{top} * a)$ 
<proof>

lemma atom-covector-regular:
  atom  $a \implies \text{regular} (\text{top} * a)$ 
<proof>

lemma atom-vector-bijective:
  atom  $a \implies \text{bijective} (a * \text{top})$ 

```

*<proof>*

**lemma** *atom-vector-regular*:  
   $atom\ a \implies regular\ (a * top)$   
*<proof>*

**lemma** *atom-rectangle-regular*:  
   $atom\ a \implies regular\ (a * top * a)$   
*<proof>*

**lemma** *atom-regular*:  
   $atom\ a \implies regular\ a$   
*<proof>*

**end**

### 3.7 Atomic, Atom-rectangular and Atom-simple

**class** *stone-relation-algebra-atomic-atomrect-atomsimple* =  
   $stone-relation-algebra-atomic + stone-relation-algebra-atomrect +$   
   $stone-relation-algebra-atomsimple$   
**begin**

**subclass** *stone-relation-algebra-atomic-atomrect* *<proof>*  
**subclass** *stone-relation-algebra-atomic-atomsimple* *<proof>*  
**subclass** *stone-relation-algebra-atomrect-atomsimple* *<proof>*

**lemma** *nAB-atom-iff*:  
   $atom\ a \longleftrightarrow nAB\ a = 1$   
*<proof>*

**end**

### 3.8 Finitely Many Atoms

**class** *stone-relation-algebra-finiteatoms* =  $stone-relation-algebra +$   
  **assumes** *finiteatoms*:  $finite\ \{ a . atom\ a \}$   
**begin**

**lemma** *finite-AB*:  
   $finite\ (AB\ x)$   
*<proof>*

**lemma** *nAB-top-finite*:  
   $nAB\ top \neq \infty$   
*<proof>*

**end**



### 3.9 Atomic and Finitely Many Atoms

```
class stone-relation-algebra-atomic-finiteatoms = stone-relation-algebra-atomic +  
stone-relation-algebra-finiteatoms  
begin
```

```
lemma finite-ideal-points:  
  finite { p . ideal-point p }  
  ⟨proof⟩
```

```
end
```

### 3.10 Atom-rectangular and Finitely Many Atoms

```
class stone-relation-algebra-atomrect-finiteatoms =  
stone-relation-algebra-atomrect + stone-relation-algebra-finiteatoms
```

### 3.11 Atomic, Atom-rectangular and Finitely Many Atoms

```
class stone-relation-algebra-atomic-atomrect-finiteatoms =  
stone-relation-algebra-atomic + stone-relation-algebra-atomrect +  
stone-relation-algebra-finiteatoms  
begin
```

```
subclass stone-relation-algebra-atomic-atomrect ⟨proof⟩  
subclass stone-relation-algebra-atomic-finiteatoms ⟨proof⟩  
subclass stone-relation-algebra-atomrect-finiteatoms ⟨proof⟩
```

```
lemma counterexample-nAB-atom-iff:  
  atom x  $\longleftrightarrow$  nAB x = 1  
  nitpick[expect=genuine,card=3]  
  ⟨proof⟩
```

```
lemma counterexample-nAB-top-iff-eq:  
  nAB x = nAB top  $\longleftrightarrow$  x = top  
  nitpick[expect=genuine,card=3]  
  ⟨proof⟩
```

```
lemma counterexample-nAB-top-iff-leq:  
  nAB top  $\leq$  nAB x  $\longleftrightarrow$  x = top  
  nitpick[expect=genuine,card=3]  
  ⟨proof⟩
```

```
end
```

### 3.12 Atom-simple and Finitely Many Atoms

```
class stone-relation-algebra-atomsimple-finiteatoms =  
stone-relation-algebra-atomsimple + stone-relation-algebra-finiteatoms
```

### 3.13 Atomic, Atom-simple and Finitely Many Atoms

```

class stone-relation-algebra-atomic-atomsimple-finiteatoms =
  stone-relation-algebra-atomic + stone-relation-algebra-atomsimple +
  stone-relation-algebra-finiteatoms
begin

subclass stone-relation-algebra-atomic-atomsimple <proof>
subclass stone-relation-algebra-atomic-finiteatoms <proof>
subclass stone-relation-algebra-atomsimple-finiteatoms <proof>

lemma nAB-top-2:
   $nAB\ 1 * nAB\ 1 \leq nAB\ top$ 
  <proof>

lemma counterexample-nAB-atom-iff-2:
   $atom\ x \longleftrightarrow nAB\ x = 1$ 
  nitpick[expect=genuine,card=6]
  <proof>

lemma counterexample-nAB-top-iff-eq-2:
   $nAB\ x = nAB\ top \longleftrightarrow x = top$ 
  nitpick[expect=genuine,card=6]
  <proof>

lemma counterexample-nAB-top-iff-leq-2:
   $nAB\ top \leq nAB\ x \longleftrightarrow x = top$ 
  nitpick[expect=genuine,card=6]
  <proof>

lemma counterexample-nAB-atom-top-iff-leq-2:
   $(atom\ x \longleftrightarrow nAB\ x = 1) \vee (nAB\ y = nAB\ top \longleftrightarrow y = top) \vee (nAB\ top \leq$ 
   $nAB\ y \longleftrightarrow y = top)$ 
  nitpick[expect=genuine,card=6]
  <proof>

end

```

### 3.14 Atom-rectangular, Atom-simple and Finitely Many Atoms

```

class stone-relation-algebra-atomrect-atomsimple-finiteatoms =
  stone-relation-algebra-atomrect + stone-relation-algebra-atomsimple +
  stone-relation-algebra-finiteatoms
begin

subclass stone-relation-algebra-atomrect-atomsimple <proof>
subclass stone-relation-algebra-atomrect-finiteatoms <proof>
subclass stone-relation-algebra-atomsimple-finiteatoms <proof>

```

end

### 3.15 Atomic, Atom-rectangular, Atom-simple and Finitely Many Atoms

```
class stone-relation-algebra-atomic-atomrect-atomsimple-finiteatoms =
  stone-relation-algebra-atomic + stone-relation-algebra-atomrect +
  stone-relation-algebra-atomsimple + stone-relation-algebra-finiteatoms
begin
```

```
subclass stone-relation-algebra-atomic-atomrect-atomsimple <proof>
subclass stone-relation-algebra-atomic-atomrect-finiteatoms <proof>
subclass stone-relation-algebra-atomic-atomsimple-finiteatoms <proof>
subclass stone-relation-algebra-atomrect-atomsimple-finiteatoms <proof>
```

lemma *all-regular*:

*regular x*  
 <proof>

```
sublocale ra: relation-algebra where minus =  $\lambda x y . x \sqcap - y$ 
<proof>
```

end

```
class stone-relation-algebra-finite = stone-relation-algebra + finite
begin
```

```
subclass stone-relation-algebra-atomic-finiteatoms
<proof>
```

end

### 3.16 Relation Algebra and Atomic

```
class relation-algebra-atomic = relation-algebra + stone-relation-algebra-atomic
begin
```

lemma *nAB-atom-iff*:

*atom a  $\longleftrightarrow$  nAB a = 1*  
 <proof>

end

### 3.17 Relation Algebra, Atomic and Finitely Many Atoms

```
class relation-algebra-atomic-finiteatoms = relation-algebra-atomic +
  stone-relation-algebra-atomic-finiteatoms
begin
```

*Sup-fin only works for non-empty finite sets.*

```

lemma atomistic:
  assumes  $x \neq \text{bot}$ 
  shows  $x = \text{Sup-fin } (AB\ x)$ 
   $\langle \text{proof} \rangle$ 

lemma counterexample-nAB-top:
   $1 \neq \text{top} \implies nAB\ \text{top} = nAB\ 1 * nAB\ 1$ 
  nitpick[expect=genuine,card=4]
   $\langle \text{proof} \rangle$ 

end

class relation-algebra-atomic-atomsimple-finiteatoms =
  relation-algebra-atomic-finiteatoms +
  stone-relation-algebra-atomic-atomsimple-finiteatoms
begin

lemma counterexample-atom-rectangle:
   $\text{atom } x \longrightarrow \text{rectangle } x$ 
  nitpick[expect=genuine,card=4]
   $\langle \text{proof} \rangle$ 

lemma counterexample-atom-univalent:
   $\text{atom } x \longrightarrow \text{univalent } x$ 
  nitpick[expect=genuine,card=4]
   $\langle \text{proof} \rangle$ 

lemma counterexample-point-dense:
  assumes  $x \neq \text{bot}$ 
  and  $x \leq 1$ 
  shows  $\exists a . a \neq \text{bot} \wedge a * \text{top} * a \leq 1 \wedge a \leq x$ 
  nitpick[expect=genuine,card=4]
   $\langle \text{proof} \rangle$ 

end

class relation-algebra-atomic-atomrect-atomsimple-finiteatoms =
  relation-algebra-atomic-atomsimple-finiteatoms +
  stone-relation-algebra-atomic-atomrect-atomsimple-finiteatoms

```

## 4 Cardinality in Stone Relation Algebras

We study various axioms for a cardinality operation in Stone relation algebras.

```

class card =
  fixes cardinality :: 'a  $\Rightarrow$  enat ( $\#$ - [100] 100)

class sra-card = stone-relation-algebra + card

```

begin

**abbreviation** *card-bot* :: 'a  $\Rightarrow$  bool **where** *card-bot* -  $\equiv$  #bot  
 $= 0$   
**abbreviation** *card-bot-iff* :: 'a  $\Rightarrow$  bool **where** *card-bot-iff* -  $\equiv$   
 $\forall x::'a . \#x = 0 \longleftrightarrow x = \text{bot}$   
**abbreviation** *card-top* :: 'a  $\Rightarrow$  bool **where** *card-top* -  $\equiv$   
 $\#top = \#1 * \#1$   
**abbreviation** *card-conv* :: 'a  $\Rightarrow$  bool **where** *card-conv* -  $\equiv$   
 $\forall x::'a . \#(x^T) = \#x$   
**abbreviation** *card-add* :: 'a  $\Rightarrow$  bool **where** *card-add* -  $\equiv \forall x$   
 $y::'a . \#x + \#y = \#(x \sqcup y) + \#(x \sqcap y)$   
**abbreviation** *card-iso* :: 'a  $\Rightarrow$  bool **where** *card-iso* -  $\equiv \forall x$   
 $y::'a . x \leq y \longrightarrow \#x \leq \#y$   
**abbreviation** *card-univ-comp-meet* :: 'a  $\Rightarrow$  bool **where** *card-univ-comp-meet* -  $\equiv \forall x y z::'a . \text{univalent } x \longrightarrow \#(x^T * y \sqcap z) \leq \#(x * z \sqcap y)$   
**abbreviation** *card-univ-meet-comp* :: 'a  $\Rightarrow$  bool **where** *card-univ-meet-comp* -  $\equiv \forall x y z::'a . \text{univalent } x \longrightarrow \#(x \sqcap y * z^T) \leq \#(x * z \sqcap y)$   
**abbreviation** *card-comp-univ* :: 'a  $\Rightarrow$  bool **where** *card-comp-univ* -  $\equiv \forall x y::'a . \text{univalent } x \longrightarrow \#(y * x) \leq \#y$   
**abbreviation** *card-univ-meet-vector* :: 'a  $\Rightarrow$  bool **where** *card-univ-meet-vector* -  $\equiv \forall x y::'a . \text{univalent } x \longrightarrow \#(x \sqcap y * top) \leq \#y$   
**abbreviation** *card-univ-meet-conv* :: 'a  $\Rightarrow$  bool **where** *card-univ-meet-conv* -  $\equiv \forall x y::'a . \text{univalent } x \longrightarrow \#(x \sqcap y * y^T) \leq \#y$   
**abbreviation** *card-domain-sym* :: 'a  $\Rightarrow$  bool **where** *card-domain-sym* -  $\equiv \forall x::'a . \#(1 \sqcap x * x^T) \leq \#x$   
**abbreviation** *card-domain-sym-conv* :: 'a  $\Rightarrow$  bool **where** *card-domain-sym-conv* -  $\equiv \forall x::'a . \#(1 \sqcap x^T * x) \leq \#x$   
**abbreviation** *card-domain* :: 'a  $\Rightarrow$  bool **where** *card-domain* -  $\equiv \forall x::'a . \#(1 \sqcap x * top) \leq \#x$   
**abbreviation** *card-domain-conv* :: 'a  $\Rightarrow$  bool **where** *card-domain-conv* -  $\equiv \forall x::'a . \#(1 \sqcap x^T * top) \leq \#x$   
**abbreviation** *card-codomain* :: 'a  $\Rightarrow$  bool **where** *card-codomain* -  $\equiv \forall x::'a . \#(1 \sqcap top * x) \leq \#x$   
**abbreviation** *card-codomain-conv* :: 'a  $\Rightarrow$  bool **where** *card-codomain-conv* -  $\equiv \forall x::'a . \#(1 \sqcap top * x^T) \leq \#x$   
**abbreviation** *card-univ* :: 'a  $\Rightarrow$  bool **where** *card-univ* -  $\equiv \forall x::'a . \text{univalent } x \longrightarrow \#x \leq \#(x * top)$   
**abbreviation** *card-atom* :: 'a  $\Rightarrow$  bool **where** *card-atom* -  $\equiv \forall x::'a . \text{atom } x \longrightarrow \#x = 1$   
**abbreviation** *card-atom-iff* :: 'a  $\Rightarrow$  bool **where** *card-atom-iff* -  $\equiv \forall x::'a . \text{atom } x \longleftrightarrow \#x = 1$   
**abbreviation** *card-top-iff-eq* :: 'a  $\Rightarrow$  bool **where** *card-top-iff-eq* -  $\equiv \forall x::'a . \#x = \#top \longleftrightarrow x = top$   
**abbreviation** *card-top-iff-leq* :: 'a  $\Rightarrow$  bool **where** *card-top-iff-leq* -  $\equiv \forall x::'a . \#top \leq \#x \longleftrightarrow x = top$   
**abbreviation** *card-top-finite* :: 'a  $\Rightarrow$  bool **where** *card-top-finite* -  $\equiv \#top \neq \infty$

**lemma** *card-domain-iff*:

*card-domain* -  $\longleftrightarrow$  *card-domain-sym* -

$\langle \text{proof} \rangle$

**lemma** *card-codomain-conv-iff*:

*card-codomain-conv* -  $\longleftrightarrow$  *card-domain* -

$\langle \text{proof} \rangle$

**lemma** *card-codomain-iff*:

**assumes** *card-conv*: *card-conv* -

**shows** *card-codomain* -  $\longleftrightarrow$  *card-codomain-conv* -

$\langle \text{proof} \rangle$

**lemma** *card-domain-conv-iff*:

*card-codomain* -  $\longleftrightarrow$  *card-domain-conv* -

$\langle \text{proof} \rangle$

**lemma** *card-domain-sym-conv-iff*:

*card-domain-conv* -  $\longleftrightarrow$  *card-domain-sym-conv* -

$\langle \text{proof} \rangle$

**lemma** *card-bot*:

**assumes** *card-bot-iff*: *card-bot-iff* -

**shows** *card-bot* -

$\langle \text{proof} \rangle$

**lemma** *card-comp-univ-implies-card-univ-comp-meet*:

**assumes** *card-conv*: *card-conv* -

**and** *card-comp-univ*: *card-comp-univ* -

**shows** *card-univ-comp-meet* -

$\langle \text{proof} \rangle$

**lemma** *card-univ-meet-conv-implies-card-domain-sym*:

**assumes** *card-univ-meet-conv*: *card-univ-meet-conv* -

**shows** *card-domain-sym* -

$\langle \text{proof} \rangle$

**lemma** *card-add-disjoint*:

**assumes** *card-bot*: *card-bot* -

**and** *card-add*: *card-add* -

**and**  $x \sqcap y = \text{bot}$

**shows**  $\#(x \sqcup y) = \#x + \#y$

$\langle \text{proof} \rangle$

**lemma** *card-dist-sup-disjoint*:

**assumes** *card-bot*: *card-bot* -

**and** *card-add*: *card-add* -

**and**  $A \neq \{\}$

**and** *finite*  $A$

**and**  $\forall x \in A . \forall y \in A . x \neq y \longrightarrow x \sqcap y = \text{bot}$   
**shows**  $\# \text{Sup-fin } A = \text{sum cardinality } A$   
 $\langle \text{proof} \rangle$

**lemma** *card-dist-sup-atoms*:  
**assumes** *card-bot*: *card-bot* -  
**and** *card-add*: *card-add* -  
**and**  $A \neq \{\}$   
**and** *finite* *A*  
**and**  $A \subseteq AB \text{ top}$   
**shows**  $\# \text{Sup-fin } A = \text{sum cardinality } A$   
 $\langle \text{proof} \rangle$

**lemma** *card-univ-meet-comp-implies-card-domain-sym*:  
**assumes** *card-univ-meet-comp*: *card-univ-meet-comp* -  
**shows** *card-domain-sym* -  
 $\langle \text{proof} \rangle$

**lemma** *card-top-greatest*:  
**assumes** *card-iso*: *card-iso* -  
**shows**  $\#x \leq \# \text{top}$   
 $\langle \text{proof} \rangle$

**lemma** *card-pp-increasing*:  
**assumes** *card-iso*: *card-iso* -  
**shows**  $\#x \leq \#(\neg\neg x)$   
 $\langle \text{proof} \rangle$

**lemma** *card-top-iff-eq-leq*:  
**assumes** *card-iso*: *card-iso* -  
**shows** *card-top-iff-eq* -  $\longleftrightarrow$  *card-top-iff-leq* -  
 $\langle \text{proof} \rangle$

**lemma** *card-univ-comp-meet-implies-card-comp-univ*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-conv*: *card-conv* -  
**and** *card-univ-comp-meet*: *card-univ-comp-meet* -  
**shows** *card-comp-univ* -  
 $\langle \text{proof} \rangle$

**lemma** *card-comp-univ-iff-card-univ-comp-meet*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-conv*: *card-conv* -  
**shows** *card-comp-univ* -  $\longleftrightarrow$  *card-univ-comp-meet* -  
 $\langle \text{proof} \rangle$

**lemma** *card-univ-meet-vector-implies-card-univ-meet-comp*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-univ-meet-vector*: *card-univ-meet-vector* -

**shows** *card-univ-meet-comp* -  
 $\langle proof \rangle$

**lemma** *card-univ-meet-comp-implies-card-univ-meet-vector*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-univ-meet-comp*: *card-univ-meet-comp* -  
**shows** *card-univ-meet-vector* -  
 $\langle proof \rangle$

**lemma** *card-univ-meet-vector-iff-card-univ-meet-comp*:  
**assumes** *card-iso*: *card-iso* -  
**shows** *card-univ-meet-vector* -  $\longleftrightarrow$  *card-univ-meet-comp* -  
 $\langle proof \rangle$

**lemma** *card-univ-meet-vector-implies-card-univ-meet-conv*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-univ-meet-vector*: *card-univ-meet-vector* -  
**shows** *card-univ-meet-conv* -  
 $\langle proof \rangle$

**lemma** *card-domain-sym-implies-card-univ-meet-vector*:  
**assumes** *card-comp-univ*: *card-comp-univ* -  
**and** *card-domain-sym*: *card-domain-sym* -  
**shows** *card-univ-meet-vector* -  
 $\langle proof \rangle$

**lemma** *card-domain-sym-iff-card-univ-meet-vector*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-comp-univ*: *card-comp-univ* -  
**shows** *card-domain-sym* -  $\longleftrightarrow$  *card-univ-meet-vector* -  
 $\langle proof \rangle$

**lemma** *card-univ-meet-conv-iff-card-univ-meet-comp*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-comp-univ*: *card-comp-univ* -  
**shows** *card-univ-meet-conv* -  $\longleftrightarrow$  *card-univ-meet-comp* -  
 $\langle proof \rangle$

**lemma** *card-domain-sym-iff-card-univ-meet-comp*:  
**assumes** *card-iso*: *card-iso* -  
**and** *card-comp-univ*: *card-comp-univ* -  
**shows** *card-domain-sym* -  $\longleftrightarrow$  *card-univ-meet-comp* -  
 $\langle proof \rangle$

**lemma** *card-univ-comp-mapping*:  
**assumes** *card-comp-univ*: *card-comp-univ* -  
**and** *card-univ-meet-comp*: *card-univ-meet-comp* -  
**and** *univalent* *x*  
**and** *mapping* *y*



**shows**  $\#(x * y) = \#x$   
 $\langle proof \rangle$

**lemma** *card-point-one*:  
**assumes** *card-comp-univ*: *card-comp-univ* -  
**and** *card-univ-meet-comp*: *card-univ-meet-comp* -  
**and** *card-conv*: *card-conv* -  
**and** *point x*  
**shows**  $\#x = \#1$   
 $\langle proof \rangle$

**lemma** *counterexample-card-univ-comp-meet-card-comp-univ*:  
**assumes** *card-add*: *card-add* -  
**and** *card-conv*: *card-conv* -  
**and** *card-bot-iff*: *card-bot-iff* -  
**and** *card-atom-iff*: *card-atom-iff* -  
**and** *card-univ-meet-comp*: *card-univ-meet-comp* -  
**shows** *card-univ-comp-meet* -  $\longleftrightarrow$  *card-comp-univ* -  
**nitpick**[*expect=genuine*]  
 $\langle proof \rangle$

**lemma** *counterexample-card-univ-meet-comp-card-univ-meet-vector*:  
**assumes** *card-add*: *card-add* -  
**and** *card-conv*: *card-conv* -  
**and** *card-bot-iff*: *card-bot-iff* -  
**and** *card-atom-iff*: *card-atom-iff* -  
**and** *card-univ-comp-meet*: *card-univ-comp-meet* -  
**shows** *card-univ-meet-comp* -  $\longleftrightarrow$  *card-univ-meet-vector* -  
**nitpick**[*expect=genuine*]  
 $\langle proof \rangle$

**lemma** *counterexample-card-univ-meet-comp-card-univ-meet-conv*:  
**assumes** *card-add*: *card-add* -  
**and** *card-conv*: *card-conv* -  
**and** *card-bot-iff*: *card-bot-iff* -  
**and** *card-atom-iff*: *card-atom-iff* -  
**and** *card-univ-comp-meet*: *card-univ-comp-meet* -  
**shows** *card-univ-meet-comp* -  $\longleftrightarrow$  *card-univ-meet-conv* -  
**nitpick**[*expect=genuine*]  
 $\langle proof \rangle$

**lemma** *counterexample-card-univ-meet-vector-card-domain-sym*:  
**assumes** *card-add*: *card-add* -  
**and** *card-conv*: *card-conv* -  
**and** *card-bot-iff*: *card-bot-iff* -  
**and** *card-atom-iff*: *card-atom-iff* -  
**and** *card-univ-comp-meet*: *card-univ-comp-meet* -  
**shows** *card-univ-meet-vector* -  $\longleftrightarrow$  *card-domain-sym* -  
**nitpick**[*expect=genuine*]

```

    <proof>

lemma counterexample-card-univ-meet-conv-card-domain-sym:
  assumes card-add: card-add -
    and card-conv: card-conv -
    and card-bot-iff: card-bot-iff -
    and card-atom-iff: card-atom-iff -
    and card-univ-comp-meet: card-univ-comp-meet -
  shows card-univ-meet-conv -  $\longleftrightarrow$  card-domain-sym -
  nitpick[expect=genuine]
  <proof>

end

```

#### 4.1 Cardinality in Relation Algebras

```

class ra-card = sra-card + relation-algebra
begin

lemma card-iso:
  assumes card-bot: card-bot -
    and card-add: card-add -
  shows card-iso -
  <proof>

lemma card-top-iff-eq:
  assumes card-bot-iff: card-bot-iff -
    and card-add: card-add -
    and card-top-finite: card-top-finite -
  shows card-top-iff-eq -
  <proof>

end

class sra-card-atomic-finiteatoms = sra-card +
  stone-relation-algebra-atomic-finiteatoms
begin

```

```

lemma counterexample-card-nAB:
  assumes card-bot-iff: card-bot-iff -
    and card-atom-iff: card-atom-iff -
    and card-conv: card-conv -
    and card-add: card-add -
    and card-iso: card-iso -
    and card-top-iff-eq: card-top-iff-eq -
    and card-top-finite: card-top-finite -
  shows #x = nAB x
  nitpick[expect=genuine]
  <proof>

```

```

end

class ra-card-atomic-finiteatoms = ra-card + relation-algebra-atomic-finiteatoms
begin

lemma card-nAB:
  assumes card-bot: card-bot -
    and card-add: card-add -
    and card-atom: card-atom -
  shows  $\#x = nAB\ x$ 
  <proof>

end

class card-ab = sra-card +
  assumes card-nAB':  $\#x = nAB\ x$ 

class sra-card-ab-atomsimple-finiteatoms = sra-card + card-ab +
  stone-relation-algebra-atomsimple-finiteatoms +
  assumes card-bot-iff: card-bot-iff -
  assumes card-top: card-top -
begin

subclass stone-relation-algebra-atomic-atomsimple-finiteatoms
  <proof>

lemma dom-cod-inj-atoms:
  inj-on dom-cod (AB top)
  <proof>

subclass stone-relation-algebra-atomic-atomrect-atomsimple-finiteatoms
  <proof>

lemma atom-rectangle-card:
  assumes atom a
  shows  $\#(a * top * a) = 1$ 
  <proof>

lemma atom-regular-rectangle:
  assumes atom a
  shows  $--a = a * top * a$ 
  <proof>

sublocale ra-atom: relation-algebra-atomic where minus =  $\lambda x\ y . x \sqcap - y$ 
  <proof>

end

```

```

class ra-card-atomic-atomssimple-finiteatoms = ra-card +
relation-algebra-atomic-atomssimple-finiteatoms +
  assumes card-bot: card-bot -
  assumes card-add: card-add -
  assumes card-atom: card-atom -
  assumes card-top: card-top -
begin

subclass ra-card-atomic-finiteatoms
  <proof>

subclass sra-card-ab-atomssimple-finiteatoms
  <proof>

subclass relation-algebra-atomic-atomrect-atomssimple-finiteatoms
  <proof>

end

```

## 4.2 Counterexamples

```

class ra-card-notop = ra-card +
  assumes card-bot-iff: card-bot-iff -
  assumes card-conv: card-conv -
  assumes card-add: card-add -
  assumes card-atom-iff: card-atom-iff -
  assumes card-univ-comp-meet: card-univ-comp-meet -
  assumes card-univ-meet-comp: card-univ-meet-comp -

class ra-card-all = ra-card-notop +
  assumes card-top: card-top -
  assumes card-top-finite: card-top-finite -

class ra-card-notop-atomic-finiteatoms = ra-card-atomic-finiteatoms +
ra-card-notop

class ra-card-all-atomic-finiteatoms = ra-card-notop-atomic-finiteatoms +
ra-card-all

abbreviation r0000 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r0000 x y  $\equiv$  False
abbreviation r1000 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r1000 x y  $\equiv$   $\neg x \wedge \neg y$ 
abbreviation r0001 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r0001 x y  $\equiv$   $x \wedge y$ 
abbreviation r1001 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r1001 x y  $\equiv$   $x = y$ 
abbreviation r0110 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r0110 x y  $\equiv$   $x \neq y$ 
abbreviation r1111 :: bool  $\Rightarrow$  bool  $\Rightarrow$  bool where r1111 x y  $\equiv$  True

lemma r-all-different:
  r0000  $\neq$  r1000 r0000  $\neq$  r0001 r0000  $\neq$  r1001 r0000  $\neq$  r0110
r0000  $\neq$  r1111

```

$r1000 \neq r0000$                        $r1000 \neq r0001$   $r1000 \neq r1001$   $r1000 \neq r0110$   
 $r1000 \neq r1111$   
 $r0001 \neq r0000$   $r0001 \neq r1000$                        $r0001 \neq r1001$   $r0001 \neq r0110$   
 $r0001 \neq r1111$   
 $r1001 \neq r0000$   $r1001 \neq r1000$   $r1001 \neq r0001$                        $r1001 \neq r0110$   
 $r1001 \neq r1111$   
 $r0110 \neq r0000$   $r0110 \neq r1000$   $r0110 \neq r0001$   $r0110 \neq r1001$   
 $r0110 \neq r1111$   
 $r1111 \neq r0000$   $r1111 \neq r1000$   $r1111 \neq r0001$   $r1111 \neq r1001$   $r1111 \neq r0110$   
 $\langle proof \rangle$

**typedef (overloaded)**  $ra1 = \{r0000, r1001, r0110, r1111\}$   
 $\langle proof \rangle$

**typedef (overloaded)**  $ra2 = \{r0000, r1000, r0001, r1001\}$   
 $\langle proof \rangle$

**setup-lifting**  $type-definition-ra1$   
**setup-lifting**  $type-definition-ra2$   
**setup-lifting**  $type-definition-prod$

**instantiation**  $Enum.finite-4 :: ra-card-atomic-finiteatoms$   
**begin**

**definition**  $one-finite-4 :: Enum.finite-4 \rightarrow \mathbf{where} \ one-finite-4 = finite-4.a_2$   
**definition**  $conv-finite-4 :: Enum.finite-4 \Rightarrow Enum.finite-4 \rightarrow \mathbf{where} \ conv-finite-4 \ x$   
 $= x$   
**definition**  $times-finite-4 :: Enum.finite-4 \Rightarrow Enum.finite-4 \Rightarrow Enum.finite-4$   
**where**  $times-finite-4 \ x \ y = (case \ (x,y) \ of \ (finite-4.a_1, -) \Rightarrow finite-4.a_1 \mid$   
 $(-, finite-4.a_1) \Rightarrow finite-4.a_1 \mid (finite-4.a_2, y) \Rightarrow y \mid (x, finite-4.a_2) \Rightarrow x \mid - \Rightarrow$   
 $finite-4.a_4)$   
**definition**  $cardinality-finite-4 :: Enum.finite-4 \Rightarrow enat \rightarrow \mathbf{where} \ cardinality-finite-4$   
 $x = (case \ x \ of \ finite-4.a_1 \Rightarrow 0 \mid finite-4.a_4 \Rightarrow 2 \mid - \Rightarrow 1)$

**instance**  
 $\langle proof \rangle$

**end**

**instantiation**  $Enum.finite-4 :: ra-card-notop-atomic-finiteatoms$   
**begin**

**instance**  
 $\langle proof \rangle$

**end**

**instantiation**  $ra1 :: ra-card-atomic-finiteatoms$   
**begin**

**lift-definition** *bot-ra1* :: *ra1* is *r0000*  $\langle \text{proof} \rangle$   
**lift-definition** *one-ra1* :: *ra1* is *r1001*  $\langle \text{proof} \rangle$   
**lift-definition** *top-ra1* :: *ra1* is *r1111*  $\langle \text{proof} \rangle$   
**lift-definition** *conv-ra1* :: *ra1*  $\Rightarrow$  *ra1* is *id*  $\langle \text{proof} \rangle$   
**lift-definition** *uminus-ra1* :: *ra1*  $\Rightarrow$  *ra1* is  $\lambda r\ x\ y . \neg\ r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *sup-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *ra1* is  $\lambda q\ r\ x\ y . q\ x\ y \vee r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *inf-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *ra1* is  $\lambda q\ r\ x\ y . q\ x\ y \wedge r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *times-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *ra1* is  $\lambda q\ r\ x\ y . \exists z . q\ x\ z \wedge r\ z\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *minus-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *ra1* is  $\lambda q\ r\ x\ y . q\ x\ y \wedge \neg\ r\ x\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *less-eq-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *bool* is  $\lambda q\ r . \forall x\ y . q\ x\ y \longrightarrow r\ x\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *less-ra1* :: *ra1*  $\Rightarrow$  *ra1*  $\Rightarrow$  *bool* is  $\lambda q\ r . (\forall x\ y . q\ x\ y \longrightarrow r\ x\ y) \wedge q \neq r$   $\langle \text{proof} \rangle$   
**lift-definition** *cardinality-ra1* :: *ra1*  $\Rightarrow$  *enat* is  $\lambda q . \text{if } q = r0000 \text{ then } 0 \text{ else if } q = r1111 \text{ then } 2 \text{ else } 1$   $\langle \text{proof} \rangle$

**instance**  
 $\langle \text{proof} \rangle$

**end**

**lemma** *four-cases*:

**assumes** *P x1 P x2 P x3 P x4*  
**shows**  $\forall y \in \{ x . x \in \{x1, x2, x3, x4\} \} . P\ y$   
 $\langle \text{proof} \rangle$

**lemma** *r-aux*:

$(\lambda x\ y . r1001\ x\ y \vee r0110\ x\ y) = r1111\ (\lambda x\ y . r1001\ x\ y \wedge r0110\ x\ y) = r0000$   
 $(\lambda x\ y . r0110\ x\ y \vee r1001\ x\ y) = r1111\ (\lambda x\ y . r0110\ x\ y \wedge r1001\ x\ y) = r0000$   
 $(\lambda x\ y . r1000\ x\ y \vee r0001\ x\ y) = r1001\ (\lambda x\ y . r1000\ x\ y \wedge r0001\ x\ y) = r0000$   
 $(\lambda x\ y . r1000\ x\ y \vee r1001\ x\ y) = r1001\ (\lambda x\ y . r1000\ x\ y \wedge r1001\ x\ y) = r1000$   
 $(\lambda x\ y . r0001\ x\ y \vee r1000\ x\ y) = r1001\ (\lambda x\ y . r0001\ x\ y \wedge r1000\ x\ y) = r0000$   
 $(\lambda x\ y . r0001\ x\ y \vee r1001\ x\ y) = r1001\ (\lambda x\ y . r0001\ x\ y \wedge r1001\ x\ y) = r0001$   
 $(\lambda x\ y . r1001\ x\ y \vee r1000\ x\ y) = r1001\ (\lambda x\ y . r1001\ x\ y \wedge r1000\ x\ y) = r1000$   
 $(\lambda x\ y . r1001\ x\ y \vee r0001\ x\ y) = r1001\ (\lambda x\ y . r1001\ x\ y \wedge r0001\ x\ y) = r0001$   
 $\langle \text{proof} \rangle$

**instantiation** *ra1* :: *ra-card-notop-atomic-finiteatoms*  
**begin**

**instance**  
 $\langle \text{proof} \rangle$

**end**

**instantiation** *ra2* :: *ra-card-atomic-finiteatoms*

**begin**

**lift-definition** *bot-ra2* :: *ra2* **is** *r0000*  $\langle \text{proof} \rangle$   
**lift-definition** *one-ra2* :: *ra2* **is** *r1001*  $\langle \text{proof} \rangle$   
**lift-definition** *top-ra2* :: *ra2* **is** *r1001*  $\langle \text{proof} \rangle$   
**lift-definition** *conv-ra2* :: *ra2*  $\Rightarrow$  *ra2* **is** *id*  $\langle \text{proof} \rangle$   
**lift-definition** *uminus-ra2* :: *ra2*  $\Rightarrow$  *ra2* **is**  $\lambda r\ x\ y . x = y \wedge \neg r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *sup-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *ra2* **is**  $\lambda q\ r\ x\ y . q\ x\ y \vee r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *inf-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *ra2* **is**  $\lambda q\ r\ x\ y . q\ x\ y \wedge r\ x\ y$   $\langle \text{proof} \rangle$   
**lift-definition** *times-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *ra2* **is**  $\lambda q\ r\ x\ y . \exists z . q\ x\ z \wedge r\ z\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *minus-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *ra2* **is**  $\lambda q\ r\ x\ y . q\ x\ y \wedge \neg r\ x\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *less-eq-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *bool* **is**  $\lambda q\ r . \forall x\ y . q\ x\ y \longrightarrow r\ x\ y$   
 $\langle \text{proof} \rangle$   
**lift-definition** *less-ra2* :: *ra2*  $\Rightarrow$  *ra2*  $\Rightarrow$  *bool* **is**  $\lambda q\ r . (\forall x\ y . q\ x\ y \longrightarrow r\ x\ y) \wedge q \neq r$   $\langle \text{proof} \rangle$   
**lift-definition** *cardinality-ra2* :: *ra2*  $\Rightarrow$  *enat* **is**  $\lambda q . \text{if } q = r0000 \text{ then } 0 \text{ else if } q = r1001 \text{ then } 2 \text{ else } 1$   $\langle \text{proof} \rangle$

**instance**  
 $\langle \text{proof} \rangle$

**end**

**instantiation** *ra2* :: *ra-card-notop-atomic-finiteatoms*  
**begin**

**instance**  
 $\langle \text{proof} \rangle$

**end**

**instantiation** *prod* :: (*stone-relation-algebra*, *stone-relation-algebra*)  
*stone-relation-algebra*  
**begin**

**lift-definition** *bot-prod* :: '*a*  $\times$  '*b* **is** (*bot*::'*a*,*bot*::'*b*)  $\langle \text{proof} \rangle$   
**lift-definition** *one-prod* :: '*a*  $\times$  '*b* **is** (*1*::'*a*,*1*::'*b*)  $\langle \text{proof} \rangle$   
**lift-definition** *top-prod* :: '*a*  $\times$  '*b* **is** (*top*::'*a*,*top*::'*b*)  $\langle \text{proof} \rangle$   
**lift-definition** *conv-prod* :: '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b* **is**  $\lambda(u,v) . (\text{conv } u, \text{conv } v)$   $\langle \text{proof} \rangle$   
**lift-definition** *uminus-prod* :: '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b* **is**  $\lambda(u,v) . (\text{uminus } u, \text{uminus } v)$   
 $\langle \text{proof} \rangle$   
**lift-definition** *sup-prod* :: '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b* **is**  $\lambda(u,v) (w,x) . (u \sqcup w, v \sqcup x)$   $\langle \text{proof} \rangle$   
**lift-definition** *inf-prod* :: '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b* **is**  $\lambda(u,v) (w,x) . (u \sqcap w, v \sqcap x)$   $\langle \text{proof} \rangle$   
**lift-definition** *times-prod* :: '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b*  $\Rightarrow$  '*a*  $\times$  '*b* **is**  $\lambda(u,v) (w,x) . (u * w, v * x)$   $\langle \text{proof} \rangle$

**lift-definition** *less-eq-prod* :: 'a × 'b ⇒ 'a × 'b ⇒ bool **is** λ(u,v) (w,x) . u ≤ w ∧ v ≤ x ⟨proof⟩

**lift-definition** *less-prod* :: 'a × 'b ⇒ 'a × 'b ⇒ bool **is** λ(u,v) (w,x) . u ≤ w ∧ v ≤ x ∧ ¬(u = w ∧ v = x) ⟨proof⟩

**instance**  
 ⟨proof⟩

**end**

**instantiation** *prod* :: (relation-algebra, relation-algebra) relation-algebra  
**begin**

**lift-definition** *minus-prod* :: 'a × 'b ⇒ 'a × 'b ⇒ 'a × 'b **is** λ(u,v) (w,x) . (u − w, v − x) ⟨proof⟩

**instance**  
 ⟨proof⟩

**end**

**instantiation** *prod* ::  
 (relation-algebra-atomic-finiteatoms, relation-algebra-atomic-finiteatoms)  
 relation-algebra-atomic-finiteatoms  
**begin**

**instance**  
 ⟨proof⟩

**end**

**instantiation** *prod* ::  
 (ra-card-notop-atomic-finiteatoms, ra-card-notop-atomic-finiteatoms)  
 ra-card-notop-atomic-finiteatoms  
**begin**

**lift-definition** *cardinality-prod* :: 'a × 'b ⇒ enat **is** λ(u,v) . #u + #v ⟨proof⟩

**instance**  
 ⟨proof⟩

**end**

**type-synonym** *finite-4-square* = Enum.finite-4 × Enum.finite-4

**interpretation** *finite-4-square*: ra-card-atomic-finiteatoms **where** *cardinality* = *cardinality* **and** *inf* = (⊔) **and** *less-eq* = (≤) **and** *less* = (<) **and** *sup* = (⊓) **and** *bot* = bot::finite-4-square **and** *top* = top **and** *uminus* = uminus **and** *one* = 1 **and** *times* = (\*) **and** *conv* = conv **and** *minus* = (−) ⟨proof⟩



**interpretation** *finite-4-square*: *ra-card-all-atomic-finiteatoms* **where** *cardinality* = *cardinality* **and** *inf* =  $(\sqcap)$  **and** *less-eq* =  $(\leq)$  **and** *less* =  $(<)$  **and** *sup* =  $(\sqcup)$  **and** *bot* = *bot::finite-4-square* **and** *top* = *top* **and** *uminus* = *uminus* **and** *one* = 1 **and** *times* =  $(*)$  **and** *conv* = *conv* **and** *minus* =  $(-)$   
 ⟨proof⟩

**lemma** *counterexample-atom-rectangle-2*:  
 $\text{atom } a \longrightarrow a * \text{top} * a \leq (a::\text{finite-4-square})$   
 nitpick[expect=genuine]  
 ⟨proof⟩

**lemma** *counterexample-atom-univalent-2*:  
 $\text{atom } a \longrightarrow \text{univalent } (a::\text{finite-4-square})$   
 nitpick[expect=genuine]  
 ⟨proof⟩

**lemma** *counterexample-point-dense-2*:  
 assumes  $x \neq \text{bot}$   
 and  $x \leq 1$   
 shows  $\exists a::\text{finite-4-square} . a \neq \text{bot} \wedge a * \text{top} * a \leq 1 \wedge a \leq x$   
 nitpick[expect=genuine]  
 ⟨proof⟩

**type-synonym** *ra11* = *ra1*  $\times$  *ra1*

**interpretation** *ra11*: *ra-card-atomic-finiteatoms* **where** *cardinality* = *cardinality* **and** *inf* =  $(\sqcap)$  **and** *less-eq* =  $(\leq)$  **and** *less* =  $(<)$  **and** *sup* =  $(\sqcup)$  **and** *bot* = *bot::ra11* **and** *top* = *top* **and** *uminus* = *uminus* **and** *one* = 1 **and** *times* =  $(*)$  **and** *conv* = *conv* **and** *minus* =  $(-)$  ⟨proof⟩

**interpretation** *ra11*: *ra-card-all-atomic-finiteatoms* **where** *cardinality* = *cardinality* **and** *inf* =  $(\sqcap)$  **and** *less-eq* =  $(\leq)$  **and** *less* =  $(<)$  **and** *sup* =  $(\sqcup)$  **and** *bot* = *bot::ra11* **and** *top* = *top* **and** *uminus* = *uminus* **and** *one* = 1 **and** *times* =  $(*)$  **and** *conv* = *conv* **and** *minus* =  $(-)$   
 ⟨proof⟩

**interpretation** *ra11*: *stone-relation-algebra-atomrect* **where** *inf* =  $(\sqcap)$  **and** *less-eq* =  $(\leq)$  **and** *less* =  $(<)$  **and** *sup* =  $(\sqcup)$  **and** *bot* = *bot::ra11* **and** *top* = *top* **and** *uminus* = *uminus* **and** *one* = 1 **and** *times* =  $(*)$  **and** *conv* = *conv*  
 ⟨proof⟩

**lemma**  $\neg (\forall a::\text{ra1} \times \text{ra1} . \text{atom } a \longrightarrow a * \text{top} * a \leq a)$   
 ⟨proof⟩

**end**

## References

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