Implementing field extensions of the form $\mathbb{Q}[\sqrt{b}]^*$

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Abstract

We apply data refinement to implement the real numbers, where we support all numbers in the field extension $\mathbb{Q}[\sqrt{b}]$, i.e., all numbers of the form $p + q\sqrt{b}$ for rational numbers p and q and some fixed natural number b. To this end, we also developed algorithms to precisely compute roots of a rational number, and to perform a factorization of natural numbers which eliminates duplicate prime factors.

Our results have been used to certify termination proofs which involve polynomial interpretations over the reals.

Contents

1 Introduction

It has been shown that polynomial interpretations over the reals are strictly more powerful for termination proving than polynomial interpretations over the rationals. To this end, also automated termination prover started to generate such interpretations. [3, 4, 5, 7, 8]. However, for all current implementations, only reals of the form $p + q \cdot \sqrt{b}$ are generated where b is some fixed natural number and p and q may be arbitrary rationals, i.e., we get numbers within $\mathbb{Q}[\sqrt{b}]$.

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To support these termination proofs in our certifier CeTA [6], we therefore required executable functions on $\mathbb{Q}[\sqrt{b}]$, which can then be used as an implementation type for the reals. Here, we used ideas from [1, 2] to provide a sufficiently powerful partial implementations via data refinement.

2 Auxiliary lemmas which might be moved into the Isabelle distribution.

theory *Real-Impl-Auxiliary*
imports
  HOL−Computational-Algebra.Primes
begin

lemma *multiplicity-prime*:
  assumes *p*: *prime* (*i* :: *nat*) **and** *ji*: $j \neq i$
  shows *multiplicity j i* = *0*
  ⟨*proof*⟩

end

3 Prime products

theory *Prime-Product*
imports
  Real-Impl-Auxiliary
  Sqrt-Babylonian.Sqrt-Babylonian
begin

Prime products are natural numbers where no prime factor occurs more than once.

definition *prime-product*
  where *prime-product* (*n* :: *nat*) = (∀ *p*. *prime p* ⟶ *multiplicity p n* ≤ *1*)

The main property is that whenever b_1 and b_2 are different prime products, then $p_1 + q_1\sqrt{b_1} = p_2 + q_2\sqrt{b_2}$ implies $(p_1, q_1, b_1) = (p_2, q_2, b_2)$ for all rational numbers p_1, q_1, p_2, q_2. This is the key property to uniquely represent numbers in $\mathbb{Q}[\sqrt{b}]$ by triples. In the following we develop an algorithm to decompose any natural number n into $n = s^2 \cdot p$ for some s and prime product p.

function *prime-product-factor-main* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* × *nat* **where**
  prime-product-factor-main factor-sq factor-pr limit n i =
    (*if i* ≤ *limit* ∧ *i* ≥ *2 then*
      (*if i dvd n*
        then (*let n′* = *n div i in*
          (*if i dvd n′ then*

let n′′ = n′ div i in
prime-product-factor-main (factor-sq ∗ i) factor-pr (nat (root-nat-floor 3 n′′)) n′′ i
else
(case sqrt-nat n′ of
Cons sn - ⇒ (factor-sq ∗ sn, factor-pr ∗ i)
| [] ⇒ prime-product-factor-main factor-sq (factor-pr ∗ i) (nat (root-nat-floor 3 n′)) n′ (Suc i)
)
)
)
else
prime-product-factor-main factor-sq factor-pr limit n (Suc i))
else
(factor-sq, factor-pr ∗ n)) ⟨*proof*⟩

termination
⟨*proof*⟩

lemma *prime-product-factor-main*: **assumes** ¬ (∃ *s. s* ∗ *s* = *n*)
and *limit* = *nat* (*root-nat-floor 3 n*)
and *m* = *factor-sq* ∗ *factor-sq* ∗ *factor-pr* ∗ *n*
and *prime-product-factor-main factor-sq factor-pr limit n i* = (*sq*, *p*)
and *i* ≥ *2*
and ⋀ *j*. *j* ≥ *2* ⟹ *j* < *i* ⟹ ¬ *j dvd n*
and ⋀ *j*. *prime j* ⟹ *j* < *i* ⟹ *multiplicity j factor-pr* ≤ *1*
and ⋀ *j*. *prime j* ⟹ *j* ≥ *i* ⟹ *multiplicity j factor-pr* = *0*
and *factor-pr* > *0*
shows *m* = *sq* ∗ *sq* ∗ *p* ∧ *prime-product p*
⟨*proof*⟩

definition *prime-product-factor* :: *nat* ⇒ *nat* × *nat* **where**
prime-product-factor n = (*case sqrt-nat n of*
(*Cons s -*) ⇒ (*s*,*1*)
| [] ⇒ *prime-product-factor-main 1 1* (*nat* (*root-nat-floor 3 n*)) *n 2*)

lemma *prime-product-one*[*simp*, *intro*]: *prime-product 1*
⟨*proof*⟩

lemma *prime-product-factor*: **assumes** *pf*: *prime-product-factor n* = (*sq*,*p*)
shows *n* = *sq* ∗ *sq* ∗ *p* ∧ *prime-product p*
⟨*proof*⟩

end

4 A representation of real numbers via triples

theory *Real-Impl*

imports
Sqrt-Babylonian.Sqrt-Babylonian
begin

We represent real numbers of the form $p + q \cdot \sqrt{b}$ for $p, q \in \mathbb{Q}$, $n \in \mathbb{N}$ by triples (p, q, b). However, we require the invariant that $\sqrt{b}$ is irrational. Most binary operations are implemented via partial functions where the common the restriction is that the numbers b in both triples have to be identical. So, we support addition of $\sqrt{2} + \sqrt{2}$, but not $\sqrt{2} + \sqrt{3}$.

The set of natural numbers whose sqrt is irrational

definition *sqrt-irrat* = { *q* :: *nat*. ¬ (∃ *p*. *p* ∗ *p* = *rat-of-nat q*)}

lemma *sqrt-irrat*: **assumes** *choice*: *q* = *0* ∨ *b* ∈ *sqrt-irrat*
and *eq*: *real-of-rat p* + *real-of-rat q* ∗ *sqrt* (*of-nat b*) = *0*
shows *q* = *0*
⟨*proof*⟩

To represent numbers of the form $p + q \cdot \sqrt{b}$, use mini algebraic numbers, i.e., triples (p, q, b) with irrational $\sqrt{b}$.

typedef *mini-alg* =
{(*p*,*q*,*b*) | (*p* :: *rat*) (*q* :: *rat*) (*b* :: *nat*).
q = *0* ∨ *b* ∈ *sqrt-irrat*}
⟨*proof*⟩

setup-lifting *type-definition-mini-alg*

lift-definition *real-of* :: *mini-alg* ⇒ *real* **is**
λ (*p*,*q*,*b*). *of-rat p* + *of-rat q* ∗ *sqrt* (*of-nat b*) ⟨*proof*⟩

lift-definition *ma-of-rat* :: *rat* ⇒ *mini-alg* **is** λ *x*. (*x*,*0*,*0*) ⟨*proof*⟩

lift-definition *ma-rat* :: *mini-alg* ⇒ *rat* **is** *fst* ⟨*proof*⟩
lift-definition *ma-base* :: *mini-alg* ⇒ *nat* **is** *snd o snd* ⟨*proof*⟩
lift-definition *ma-coeff* :: *mini-alg* ⇒ *rat* **is** *fst o snd* ⟨*proof*⟩

lift-definition *ma-uminus* :: *mini-alg* ⇒ *mini-alg* **is**
λ (*p1*,*q1*,*b1*). (− *p1*, − *q1*, *b1*) ⟨*proof*⟩

lift-definition *ma-compatible* :: *mini-alg* ⇒ *mini-alg* ⇒ *bool* **is**
λ (*p1*,*q1*,*b1*) (*p2*,*q2*,*b2*). *q1* = *0* ∨ *q2* = *0* ∨ *b1* = *b2* ⟨*proof*⟩

definition *ma-normalize* :: *rat* × *rat* × *nat* ⇒ *rat* × *rat* × *nat* **where**
ma-normalize x ≡ *case x of* (*a*,*b*,*c*) ⇒ *if b* = *0 then* (*a*,*0*,*0*) *else* (*a*,*b*,*c*)

lemma *ma-normalize-case*[*simp*]: (*case ma-normalize r of* (*a*,*b*,*c*) ⇒ *real-of-rat a* + *real-of-rat b* ∗ *sqrt* (*of-nat c*))
= (*case r of* (*a*,*b*,*c*) ⇒ *real-of-rat a* + *real-of-rat b* ∗ *sqrt* (*of-nat c*))
⟨*proof*⟩

lift-definition *ma-plus* :: *mini-alg* ⇒ *mini-alg* ⇒ *mini-alg* **is**
λ (*p1*,*q1*,*b1*) (*p2*,*q2*,*b2*). *if q1* = *0 then*
(*p1* + *p2*, *q2*, *b2*) *else ma-normalize* (*p1* + *p2*, *q1* + *q2*, *b1*) ⟨*proof*⟩

lift-definition *ma-times* :: *mini-alg* ⇒ *mini-alg* ⇒ *mini-alg* **is**
λ (*p1*,*q1*,*b1*) (*p2*,*q2*,*b2*). *if q1* = *0 then*
ma-normalize (*p1*∗*p2*, *p1*∗*q2*, *b2*) *else*
ma-normalize (*p1*∗*p2* + *of-nat b2*∗*q1*∗*q2*, *p1*∗*q2* + *q1*∗*p2*, *b1*) ⟨*proof*⟩

lift-definition *ma-inverse* :: *mini-alg* ⇒ *mini-alg* **is**
λ (*p*,*q*,*b*). *let d* = *inverse* (*p* ∗ *p* − *of-nat b* ∗ *q* ∗ *q*) *in*
ma-normalize (*p* ∗ *d*, − *q* ∗ *d*, *b*) ⟨*proof*⟩

lift-definition *ma-floor* :: *mini-alg* ⇒ *int* **is**
λ (*p*,*q*,*b*). *case* (*quotient-of p*,*quotient-of q*) *of* ((*z1*,*n1*),(*z2*,*n2*)) ⇒
let z2n1 = *z2* ∗ *n1*; *z1n2* = *z1* ∗ *n2*; *n12* = *n1* ∗ *n2*; *prod* = *z2n1* ∗ *z2n1* ∗ *int b in*
(*z1n2* + (*if z2n1* ≥ *0 then sqrt-int-floor-pos prod else* − *sqrt-int-ceiling-pos prod*)) *div n12* ⟨*proof*⟩

lift-definition *ma-sqrt* :: *mini-alg* ⇒ *mini-alg* **is**
λ (*p*,*q*,*b*). *let* (*a*,*b*) = *quotient-of p*; *aa* = *abs* (*a* ∗ *b*) *in*
case sqrt-int aa of [] ⇒ (*0*,*inverse* (*of-int b*),*nat aa*) | (*Cons s* -) ⇒ (*of-int s* / *of-int b*,*0*,*0*)
⟨*proof*⟩

lift-definition *ma-equal* :: *mini-alg* ⇒ *mini-alg* ⇒ *bool* **is**
λ (*p1*,*q1*,*b1*) (*p2*,*q2*,*b2*).
p1 = *p2* ∧ *q1* = *q2* ∧ (*q1* = *0* ∨ *b1* = *b2*) ⟨*proof*⟩

lift-definition *ma-ge-0* :: *mini-alg* ⇒ *bool* **is**
λ (*p*,*q*,*b*). *let bqq* = *of-nat b* ∗ *q* ∗ *q*; *pp* = *p* ∗ *p in*
0 ≤ *p* ∧ *bqq* ≤ *pp* ∨ *0* ≤ *q* ∧ *pp* ≤ *bqq* ⟨*proof*⟩

lift-definition *ma-is-rat* :: *mini-alg* ⇒ *bool* **is**
λ (*p*,*q*,*b*). *q* = *0* ⟨*proof*⟩

definition *ge-0* :: *real* ⇒ *bool* **where** [*code del*]: *ge-0 x* = (*x* ≥ *0*)

lemma *ma-ge-0*: *ge-0* (*real-of x*) = *ma-ge-0 x*
⟨*proof*⟩

lemma *ma-0*: *0* = *real-of* (*ma-of-rat 0*) ⟨*proof*⟩

lemma *ma-1*: *1* = *real-of* (*ma-of-rat 1*) ⟨*proof*⟩

lemma *ma-uminus*:
− (*real-of x*) = *real-of* (*ma-uminus x*)

⟨*proof*⟩

lemma *ma-inverse*: *inverse* (*real-of r*) = *real-of* (*ma-inverse r*)
⟨*proof*⟩

lemma *ma-sqrt-main*: *ma-rat r* ≥ *0* ⟹ *ma-coeff r* = *0* ⟹ *sqrt* (*real-of r*) = *real-of* (*ma-sqrt r*)
⟨*proof*⟩

lemma *ma-sqrt*: *sqrt* (*real-of r*) = (*if ma-coeff r* = *0 then*
(*if ma-rat r* ≥ *0 then real-of* (*ma-sqrt r*) *else* − *real-of* (*ma-sqrt* (*ma-uminus r*)))
else Code.abort (*STR ''cannot represent sqrt of irrational number''*) (λ -. *sqrt* (*real-of r*)))
⟨*proof*⟩

lemma *ma-plus*:
(*real-of r1* + *real-of r2*) = (*if ma-compatible r1 r2*
then real-of (*ma-plus r1 r2*) *else*
Code.abort (*STR ''different base''*) (λ -. *real-of r1* + *real-of r2*))
⟨*proof*⟩

lemma *ma-times*:
(*real-of r1* ∗ *real-of r2*) = (*if ma-compatible r1 r2*
then real-of (*ma-times r1 r2*) *else*
Code.abort (*STR ''different base''*) (λ -. *real-of r1* ∗ *real-of r2*))
⟨*proof*⟩

lemma *ma-equal*:
HOL.equal (*real-of r1*) (*real-of r2*) = (*if ma-compatible r1 r2*
then ma-equal r1 r2 else
Code.abort (*STR ''different base''*) (λ -. *HOL.equal* (*real-of r1*) (*real-of r2*)))
⟨*proof*⟩

lemma *ma-floor*: *floor* (*real-of r*) = *ma-floor r*
⟨*proof*⟩

lemma *comparison-impl*:
(*x* :: *real*) ≤ (*y* :: *real*) = *ge-0* (*y* − *x*)
(*x* :: *real*) < (*y* :: *real*) = (*x* ≠ *y* ∧ *ge-0* (*y* − *x*))
⟨*proof*⟩

lemma *ma-of-rat*: *real-of-rat r* = *real-of* (*ma-of-rat r*)
⟨*proof*⟩

definition *is-rat* :: *real* ⇒ *bool* **where**
[*code-abbrev*]: *is-rat x* ⟷ *x* ∈ ℚ

lemma *ma-is-rat*: *is-rat* (*real-of x*) = *ma-is-rat x*
⟨*proof*⟩

definition *sqrt-real* x = (*if* $x \in \mathbb{Q} \wedge x \geq 0$ *then* (*if* $x = 0$ *then* [*0*] *else* (*let sx* = *sqrt x in* [*sx*,−*sx*])) *else* [])

lemma *sqrt-real*[*simp*]: **assumes** *x*: $x \in \mathbb{Q}$
shows *set* (*sqrt-real x*) = {*y* . *y* ∗ *y* = *x*}
⟨*proof*⟩

code-datatype *real-of*

lemma [*code*]:
Ratreal = *real-of* ∘ *ma-of-rat*
⟨*proof*⟩

lemmas *ma-code-eqns* [*code*] = *ma-ge-0 ma-floor ma-0 ma-1 ma-uminus ma-inverse ma-sqrt ma-plus ma-times ma-equal ma-is-rat*
comparison-impl

lemma [*code*]:
(*x* :: *real*) / (*y* :: *real*) = *x* ∗ *inverse y*
(*x* :: *real*) − (*y* :: *real*) = *x* + (− *y*)
⟨*proof*⟩

Some tests with small numbers. To work on larger number, one should additionally import the theories for efficient calculation on numbers

value ⌊*101.1* ∗ (*3* ∗ *sqrt 2* + *6* ∗ *sqrt 0.5*)⌋
value ⌊*606.2* ∗ *sqrt 2* + *0.001*⌋
value *101.1* ∗ (*3* ∗ *sqrt 2* + *6* ∗ *sqrt 0.5*) = *606.2* ∗ *sqrt 2* + *0.001*
value *101.1* ∗ (*3* ∗ *sqrt 2* + *6* ∗ *sqrt 0.5*) > *606.2* ∗ *sqrt 2* + *0.001*
value (*sqrt 0.1* ∈ ℚ, *sqrt* (− *0.09*) ∈ ℚ)

end

5 A unique representation of real numbers via triples

theory *Real-Unique-Impl*
imports
Prime-Product
Real-Impl
Show.Show-Instances
Show.Show-Real
begin

We implement the real numbers again using triples, but now we require an additional invariant on the triples, namely that the base has to be a prime product. This has the consequence that the mapping of triples into $\mathbb{R}$ is injective. Hence, equality on reals is now equality on triples, which can even be executed in case of different bases. Similarly, we now also allow different basis in comparisons. Ultimately, injectivity allows us to define a show-function for real numbers, which pretty prints real numbers into strings.

typedef *mini-alg-unique* =
{ *r* :: *mini-alg* . *ma-coeff r* = *0* ∧ *ma-base r* = *0* ∨ *ma-coeff r* ≠ *0* ∧ *prime-product* (*ma-base r*)}
⟨*proof*⟩

setup-lifting *type-definition-mini-alg-unique*

lift-definition *real-of-u* :: *mini-alg-unique* ⇒ *real* **is** *real-of* ⟨*proof*⟩
lift-definition *mau-floor* :: *mini-alg-unique* ⇒ *int* **is** *ma-floor* ⟨*proof*⟩
lift-definition *mau-of-rat* :: *rat* ⇒ *mini-alg-unique* **is** *ma-of-rat* ⟨*proof*⟩
lift-definition *mau-rat* :: *mini-alg-unique* ⇒ *rat* **is** *ma-rat* ⟨*proof*⟩
lift-definition *mau-base* :: *mini-alg-unique* ⇒ *nat* **is** *ma-base* ⟨*proof*⟩
lift-definition *mau-coeff* :: *mini-alg-unique* ⇒ *rat* **is** *ma-coeff* ⟨*proof*⟩
lift-definition *mau-uminus* :: *mini-alg-unique* ⇒ *mini-alg-unique* **is** *ma-uminus* ⟨*proof*⟩
lift-definition *mau-compatible* :: *mini-alg-unique* ⇒ *mini-alg-unique* ⇒ *bool* **is** *ma-compatible* ⟨*proof*⟩
lift-definition *mau-ge-0* :: *mini-alg-unique* ⇒ *bool* **is** *ma-ge-0* ⟨*proof*⟩
lift-definition *mau-inverse* :: *mini-alg-unique* ⇒ *mini-alg-unique* **is** *ma-inverse*
⟨*proof*⟩
lift-definition *mau-plus* :: *mini-alg-unique* ⇒ *mini-alg-unique* ⇒ *mini-alg-unique* **is** *ma-plus*
⟨*proof*⟩
lift-definition *mau-times* :: *mini-alg-unique* ⇒ *mini-alg-unique* ⇒ *mini-alg-unique* **is** *ma-times*
⟨*proof*⟩
lift-definition *ma-identity* :: *mini-alg* ⇒ *mini-alg* ⇒ *bool* **is** (=) ⟨*proof*⟩
lift-definition *mau-equal* :: *mini-alg-unique* ⇒ *mini-alg-unique* ⇒ *bool* **is** *ma-identity* ⟨*proof*⟩
lift-definition *mau-is-rat* :: *mini-alg-unique* ⇒ *bool* **is** *ma-is-rat* ⟨*proof*⟩

lemma *Ratreal-code*[*code*]:
Ratreal = *real-of-u* ∘ *mau-of-rat*
⟨*proof*⟩

lemma *mau-floor*: *floor* (*real-of-u r*) = *mau-floor r*
⟨*proof*⟩
lemma *mau-inverse*: *inverse* (*real-of-u r*) = *real-of-u* (*mau-inverse r*)
⟨*proof*⟩
lemma *mau-uminus*: − (*real-of-u r*) = *real-of-u* (*mau-uminus r*)
⟨*proof*⟩
lemma *mau-times*:
(*real-of-u r1* ∗ *real-of-u r2*) = (*if mau-compatible r1 r2*
then real-of-u (*mau-times r1 r2*) *else*
Code.abort (*STR ''different base''*) (λ -. *real-of-u r1* ∗ *real-of-u r2*))
⟨*proof*⟩

lemma *mau-plus*:
  (*real-of-u r1* + *real-of-u r2*) = (*if mau-compatible r1 r2*
    then real-of-u (*mau-plus r1 r2*) *else*
    Code.abort (*STR ''different base''*) (λ -. *real-of-u r1* + *real-of-u r2*))
  ⟨*proof*⟩

lemma *real-of-u-inj*[*simp*]: *real-of-u x* = *real-of-u y* ⟷ *x* = *y*
⟨*proof*⟩

lift-definition *mau-sqrt* :: *mini-alg-unique* ⇒ *mini-alg-unique* **is**
  λ *ma*. *let* (*a*,*b*) = *quotient-of* (*ma-rat ma*); (*sq*,*fact*) = *prime-product-factor* (*nat* (*abs a* ∗ *b*));
      ma′ = *ma-of-rat* (*of-int* (*sgn*(*a*)) ∗ *of-nat sq* / *of-int b*)
      in ma-times ma′ (*ma-sqrt* (*ma-of-rat* (*of-nat fact*)))
⟨*proof*⟩

lemma *sqrt-sgn*[*simp*]: *sqrt* (*of-int* (*sgn a*)) = *of-int* (*sgn a*)
  ⟨*proof*⟩

lemma *mau-sqrt-main*: *mau-coeff r* = *0* ⟹ *sqrt* (*real-of-u r*) = *real-of-u* (*mau-sqrt r*)
⟨*proof*⟩

lemma *mau-sqrt*: *sqrt* (*real-of-u r*) = (*if mau-coeff r* = *0 then*
  real-of-u (*mau-sqrt r*)
  else Code.abort (*STR ''cannot represent sqrt of irrational number''*) (λ -. *sqrt* (*real-of-u r*)))
  ⟨*proof*⟩

lemma *mau-0*: *0* = *real-of-u* (*mau-of-rat 0*) ⟨*proof*⟩

lemma *mau-1*: *1* = *real-of-u* (*mau-of-rat 1*) ⟨*proof*⟩

lemma *mau-equal*:
  HOL.equal (*real-of-u r1*) (*real-of-u r2*) = *mau-equal r1 r2* ⟨*proof*⟩

lemma *mau-ge-0*: *ge-0* (*real-of-u x*) = *mau-ge-0 x* ⟨*proof*⟩

definition *real-lt* :: *real* ⇒ *real* ⇒ *bool* **where** *real-lt* = (<)

The following code equation terminates if it is started on two different inputs.

lemma *real-lt* [*code*]: *real-lt x y* = (*let fx* = *floor x*; *fy* = *floor y in*
  (*if fx* < *fy then True else if fx* > *fy then False else real-lt* (*x* ∗ *1024*) (*y* ∗ *1024*)))
⟨*proof*⟩

For comparisons we first check for equality. Then, if the bases are compatible we can just compare the differences with 0. Otherwise, we start the recursive algorithm *real-lt* which works on arbitrary bases. In this way, we have an implementation of comparisons which can compare all representable numbers.

Note that in *Real-Impl.Real-Impl* we did not use *real-lt* as there the code-equations for equality already require identical bases.

lemma *comparison-impl*:
real-of-u x $\leq$ *real-of-u y* $\longleftrightarrow$ *real-of-u x* $=$ *real-of-u y* $\vee$
(*if mau-compatible x y then ge-0* (*real-of-u y* $-$ *real-of-u x*) *else real-lt* (*real-of-u x*) (*real-of-u y*))
real-of-u x $<$ *real-of-u y* $\longleftrightarrow$ *real-of-u x* $\neq$ *real-of-u y* $\wedge$
(*if mau-compatible x y then ge-0* (*real-of-u y* $-$ *real-of-u x*) *else real-lt* (*real-of-u x*) (*real-of-u y*))
⟨*proof*⟩

lemma *mau-is-rat*: *is-rat* (*real-of-u x*) $=$ *mau-is-rat x* ⟨*proof*⟩

lift-definition *ma-show-real* :: *mini-alg* $\Rightarrow$ *string* **is**
λ (*p*,*q*,*b*). *let sb* $=$ *shows* ''*sqrt*('' $\circ$ *shows b* $\circ$ *shows* '')'';
qb $=$ (*if q* $=$ *1 then sb else if q* $=$ $-$*1 then shows* ''$-$'' $\circ$ *sb else shows q* $\circ$ *shows* ''$*$'' $\circ$ *sb*) *in*
if q $=$ *0 then shows p* [] *else*
if p $=$ *0 then qb* [] *else*
if q $<$ *0 then* ((*shows p* $\circ$ *qb*) [])
else ((*shows p* $\circ$ *shows* ''$+$'' $\circ$ *qb*) []) ⟨*proof*⟩

lift-definition *mau-show-real* :: *mini-alg-unique* $\Rightarrow$ *string* **is** *ma-show-real* ⟨*proof*⟩

overloading *show-real* $\equiv$ *show-real*
begin
definition *show-real*
where *show-real x* $\equiv$
(*if* ($\exists$ *y*. *x* $=$ *real-of-u y*) *then mau-show-real* (*THE y*. *x* $=$ *real-of-u y*) *else* [])
end

lemma *mau-show-real*: *show-real* (*real-of-u x*) $=$ *mau-show-real x*
⟨*proof*⟩

code-datatype *real-of-u*

lemmas *mau-code-eqns* [*code*] $=$ *mau-floor mau-0 mau-1 mau-uminus mau-inverse mau-sqrt mau-plus mau-times mau-equal mau-ge-0 mau-is-rat*
mau-show-real comparison-impl

Some tests with small numbers. To work on larger number, one should additionally import the theories for efficient calculation on numbers

value $\lfloor$*101.1* $*$ (*sqrt 18* $+$ *6* $*$ *sqrt 0.5*)$\rfloor$
value $\lfloor$*324* $*$ *sqrt 7* $+$ *0.001*$\rfloor$
value *101.1* $*$ (*sqrt 18* $+$ *6* $*$ *sqrt 0.5*) $=$ *324* $*$ *sqrt 7* $+$ *0.001*
value *101.1* $*$ (*sqrt 18* $+$ *6* $*$ *sqrt 0.5*) $>$ *324* $*$ *sqrt 7* $+$ *0.001*
value *show* (*101.1* $*$ (*sqrt 18* $+$ *6* $*$ *sqrt 0.5*))

value (*sqrt 0.1* ∈ ℚ, *sqrt* (− *0.09*) ∈ ℚ)

end

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