

Randomised Binary Search Trees

Manuel Eberl

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Abstract

This work is a formalisation of the Randomised Binary Search Trees introduced by Martínez and Roura [1], including definitions and correctness proofs. Like randomised treaps, they are a probabilistic data structure that behaves exactly as if elements were inserted into a non-balancing BST in random order. However, unlike treaps, they only use discrete probability distributions, but their use of randomness is more complicated.

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1 Randomised Binary Search Trees

```
theory Randomised-BSTs
imports Random-BSTs.Random-BSTs Monad-Normalisation.Monad-Normalisation
begin
```

1.1 Auxiliary facts

First of all, we need some fairly simple auxiliary lemmas.

```
lemma return-pmf-if: return-pmf (if P then a else b) = (if P then return-pmf a
else return-pmf b)
  <proof>
```

```
context
begin
```

```
interpretation pmf-as-function <proof>
```

```
lemma True-in-set-bernoulli-pmf-iff [simp]:
  True  $\in$  set-pmf (bernoulli-pmf p)  $\longleftrightarrow$   $p > 0$ 
  <proof>
```

```
lemma False-in-set-bernoulli-pmf-iff [simp]:
  False  $\in$  set-pmf (bernoulli-pmf p)  $\longleftrightarrow$   $p < 1$ 
  <proof>
```

```
end
```

```
lemma in-set-pmf-of-setD:  $x \in$  set-pmf (pmf-of-set A)  $\implies$  finite A  $\implies$   $A \neq \{\}$ 
 $\implies x \in A$ 
  <proof>
```

```
lemma random-bst-reduce:
  finite A  $\implies$   $A \neq \{\}$   $\implies$ 
  random-bst A = do {x  $\leftarrow$  pmf-of-set A; l  $\leftarrow$  random-bst {y  $\in$  A. y < x};
  r  $\leftarrow$  random-bst {y  $\in$  A. y > x}; return-pmf {l, x, r}
  <proof>
```

```
lemma pmf-bind-bernoulli:
  assumes  $x \in \{0..1\}$ 
  shows pmf (bernoulli-pmf x  $\gg$  f) y =  $x * \text{pmf } (f \text{ True}) y + (1 - x) * \text{pmf}$ 
   $(f \text{ False}) y$ 
  <proof>
```

```
lemma image-bool-pair:
  f - '  $A = (\bigcup x \in \{\text{True}, \text{False}\}. \bigcup y \in \{\text{True}, \text{False}\}. \text{if } f(x, y) \in A \text{ then } \{(x, y)\}$ 
  else  $\{\}$ )
  (is ?lhs = ?rhs) <proof>
```

lemma *Leaf-in-set-random-bst-iff* [*simp*]:
 $Leaf \in \text{set-pmf} (\text{random-bst } A) \longleftrightarrow A = \{\} \vee \neg \text{finite } A$
 <proof>

lemma *bst-insert* [*intro*]: $\text{bst } t \implies \text{bst } (\text{Tree-Set.insert } x \ t)$
 <proof>

lemma *bst-bst-of-list* [*intro*]: $\text{bst } (\text{bst-of-list } xs)$
 <proof>

lemma *bst-random-bst*:
 assumes $t \in \text{set-pmf} (\text{random-bst } A)$
 shows $\text{bst } t$
 <proof>

lemma *set-random-bst*:
 assumes $t \in \text{set-pmf} (\text{random-bst } A)$ $\text{finite } A$
 shows $\text{set-tree } t = A$
 <proof>

lemma *isin-bst*:
 assumes $\text{bst } t$
 shows $\text{isin } t \ x \longleftrightarrow x \in \text{set-tree } t$
 <proof>

lemma *isin-random-bst*:
 assumes $\text{finite } A$ $t \in \text{set-pmf} (\text{random-bst } A)$
 shows $\text{isin } t \ x \longleftrightarrow x \in A$
 <proof>

lemma *card-3way-split*:
 assumes $x \in (A :: 'a :: \text{linorder set})$ $\text{finite } A$
 shows $\text{card } A = \text{card } \{y \in A. y < x\} + \text{card } \{y \in A. y > x\} + 1$
 <proof>

The following theorem allows splitting a uniformly random choice from a union of two disjoint sets to first tossing a coin to decide on one of the constituent sets and then choosing an element from it uniformly at random.

lemma *pmf-of-set-union-split*:
 assumes $\text{finite } A$ $\text{finite } B$ $A \cap B = \{\}$ $A \cup B \neq \{\}$
 assumes $p = \text{card } A / (\text{card } A + \text{card } B)$
 shows $\text{do } \{b \leftarrow \text{bernoulli-pmf } p; \text{ if } b \text{ then pmf-of-set } A \text{ else pmf-of-set } B\} =$
 $\text{pmf-of-set } (A \cup B)$
 (is ?lhs = ?rhs)
 <proof>

lemma *pmf-of-set-split-inter-diff*:
 assumes $\text{finite } A$ $\text{finite } B$ $A \neq \{\}$ $B \neq \{\}$
 assumes $p = \text{card } (A \cap B) / \text{card } B$

shows $\{b \leftarrow \text{bernoulli-pmf } p; \text{ if } b \text{ then pmf-of-set } (A \cap B) \text{ else pmf-of-set } (B - A)\} =$
 $\text{pmf-of-set } B$ (**is** ?lhs = ?rhs)
 <proof>

Similarly to the above rule, we can split up a uniformly random choice from the disjoint union of three sets. This could be done with two coin flips, but it is more convenient to choose a natural number uniformly at random instead and then do a case distinction on it.

lemma *pmf-of-set-3way-split*:

fixes $f\ g\ h :: 'a \Rightarrow 'b\ \text{pmf}$
assumes $\text{finite } A\ A \neq \{\}\ A1 \cap A2 = \{\}\ A1 \cap A3 = \{\}\ A2 \cap A3 = \{\}\ A1 \cup A2 \cup A3 = A$
shows $\{x \leftarrow \text{pmf-of-set } A; \text{ if } x \in A1 \text{ then } f\ x \text{ else if } x \in A2 \text{ then } g\ x \text{ else } h\ x\} =$
 $\text{do } \{i \leftarrow \text{pmf-of-set } \{..<\text{card } A\};$
 $\text{ if } i < \text{card } A1 \text{ then pmf-of-set } A1 \ggg f$
 $\text{ else if } i < \text{card } A1 + \text{card } A2 \text{ then pmf-of-set } A2 \ggg g$
 $\text{ else pmf-of-set } A3 \ggg h\}$ (**is** ?lhs = ?rhs)
 <proof>

1.2 Partitioning a BST

The split operation takes a search parameter x and partitions a BST into two BSTs containing all the values that are smaller than x and those that are greater than x , respectively. Note that x need not be an element of the tree.

fun *split-bst* :: $'a :: \text{linorder} \Rightarrow 'a\ \text{tree} \Rightarrow 'a\ \text{tree} \times 'a\ \text{tree}$ **where**

$\text{split-bst } - \langle \rangle = (\langle \rangle, \langle \rangle)$
 $| \text{split-bst } x \langle l, y, r \rangle =$
 $\text{ (if } y < x \text{ then}$
 $\text{ case split-bst } x\ r \text{ of } (t1, t2) \Rightarrow (\langle l, y, t1 \rangle, t2)$
 $\text{ else if } y > x \text{ then}$
 $\text{ case split-bst } x\ l \text{ of } (t1, t2) \Rightarrow (t1, \langle t2, y, r \rangle)$
 else
 $\text{ (} l, r \text{))}$

fun *split-bst'* :: $'a :: \text{linorder} \Rightarrow 'a\ \text{tree} \Rightarrow \text{bool} \times 'a\ \text{tree} \times 'a\ \text{tree}$ **where**

$\text{split-bst}' - \langle \rangle = (\text{False}, \langle \rangle, \langle \rangle)$
 $| \text{split-bst}' x \langle l, y, r \rangle =$
 $\text{ (if } y < x \text{ then}$
 $\text{ case split-bst}' x\ r \text{ of } (b, t1, t2) \Rightarrow (b, \langle l, y, t1 \rangle, t2)$
 $\text{ else if } y > x \text{ then}$
 $\text{ case split-bst}' x\ l \text{ of } (b, t1, t2) \Rightarrow (b, t1, \langle t2, y, r \rangle)$
 else
 $\text{ (True, } l, r \text{))}$

lemma *split-bst'-altdef*: $\text{split-bst}' x\ t = (\text{isin } t\ x, \text{split-bst } x\ t)$

<proof>

lemma *fst-split-bst'* [simp]: $\text{fst } (\text{split-bst}' x t) = \text{isin } t x$
and *snd-split-bst'* [simp]: $\text{snd } (\text{split-bst}' x t) = \text{split-bst } x t$
<proof>

lemma *size-fst-split-bst* [termination-simp]: $\text{size } (\text{fst } (\text{split-bst } x t)) \leq \text{size } t$
<proof>

lemma *size-snd-split-bst* [termination-simp]: $\text{size } (\text{snd } (\text{split-bst } x t)) \leq \text{size } t$
<proof>

lemmas *size-split-bst = size-fst-split-bst size-snd-split-bst*

lemma *set-split-bst1*: $\text{bst } t \implies \text{set-tree } (\text{fst } (\text{split-bst } x t)) = \{y \in \text{set-tree } t. y < x\}$
<proof>

lemma *set-split-bst2*: $\text{bst } t \implies \text{set-tree } (\text{snd } (\text{split-bst } x t)) = \{y \in \text{set-tree } t. y > x\}$
<proof>

lemma *bst-split-bst1* [intro]: $\text{bst } t \implies \text{bst } (\text{fst } (\text{split-bst } x t))$
<proof>

lemma *bst-split-bst2* [intro]: $\text{bst } t \implies \text{bst } (\text{snd } (\text{split-bst } x t))$
<proof>

Splitting a random BST produces two random BSTs:

theorem *split-random-bst*:
assumes *finite A*
shows $\text{map-pmf } (\text{split-bst } x) (\text{random-bst } A) =$
 $\text{pair-pmf } (\text{random-bst } \{y \in A. y < x\}) (\text{random-bst } \{y \in A. y > x\})$
<proof>
include *monad-normalisation*

<proof>

1.3 Joining

The “join” operation computes the union of two BSTs l and r where all the values in l are strictly smaller than those in r .

fun *mrbst-join* :: $'a \text{ tree} \Rightarrow 'a \text{ tree} \Rightarrow 'a \text{ tree pmf}$ **where**
mrbst-join $t1 t2 =$
 (if $t1 = \langle \rangle$ *then* *return-pmf* $t2$
 else if $t2 = \langle \rangle$ *then* *return-pmf* $t1$
 else do {

```

    b ← bernoulli-pmf (size t1 / (size t1 + size t2));
    if b then
      (case t1 of ⟨l, x, r⟩ ⇒ map-pmf (λr'. ⟨l, x, r^⟩) (mrbst-join r t2))
    else
      (case t2 of ⟨l, x, r⟩ ⇒ map-pmf (λl'. ⟨l', x, r⟩) (mrbst-join t1 l))
  })

```

lemma *mrbst-join-Leaf-left* [simp]: *mrbst-join* ⟨⟩ = *return-pmf* ⟨proof⟩

lemma *mrbst-join-Leaf-right* [simp]: *mrbst-join* t ⟨⟩ = *return-pmf* t ⟨proof⟩

lemma *mrbst-join-reduce*:

```

t1 ≠ ⟨⟩ ⇒ t2 ≠ ⟨⟩ ⇒ mrbst-join t1 t2 =
do {
  b ← bernoulli-pmf (size t1 / (size t1 + size t2));
  if b then
    (case t1 of ⟨l, x, r⟩ ⇒ map-pmf (λr'. ⟨l, x, r^⟩) (mrbst-join r t2))
  else
    (case t2 of ⟨l, x, r⟩ ⇒ map-pmf (λl'. ⟨l', x, r⟩) (mrbst-join t1 l))
}
⟨proof⟩

```

lemmas [simp del] = *mrbst-join.simps*

lemma

```

assumes t' ∈ set-pmf (mrbst-join t1 t2) bst t1 bst t2
assumes ∧x y. x ∈ set-tree t1 ⇒ y ∈ set-tree t2 ⇒ x < y
shows bst-mrbst-join: bst t'
and set-mrbst-join: set-tree t' = set-tree t1 ∪ set-tree t2
⟨proof⟩

```

Joining two random BSTs that satisfy the necessary preconditions again yields a random BST.

theorem *mrbst-join-correct*:

```

fixes A B :: 'a :: linorder set
assumes finite A finite B ∧x y. x ∈ A ⇒ y ∈ B ⇒ x < y
shows do {t1 ← random-bst A; t2 ← random-bst B; mrbst-join t1 t2} =
random-bst (A ∪ B)
⟨proof⟩

```

```

include monad-normalisation
⟨proof⟩

```

1.4 Pushdown

The “push down” operation “forgets” information about the root of a tree in the following sense: It takes a non-empty tree whose root is some known

fixed value and whose children are random BSTs and shuffles the root in such a way that the resulting tree is a random BST.

```
fun mrbst-push-down :: 'a tree  $\Rightarrow$  'a  $\Rightarrow$  'a tree  $\Rightarrow$  'a tree pmf where
  mrbst-push-down l x r =
  do {
    k  $\leftarrow$  pmf-of-set {0..size l + size r};
    if k < size l then
      case l of
        <ll, y, lr>  $\Rightarrow$  map-pmf ( $\lambda r'$ . <ll, y, r'>) (mrbst-push-down lr x r)
    else if k < size l + size r then
      case r of
        <rl, y, rr>  $\Rightarrow$  map-pmf ( $\lambda l'$ . <l', y, rr>) (mrbst-push-down l x rl)
    else
      return-pmf <l, x, r>
  }
```

lemmas [*simp del*] = *mrbst-push-down.simps*

lemma

```
assumes  $t' \in$  set-pmf (mrbst-push-down t1 x t2) bst t1 bst t2
assumes  $\bigwedge y. y \in$  set-tree t1  $\implies y < x \wedge y. y \in$  set-tree t2  $\implies y > x$ 
shows bst-mrbst-push-down: bst t'
and set-mrbst-push-down: set-tree t' = {x}  $\cup$  set-tree t1  $\cup$  set-tree t2
<proof>
```

theorem *mrbst-push-down-correct*:

```
fixes A B :: 'a :: linorder set
assumes finite A finite B  $\bigwedge y. y \in A \implies y < x \wedge y. y \in B \implies x < y$ 
shows do {l  $\leftarrow$  random-bst A; r  $\leftarrow$  random-bst B; mrbst-push-down l x r} =
  random-bst ({x}  $\cup$  A  $\cup$  B)
<proof>
```

```
include monad-normalisation
<proof>
```

lemma *mrbst-push-down-correct'*:

```
assumes finite (A :: 'a :: linorder set) x  $\in$  A
shows do {l  $\leftarrow$  random-bst {y $\in$ A. y < x}; r  $\leftarrow$  random-bst {y $\in$ A. y > x};
mrbst-push-down l x r} =
  random-bst A (is ?lhs = ?rhs)
<proof>
```

1.5 Intersection and Difference

The algorithms for intersection and difference of two trees are almost identical; the only difference is that the “if” statement at the end of the recursive case is flipped. We therefore introduce a generic intersection/difference operation first and prove its correctness to avoid duplication.

```

fun mrbst-inter-diff where
  mrbst-inter-diff - ⟨⟩ - = return-pmf ⟨⟩
| mrbst-inter-diff b ⟨l1, x, r1⟩ t2 =
  (case split-bst' x t2 of (sep, l2, r2) ⇒
    do {
      l ← mrbst-inter-diff b l1 l2;
      r ← mrbst-inter-diff b r1 r2;
      if sep = b then return-pmf ⟨l, x, r⟩ else mrbst-join l r
    })

```

```

lemma mrbst-inter-diff-reduce:
  mrbst-inter-diff b ⟨l1, x, r1⟩ =
    (λt2. case split-bst' x t2 of (sep, l2, r2) ⇒
      do {
        l ← mrbst-inter-diff b l1 l2;
        r ← mrbst-inter-diff b r1 r2;
        if sep = b then return-pmf ⟨l, x, r⟩ else mrbst-join l r
      })
  ⟨proof⟩

```

```

lemma mrbst-inter-diff-Leaf-left [simp]:
  mrbst-inter-diff b ⟨⟩ = (λ-. return-pmf ⟨⟩)
  ⟨proof⟩

```

```

lemma mrbst-inter-diff-Leaf-right [simp]:
  mrbst-inter-diff b (t1 :: 'a :: linorder tree) ⟨⟩ = return-pmf (if b then ⟨⟩ else t1)
  ⟨proof⟩

```

```

lemma
  fixes t1 t2 :: 'a :: linorder tree and b :: bool
  defines setop ≡ (if b then (∩) else (-) :: 'a set ⇒ -)
  assumes t' ∈ set-pmf (mrbst-inter-diff b t1 t2) bst t1 bst t2
  shows bst-mrbst-inter-diff: bst t'
    and set-mrbst-inter-diff: set-tree t' = setop (set-tree t1) (set-tree t2)
  ⟨proof⟩

```

```

theorem mrbst-inter-diff-correct:
  fixes A B :: 'a :: linorder set and b :: bool
  defines setop ≡ (if b then (∩) else (-) :: 'a set ⇒ -)
  assumes finite A finite B
  shows do {t1 ← random-bst A; t2 ← random-bst B; mrbst-inter-diff b t1 t2}
  =
    random-bst (setop A B)
  ⟨proof⟩
  include monad-normalisation
  ⟨proof⟩

```

We now derive the intersection and difference from the generic operation:

```

fun mrbst-inter where

```



```

  mrbst-inter ⟨⟩ - = return-pmf ⟨⟩
| mrbst-inter ⟨l1, x, r1⟩ t2 =
  (case split-bst' x t2 of (sep, l2, r2) ⇒
    do {
      l ← mrbst-inter l1 l2;
      r ← mrbst-inter r1 r2;
      if sep then return-pmf ⟨l, x, r⟩ else mrbst-join l r
    })

```

lemma *mrbst-inter-Leaf-left* [simp]:
 mrbst-inter ⟨⟩ = (λ-. return-pmf ⟨⟩)
 ⟨proof⟩

lemma *mrbst-inter-Leaf-right* [simp]:
 mrbst-inter (t1 :: 'a :: linorder tree) ⟨⟩ = return-pmf ⟨⟩
 ⟨proof⟩

lemma *mrbst-inter-reduce*:
 mrbst-inter ⟨l1, x, r1⟩ =
 (λt2. case split-bst' x t2 of (sep, l2, r2) ⇒
 do {
 l ← mrbst-inter l1 l2;
 r ← mrbst-inter r1 r2;
 if sep then return-pmf ⟨l, x, r⟩ else mrbst-join l r
 })
 ⟨proof⟩

lemma *mrbst-inter-altdef*: mrbst-inter = mrbst-inter-diff True
 ⟨proof⟩

corollary
 fixes t1 t2 :: 'a :: linorder tree
 assumes t' ∈ set-pmf (mrbst-inter t1 t2) bst t1 bst t2
 shows bst-mrbst-inter: bst t'
 and set-mrbst-inter: set-tree t' = set-tree t1 ∩ set-tree t2
 ⟨proof⟩

corollary *mrbst-inter-correct*:
 fixes A B :: 'a :: linorder set
 assumes finite A finite B
 shows do {t1 ← random-bst A; t2 ← random-bst B; mrbst-inter t1 t2} =
 random-bst (A ∩ B)
 ⟨proof⟩

fun *mrbst-diff* **where**
 mrbst-diff ⟨⟩ - = return-pmf ⟨⟩
| mrbst-diff ⟨l1, x, r1⟩ t2 =
 (case split-bst' x t2 of (sep, l2, r2) ⇒

```

do {
  l ← mrbst-diff l1 l2;
  r ← mrbst-diff r1 r2;
  if sep then mrbst-join l r else return-pmf ⟨l, x, r⟩
})

```

lemma *mrbst-diff-Leaf-left* [simp]:
mrbst-diff ⟨⟩ = (λ-. return-pmf ⟨⟩)
 ⟨proof⟩

lemma *mrbst-diff-Leaf-right* [simp]:
mrbst-diff (t1 :: 'a :: linorder tree) ⟨⟩ = return-pmf t1
 ⟨proof⟩

lemma *mrbst-diff-reduce*:
mrbst-diff ⟨l1, x, r1⟩ =
 (λt2. case split-bst' x t2 of (sep, l2, r2) ⇒
 do {
 l ← mrbst-diff l1 l2;
 r ← mrbst-diff r1 r2;
 if sep then mrbst-join l r else return-pmf ⟨l, x, r⟩
 })
 ⟨proof⟩

lemma *If-not*: (if ¬b then x else y) = (if b then y else x)
 ⟨proof⟩

lemma *mrbst-diff-altdef*: *mrbst-diff* = *mrbst-inter-diff* False
 ⟨proof⟩

corollary

```

fixes t1 t2 :: 'a :: linorder tree
assumes t' ∈ set-pmf (mrbst-diff t1 t2) bst t1 bst t2
shows bst-mrbst-diff: bst t'
and set-mrbst-diff: set-tree t' = set-tree t1 - set-tree t2
⟨proof⟩

```

corollary *mrbst-diff-correct*:

```

fixes A B :: 'a :: linorder set
assumes finite A finite B
shows do {t1 ← random-bst A; t2 ← random-bst B; mrbst-diff t1 t2} =
random-bst (A - B)
⟨proof⟩

```

1.6 Union

The algorithm for the union of two trees is by far the most complicated one. It involves a

```

fun mrbst-union where
  mrbst-union ⟨⟩ t2 = return-pmf t2
| mrbst-union t1 ⟨⟩ = return-pmf t1
| mrbst-union ⟨l1, x, r1⟩ ⟨l2, y, r2⟩ =
  do {
    let m = size ⟨l1, x, r1⟩; let n = size ⟨l2, y, r2⟩;
    b ← bernoulli-pmf (m / (m + n));
    if b then do {
      let ⟨l2', r2'⟩ = split-bst x ⟨l2, y, r2⟩;
      l ← mrbst-union l1 l2';
      r ← mrbst-union r1 r2';
      return-pmf ⟨l, x, r⟩
    } else do {
      let ⟨sep, l1', r1'⟩ = split-bst' y ⟨l1, x, r1⟩;
      l ← mrbst-union l1' l2;
      r ← mrbst-union r1' r2;
      if sep then
        mrbst-push-down l y r
      else
        return-pmf ⟨l, y, r⟩
    }
  }

```

lemma *mrbst-union-Leaf-left* [*simp*]: *mrbst-union* ⟨⟩ = *return-pmf* ⟨*proof*⟩

lemma *mrbst-union-Leaf-right* [*simp*]: *mrbst-union* *t1* ⟨⟩ = *return-pmf* *t1* ⟨*proof*⟩

lemma
fixes *t1 t2* :: '*a* :: *linorder tree* **and** *b* :: *bool*
assumes *t' ∈ set-pmf* (*mrbst-union* *t1 t2*) *bst t1 bst t2*
shows *bst-mrbst-union*: *bst t'*
and *set-mrbst-union*: *set-tree t' = set-tree t1 ∪ set-tree t2*
 ⟨*proof*⟩

theorem *mrbst-union-correct*:
assumes *finite A finite B*
shows *do* {*t1* ← *random-bst A*; *t2* ← *random-bst B*; *mrbst-union* *t1 t2*} =
random-bst (*A ∪ B*)
 ⟨*proof*⟩ **including** *monad-normalisation* ⟨*proof*⟩

include *monad-normalisation*
 ⟨*proof*⟩

1.7 Insertion and Deletion

The insertion and deletion operations are simple special cases of the union and difference operations where one of the trees is a singleton tree.

```

fun mrbst-insert where
  mrbst-insert  $x \langle \rangle = \text{return-pmf } \langle \rangle, x, \langle \rangle$ 
| mrbst-insert  $x \langle l, y, r \rangle =$ 
  do {
     $b \leftarrow \text{bernoulli-pmf } (1 / \text{real } (\text{size } l + \text{size } r + 2));$ 
    if  $b$  then do {
       $\text{let } (l', r') = \text{split-bst } x \langle l, y, r \rangle;$ 
       $\text{return-pmf } \langle l', x, r' \rangle$ 
    } else if  $x < y$  then do {
       $\text{map-pmf } (\lambda l'. \langle l', y, r \rangle) (\text{mrbst-insert } x \ l)$ 
    } else if  $x > y$  then do {
       $\text{map-pmf } (\lambda r'. \langle l, y, r' \rangle) (\text{mrbst-insert } x \ r)$ 
    } else do {
       $\text{mrbst-push-down } l \ y \ r$ 
    }
  }

```

lemma *mrbst-insert-altdef*: $\text{mrbst-insert } x \ t = \text{mrbst-union } \langle \rangle, x, \langle \rangle \ t$
<proof>

corollary

```

fixes  $t :: 'a :: \text{linorder tree}$ 
assumes  $t' \in \text{set-pmf } (\text{mrbst-insert } x \ t) \ \text{bst } t$ 
shows  $\text{bst-mrbst-insert}: \text{bst } t'$ 
  and  $\text{set-mrbst-insert}: \text{set-tree } t' = \text{insert } x \ (\text{set-tree } t)$ 
<proof>

```

corollary *mrbst-insert-correct*:

```

assumes finite  $A$ 
shows  $\text{random-bst } A \ggg \text{mrbst-insert } x = \text{random-bst } (\text{insert } x \ A)$ 
<proof>

```

fun *mrbst-delete* :: $'a :: \text{ord} \Rightarrow 'a \ \text{tree} \Rightarrow 'a \ \text{tree pmf}$ **where**

```

  mrbst-delete  $x \langle \rangle = \text{return-pmf } \langle \rangle$ 
| mrbst-delete  $x \langle l, y, r \rangle =$ 
  if  $x < y$  then
     $\text{map-pmf } (\lambda l'. \langle l', y, r \rangle) (\text{mrbst-delete } x \ l)$ 
  else if  $x > y$  then
     $\text{map-pmf } (\lambda r'. \langle l, y, r' \rangle) (\text{mrbst-delete } x \ r)$ 
  else
     $\text{mrbst-join } l \ r$ 

```

lemma *mrbst-delete-altdef*: $\text{mrbst-delete } x \ t = \text{mrbst-diff } t \ \langle \rangle, x, \langle \rangle$
<proof>

corollary

```

fixes  $t :: 'a :: \text{linorder tree}$ 
assumes  $t' \in \text{set-pmf } (\text{mrbst-delete } x \ t) \ \text{bst } t$ 

```

shows *bst-mrbst-delete*: *bst t'*
and *set-mrbst-delete*: *set-tree t' = set-tree t - {x}*
<proof>

corollary *mrbst-delete-correct*:
finite A \implies *do {t ← random-bst A; mrbst-delete x t} = random-bst (A - {x})*
<proof>

end

References

- [1] C. Martínez and S. Roura. Randomized binary search trees. *Journal of the ACM*, 45, 1997.