

Ramsey's Theorem

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

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1 Ramsey's Theorem

```
theory Ramsey
imports Main HOL-Library.Infinite-Set HOL-Library.Ramsey
```

```
begin
```

Please note: this entire development has been updated and incorporated into *HOL-Library.Ramsey* above. Below, some of the results of the original development are linked to their current versions elsewhere in the Isabelle libraries.

1.1 Library lemmas

```
lemma infinite-inj-infinite-image: infinite Z ==> inj-on f Z ==> infinite (f ` Z)
  ⟨proof⟩
```

```
lemma infinite-dom-finite-rng: [| infinite A; finite (f ` A) |] ==> ∃ b ∈ f ` A.
  infinite {a : A. f a = b}
  ⟨proof⟩
```

```
lemma infinite-mem: infinite X ==> ∃ x. x ∈ X
  ⟨proof⟩
```

lemma *not-empty-least*: $(Y::nat\ set) \neq \{\} \implies \exists m. m \in Y \wedge (\forall m'. m' \in Y \longrightarrow m \leq m')$
 $\langle proof \rangle$

1.2 Dependent Choice Variant

lemma *dc*:

assumes *trans*: *trans r*
and *P0*: *P x0*
and *Pstep*: $\bigwedge x. P x \implies \exists y. P y \wedge (x, y) \in r$
obtains *f :: nat* \Rightarrow *'a where* $\bigwedge n. P(f n)$ **and** $\bigwedge n m. n < m \implies (f n, f m) \in r$
 $\langle proof \rangle$

1.3 Ramsey's theorem

lemma *ramsey*: $\forall (s::nat) (r::nat) (YY::'a\ set) (f::'a\ set \Rightarrow nat).$
infinite YY
 $\wedge (\forall X. X \subseteq YY \wedge finite X \wedge card X = r \longrightarrow f X < s)$
 $\longrightarrow (\exists Y' t').$
 $Y' \subseteq YY$
 $\wedge infinite Y'$
 $\wedge t' < s$
 $\wedge (\forall X. X \subseteq Y' \wedge finite X \wedge card X = r \longrightarrow f X = t')$
 $\langle proof \rangle$

end