Quasi-Borel Spaces

Michikazu Hirata, Yasuhiko Minamide, Tetsuya Sato

March 29, 2023

Abstract

The notion of quasi-Borel spaces was introduced by Heunen et al. [1]. The theory provides a suitable denotational model for higher-order probabilistic programming languages with continuous distributions.

This entry is a formalization of the theory of quasi-Borel spaces, including construction of quasi-Borel spaces (product, coproduct, function spaces), the adjunction between the category of measurable spaces and the category of quasi-Borel spaces, and the probability monad on quasi-Borel spaces. This entry also contains the formalization of the Bayesian regression presented in the work of Heunen et al.

This work is a part of the work by same authors, *Program Logic for Higher-Order Probabilistic Programs in Isabelle/HOL*, which will be published in proceedings of the 16th International Symposium on Functional and Logic Programming (FLOPS 2022).

Contents

1 Standard Borel Spaces

theory *StandardBorel*
imports *HOL−Probability.Probability*
begin

A standard Borel space is the Borel space associated with a Polish space. Here, we define standard Borel spaces in another, but equivallent, way. See [1] Proposition 5.

abbreviation *real-borel* $\equiv$ *borel* :: *real measure*
abbreviation *nat-borel* $\equiv$ *borel* :: *nat measure*
abbreviation *ennreal-borel* $\equiv$ *borel* :: *ennreal measure*
abbreviation *bool-borel* $\equiv$ *borel* :: *bool measure*

1.1 Definition

locale *standard-borel* =
  fixes M :: *'a measure*
  assumes *exist-fg*: $\exists f \in M \rightarrow_M$ *real-borel*. $\exists g \in$ *real-borel* $\rightarrow_M M$.
    $\forall x \in$ *space* M. $(g \circ f)\ x = x$
begin

abbreviation *fg* $\equiv$ (*SOME k*. (*fst k*) $\in M \rightarrow_M$ *real-borel* $\wedge$
    (*snd k*) $\in$ *real-borel* $\rightarrow_M M$ $\wedge$
    ($\forall x \in$ *space* M. ((*snd k*) $\circ$ (*fst k*)) $x = x$))

definition $f \equiv$ (*fst fg*)
definition $g \equiv$ (*snd fg*)

lemma
  shows *f-meas*[*simp*,*measurable*] : $f \in M \rightarrow_M$ *real-borel*
    and *g-meas*[*simp*,*measurable*] : $g \in$ *real-borel* $\rightarrow_M M$
    and *gf-comp-id*[*simp*]: $\bigwedge x.\ x \in$ *space* $M \Longrightarrow (g \circ f)\ x = x$
      $\bigwedge x.\ x \in$ *space* $M \Longrightarrow g\ (f\ x) = x$
⟨*proof*⟩

lemma *standard-borel-sets*[*simp*]:
  assumes *sets* M = *sets* Y
  shows *standard-borel* Y
  ⟨*proof*⟩

lemma *f-inj*:
  inj-on f (*space* M)
  ⟨*proof*⟩

lemma *singleton-sets*:
  assumes $x \in$ *space* M
    shows $\{x\} \in$ *sets* M
⟨*proof*⟩

lemma *countable-space-discrete*:
  assumes *countable* (*space* M)
  shows *sets* M = *sets* (*count-space* (*space* M))
⟨*proof*⟩

end

lemma *standard-borelI*:
assumes $f \in Y \to_M$ *real-borel*
$g \in$ *real-borel* $\to_M Y$
and $\bigwedge y.\ y \in$ *space* $Y \Longrightarrow (g \circ f)\ y = y$
shows *standard-borel Y*
⟨*proof*⟩

locale *standard-borel-space-UNIV* = *standard-borel* +
assumes *space-UNIV*:*space M* = *UNIV*
begin

lemma *gf-comp-id′*[*simp*]:
$g \circ f = id\ g\ (f\ x) = x$
⟨*proof*⟩

lemma *f-inj′*:
inj f
⟨*proof*⟩

lemma *g-surj′*:
surj g
⟨*proof*⟩

end

lemma *standard-borel-space-UNIVI*:
assumes $f \in Y \to_M$ *real-borel*
$g \in$ *real-borel* $\to_M Y$
$(g \circ f) = id$
and *space Y* = *UNIV*
shows *standard-borel-space-UNIV Y*
⟨*proof*⟩

lemma *standard-borel-space-UNIVI′*:
assumes *standard-borel Y*
and *space Y* = *UNIV*
shows *standard-borel-space-UNIV Y*
⟨*proof*⟩

1.2 $\mathbb{R}$, $\mathbb{N}$, Boolean, $[0, \infty]$

$\mathbb{R}$ is a standard Borel space.

interpretation *real* : *standard-borel-space-UNIV real-borel*
⟨*proof*⟩

A non-empty Borel subspace of $\mathbb{R}$ is also a standard Borel space.

lemma *real-standard-borel-subset*:
  assumes $U \in$ *sets real-borel*
    and $U \neq \{\}$
  shows *standard-borel* (*restrict-space real-borel* U)
⟨*proof*⟩

A non-empty measurable subset of a standard Borel space is also a standard Borel space.

lemma(**in** *standard-borel*) *standard-borel-subset*:
  assumes $U \in$ *sets* M
    $U \neq \{\}$
  shows *standard-borel* (*restrict-space* M U)
⟨*proof*⟩

$\mathbb{N}$ is a standard Borel space.

interpretation *nat* : *standard-borel-space-UNIV nat-borel*
⟨*proof*⟩

For a countable space X, X is a standard Borel space iff X is a discrete space.

lemma *countable-standard-iff*:
  assumes *space* $X \neq \{\}$
    and *countable* (*space* X)
  shows *standard-borel* $X \longleftrightarrow$ *sets* X = *sets* (*count-space* (*space* X))
⟨*proof*⟩

$\mathbb{B}$ is a standard Borel space.

lemma *to-bool-measurable*:
  assumes f −‘ {*True*} ∩ *space* M ∈ *sets* M
  shows $f \in M \rightarrow_M$ *bool-borel*
⟨*proof*⟩

interpretation *bool* : *standard-borel-space-UNIV bool-borel*
  ⟨*proof*⟩

$[0, \infty]$ (the set of extended non-negative real numbers) is a standard Borel space.

interpretation *ennreal* : *standard-borel-space-UNIV ennreal-borel*
⟨*proof*⟩

1.3 $\mathbb{R} \times \mathbb{R}$

definition *real-to-01open* :: *real* ⇒ *real* **where**
real-to-01open r ≡ *arctan* r / *pi* + 1 / 2

definition *real-to-01open-inverse* :: *real* ⇒ *real* **where**
real-to-01open-inverse r ≡ *tan* (*pi* ∗ r − (*pi* / 2))

lemma *real-to-01open-inverse-correct*:
real-to-01open-inverse ∘ *real-to-01open* = *id*
⟨*proof*⟩

lemma *real-to-01open-inverse-correct′*:
assumes *0* < *r r* < *1*
shows *real-to-01open* (*real-to-01open-inverse r*) = *r*
⟨*proof*⟩

lemma *real-to-01open-01* :
0 < *real-to-01open r* ∧ *real-to-01open r* < *1*
⟨*proof*⟩

lemma *real-to-01open-continuous*:
continuous-on UNIV real-to-01open
⟨*proof*⟩

lemma *real-to-01open-inverse-continuous*:
continuous-on {*0*<..<*1*} *real-to-01open-inverse*
⟨*proof*⟩

lemma *real-to-01open-inverse-measurable*:
real-to-01open-inverse ∈ *restrict-space real-borel* {*0*<..<*1*} $\rightarrow_M$ *real-borel*
⟨*proof*⟩

fun *r01-binary-expansion′′* :: *real* ⇒ *nat* ⇒ *nat* × *real* × *real* **where**
r01-binary-expansion′′ r 0 = (*if 1/2* ≤ *r then* (*1*,*1* ,*1/2*)
else (*0*,*1/2*, *0*)) |
r01-binary-expansion′′ r (*Suc n*) = (*let* (-,*ur*,*lr*) = *r01-binary-expansion′′ r n*;
k = (*ur* + *lr*)/*2 in*
(*if k* ≤ *r then* (*1*,*ur*,*k*)
else (*0*,*k*,*lr*)))

a_n where $r = 0.a_0a_1a_2....$ for $0 < r < 1$.

definition *r01-binary-expansion′* :: *real* ⇒ *nat* ⇒ *nat* **where**
r01-binary-expansion′ r n ≡ *fst* (*r01-binary-expansion′′ r n*)

$a_n = 0$ or 1.

lemma *real01-binary-expansion′-0or1*:
r01-binary-expansion′ r n ∈ {*0*,*1*}
⟨*proof*⟩

definition *r01-binary-sum* :: (*nat* ⇒ *nat*) ⇒ *nat* ⇒ *real* **where**
r01-binary-sum a n ≡ (∑ *i*=*0*..*n*. *real* (*a i*) ∗ ((*1/2*)^(*Suc i*)))

definition *r01-binary-sum-lim* :: (*nat* ⇒ *nat*) ⇒ *real* **where**
r01-binary-sum-lim ≡ *lim* ∘ *r01-binary-sum*

definition *r01-binary-expression* :: *real* ⇒ *nat* ⇒ *real* **where**
r01-binary-expression ≡ *r01-binary-sum* ∘ *r01-binary-expansion′*

lemma *r01-binary-expansion-lr-r-ur*:
assumes *0* < *r r* < *1*
shows (*snd* (*snd* (*r01-binary-expansion″ r n*))) ≤ *r* ∧
r < (*fst* (*snd* (*r01-binary-expansion″ r n*)))
⟨*proof*⟩

0 ≤ *lr* ∧ *lr* < *ur* ∧ *ur* ≤ *1*.

lemma *r01-binary-expansion-lr-ur-nn*:
shows *0* ≤ *snd* (*snd* (*r01-binary-expansion″ r n*)) ∧
snd (*snd* (*r01-binary-expansion″ r n*)) < *fst* (*snd* (*r01-binary-expansion″ r n*)) ∧
fst (*snd* (*r01-binary-expansion″ r n*)) ≤ *1*
⟨*proof*⟩

lemma *r01-binary-expansion-diff*:
shows (*fst* (*snd* (*r01-binary-expansion″ r n*))) − (*snd* (*snd* (*r01-binary-expansion″ r n*))) = (*1/2*)ˆ(*Suc n*)
⟨*proof*⟩

lrn = *Sn*.

lemma *r01-binary-expression-eq-lr*:
snd (*snd* (*r01-binary-expansion″ r n*)) = *r01-binary-expression r n*
⟨*proof*⟩

lemma *r01-binary-expression′-sum-range*:
∃ *k*::*nat*. (*snd* (*snd* (*r01-binary-expansion″ r n*))) = *real k*/*2*ˆ(*Suc n*) ∧
k < *2*ˆ(*Suc n*) ∧
((*r01-binary-expansion′ r n*) = *0* ⟶ *even k*) ∧
((*r01-binary-expansion′ r n*) = *1* ⟶ *odd k*)
⟨*proof*⟩

an = *bn* ↔ *Sn* = *S′n*.

lemma *r01-binary-expansion′-expression-eq*:
r01-binary-expansion′ r1 = *r01-binary-expansion′ r2* ⟷
r01-binary-expression r1 = *r01-binary-expression r2*
⟨*proof*⟩

lemma *power2-e*:
⋀*e*::*real*. *0* < *e* ⟹ ∃ *n*::*nat*. *real-of-rat* (*1/2*)ˆ*n* < *e*
⟨*proof*⟩

lemma *r01-binary-expression-converges-to-r*:
assumes *0* < *r*
and *r* < *1*
shows *LIMSEQ* (*r01-binary-expression r*) *r*

$\langle proof \rangle$

lemma *r01-binary-expression-correct*:
  assumes $0 < r$
    and $r < 1$
  shows $r = (\sum n.\ real\ (r01\text{-}binary\text{-}expansion'\ r\ n) * (1/2)\hat{}(Suc\ n))$
$\langle proof \rangle$

$S0 \leq S1 \leq S2 \leq ....$

lemma *binary-sum-incseq*:
  $incseq\ (r01\text{-}binary\text{-}sum\ a)$
  $\langle proof \rangle$

lemma *r01-eq-iff*:
  assumes $0 < r1\ r1 < 1$
    $0 < r2\ r2 < 1$
  shows $r1 = r2 \longleftrightarrow r01\text{-}binary\text{-}expansion'\ r1 = r01\text{-}binary\text{-}expansion'\ r2$
$\langle proof \rangle$

lemma *power-half-summable*:
 $summable\ (\lambda n.\ ((1::real)\ /\ 2)\ \hat{}\ Suc\ n)$
  $\langle proof \rangle$

lemma *binary-expression-summable*:
  assumes $\bigwedge n.\ a\ n \in \{0,1 :: nat\}$
  shows $summable\ (\lambda n.\ real\ (a\ n) * (1/2)\hat{}(Suc\ n))$
$\langle proof \rangle$

lemma *binary-expression-gteq0*:
  assumes $\bigwedge n.\ a\ n \in \{0,1 :: nat\}$
  shows $0 \leq (\sum n.\ real\ (a\ (n + k)) * (1\ /\ 2)\ \hat{}\ Suc\ (n + k))$
$\langle proof \rangle$

lemma *binary-expression-leeq1*:
  assumes $\bigwedge n.\ a\ n \in \{0,1 :: nat\}$
  shows $(\sum n.\ real\ (a\ (n + k)) * (1\ /\ 2)\ \hat{}\ Suc\ (n + k)) \leq 1$
$\langle proof \rangle$

lemma *binary-expression-less-than*:
  assumes $\bigwedge n.\ a\ n \in \{0,1 :: nat\}$
  shows $(\sum n.\ real\ (a\ (n + k)) * (1\ /\ 2)\ \hat{}\ Suc\ (n + k)) \leq (\sum n.\ (1\ /\ 2)\ \hat{}\ Suc\ (n + k))$
$\langle proof \rangle$

lemma *lim-sum-ai*:
  assumes $\bigwedge n.\ a\ n \in \{0,1 :: nat\}$
  shows $lim\ (\lambda n.\ (\sum i{=}0..n.\ real\ (a\ i) * (1/2)\hat{}(Suc\ i))) = (\sum n::nat.\ real\ (a\ n) * (1/2)\hat{}(Suc\ n))$

⟨*proof*⟩

lemma *half-1-minus-sum*:
$1 - (\sum i{<}k.\ ((1::real)\ /\ 2)\ \hat{}\ Suc\ i) = (1/2)\hat{}k$
⟨*proof*⟩

lemma *half-sum*:
$(\sum n.\ ((1::real)\ /\ 2)\ \hat{}\,(Suc\ (n + k))) = (1/2)\hat{}k$
⟨*proof*⟩

lemma *ai-exists0-less-than-sum*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$
$i \geq m$
and $a\ i = 0$
shows $(\sum n::nat.\ real\ (a\ (n + m)) * (1/2)\hat{}(Suc\ (n + m))) < (1\ /\ 2)\ \hat{}\ m$
⟨*proof*⟩

lemma *ai-exists0-less-than1*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$
and $\exists i.\ a\ i = 0$
shows $(\sum n::nat.\ real\ (a\ n) * (1/2)\hat{}(Suc\ n)) < 1$
⟨*proof*⟩

lemma *ai-1-gt*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$
and $a\ i = 1$
shows $(1/2)\hat{}(Suc\ i) \leq (\sum n::nat.\ real\ (a\ (n+i)) * (1/2)\hat{}(Suc\ (n+i)))$
⟨*proof*⟩

lemma *ai-exists1-gt0*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$
and $\exists i.\ a\ i = 1$
shows $0 < (\sum n::nat.\ real\ (a\ n) * (1/2)\hat{}(Suc\ n))$
⟨*proof*⟩

lemma *r01-binary-expression-ex0*:
assumes $0 < r\ r < 1$
shows $\exists i.\ \ r01\text{-}binary\text{-}expansion'\ r\ i = 0$
⟨*proof*⟩

lemma *r01-binary-expression-ex1*:
assumes $0 < r\ r < 1$
shows $\exists i.\ \ r01\text{-}binary\text{-}expansion'\ r\ i = 1$
⟨*proof*⟩

lemma *r01-binary-expansion′-gt1*:
$1 \leq r \longleftrightarrow (\forall n.\ r01\text{-}binary\text{-}expansion'\ r\ n = 1)$
⟨*proof*⟩

lemma *r01-binary-expansion′-lt0*:
$r \leq 0 \longleftrightarrow (\forall n.\ r01\text{-}binary\text{-}expansion'\ r\ n = 0)$
⟨*proof*⟩

The sequence $111111\ldots$ does not appear in $r = 0.a_1a_2\ldots$.

lemma *r01-binary-expression-ex0-strong*:
assumes $0 < r\ r < 1$
shows $\exists\, i \geq n.\ r01\text{-}binary\text{-}expansion'\ r\ i = 0$
⟨*proof*⟩

A binary expression is well-formed when $111\ldots$ does not appear in the tail of the sequence

definition *biexp01-well-formed* :: $(nat \Rightarrow nat) \Rightarrow bool$ **where**
$biexp01\text{-}well\text{-}formed\ a \equiv (\forall n.\ a\ n \in \{0,1\}) \wedge (\forall n.\ \exists\, m \geq n.\ a\ m = 0)$

lemma *biexp01-well-formedE*:
assumes *biexp01-well-formed a*
shows $(\forall n.\ a\ n \in \{0,1\}) \wedge (\forall n.\ \exists\, m \geq n.\ a\ m = 0)$
⟨*proof*⟩

lemma *biexp01-well-formedI*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$
and $\bigwedge n.\ \exists\, m \geq n.\ a\ m = 0$
shows *biexp01-well-formed a*
⟨*proof*⟩

lemma *r01-binary-expansion-well-formed*:
assumes $0 < r\ r < 1$
shows *biexp01-well-formed* (*r01-binary-expansion′ r*)
⟨*proof*⟩

lemma *biexp01-well-formed-comb*:
assumes *biexp01-well-formed a*
and *biexp01-well-formed b*
shows *biexp01-well-formed* $(\lambda n.\ if\ even\ n\ then\ a\ (n\ div\ 2)$
$else\ b\ ((n-1)\ div\ 2))$
⟨*proof*⟩

lemma *nat-complete-induction*:
assumes $P\ (0 :: nat)$
and $\bigwedge n.\ (\bigwedge m.\ m \leq n \Longrightarrow P\ m) \Longrightarrow P\ (Suc\ n)$
shows $P\ n$
⟨*proof*⟩

$(\sum m.\ real\ (a\ m) * (1\ /\ 2)\ \hat{}\ Suc\ m)\ n = a\ n.$

lemma *biexp01-well-formed-an*:

assumes *biexp01-well-formed a*
shows $r01\text{-}binary\text{-}expansion'$ $(\sum m.\ real\ (a\ m) * (1\ /\ 2)\ \hat{}\ Suc\ m)\ n = a\ n$
⟨*proof*⟩

lemma *f01-borel-measurable*:
assumes $f -`\ \{0::real\} \in sets\ real\text{-}borel$
$f -`\ \{1\} \in sets\ borel$
and $\bigwedge r::real.\ f\ r \in \{0,1\}$
shows $f \in borel\text{-}measurable\ real\text{-}borel$
⟨*proof*⟩

lemma *r01-binary-expansion′-measurable*:
$(\lambda r.\ real\ (r01\text{-}binary\text{-}expansion'\ r\ n)) \in borel\text{-}measurable\ (borel :: real\ measure)$
⟨*proof*⟩

definition $r01\text{-}to\text{-}r01\text{-}r01\text{-}fst' :: real \Rightarrow nat \Rightarrow nat$ **where**
$r01\text{-}to\text{-}r01\text{-}r01\text{-}fst'\ r\ n \equiv r01\text{-}binary\text{-}expansion'\ r\ (2*n)$

lemma *r01-to-r01-r01-fst′in01*:
$\bigwedge n.\ r01\text{-}to\text{-}r01\text{-}r01\text{-}fst'\ r\ n \in \{0,1\}$
⟨*proof*⟩

definition $r01\text{-}to\text{-}r01\text{-}r01\text{-}fst\text{-}sum :: real \Rightarrow nat \Rightarrow real$ **where**
$r01\text{-}to\text{-}r01\text{-}r01\text{-}fst\text{-}sum \equiv r01\text{-}binary\text{-}sum \circ r01\text{-}to\text{-}r01\text{-}r01\text{-}fst'$

definition $r01\text{-}to\text{-}r01\text{-}r01\text{-}fst :: real \Rightarrow real$ **where**
$r01\text{-}to\text{-}r01\text{-}r01\text{-}fst = lim \circ r01\text{-}to\text{-}r01\text{-}r01\text{-}fst\text{-}sum$

lemma *r01-to-r01-r01-fst-def′*:
$r01\text{-}to\text{-}r01\text{-}r01\text{-}fst\ r = (\sum n.\ real\ (r01\text{-}binary\text{-}expansion'\ r\ (2*n)) * (1/2)\hat{}(n+1))$
⟨*proof*⟩

lemma *r01-to-r01-r01-fst-measurable*:
$r01\text{-}to\text{-}r01\text{-}r01\text{-}fst \in borel\text{-}measurable\ borel$
⟨*proof*⟩

definition $r01\text{-}to\text{-}r01\text{-}r01\text{-}snd' :: real \Rightarrow nat \Rightarrow nat$ **where**
$r01\text{-}to\text{-}r01\text{-}r01\text{-}snd'\ r\ n = r01\text{-}binary\text{-}expansion'\ r\ (2*n + 1)$

lemma *r01-to-r01-r01-snd′in01*:
$\bigwedge n.\ r01\text{-}to\text{-}r01\text{-}r01\text{-}snd'\ r\ n \in \{0,1\}$
⟨*proof*⟩

definition *r01-to-r01-r01-snd-sum* :: *real* ⇒ *nat* ⇒ *real* **where**
r01-to-r01-r01-snd-sum ≡ *r01-binary-sum* ∘ *r01-to-r01-r01-snd′*

definition *r01-to-r01-r01-snd* :: *real* ⇒ *real* **where**
r01-to-r01-r01-snd = *lim* ∘ *r01-to-r01-r01-snd-sum*

lemma *r01-to-r01-r01-snd-def′*:
r01-to-r01-r01-snd r = ($\sum$ *n*. *real* (*r01-binary-expansion′ r* (*2*∗*n* + *1*)) ∗ (*1*/*2*)^(*n*+*1*))
⟨*proof*⟩

lemma *r01-to-r01-r01-snd-measurable*:
r01-to-r01-r01-snd ∈ *borel-measurable borel*
⟨*proof*⟩

definition *r01-to-r01-r01* :: *real* ⇒ *real* × *real* **where**
r01-to-r01-r01 r = (*r01-to-r01-r01-fst r*,*r01-to-r01-r01-snd r*)

lemma *r01-to-r01-r01-image*:
r01-to-r01-r01 r ∈ {*0*..*1*}×{*0*..*1*}
⟨*proof*⟩

lemma *r01-to-r01-r01-measurable*:
r01-to-r01-r01 ∈ *real-borel* $\rightarrow_M$ *real-borel* $\bigotimes_M$ *real-borel*
⟨*proof*⟩

lemma *r01-to-r01-r01-3over4*:
r01-to-r01-r01 (*3*/*4*) = (*1*/*2*,*1*/*2*)
⟨*proof*⟩

definition *r01-r01-to-r01′* :: *real* × *real* ⇒ *nat* ⇒ *nat* **where**
r01-r01-to-r01′ rs ≡ (λ*n*. *if even n then r01-binary-expansion′* (*fst rs*) (*n div 2*)
else r01-binary-expansion′ (*snd rs*) ((*n*−*1*) *div 2*))

lemma *r01-r01-to-r01′in01*:
⋀*n*. *r01-r01-to-r01′ rs n* ∈ {*0*,*1*}
⟨*proof*⟩

lemma *r01-r01-to-r01′-well-formed*:
assumes *0* < *r1 r1* < *1*
and *0* < *r2 r2* < *1*
shows *biexp01-well-formed* (*r01-r01-to-r01′* (*r1*,*r2*))
⟨*proof*⟩

definition *r01-r01-to-r01-sum* :: *real* × *real* ⇒ *nat* ⇒ *real* **where**
r01-r01-to-r01-sum ≡ *r01-binary-sum* ∘ *r01-r01-to-r01′*

definition *r01-r01-to-r01* :: *real* × *real* ⇒ *real* **where**
r01-r01-to-r01 ≡ *lim* ∘ *r01-r01-to-r01-sum*

lemma *r01-r01-to-r01-def′*:
r01-r01-to-r01 (*r1*,*r2*) = ($\sum$ *n*. *real* (*r01-r01-to-r01′* (*r1*,*r2*) *n*) ∗ (*1*/*2*)^(*n*+*1*))
⟨*proof*⟩

lemma *r01-r01-to-r01-measurable*:
r01-r01-to-r01 ∈ *real-borel* $\bigotimes_M$ *real-borel* $\rightarrow_M$ *real-borel*
⟨*proof*⟩

lemma *r01-r01-to-r01-image*:
assumes *0* < *r1* *r1* < *1*
shows *r01-r01-to-r01* (*r1*,*r2*) ∈ {*0*<..<*1*}
⟨*proof*⟩

lemma *r01-r01-to-r01-image′*:
assumes *0* < *r2* *r2* < *1*
shows *r01-r01-to-r01* (*r1*,*r2*) ∈ {*0*<..<*1*}
⟨*proof*⟩

lemma *r01-r01-to-r01-binary-nth*:
assumes *0* < *r1* *r1* < *1*
and *0* < *r2* *r2* < *1*
shows *r01-binary-expansion′* *r1* *n* = *r01-binary-expansion′* (*r01-r01-to-r01* (*r1*,*r2*)) (*2*∗*n*) ∧
r01-binary-expansion′ *r2* *n* = *r01-binary-expansion′* (*r01-r01-to-r01* (*r1*,*r2*)) (*2*∗*n* + *1*)
⟨*proof*⟩

lemma *r01-r01--r01--r01-r01-id*:
assumes *0* < *r1* *r1* < *1*
0 < *r2* *r2* < *1*
shows (*r01-to-r01-r01* ∘ *r01-r01-to-r01*) (*r1*,*r2*) = (*r1*,*r2*)
⟨*proof*⟩

We first show that $M \bigotimes_M N$ is a standard Borel space for standard Borel spaces M and N.

lemma *pair-measurable*[*measurable*]:
assumes *f* ∈ $X \rightarrow_M Y$
and *g* ∈ $X' \rightarrow_M Y'$
shows *map-prod f g* ∈ $X \bigotimes_M X' \rightarrow_M Y \bigotimes_M Y'$
⟨*proof*⟩

lemma *pair-standard-borel-standard*:
assumes *standard-borel M*
and *standard-borel N*

shows *standard-borel* $(M \bigotimes_M N)$
⟨*proof*⟩

lemma *pair-standard-borel-spaceUNIV*:
assumes *standard-borel-space-UNIV M*
and *standard-borel-space-UNIV N*
shows *standard-borel-space-UNIV* $(M \bigotimes_M N)$
⟨*proof*⟩

locale *pair-standard-borel* = *s1*: *standard-borel M* + *s2*: *standard-borel N*
for M :: *'a measure* **and** N :: *'b measure*
begin

sublocale *standard-borel* $M \bigotimes_M N$
⟨*proof*⟩

end

locale *pair-standard-borel-space-UNIV* = *s1*: *standard-borel-space-UNIV M* + *s2*: *standard-borel-space-UNIV N*
for M :: *'a measure* **and** N :: *'b measure*
begin

sublocale *pair-standard-borel M N*
⟨*proof*⟩

sublocale *standard-borel-space-UNIV* $M \bigotimes_M N$
⟨*proof*⟩

end

$\mathbb{R} \times \mathbb{R}$ is a standard Borel space.

interpretation *real-real* : *pair-standard-borel-space-UNIV real-borel real-borel*
⟨*proof*⟩

1.4 $\mathbb{N} \times \mathbb{R}$

$\mathbb{N} \times \mathbb{R}$ is a standard Borel space.

interpretation *nat-real*: *pair-standard-borel-space-UNIV nat-borel real-borel*
⟨*proof*⟩

end

2 Quasi-Borel Spaces

theory *QuasiBorel*
imports *StandardBorel*

begin

2.1 Definitions

We formalize quasi-Borel spaces introduced by Heunen et al. [1].

2.1.1 Quasi-Borel Spaces

definition *qbs-closed1* :: $(real \Rightarrow 'a)$ *set* $\Rightarrow$ *bool*
 where *qbs-closed1* $Mx \equiv (\forall a \in Mx.\ \forall f \in \textit{real-borel} \rightarrow_M \textit{real-borel}.\ a \circ f \in Mx)$

definition *qbs-closed2* :: $['a\ set, (real \Rightarrow 'a)\ set] \Rightarrow bool$
 where *qbs-closed2* $X\ Mx \equiv (\forall x \in X.\ (\lambda r.\ x) \in Mx)$

definition *qbs-closed3* :: $(real \Rightarrow 'a)$ *set* $\Rightarrow$ *bool*
 where *qbs-closed3* $Mx \equiv (\forall P{::}real \Rightarrow nat.\ \forall Fi{::}nat \Rightarrow real \Rightarrow 'a.$
$(\forall i.\ P -` \{i\} \in \textit{sets real-borel})$
$\longrightarrow (\forall i.\ Fi\ i \in Mx)$
$\longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx)$

lemma *separate-measurable*:
 fixes P :: $real \Rightarrow nat$
 assumes $\bigwedge i.\ P -` \{i\} \in \textit{sets real-borel}$
 shows $P \in \textit{real-borel} \rightarrow_M \textit{nat-borel}$
⟨*proof*⟩

lemma *measurable-separate*:
 fixes P :: $real \Rightarrow nat$
 assumes $P \in \textit{real-borel} \rightarrow_M \textit{nat-borel}$
 shows $P -` \{i\} \in \textit{sets real-borel}$
 ⟨*proof*⟩

definition *is-quasi-borel* $X\ Mx \longleftrightarrow Mx \subseteq UNIV \rightarrow X \wedge$ *qbs-closed1* $Mx \wedge$ *qbs-closed2* $X\ Mx \wedge$ *qbs-closed3* Mx

lemma *is-quasi-borel-intro*[*simp*]:
 assumes $Mx \subseteq UNIV \rightarrow X$
  and *qbs-closed1* Mx *qbs-closed2* $X\ Mx$ *qbs-closed3* Mx
 shows *is-quasi-borel* $X\ Mx$
 ⟨*proof*⟩

typedef $'a$ *quasi-borel* $= \{(X{::}'a\ set, Mx).\ \textit{is-quasi-borel}\ X\ Mx\}$
⟨*proof*⟩

definition *qbs-space* :: $'a$ *quasi-borel* $\Rightarrow 'a$ *set* **where**
 qbs-space $X \equiv fst\ (\textit{Rep-quasi-borel}\ X)$

definition *qbs-Mx* :: $'a$ *quasi-borel* $\Rightarrow (real \Rightarrow 'a)$ *set* **where**
 qbs-Mx $X \equiv snd\ (\textit{Rep-quasi-borel}\ X)$

lemma *qbs-decomp* :
$(qbs\text{-}space\ X, qbs\text{-}Mx\ X) \in \{(X{::}'a\ set,\ Mx).\ is\text{-}quasi\text{-}borel\ X\ Mx\}$
⟨*proof*⟩

lemma *qbs-Mx-to-X*[*dest*]:
assumes $\alpha \in qbs\text{-}Mx\ X$
shows $\alpha \in UNIV \rightarrow qbs\text{-}space\ X$
$\alpha\ r \in qbs\text{-}space\ X$
⟨*proof*⟩

lemma *qbs-closed1I*:
assumes $\bigwedge\alpha\ f.\ \alpha \in Mx \Longrightarrow f \in real\text{-}borel \rightarrow_M real\text{-}borel \Longrightarrow \alpha \circ f \in Mx$
shows *qbs-closed1 Mx*
⟨*proof*⟩

lemma *qbs-closed1-dest*[*simp*]:
assumes $\alpha \in qbs\text{-}Mx\ X$
and $f \in real\text{-}borel \rightarrow_M real\text{-}borel$
shows $\alpha \circ f \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-closed2I*:
assumes $\bigwedge x.\ x \in X \Longrightarrow (\lambda r.\ x) \in Mx$
shows *qbs-closed2 X Mx*
⟨*proof*⟩

lemma *qbs-closed2-dest*[*simp*]:
assumes $x \in qbs\text{-}space\ X$
shows $(\lambda r.\ x) \in qbs\text{-}Mx\ X$
⟨*proof*⟩

lemma *qbs-closed3I*:
assumes $\bigwedge(P :: real \Rightarrow nat)\ Fi.\ (\bigwedge i.\ P -` \{i\} \in sets\ real\text{-}borel) \Longrightarrow (\bigwedge i.\ Fi\ i \in Mx)$
$\Longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx$
shows *qbs-closed3 Mx*
⟨*proof*⟩

lemma *qbs-closed3I'*:
assumes $\bigwedge(P :: real \Rightarrow nat)\ Fi.\ P \in real\text{-}borel \rightarrow_M nat\text{-}borel \Longrightarrow (\bigwedge i.\ Fi\ i \in Mx)$
$\Longrightarrow (\lambda r.\ Fi\ (P\ r)\ r) \in Mx$
shows *qbs-closed3 Mx*
⟨*proof*⟩

lemma *qbs-closed3-dest*[*simp*]:
fixes $P{::}real \Rightarrow nat$ **and** $Fi :: nat \Rightarrow real \Rightarrow$ -

assumes $\bigwedge i.\ P -` \{i\} \in$ *sets real-borel*
and $\bigwedge i.$ *Fi i* $\in$ *qbs-Mx X*
shows $(\lambda r.$ *Fi* $(P\ r)\ r) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed3-dest′*:
fixes *P*::*real* $\Rightarrow$ *nat* **and** *Fi* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ -
assumes $P \in$ *real-borel* $\rightarrow_M$ *nat-borel*
and $\bigwedge i.$ *Fi i* $\in$ *qbs-Mx X*
shows $(\lambda r.$ *Fi* $(P\ r)\ r) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed3-dest2*:
assumes *countable I*
and [*measurable*]: $P \in$ *real-borel* $\rightarrow_M$ *count-space I*
and $\bigwedge i.\ i \in I \Longrightarrow$ *Fi i* $\in$ *qbs-Mx X*
shows $(\lambda r.$ *Fi* $(P\ r)\ r) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-closed3-dest2′*:
assumes *countable I*
and [*measurable*]: $P \in$ *real-borel* $\rightarrow_M$ *count-space I*
and $\bigwedge i.\ i \in$ *range P* $\Longrightarrow$ *Fi i* $\in$ *qbs-Mx X*
shows $(\lambda r.$ *Fi* $(P\ r)\ r) \in$ *qbs-Mx X*
⟨*proof*⟩

lemma *qbs-space-Mx*:
qbs-space X $= \{\alpha\ x\ | x\ \alpha.\ \alpha \in$ *qbs-Mx X*$\}$
⟨*proof*⟩

lemma *qbs-space-eq-Mx*:
assumes *qbs-Mx X* = *qbs-Mx Y*
shows *qbs-space X* = *qbs-space Y*
⟨*proof*⟩

lemma *qbs-eqI*:
assumes *qbs-Mx X* = *qbs-Mx Y*
shows *X* = *Y*
⟨*proof*⟩

2.1.2 Morphism of Quasi-Borel Spaces

definition *qbs-morphism* :: [*′a quasi-borel*, *′b quasi-borel*] $\Rightarrow$ (*′a* $\Rightarrow$ *′b*) *set* (**infixr** $\rightarrow_Q$ *60*) **where**
$X \rightarrow_Q Y \equiv \{f \in$ *qbs-space X* $\rightarrow$ *qbs-space Y*. $\forall\alpha \in$ *qbs-Mx X*. $f \circ \alpha \in$ *qbs-Mx Y*$\}$

lemma *qbs-morphismI*:
assumes $\bigwedge\alpha.\ \alpha \in$ *qbs-Mx X* $\Longrightarrow f \circ \alpha \in$ *qbs-Mx Y*

shows $f \in X \rightarrow_Q Y$
⟨*proof*⟩

lemma *qbs-morphismE*[*dest*]:
assumes $f \in X \rightarrow_Q Y$
shows $f \in \textit{qbs-space } X \rightarrow \textit{qbs-space } Y$
$\bigwedge x.\ x \in \textit{qbs-space } X \Longrightarrow f\ x \in \textit{qbs-space } Y$
$\bigwedge \alpha.\ \alpha \in \textit{qbs-Mx } X \Longrightarrow f \circ \alpha \in \textit{qbs-Mx } Y$
⟨*proof*⟩

lemma *qbs-morphism-ident*[*simp*]:
$id \in X \rightarrow_Q X$
⟨*proof*⟩

lemma *qbs-morphism-ident'*[*simp*]:
$(\lambda x.\ x) \in X \rightarrow_Q X$
⟨*proof*⟩

lemma *qbs-morphism-comp*:
assumes $f \in X \rightarrow_Q Y$ $g \in Y \rightarrow_Q Z$
shows $g \circ f \in X \rightarrow_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-cong*:
assumes $\bigwedge x.\ x \in \textit{qbs-space } X \Longrightarrow f\ x = g\ x$
and $f \in X \rightarrow_Q Y$
shows $g \in X \rightarrow_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-const*:
assumes $y \in \textit{qbs-space } Y$
shows $(\lambda\text{-}.\ y) \in X \rightarrow_Q Y$
⟨*proof*⟩

2.1.3 Empty Space

definition *empty-quasi-borel* :: *'a quasi-borel* **where**
empty-quasi-borel $\equiv$ *Abs-quasi-borel* $(\{\},\{\})$

lemma *eqb-correct*: *Rep-quasi-borel empty-quasi-borel* $= (\{\}, \{\})$
⟨*proof*⟩

lemma *eqb-space*[*simp*]: *qbs-space empty-quasi-borel* $= \{\}$
⟨*proof*⟩

lemma *eqb-Mx*[*simp*]: *qbs-Mx empty-quasi-borel* $= \{\}$
⟨*proof*⟩

lemma *qbs-empty-equiv* :*qbs-space* $X = \{\} \longleftrightarrow$ *qbs-Mx* $X = \{\}$
⟨*proof*⟩

lemma *empty-quasi-borel-iff*:
qbs-space $X = \{\} \longleftrightarrow X =$ *empty-quasi-borel*
⟨*proof*⟩

2.1.4 Unit Space

definition *unit-quasi-borel* :: *unit quasi-borel* (1_Q) **where**
unit-quasi-borel ≡ *Abs-quasi-borel* (*UNIV*,*UNIV*)

lemma *uqb-correct*: *Rep-quasi-borel unit-quasi-borel* = (*UNIV*,*UNIV*)
⟨*proof*⟩

lemma *uqb-space*[*simp*]: *qbs-space unit-quasi-borel* = {()}
⟨*proof*⟩

lemma *uqb-Mx*[*simp*]: *qbs-Mx unit-quasi-borel* = $\{\lambda r.\ ()\}$
⟨*proof*⟩

lemma *unit-quasi-borel-terminal*:
$\exists!\ f.\ f \in X \rightarrow_Q$ *unit-quasi-borel*
⟨*proof*⟩

definition *to-unit-quasi-borel* :: $'a \Rightarrow$ *unit* ($!_Q$) **where**
to-unit-quasi-borel ≡ (λ-.())

lemma *to-unit-quasi-borel-morphism* :
$!_Q \in X \rightarrow_Q$ *unit-quasi-borel*
⟨*proof*⟩

2.1.5 Subspaces

definition *sub-qbs* :: [$'a$ *quasi-borel*, $'a$ *set*] ⇒ $'a$ *quasi-borel* **where**
sub-qbs X U ≡ *Abs-quasi-borel* (*qbs-space* $X \cap U$,{$f \in$ *UNIV* → *qbs-space* $X \cap U$. $f \in$ *qbs-Mx* X})

lemma *sub-qbs-closed*:
qbs-closed1 {$f \in$ *UNIV* → *qbs-space* $X \cap U$. $f \in$ *qbs-Mx* X}
qbs-closed2 (*qbs-space* $X \cap U$) {$f \in$ *UNIV* → *qbs-space* $X \cap U$. $f \in$ *qbs-Mx* X}
qbs-closed3 {$f \in$ *UNIV* → *qbs-space* $X \cap U$. $f \in$ *qbs-Mx* X}
⟨*proof*⟩

lemma *sub-qbs-correct*[*simp*]: *Rep-quasi-borel* (*sub-qbs* X U) = (*qbs-space* $X \cap U$,{$f \in$ *UNIV* → *qbs-space* $X \cap U$. $f \in$ *qbs-Mx* X})
⟨*proof*⟩

lemma *sub-qbs-space*[*simp*]: *qbs-space* (*sub-qbs* X U) = *qbs-space* $X \cap U$
⟨*proof*⟩

lemma *sub-qbs-Mx*[*simp*]: *qbs-Mx* (*sub-qbs X U*) = {*f* ∈ *UNIV* → *qbs-space X* ∩ *U*. *f* ∈ *qbs-Mx X*}
⟨*proof*⟩

lemma *sub-qbs*:
assumes *U* ⊆ *qbs-space X*
shows (*qbs-space* (*sub-qbs X U*), *qbs-Mx* (*sub-qbs X U*)) = (*U*, {*f* ∈ *UNIV* → *U*. *f* ∈ *qbs-Mx X*})
⟨*proof*⟩

2.1.6 Image Spaces

definition *map-qbs* :: [′*a* ⇒ ′*b*] ⇒ ′*a quasi-borel* ⇒ ′*b quasi-borel* **where**
map-qbs f X = *Abs-quasi-borel* (*f* ‘ (*qbs-space X*),{β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α})

lemma *map-qbs-f*:
{β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α} ⊆ *UNIV* → *f* ‘ (*qbs-space X*)
⟨*proof*⟩

lemma *map-qbs-closed1*:
qbs-closed1 {β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α}
⟨*proof*⟩

lemma *map-qbs-closed2*:
qbs-closed2 (*f* ‘ (*qbs-space X*)) {β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α}
⟨*proof*⟩

lemma *map-qbs-closed3*:
qbs-closed3 {β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α}
⟨*proof*⟩

lemma *map-qbs-correct*[*simp*]:
Rep-quasi-borel (*map-qbs f X*) = (*f* ‘ (*qbs-space X*),{β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α})
⟨*proof*⟩

lemma *map-qbs-space*[*simp*]:
qbs-space (*map-qbs f X*) = *f* ‘ (*qbs-space X*)
⟨*proof*⟩

lemma *map-qbs-Mx*[*simp*]:
qbs-Mx (*map-qbs f X*) = {β. ∃α∈ *qbs-Mx X*. β = *f* ∘ α}
⟨*proof*⟩

inductive-set *generating-Mx* :: ′*a set* ⇒ (*real* ⇒ ′*a*) *set* ⇒ (*real* ⇒ ′*a*) *set*
for *X* :: ′*a set* **and** *Mx* :: (*real* ⇒ ′*a*) *set*
where

Basic: $\alpha \in Mx \Longrightarrow \alpha \in$ *generating-Mx X Mx*
| *Const*: $x \in X \Longrightarrow (\lambda r.\ x) \in$ *generating-Mx X Mx*
| *Comp* : $f \in$ *real-borel* $\rightarrow_M$ *real-borel* $\Longrightarrow \alpha \in$ *generating-Mx X Mx* $\Longrightarrow \alpha \circ f \in$ *generating-Mx X Mx*
| *Part* : $(\bigwedge i.\ Fi\ i \in$ *generating-Mx X Mx*$) \Longrightarrow P \in$ *real-borel* $\rightarrow_M$ *nat-borel* $\Longrightarrow$ $(\lambda r.\ Fi\ (P\ r)\ r) \in$ *generating-Mx X Mx*

lemma *generating-Mx-to-space*:
assumes $Mx \subseteq UNIV \rightarrow X$
shows *generating-Mx X Mx* $\subseteq UNIV \rightarrow X$
⟨*proof*⟩

lemma *generating-Mx-closed1*:
qbs-closed1 (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-closed2*:
qbs-closed2 X (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-closed3*:
qbs-closed3 (*generating-Mx X Mx*)
⟨*proof*⟩

lemma *generating-Mx-Mx*:
generating-Mx (*qbs-space X*) (*qbs-Mx X*) = *qbs-Mx X*
⟨*proof*⟩

2.1.7 Ordering of Quasi-Borel Spaces

instantiation *quasi-borel* :: (*type*) *order-bot*
begin

inductive *less-eq-quasi-borel* :: *'a quasi-borel* ⇒ *'a quasi-borel* ⇒ *bool* **where**
qbs-space X ⊂ *qbs-space Y* ⟹ *less-eq-quasi-borel X Y*
| *qbs-space X* = *qbs-space Y* ⟹ *qbs-Mx Y* ⊆ *qbs-Mx X* ⟹ *less-eq-quasi-borel X Y*

lemma *le-quasi-borel-iff*:
$X \le Y \longleftrightarrow$ (*if qbs-space X* = *qbs-space Y then qbs-Mx Y* ⊆ *qbs-Mx X else qbs-space X* ⊂ *qbs-space Y*)
⟨*proof*⟩

definition *less-quasi-borel* :: *'a quasi-borel* ⇒ *'a quasi-borel* ⇒ *bool* **where**
less-quasi-borel X Y $\longleftrightarrow (X \le Y \land \neg\ Y \le X)$

definition *bot-quasi-borel* :: *'a quasi-borel* **where**
bot-quasi-borel = *empty-quasi-borel*

instance
⟨*proof*⟩
end

definition *inf-quasi-borel* :: $['a\ quasi\text{-}borel, 'a\ quasi\text{-}borel] \Rightarrow 'a\ quasi\text{-}borel$ **where**
$inf\text{-}quasi\text{-}borel\ X\ X' = Abs\text{-}quasi\text{-}borel\ (qbs\text{-}space\ X \cap qbs\text{-}space\ X',\ qbs\text{-}Mx\ X \cap qbs\text{-}Mx\ X')$

lemma *inf-quasi-borel-correct*: $Rep\text{-}quasi\text{-}borel\ (inf\text{-}quasi\text{-}borel\ X\ X') = (qbs\text{-}space\ X \cap qbs\text{-}space\ X',\ qbs\text{-}Mx\ X \cap qbs\text{-}Mx\ X')$
⟨*proof*⟩

lemma *inf-qbs-space*[*simp*]: $qbs\text{-}space\ (inf\text{-}quasi\text{-}borel\ X\ X') = qbs\text{-}space\ X \cap qbs\text{-}space\ X'$
⟨*proof*⟩

lemma *inf-qbs-Mx*[*simp*]: $qbs\text{-}Mx\ (inf\text{-}quasi\text{-}borel\ X\ X') = qbs\text{-}Mx\ X \cap qbs\text{-}Mx\ X'$
⟨*proof*⟩

definition *max-quasi-borel* :: $'a\ set \Rightarrow 'a\ quasi\text{-}borel$ **where**
$max\text{-}quasi\text{-}borel\ X = Abs\text{-}quasi\text{-}borel\ (X,\ UNIV \to X)$

lemma *max-quasi-borel-correct*: $Rep\text{-}quasi\text{-}borel\ (max\text{-}quasi\text{-}borel\ X) = (X,\ UNIV \to X)$
⟨*proof*⟩

lemma *max-qbs-space*[*simp*]: $qbs\text{-}space\ (max\text{-}quasi\text{-}borel\ X) = X$
⟨*proof*⟩

lemma *max-qbs-Mx*[*simp*]: $qbs\text{-}Mx\ (max\text{-}quasi\text{-}borel\ X) = UNIV \to X$
⟨*proof*⟩

instantiation *quasi-borel* :: (*type*) *semilattice-sup*
begin

definition *sup-quasi-borel* :: $'a\ quasi\text{-}borel \Rightarrow 'a\ quasi\text{-}borel \Rightarrow 'a\ quasi\text{-}borel$ **where**
$sup\text{-}quasi\text{-}borel\ X\ Y \equiv (if\ qbs\text{-}space\ X = qbs\text{-}space\ Y\quad then\ inf\text{-}quasi\text{-}borel\ X\ Y$
$else\ if\ qbs\text{-}space\ X \subset qbs\text{-}space\ Y\ then\ Y$
$else\ if\ qbs\text{-}space\ Y \subset qbs\text{-}space\ X\ then\ X$
$else\ max\text{-}quasi\text{-}borel\ (qbs\text{-}space\ X \cup qbs\text{-}space\ Y))$

instance
⟨*proof*⟩
end

end

2.2 Relation to Measurable Spaces

theory *Measure-QuasiBorel-Adjunction*
imports *QuasiBorel*
begin

We construct the adjunction between **Meas** and **QBS**, where **Meas** is the category of measurable spaces and measurable functions and **QBS** is the category of quasi-Borel spaces and morphisms.

2.2.1 The Functor R

definition *measure-to-qbs* :: $'a$ *measure* $\Rightarrow$ $'a$ *quasi-borel* **where**
measure-to-qbs $X \equiv$ *Abs-quasi-borel* (*space* X, *real-borel* $\rightarrow_M X$)

lemma *R-Mx-correct*: *real-borel* $\rightarrow_M X \subseteq$ *UNIV* $\rightarrow$ *space* X
⟨*proof*⟩

lemma *R-qbs-closed1*: *qbs-closed1* (*real-borel* $\rightarrow_M X$)
⟨*proof*⟩

lemma *R-qbs-closed2*: *qbs-closed2* (*space* X) (*real-borel* $\rightarrow_M X$)
⟨*proof*⟩

lemma *R-qbs-closed3*: *qbs-closed3* (*real-borel* $\rightarrow_M$ (X :: $'a$ *measure*))
⟨*proof*⟩

lemma *R-correct*[*simp*]: *Rep-quasi-borel* (*measure-to-qbs* X) = (*space* X, *real-borel* $\rightarrow_M X$)
⟨*proof*⟩

lemma *qbs-space-R*[*simp*]: *qbs-space* (*measure-to-qbs* X) = *space* X
⟨*proof*⟩

lemma *qbs-Mx-R*[*simp*]: *qbs-Mx* (*measure-to-qbs* X) = *real-borel* $\rightarrow_M X$
⟨*proof*⟩

The following lemma says that *measure-to-qbs* is a functor from **Meas** to **QBS**.

lemma *r-preserves-morphisms*:
$X \rightarrow_M Y \subseteq$ (*measure-to-qbs* X) $\rightarrow_Q$ (*measure-to-qbs* Y)
⟨*proof*⟩

2.2.2 The Functor L

definition *sigma-Mx* :: $'a$ *quasi-borel* $\Rightarrow$ $'a$ *set set* **where**
sigma-Mx $X \equiv \{U \cap$ *qbs-space* X $|U.\ \forall \alpha \in$ *qbs-Mx* $X.\ \alpha$ −‘ $U \in$ *sets real-borel*$\}$

definition *qbs-to-measure* :: $'a$ *quasi-borel* $\Rightarrow$ $'a$ *measure* **where**

qbs-to-measure $X \equiv$ *Abs-measure* (*qbs-space* X, *sigma-Mx* X, λA. (*if* $A = \{\}$ *then 0 else if* $A \in -$ *sigma-Mx* X *then 0 else* ∞))

lemma *measure-space-L*: *measure-space* (*qbs-space* X) (*sigma-Mx* X) (λA. (*if* $A = \{\}$ *then 0 else if* $A \in -$ *sigma-Mx* X *then 0 else* ∞))
⟨*proof*⟩

lemma *L-correct*[*simp*]:*Rep-measure* (*qbs-to-measure* X) = (*qbs-space* X, *sigma-Mx* X, λA. (*if* $A = \{\}$ *then 0 else if* $A \in -$ *sigma-Mx* X *then 0 else* ∞))
⟨*proof*⟩

lemma *space-L*[*simp*]: *space* (*qbs-to-measure* X) = *qbs-space* X
⟨*proof*⟩

lemma *sets-L*[*simp*]: *sets* (*qbs-to-measure* X) = *sigma-Mx* X
⟨*proof*⟩

lemma *emeasure-L*[*simp*]: *emeasure* (*qbs-to-measure* X) = (λA. *if* $A = \{\} \vee A \notin$ *sigma-Mx* X *then 0 else* ∞)
⟨*proof*⟩

lemma *qbs-Mx-sigma-Mx-contra*:
assumes *qbs-space* X = *qbs-space* Y
and *qbs-Mx* $X \subseteq$ *qbs-Mx* Y
shows *sigma-Mx* $Y \subseteq$ *sigma-Mx* X
⟨*proof*⟩

The following lemma says that *qbs-to-measure* is a functor from **QBS** to **Meas**.

lemma *l-preserves-morphisms*:
$X \to_Q Y \subseteq$ (*qbs-to-measure* X) $\to_M$ (*qbs-to-measure* Y)
⟨*proof*⟩

abbreviation *qbs-borel* $\equiv$ *measure-to-qbs borel*

declare [[*coercion measure-to-qbs*]]

abbreviation *real-quasi-borel* :: *real quasi-borel* ($\mathbb{R}_Q$) **where**
real-quasi-borel $\equiv$ *qbs-borel*
abbreviation *nat-quasi-borel* :: *nat quasi-borel* ($\mathbb{N}_Q$) **where**
nat-quasi-borel $\equiv$ *qbs-borel*
abbreviation *ennreal-quasi-borel* :: *ennreal quasi-borel* ($\mathbb{R}_{Q\geq 0}$) **where**
ennreal-quasi-borel $\equiv$ *qbs-borel*
abbreviation *bool-quasi-borel* :: *bool quasi-borel* ($\mathbb{B}_Q$) **where**
bool-quasi-borel $\equiv$ *qbs-borel*

lemma *qbs-Mx-is-morphisms*:
$\textit{qbs-Mx}\ X = \textit{real-quasi-borel} \to_Q X$
⟨*proof*⟩

lemma *qbs-Mx-subset-of-measurable*:
$\textit{qbs-Mx}\ X \subseteq \textit{real-borel} \to_M \textit{qbs-to-measure}\ X$
⟨*proof*⟩

lemma *L-max-of-measurables*:
assumes $\textit{space}\ M = \textit{qbs-space}\ X$
and $\textit{qbs-Mx}\ X \subseteq \textit{real-borel} \to_M M$
shows $\textit{sets}\ M \subseteq \textit{sets}\ (\textit{qbs-to-measure}\ X)$
⟨*proof*⟩

lemma *qbs-Mx-are-measurable*[*simp*,*measurable*]:
assumes $\alpha \in \textit{qbs-Mx}\ X$
shows $\alpha \in \textit{real-borel} \to_M \textit{qbs-to-measure}\ X$
⟨*proof*⟩

lemma *measure-to-qbs-cong-sets*:
assumes $\textit{sets}\ M = \textit{sets}\ N$
shows $\textit{measure-to-qbs}\ M = \textit{measure-to-qbs}\ N$
⟨*proof*⟩

lemma *lr-sets*[*simp*,*measurable-cong*]:
$\textit{sets}\ X \subseteq \textit{sets}\ (\textit{qbs-to-measure}\ (\textit{measure-to-qbs}\ X))$
⟨*proof*⟩

lemma(**in** *standard-borel*) *standard-borel-lr-sets-ident*[*simp*, *measurable-cong*]:
$\textit{sets}\ (\textit{qbs-to-measure}\ (\textit{measure-to-qbs}\ M)) = \textit{sets}\ M$
⟨*proof*⟩

2.2.3 The Adjunction

lemma *lr-adjunction-correspondence* :
$X \to_Q (\textit{measure-to-qbs}\ Y) = (\textit{qbs-to-measure}\ X) \to_M Y$
⟨*proof*⟩

lemma(**in** *standard-borel*) *standard-borel-r-full-faithful*:
$M \to_M Y = \textit{measure-to-qbs}\ M \to_Q \textit{measure-to-qbs}\ Y$
⟨*proof*⟩

lemma *qbs-morphism-dest*[*dest*]:
assumes $f \in X \to_Q \textit{measure-to-qbs}\ Y$
shows $f \in \textit{qbs-to-measure}\ X \to_M Y$
⟨*proof*⟩

lemma(**in** *standard-borel*) *qbs-morphism-dest*[*dest*]:

assumes $k \in$ *measure-to-qbs* $M \rightarrow_Q$ *measure-to-qbs* Y
shows $k \in M \rightarrow_M Y$
⟨*proof*⟩

lemma *qbs-morphism-measurable-intro*[*intro*!]:
assumes $f \in$ *qbs-to-measure* $X \rightarrow_M Y$
shows $f \in X \rightarrow_Q$ *measure-to-qbs* Y
⟨*proof*⟩

lemma(**in** *standard-borel*) *qbs-morphism-measurable-intro*[*intro*!]:
assumes $k \in M \rightarrow_M Y$
shows $k \in$ *measure-to-qbs* $M \rightarrow_Q$ *measure-to-qbs* Y
⟨*proof*⟩

We can use the measurability prover when we reason about morphisms.

lemma
assumes $f \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$
shows $(\lambda x.\ 2 * f\ x + (f\ x)\hat{\ }2) \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma
assumes $f \in X \rightarrow_Q \mathbb{R}_Q$
and $\alpha \in$ *qbs-Mx* X
shows $(\lambda x.\ 2 * f\ (\alpha\ x) + (f\ (\alpha\ x))\hat{\ }2) \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-morphisn-from-countable*:
fixes X :: *'a quasi-borel*
assumes *countable* (*qbs-space* X)
qbs-Mx $X \subseteq$ *real-borel* $\rightarrow_M$ *count-space* (*qbs-space* X)
and $\bigwedge i.\ i \in$ *qbs-space* $X \Longrightarrow f\ i \in$ *qbs-space* Y
shows $f \in X \rightarrow_Q Y$
⟨*proof*⟩

corollary *nat-qbs-morphism*:
assumes $\bigwedge n.\ f\ n \in$ *qbs-space* Y
shows $f \in \mathbb{N}_Q \rightarrow_Q Y$
⟨*proof*⟩

corollary *bool-qbs-morphism*:
assumes $\bigwedge b.\ f\ b \in$ *qbs-space* Y
shows $f \in \mathbb{B}_Q \rightarrow_Q Y$
⟨*proof*⟩

2.2.4 The Adjunction w.r.t. Ordering

lemma *l-mono*:
mono qbs-to-measure

⟨*proof*⟩

lemma *r-mono*:
mono measure-to-qbs
⟨*proof*⟩

lemma *rl-order-adjunction*:
$X \leq$ *qbs-to-measure* $Y \longleftrightarrow$ *measure-to-qbs* $X \leq Y$
⟨*proof*⟩

end

2.3 Product Spaces

theory *Binary-Product-QuasiBorel*
imports *Measure-QuasiBorel-Adjunction*
begin

2.3.1 Binary Product Spaces

definition *pair-qbs-Mx* :: $['a$ *quasi-borel*, $'b$ *quasi-borel*$] \Rightarrow$ (*real* => $'a \times 'b$) *set*
where
pair-qbs-Mx X $Y \equiv \{f.$ *fst* $\circ f \in$ *qbs-Mx* $X \wedge$ *snd* $\circ f \in$ *qbs-Mx* $Y\}$

definition *pair-qbs* :: $['a$ *quasi-borel*, $'b$ *quasi-borel*$] \Rightarrow ('a \times 'b)$ *quasi-borel* (**infixr** $\bigotimes_Q$ *80*) **where**
pair-qbs X Y = *Abs-quasi-borel* (*qbs-space* $X \times$ *qbs-space* Y, *pair-qbs-Mx* X Y)

lemma *pair-qbs-f*[*simp*]: *pair-qbs-Mx* X $Y \subseteq$ *UNIV* $\rightarrow$ *qbs-space* $X \times$ *qbs-space* Y
⟨*proof*⟩

lemma *pair-qbs-closed1*: *qbs-closed1* (*pair-qbs-Mx* (X::$'a$ *quasi-borel*) (Y::$'b$ *quasi-borel*))
⟨*proof*⟩

lemma *pair-qbs-closed2*: *qbs-closed2* (*qbs-space* $X \times$ *qbs-space* Y) (*pair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *pair-qbs-closed3*: *qbs-closed3* (*pair-qbs-Mx* (X::$'a$ *quasi-borel*) (Y::$'b$ *quasi-borel*))
⟨*proof*⟩

lemma *pair-qbs-correct*: *Rep-quasi-borel* ($X \bigotimes_Q Y$) = (*qbs-space* $X \times$ *qbs-space* Y, *pair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *pair-qbs-space*[*simp*]: *qbs-space* ($X \bigotimes_Q Y$) = *qbs-space* $X \times$ *qbs-space* Y
⟨*proof*⟩

lemma *pair-qbs-Mx*[*simp*]: *qbs-Mx* ($X \bigotimes_Q Y$) = *pair-qbs-Mx* X Y

$\langle proof \rangle$

lemma *pair-qbs-morphismI*:
assumes $\bigwedge\alpha\ \beta.\ \alpha \in qbs\text{-}Mx\ X \Longrightarrow \beta \in qbs\text{-}Mx\ Y$
$\Longrightarrow f \circ (\lambda r.\ (\alpha\ r,\ \beta\ r)) \in qbs\text{-}Mx\ Z$
shows $f \in (X \bigotimes_Q Y) \rightarrow_Q Z$
$\langle proof \rangle$

lemma *fst-qbs-morphism*:
$fst \in X \bigotimes_Q Y \rightarrow_Q X$
$\langle proof \rangle$

lemma *snd-qbs-morphism*:
$snd \in X \bigotimes_Q Y \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-pair-iff*:
$f \in X \rightarrow_Q Y \bigotimes_Q Z \longleftrightarrow fst \circ f \in X \rightarrow_Q Y \wedge snd \circ f \in X \rightarrow_Q Z$
$\langle proof \rangle$

lemma *qbs-morphism-Pair1*:
assumes $x \in qbs\text{-}space\ X$
shows $Pair\ x \in Y \rightarrow_Q X \bigotimes_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-Pair1'*:
assumes $x \in qbs\text{-}space\ X$
and $f \in X \bigotimes_Q Y \rightarrow_Q Z$
shows $(\lambda y.\ f\ (x,y)) \in Y \rightarrow_Q Z$
$\langle proof \rangle$

lemma *qbs-morphism-Pair2*:
assumes $y \in qbs\text{-}space\ Y$
shows $(\lambda x.\ (x,y)) \in X \rightarrow_Q X \bigotimes_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-Pair2'*:
assumes $y \in qbs\text{-}space\ Y$
and $f \in X \bigotimes_Q Y \rightarrow_Q Z$
shows $(\lambda x.\ f\ (x,y)) \in X \rightarrow_Q Z$
$\langle proof \rangle$

lemma *qbs-morphism-fst''*:
assumes $f \in X \rightarrow_Q Y$
shows $(\lambda k.\ f\ (fst\ k)) \in X \bigotimes_Q Z \rightarrow_Q Y$
$\langle proof \rangle$

lemma *qbs-morphism-snd''*:
  assumes $f \in X \to_Q Y$
  shows $(\lambda k.\ f\ (snd\ k)) \in Z \bigotimes_Q X \to_Q Y$
  ⟨*proof*⟩

lemma *qbs-morphism-tuple*:
  assumes $f \in Z \to_Q X$
    and $g \in Z \to_Q Y$
   shows $(\lambda z.\ (f\ z,\ g\ z)) \in Z \to_Q X \bigotimes_Q Y$
⟨*proof*⟩

lemma *qbs-morphism-map-prod*:
  assumes $f \in X \to_Q Y$
    and $g \in X' \to_Q Y'$
   shows $map\text{-}prod\ f\ g \in X \bigotimes_Q X' \to_Q Y \bigotimes_Q Y'$
⟨*proof*⟩

lemma *qbs-morphism-pair-swap'*:
  $(\lambda(x,y).\ (y,x)) \in (X::'a\ quasi\text{-}borel) \bigotimes_Q (Y::'b\ quasi\text{-}borel) \to_Q Y \bigotimes_Q X$
  ⟨*proof*⟩

lemma *qbs-morphism-pair-swap*:
  assumes $f \in X \bigotimes_Q Y \to_Q Z$
  shows $(\lambda(x,y).\ f\ (y,x)) \in Y \bigotimes_Q X \to_Q Z$
⟨*proof*⟩

lemma *qbs-morphism-pair-assoc1*:
 $(\lambda((x,y),z).\ (x,(y,z))) \in (X \bigotimes_Q Y) \bigotimes_Q Z \to_Q X \bigotimes_Q (Y \bigotimes_Q Z)$
  ⟨*proof*⟩

lemma *qbs-morphism-pair-assoc2*:
 $(\lambda(x,(y,z)).\ ((x,y),z)) \in X \bigotimes_Q (Y \bigotimes_Q Z) \to_Q (X \bigotimes_Q Y) \bigotimes_Q Z$
  ⟨*proof*⟩

lemma *pair-qbs-fst*:
  assumes $qbs\text{-}space\ Y \neq \{\}$
  shows $map\text{-}qbs\ fst\ (X \bigotimes_Q Y) = X$
⟨*proof*⟩

lemma *pair-qbs-snd*:
  assumes $qbs\text{-}space\ X \neq \{\}$
  shows $map\text{-}qbs\ snd\ (X \bigotimes_Q Y) = Y$
⟨*proof*⟩

The following lemma corresponds to [1] Proposition 19(1).

lemma *r-preserves-product* :
  $measure\text{-}to\text{-}qbs\ (X \bigotimes_M Y) = measure\text{-}to\text{-}qbs\ X \bigotimes_Q measure\text{-}to\text{-}qbs\ Y$
  ⟨*proof*⟩

lemma *l-product-sets*[*simp*,*measurable-cong*]:
sets (*qbs-to-measure* $X \bigotimes_M$ *qbs-to-measure* Y) $\subseteq$ *sets* (*qbs-to-measure* ($X \bigotimes_Q$ Y))
⟨*proof*⟩

lemma(**in** *pair-standard-borel*) *l-r-r-sets*[*simp*,*measurable-cong*]:
sets (*qbs-to-measure* (*measure-to-qbs* $M \bigotimes_Q$ *measure-to-qbs* N)) $=$ *sets* ($M \bigotimes_M$ N)
⟨*proof*⟩

end

2.3.2 Product Spaces

theory *Product-QuasiBorel*

imports *Binary-Product-QuasiBorel*

begin

definition *prod-qbs-Mx* :: $['a\ set,\ 'a \Rightarrow 'b\ quasi\text{-}borel] \Rightarrow (real \Rightarrow 'a \Rightarrow 'b)\ set$ **where**
prod-qbs-Mx I $M \equiv \{\ \alpha \mid \alpha.\ \forall i.\ (i \in I \longrightarrow (\lambda r.\ \alpha\ r\ i) \in qbs\text{-}Mx\ (M\ i)) \wedge (i \notin I \longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ undefined))\}$

lemma *prod-qbs-MxI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow (\lambda r.\ \alpha\ r\ i) \in qbs\text{-}Mx\ (M\ i)$
and $\bigwedge i.\ i \notin I \Longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ undefined)$
shows $\alpha \in$ *prod-qbs-Mx* I M
⟨*proof*⟩

lemma *prod-qbs-MxE*:
assumes $\alpha \in$ *prod-qbs-Mx* I M
shows $\bigwedge i.\ i \in I \Longrightarrow (\lambda r.\ \alpha\ r\ i) \in qbs\text{-}Mx\ (M\ i)$
and $\bigwedge i.\ i \notin I \Longrightarrow (\lambda r.\ \alpha\ r\ i) = (\lambda r.\ undefined)$
and $\bigwedge i\ r.\ i \notin I \Longrightarrow \alpha\ r\ i = undefined$
⟨*proof*⟩

definition *PiQ* :: $'a\ set \Rightarrow ('a \Rightarrow 'b\ quasi\text{-}borel) \Rightarrow ('a \Rightarrow 'b)\ quasi\text{-}borel$ **where**
PiQ I $M \equiv$ *Abs-quasi-borel* ($\Pi_E\ i{\in}I.$ *qbs-space* (M i), *prod-qbs-Mx* I M)

syntax
-PiQ :: *pttrn* $\Rightarrow$ *'i set* $\Rightarrow$ *'a quasi-borel* $\Rightarrow$ (*'i* => *'a*) *quasi-borel* ((*3*Π_Q -∈-./ -) *10*)
translations
$\Pi_Q\ x{\in}I.\ M$ == *CONST PiQ* I ($\lambda x.\ M$)

lemma *PiQ-f*: *prod-qbs-Mx* I $M \subseteq$ *UNIV* $\rightarrow$ ($\Pi_E\ i{\in}I.$ *qbs-space* (M i))

⟨*proof*⟩

lemma *PiQ-closed1*: *qbs-closed1* (*prod-qbs-Mx I M*)
⟨*proof*⟩

lemma *PiQ-closed2*: *qbs-closed2* (Π_E *i*∈*I*. *qbs-space* (*M i*)) (*prod-qbs-Mx I M*)
⟨*proof*⟩

lemma *PiQ-closed3*: *qbs-closed3* (*prod-qbs-Mx I M*)
⟨*proof*⟩

lemma *PiQ-correct*: *Rep-quasi-borel* (*PiQ I M*) = (Π_E *i*∈*I*. *qbs-space* (*M i*), *prod-qbs-Mx I M*)
⟨*proof*⟩

lemma *PiQ-space*[*simp*]: *qbs-space* (*PiQ I M*) = (Π_E *i*∈*I*. *qbs-space* (*M i*))
⟨*proof*⟩

lemma *PiQ-Mx*[*simp*]: *qbs-Mx* (*PiQ I M*) = *prod-qbs-Mx I M*
⟨*proof*⟩

lemma *qbs-morphism-component-singleton*:
assumes $i \in I$
shows $(\lambda x.\ x\ i) \in (\Pi_Q\ i{\in}I.\ (M\ i)) \rightarrow_Q M\ i$
⟨*proof*⟩

lemma *product-qbs-canonical1*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in Y \rightarrow_Q X\ i$
and $\bigwedge i.\ i \notin I \Longrightarrow f\ i = (\lambda y.\ \mathit{undefined})$
shows $(\lambda y\ i.\ f\ i\ y) \in Y \rightarrow_Q (\Pi_Q\ i{\in}I.\ X\ i)$
⟨*proof*⟩

lemma *product-qbs-canonical2*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in Y \rightarrow_Q X\ i$
$\bigwedge i.\ i \notin I \Longrightarrow f\ i = (\lambda y.\ \mathit{undefined})$
$g \in Y \rightarrow_Q (\Pi_Q\ i{\in}I.\ X\ i)$
$\bigwedge i.\ i \in I \Longrightarrow f\ i = (\lambda x.\ x\ i) \circ g$
and $y \in \mathit{qbs\text{-}space}\ Y$
shows $g\ y = (\lambda i.\ f\ i\ y)$
⟨*proof*⟩

lemma *merge-qbs-morphism*:
$\mathit{merge}\ I\ J \in (\Pi_Q\ i{\in}I.\ (M\ i)) \bigotimes_Q (\Pi_Q\ j{\in}J.\ (M\ j)) \rightarrow_Q (\Pi_Q\ i{\in}I{\cup}J.\ (M\ i))$
⟨*proof*⟩

The following lemma corresponds to [1] Proposition 19(1).

lemma *r-preserves-product′*:
$\mathit{measure\text{-}to\text{-}qbs}\ (\Pi_M\ i{\in}I.\ M\ i) = (\Pi_Q\ i{\in}I.\ \mathit{measure\text{-}to\text{-}qbs}\ (M\ i))$

$\langle proof \rangle$

$\prod_{i=0,1} X_i \cong X_1 \times X_2$.

lemma *product-binary-product*:
$\exists f\ g.\ f \in (\Pi_Q\ i{\in}UNIV.\ if\ i\ then\ X\ else\ Y) \to_Q X \bigotimes_Q Y \wedge g \in X \bigotimes_Q Y \to_Q (\Pi_Q\ i{\in}UNIV.\ if\ i\ then\ X\ else\ Y) \wedge$
$g \circ f = id \wedge f \circ g = id$
$\langle proof \rangle$

end

2.4 Coproduct Spaces

theory *Binary-CoProduct-QuasiBorel*
imports *Measure-QuasiBorel-Adjunction*
begin

2.4.1 Binary Coproduct Spaces

definition *copair-qbs-Mx* :: $['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel] \Rightarrow (real => 'a + 'b)\ set$
where
$copair\text{-}qbs\text{-}Mx\ X\ Y \equiv$
$\{g.\ \exists\ S \in sets\ real\text{-}borel.$
$(S = \{\} \longrightarrow (\exists\ \alpha 1{\in}\ qbs\text{-}Mx\ X.\ g = (\lambda r.\ Inl\ (\alpha 1\ r)))) \wedge$
$(S = UNIV \longrightarrow (\exists\ \alpha 2{\in}\ qbs\text{-}Mx\ Y.\ g = (\lambda r.\ Inr\ (\alpha 2\ r)))) \wedge$
$((S \neq \{\} \wedge S \neq UNIV) \longrightarrow$
$(\exists\ \alpha 1{\in}\ qbs\text{-}Mx\ X.$
$\exists\ \alpha 2{\in}\ qbs\text{-}Mx\ Y.$
$g = (\lambda r{::}real.\ (if\ (r \in S)\ then\ Inl\ (\alpha 1\ r)\ else\ Inr\ (\alpha 2\ r)))))\}$

definition *copair-qbs* :: $['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel] \Rightarrow ('a + 'b)\ quasi\text{-}borel$
(**infixr** $<+>_Q$ *65*) **where**
$copair\text{-}qbs\ X\ Y \equiv Abs\text{-}quasi\text{-}borel\ (qbs\text{-}space\ X <+> qbs\text{-}space\ Y,\ copair\text{-}qbs\text{-}Mx\ X\ Y)$

The followin is an equivalent definition of *copair-qbs-Mx*.

definition *copair-qbs-Mx2* :: $['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel] \Rightarrow (real => 'a + 'b)\ set$ **where**
$copair\text{-}qbs\text{-}Mx2\ X\ Y \equiv$
$\{g.\ (if\ qbs\text{-}space\ X = \{\} \wedge qbs\text{-}space\ Y = \{\}\ then\ False$
$else\ if\ qbs\text{-}space\ X \neq \{\} \wedge qbs\text{-}space\ Y = \{\}\ then$
$(\exists \alpha 1{\in}\ qbs\text{-}Mx\ X.\ g = (\lambda r.\ Inl\ (\alpha 1\ r)))$
$else\ if\ qbs\text{-}space\ X = \{\} \wedge qbs\text{-}space\ Y \neq \{\}\ then$
$(\exists \alpha 2{\in}\ qbs\text{-}Mx\ Y.\ g = (\lambda r.\ Inr\ (\alpha 2\ r)))$
$else$
$(\exists S \in sets\ real\text{-}borel.\ \exists \alpha 1{\in}\ qbs\text{-}Mx\ X.\ \exists \alpha 2{\in}\ qbs\text{-}Mx\ Y.$
$g = (\lambda r{::}real.\ (if\ (r \in S)\ then\ Inl\ (\alpha 1\ r)\ else\ Inr\ (\alpha 2\ r)))))\ \}$

lemma *copair-qbs-Mx-equiv* :*copair-qbs-Mx* (X :: *'a quasi-borel*) (Y :: *'b quasi-borel*) = *copair-qbs-Mx2* X Y
⟨*proof*⟩

lemma *copair-qbs-f*[*simp*]: *copair-qbs-Mx* X Y $\subseteq$ *UNIV* $\rightarrow$ *qbs-space* X <+> *qbs-space* Y
⟨*proof*⟩

lemma *copair-qbs-closed1*: *qbs-closed1* (*copair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *copair-qbs-closed2*: *qbs-closed2* (*qbs-space* X <+> *qbs-space* Y) (*copair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *copair-qbs-closed3*: *qbs-closed3* (*copair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *copair-qbs-correct*: *Rep-quasi-borel* (*copair-qbs* X Y) = (*qbs-space* X <+> *qbs-space* Y, *copair-qbs-Mx* X Y)
⟨*proof*⟩

lemma *copair-qbs-space*[*simp*]: *qbs-space* (*copair-qbs* X Y) = *qbs-space* X <+> *qbs-space* Y
⟨*proof*⟩

lemma *copair-qbs-Mx*[*simp*]: *qbs-Mx* (*copair-qbs* X Y) = *copair-qbs-Mx* X Y
⟨*proof*⟩

lemma *Inl-qbs-morphism*:
Inl $\in X \rightarrow_Q X$ <+>$_Q$ Y
⟨*proof*⟩

lemma *Inr-qbs-morphism*:
Inr $\in Y \rightarrow_Q X$ <+>$_Q$ Y
⟨*proof*⟩

lemma *case-sum-preserves-morphisms*:
assumes $f \in X \rightarrow_Q Z$
and $g \in Y \rightarrow_Q Z$
shows *case-sum* f $g \in X$ <+>$_Q$ $Y \rightarrow_Q Z$
⟨*proof*⟩

lemma *map-sum-preserves-morphisms*:
assumes $f \in X \rightarrow_Q Y$
and $g \in X' \rightarrow_Q Y'$

shows *map-sum f g* $\in X$ <+>$_Q$ $X' \to_Q Y$ <+>$_Q$ Y'
$\langle proof \rangle$

end

2.4.2 Countable Coproduct Spaces

theory *CoProduct-QuasiBorel*

imports
Product-QuasiBorel
Binary-CoProduct-QuasiBorel
begin

definition *coprod-qbs-Mx* :: $['a\ set,\ 'a \Rightarrow 'b\ quasi\text{-}borel] \Rightarrow (real \Rightarrow 'a \times 'b)\ set$ **where**
coprod-qbs-Mx I X $\equiv \{\ \lambda r.\ (f\ r,\ \alpha\ (f\ r)\ r)\ |f\ \alpha.\ f \in real\text{-}borel \to_M count\text{-}space\ I \wedge (\forall\ i{\in}range\ f.\ \alpha\ i \in qbs\text{-}Mx\ (X\ i))\}$

lemma *coprod-qbs-MxI*:
assumes $f \in real\text{-}borel \to_M count\text{-}space\ I$
and $\bigwedge i.\ i \in range\ f \Longrightarrow \alpha\ i \in qbs\text{-}Mx\ (X\ i)$
shows $(\lambda r.\ (f\ r,\ \alpha\ (f\ r)\ r)) \in coprod\text{-}qbs\text{-}Mx\ I\ X$
$\langle proof \rangle$

definition *coprod-qbs-Mx'* :: $['a\ set,\ 'a \Rightarrow 'b\ quasi\text{-}borel] \Rightarrow (real \Rightarrow 'a \times 'b)\ set$ **where**
coprod-qbs-Mx' I X $\equiv \{\ \lambda r.\ (f\ r,\ \alpha\ (f\ r)\ r)\ |f\ \alpha.\ f \in real\text{-}borel \to_M count\text{-}space\ I \wedge (\forall\ i.\ (i \in range\ f \vee qbs\text{-}space\ (X\ i) \neq \{\}) \longrightarrow \alpha\ i \in qbs\text{-}Mx\ (X\ i))\}$

lemma *coproduct-qbs-Mx-eq*:
coprod-qbs-Mx I X = *coprod-qbs-Mx' I X*
$\langle proof \rangle$

definition *coprod-qbs* :: $['a\ set,\ 'a \Rightarrow 'b\ quasi\text{-}borel] \Rightarrow ('a \times 'b)\ quasi\text{-}borel$ **where**
coprod-qbs I X $\equiv$ *Abs-quasi-borel* (*SIGMA i:I. qbs-space* (*X i*), *coprod-qbs-Mx I X*)

syntax
-coprod-qbs :: $pttrn \Rightarrow 'i\ set \Rightarrow 'a\ quasi\text{-}borel \Rightarrow ('i \times 'a)\ quasi\text{-}borel$ ((*3*$\amalg_Q$ -$\in$-./ -) *10*)
translations
$\amalg_Q\ x{\in}I.\ M \rightleftharpoons CONST\ coprod\text{-}qbs\ I\ (\lambda x.\ M)$

lemma *coprod-qbs-f*[*simp*]: *coprod-qbs-Mx I X* $\subseteq UNIV \to$ (*SIGMA i:I. qbs-space* (*X i*))
$\langle proof \rangle$

lemma *coprod-qbs-closed1*: *qbs-closed1* (*coprod-qbs-Mx I X*)

⟨*proof*⟩

lemma *coprod-qbs-closed2*: *qbs-closed2* (*SIGMA i*:*I*. *qbs-space* (*X i*)) (*coprod-qbs-Mx I X*)
⟨*proof*⟩

lemma *coprod-qbs-closed3*:
qbs-closed3 (*coprod-qbs-Mx I X*)
⟨*proof*⟩

lemma *coprod-qbs-correct*: *Rep-quasi-borel* (*coprod-qbs I X*) = (*SIGMA i*:*I*. *qbs-space* (*X i*), *coprod-qbs-Mx I X*)
⟨*proof*⟩

lemma *coproduct-qbs-space*[*simp*]: *qbs-space* (*coprod-qbs I X*) = (*SIGMA i*:*I*. *qbs-space* (*X i*))
⟨*proof*⟩

lemma *coproduct-qbs-Mx*[*simp*]: *qbs-Mx* (*coprod-qbs I X*) = *coprod-qbs-Mx I X*
⟨*proof*⟩

lemma *ini-morphism*:
assumes $j \in I$
shows $(\lambda x.\ (j,x)) \in X\ j \rightarrow_Q (\Pi_Q\ i{\in}I.\ X\ i)$
⟨*proof*⟩

lemma *coprod-qbs-canonical1*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow f\ i \in X\ i \rightarrow_Q Y$
shows $(\lambda(i,x).\ f\ i\ x) \in (\Pi_Q\ i \in I.\ X\ i) \rightarrow_Q Y$
⟨*proof*⟩

lemma *coprod-qbs-canonical1'*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow (\lambda x.\ f\ (i,x)) \in X\ i \rightarrow_Q Y$
shows $f \in (\Pi_Q\ i \in I.\ X\ i) \rightarrow_Q Y$
⟨*proof*⟩

$\coprod_{i=0,1} X_i \cong X_1 + X_2$.

lemma *coproduct-binary-coproduct*:
$\exists f\ g.\ f \in (\Pi_Q\ i{\in}UNIV.\ \mathit{if}\ i\ \mathit{then}\ X\ \mathit{else}\ Y) \rightarrow_Q X <\!+\!>_Q Y \wedge g \in X <\!+\!>_Q Y \rightarrow_Q (\Pi_Q\ i{\in}UNIV.\ \mathit{if}\ i\ \mathit{then}\ X\ \mathit{else}\ Y) \wedge$
$g \circ f = id \wedge f \circ g = id$
⟨*proof*⟩

2.4.3 Lists

abbreviation *list-of* $X \equiv \Pi_Q\ n{\in}(UNIV :: nat\ set).\ (\Pi_Q\ i{\in}\{..{<}n\}.\ X)$

abbreviation *list-nil* :: $nat \times (nat \Rightarrow 'a)$ **where**
list-nil $\equiv$ (*0*, $\lambda n.$ *undefined*)
abbreviation *list-cons* :: $['a, nat \times (nat \Rightarrow 'a)] \Rightarrow nat \times (nat \Rightarrow 'a)$ **where**
list-cons x l $\equiv$ (*Suc* (*fst l*), ($\lambda n.$ *if n* = *0 then x else* (*snd l*) ($n - 1$)))

definition *list-head* :: $nat \times (nat \Rightarrow 'a) \Rightarrow 'a$ **where**
list-head l = *snd l 0*
definition *list-tail* :: $nat \times (nat \Rightarrow 'a) \Rightarrow nat \times (nat \Rightarrow 'a)$ **where**
list-tail l = (*fst l* $-$ *1*, $\lambda m.$ (*snd l*) (*Suc m*))

lemma *list-simp1*:
list-nil $\neq$ *list-cons x l*
⟨*proof*⟩

lemma *list-simp2*:
assumes *list-cons a al* = *list-cons b bl*
shows *a* = *b al* = *bl*
⟨*proof*⟩

lemma *list-simp3*:
shows *list-head* (*list-cons a l*) = *a*
⟨*proof*⟩

lemma *list-simp4*:
assumes *l* $\in$ *qbs-space* (*list-of X*)
shows *list-tail* (*list-cons a l*) = *l*
⟨*proof*⟩

lemma *list-decomp1*:
assumes *l* $\in$ *qbs-space* (*list-of X*)
shows *l* = *list-nil* $\vee$
($\exists$ *a l'*. *a* $\in$ *qbs-space X* $\wedge$ *l'* $\in$ *qbs-space* (*list-of X*) $\wedge$ *l* = *list-cons a l'*)
⟨*proof*⟩

lemma *list-simp5*:
assumes *l* $\in$ *qbs-space* (*list-of X*)
and *l* $\neq$ *list-nil*
shows *l* = *list-cons* (*list-head l*) (*list-tail l*)
⟨*proof*⟩

lemma *list-simp6*:
list-nil $\in$ *qbs-space* (*list-of X*)
⟨*proof*⟩

lemma *list-simp7*:
assumes *a* $\in$ *qbs-space X*
and *l* $\in$ *qbs-space* (*list-of X*)
shows *list-cons a l* $\in$ *qbs-space* (*list-of X*)

⟨*proof*⟩

lemma *list-destruct-rule*:
assumes $l \in$ *qbs-space* (*list-of* X)
P list-nil
and $\bigwedge a\ l'.\ a \in$ *qbs-space* $X \Longrightarrow l' \in$ *qbs-space* (*list-of* X) $\Longrightarrow P$ (*list-cons* a l')
shows $P\ l$
⟨*proof*⟩

lemma *list-induct-rule*:
assumes $l \in$ *qbs-space* (*list-of* X)
P list-nil
and $\bigwedge a\ l'.\ a \in$ *qbs-space* $X \Longrightarrow l' \in$ *qbs-space* (*list-of* X) $\Longrightarrow P\ l' \Longrightarrow P$ (*list-cons* a l')
shows $P\ l$
⟨*proof*⟩

fun *from-list* :: $'a$ *list* $\Rightarrow$ *nat* $\times$ (*nat* $\Rightarrow$ $'a$) **where**
from-list [] = *list-nil* |
from-list ($a\#l$) = *list-cons* a (*from-list* l)

fun *to-list'* :: *nat* $\Rightarrow$ (*nat* $\Rightarrow$ $'a$) $\Rightarrow$ $'a$ *list* **where**
to-list' 0 - = [] |
to-list' (*Suc* n) f = f 0 # *to-list'* n ($\lambda n.\ f$ (*Suc* n))

definition *to-list* :: *nat* $\times$ (*nat* $\Rightarrow$ $'a$) $\Rightarrow$ $'a$ *list* **where**
to-list $\equiv$ *case-prod to-list'*

lemma *to-list-simp1*:
shows *to-list list-nil* = []
⟨*proof*⟩

lemma *to-list-simp2*:
assumes $l \in$ *qbs-space* (*list-of* X)
shows *to-list* (*list-cons* a l) = a # *to-list* l
⟨*proof*⟩

lemma *from-list-length*:
fst (*from-list* l) = *length* l
⟨*proof*⟩

lemma *from-list-in-list-of*:
assumes *set* $l \subseteq$ *qbs-space* X
shows *from-list* $l \in$ *qbs-space* (*list-of* X)
⟨*proof*⟩

lemma *from-list-in-list-of'*:

shows *from-list* $l \in$ *qbs-space* (*list-of* (*Abs-quasi-borel* (*UNIV*,*UNIV*)))
⟨*proof*⟩

lemma *list-cons-in-list-of*:
assumes *set* $(a\#l) \subseteq$ *qbs-space* X
shows *list-cons* a (*from-list* l) $\in$ *qbs-space* (*list-of* X)
⟨*proof*⟩

lemma *from-list-to-list-ident*:
(*to-list* $\circ$ *from-list*) $l = l$
⟨*proof*⟩

lemma *to-list-from-list-ident*:
assumes $l \in$ *qbs-space* (*list-of* X)
shows (*from-list* $\circ$ *to-list*) $l = l$
⟨*proof*⟩

definition *rec-list'* :: $'b \Rightarrow ('a \Rightarrow (nat \times (nat \Rightarrow 'a)) \Rightarrow 'b \Rightarrow 'b) \Rightarrow (nat \times (nat \Rightarrow 'a)) \Rightarrow 'b$ **where**
rec-list' $t0\ f\ l \equiv$ (*rec-list* $t0$ ($\lambda x\ l'.\ f\ x$ (*from-list* l')) (*to-list* l))

lemma *rec-list'-simp1*:
rec-list' $t\ f$ *list-nil* $= t$
⟨*proof*⟩

lemma *rec-list'-simp2*:
assumes $l \in$ *qbs-space* (*list-of* X)
shows *rec-list'* $t\ f$ (*list-cons* $x\ l$) $= f\ x\ l$ (*rec-list'* $t\ f\ l$)
⟨*proof*⟩

end

2.5 Function Spaces

theory *Exponent-QuasiBorel*
imports *CoProduct-QuasiBorel*
begin

2.5.1 Function Spaces

definition *exp-qbs-Mx* :: ['a *quasi-borel*, 'b *quasi-borel*] $\Rightarrow$ (*real* $\Rightarrow$ 'a => 'b) *set*
where
exp-qbs-Mx $X\ Y \equiv \{g :: real \Rightarrow 'a \Rightarrow 'b.\ case\text{-}prod\ g \in \mathbb{R}_Q \bigotimes_Q X \to_Q Y\}$

definition *exp-qbs* :: ['a *quasi-borel*, 'b *quasi-borel*] $\Rightarrow$ ('a $\Rightarrow$ 'b) *quasi-borel* (**infixr** $\Rightarrow_Q$ *61*) **where**
$X \Rightarrow_Q Y \equiv$ *Abs-quasi-borel* ($X \to_Q Y$, *exp-qbs-Mx* $X\ Y$)

lemma *exp-qbs-f*[*simp*]: $exp\text{-}qbs\text{-}Mx\ X\ Y \subseteq UNIV \rightarrow (X :: 'a\ quasi\text{-}borel) \rightarrow_Q (Y :: 'b\ quasi\text{-}borel)$
⟨*proof*⟩

lemma *exp-qbs-closed1*: $qbs\text{-}closed1\ (exp\text{-}qbs\text{-}Mx\ X\ Y)$
⟨*proof*⟩

lemma *exp-qbs-closed2*: $qbs\text{-}closed2\ (X \rightarrow_Q Y)\ (exp\text{-}qbs\text{-}Mx\ X\ Y)$
⟨*proof*⟩

lemma *exp-qbs-closed3*:$qbs\text{-}closed3\ (exp\text{-}qbs\text{-}Mx\ X\ Y)$
⟨*proof*⟩

lemma *exp-qbs-correct*: $Rep\text{-}quasi\text{-}borel\ (exp\text{-}qbs\ X\ Y) = (X \rightarrow_Q Y,\ exp\text{-}qbs\text{-}Mx\ X\ Y)$
⟨*proof*⟩

lemma *exp-qbs-space*[*simp*]: $qbs\text{-}space\ (exp\text{-}qbs\ X\ Y) = X \rightarrow_Q Y$
⟨*proof*⟩

lemma *exp-qbs-Mx*[*simp*]: $qbs\text{-}Mx\ (exp\text{-}qbs\ X\ Y) = exp\text{-}qbs\text{-}Mx\ X\ Y$
⟨*proof*⟩

lemma *qbs-exp-morphismI*:
assumes $\bigwedge\alpha\ \beta.\ \alpha \in qbs\text{-}Mx\ X \Longrightarrow$
$\beta \in pair\text{-}qbs\text{-}Mx\ real\text{-}quasi\text{-}borel\ Y \Longrightarrow$
$(\lambda(r,x).\ (f \circ \alpha)\ r\ x) \circ \beta \in qbs\text{-}Mx\ Z$
shows $f \in X \rightarrow_Q exp\text{-}qbs\ Y\ Z$
⟨*proof*⟩

definition $qbs\text{-}eval :: (('a \Rightarrow 'b) \times 'a) \Rightarrow 'b$ **where**
$qbs\text{-}eval\ a \equiv (fst\ a)\ (snd\ a)$

lemma *qbs-eval-morphism*:
$qbs\text{-}eval \in (exp\text{-}qbs\ X\ Y) \bigotimes_Q X \rightarrow_Q Y$
⟨*proof*⟩

lemma *curry-morphism*:
$curry \in exp\text{-}qbs\ (X \bigotimes_Q Y)\ Z \rightarrow_Q exp\text{-}qbs\ X\ (exp\text{-}qbs\ Y\ Z)$
⟨*proof*⟩

lemma *curry-preserves-morphisms*:
assumes $f \in X \bigotimes_Q Y \rightarrow_Q Z$
shows $curry\ f \in X \rightarrow_Q exp\text{-}qbs\ Y\ Z$
⟨*proof*⟩

lemma *uncurry-morphism*:

case-prod $\in$ *exp-qbs* X (*exp-qbs* Y Z) $\rightarrow_Q$ *exp-qbs* ($X \bigotimes_Q Y$) Z
⟨*proof*⟩

lemma *uncurry-preserves-morphisms*:
assumes $f \in X \rightarrow_Q$ *exp-qbs* Y Z
shows *case-prod* $f \in X \bigotimes_Q Y \rightarrow_Q Z$
⟨*proof*⟩

lemma *arg-swap-morphism*:
assumes $f \in X \rightarrow_Q$ *exp-qbs* Y Z
shows $(\lambda y\ x.\ f\ x\ y) \in Y \rightarrow_Q$ *exp-qbs* X Z
⟨*proof*⟩

lemma *exp-qbs-comp-morphism*:
assumes $f \in W \rightarrow_Q$ *exp-qbs* X Y
and $g \in W \rightarrow_Q$ *exp-qbs* Y Z
shows $(\lambda w.\ g\ w \circ f\ w) \in W \rightarrow_Q$ *exp-qbs* X Z
⟨*proof*⟩

lemma *case-sum-morphism*:
case-prod case-sum $\in$ *exp-qbs* X $Z \bigotimes_Q$ *exp-qbs* Y Z $\rightarrow_Q$ *exp-qbs* (X <+>$_Q$ Y) Z
⟨*proof*⟩

lemma *not-qbs-morphism*:
Not $\in \mathbb{B}_Q \rightarrow_Q \mathbb{B}_Q$
⟨*proof*⟩

lemma *or-qbs-morphism*:
$(\vee) \in \mathbb{B}_Q \rightarrow_Q$ *exp-qbs* $\mathbb{B}_Q$ $\mathbb{B}_Q$
⟨*proof*⟩

lemma *and-qbs-morphism*:
$(\wedge) \in \mathbb{B}_Q \rightarrow_Q$ *exp-qbs* $\mathbb{B}_Q$ $\mathbb{B}_Q$
⟨*proof*⟩

lemma *implies-qbs-morphism*:
$(\longrightarrow) \in \mathbb{B}_Q \rightarrow_Q \mathbb{B}_Q \Rightarrow_Q \mathbb{B}_Q$
⟨*proof*⟩

lemma *less-nat-qbs-morphism*:
$(<) \in \mathbb{N}_Q \rightarrow_Q$ *exp-qbs* $\mathbb{N}_Q$ $\mathbb{B}_Q$
⟨*proof*⟩

lemma *less-real-qbs-morphism*:
$(<) \in \mathbb{R}_Q \rightarrow_Q$ *exp-qbs* $\mathbb{R}_Q$ $\mathbb{B}_Q$
⟨*proof*⟩

lemma *rec-list-morphism'*:
rec-list' $\in$ *qbs-space* (*exp-qbs* *Y* (*exp-qbs* (*exp-qbs* *X* (*exp-qbs* (*list-of* *X*) (*exp-qbs* *Y* *Y*))) (*exp-qbs* (*list-of* *X*) *Y*)))
⟨*proof*⟩

end

3 Probability Spaces

3.1 Probability Measures

theory *Probability-Space-QuasiBorel*
imports *Exponent-QuasiBorel*
begin

3.1.1 Probability Measures

type-synonym *'a qbs-prob-t* = *'a quasi-borel* $*$ (*real* $\Rightarrow$ *'a*) $*$ *real measure*

locale *in-Mx* =
fixes *X* :: *'a quasi-borel*
and α :: *real* $\Rightarrow$ *'a*
assumes *in-Mx*[*simp*]:α $\in$ *qbs-Mx X*

locale *qbs-prob* = *in-Mx X* α + *real-distribution* μ
for *X* :: *'a quasi-borel* **and** α **and** μ
begin
declare *prob-space-axioms*[*simp*]

lemma *m-in-space-prob-algebra*[*simp*]:
μ $\in$ *space* (*prob-algebra real-borel*)
⟨*proof*⟩
end

locale *pair-qbs-probs* = *qp1*:*qbs-prob X* α μ + *qp2*:*qbs-prob Y* β ν
for *X* :: *'a quasi-borel***and** α μ **and** *Y* :: *'b quasi-borel* **and** β ν
begin

sublocale *pair-prob-space* μ ν
⟨*proof*⟩

lemma *ab-measurable*[*measurable*]:
map-prod α β $\in$ *real-borel* $\bigotimes_M$ *real-borel* $\rightarrow_M$ *qbs-to-measure* (*X* $\bigotimes_Q$ *Y*)
⟨*proof*⟩

lemma *ab-g-in-Mx*[*simp*]:
map-prod α β $\circ$ *real-real.g* $\in$ *pair-qbs-Mx X Y*

⟨*proof*⟩

sublocale *qbs-prob* $X \bigotimes_Q Y$ *map-prod* α β $\circ$ *real-real.g* *distr* $(\mu \bigotimes_M \nu)$ *real-borel* *real-real.f*
⟨*proof*⟩

end

locale *pair-qbs-prob* = *qp1*:*qbs-prob* X α μ + *qp2*:*qbs-prob* Y β ν
for X :: *'a quasi-borel***and** α μ **and** Y :: *'a quasi-borel* **and** β ν
begin

sublocale *pair-qbs-probs*
⟨*proof*⟩

lemma *same-spaces*[*simp*]:
assumes $Y = X$
shows $\beta \in$ *qbs-Mx* X
⟨*proof*⟩

end

lemma *prob-algebra-real-prob-measure*:
p $\in$ *space* (*prob-algebra* (*real-borel*)) = *real-distribution* *p*
⟨*proof*⟩

lemma *qbs-probI*:
assumes $\alpha \in$ *qbs-Mx* X
and *sets* μ = *sets borel*
and *prob-space* μ
shows *qbs-prob* X α μ
⟨*proof*⟩

lemma *qbs-empty-not-qbs-prob* :¬ *qbs-prob* (*empty-quasi-borel*) *f M*
⟨*proof*⟩

definition *qbs-prob-eq* :: [*'a qbs-prob-t*, *'a qbs-prob-t*] ⇒ *bool* **where**
qbs-prob-eq p1 p2 ≡
(*let* (*qbs1*, *a1*, *m1*) = *p1*;
(*qbs2*, *a2*, *m2*) = *p2* *in*
qbs-prob qbs1 a1 m1 ∧ *qbs-prob qbs2 a2 m2* ∧ *qbs1* = *qbs2* ∧
distr m1 (*qbs-to-measure qbs1*) *a1* = *distr m2* (*qbs-to-measure qbs2*) *a2*)

definition *qbs-prob-eq2* :: [*'a qbs-prob-t*, *'a qbs-prob-t*] ⇒ *bool* **where**
qbs-prob-eq2 p1 p2 ≡
(*let* (*qbs1*, *a1*, *m1*) = *p1*;
(*qbs2*, *a2*, *m2*) = *p2* *in*
qbs-prob qbs1 a1 m1 ∧ *qbs-prob qbs2 a2 m2* ∧ *qbs1* = *qbs2* ∧
(∀ *f* ∈ *qbs1* $\rightarrow_Q$ *real-quasi-borel*.

$(\int x.\ f\ (a1\ x)\ \partial\ m1) = (\int x.\ f\ (a2\ x)\ \partial\ m2)))$

definition *qbs-prob-eq3* :: [*'a qbs-prob-t*, *'a qbs-prob-t*] ⇒ *bool* **where**
qbs-prob-eq3 p1 p2 ≡
(*let* (*qbs1*, *a1*, *m1*) = *p1*;
(*qbs2*, *a2*, *m2*) = *p2 in*
(*qbs-prob qbs1 a1 m1* ∧ *qbs-prob qbs2 a2 m2* ∧ *qbs1* = *qbs2* ∧
$(\forall f \in qbs1 \rightarrow_Q$ *real-quasi-borel*.
(∀ *k* ∈ *qbs-space qbs1*. *0* ≤ *f k*) ⟶
$(\int x.\ f\ (a1\ x)\ \partial\ m1) = (\int x.\ f\ (a2\ x)\ \partial\ m2))))$

definition *qbs-prob-eq4* :: [*'a qbs-prob-t*, *'a qbs-prob-t*] ⇒ *bool* **where**
qbs-prob-eq4 p1 p2 ≡
(*let* (*qbs1*, *a1*, *m1*) = *p1*;
(*qbs2*, *a2*, *m2*) = *p2 in*
(*qbs-prob qbs1 a1 m1* ∧ *qbs-prob qbs2 a2 m2* ∧ *qbs1* = *qbs2* ∧
$(\forall f \in qbs1 \rightarrow_Q \mathbb{R}_{Q\geq 0}.$
$(\int^+ x.\ f\ (a1\ x)\ \partial\ m1) = (\int^+ x.\ f\ (a2\ x)\ \partial\ m2))))$

lemma(**in** *qbs-prob*) *qbs-prob-eq-refl*[*simp*]:
qbs-prob-eq (*X*,α,μ) (*X*,α,μ)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-eq2-refl*[*simp*]:
qbs-prob-eq2 (*X*,α,μ) (*X*,α,μ)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-eq3-refl*[*simp*]:
qbs-prob-eq3 (*X*,α,μ) (*X*,α,μ)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-eq4-refl*[*simp*]:
qbs-prob-eq4 (*X*,α,μ) (*X*,α,μ)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-eq-intro*:
assumes *X* = *Y*
and *distr* μ (*qbs-to-measure X*) α = *distr* ν (*qbs-to-measure X*) β
shows *qbs-prob-eq* (*X*,α,μ) (*Y*,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-eq2-intro*:
assumes *X* = *Y*
and $\bigwedge f.\ f \in qbs\text{-}to\text{-}measure\ X \rightarrow_M real\text{-}borel$
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-eq2* (*X*,α,μ) (*Y*,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-eq3-intro*:

assumes $X = Y$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *real-borel* $\Longrightarrow (\forall\ k \in$ *qbs-space* $X.\ 0 \le f$ $k)$
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-eq3* (X,α,μ) (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-eq4-intro*:
assumes $X = Y$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *ennreal-borel*
$\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-eq4* (X,α,μ) (Y,β,ν)
⟨*proof*⟩

lemma *qbs-prob-eq-dest*:
assumes *qbs-prob-eq* (X,α,μ) (Y,β,ν)
shows *qbs-prob* $X\ \alpha\ \mu$
qbs-prob $Y\ \beta\ \nu$
$Y = X$
and *distr* μ (*qbs-to-measure* X) α = *distr* ν (*qbs-to-measure* X) β
⟨*proof*⟩

lemma *qbs-prob-eq2-dest*:
assumes *qbs-prob-eq2* (X,α,μ) (Y,β,ν)
shows *qbs-prob* $X\ \alpha\ \mu$
qbs-prob $Y\ \beta\ \nu$
$Y = X$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *real-borel*
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
⟨*proof*⟩

lemma *qbs-prob-eq3-dest*:
assumes *qbs-prob-eq3* (X,α,μ) (Y,β,ν)
shows *qbs-prob* $X\ \alpha\ \mu$
qbs-prob $Y\ \beta\ \nu$
$Y = X$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *real-borel* $\Longrightarrow (\forall\ k \in$ *qbs-space* $X.\ 0 \le f\, k)$
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
⟨*proof*⟩

lemma *qbs-prob-eq4-dest*:
assumes *qbs-prob-eq4* (X,α,μ) (Y,β,ν)
shows *qbs-prob* $X\ \alpha\ \mu$
qbs-prob $Y\ \beta\ \nu$
$Y = X$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *ennreal-borel*
$\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)$
⟨*proof*⟩

definition *qbs-prob-t-ennintegral* :: $['a\ qbs\text{-}prob\text{-}t,\ 'a \Rightarrow ennreal] \Rightarrow ennreal$ **where**
qbs-prob-t-ennintegral p f $\equiv$
  (*if f* $\in$ (*fst p*) $\rightarrow_Q$ *ennreal-quasi-borel*
    then ($\int^+ x.\ f\ (fst\ (snd\ p)\ x)\ \partial\ (snd\ (snd\ p))$) *else 0*)

definition *qbs-prob-t-integral* :: $['a\ qbs\text{-}prob\text{-}t,\ 'a \Rightarrow real] \Rightarrow real$ **where**
qbs-prob-t-integral p f $\equiv$
  (*if f* $\in$ (*fst p*) $\rightarrow_Q$ $\mathbb{R}_Q$
    then ($\int x.\ f\ (fst\ (snd\ p)\ x)\ \partial\ (snd\ (snd\ p))$)
    else 0)

definition *qbs-prob-t-integrable* :: $['a\ qbs\text{-}prob\text{-}t,\ 'a \Rightarrow real] \Rightarrow bool$ **where**
qbs-prob-t-integrable p f $\equiv$ *f* $\in$ *fst p* $\rightarrow_Q$ *real-quasi-borel* $\wedge$ *integrable* (*snd* (*snd p*)) (*f* $\circ$ (*fst* (*snd p*)))

definition *qbs-prob-t-measure* :: *'a qbs-prob-t* $\Rightarrow$ *'a measure* **where**
qbs-prob-t-measure p $\equiv$ *distr* (*snd* (*snd p*)) (*qbs-to-measure* (*fst p*)) (*fst* (*snd p*))

lemma *qbs-prob-eq-symp*:
 symp qbs-prob-eq
  ⟨*proof*⟩

lemma *qbs-prob-eq-transp*:
 transp qbs-prob-eq
  ⟨*proof*⟩

quotient-type *'a qbs-prob-space* = *'a qbs-prob-t* / *partial*: *qbs-prob-eq*
  morphisms *rep-qbs-prob-space qbs-prob-space*
⟨*proof*⟩

interpretation *qbs-prob-space* : *quot-type qbs-prob-eq Abs-qbs-prob-space Rep-qbs-prob-space*
  ⟨*proof*⟩

lemma *qbs-prob-space-induct*:
  assumes $\bigwedge X\ \alpha\ \mu.\ qbs\text{-}prob\ X\ \alpha\ \mu \Longrightarrow P\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu))$
  shows *P s*
  ⟨*proof*⟩

lemma *qbs-prob-space-induct'*:
  assumes $\bigwedge X\ \alpha\ \mu.\ qbs\text{-}prob\ X\ \alpha\ \mu \Longrightarrow s = qbs\text{-}prob\text{-}space\ (X,\alpha,\mu) \Longrightarrow P\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu))$
  shows *P s*
  ⟨*proof*⟩

lemma *rep-qbs-prob-space*:
 $\exists X\ \alpha\ \mu.\ p = qbs\text{-}prob\text{-}space\ (X,\ \alpha,\ \mu) \wedge qbs\text{-}prob\ X\ \alpha\ \mu$
  ⟨*proof*⟩

lemma(**in** *qbs-prob*) *in-Rep*:
$(X,\ \alpha,\ \mu) \in$ *Rep-qbs-prob-space* (*qbs-prob-space* (X,α,μ))
⟨*proof*⟩

lemma(**in** *qbs-prob*) *if-in-Rep*:
assumes $(X',\alpha',\mu') \in$ *Rep-qbs-prob-space* (*qbs-prob-space* (X,α,μ))
shows $X' = X$
qbs-prob $X'\ \alpha'\ \mu'$
qbs-prob-eq $(X,\alpha,\mu)\ (X',\alpha',\mu')$
⟨*proof*⟩

lemma(**in** *qbs-prob*) *in-Rep-induct*:
assumes $\bigwedge Y\ \beta\ \nu.\ (Y,\beta,\nu) \in$ *Rep-qbs-prob-space* (*qbs-prob-space* (X,α,μ)) $\Longrightarrow P$
(Y,β,ν)
shows P (*rep-qbs-prob-space* (*qbs-prob-space* (X,α,μ)))
⟨*proof*⟩

lemma *qbs-prob-eq-2-implies-3* :
assumes *qbs-prob-eq2 p1 p2*
shows *qbs-prob-eq3 p1 p2*
⟨*proof*⟩

lemma *qbs-prob-eq-3-implies-1* :
assumes *qbs-prob-eq3* (*p1* :: *'a qbs-prob-t*) *p2*
shows *qbs-prob-eq p1 p2*
⟨*proof*⟩

lemma *qbs-prob-eq-1-implies-2* :
assumes *qbs-prob-eq p1* (*p2* :: *'a qbs-prob-t*)
shows *qbs-prob-eq2 p1 p2*
⟨*proof*⟩

lemma *qbs-prob-eq-1-implies-4* :
assumes *qbs-prob-eq p1 p2*
shows *qbs-prob-eq4 p1 p2*
⟨*proof*⟩

lemma *qbs-prob-eq-4-implies-3* :
assumes *qbs-prob-eq4 p1 p2*
shows *qbs-prob-eq3 p1 p2*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv12* :
qbs-prob-eq = *qbs-prob-eq2*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv13* :
qbs-prob-eq = *qbs-prob-eq3*

⟨*proof*⟩

lemma *qbs-prob-eq-equiv14* :
qbs-prob-eq = *qbs-prob-eq4*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv23* :
qbs-prob-eq2 = *qbs-prob-eq3*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv24* :
qbs-prob-eq2 = *qbs-prob-eq4*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv34*:
qbs-prob-eq3 = *qbs-prob-eq4*
⟨*proof*⟩

lemma *qbs-prob-eq-equiv31* :
qbs-prob-eq = *qbs-prob-eq3*
⟨*proof*⟩

lemma *qbs-prob-space-eq*:
assumes *qbs-prob-eq* (X,α,μ) (Y,β,ν)
shows *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-space-eq*:
assumes $Y = X$
and *distr* μ (*qbs-to-measure* X) α = *distr* ν (*qbs-to-measure* X) β
shows *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-space-eq2*:
assumes $Y = X$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \rightarrow_M$ *real-borel*
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-space-eq3*:
assumes $Y = X$
and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \rightarrow_M$ *real-borel* $\Longrightarrow (\forall k\in$ *qbs-space* $X.\ 0 \le f\,k)$
$\Longrightarrow (\int x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-space-eq4*:
assumes $Y = X$

and $\bigwedge f.\ f \in$ *qbs-to-measure* $X \to_M$ *ennreal-borel*
$\Longrightarrow (\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu) = (\int^+ x.\ f\ (\beta\ x)\ \partial\ \nu)$
shows *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
⟨*proof*⟩

lemma(**in** *pair-qbs-prob*) *qbs-prob-space-eq-inverse*:
assumes *qbs-prob-space* (X,α,μ) = *qbs-prob-space* (Y,β,ν)
shows *qbs-prob-eq* (X,α,μ) (Y,β,ν)
and *qbs-prob-eq2* (X,α,μ) (Y,β,ν)
and *qbs-prob-eq3* (X,α,μ) (Y,β,ν)
and *qbs-prob-eq4* (X,α,μ) (Y,β,ν)
⟨*proof*⟩

lift-definition *qbs-prob-space-qbs* :: *'a qbs-prob-space* ⇒ *'a quasi-borel*
is *fst* ⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-space-qbs-computation*[*simp*]:
qbs-prob-space-qbs (*qbs-prob-space* (X,α,μ)) = X
⟨*proof*⟩

lemma *rep-qbs-prob-space'*:
assumes *qbs-prob-space-qbs* $s = X$
shows $\exists \alpha\ \mu.\ s =$ *qbs-prob-space* $(X,\alpha,\mu) \wedge$ *qbs-prob* $X\ \alpha\ \mu$
⟨*proof*⟩

lift-definition *qbs-prob-ennintegral* :: [*'a qbs-prob-space*, *'a* ⇒ *ennreal*] ⇒ *ennreal*
is *qbs-prob-t-ennintegral*
⟨*proof*⟩

lift-definition *qbs-prob-integral* :: [*'a qbs-prob-space*, *'a* ⇒ *real*] ⇒ *real*
is *qbs-prob-t-integral*
⟨*proof*⟩

syntax
-qbs-prob-ennintegral :: *pttrn* ⇒ *ennreal* ⇒ *'a qbs-prob-space* ⇒ *ennreal* ($\int^+{}_Q$((*2* -./ -)/ ∂-) [*60*,*61*] *110*)

translations
$\int^+{}_Q\ x.\ f\ \partial p \rightleftharpoons$ *CONST qbs-prob-ennintegral* p $(\lambda x.\ f)$

syntax
-qbs-prob-integral :: *pttrn* ⇒ *real* ⇒ *'a qbs-prob-space* ⇒ *real* ($\int_Q$((*2* -./ -)/ ∂-) [*60*,*61*] *110*)

translations
$\int_Q\ x.\ f\ \partial p \rightleftharpoons$ *CONST qbs-prob-integral* p $(\lambda x.\ f)$

We define the function $l_X \in L(P(X)) \to_M G(X)$.

lift-definition *qbs-prob-measure* :: *'a qbs-prob-space* ⇒ *'a measure*
is *qbs-prob-t-measure*
⟨*proof*⟩

declare [[*coercion qbs-prob-measure*]]

lemma(**in** *qbs-prob*) *qbs-prob-measure-computation*[*simp*]:
qbs-prob-measure (*qbs-prob-space* (X,α,μ)) = *distr* μ (*qbs-to-measure* X) α
⟨*proof*⟩

definition *qbs-emeasure* ::*'a qbs-prob-space* ⇒ *'a set* ⇒ *ennreal* **where**
qbs-emeasure s ≡ *emeasure* (*qbs-prob-measure s*)

lemma(**in** *qbs-prob*) *qbs-emeasure-computation*[*simp*]:
assumes $U \in$ *sets* (*qbs-to-measure* X)
shows *qbs-emeasure* (*qbs-prob-space* (X,α,μ)) U = *emeasure* μ (α −' U)
⟨*proof*⟩

definition *qbs-measure* ::*'a qbs-prob-space* ⇒ *'a set* ⇒ *real* **where**
qbs-measure s ≡ *measure* (*qbs-prob-measure s*)

interpretation *qbs-prob-measure-prob-space* : *prob-space qbs-prob-measure* (*s*::*'a qbs-prob-space*) **for** *s*
⟨*proof*⟩

lemma *qbs-prob-measure-space*:
qbs-space (*qbs-prob-space-qbs s*) = *space* (*qbs-prob-measure s*)
⟨*proof*⟩

lemma *qbs-prob-measure-sets*[*measurable-cong*]:
sets (*qbs-to-measure* (*qbs-prob-space-qbs s*)) = *sets* (*qbs-prob-measure s*)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-ennintegral-def*:
assumes $f \in X \to_Q \mathbb{R}_{Q\geq 0}$
shows *qbs-prob-ennintegral* (*qbs-prob-space* (X,α,μ)) f = ($\int^+ x.\ f\ (\alpha\ x)\ \partial\ \mu$)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-ennintegral-def2*:
assumes $f \in X \to_Q \mathbb{R}_{Q\geq 0}$
shows *qbs-prob-ennintegral* (*qbs-prob-space* (X,α,μ)) f = $integral^N$ (*distr* μ (*qbs-to-measure* X) α) f
⟨*proof*⟩

lemma (**in** *qbs-prob*) *qbs-prob-ennintegral-not-morphism*:
assumes $f \notin X \to_Q \mathbb{R}_{Q\geq 0}$

shows *qbs-prob-ennintegral* (*qbs-prob-space* (X,α,μ)) f = *0*
⟨*proof*⟩

lemma *qbs-prob-ennintegral-def2*:
assumes *qbs-prob-space-qbs* s = (X :: *'a quasi-borel*)
and $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$
shows *qbs-prob-ennintegral* s f = *integral*^{N} (*qbs-prob-measure* s) f
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-integral-def*:
assumes $f \in X \rightarrow_Q$ *real-quasi-borel*
shows *qbs-prob-integral* (*qbs-prob-space* (X,α,μ)) f = ($\int x.\ f\ (\alpha\ x)\ \partial\ \mu$)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-prob-integral-def2*:
qbs-prob-integral (*qbs-prob-space* (X,α,μ)) f = *integral*^{L} (*distr* μ (*qbs-to-measure* X) α) f
⟨*proof*⟩

lemma *qbs-prob-integral-def2*:
qbs-prob-integral (s::*'a qbs-prob-space*) f = *integral*^{L} (*qbs-prob-measure* s) f
⟨*proof*⟩

definition *qbs-prob-var* :: *'a qbs-prob-space* ⇒ (*'a* ⇒ *real*) ⇒ *real* **where**
qbs-prob-var s f ≡ *qbs-prob-integral* s ($\lambda x.$ ($f\ x$ − *qbs-prob-integral* s f)^2)

lemma(**in** *qbs-prob*) *qbs-prob-var-computation*:
assumes $f \in X \rightarrow_Q$ *real-quasi-borel*
shows *qbs-prob-var* (*qbs-prob-space* (X,α,μ)) f = ($\int x.\ (f\ (\alpha\ x) - (\int x.\ f\ (\alpha\ x)\ \partial\ \mu))^2\ \partial\ \mu$)
⟨*proof*⟩

lift-definition *qbs-integrable* :: [*'a qbs-prob-space*, *'a* ⇒ *real*] ⇒ *bool*
is *qbs-prob-t-integrable*
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-integrable-def*:
qbs-integrable (*qbs-prob-space* (X, α, μ)) f = ($f \in X \rightarrow_Q \mathbb{R}_Q$ ∧ *integrable* μ ($f \circ \alpha$))
⟨*proof*⟩

lemma *qbs-integrable-morphism*:
assumes *qbs-prob-space-qbs* s = X
and *qbs-integrable* s f
shows $f \in X \rightarrow_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-integrable-measurable*[*simp*,*measurable*]:
assumes *qbs-integrable* (*qbs-prob-space* (X,α,μ)) f

shows $f \in$ *qbs-to-measure* $X \rightarrow_M$ *real-borel*
⟨*proof*⟩

lemma *qbs-integrable-iff-integrable*:
(*qbs-integrable* (*s*::′*a qbs-prob-space*) *f*) = (*integrable* (*qbs-prob-measure s*) *f*)
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-integrable-iff-integrable-distr*:
qbs-integrable (*qbs-prob-space* (X,α,μ)) f = *integrable* (*distr* μ (*qbs-to-measure* X) α) f
⟨*proof*⟩

lemma(**in** *qbs-prob*) *qbs-integrable-iff-integrable*:
assumes $f \in$ *qbs-to-measure* $X \rightarrow_M$ *real-borel*
shows *qbs-integrable* (*qbs-prob-space* (X,α,μ)) f = *integrable* μ ($\lambda x.\ f\ (\alpha\ x)$)
⟨*proof*⟩

lemma *qbs-integrable-if-integrable*:
assumes *integrable* (*qbs-prob-measure s*) *f*
shows *qbs-integrable* (*s*::′*a qbs-prob-space*) *f*
⟨*proof*⟩

lemma *integrable-if-qbs-integrable*:
assumes *qbs-integrable* (*s*::′*a qbs-prob-space*) *f*
shows *integrable* (*qbs-prob-measure s*) *f*
⟨*proof*⟩

lemma *qbs-integrable-iff-bounded*:
assumes *qbs-prob-space-qbs* $s = X$
shows *qbs-integrable s f* $\longleftrightarrow f \in X \rightarrow_Q \mathbb{R}_Q \wedge$ *qbs-prob-ennintegral s* ($\lambda x.$ *ennreal* $|f\ x\ |) < \infty$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-integrable-cong*:
assumes *qbs-prob-space-qbs* $s = X$
$\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
and *qbs-integrable s f*
shows *qbs-integrable s g*
⟨*proof*⟩

lemma *qbs-integrable-const*[*simp*]:
qbs-integrable s ($\lambda x.\ c$)
⟨*proof*⟩

lemma *qbs-integrable-add*[*simp*]:
assumes *qbs-integrable s f*
and *qbs-integrable s g*
shows *qbs-integrable s* ($\lambda x.\ f\ x + g\ x$)

⟨*proof*⟩

lemma *qbs-integrable-diff*[*simp*]:
assumes *qbs-integrable s f*
and *qbs-integrable s g*
shows *qbs-integrable s* ($\lambda x.$ *f x* − *g x*)
⟨*proof*⟩

lemma *qbs-integrable-mult-iff*[*simp*]:
(*qbs-integrable s* ($\lambda x.$ *c* ∗ *f x*)) = (*c* = *0* ∨ *qbs-integrable s f*)
⟨*proof*⟩

lemma *qbs-integrable-mult*[*simp*]:
assumes *qbs-integrable s f*
shows *qbs-integrable s* ($\lambda x.$ *c* ∗ *f x*)
⟨*proof*⟩

lemma *qbs-integrable-abs*[*simp*]:
assumes *qbs-integrable s f*
shows *qbs-integrable s* ($\lambda x.$ |*f x*|)
⟨*proof*⟩

lemma *qbs-integrable-sq*[*simp*]:
assumes *qbs-integrable s f*
and *qbs-integrable s* ($\lambda x.$ $(f\ x)^2$)
shows *qbs-integrable s* ($\lambda x.$ $(f\ x - c)^2$)
⟨*proof*⟩

lemma *qbs-ennintegral-eq-qbs-integral*:
assumes *qbs-prob-space-qbs s* = *X*
qbs-integrable s f
and $\bigwedge x.\ x \in$ *qbs-space X* ⟹ *0* ≤ *f x*
shows *qbs-prob-ennintegral s* ($\lambda x.$ *ennreal* (*f x*)) = *ennreal* (*qbs-prob-integral s f*)
⟨*proof*⟩

lemma *qbs-prob-ennintegral-cong*:
assumes *qbs-prob-space-qbs s* = *X*
and $\bigwedge x.\ x \in$ *qbs-space X* ⟹ *f x* = *g x*
shows *qbs-prob-ennintegral s f* = *qbs-prob-ennintegral s g*
⟨*proof*⟩

lemma *qbs-prob-ennintegral-const*:
qbs-prob-ennintegral (*s*::′*a qbs-prob-space*) ($\lambda x.$ *c*) = *c*
⟨*proof*⟩

lemma *qbs-prob-ennintegral-add*:
assumes *qbs-prob-space-qbs s* = *X*

$f \in (X{::}'a\ quasi\text{-}borel) \rightarrow_Q \mathbb{R}_{Q\geq 0}$
and $g \in X \rightarrow_Q \mathbb{R}_{Q\geq 0}$
shows *qbs-prob-ennintegral s* ($\lambda x.\ f\ x + g\ x$) = *qbs-prob-ennintegral s f* + *qbs-prob-ennintegral s g*
⟨*proof*⟩

lemma *qbs-prob-ennintegral-cmult*:
assumes *qbs-prob-space-qbs s* = X
and $f \in X \rightarrow_Q \mathbb{R}_{Q\geq 0}$
shows *qbs-prob-ennintegral s* ($\lambda x.\ c * f\ x$) = c ∗ *qbs-prob-ennintegral s f*
⟨*proof*⟩

lemma *qbs-prob-ennintegral-cmult-noninfty*:
assumes $c \neq \infty$
shows *qbs-prob-ennintegral s* ($\lambda x.\ c * f\ x$) = c ∗ *qbs-prob-ennintegral s f*
⟨*proof*⟩

lemma *qbs-prob-integral-cong*:
assumes *qbs-prob-space-qbs s* = X
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
shows *qbs-prob-integral s f* = *qbs-prob-integral s g*
⟨*proof*⟩

lemma *qbs-prob-integral-nonneg*:
assumes *qbs-prob-space-qbs s* = X
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow 0 \leq f\ x$
shows $0 \leq$ *qbs-prob-integral s f*
⟨*proof*⟩

lemma *qbs-prob-integral-mono*:
assumes *qbs-prob-space-qbs s* = X
qbs-integrable (s :: $'a$ *qbs-prob-space*) f
qbs-integrable s g
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x \leq g\ x$
shows *qbs-prob-integral s f* $\leq$ *qbs-prob-integral s g*
⟨*proof*⟩

lemma *qbs-prob-integral-const*:
qbs-prob-integral ($s{::}'a$ *qbs-prob-space*) ($\lambda x.\ c$) = c
⟨*proof*⟩

lemma *qbs-prob-integral-add*:
assumes *qbs-integrable* ($s{::}'a$ *qbs-prob-space*) f
and *qbs-integrable s g*
shows *qbs-prob-integral s* ($\lambda x.\ f\ x + g\ x$) = *qbs-prob-integral s f* + *qbs-prob-integral s g*
⟨*proof*⟩

lemma *qbs-prob-integral-diff*:

assumes *qbs-integrable* (*s*::′*a qbs-prob-space*) *f*
and *qbs-integrable s g*
shows *qbs-prob-integral s* ($\lambda x.$ *f x* − *g x*) = *qbs-prob-integral s f* − *qbs-prob-integral s g*
⟨*proof*⟩

lemma *qbs-prob-integral-cmult*:
qbs-prob-integral s ($\lambda x.$ *c* ∗ *f x*) = *c* ∗ *qbs-prob-integral s f*
⟨*proof*⟩

lemma *real-qbs-prob-integral-def*:
assumes *qbs-integrable* (*s*::′*a qbs-prob-space*) *f*
shows *qbs-prob-integral s f* = *enn2real* (*qbs-prob-ennintegral s* ($\lambda x.$ *ennreal* (*f x*))) − *enn2real* (*qbs-prob-ennintegral s* ($\lambda x.$ *ennreal* (− *f x*)))
⟨*proof*⟩

lemma *qbs-prob-var-eq*:
assumes *qbs-integrable* (*s*::′*a qbs-prob-space*) *f*
and *qbs-integrable s* ($\lambda x.$ $(f\,x)^2$)
shows *qbs-prob-var s f* = *qbs-prob-integral s* ($\lambda x.$ $(f\,x)^2$) − (*qbs-prob-integral s f*)^2
⟨*proof*⟩

lemma *qbs-prob-var-affine*:
assumes *qbs-integrable s f*
shows *qbs-prob-var s* ($\lambda x.$ *a* ∗ *f x* + *b*) = a^2 ∗ *qbs-prob-var s f*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-integral-Markov-inequality*:
assumes *qbs-prob-space-qbs s* = *X*
and *qbs-integrable s f*
$\bigwedge x.$ *x* ∈ *qbs-space X* ⟹ *0* ≤ *f x*
and *0* < *c*
shows *qbs-emeasure s* {*x* ∈ *qbs-space X*. *c* ≤ *f x*} ≤ *ennreal* (*1*/*c* ∗ *qbs-prob-integral s f*)
⟨*proof*⟩

lemma *qbs-prob-integral-Markov-inequality*′:
assumes *qbs-prob-space-qbs s* = *X*
qbs-integrable s f
$\bigwedge x.$ *x* ∈ *qbs-space* (*qbs-prob-space-qbs s*) ⟹ *0* ≤ *f x*
and *0* < *c*
shows *qbs-measure s* {*x* ∈ *qbs-space* (*qbs-prob-space-qbs s*). *c* ≤ *f x*} ≤ (*1*/*c* ∗ *qbs-prob-integral s f*)
⟨*proof*⟩

lemma *qbs-prob-integral-Markov-inequality-abs*:
assumes *qbs-prob-space-qbs s* = *X*

qbs-integrable s f
and $0 < c$
shows *qbs-emeasure s* $\{x \in$ *qbs-space X*. $c \le |f\,x|\} \le$ *ennreal* $(1/c *$ *qbs-prob-integral s* $(\lambda x.\ |f\,x|))$
⟨*proof*⟩

lemma *qbs-prob-integral-Markov-inequality-abs′*:
assumes *qbs-prob-space-qbs* $s = X$
qbs-integrable s f
and $0 < c$
shows *qbs-measure s* $\{x \in$ *qbs-space X*. $c \le |f\,x|\} \le (1/c *$ *qbs-prob-integral s* $(\lambda x.\ |f\,x|))$
⟨*proof*⟩

lemma *qbs-prob-integral-real-Markov-inequality*:
assumes *qbs-prob-space-qbs* $s = \mathbb{R}_Q$
qbs-integrable s f
and $0 < c$
shows *qbs-emeasure s* $\{r.\ c \le |f\,r|\} \le$ *ennreal* $(1/c *$ *qbs-prob-integral s* $(\lambda x.\ |f\,x|))$
⟨*proof*⟩

lemma *qbs-prob-integral-real-Markov-inequality′*:
assumes *qbs-prob-space-qbs* $s = \mathbb{R}_Q$
qbs-integrable s f
and $0 < c$
shows *qbs-measure s* $\{r.\ c \le |f\,r|\} \le 1/c *$ *qbs-prob-integral s* $(\lambda x.\ |f\,x|)$
⟨*proof*⟩

lemma *qbs-prob-integral-Chebyshev-inequality*:
assumes *qbs-prob-space-qbs* $s = X$
qbs-integrable s f
qbs-integrable s $(\lambda x.\ (f\,x)^2)$
and $0 < b$
shows *qbs-measure s* $\{x \in$ *qbs-space X*. $b \le |f\,x -$ *qbs-prob-integral s f*$|\} \le 1 / b^2 *$ *qbs-prob-var s f*
⟨*proof*⟩

end

3.2 The Probability Monad

theory *Monad-QuasiBorel*
imports *Probability-Space-QuasiBorel*
begin

3.2.1 The Probability Monad *P*

definition *monadP-qbs-Px* :: *′a quasi-borel* ⇒ *′a qbs-prob-space set* **where**
monadP-qbs-Px X ≡ $\{s.$ *qbs-prob-space-qbs* $s = X\}$

locale *in-Px* =
  fixes X :: *'a quasi-borel* **and** *s* :: *'a qbs-prob-space*
  assumes *in-Px*:$s \in$ *monadP-qbs-Px X*
begin

lemma *qbs-prob-space-X*[*simp*]:
  qbs-prob-space-qbs s = X
  ⟨*proof*⟩

end

locale *in-MPx* =
  fixes X :: *'a quasi-borel* **and** β :: *real* ⇒ *'a qbs-prob-space*
  assumes *ex*:$\exists\,\alpha\in$ *qbs-Mx X*. $\exists\, g \in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel*.
    $\forall\, r.\ \beta\ r$ = *qbs-prob-space* $(X,\alpha,g\ r)$
begin

lemma *rep-inMPx*:
  $\exists\,\alpha\ g.\ \alpha \in$ *qbs-Mx X* $\wedge\ g \in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel* $\wedge$
    $\beta = (\lambda r.$ *qbs-prob-space* $(X,\alpha,g\ r))$
⟨*proof*⟩

end

definition *monadP-qbs-MPx* :: *'a quasi-borel* ⇒ (*real* ⇒ *'a qbs-prob-space*) *set*
where
monadP-qbs-MPx X ≡ $\{\beta.$ *in-MPx X* $\beta\}$

definition *monadP-qbs* :: *'a quasi-borel* ⇒ *'a qbs-prob-space quasi-borel* **where**
monadP-qbs X ≡ *Abs-quasi-borel* (*monadP-qbs-Px X*, *monadP-qbs-MPx X*)

lemma(**in** *qbs-prob*) *qbs-prob-space-in-Px*:
  qbs-prob-space $(X,\alpha,\mu) \in$ *monadP-qbs-Px X*
  ⟨*proof*⟩

lemma *rep-monadP-qbs-Px*:
  assumes $s \in$ *monadP-qbs-Px X*
  shows $\exists\,\alpha\ \mu.\ s$ = *qbs-prob-space* $(X, \alpha, \mu)\ \wedge$ *qbs-prob X* $\alpha\ \mu$
  ⟨*proof*⟩

lemma *rep-monadP-qbs-MPx*:
  assumes $\beta \in$ *monadP-qbs-MPx X*
  shows $\exists\,\alpha\ g.\ \alpha \in$ *qbs-Mx X* $\wedge\ g \in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel* $\wedge$
    $\beta = (\lambda r.$ *qbs-prob-space* $(X,\alpha,g\ r))$
  ⟨*proof*⟩

lemma *qbs-prob-MPx*:
  assumes $\alpha \in$ *qbs-Mx X*

and $g \in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel*
shows *qbs-prob X* α $(g\ r)$
⟨*proof*⟩

lemma *monadP-qbs-f*[*simp*]: *monadP-qbs-MPx X* $\subseteq$ *UNIV* $\rightarrow$ *monadP-qbs-Px X*
⟨*proof*⟩

lemma *monadP-qbs-closed1*: *qbs-closed1* (*monadP-qbs-MPx X*)
⟨*proof*⟩

lemma *monadP-qbs-closed2*: *qbs-closed2* (*monadP-qbs-Px X*) (*monadP-qbs-MPx X*)
⟨*proof*⟩

lemma *monadP-qbs-closed3*: *qbs-closed3* (*monadP-qbs-MPx* (*X* :: *'a quasi-borel*))
⟨*proof*⟩

lemma *monadP-qbs-correct*: *Rep-quasi-borel* (*monadP-qbs X*) = (*monadP-qbs-Px X*, *monadP-qbs-MPx X*)
⟨*proof*⟩

lemma *monadP-qbs-space*[*simp*] : *qbs-space* (*monadP-qbs X*) = *monadP-qbs-Px X*
⟨*proof*⟩

lemma *monadP-qbs-Mx*[*simp*] : *qbs-Mx* (*monadP-qbs X*) = *monadP-qbs-MPx X*
⟨*proof*⟩

lemma *monadP-qbs-empty-iff*:
qbs-space X = {} $\longleftrightarrow$ *qbs-space* (*monadP-qbs X*) = {}
⟨*proof*⟩

If $\beta \in MPx$, there exists $X\ \alpha\ g$ s.t.$\beta\ r = [X,\alpha,g\ r]$. We define a function which picks $X\ \alpha\ g$ from $\beta \in MPx$.

definition *rep-monadP-qbs-MPx* :: (*real* $\Rightarrow$ *'a qbs-prob-space*) $\Rightarrow$ *'a quasi-borel* $\times$ (*real* $\Rightarrow$ *'a*) $\times$ (*real* $\Rightarrow$ *real measure*) **where**
rep-monadP-qbs-MPx $\beta \equiv$ *let X* = *qbs-prob-space-qbs* (β *undefined*);
αg = (*SOME k.* (*fst k*) $\in$ *qbs-Mx X* $\wedge$ (*snd k*) $\in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel*
$\wedge\ \beta = (\lambda r.$ *qbs-prob-space* (*X*,*fst k*,*snd k r*)))
in (X,αg)

lemma *qbs-prob-measure-measurable*[*measurable*]:
qbs-prob-measure $\in$ *qbs-to-measure* (*monadP-qbs* (*X* :: *'a quasi-borel*)) $\rightarrow_M$ *prob-algebra* (*qbs-to-measure X*)
⟨*proof*⟩

lemma *qbs-l-inj*:
inj-on qbs-prob-measure (*monadP-qbs-Px X*)
⟨*proof*⟩

lemma *qbs-prob-measure-measurable'*[*measurable*]:
qbs-prob-measure $\in$ *qbs-to-measure* (*monadP-qbs* (X :: *'a quasi-borel*)) $\rightarrow_M$ *sub-prob-algebra* (*qbs-to-measure* X)
⟨*proof*⟩

3.2.2 Return

definition *qbs-return* :: [*'a quasi-borel*, *'a*] $\Rightarrow$ *'a qbs-prob-space* **where**
qbs-return X x $\equiv$ *qbs-prob-space* (X,λr. x,*Eps real-distribution*)

lemma(**in** *real-distribution*) *qbs-return-qbs-prob*:
assumes $x \in$ *qbs-space* X
shows *qbs-prob* X (λr. x) M
⟨*proof*⟩

lemma(**in** *real-distribution*) *qbs-return-computation* :
assumes $x \in$ *qbs-space* X
shows *qbs-return* X x = *qbs-prob-space* (X,λr. x,M)
⟨*proof*⟩

lemma *qbs-return-morphism*:
qbs-return $X \in X \rightarrow_Q$ *monadP-qbs* X
⟨*proof*⟩

lemma *qbs-return-morphism'*:
assumes $f \in X \rightarrow_Q Y$
shows (λx. *qbs-return* Y (f x)) $\in X \rightarrow_Q$ *monadP-qbs* Y
⟨*proof*⟩

3.2.3 Bind

definition *qbs-bind* :: *'a qbs-prob-space* $\Rightarrow$ (*'a* $\Rightarrow$ *'b qbs-prob-space*) $\Rightarrow$ *'b qbs-prob-space*
where
qbs-bind s f $\equiv$ (*let* (*qbsx*,α,μ) = *rep-qbs-prob-space* s;
(*qbsy*,β,g) = *rep-monadP-qbs-MPx* ($f \circ \alpha$)
in *qbs-prob-space* (*qbsy*,β,$\mu \gg= g$))

adhoc-overloading *Monad-Syntax.bind qbs-bind*

lemma(**in** *qbs-prob*) *qbs-bind-computation*:
assumes$s$ = *qbs-prob-space* (X,α,μ)
$f \in X \rightarrow_Q$ *monadP-qbs* Y
$\beta \in$ *qbs-Mx* Y
and [*measurable*]: $g \in$ *real-borel* $\rightarrow_M$ *prob-algebra real-borel*
and ($f \circ \alpha$) = (λr. *qbs-prob-space* (Y,β, g r))
shows *qbs-prob* Y β ($\mu \gg= g$)
$s \gg= f$ = *qbs-prob-space* (Y,β,$\mu \gg= g$)
⟨*proof*⟩

lemma *qbs-bind-morphism′*:
 assumes $f \in X \rightarrow_Q$ *monadP-qbs* Y
 shows $(\lambda x.\ x \gg= f) \in$ *monadP-qbs* $X \rightarrow_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *qbs-return-comp*:
 assumes $\alpha \in$ *qbs-Mx* X
 shows (*qbs-return* $X \circ \alpha$) = ($\lambda r.$ *qbs-prob-space* (X,α,*return real-borel* r))
⟨*proof*⟩

lemma *qbs-bind-return′*:
 assumes $x \in$ *monadP-qbs-Px* X
 shows $x \gg=$ *qbs-return* $X = x$
⟨*proof*⟩

lemma *qbs-bind-return*:
 assumes $f \in X \rightarrow_Q$ *monadP-qbs* Y
  and $x \in$ *qbs-space* X
  shows *qbs-return* $X\ x \gg= f = f\ x$
⟨*proof*⟩

lemma *qbs-bind-assoc*:
 assumes $s \in$ *monadP-qbs-Px* X
  $f \in X \rightarrow_Q$ *monadP-qbs* Y
  and $g \in Y \rightarrow_Q$ *monadP-qbs* Z
 shows $s \gg= (\lambda x.\ f\ x \gg= g) = (s \gg= f) \gg= g$
⟨*proof*⟩

lemma *qbs-bind-cong*:
 assumes $s \in$ *monadP-qbs-Px* X
  $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow f\ x = g\ x$
  and $f \in X \rightarrow_Q$ *monadP-qbs* Y
  shows $s \gg= f = s \gg= g$
⟨*proof*⟩

3.2.4 The Functorial Action $P(f)$

definition *monadP-qbs-Pf* :: [*′a quasi-borel*, *′b quasi-borel*,*′a* ⇒ *′b*,*′a qbs-prob-space*] ⇒ *′b qbs-prob-space* **where**
monadP-qbs-Pf - $Y\ f\ sx \equiv sx \gg=$ *qbs-return* $Y \circ f$

lemma *monadP-qbs-Pf-morphism*:
 assumes $f \in X \rightarrow_Q Y$
 shows *monadP-qbs-Pf* $X\ Y\ f \in$ *monadP-qbs* $X \rightarrow_Q$ *monadP-qbs* Y
 ⟨*proof*⟩

lemma(**in** *qbs-prob*) *monadP-qbs-Pf-computation*:
 assumes $s =$ *qbs-prob-space* (X,α,μ)
  and $f \in X \rightarrow_Q Y$

shows *qbs-prob* Y $(f \circ \alpha)$ μ
and *monadP-qbs-Pf* X Y f s = *qbs-prob-space* $(Y,f \circ \alpha,\mu)$
⟨*proof*⟩

We show that P is a functor i.e. P preserves identity and composition.

lemma *monadP-qbs-Pf-id*:
assumes $s \in$ *monadP-qbs-Px* X
shows *monadP-qbs-Pf* X X *id* $s = s$
⟨*proof*⟩

lemma *monadP-qbs-Pf-comp*:
assumes $s \in$ *monadP-qbs-Px* X
$f \in X \to_Q Y$
and $g \in Y \to_Q Z$
shows ((*monadP-qbs-Pf* Y Z g) $\circ$ (*monadP-qbs-Pf* X Y f)) s = *monadP-qbs-Pf* X Z $(g \circ f)$ s
⟨*proof*⟩

3.2.5 Join

definition *qbs-join* :: *'a qbs-prob-space qbs-prob-space* ⇒ *'a qbs-prob-space* **where**
qbs-join ≡ (λ*sst. sst* $\gg\!=$ *id*)

lemma *qbs-join-morphism*:
qbs-join $\in$ *monadP-qbs* (*monadP-qbs* X) $\to_Q$ *monadP-qbs* X
⟨*proof*⟩

lemma *qbs-join-computation*:
assumes *qbs-prob* (*monadP-qbs* X) β μ
ssx = *qbs-prob-space* (*monadP-qbs* X,β,μ)
$\alpha \in$ *qbs-Mx* X
$g \in$ *real-borel* $\to_M$ *prob-algebra real-borel*
and β =(λr. *qbs-prob-space* (X,α,g r))
shows *qbs-prob* X α ($\mu \gg\!= g$) *qbs-join ssx* = *qbs-prob-space* (X,α, $\mu \gg\!= g$)
⟨*proof*⟩

3.2.6 Strength

definition *qbs-strength* :: [*'a quasi-borel*,*'b quasi-borel*,*'a* × *'b qbs-prob-space*] ⇒ (*'a* × *'b*) *qbs-prob-space* **where**
qbs-strength W X = (λ(w,*sx*). *let* (-,α,μ) = *rep-qbs-prob-space sx*
in qbs-prob-space ($W \bigotimes_Q X$, λr. (w,α r), μ))

lemma(**in** *qbs-prob*) *qbs-strength-computation*:
assumes $w \in$ *qbs-space* W
and *sx* = *qbs-prob-space* (X,α,μ)
shows *qbs-prob* ($W \bigotimes_Q X$) (λr. (w,α r)) μ
qbs-strength W X (w,*sx*) = *qbs-prob-space* ($W \bigotimes_Q X$, λr. (w,α r), μ)
⟨*proof*⟩

lemma *qbs-strength-natural*:
assumes $f \in X \to_Q X'$
$g \in Y \to_Q Y'$
$x \in$ *qbs-space* X
and $sy \in$ *monadP-qbs-Px* Y
shows (*monadP-qbs-Pf* $(X \bigotimes_Q Y)$ $(X' \bigotimes_Q Y')$ (*map-prod* f g) $\circ$ *qbs-strength* X Y) (x,sy) = (*qbs-strength* X' Y' $\circ$ *map-prod* f (*monadP-qbs-Pf* Y Y' g)) (x,sy)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-strength-ab-r*:
assumes $\alpha \in$ *qbs-Mx* X
$\beta \in$ *monadP-qbs-MPx* Y
$\gamma \in$ *qbs-Mx* Y
and [*measurable*]:$g \in$ *real-borel* $\to_M$ *prob-algebra real-borel*
and $\beta = (\lambda r.$ *qbs-prob-space* $(Y,\gamma,g\ r))$
shows *qbs-prob* $(X \bigotimes_Q Y)$ (*map-prod* α γ $\circ$ *real-real.g*) (*distr* (*return real-borel* r $\bigotimes_M$ g r) *real-borel real-real.f*)
qbs-strength X Y $(\alpha\ r,\ \beta\ r)$ = *qbs-prob-space* $(X \bigotimes_Q Y$, *map-prod* α γ $\circ$ *real-real.g*, *distr* (*return real-borel* r $\bigotimes_M$ g r) *real-borel real-real.f*)
⟨*proof*⟩

lemma *qbs-strength-morphism*:
qbs-strength X $Y \in X \bigotimes_Q$ *monadP-qbs* $Y \to_Q$ *monadP-qbs* $(X \bigotimes_Q Y)$
⟨*proof*⟩

lemma *qbs-bind-morphism''*:
$(\lambda(f,x).\ x \gg= f) \in$ *exp-qbs* X (*monadP-qbs* Y) $\bigotimes_Q$ (*monadP-qbs* X) $\to_Q$ (*monadP-qbs* Y)
⟨*proof*⟩

lemma *qbs-bind-morphism'''*:
$(\lambda f\ x.\ x \gg= f) \in$ *exp-qbs* X (*monadP-qbs* Y) $\to_Q$ *exp-qbs* (*monadP-qbs* X) (*monadP-qbs* Y)
⟨*proof*⟩

lemma *qbs-bind-morphism*:
assumes $f \in X \to_Q$ *monadP-qbs* Y
and $g \in X \to_Q$ *exp-qbs* Y (*monadP-qbs* Z)
shows $(\lambda x.\ f\ x \gg= g\ x) \in X \to_Q$ *monadP-qbs* Z
⟨*proof*⟩

lemma *qbs-bind-morphism''''*:
assumes $x \in$ *monadP-qbs-Px* X
shows $(\lambda f.\ x \gg= f) \in$ *exp-qbs* X (*monadP-qbs* Y) $\to_Q$ *monadP-qbs* Y
⟨*proof*⟩

lemma *qbs-strength-law1*:

assumes $x \in$ *qbs-space* (*unit-quasi-borel* $\bigotimes_Q$ *monadP-qbs* X)
shows *snd* $x =$ (*monadP-qbs-Pf* (*unit-quasi-borel* $\bigotimes_Q$ X) X *snd* $\circ$ *qbs-strength* *unit-quasi-borel* X) x
⟨*proof*⟩

lemma *qbs-strength-law2*:
assumes $x \in$ *qbs-space* (($X \bigotimes_Q Y$) $\bigotimes_Q$ *monadP-qbs* Z)
shows (*qbs-strength* X ($Y \bigotimes_Q Z$) $\circ$ (*map-prod id* (*qbs-strength* Y Z)) $\circ$ ($\lambda((x,y),z).$ $(x,(y,z))$)) $x =$
(*monadP-qbs-Pf* (($X \bigotimes_Q Y$) $\bigotimes_Q Z$) ($X \bigotimes_Q$ ($Y \bigotimes_Q Z$)) ($\lambda((x,y),z).$ $(x,(y,z))$) $\circ$ *qbs-strength* ($X \bigotimes_Q Y$) Z) x
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-strength-law3*:
assumes $x \in$ *qbs-space* ($X \bigotimes_Q Y$)
shows *qbs-return* ($X \bigotimes_Q Y$) $x =$ (*qbs-strength* X Y $\circ$ (*map-prod id* (*qbs-return* Y))) x
⟨*proof*⟩

lemma *qbs-strength-law4*:
assumes $x \in$ *qbs-space* ($X \bigotimes_Q$ *monadP-qbs* (*monadP-qbs* Y))
shows (*qbs-strength* X Y $\circ$ *map-prod id qbs-join*) $x =$ (*qbs-join* $\circ$ *monadP-qbs-Pf* ($X \bigotimes_Q$ *monadP-qbs* Y) (*monadP-qbs* ($X \bigotimes_Q Y$))(*qbs-strength* X Y) $\circ$ *qbs-strength* X (*monadP-qbs* Y)) x
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-return-Mxpair*:
assumes $\alpha \in$ *qbs-Mx* X
and $\beta \in$ *qbs-Mx* Y
shows *qbs-return* ($X \bigotimes_Q Y$) (α r, β k) = *qbs-prob-space* ($X \bigotimes_Q Y$,*map-prod* α β $\circ$ *real-real.g*, *distr* (*return real-borel* r $\bigotimes_M$ *return real-borel* k) *real-borel* *real-real.f*)
qbs-prob ($X \bigotimes_Q Y$) (*map-prod* α β $\circ$ *real-real.g*) (*distr* (*return real-borel* r $\bigotimes_M$ *return real-borel* k) *real-borel real-real.f*)
⟨*proof*⟩

lemma *pair-return-return*:
assumes $l \in$ *space* M
and $r \in$ *space* N
shows *return* M l $\bigotimes_M$ *return* N r = *return* ($M \bigotimes_M N$) (l,r)
⟨*proof*⟩

lemma *bind-bind-return-distr*:
assumes *real-distribution* μ
and *real-distribution* ν

shows $\mu \gg= (\lambda r.\ \nu \gg= (\lambda l.\ distr\ (return\ real\text{-}borel\ r \bigotimes_M return\ real\text{-}borel\ l)\ real\text{-}borel\ real\text{-}real.f))$
$= distr\ (\mu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-bind-return-qp*:
shows $qbs\text{-}prob\text{-}space\ (Y,\ \beta,\ \nu) \gg= (\lambda y.\ qbs\text{-}prob\text{-}space\ (X,\ \alpha,\ \mu) \gg= (\lambda x.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y))) = qbs\text{-}prob\text{-}space\ (X \bigotimes_Q Y,\ map\text{-}prod\ \alpha\ \beta \circ real\text{-}real.g,\ distr\ (\mu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
$qbs\text{-}prob\ (X \bigotimes_Q Y)\ (map\text{-}prod\ \alpha\ \beta \circ real\text{-}real.g)\ (distr\ (\mu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-bind-return-pq*:
shows $qbs\text{-}prob\text{-}space\ (X,\ \alpha,\ \mu) \gg= (\lambda x.\ qbs\text{-}prob\text{-}space\ (Y,\ \beta,\ \nu) \gg= (\lambda y.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y))) = qbs\text{-}prob\text{-}space\ (X \bigotimes_Q Y,\ map\text{-}prod\ \alpha\ \beta \circ real\text{-}real.g,\ distr\ (\mu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
$qbs\text{-}prob\ (X \bigotimes_Q Y)\ (map\text{-}prod\ \alpha\ \beta \circ real\text{-}real.g)\ (distr\ (\mu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
⟨*proof*⟩

lemma *qbs-bind-return-rotate*:
assumes $p \in monadP\text{-}qbs\text{-}Px\ X$
and $q \in monadP\text{-}qbs\text{-}Px\ Y$
shows $q \gg= (\lambda y.\ p \gg= (\lambda x.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y))) = p \gg= (\lambda x.\ q \gg= (\lambda y.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y)))$
⟨*proof*⟩

lemma *qbs-pair-bind-return1*:
assumes $f \in X \bigotimes_Q Y \to_Q monadP\text{-}qbs\ Z$
$p \in monadP\text{-}qbs\text{-}Px\ X$
and $q \in monadP\text{-}qbs\text{-}Px\ Y$
shows $q \gg= (\lambda y.\ p \gg= (\lambda x.\ f\ (x,y))) = (q \gg= (\lambda y.\ p \gg= (\lambda x.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y)))) \gg= f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-pair-bind-return2*:
assumes $f \in X \bigotimes_Q Y \to_Q monadP\text{-}qbs\ Z$
$p \in monadP\text{-}qbs\text{-}Px\ X$
and $q \in monadP\text{-}qbs\text{-}Px\ Y$
shows $p \gg= (\lambda x.\ q \gg= (\lambda y.\ f\ (x,y))) = (p \gg= (\lambda x.\ q \gg= (\lambda y.\ qbs\text{-}return\ (X \bigotimes_Q Y)\ (x,y)))) \gg= f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-bind-rotate*:
assumes $f \in X \bigotimes_Q Y \to_Q monadP\text{-}qbs\ Z$

$p \in$ *monadP-qbs-Px X*
and $q \in$ *monadP-qbs-Px Y*
shows $q \ggg= (\lambda y.\ p \ggg= (\lambda x.\ f\ (x,y))) = p \ggg= (\lambda x.\ q \ggg= (\lambda y.\ f\ (x,y)))$
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-bind-bind-return*:
assumes $f \in X \bigotimes_Q Y \to_Q Z$
shows *qbs-prob Z* $(f \circ (\textit{map-prod}\ \alpha\ \beta \circ \textit{real-real.g}))$ $(\textit{distr}\ (\mu \bigotimes_M \nu)\ \textit{real-borel}\ \textit{real-real.f})$
and *qbs-prob-space* $(X,\alpha,\mu) \ggg= (\lambda x.\ \textit{qbs-prob-space}\ (Y,\beta,\nu) \ggg= (\lambda y.\ \textit{qbs-return}\ Z\ (f\ (x,y)))) = \textit{qbs-prob-space}\ (Z,f \circ (\textit{map-prod}\ \alpha\ \beta \circ \textit{real-real.g}),\textit{distr}\ (\mu \bigotimes_M \nu)\ \textit{real-borel}\ \textit{real-real.f})$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

3.2.7 Properties of Return and Bind

lemma *qbs-prob-measure-return*:
assumes $x \in$ *qbs-space X*
shows *qbs-prob-measure* (*qbs-return X x*) = *return* (*qbs-to-measure X*) *x*
⟨*proof*⟩

lemma *qbs-prob-measure-bind*:
assumes $s \in$ *monadP-qbs-Px X*
and $f \in X \to_Q$ *monadP-qbs Y*
shows *qbs-prob-measure* $(s \ggg= f)$ = *qbs-prob-measure s* $\ggg=$ *qbs-prob-measure* $\circ f$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *qbs-of-return*:
assumes $x \in$ *qbs-space X*
shows *qbs-prob-space-qbs* (*qbs-return X x*) = *X*
⟨*proof*⟩

lemma *qbs-of-bind*:
assumes $s \in$ *monadP-qbs-Px X*
and $f \in X \to_Q$ *monadP-qbs Y*
shows *qbs-prob-space-qbs* $(s \ggg= f) = Y$
⟨*proof*⟩

3.2.8 Properties of Integrals

lemma *qbs-integrable-return*:
assumes $x \in$ *qbs-space X*
and $f \in X \to_Q \mathbb{R}_Q$
shows *qbs-integrable* (*qbs-return X x*) *f*
⟨*proof*⟩

lemma *qbs-integrable-bind-return*:
assumes $s \in$ *monadP-qbs-Px* Y
$f \in Z \to_Q \mathbb{R}_Q$
and $g \in Y \to_Q Z$
shows *qbs-integrable* $(s \gg= (\lambda y.\ \textit{qbs-return}\ Z\ (g\ y)))\ f = \textit{qbs-integrable}\ s\ (f \circ g)$
$\langle \textit{proof} \rangle$

lemma *qbs-prob-ennintegral-morphism*:
assumes $L \in X \to_Q$ *monadP-qbs* Y
and $f \in X \to_Q$ *exp-qbs* $Y\ \mathbb{R}_{Q\geq 0}$
shows $(\lambda x.\ \textit{qbs-prob-ennintegral}\ (L\ x)\ (f\ x)) \in X \to_Q \mathbb{R}_{Q\geq 0}$
$\langle \textit{proof} \rangle$

lemma *qbs-morphism-ennintegral-fst*:
assumes $q \in$ *monadP-qbs-Px* Y
and $f \in X \bigotimes_Q Y \to_Q \mathbb{R}_{Q\geq 0}$
shows $(\lambda x.\ \int^{+}_Q y.\ f\ (x,\ y)\ \partial q) \in X \to_Q \mathbb{R}_{Q\geq 0}$
$\langle \textit{proof} \rangle$

lemma *qbs-morphism-ennintegral-snd*:
assumes $p \in$ *monadP-qbs-Px* X
and $f \in X \bigotimes_Q Y \to_Q \mathbb{R}_{Q\geq 0}$
shows $(\lambda y.\ \int^{+}_Q x.\ f\ (x,\ y)\ \partial p) \in Y \to_Q \mathbb{R}_{Q\geq 0}$
$\langle \textit{proof} \rangle$

lemma *qbs-prob-ennintegral-morphism'*:
assumes $f \in X \to_Q \mathbb{R}_{Q\geq 0}$
shows $(\lambda s.\ \textit{qbs-prob-ennintegral}\ s\ f) \in \textit{monadP-qbs}\ X \to_Q \mathbb{R}_{Q\geq 0}$
$\langle \textit{proof} \rangle$

lemma *qbs-prob-ennintegral-return*:
assumes $f \in X \to_Q \mathbb{R}_{Q\geq 0}$
and $x \in$ *qbs-space* X
shows *qbs-prob-ennintegral* $(\textit{qbs-return}\ X\ x)\ f = f\ x$
$\langle \textit{proof} \rangle$

lemma *qbs-prob-ennintegral-bind*:
assumes $s \in$ *monadP-qbs-Px* X
$f \in X \to_Q$ *monadP-qbs* Y
and $g \in Y \to_Q \mathbb{R}_{Q\geq 0}$
shows *qbs-prob-ennintegral* $(s \gg= f)\ g = \textit{qbs-prob-ennintegral}\ s\ (\lambda y.\ (\textit{qbs-prob-ennintegral}\ (f\ y)\ g))$
(**is** *?lhs* = *?rhs*)
$\langle \textit{proof} \rangle$

lemma *qbs-prob-ennintegral-bind-return*:
assumes $s \in$ *monadP-qbs-Px* Y

$f \in Z \to_Q \mathbb{R}_{Q\geq 0}$
and $g \in Y \to_Q Z$
shows *qbs-prob-ennintegral* $(s \gg= (\lambda y.\ \textit{qbs-return}\ Z\ (g\ y)))\ f = \textit{qbs-prob-ennintegral}\ s\ (f \circ g)$
⟨*proof*⟩

lemma *qbs-prob-integral-morphism′*:
assumes $f \in X \to_Q \mathbb{R}_Q$
shows $(\lambda s.\ \textit{qbs-prob-integral}\ s\ f) \in \textit{monadP-qbs}\ X \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-morphism-integral-fst*:
assumes $q \in \textit{monadP-qbs-Px}\ Y$
and $f \in X \bigotimes_Q Y \to_Q \mathbb{R}_Q$
shows $(\lambda x.\ \int_Q y.\ f\ (x,\ y)\ \partial q) \in X \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-morphism-integral-snd*:
assumes $p \in \textit{monadP-qbs-Px}\ X$
and $f \in X \bigotimes_Q Y \to_Q \mathbb{R}_Q$
shows $(\lambda y.\ \int_Q x.\ f\ (x,\ y)\ \partial p) \in Y \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-prob-integral-morphism*:
assumes $L \in X \to_Q \textit{monadP-qbs}\ Y$
$f \in X \to_Q \textit{exp-qbs}\ Y\ \mathbb{R}_Q$
and $\bigwedge x.\ x \in \textit{qbs-space}\ X \Longrightarrow \textit{qbs-integrable}\ (L\ x)\ (f\ x)$
shows $(\lambda x.\ \textit{qbs-prob-integral}\ (L\ x)\ (f\ x)) \in X \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-prob-integral-morphism″*:
assumes $f \in X \to_Q \mathbb{R}_Q$
and $L \in Y \to_Q \textit{monadP-qbs}\ X$
shows $(\lambda y.\ \textit{qbs-prob-integral}\ (L\ y)\ f) \in Y \to_Q \mathbb{R}_Q$
⟨*proof*⟩

lemma *qbs-prob-integral-return*:
assumes $f \in X \to_Q \mathbb{R}_Q$
and $x \in \textit{qbs-space}\ X$
shows $\textit{qbs-prob-integral}\ (\textit{qbs-return}\ X\ x)\ f = f\ x$
⟨*proof*⟩

lemma *qbs-prob-integral-bind*:
assumes $s \in \textit{monadP-qbs-Px}\ X$
$f \in X \to_Q \textit{monadP-qbs}\ Y$
$g \in Y \to_Q \mathbb{R}_Q$
and $\exists K.\ \forall y \in \textit{qbs-space}\ Y. |g\ y| \leq K$
shows $\textit{qbs-prob-integral}\ (s \gg= f)\ g = \textit{qbs-prob-integral}\ s\ (\lambda y.\ (\textit{qbs-prob-integral}\ (f\ y)\ g))$

(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-integral-bind-return*:
assumes $s \in$ *monadP-qbs-Px Y*
$f \in Z \to_Q \mathbb{R}_Q$
and $g \in Y \to_Q Z$
shows *qbs-prob-integral* ($s \gg\!= (\lambda y.$ *qbs-return Z* $(g\ y)))\ f =$ *qbs-prob-integral s* $(f \circ g)$
⟨*proof*⟩

lemma *qbs-prob-var-bind-return*:
assumes $s \in$ *monadP-qbs-Px Y*
$f \in Z \to_Q \mathbb{R}_Q$
and $g \in Y \to_Q Z$
shows *qbs-prob-var* ($s \gg\!= (\lambda y.$ *qbs-return Z* $(g\ y)))\ f =$ *qbs-prob-var s* $(f \circ g)$
⟨*proof*⟩

end

3.3 Binary Product Measure

theory *Pair-QuasiBorel-Measure*
imports *Monad-QuasiBorel*
begin

3.3.1 Binary Product Measure

Special case of [1] Proposition 23 where $\Omega = \mathbb{R} \times \mathbb{R}$ and $X = X \times Y$. Let $[\alpha, \mu] \in P(X)$ and $[\beta, \nu] \in P(Y)$. $\alpha \times \beta$ is the α in Proposition 23.

definition *qbs-prob-pair-measure-t* :: ['*a qbs-prob-t*, '*b qbs-prob-t*] ⇒ ('*a* × '*b*) *qbs-prob-t* **where**
qbs-prob-pair-measure-t p q ≡ (*let* $(X,\alpha,\mu) = p$;
$(Y,\beta,\nu) = q$ *in*
$(X \bigotimes_Q Y$, *map-prod* $\alpha\ \beta \circ$ *real-real.g*, *distr* $(\mu \bigotimes_M \nu)$ *real-borel real-real.f*))

lift-definition *qbs-prob-pair-measure* :: ['*a qbs-prob-space*, '*b qbs-prob-space*] ⇒ ('*a* × '*b*) *qbs-prob-space* (**infix** $\bigotimes_{Qmes}$ *80*)
is *qbs-prob-pair-measure-t*
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-computation*:
(*qbs-prob-space* (X,α,μ)) $\bigotimes_{Qmes}$ (*qbs-prob-space* (Y,β,ν)) = *qbs-prob-space* $(X \bigotimes_Q Y$, *map-prod* $\alpha\ \beta \circ$ *real-real.g* , *distr* $(\mu \bigotimes_M \nu)$ *real-borel real-real.f*)
qbs-prob $(X \bigotimes_Q Y)$ (*map-prod* $\alpha\ \beta \circ$ *real-real.g*) (*distr* $(\mu \bigotimes_M \nu)$ *real-borel real-real.f*)
⟨*proof*⟩

lemma *qbs-prob-pair-measure-qbs*:
qbs-prob-space-qbs $(p \bigotimes_{Qmes} q)$ = *qbs-prob-space-qbs* $p \bigotimes_Q$ *qbs-prob-space-qbs* q
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-measure*:
shows *qbs-prob-measure* (*qbs-prob-space* $(X,\alpha,\mu) \bigotimes_{Qmes}$ *qbs-prob-space* (Y,β,ν)) = *distr* $(\mu \bigotimes_M \nu)$ (*qbs-to-measure* $(X \bigotimes_Q Y)$) (*map-prod* α β)
⟨*proof*⟩

lemma *qbs-prob-pair-measure-morphism*:
case-prod qbs-prob-pair-measure $\in$ *monadP-qbs* $X \bigotimes_Q$ *monadP-qbs* $Y \rightarrow_Q$ *monadP-qbs* $(X \bigotimes_Q Y)$
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-nnintegral*:
assumes $f \in X \bigotimes_Q Y \rightarrow_Q \mathbb{R}_{Q\geq 0}$
shows $(\int^+_Q z.\ f\ z\ \partial($*qbs-prob-space* $(X,\alpha,\mu) \bigotimes_{Qmes}$ *qbs-prob-space* $(Y,\beta,\nu)))$ $= (\int^+ z.\ (f \circ$ *map-prod* $\alpha\ \beta)\ z\ \partial(\mu \bigotimes_M \nu))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-integral*:
assumes $f \in X \bigotimes_Q Y \rightarrow_Q \mathbb{R}_Q$
shows $(\int_Q z.\ f\ z\ \partial($*qbs-prob-space* $(X,\alpha,\mu) \bigotimes_{Qmes}$ *qbs-prob-space* $(Y,\beta,\nu)))$ $= (\int z.\ (f \circ$ *map-prod* $\alpha\ \beta)\ z\ \partial(\mu \bigotimes_M \nu))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-pair-measure-eq-bind*:
assumes $p \in$ *monadP-qbs-Px* X
and $q \in$ *monadP-qbs-Px* Y
shows $p \bigotimes_{Qmes} q = p \gg= (\lambda x.\ q \gg= (\lambda y.$ *qbs-return* $(X \bigotimes_Q Y)\ (x,y)))$
⟨*proof*⟩

3.3.2 Fubini Theorem

lemma *qbs-prob-ennintegral-Fubini-fst*:
assumes $p \in$ *monadP-qbs-Px* X
$q \in$ *monadP-qbs-Px* Y
and $f \in X \bigotimes_Q Y \rightarrow_Q \mathbb{R}_{Q\geq 0}$
shows $(\int^+_Q x.\ \int^+_Q y.\ f\ (x,y)\ \partial q\ \partial p) = (\int^+_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-ennintegral-Fubini-snd*:
assumes $p \in$ *monadP-qbs-Px* X
$q \in$ *monadP-qbs-Px* Y
and $f \in X \bigotimes_Q Y \rightarrow_Q \mathbb{R}_{Q\geq 0}$

shows $(\int^{+}{}_{Q}\ y.\ \int^{+}{}_{Q}\ x.\ f\ (x,y)\ \partial p\ \partial q) = (\int^{+}{}_{Q}\ x.\ f\ x\ \partial(p \bigotimes_{Qmes}\ q))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-ennintegral-indep1*:
assumes $p \in$ *monadP-qbs-Px* X
and $f \in X \rightarrow_{Q} \mathbb{R}_{Q\geq 0}$
shows $(\int^{+}{}_{Q}\ z.\ f\ (\mathit{fst}\ z)\ \partial(p \bigotimes_{Qmes}\ q)) = (\int^{+}{}_{Q}\ x.\ f\ x\ \partial p)$
(**is** *?lhs* = -)
⟨*proof*⟩

lemma *qbs-prob-ennintegral-indep2*:
assumes $q \in$ *monadP-qbs-Px* Y
and $f \in Y \rightarrow_{Q} \mathbb{R}_{Q\geq 0}$
shows $(\int^{+}{}_{Q}\ z.\ f\ (\mathit{snd}\ z)\ \partial(p \bigotimes_{Qmes}\ q)) = (\int^{+}{}_{Q}\ y.\ f\ y\ \partial q)$
(**is** *?lhs* = -)
⟨*proof*⟩

lemma *qbs-ennintegral-indep-mult*:
assumes $p \in$ *monadP-qbs-Px* X
$q \in$ *monadP-qbs-Px* Y
$f \in X \rightarrow_{Q} \mathbb{R}_{Q\geq 0}$
and $g \in Y \rightarrow_{Q} \mathbb{R}_{Q\geq 0}$
shows $(\int^{+}{}_{Q}\ z.\ f\ (\mathit{fst}\ z) * g\ (\mathit{snd}\ z)\ \partial(p \bigotimes_{Qmes}\ q)) = (\int^{+}{}_{Q}\ x.\ f\ x\ \partial p) *$
$(\int^{+}{}_{Q}\ y.\ g\ y\ \partial q)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-integrable*:
assumes *qbs-integrable* (*qbs-prob-space* (X,α,μ) $\bigotimes_{Qmes}$ *qbs-prob-space* (Y,β,ν))
f
shows $f \in X \bigotimes_{Q} Y \rightarrow_{Q} \mathbb{R}_{Q}$
integrable $(\mu \bigotimes_{M} \nu)$ $(f \circ$ (*map-prod* $\alpha\ \beta$))
⟨*proof*⟩

lemma(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-integrable'*:
assumes $f \in X \bigotimes_{Q} Y \rightarrow_{Q} \mathbb{R}_{Q}$
and *integrable* $(\mu \bigotimes_{M} \nu)$ $(f \circ$ (*map-prod* $\alpha\ \beta$))
shows *qbs-integrable* (*qbs-prob-space* (X,α,μ) $\bigotimes_{Qmes}$ *qbs-prob-space* (Y,β,ν))
f
⟨*proof*⟩

lemma *qbs-integrable-pair-swap*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes}\ q)$ f
shows *qbs-integrable* $(q \bigotimes_{Qmes}\ p)$ $(\lambda(x,y).\ f\ (y,x))$
⟨*proof*⟩

lemma *qbs-integrable-pair1*:

assumes $p \in$ *monadP-qbs-Px* X
$q \in$ *monadP-qbs-Px* Y
$f \in X \bigotimes_Q Y \to_Q \mathbb{R}_Q$
qbs-integrable p $(\lambda x.\ \int_Q y.\ |f\ (x,y)|\ \partial q)$
and $\bigwedge x.\ x \in$ *qbs-space* $X \Longrightarrow$ *qbs-integrable* q $(\lambda y.\ f\ (x,y))$
shows *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
⟨*proof*⟩

lemma *qbs-integrable-pair2*:
assumes $p \in$ *monadP-qbs-Px* X
$q \in$ *monadP-qbs-Px* Y
$f \in X \bigotimes_Q Y \to_Q \mathbb{R}_Q$
qbs-integrable q $(\lambda y.\ \int_Q x.\ |f\ (x,y)|\ \partial p)$
and $\bigwedge y.\ y \in$ *qbs-space* $Y \Longrightarrow$ *qbs-integrable* p $(\lambda x.\ f\ (x,y))$
shows *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
⟨*proof*⟩

lemma *qbs-integrable-fst*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
shows *qbs-integrable* p $(\lambda x.\ \int_Q y.\ f\ (x,y)\ \partial q)$
⟨*proof*⟩

lemma *qbs-integrable-snd*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
shows *qbs-integrable* q $(\lambda y.\ \int_Q x.\ f\ (x,y)\ \partial p)$
⟨*proof*⟩

lemma *qbs-integrable-indep-mult*:
assumes *qbs-integrable* p f
and *qbs-integrable* q g
shows *qbs-integrable* $(p \bigotimes_{Qmes} q)$ $(\lambda x.\ f\ (fst\ x) * g\ (snd\ x))$
⟨*proof*⟩

lemma *qbs-integrable-indep1*:
assumes *qbs-integrable* p f
shows *qbs-integrable* $(p \bigotimes_{Qmes} q)$ $(\lambda x.\ f\ (fst\ x))$
⟨*proof*⟩

lemma *qbs-integrable-indep2*:
assumes *qbs-integrable* q g
shows *qbs-integrable* $(p \bigotimes_{Qmes} q)$ $(\lambda x.\ g\ (snd\ x))$
⟨*proof*⟩

lemma *qbs-prob-integral-Fubini-fst*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
shows $(\int_Q x.\ \int_Q y.\ f\ (x,y)\ \partial q\ \partial p) = (\int_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-integral-Fubini-snd*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
shows $(\int_Q y.\ \int_Q x.\ f\ (x,y)\ \partial p\ \partial q) = (\int_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-integral-indep1*:
assumes *qbs-integrable* p f
shows $(\int_Q z.\ f\ (fst\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q x.\ f\ x\ \partial p)$
⟨*proof*⟩

lemma *qbs-prob-integral-indep2*:
assumes *qbs-integrable* q g
shows $(\int_Q z.\ g\ (snd\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q y.\ g\ y\ \partial q)$
⟨*proof*⟩

lemma *qbs-prob-integral-indep-mult*:
assumes *qbs-integrable* p f
and *qbs-integrable* q g
shows $(\int_Q z.\ f\ (fst\ z) * g\ (snd\ z)\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q x.\ f\ x\ \partial p) * (\int_Q y.\ g\ y\ \partial q)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *qbs-prob-var-indep-plus*:
assumes *qbs-integrable* $(p \bigotimes_{Qmes} q)$ f
qbs-integrable $(p \bigotimes_{Qmes} q)$ $(\lambda z.\ (f\ z)^2)$
qbs-integrable $(p \bigotimes_{Qmes} q)$ g
qbs-integrable $(p \bigotimes_{Qmes} q)$ $(\lambda z.\ (g\ z)^2)$
qbs-integrable $(p \bigotimes_{Qmes} q)$ $(\lambda z.\ (f\ z) * (g\ z))$
and $(\int_Q z.\ f\ z * g\ z\ \partial(p \bigotimes_{Qmes} q)) = (\int_Q z.\ f\ z\ \partial(p \bigotimes_{Qmes} q)) * (\int_Q z.\ g\ z\ \partial(p \bigotimes_{Qmes} q))$
shows *qbs-prob-var* $(p \bigotimes_{Qmes} q)$ $(\lambda z.\ f\ z + g\ z)$ = *qbs-prob-var* $(p \bigotimes_{Qmes} q)$ f + *qbs-prob-var* $(p \bigotimes_{Qmes} q)$ g
⟨*proof*⟩

lemma *qbs-prob-var-indep-plus′*:
assumes *qbs-integrable* p f
qbs-integrable p $(\lambda x.\ (f\ x)^2)$
qbs-integrable q g
and *qbs-integrable* q $(\lambda x.\ (g\ x)^2)$
shows *qbs-prob-var* $(p \bigotimes_{Qmes} q)$ $(\lambda z.\ f\ (fst\ z) + g\ (snd\ z))$ = *qbs-prob-var* p f + *qbs-prob-var* q g
⟨*proof*⟩

end

3.4 Measure as QBS Measure

theory *Measure-as-QuasiBorel-Measure*
  imports *Pair-QuasiBorel-Measure*

begin

lemma *distr-id'*:
  assumes $sets\ N = sets\ M$
    $f \in N \rightarrow_M N$
    and $\bigwedge x.\ x \in space\ N \Longrightarrow f\ x = x$
  shows $distr\ N\ M\ f = N$
$\langle proof \rangle$

Every probability measure on a standard Borel space can be represented as a measure on a quasi-Borel space [1], Proposition 23.

locale *standard-borel-prob-space* = *standard-borel P* + *p:prob-space P*
  for P :: *'a measure*
begin

sublocale *qbs-prob measure-to-qbs P g distr P real-borel f*
  $\langle proof \rangle$

lift-definition *as-qbs-measure* :: *'a qbs-prob-space* **is**
(*measure-to-qbs P*, *g*, *distr P real-borel f*)
  $\langle proof \rangle$

lemma *as-qbs-measure-retract*:
  assumes [*measurable*]:$a \in P \rightarrow_M real\text{-}borel$
    and [*measurable*]:$b \in real\text{-}borel \rightarrow_M P$
    and [*simp*]:$\bigwedge x.\ x \in space\ P \Longrightarrow (b \circ a)\ x = x$
  shows *qbs-prob* (*measure-to-qbs P*) *b* (*distr P real-borel a*)
    as-qbs-measure = *qbs-prob-space* (*measure-to-qbs P*, *b*, *distr P real-borel a*)
$\langle proof \rangle$

lemma *measure-as-qbs-measure-qbs*:
 qbs-prob-space-qbs as-qbs-measure = *measure-to-qbs P*
  $\langle proof \rangle$

lemma *measure-as-qbs-measure-image*:
 as-qbs-measure $\in$ *monadP-qbs-Px* (*measure-to-qbs P*)
  $\langle proof \rangle$

lemma *as-qbs-measure-as-measure*[*simp*]:
 distr (*distr P real-borel f*) (*qbs-to-measure* (*measure-to-qbs P*)) *g* = *P*
  $\langle proof \rangle$

lemma *measure-as-qbs-measure-recover*:
 qbs-prob-measure as-qbs-measure = *P*

⟨*proof*⟩

end

lemma(**in** *standard-borel*) *qbs-prob-measure-recover*:
assumes $q \in$ *monadP-qbs-Px* (*measure-to-qbs M*)
shows *standard-borel-prob-space.as-qbs-measure* (*qbs-prob-measure q*) = *q*
⟨*proof*⟩

lemma(**in** *standard-borel-prob-space*) *ennintegral-as-qbs-ennintegral*:
assumes $k \in$ *borel-measurable P*
shows ($\int^{+}_{Q}$ *x. k x* ∂*as-qbs-measure*) = ($\int^{+}$ *x. k x* ∂*P*)
⟨*proof*⟩

lemma(**in** *standard-borel-prob-space*) *integral-as-qbs-integral*:
($\int_{Q}$ *x. k x* ∂*as-qbs-measure*) = ($\int$ *x. k x* ∂*P*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *measure-with-args-morphism*:
assumes [*measurable*]:$\mu \in X \to_M$ *prob-algebra M*
shows *standard-borel-prob-space.as-qbs-measure* ∘ $\mu \in$ *measure-to-qbs* $X \to_Q$ *monadP-qbs* (*measure-to-qbs M*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *measure-with-args-recover*:
assumes $\mu \in$ *space X* → *space* (*prob-algebra M*)
and $x \in$ *space X*
shows *qbs-prob-measure* (*standard-borel-prob-space.as-qbs-measure* (μ *x*)) = μ *x*
⟨*proof*⟩

3.5 Example of Probability Measures

Probability measures on $\mathbb{R}$ can be represented as probability measures on the quasi-Borel space $\mathbb{R}$.

3.5.1 Normal Distribution

definition *normal-distribution* :: *real* × *real* ⇒ *real measure* **where**
normal-distribution $\mu\sigma$ = (*if 0* < (*snd* $\mu\sigma$) *then density lborel* (λx. *ennreal* (*normal-density* (*fst* $\mu\sigma$) (*snd* $\mu\sigma$) *x*))
else return lborel 0)

lemma *normal-distribution-measurable*:
normal-distribution ∈ *real-borel* $\bigotimes_M$ *real-borel* $\to_M$ *prob-algebra real-borel*
⟨*proof*⟩

definition *qbs-normal-distribution* :: *real* ⇒ *real* ⇒ *real qbs-prob-space* **where**

qbs-normal-distribution $\equiv$ *curry* (*standard-borel-prob-space.as-qbs-measure* $\circ$ *normal-distribution*)

lemma *qbs-normal-distribution-morphism*:
qbs-normal-distribution $\in \mathbb{R}_Q \to_Q$ *exp-qbs* $\mathbb{R}_Q$ (*monadP-qbs* $\mathbb{R}_Q$)
$\langle proof \rangle$

context
fixes μ σ :: *real*
assumes *sigma*:$\sigma > 0$
begin

interpretation *n-dist*:*standard-borel-prob-space normal-distribution* (μ,σ)
$\langle proof \rangle$

lemma *qbs-normal-distribution-def2*:
qbs-normal-distribution μ σ = *n-dist.as-qbs-measure*
$\langle proof \rangle$

lemma *qbs-normal-distribution-integral*:
($\int_Q$ *x*. *f x* ∂ (*qbs-normal-distribution* μ σ)) = ($\int$ *x*. *f x* ∂ (*density lborel* (λx. *ennreal* (*normal-density* μ σ *x*))))
$\langle proof \rangle$

lemma *qbs-normal-distribution-expectation*:
assumes $f \in$ *real-borel* $\to_M$ *real-borel*
shows ($\int_Q$ *x*. *f x* ∂ (*qbs-normal-distribution* μ σ)) = ($\int$ *x*. *normal-density* μ σ *x* $*$ *f x* ∂ *lborel*)
$\langle proof \rangle$

end

3.5.2 Uniform Distribution

definition *interval-uniform-distribution* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real measure* **where**
interval-uniform-distribution a b $\equiv$ (*if a* < *b then uniform-measure lborel* {*a*<..<*b*}
else return lborel 0)

lemma *sets-interval-uniform-distribution*[*measurable-cong*]:
sets (*interval-uniform-distribution a b*) = *borel*
$\langle proof \rangle$

lemma *interval-uniform-distribution-meaurable*:
(λr. *interval-uniform-distribution* (*fst r*) (*snd r*)) $\in$ *real-borel* $\bigotimes_M$ *real-borel* $\to_M$ *prob-algebra real-borel*
$\langle proof \rangle$

definition *qbs-interval-uniform-distribution* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real qbs-prob-space*

where
qbs-interval-uniform-distribution ≡ *curry* (*standard-borel-prob-space.as-qbs-measure* ∘ (λr. *interval-uniform-distribution* (*fst r*) (*snd r*)))

lemma *qbs-interval-uniform-distribution-morphism*:
qbs-interval-uniform-distribution $\in \mathbb{R}_Q \to_Q$ *exp-qbs* $\mathbb{R}_Q$ (*monadP-qbs* $\mathbb{R}_Q$)
⟨*proof*⟩

context
fixes *a b* :: *real*
assumes *a-less-than-b*: $a < b$
begin

definition *ab-qbs-uniform-distribution* ≡ *qbs-interval-uniform-distribution a b*

interpretation *ab-u-dist*: *standard-borel-prob-space interval-uniform-distribution a b*
⟨*proof*⟩

lemma *qbs-interval-uniform-distribution-def2*:
ab-qbs-uniform-distribution = *ab-u-dist.as-qbs-measure*
⟨*proof*⟩

lemma *qbs-uniform-distribution-expectation*:
assumes $f \in \mathbb{R}_Q \to_Q \mathbb{R}_{Q\geq 0}$
shows ($\int^{+}_Q x.\ f\, x\ \partial$*ab-qbs-uniform-distribution*) = ($\int^{+} x \in \{a{<}..{<}b\}.\ f\, x\ \partial$*lborel*) / ($b - a$)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

3.5.3 Bernoulli Distribution

definition *qbs-bernoulli* :: *real* ⇒ *bool qbs-prob-space* **where**
qbs-bernoulli ≡ *standard-borel-prob-space.as-qbs-measure* ∘ (λx. *measure-pmf* (*bernoulli-pmf x*))

lemma *bernoulli-measurable*:
(λx. *measure-pmf* (*bernoulli-pmf x*)) ∈ *real-borel* $\to_M$ *prob-algebra bool-borel*
⟨*proof*⟩

lemma *qbs-bernoulli-morphism*:
qbs-bernoulli $\in \mathbb{R}_Q \to_Q$ *monadP-qbs* $\mathbb{B}_Q$
⟨*proof*⟩

lemma *qbs-bernoulli-measure*:
qbs-prob-measure (*qbs-bernoulli p*) = *measure-pmf* (*bernoulli-pmf p*)

⟨*proof*⟩

context
fixes p :: *real*
assumes *pgeq-0*[*simp*]:$0 \leq p$ **and** *pleq-1*[*simp*]:$p \leq 1$
begin

lemma *qbs-bernoulli-expectation*:
$(\int_Q x.\ f\ x\ \partial qbs\text{-}bernoulli\ p) = f\ True * p + f\ False * (1 - p)$
⟨*proof*⟩

end

end

3.6 Bayesian Linear Regression

theory *Bayesian-Linear-Regression*
imports *Measure-as-QuasiBorel-Measure*
begin

We formalize the Bayesian linear regression presented in [1] section VI.

3.6.1 Prior

abbreviation $\nu \equiv$ *density lborel* ($\lambda x.$ *ennreal* (*normal-density 0 3 x*))

interpretation ν: *standard-borel-prob-space* ν
⟨*proof*⟩

term ν*.as-qbs-measure* :: *real qbs-prob-space*
definition *prior* :: (*real* ⇒ *real*) *qbs-prob-space* **where**
prior ≡ *do* { $s \leftarrow \nu$*.as-qbs-measure* ;
$b \leftarrow \nu$*.as-qbs-measure* ;
qbs-return $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)\ (\lambda r.\ s * r + b)$}

lemma ν*-as-qbs-measure-eq*:
ν*.as-qbs-measure* = *qbs-prob-space* $(\mathbb{R}_Q, id, \nu)$
⟨*proof*⟩

interpretation ν*-qp*: *pair-qbs-prob* $\mathbb{R}_Q$ *id* ν $\mathbb{R}_Q$ *id* ν
⟨*proof*⟩

lemma ν*-as-qbs-measure-in-Pr*:
ν*.as-qbs-measure* ∈ *monadP-qbs-Px* $\mathbb{R}_Q$
⟨*proof*⟩

lemma *sets-real-real-real*[*measurable-cong*]:
sets (*qbs-to-measure* $((\mathbb{R}_Q \bigotimes_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_Q)$) = *sets* $((borel \bigotimes_M borel) \bigotimes_M borel)$

$\langle proof \rangle$

lemma *lin-morphism*:
$(\lambda(s,\ b)\ r.\ s * r + b) \in \mathbb{R}_Q \bigotimes_Q \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$
$\langle proof \rangle$

lemma *lin-measurable*[*measurable*]:
$(\lambda(s,\ b)\ r.\ s * r + b) \in \mathit{real\text{-}borel} \bigotimes_M \mathit{real\text{-}borel} \rightarrow_M \mathit{qbs\text{-}to\text{-}measure}\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)$
$\langle proof \rangle$

lemma *prior-computation*:
$\mathit{qbs\text{-}prob}\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)\ ((\lambda(s,\ b)\ r.\ s * r + b) \circ \mathit{real\text{-}real.g})\ (\mathit{distr}\ (\nu \bigotimes_M \nu)\ \mathit{real\text{-}borel}\ \mathit{real\text{-}real.f})$
$\mathit{prior} = \mathit{qbs\text{-}prob\text{-}space}\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q,\ (\lambda(s,\ b)\ r.\ s * r + b) \circ \mathit{real\text{-}real.g},\ \mathit{distr}\ (\nu \bigotimes_M \nu)\ \mathit{real\text{-}borel}\ \mathit{real\text{-}real.f})$
$\langle proof \rangle$

The following lemma corresponds to the equation (5).

lemma *prior-measure*:
$\mathit{qbs\text{-}prob\text{-}measure}\ \mathit{prior} = \mathit{distr}\ (\nu \bigotimes_M \nu)\ (\mathit{qbs\text{-}to\text{-}measure}\ (\mathit{exp\text{-}qbs}\ \mathbb{R}_Q\ \mathbb{R}_Q))\ (\lambda(s,b)\ r.\ s * r + b)$
$\langle proof \rangle$

lemma *prior-in-space*:
$\mathit{prior} \in \mathit{qbs\text{-}space}\ (\mathit{monadP\text{-}qbs}\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q))$
$\langle proof \rangle$

3.6.2 Likelihood

abbreviation $d\ \mu\ x \equiv \mathit{normal\text{-}density}\ \mu\ (1/2)\ x$

lemma *d-positive* : $0 < d\ \mu\ x$
$\langle proof \rangle$

definition $\mathit{obs} :: (\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{ennreal}$ **where**
$\mathit{obs}\ f \equiv d\ (f\ 1)\ 2.5 * d\ (f\ 2)\ 3.8 * d\ (f\ 3)\ 4.5 * d\ (f\ 4)\ 6.2 * d\ (f\ 5)\ 8$

lemma *obs-morphism*:
$\mathit{obs} \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \rightarrow_Q \mathbb{R}_{Q\geq 0}$
$\langle proof \rangle$

lemma *obs-measurable*[*measurable*]:
$\mathit{obs} \in \mathit{qbs\text{-}to\text{-}measure}\ (\mathit{exp\text{-}qbs}\ \mathbb{R}_Q\ \mathbb{R}_Q) \rightarrow_M \mathit{ennreal\text{-}borel}$
$\langle proof \rangle$

3.6.3 Posterior

lemma *id-obs-morphism*:
$(\lambda f.\ (f,\mathit{obs}\ f)) \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \rightarrow_Q (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0}$

⟨*proof*⟩

lemma *push-forward-measure-in-space*:
monadP-qbs-Pf $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)$ $((\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $(\lambda f.\ (f, obs\ f))$ *prior* $\in$ *qbs-space* (*monadP-qbs* $((\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0})$)
⟨*proof*⟩

lemma *push-forward-measure-computation*:
qbs-prob $((\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $(\lambda l.\ (((\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ l,\ ((obs \circ (\lambda(s,\ b)\ r.\ s * r + b)) \circ real\text{-}real.g)\ l))$ $(distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
monadP-qbs-Pf $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)$ $((\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $(\lambda f.\ (f,\ obs\ f))$ *prior* = *qbs-prob-space* $((\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \bigotimes_Q \mathbb{R}_{Q\geq 0},\ (\lambda l.\ (((\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ l,\ ((obs \circ (\lambda(s,\ b)\ r.\ s * r + b)) \circ real\text{-}real.g)\ l)),\ distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)$
⟨*proof*⟩

3.6.4 Normalizer

We use the unit space for an error.

definition *norm-qbs-measure* :: $('a \times ennreal)$ *qbs-prob-space* $\Rightarrow$ $'a$ *qbs-prob-space* + *unit* **where**
norm-qbs-measure p $\equiv$ (*let* $(XR,\alpha\beta,\nu)$ = *rep-qbs-prob-space p in*
if emeasure $(density\ \nu\ (snd \circ \alpha\beta))$ *UNIV* = *0 then Inr* ()
else if emeasure $(density\ \nu\ (snd \circ \alpha\beta))$ *UNIV* = ∞ *then Inr* ()
else Inl (*qbs-prob-space* (*map-qbs fst XR*, $fst \circ \alpha\beta$, *density* ν $(\lambda r.\ snd\ (\alpha\beta\ r)\ /\ emeasure\ (density\ \nu\ (snd \circ \alpha\beta))\ UNIV))))$

lemma *norm-qbs-measure-qbs-prob*:
assumes *qbs-prob* $(X \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $(\lambda r.\ (\alpha\ r,\ \beta\ r))$ μ
emeasure $(density\ \mu\ \beta)$ *UNIV* $\neq$ *0*
and *emeasure* $(density\ \mu\ \beta)$ *UNIV* $\neq \infty$
shows *qbs-prob X* α $(density\ \mu\ (\lambda r.\ (\beta\ r)\ /\ emeasure\ (density\ \mu\ \beta)\ UNIV))$
⟨*proof*⟩

lemma *norm-qbs-measure-computation*:
assumes *qbs-prob* $(X \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $(\lambda r.\ (\alpha\ r,\ \beta\ r))$ μ
shows *norm-qbs-measure* (*qbs-prob-space* $(X \bigotimes_Q \mathbb{R}_{Q\geq 0},\ (\lambda r.\ (\alpha\ r,\ \beta\ r)),\ \mu))$ =
(*if emeasure* $(density\ \mu\ \beta)$ *UNIV* = *0 then Inr* ()
else if emeasure $(density\ \mu\ \beta)$ *UNIV* = ∞ *then Inr* ()
else Inl (*qbs-prob-space* $(X,\ \alpha,\ density\ \mu\ (\lambda r.\ (\beta\ r)\ /\ emeasure\ (density\ \mu\ \beta)\ UNIV))))$
⟨*proof*⟩

lemma *norm-qbs-measure-morphism*:
norm-qbs-measure $\in$ *monadP-qbs* $(X \bigotimes_Q \mathbb{R}_{Q\geq 0})$ $\rightarrow_Q$ *monadP-qbs X* $<+>_Q$ 1_Q
⟨*proof*⟩

The following is the semantics of the entire program.

definition *program* :: (*real* ⇒ *real*) *qbs-prob-space* + *unit* **where**
program ≡ *norm-qbs-measure* (*monadP-qbs-Pf* ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) (($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) $\bigotimes_Q$ $\mathbb{R}_{Q\geq 0}$) ($\lambda f.$ (f,*obs* f)) *prior*)

lemma *program-in-space*:
program ∈ *qbs-space* (*monadP-qbs* ($\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$) <+>$_Q$ 1_Q)
⟨*proof*⟩

We calculate the normalizing constant.

lemma *complete-the-square*:
fixes *a b c x* :: *real*
assumes $a \neq 0$
shows $a*x^2 + b * x + c = a * (x + (b\ /\ (2*a)))^2 - ((b^2 - 4* a * c)/(4*a))$
⟨*proof*⟩

lemma *complete-the-square2'*:
fixes *a b c x* :: *real*
assumes $a \neq 0$
shows $a*x^2 - 2 * b * x + c = a * (x - (b\ /\ a))^2 - ((b^2 - a*c)/a)$
⟨*proof*⟩

lemma *normal-density-mu-x-swap*:
normal-density $\mu\ \sigma\ x$ = *normal-density* $x\ \sigma\ \mu$
⟨*proof*⟩

lemma *normal-density-plus-shift*:
normal-density $\mu\ \sigma\ (x + y)$ = *normal-density* $(\mu - x)\ \sigma\ y$
⟨*proof*⟩

lemma *normal-density-times*:
assumes $\sigma > 0$ $\sigma' > 0$
shows *normal-density* $\mu\ \sigma\ x$ ∗ *normal-density* $\mu'\ \sigma'\ x = (1\ /\ sqrt\ (2 * pi * (\sigma^2 + \sigma'^2))) * exp\ (-\ (\mu - \mu')^2\ /\ (2 * (\sigma^2 + \sigma'^2)))$ ∗ *normal-density* $((\mu*\sigma'^2 + \mu'*\sigma^2)/(\sigma^2 + \sigma'^2))\ (\sigma * \sigma'\ /\ sqrt\ (\sigma^2 + \sigma'^2))\ x$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *normal-density-times'*:
assumes $\sigma > 0$ $\sigma' > 0$
shows a ∗ *normal-density* $\mu\ \sigma\ x$ ∗ *normal-density* $\mu'\ \sigma'\ x = a * (1\ /\ sqrt\ (2 * pi * (\sigma^2 + \sigma'^2))) * exp\ (-\ (\mu - \mu')^2\ /\ (2 * (\sigma^2 + \sigma'^2)))$ ∗ *normal-density* $((\mu*\sigma'^2 + \mu'*\sigma^2)/(\sigma^2 + \sigma'^2))\ (\sigma * \sigma'\ /\ sqrt\ (\sigma^2 + \sigma'^2))\ x$
⟨*proof*⟩

lemma *normal-density-times-minusx*:
assumes $\sigma > 0$ $\sigma' > 0$ $a \neq a'$
shows *normal-density* $(\mu - a*x)\ \sigma\ y$ ∗ *normal-density* $(\mu' - a'*x)\ \sigma'\ y = (1$

$/ \ |a' - a|) * normal\text{-}density\ ((\mu' - \mu)/(a' - a))\ (sqrt\ ((\sigma^2 + \sigma'^2)/(a' - a)^2))\ x * normal\text{-}density\ (((\mu - a*x)*\sigma'^2 + (\mu' - a'*x)*\sigma^2)/(\sigma^2 + \sigma'^2))\ (\sigma * \sigma' / sqrt\ (\sigma^2 + \sigma'^2))\ y$
$\langle proof \rangle$

The following is the normalizing constant of the program.

abbreviation $C \equiv ennreal\ ((4 * sqrt\ 2\ /\ (pi^2 * sqrt\ (66961 * pi))) * (exp\ (-(1674761\ /\ 1674025))))$

lemma *program-normalizing-constant*:
$emeasure\ (density\ (distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)\ (obs \circ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g))\ UNIV = C$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

The program returns a probability measure, rather than error.

lemma *program-result*:
$qbs\text{-}prob\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)\ ((\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ (density\ (distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)\ (\lambda r.\ (obs \circ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ r\ /\ C))$
$program = Inl\ (qbs\text{-}prob\text{-}space\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q,\ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g,\ density\ (distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)\ (\lambda r.\ (obs \circ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ r\ /\ C)))$
$\langle proof \rangle$

lemma *program-inl*:
$program \in Inl\ `\ (qbs\text{-}space\ (monadP\text{-}qbs\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)))$
$\langle proof \rangle$

lemma *program-result-measure*:
$qbs\text{-}prob\text{-}measure\ (qbs\text{-}prob\text{-}space\ (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q,\ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g,\ density\ (distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)\ (\lambda r.\ (obs \circ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ r\ /\ C)))$
$= density\ (qbs\text{-}prob\text{-}measure\ prior)\ (\lambda k.\ obs\ k\ /\ C)$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

lemma *program-result-measure'*:
$qbs\text{-}prob\text{-}measure\ (qbs\text{-}prob\text{-}space\ (exp\text{-}qbs\ \mathbb{R}_Q\ \mathbb{R}_Q,\ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g,\ density\ (distr\ (\nu \bigotimes_M \nu)\ real\text{-}borel\ real\text{-}real.f)\ (\lambda r.\ (obs \circ (\lambda(s,\ b)\ r.\ s * r + b) \circ real\text{-}real.g)\ r\ /\ C)))$
$= distr\ (density\ (\nu \bigotimes_M \nu)\ (\lambda(s,b).\ obs\ (\lambda r.\ s * r + b)\ /\ C))\ (qbs\text{-}to\text{-}measure\ (exp\text{-}qbs\ \mathbb{R}_Q\ \mathbb{R}_Q))\ (\lambda(s,\ b)\ r.\ s * r + b)$
$\langle proof \rangle$

end

References

[1] C. Heunen, O. Kammar, S. Staton, and H. Yang. A convenient category for higher-order probability theory. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '17. IEEE Press, 2017.