

# Soundness of the Q0 proof system for higher-order logic

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## Abstract

This entry formalizes the Q0 proof system for higher-order logic (also known as simple type theory) from the book “An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof” by Peter B. Andrews [And13] together with the system’s soundness. Most of the used theorems and lemmas are also from his book. The soundness proof is with respect to general models and as a consequence also holds for standard models. Higher-order logic is given a semantics by porting to Isabelle/HOL the specification of set theory from the CakeML project [KAMO14, KAMO16]. The independently developed AFP entry “Metatheory of Q0” by Javier Díaz [Día23] also formalizes Q0 in Isabelle/HOL. I highly recommend the reader to also take a look at his formalization!

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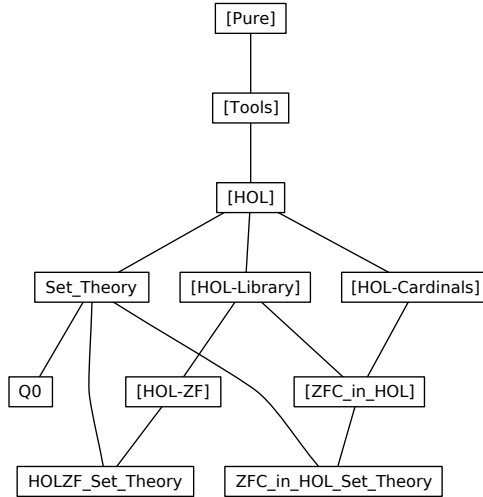


Figure 1: Theory dependency graph

## 1 Introduction

This entry formalizes the Q0 proof system for higher-order logic (also called simple type theory) and its soundness. Both the system and most of the proven lemmas and theorems are from Peter B. Andrews’ book [And13]. In the book’s chapter on type theory, Andrews explains that type theory was invented by Russell [Rus08] and that Whitehead and Russell [WR13] showed that fundamental parts of mathematics could be formalized in type theory. As influences on the type theory presented in Andrews’ chapter he mentions Church [Chu40], Henkin [Hen50, Hen63] and earlier works by himself [And63, And72]. The present Isabelle formalization of higher-order logic is given a semantics by using a port to Isabelle/HOL of CakeML’s specification of set theory [KAMO16]. The specification is formalized as a locale that fixes a type of set theoretic sets and a number of functions (powerset, union, separation) and the set membership predicate. The soundness proof is with respect to general models and as a consequence it also holds for standard models. Variables are implemented simply using named variables rather than e.g. De Bruijn indices or nominal techniques. It is well-known that named variables require the definition of substitution to rename variables, but since substitution is derived in Q0 rather than built into the proof system this complication is essentially circumvented in the present formalization. As a curiosity the present AFP entry also proves that the set theory specification is fulfilled by the sets axiomatized in Isabelle/HOLZF [Obu06] and also by the sets axiomatized by the AFP entry on Zermelo Fraenkel Set Theory in Higher-Order Logic [Pau19]. The theory files of the present entry appeared in the IsaFoL effort [Sch23].

In the literature we find other formalizations of the metatheory of higher-order logics and type theories. Independently from my work here, Javier Díaz also formalizes Q0 in Isabelle/HOL. His work is available in the AFP entry “Metatheory of Q0” [Día23]. I highly recommend the reader to take a look also at his very thorough formalization! Arthan specifies HOL [Art14a, Art14b, Art02] in ProofPower but does not prove soundness. John Harrison formalizes the soundness of the proof system of HOL Light [Har06] in HOL Light and Kumar et al. [KAMO14, KAMO16, AMKS22, AM23] formalize it in HOL4 as part of the CakeML project including also definitions and a verified implementation. Roßkopf and Nipkow [NR21a, NR21b, RN23] formalize a proof system for terms in Isabelle’s metalogic together with an implementation of a proof checker and prove that the proof checker indeed implements this proof system. There are also a number of works using Coq to formalize metatheory of the calculus of constructions and the calculus of inductive constructions [Bar96b, Bar96a, BW96, Bar97, SBF+20]. Worth mentioning is also the formalization in Coq of second-order logic which also formalizes general models [KK22].

There are plenty of opportunities to go further with the present formalization. The present formalization proves soundness of Q0 theorems (i.e.  $\vdash A$  implies  $\models A$ ), but not soundness of Q0 derivability (i.e.  $M \models G$  and  $G \vdash A$  implies  $M \models A$ ). Other interesting lemmas and theorems from Andrews’ book could also be proved such as e.g. derived inference rules and completeness.

## 2 Set Theory

theory Set\_Theory imports Main begin

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### 2.2 Set theory specification

This formal document is the set theory from <https://github.com/CakeML/cakeml/blob/master/candle/set-theory/set-SpecScript.sml> ported to Isabelle/HOL and extended.

```
locale set_theory =
  fixes mem :: "'s ⇒ 's ⇒ bool" (infix "∈:" 67)
  fixes sub :: "'s ⇒ ('s ⇒ bool) ⇒ 's" (infix "suchthat" 67)
  fixes pow :: "'s ⇒ 's"
  fixes uni :: "'s ⇒ 's" ("⋃:_" [900] 900)
  fixes upair :: "'s ⇒ 's ⇒ 's" (infix "+:" 67)
  assumes extensional: "⋀x y. x = y ⟷ (∀a. a ∈: x ⟷ a ∈: y)"
  assumes mem_sub[simp]: "⋀x P a. a ∈: (x suchthat P) ⟷ a ∈: x ∧ P a"
  assumes mem_pow[simp]: "⋀x a. a ∈: (pow x) ⟷ (∀b. b ∈: a ⟶ b ∈: x)"
  assumes mem_uni[simp]: "⋀x a. a ∈: ⋃: x ⟷ (∃b. a ∈: b ∧ b ∈: x)"
  assumes mem_upair[simp]: "⋀x y a. a ∈: (x +: y) ⟷ a = x ∨ a = y"
begin
```

```
lemma seperation_unique:
  assumes "∀x P a. a ∈: (sub2 x P) ⟷ a ∈: x ∧ P a"
  shows "sub2 = sub"
  <proof>
```

```
lemma pow_unique:
  assumes "∀x a. a ∈: (pow2 x) ⟷ (∀b. b ∈: a ⟶ b ∈: x)"
  shows "pow2 = pow"
  <proof>
```

```
lemma uni_unique:
```

```

assumes " $\forall x a. a \in: \text{uni2 } x \longleftrightarrow (\exists b. a \in: b \wedge b \in: x)$ "
shows " $\text{uni2} = \text{uni}$ "
<proof>

```

```

lemma upair_unique:
  assumes " $\forall x y a. a \in: \text{upair2 } x y \longleftrightarrow a = x \vee a = y$ "
  shows " $\text{upair2} = \text{upair}$ "
<proof>

```

```

definition empty :: 's ("∅") where
  "∅ = undefined suchthat ( $\lambda x. \text{False}$ )"

```

```

lemma mem_empty[simp]: " $\neg x \in: \emptyset$ "
<proof>

```

```

definition unit :: "'s  $\Rightarrow$  's" where
  "unit x = x +: x"

```

```

lemma mem_unit[simp]: " $x \in: (\text{unit } y) \longleftrightarrow x = y$ "
<proof>

```

```

lemma unit_inj: "unit x = unit y  $\longleftrightarrow$  x = y"
<proof>

```

```

definition one :: 's where
  "one = unit ∅"

```

```

lemma mem_one[simp]: " $x \in: \text{one} \longleftrightarrow x = \emptyset$ "
<proof>

```

```

definition two :: 's where
  "two = ∅ +: one"

```

```

lemma mem_two[simp]: " $\forall x. x \in: \text{two} \longleftrightarrow x = \emptyset \vee x = \text{one}$ "
<proof>

```

```

definition pair :: "'s  $\Rightarrow$  's  $\Rightarrow$  's" (infix " ,:" 50) where
  "(x, :y) = (unit x) +: (x +: y)"

```

```

lemma upair_inj:
  "a +: b = c +: d  $\longleftrightarrow$  a = c  $\wedge$  b = d  $\vee$  a = d  $\wedge$  b = c"
<proof>

```

```

lemma unit_eq_upair:
  "unit x = y +: z  $\longleftrightarrow$  x = y  $\wedge$  y = z"
<proof>

```

```

lemma pair_inj:
  "(a, :b) = (c, :d)  $\longleftrightarrow$  a = c  $\wedge$  b = d"
<proof>

```

**definition** binary\_uni (infix "∪:" 67) where

"x ∪: y =  $\bigcup$ : (x +: y)"

**lemma** mem\_binary\_uni[simp]:

"a ∈: (x ∪: y)  $\longleftrightarrow$  a ∈: x  $\vee$  a ∈: y"

*<proof>*

**definition** product :: "'s  $\Rightarrow$  's  $\Rightarrow$  's" (infix "×:" 67) where

"x ×: y = (pow (pow (x ∪: y)) suchthat ( $\lambda$ a.  $\exists$  b c. b ∈: x  $\wedge$  c ∈: y  $\wedge$  a = (b,:c)))"

**lemma** mem\_product[simp]:

"a ∈: (x ×: y)  $\longleftrightarrow$  ( $\exists$  b c. a = (b,:c)  $\wedge$  b ∈: x  $\wedge$  c ∈: y)"

*<proof>*

**definition** relspace where

"relspace x y = pow (x ×: y)"

**definition** funspace where

"funspace x y =

(relspace x y suchthat

( $\lambda$ f.  $\forall$ a. a ∈: x  $\longrightarrow$  ( $\exists!$ b. (a,:b) ∈: f)))"

**definition** "apply" :: "'s  $\Rightarrow$  's  $\Rightarrow$  's" (infixl "." 68) where

"(x.y) = (SOME a. (y,:a) ∈: x)"

**definition** boolset where

"boolset  $\equiv$  two"

**definition** true where

"true =  $\emptyset$ "

**definition** false where

"false = one"

**lemma** true\_neq\_false:

"true  $\neq$  false"

*<proof>*

**lemma** mem\_boolset[simp]:

"x ∈: boolset  $\longleftrightarrow$  ((x = true)  $\vee$  (x = false))"

*<proof>*

**definition** boolean :: "bool  $\Rightarrow$  's" where

"boolean b = (if b then true else false)"

**lemma** boolean\_in\_boolset:

"boolean b ∈: boolset"

*<proof>*

```
lemma boolean_eq_true:
  "boolean b = true  $\longleftrightarrow$  b"
  <proof>
```

```
definition "holds s x  $\longleftrightarrow$  s · x = true"
```

```
definition abstract where
  "abstract doma rang f = ((doma  $\times$ : rang) suchthat ( $\lambda$ x.  $\exists$ a. x = (a, :f a)))"
```

```
lemma apply_abstract[simp]:
  "x  $\in$ : s  $\implies$  f x  $\in$ : t  $\implies$  abstract s t f · x = f x"
  <proof>
```

```
lemma apply_abstract_matchable:
  "x  $\in$ : s  $\implies$  f x  $\in$ : t  $\implies$  f x = u  $\implies$  abstract s t f · x = u"
  <proof>
```

```
lemma apply_in_rng:
  assumes "x  $\in$ : s"
  assumes "f  $\in$ : funspace s t"
  shows "f · x  $\in$ : t"
  <proof>
```

```
lemma abstract_in_funspace[simp]:
  "( $\forall$ x. x  $\in$ : s  $\longrightarrow$  f x  $\in$ : t)  $\implies$  abstract s t f  $\in$ : funspace s t"
  <proof>
```

```
lemma abstract_in_funspace_matchable:
  "( $\forall$ x. x  $\in$ : s  $\longrightarrow$  f x  $\in$ : t)  $\implies$  fs = funspace s t  $\implies$  abstract s t f  $\in$ : fs"
  <proof>
```

```
lemma abstract_eq:
  assumes " $\forall$ x. x  $\in$ : s  $\longrightarrow$  f x  $\in$ : t1  $\wedge$  g x  $\in$ : t2  $\wedge$  f x = g x"
  shows "abstract s t1 f = abstract s t2 g"
  <proof>
```

```
lemma abstract_extensional:
  assumes " $\forall$ x. x  $\in$ : s  $\longrightarrow$  f x  $\in$ : t  $\wedge$  f x = g x"
  shows "abstract s t f = abstract s t g"
  <proof>
```

```
lemma abstract_extensional':
  assumes " $\bigwedge$ x. x  $\in$ : s  $\implies$  f x  $\in$ : t"
  assumes " $\bigwedge$ x. x  $\in$ : s  $\implies$  f x = g x"
  shows "abstract s t f = abstract s t g"
  <proof>
```

```
lemma abstract_cong:
  assumes " $\forall$ x. x  $\in$ : s  $\longrightarrow$  f x  $\in$ : t  $\wedge$  g x  $\in$ : t"
  assumes "abstract s t f = abstract s t g"
  assumes "x  $\in$ : s"
  shows "f x = g x"
  <proof>
```

```
lemma abstract_cong_specific:
```

```

assumes "x ∈ s"
assumes "f x ∈ t"
assumes "abstract s t f = abstract s t g"
assumes "g x ∈ t"
shows "f x = g x"
⟨proof⟩

```

```

lemma abstract_iff_extensional:
assumes "∀x. x ∈ s ⟶ f x ∈ t ∧ g x ∈ t"
shows "(abstract s t f = abstract s t g) ⟷ (∀x. x ∈ s ⟶ f x ∈ t ∧ f x = g x)"
⟨proof⟩

```

```

lemma in_funspace_abstract[simp]:
assumes "z ∈ funspace s t"
shows "∃f. z = abstract s t f ∧ (∀x. x ∈ s ⟶ f x ∈ t)"
⟨proof⟩

```

```

theorem axiom_of_choice:
assumes "∀a. a ∈ x ⟶ (∃b. b ∈ a)"
shows "∃f. ∀a. a ∈ x ⟶ f · a ∈ a"
⟨proof⟩

```

```

definition is_infinite where
  "is_infinite s = infinite {a. a ∈ s}"

```

```

lemma funspace_inhabited:
  "(∃x. x ∈ s) ⟹ (∃x. x ∈ t) ⟹ (∃f. f ∈ funspace s t)"
⟨proof⟩

```

```

fun tuple where
  "tuple [] = 0" |
  "tuple (a#as) = (a, : tuple as)"

```

```

lemma pair_not_empty:
  "(x, :y) ≠ 0"
⟨proof⟩

```

```

lemma tuple_empty:
  "tuple ls = 0 ⟷ ls = []"
⟨proof⟩

```

```

lemma tuple_inj:
  "tuple l1 = tuple l2 ⟷ l1 = l2"
⟨proof⟩

```

```

fun bigcross where
  "bigcross [] = one" |
  "bigcross (a#as) = a ×: (bigcross as)"

```

```

lemma mem_bigcross[simp]:
  "x ∈ (bigcross ls) ⟷ (∃xs. x = tuple xs ∧ list_all2 mem xs ls)"
⟨proof⟩

```

```

definition subs :: "'s ⇒ 's ⇒ bool" (infix "⊆:" 67) where

```

"x  $\subseteq$ : y  $\longleftrightarrow$  x  $\in$ : pow y"

**definition** one\_elem\_fun :: "'s  $\Rightarrow$  's  $\Rightarrow$  's" where  
"one\_elem\_fun x d = abstract d boolset ( $\lambda$ y. boolean (x=y))"

**definition** iden :: "'s  $\Rightarrow$  's" where  
"iden D = abstract D (funspace D boolset) ( $\lambda$ x. one\_elem\_fun x D)"

**lemma** apply\_id[simp]:  
assumes A\_in\_D: "A  $\in$ : D"  
assumes B\_in\_D: "B  $\in$ : D"  
shows "iden D  $\cdot$  A  $\cdot$  B = boolean (A = B)"  
<proof>

**lemma** apply\_id\_true[simp]:  
assumes A\_in\_D: "A  $\in$ : D"  
assumes B\_in\_D: "B  $\in$ : D"  
shows "iden D  $\cdot$  A  $\cdot$  B = true  $\longleftrightarrow$  A = B"  
<proof>

**lemma** apply\_if\_pair\_in:  
assumes "(a1, : a2)  $\in$ : f"  
assumes "f  $\in$ : funspace s t"  
shows "f  $\cdot$  a1 = a2"  
<proof>

**lemma** funspace\_app\_unique:  
assumes "f  $\in$ : funspace s t"  
assumes "(a1, : a2)  $\in$ : f"  
assumes "(a1, : a3)  $\in$ : f"  
shows "a3 = a2"  
<proof>

**lemma** funspace\_extensional:  
assumes "f  $\in$ : funspace s t"  
assumes "g  $\in$ : funspace s t"  
assumes " $\forall$ x. x  $\in$ : s  $\longrightarrow$  f  $\cdot$  x = g  $\cdot$  x"  
shows "f = g"  
<proof>

**lemma** funspace\_difference\_witness:  
assumes "f  $\in$ : funspace s t"  
assumes "g  $\in$ : funspace s t"  
assumes "f  $\neq$  g"  
shows " $\exists$ z. z  $\in$ : s  $\wedge$  f  $\cdot$  z  $\neq$  g  $\cdot$  z"  
<proof>

end

end

### 3 Isabelle/HOLZF lives up to CakeML's set theory specification

**theory** HOLZF\_Set\_Theory imports "HOL-ZF.MainZF" Set\_Theory begin

**interpretation** set\_theory Elem Sep Power Sum Upair  
<proof>

end



## 4 ZFC\_in\_HOL lives up to CakeML's set theory specification

```
theory ZFC_in_HOL_Set_Theory imports ZFC_in_HOL.ZFC_in_HOL Set_Theory begin

interpretation set_theory "λx y. x ∈ elts y" "λx::V. λP. (inv elts) ({y. y ∈ elts x ∧ P y})" VPow
  "(λY. SOME Z. elts Z = ⋃(elts ' (elts Y)))" "λx y::V. (inv elts) {x, y}"
  ⟨proof⟩

end
```

## 5 Q0 abbreviations

```
theory Q0
  imports Set_Theory
  abbrevs "App" = "."
  and "Abs" = "[λ_:_. _]"
  and "Eq1" = "[_ =_ _]"
  and "Con" = "^"
  and "Forall1" = "[∀_:_. _]"
  and "Imp" = "→"
  and "Fun" = "←"
begin

lemma arg_cong3: "a = b ⇒ c = d ⇒ e = f ⇒ h a c e = h b d f"
  ⟨proof⟩
```

## 6 Syntax and typing

```
datatype type_sym =
  Ind |
  Tv |
  Fun type_sym type_sym (infixl "←" 80)

type_synonym var_sym = string
type_synonym cst_sym = string

datatype trm =
  Var var_sym type_sym |
  Cst cst_sym type_sym |
  App trm trm (infixl "." 80) |
  Abs var_sym type_sym trm ("[λ_:_. _]" [80,80,80])

fun vars :: "trm ⇒ (var_sym * type_sym) set" where
  "vars (Var x α) = {(x,α)}"
| "vars (Cst _ _) = {}"
| "vars (A · B) = vars A ∪ vars B"
| "vars ([λx:α. A]) = {(x,α)} ∪ vars A"

fun frees :: "trm ⇒ (var_sym * type_sym) set" where
  "frees (Var x α) = {(x,α)}"
| "frees (Cst _ _) = {}"
| "frees (A · B) = frees A ∪ frees B"
| "frees ([λx:α. A]) = frees A - {(x,α)}"

lemma frees_subset_vars:
  "frees A ⊆ vars A"
  ⟨proof⟩

inductive wff :: "type_sym ⇒ trm ⇒ bool" where
  wff_Var: "wff α (Var _ α)"
| wff_Cst: "wff α (Cst _ α)"
| wff_App: "wff (α ← β) A ⇒ wff β B ⇒ wff α (A · B)"
| wff_Abs: "wff α A ⇒ wff (α ← β) [λx:β. A]"
```

```

fun type_of :: "trm  $\Rightarrow$  type_sym" where
  "type_of (Var x  $\alpha$ ) =  $\alpha$ "
| "type_of (Cst c  $\alpha$ ) =  $\alpha$ "
| "type_of (A  $\cdot$  B) =
  (case type_of A of  $\beta \Leftarrow \alpha \Rightarrow \beta$ )"
| "type_of [ $\lambda x:\alpha$ . A] = (type_of A)  $\Leftarrow \alpha$ "

lemma type_of[simp]:
  "wff  $\alpha$  A  $\Longrightarrow$  type_of A =  $\alpha$ "
  <proof>

lemma wff_Var'[simp, code]:
  "wff  $\beta$  (Var x  $\alpha$ )  $\longleftrightarrow \beta = \alpha$ "
  <proof>

lemma wff_Cst'[simp, code]:
  "wff  $\beta$  (Cst c  $\alpha$ )  $\longleftrightarrow \beta = \alpha$ "
  <proof>

lemma wff_App'[simp]:
  "wff  $\alpha$  (A  $\cdot$  B)  $\longleftrightarrow (\exists \beta. \text{wff } (\alpha \Leftarrow \beta) A \wedge \text{wff } \beta B)$ "
  <proof>

lemma wff_Abs'[simp]:
  "wff  $\gamma$  ([ $\lambda x:\alpha$ . A])  $\longleftrightarrow (\exists \beta. \text{wff } \beta A \wedge \gamma = \beta \Leftarrow \alpha)$ "
  <proof>

lemma wff_Abs_type_of[code]:
  "wff  $\gamma$  [ $\lambda x:\alpha$ . A]  $\longleftrightarrow (\text{wff } (\text{type\_of } A) A \wedge \gamma = (\text{type\_of } A) \Leftarrow \alpha)$ "
  <proof>

lemma wff_App_type_of[code]:
  "wff  $\gamma$  ((A  $\cdot$  B))  $\longleftrightarrow (\text{wff } (\text{type\_of } A) A \wedge \text{wff } (\text{type\_of } B) B \wedge \text{type\_of } A = \gamma \Leftarrow (\text{type\_of } B))$ "
  <proof>

lemma unique_type:
  "wff  $\beta$  A  $\Longrightarrow \text{wff } \alpha A \Longrightarrow \alpha = \beta$ "
  <proof>

```

## 7 Replacement

```

inductive replacement :: "trm  $\Rightarrow$  trm  $\Rightarrow$  trm  $\Rightarrow$  trm  $\Rightarrow$  bool" where
  replace: "replacement A B A B"
| replace_App_left: "replacement A B C E  $\Longrightarrow$  replacement A B (C  $\cdot$  D) (E  $\cdot$  D)"
| replace_App_right: "replacement A B D E  $\Longrightarrow$  replacement A B (C  $\cdot$  D) (C  $\cdot$  E)"
| replace_Abs: "replacement A B C D  $\Longrightarrow$  replacement A B [ $\lambda x:\alpha$ . C] [ $\lambda x:\alpha$ . D]"

```

```

lemma replacement_preserves_type:
  assumes "replacement A B C D"
  assumes "wff  $\alpha$  A"
  assumes "wff  $\alpha$  B"
  assumes "wff  $\beta$  C"
  shows "wff  $\beta$  D"
  <proof>

```

```

lemma replacement_preserved_type:
  assumes "replacement A B C D"
  assumes "wff  $\alpha$  A"
  assumes "wff  $\alpha$  B"
  assumes "wff  $\beta$  D"
  shows "wff  $\beta$  C"
  <proof>

```

## 8 Defined wffs

### 8.1 Common expressions

abbreviation (input) Var\_yi ("y<sub>i</sub>") where  
"y<sub>i</sub> == Cst 'y' Ind"

abbreviation (input) Var\_xo ("x<sub>o</sub>") where  
"x<sub>o</sub> == Var 'x' Tv"

abbreviation (input) Var\_yo ("y<sub>o</sub>") where  
"y<sub>o</sub> == Var 'y' Tv"

abbreviation (input) Fun\_oo ("oo") where  
"oo == Tv  $\Leftarrow$  Tv"

abbreviation (input) Fun\_ooo ("ooo") where  
"ooo == oo  $\Leftarrow$  Tv"

abbreviation (input) Var\_goo ("g<sub>oo</sub>") where  
"g<sub>oo</sub> == Var 'g' oo"

abbreviation (input) Var\_gooo ("g<sub>ooo</sub>") where  
"g<sub>ooo</sub> == Var 'g' ooo"

### 8.2 Equality symbol

abbreviation QQ :: "type\_sym  $\Rightarrow$  trm" ("Q") where  
"Q  $\alpha \equiv$  Cst 'Q'  $\alpha$ "

### 8.3 Description or selection operator

abbreviation  $\iota$  :: "trm" (" $\iota$ ") where  
" $\iota \equiv$  Cst 'i' (Ind  $\Leftarrow$  (Tv  $\Leftarrow$  Ind))"

### 8.4 Equality

definition Eq1 :: "trm  $\Rightarrow$  trm  $\Rightarrow$  type\_sym  $\Rightarrow$  trm" where  
"Eq1 A B  $\alpha \equiv$  (Q (Tv  $\Leftarrow$   $\alpha \Leftarrow$   $\alpha$ ))  $\cdot$  A  $\cdot$  B"

abbreviation Eq1' :: "trm  $\Rightarrow$  type\_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" ("[\_ =<sub>α</sub> \_]" [89]) where  
"[A =<sub>α</sub> B]  $\equiv$  Eq1 A B  $\alpha$ "

definition LHS where  
"LHS EqLAB = (case EqLAB of ( \_  $\cdot$  A  $\cdot$  \_ )  $\Rightarrow$  A)"

lemma LHS\_def2[simp]: "LHS [A =<sub>α</sub> B] = A"  
*<proof>*

definition RHS where  
"RHS EqLAB = (case EqLAB of ( \_  $\cdot$  B )  $\Rightarrow$  B)"

lemma RHS\_def2[simp]: "RHS ([A =<sub>α</sub> B]) = B"  
*<proof>*

lemma wff\_Eq1[simp]:  
"wff  $\alpha$  A  $\Longrightarrow$  wff  $\alpha$  B  $\Longrightarrow$  wff Tv [A =<sub>α</sub> B]"  
*<proof>*

lemma wff\_Eq1\_iff[simp]:  
"wff  $\beta$  [A =<sub>α</sub> B]  $\longleftrightarrow$  wff  $\alpha$  A  $\wedge$  wff  $\alpha$  B  $\wedge$   $\beta =$  Tv"  
*<proof>*

## 8.5 Truth

**definition** T :: trm where

"T  $\equiv$  [(Q ooo) =ooo= (Q ooo)]"

**lemma** wff\_T[simp]: "wff Tv T"

*<proof>*

**lemma** wff\_T\_iff[simp]: "wff  $\alpha$  T  $\longleftrightarrow$   $\alpha = \text{Tv}$ "

*<proof>*

## 8.6 Falsity

**abbreviation** F :: trm where

"F  $\equiv$  [[ $\lambda$ 'x':Tv. T] = oo= [ $\lambda$ 'x':Tv. x<sub>o</sub>]]"

**lemma** wff\_F[simp]: "wff Tv F"

*<proof>*

**lemma** wff\_F\_iff[simp]: "wff  $\alpha$  F  $\longleftrightarrow$   $\alpha = \text{Tv}$ "

*<proof>*

## 8.7 Pi

**definition** PI :: "type\_sym  $\Rightarrow$  trm" where

"PI  $\alpha \equiv$  (Q (Tv  $\Leftarrow$  (Tv  $\Leftarrow$   $\alpha$ )  $\Leftarrow$  (Tv  $\Leftarrow$   $\alpha$ )))  $\cdot$  [ $\lambda$ 'x': $\alpha$ . T]"

**lemma** wff\_PI[simp]: "wff (Tv  $\Leftarrow$  (Tv  $\Leftarrow$   $\alpha$ )) (PI  $\alpha$ )"

*<proof>*

**lemma** wff\_PI\_subterm[simp]: "wff (Tv  $\Leftarrow$   $\alpha$ ) [ $\lambda$ 'x': $\alpha$ . T]"

*<proof>*

**lemma** wff\_PI\_subterm\_iff[simp]:

"wff  $\beta$  [ $\lambda$ 'x': $\alpha$ . T]  $\longleftrightarrow$   $\beta = (\text{Tv} \Leftarrow \alpha)$ "

*<proof>*

## 8.8 Forall

**definition** Forall :: "string  $\Rightarrow$  type\_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" ("[" $\forall$ \_:\_. \_]" [80,80,80]) where

"[" $\forall$ x: $\alpha$ . A] = (PI  $\alpha$ )  $\cdot$  [ $\lambda$ x: $\alpha$ . A]"

**lemma** wff\_Forall[simp]: "wff Tv A  $\Longrightarrow$  wff Tv [" $\forall$ x: $\alpha$ . A]"

*<proof>*

**lemma** wff\_Forall\_iff[simp]: "wff  $\beta$  [" $\forall$ x: $\alpha$ . A]  $\longleftrightarrow$  wff Tv A  $\wedge$   $\beta = \text{Tv}$ "

*<proof>*

## 8.9 Conjunction symbol

**definition** Con\_sym :: trm where

"Con\_sym  $\equiv$

[ $\lambda$ 'x':Tv. [ $\lambda$ 'y':Tv.

[[" $\lambda$ 'g':ooo. g<sub>ooo</sub>  $\cdot$  T  $\cdot$  T] =Tv  $\Leftarrow$  ooo= [ $\lambda$ 'g':ooo. g<sub>ooo</sub>  $\cdot$  x<sub>o</sub>  $\cdot$  y<sub>o</sub>]]

]"

**lemma** wff\_Con\_sym[simp]: "wff ooo Con\_sym"

*<proof>*

**lemma** wff\_Con\_sym'[simp]: "wff  $\alpha$  Con\_sym  $\longleftrightarrow$   $\alpha = \text{ooo}$ "

*<proof>*

**lemma** wff\_Con\_sym\_subterm0[simp]:

"wff Tv A  $\Longrightarrow$  wff Tv B  $\Longrightarrow$  wff (Tv  $\Leftarrow$  ooo) [ $\lambda$ 'g':ooo. g<sub>ooo</sub>  $\cdot$  A  $\cdot$  B]"

*<proof>*

**lemma** wff\_Con\_sym\_subterm0\_iff[simp]:

"wff  $\beta$   $[\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot A \cdot B] \longleftrightarrow \text{wff Tv } A \wedge \text{wff Tv } B \wedge \beta = (\text{Tv} \Leftarrow \text{ooo})$ "

*<proof>*

**lemma** wff\_Con\_sym\_subterm1[simp]:

"wff Tv  $[[\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot T \cdot T] = (\text{Tv} \Leftarrow \text{ooo}) = [\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot x_o \cdot y_o]$ "

*<proof>*

**lemma** wff\_Con\_sym\_subterm1\_iff[simp]:

"wff  $\alpha$   $[[\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot T \cdot T] = (\text{Tv} \Leftarrow \text{ooo}) = [\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot x_o \cdot y_o] \longleftrightarrow \alpha = \text{Tv}$ "

*<proof>*

**lemma** wff\_Con\_sym\_subterm2[simp]:

"wff oo  $[\lambda''y'':\text{Tv}. [[\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot T \cdot T] = (\text{Tv} \Leftarrow \text{ooo}) = [\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot x_o \cdot y_o]]$ "

*<proof>*

**lemma** wff\_Con\_sym\_subterm2\_iff[simp]:

"wff  $\alpha$   $[\lambda''y'':\text{Tv}. [[\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot T \cdot T] = (\text{Tv} \Leftarrow \text{ooo}) = [\lambda''g'':\text{ooo}. g_{\text{ooo}} \cdot x_o \cdot y_o]] \longleftrightarrow \alpha = \text{oo}$ "

*<proof>*

## 8.10 Conjunction

**definition** Con :: "trm  $\Rightarrow$  trm  $\Rightarrow$  trm" (infix " $\wedge$ " 80) where

"A  $\wedge$  B = Con\_sym  $\cdot$  A  $\cdot$  B"

**lemma** wff\_Con[simp]: "wff Tv A  $\Longrightarrow$  wff Tv B  $\Longrightarrow$  wff Tv (A  $\wedge$  B)"

*<proof>*

**lemma** wff\_Con\_iff[simp]: "wff  $\alpha$  (A  $\wedge$  B)  $\longleftrightarrow$  wff Tv A  $\wedge$  wff Tv B  $\wedge$   $\alpha = \text{Tv}$ "

*<proof>*

## 8.11 Implication symbol

**definition** Imp\_sym :: trm where

"Imp\_sym  $\equiv [\lambda''x'':\text{Tv}. [\lambda''y'':\text{Tv}. [x_o =\text{Tv}= (x_o \wedge y_o)]]]$ "

**lemma** wff\_Imp\_sym[simp]:

"wff oo Imp\_sym"

*<proof>*

**lemma** wff\_Imp\_sym\_iff[simp]:

"wff  $\alpha$  Imp\_sym  $\longleftrightarrow$   $\alpha = \text{ooo}$ "

*<proof>*

**lemma** wff\_Imp\_sym\_subterm0[simp]:

"wff Tv (x\_o  $\wedge$  y\_o)"

*<proof>*

**lemma** wff\_Imp\_sym\_subterm0\_iff[simp]:

"wff  $\alpha$  (x\_o  $\wedge$  y\_o)  $\longleftrightarrow$   $\alpha = \text{Tv}$ "

*<proof>*

**lemma** wff\_Imp\_sym\_subterm1[simp]:

"wff Tv [x\_o =\text{Tv}= (x\_o  $\wedge$  y\_o)]"

*<proof>*

**lemma** wff\_Imp\_sym\_subterm1\_iff[simp]:

"wff  $\alpha$  [x\_o =\text{Tv}= (x\_o  $\wedge$  y\_o)]  $\longleftrightarrow$   $\alpha = \text{Tv}$ "

*<proof>*

**lemma** wff\_Imp\_sym\_subterm2[simp]:

"wff oo  $[\lambda''y'':\text{Tv}. [x_o =\text{Tv}= (x_o \wedge y_o)]]$ "

*<proof>*

```
lemma wff_Imp_sym_subterm2_iff[simp]:
  "wff  $\alpha$  [ $\lambda$  'y':Tv. [ $x_o$  =Tv= ( $x_o \wedge y_o$ )]]  $\longleftrightarrow$   $\alpha = oo$ "
  <proof>
```

## 8.12 Implication

```
definition Imp :: "trm  $\Rightarrow$  trm  $\Rightarrow$  trm" (infix " $\longrightarrow$ " 80) where
  "A  $\longrightarrow$  B = Imp_sym  $\cdot$  A  $\cdot$  B"
```

```
lemma wff_Imp[simp]: "wff Tv A  $\implies$  wff Tv B  $\implies$  wff Tv (A  $\longrightarrow$  B)"
  <proof>
```

```
lemma wff_Imp_iff[simp]: "wff  $\alpha$  (A  $\longrightarrow$  B)  $\longleftrightarrow$  wff Tv A  $\wedge$  wff Tv B  $\wedge$   $\alpha = Tv$ "
  <proof>
```

## 9 The Q0 proof system

```
definition axiom_1 :: trm where
  "axiom_1  $\equiv$  [( $g_{oo} \cdot T$ )  $\wedge$  ( $g_{oo} \cdot F$ )] =Tv= [ $\forall$  'x':Tv.  $g_{oo} \cdot x_o$ ]"
```

```
lemma wff_axiom_1[simp]: "wff Tv axiom_1"
  <proof>
```

```
definition axiom_2 :: "type_sym  $\Rightarrow$  trm" where
  "axiom_2  $\alpha \equiv$ 
  [( $\text{Var}$  'x'  $\alpha$ ) = $\alpha$ = ( $\text{Var}$  'y'  $\alpha$ )]  $\longrightarrow$ 
  [( $\text{Var}$  'h' (Tv  $\leftarrow$   $\alpha$ ))  $\cdot$  ( $\text{Var}$  'x'  $\alpha$ )] =Tv= (( $\text{Var}$  'h' (Tv  $\leftarrow$   $\alpha$ ))  $\cdot$  ( $\text{Var}$  'y'  $\alpha$ ))]"
```

```
definition axiom_3 :: "type_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm" where
  "axiom_3  $\alpha \beta \equiv$ 
  [( $\text{Var}$  'f' ( $\alpha \leftarrow \beta$ )) = $\alpha \leftarrow \beta$ = ( $\text{Var}$  'g' ( $\alpha \leftarrow \beta$ ))] =Tv=
  [ $\forall$  'x': $\beta$ . [( $\text{Var}$  'f' ( $\alpha \leftarrow \beta$ ))  $\cdot$  ( $\text{Var}$  'x'  $\beta$ )] = $\alpha$ = (( $\text{Var}$  'g' ( $\alpha \leftarrow \beta$ ))  $\cdot$  ( $\text{Var}$  'x'  $\beta$ ))]]]"
```

```
definition axiom_4_1 :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" where
  "axiom_4_1 x  $\alpha$  B  $\beta$  A  $\equiv$  [( $\lambda x:\alpha. B$ )  $\cdot$  A] = $\beta$ = B]"
```

```
definition axiom_4_1_side_condition :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  bool" where
  "axiom_4_1_side_condition x  $\alpha$  B  $\beta$  A  $\equiv$  ( $\exists c. B = \text{Cst } c \beta$ )  $\vee$  ( $\exists y. B = \text{Var } y \beta \wedge (x \neq y \vee \alpha \neq \beta)$ )"
```

```
definition axiom_4_2 :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" where
  "axiom_4_2 x  $\alpha$  A = [( $\lambda x:\alpha. \text{Var } x \alpha$ )  $\cdot$  A] = $\alpha$ = A]"
```

```
definition axiom_4_3 :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$ 
  type_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  trm  $\Rightarrow$  trm" where
  "axiom_4_3 x  $\alpha$  B  $\beta$   $\gamma$  C A = [( $\lambda x:\alpha. B \cdot C$ )  $\cdot$  A] = $\beta$ = (( $\lambda x:\alpha. B$ )  $\cdot$  A)  $\cdot$  (( $\lambda x:\alpha. C$ )  $\cdot$  A))]"
```

```
definition axiom_4_4 :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" where
  "axiom_4_4 x  $\alpha$  y  $\gamma$  B  $\delta$  A = [( $\lambda x:\alpha. [\lambda y:\gamma. B]$ )  $\cdot$  A] = $\delta \leftarrow \gamma$ = [ $\lambda y:\gamma. [\lambda x:\alpha. B]$ ]  $\cdot$  A]"
```

```
definition axiom_4_4_side_condition :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  bool" where
  "axiom_4_4_side_condition x  $\alpha$  y  $\gamma$  B  $\delta$  A  $\equiv$  ( $x \neq y \vee \alpha \neq \gamma$ )  $\wedge$  ( $y, \gamma \notin \text{vars } A$ )"
```

```
definition axiom_4_5 :: "var_sym  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  type_sym  $\Rightarrow$  trm  $\Rightarrow$  trm" where
  "axiom_4_5 x  $\alpha$  B  $\delta$  A = [( $\lambda x:\alpha. [\lambda x:\alpha. B]$ )  $\cdot$  A] = $\delta \leftarrow \alpha$ = [ $\lambda x:\alpha. B$ ]"
```

```
definition axiom_5 where
  "axiom_5 = [( $\iota \cdot ((Q (Tv \leftarrow \text{Ind} \leftarrow \text{Ind})) \cdot y_i)) =\text{Ind}= y_i]"$ 
```

```
inductive axiom :: "trm  $\Rightarrow$  bool" where
  by_axiom_1:
  "axiom axiom_1"
| by_axiom_2:
```

```

"axiom (axiom_2  $\alpha$ )"
| by_axiom_3:
"axiom (axiom_3  $\alpha$   $\beta$ )"
| by_axiom_4_1:
"wff  $\alpha$  A  $\implies$ 
wff  $\beta$  B  $\implies$ 
axiom_4_1_side_condition x  $\alpha$  B  $\beta$  A  $\implies$ 
axiom (axiom_4_1 x  $\alpha$  B  $\beta$  A)"
| by_axiom_4_2:
"wff  $\alpha$  A  $\implies$ 
axiom (axiom_4_2 x  $\alpha$  A)"
| by_axiom_4_3:
"wff  $\alpha$  A  $\implies$ 
wff ( $\beta \leftarrow \gamma$ ) B  $\implies$ 
wff  $\gamma$  C  $\implies$ 
axiom (axiom_4_3 x  $\alpha$  B  $\beta$   $\gamma$  C A)"
| by_axiom_4_4:
"wff  $\alpha$  A  $\implies$ 
wff  $\delta$  B  $\implies$ 
axiom_4_4_side_condition x  $\alpha$  y  $\gamma$  B  $\delta$  A  $\implies$ 
axiom (axiom_4_4 x  $\alpha$  y  $\gamma$  B  $\delta$  A)"
| by_axiom_4_5:
"wff  $\alpha$  A  $\implies$ 
wff  $\delta$  B  $\implies$ 
axiom (axiom_4_5 x  $\alpha$  B  $\delta$  A)"
| by_axiom_5:
"axiom (axiom_5)"

inductive rule_R :: "trm  $\implies$  trm  $\implies$  trm  $\implies$  bool" where
"replacement A B C D  $\implies$  rule_R C ([A = $\alpha$ = B]) D"

definition "proof" :: "trm  $\implies$  trm list  $\implies$  bool" where
"proof A p  $\longleftrightarrow$  (p  $\neq$  []  $\wedge$  last p = A  $\wedge$ 
( $\forall i < \text{length p. axiom (p ! i)}$ 
 $\vee (\exists j < i. \exists k < i. \text{rule\_R (p ! j) (p ! k) (p ! i)}))$ )"

inductive "theorem" :: "trm  $\implies$  bool" where
by_axiom: "axiom A  $\implies$  theorem A"
| by_rule_R: "theorem A  $\implies$  theorem B  $\implies$  rule_R A B C  $\implies$  theorem C"

definition axiom_4_1_variant_cst :: "var_sym  $\implies$  type_sym  $\implies$  var_sym  $\implies$  type_sym  $\implies$  trm  $\implies$  trm" where
"axiom_4_1_variant_cst x  $\alpha$  c  $\beta$  A  $\equiv$  [( $\lambda x:\alpha. \text{Cst c } \beta$ )  $\cdot$  A] = $\beta$ = (Cst c  $\beta$ )]"

definition axiom_4_1_variant_var :: "var_sym  $\implies$  type_sym  $\implies$  var_sym  $\implies$  type_sym  $\implies$  trm  $\implies$  trm" where
"axiom_4_1_variant_var x  $\alpha$  y  $\beta$  A  $\equiv$  [( $\lambda x:\alpha. \text{Var y } \beta$ )  $\cdot$  A] = $\beta$ = Var y  $\beta$ ]"

definition axiom_4_1_variant_var_side_condition :: "var_sym  $\implies$  type_sym  $\implies$  var_sym  $\implies$  type_sym  $\implies$  trm  $\implies$  bool"
where
"axiom_4_1_variant_var_side_condition x  $\alpha$  y  $\beta$  A  $\equiv$  x  $\neq$  y  $\vee$   $\alpha \neq \beta$ "



## 10 Semantics



type_synonym 's frame = "type_sym  $\implies$  's"

type_synonym 's denotation = "cst_sym  $\implies$  type_sym  $\implies$  's"

type_synonym 's asg = "var_sym * type_sym  $\implies$  's"

definition agree_off_asg :: "'s asg  $\implies$  's asg  $\implies$  var_sym  $\implies$  type_sym  $\implies$  bool" where
"agree_off_asg  $\varphi$   $\psi$  x  $\alpha$   $\longleftrightarrow$  ( $\forall y \beta. (y \neq x \vee \beta \neq \alpha) \longrightarrow \varphi (y, \beta) = \psi (y, \beta)$ )"

lemma agree_off_asg_def2:

```

```

"agree_off_asg  $\psi \varphi x \alpha \longleftrightarrow (\exists xa. \varphi((x, \alpha) := xa) = \psi)$ "
<proof>

lemma agree_off_asg_disagree_var_sym[simp]:
"agree_off_asg  $\psi \varphi x \alpha \implies x \neq y \implies \psi(y, \beta) = \varphi(y, \beta)$ "
<proof>

lemma agree_off_asg_disagree_type_sym[simp]:
"agree_off_asg  $\psi \varphi x \alpha \implies \alpha \neq \beta \implies \psi(y, \beta) = \varphi(y, \beta)$ "
<proof>

context set_theory
begin

definition wf_frame :: "'s frame  $\Rightarrow$  bool" where
"wf_frame D  $\longleftrightarrow$  D Tv = boolset  $\wedge$  ( $\forall \alpha \beta. D (\alpha \Leftarrow \beta) \subseteq$ : funspace (D  $\beta$ ) (D  $\alpha$ ))  $\wedge$  ( $\forall \alpha. D \alpha \neq \emptyset$ )"

definition inds :: "'s frame  $\Rightarrow$  's" where
"inds Fr = Fr Ind"

inductive wf_interp :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  bool" where
"wf_frame D  $\implies$ 
 $\forall c \alpha. I c \alpha \in$ : D  $\alpha \implies$ 
 $\forall \alpha. I \text{'Q'}$  (Tv  $\Leftarrow \alpha \Leftarrow \alpha$ ) = iden (D  $\alpha$ )  $\implies$ 
(I  $\text{'i'}$  (Ind  $\Leftarrow$  (Tv  $\Leftarrow$  Ind)))  $\in$ : funspace (D (Tv  $\Leftarrow$  Ind)) (D Ind)  $\implies$ 
 $\forall x. x \in$ : D Ind  $\longrightarrow$  (I  $\text{'i'}$  (Ind  $\Leftarrow$  (Tv  $\Leftarrow$  Ind)))  $\cdot$  one_elem_fun x (D Ind) = x  $\implies$ 
wf_interp D I"

definition asg_into_frame :: "'s asg  $\Rightarrow$  's frame  $\Rightarrow$  bool" where
"asg_into_frame  $\varphi D \longleftrightarrow$  ( $\forall x \alpha. \varphi(x, \alpha) \in$ : D  $\alpha$ )"

abbreviation(input) asg_into_interp :: "'s asg  $\Rightarrow$  's frame  $\Rightarrow$  's denotation  $\Rightarrow$  bool" where
"asg_into_interp  $\varphi D I \equiv$  asg_into_frame  $\varphi D$ "

fun val :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  's asg  $\Rightarrow$  trm  $\Rightarrow$  's" where
"val D I  $\varphi$  (Var x  $\alpha$ ) =  $\varphi(x, \alpha)$ "
| "val D I  $\varphi$  (Cst c  $\alpha$ ) = I c  $\alpha$ "
| "val D I  $\varphi$  (A  $\cdot$  B) = val D I  $\varphi$  A  $\cdot$  val D I  $\varphi$  B"
| "val D I  $\varphi$  [ $\lambda x:\alpha. B$ ] = abstract (D  $\alpha$ ) (D (type_of B)) ( $\lambda z. \text{val D I } (\varphi((x, \alpha) := z)) B$ )"

fun general_model :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  bool" where
"general_model D I  $\longleftrightarrow$  wf_interp D I  $\wedge$  ( $\forall \varphi A \alpha. \text{asg\_into\_interp } \varphi D I \longrightarrow \text{wff } \alpha A \longrightarrow \text{val D I } \varphi A \in$ : D  $\alpha$ )"

fun standard_model :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  bool" where
"standard_model D I  $\longleftrightarrow$  wf_interp D I  $\wedge$  ( $\forall \alpha \beta. D (\alpha \Leftarrow \beta) =$  funspace (D  $\beta$ ) (D  $\alpha$ ))"

lemma asg_into_frame_fun_upd:
assumes "asg_into_frame  $\varphi D$ "
assumes "xa  $\in$ : D  $\alpha$ "
shows "asg_into_frame ( $\varphi((x, \alpha) := xa)$ ) D"
<proof>

lemma asg_into_interp_fun_upd:
assumes "general_model D I"
assumes "asg_into_interp  $\varphi D I$ "
assumes "wff  $\alpha A$ "
shows "asg_into_interp ( $\varphi((x, \alpha) := \text{val D I } \varphi A)$ ) D I"
<proof>

lemma standard_model_is_general_model:
assumes "standard_model D I"
shows "general_model D I"

```



*<proof>*

**abbreviation** agree\_on\_asg :: "'s asg  $\Rightarrow$  's asg  $\Rightarrow$  var\_sym  $\Rightarrow$  type\_sym  $\Rightarrow$  bool" where  
"agree\_on\_asg  $\varphi \psi x \alpha == (\varphi (x, \alpha) = \psi (x, \alpha))$ "

**proposition** prop\_5400:

assumes "general\_model D I"  
assumes "asg\_into\_interp  $\varphi$  D I"  
assumes "asg\_into\_interp  $\psi$  D I"  
assumes "wff  $\alpha$  A"  
assumes " $\forall (x, \alpha) \in \text{frees } A. \text{agree\_on\_asg } \varphi \psi x \alpha$ "  
shows "val D I  $\varphi$  A = val D I  $\psi$  A"  
*<proof>*

**abbreviation** satisfies :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  's asg  $\Rightarrow$  trm  $\Rightarrow$  bool" where  
"satisfies D I  $\varphi$  A  $\equiv (\text{val D I } \varphi \text{ A} = \text{true})$ "

**definition** valid\_in\_model :: "'s frame  $\Rightarrow$  's denotation  $\Rightarrow$  trm  $\Rightarrow$  bool" where  
"valid\_in\_model D I A  $\equiv (\forall \varphi. \text{asg\_into\_interp } \varphi \text{ D I} \longrightarrow \text{val D I } \varphi \text{ A} = \text{true})$ "

**definition** valid\_general :: "trm  $\Rightarrow$  bool" where  
"valid\_general A  $\equiv \forall D I. \text{general\_model D I} \longrightarrow \text{valid\_in\_model D I A}$ "

**definition** valid\_standard :: "trm  $\Rightarrow$  bool" where  
"valid\_standard A  $\equiv \forall D I. \text{standard\_model D I} \longrightarrow \text{valid\_in\_model D I A}$ "

## 11 Semantics of defined wffs

**lemma** lemma\_5401\_a:

assumes "general\_model D I"  
assumes "asg\_into\_interp  $\varphi$  D I"  
assumes "wff  $\alpha$  A" "wff  $\beta$  B"  
shows "val D I  $\varphi$  ( $[\lambda x:\alpha. B] \cdot A$ ) = val D I ( $\varphi((x, \alpha) := \text{val D I } \varphi \text{ A})$ ) B"  
*<proof>*

**lemma** lemma\_5401\_b\_variant\_1:

assumes "general\_model D I"  
assumes "asg\_into\_interp  $\varphi$  D I"  
assumes "wff  $\alpha$  A" "wff  $\alpha$  B"  
shows "val D I  $\varphi$  ( $[A = \alpha = B]$ ) = (boolean (val D I  $\varphi$  A = val D I  $\varphi$  B))"  
*<proof>*

**lemma** lemma\_5401\_b:

assumes "general\_model D I"  
assumes "asg\_into\_interp  $\varphi$  D I"  
assumes "wff  $\alpha$  A" "wff  $\alpha$  B"  
shows "val D I  $\varphi$  ( $[A = \alpha = B]$ ) = true  $\longleftrightarrow$  val D I  $\varphi$  A = val D I  $\varphi$  B"  
*<proof>*

**lemma** lemma\_5401\_b\_variant\_2: — Just a reformulation of lemma\_5401\_b's directions

assumes "general\_model D I"  
assumes "asg\_into\_interp  $\varphi$  D I"  
assumes "wff  $\alpha$  A" "wff  $\alpha$  B"  
assumes "val D I  $\varphi$  A = val D I  $\varphi$  B"  
shows "val D I  $\varphi$  ( $[A = \alpha = B]$ ) = true"  
*<proof>*

**lemma** lemma\_5401\_b\_variant\_3: — Just a reformulation of lemma\_5401\_b's directions

```

assumes "general_model D I"
assumes "asg_into_interp  $\varphi$  D I"
assumes "wff  $\alpha$  A" "wff  $\alpha$  B"
assumes "val D I  $\varphi$  A  $\neq$  val D I  $\varphi$  B"
shows "val D I  $\varphi$  ([A = $\alpha$  B]) = false"
<proof>

lemma lemma_5401_c:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "val D I  $\varphi$  T = true"
  <proof>

lemma lemma_5401_d:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "val D I  $\varphi$  F = false"
  <proof>

lemma asg_into_interp_fun_upd_true:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "asg_into_interp ( $\varphi$ ((x, Tv) := true)) D I"
  <proof>

lemma asg_into_interp_fun_upd_false:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "asg_into_interp ( $\varphi$ ((x, Tv) := false)) D I"
  <proof>

lemma lemma_5401_e_1:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "(val D I  $\varphi$  Con_sym)  $\cdot$  true  $\cdot$  true = true"
  <proof>

lemma lemma_5401_e_2:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "y = true  $\vee$  y = false"
  shows "(val D I  $\varphi$  Con_sym)  $\cdot$  false  $\cdot$  y = false"
  <proof>

lemma lemma_5401_e_3:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "x = true  $\vee$  x = false"
  shows "(val D I  $\varphi$  Con_sym)  $\cdot$  x  $\cdot$  false = false"
  <proof>

lemma lemma_5401_e_variant_1:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "y = true  $\vee$  y = false"
  assumes "x = true  $\vee$  x = false"
  shows "(val D I  $\varphi$  Con_sym)  $\cdot$  x  $\cdot$  y = boolean (x = true  $\wedge$  y = true)"
  <proof>

```

```

lemma asg_into_interp_is_true_or_false:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows " $\varphi$  (x, Tv) = true  $\vee$   $\varphi$  (x, Tv) = false"
<proof>

lemma wff_Tv_is_true_or_false:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "wff Tv A"
  shows "val D I  $\varphi$  A = true  $\vee$  val D I  $\varphi$  A = false"
<proof>

lemma lemma_5401_e_variant_2:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "wff Tv A"
  assumes "wff Tv B"
  shows "(val D I  $\varphi$  (A  $\wedge$  B)) = boolean (satisfies D I  $\varphi$  A  $\wedge$  satisfies D I  $\varphi$  B)"
<proof>

lemma lemma_5401_f_1:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "y = true  $\vee$  y = false"
  shows "(val D I  $\varphi$  Imp_sym)  $\cdot$  false  $\cdot$  y = true"
<proof>

lemma lemma_5401_f_2:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "x = true  $\vee$  x = false"
  shows "(val D I  $\varphi$  Imp_sym)  $\cdot$  x  $\cdot$  true = true"
<proof>

lemma lemma_5401_f_3:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "(val D I  $\varphi$  Imp_sym)  $\cdot$  true  $\cdot$  false = false"
<proof>

lemma lemma_5401_f_variant_1:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "x = true  $\vee$  x = false"
  assumes "y = true  $\vee$  y = false"
  shows "(val D I  $\varphi$  Imp_sym)  $\cdot$  x  $\cdot$  y = boolean (x = true  $\longrightarrow$  y = true)"
<proof>

lemma lemma_5401_f_variant_2:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "wff Tv A"
  assumes "wff Tv B"
  shows "(val D I  $\varphi$  (A  $\longrightarrow$  B)) = boolean (satisfies D I  $\varphi$  A  $\longrightarrow$  satisfies D I  $\varphi$  B)"
<proof>

```

```

lemma lemma_5401_g:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "wff Tv A"
  shows "satisfies D I  $\varphi$  [ $\forall x:\alpha. A$ ]  $\longleftrightarrow$ 
        ( $\forall \psi. \text{asg\_into\_interp } \psi \text{ D I} \longrightarrow \text{agree\_off\_asg } \psi \varphi x \alpha \longrightarrow \text{satisfies D I } \psi A$ )"
<proof>

```

```

theorem lemma_5401_g_variant_1:
  assumes "asg_into_interp  $\varphi$  D I"
  assumes "general_model D I"
  assumes "wff Tv A"
  shows "val D I  $\varphi$  [ $\forall x:\alpha. A$ ] =
        boolean ( $\forall \psi. \text{asg\_into\_interp } \psi \text{ D I} \longrightarrow \text{agree\_off\_asg } \psi \varphi x \alpha \longrightarrow \text{satisfies D I } \psi A$ )"
<proof>

```

## 12 Soundness

```

lemma fun_sym_asg_to_funspace:
  assumes "asg_into_frame  $\varphi$  D"
  assumes "general_model D I"
  shows " $\varphi (f, \alpha \Leftarrow \beta) \in \text{funspace } (D \beta) (D \alpha)$ "
<proof>

```

```

lemma fun_sym_interp_to_funspace:
  assumes "asg_into_frame  $\varphi$  D"
  assumes "general_model D I"
  shows " $I f (\alpha \Leftarrow \beta) \in \text{funspace } (D \beta) (D \alpha)$ "
<proof>

```

```

theorem theorem_5402_a_rule_R:
  assumes A_eql_B: "valid_general ([A = $\alpha$ = B])"
  assumes "valid_general C"
  assumes "rule_R C ([A = $\alpha$ = B]) C'"
  assumes "wff  $\alpha$  A"
  assumes "wff  $\alpha$  B"
  assumes "wff  $\beta$  C"
  shows "valid_general C'"
<proof>

```

```

theorem Fun_Tv_Tv_frame_subs_funspace:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "D oo  $\subseteq$ : funspace (boolset) (boolset)"
<proof>

```

```

theorem theorem_5402_a_axiom_1_variant:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"
  shows "satisfies D I  $\varphi$  axiom_1"
<proof>

```

```

theorem theorem_5402_a_axiom_1: "valid_general axiom_1"
<proof>

```

```

theorem theorem_5402_a_axiom_2_variant:
  assumes "general_model D I"
  assumes "asg_into_interp  $\varphi$  D I"

```

shows "satisfies D I  $\varphi$  (axiom\_2  $\alpha$ )"  
<proof>

theorem theorem\_5402\_a\_axiom\_2: "valid\_general (axiom\_2  $\alpha$ )"  
<proof>

theorem theorem\_5402\_a\_axiom\_3\_variant:  
 assumes "general\_model D I"  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 shows "satisfies D I  $\varphi$  (axiom\_3  $\alpha$   $\beta$ )"  
<proof>

theorem theorem\_5402\_a\_axiom\_3: "valid\_general (axiom\_3  $\alpha$   $\beta$ )"  
<proof>

theorem theorem\_5402\_a\_axiom\_4\_1\_variant\_cst:  
 assumes "general\_model D I"  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 assumes "wff  $\alpha$  A"  
 shows "satisfies D I  $\varphi$  (axiom\_4\_1\_variant\_cst x  $\alpha$  c  $\beta$  A)"  
<proof>

theorem theorem\_5402\_a\_axiom\_4\_1\_variant\_var:  
 assumes "general\_model D I"  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 assumes "wff  $\alpha$  A"  
 assumes "axiom\_4\_1\_variant\_var\_side\_condition x  $\alpha$  y  $\beta$  A"  
 shows "satisfies D I  $\varphi$  (axiom\_4\_1\_variant\_var x  $\alpha$  y  $\beta$  A)"  
<proof>

theorem theorem\_5402\_a\_axiom\_4\_1:  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 assumes "general\_model D I"  
 assumes "axiom\_4\_1\_side\_condition x  $\alpha$  y  $\beta$  A"  
 assumes "wff  $\alpha$  A"  
 shows "satisfies D I  $\varphi$  (axiom\_4\_1 x  $\alpha$  y  $\beta$  A)"  
<proof>

theorem theorem\_5402\_a\_axiom\_4\_2:  
 assumes "general\_model D I"  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 assumes "wff  $\alpha$  A"  
 shows "satisfies D I  $\varphi$  (axiom\_4\_2 x  $\alpha$  A)"  
<proof>

theorem theorem\_5402\_a\_axiom\_4\_3:  
 assumes "general\_model D I"  
 assumes "asg\_into\_interp  $\varphi$  D I"  
 assumes "wff  $\alpha$  A"  
 assumes "wff ( $\beta \Leftarrow \gamma$ ) B"  
 assumes "wff  $\gamma$  C"  
 shows "satisfies D I  $\varphi$  (axiom\_4\_3 x  $\alpha$  B  $\beta$   $\gamma$  C A)"  
<proof>

lemma lemma\_to\_help\_with\_theorem\_5402\_a\_axiom\_4\_4:  
 assumes lambda\_eql\_lambda\_lambda:

```

  "\^z. z ∈: D γ ⇒ val D I ψ [λy:γ. B] · z = val D I φ [λy:γ. [λx:α. B] · A] · z"
  assumes ψ_eq1: "ψ = φ((x, α) := val D I φ A)"
  assumes "asg_into_frame φ D"
  assumes "general_model D I"
  assumes "axiom_4_4_side_condition x α y γ B δ A"
  assumes "wff α A"
  assumes "wff δ B"
  shows "val D I ψ [λy:γ. B] = val D I φ [λy:γ. [λx:α. B] · A]"
  <proof>

```

```

theorem theorem_5402_a_axiom_4_4:
  assumes "general_model D I"
  assumes "asg_into_interp φ D I"
  assumes "axiom_4_4_side_condition x α y γ B δ A"
  assumes "wff α A"
  assumes "wff δ B"
  shows "satisfies D I φ (axiom_4_4 x α y γ B δ A)"
  <proof>

```

```

theorem theorem_5402_a_axiom_4_5:
  assumes "general_model D I"
  assumes "asg_into_interp φ D I"
  assumes "wff α A"
  assumes "wff δ B"
  shows "satisfies D I φ (axiom_4_5 x α B δ A)"
  <proof>

```

```

theorem theorem_5402_a_axiom_5:
  assumes "general_model D I"
  assumes "asg_into_interp φ D I"
  shows "satisfies D I φ (axiom_5)"
  <proof>

```

```

lemma theorem_isa_Tv:
  assumes "theorem A"
  shows "wff Tv A"
  <proof>

```

```

theorem theorem_5402_a_general:
  assumes "theorem A"
  shows "valid_general A"
  <proof>

```

```

theorem theorem_5402_a_standard:
  assumes "theorem A"
  shows "valid_standard A"
  <proof>

```

end

end

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