Pushdown Systems

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Abstract

We formalize pushdown systems and the correctness of the pushdown reachability algorithms post* (forward search), pre* (backward search) and dual* (bi-directional search). For pre* we refine the algorithm to an executable version for which one can generate code using Isabelle's code generator. For pre* and post* we follow Stefan Schwoon's PhD thesis [Sch02a]. The dual* algorithm is from a paper by Jensen et. al presented at ATVA2021 [JSS+21]. The formalization is described in our FMCAD2022 paper [SSST22] in which we also document how we have used it to do differential testing against a C++ implementation of pushdown reachability called PDAAAL. Lammich et al. [Lam09, LMW09] formalized the pre* algorithm for dynamic pushdown networks (DPN) which is a generalization of pushdown systems. Our work is independent from that because the post* of DPNs is not regular and additionally the DPN formalization does not support epsilon transitions which we use for post* and dual*.

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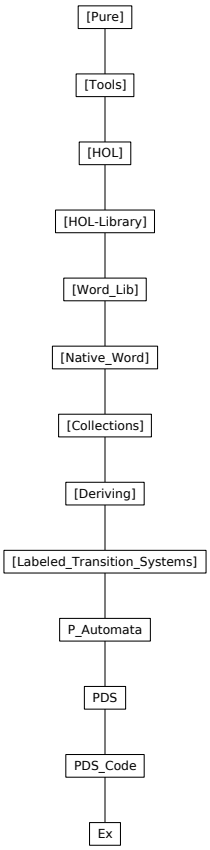

Figure 1: Theory dependency graph

1 Introduction

Pushdown reachability was studied by Büchi in 1964 [Büc64] and has been used for, among other things, interprocedural control-flow analysis of recursive programs [EK99, CNDE05], model checking [ES01, Sch02b, SSE05, BEM97] and communication network analysis [JKM+18, JKS+20, vDJJ+21]. In this formalization we formalize the pre* and post* algorithms [Sch02a] and the dual* algorithm [JSS+21]. For pre* we have also an executable version. In our FMCAD2022 paper [SSST22] we describe the formalization and use it to do differential testing against a C++ implementation of pushdown reachability called PDAAAL [JSS+21]. The differential testing revealed a number of bugs in PDAAAL that we were then able to fix.

theory *P_Automata* **imports** *Labeled_Transition_Systems.LTS* **begin**

2 Automata

2.1 P-Automaton locale

locale *P_Automaton* = *LTS transition_relation*
  for *transition_relation* :: "('state::finite, 'label) transition set" +
  fixes *Init* :: "'ctr_loc::enum ⇒ 'state"
    and *finals* :: "'state set"
begin

definition *initials* :: "'state set" **where**
  "*initials* ≡ *Init* ‘ *UNIV*"

lemma *initials_list*:
  "*initials* = *set* (*map Init Enum.enum*)"
  ⟨*proof*⟩

definition *accepts_aut* :: "'ctr_loc ⇒ 'label list ⇒ *bool*" **where**
  "*accepts_aut* ≡ λ*p w*. (∃ *q* ∈ *finals*. (*Init p*, *w*, *q*) ∈ *trans_star*)"

definition *lang_aut* :: "('ctr_loc ∗ 'label list) *set*" **where**
  "*lang_aut* = {(*p*,*w*). *accepts_aut p w*}"

definition *nonempty* **where**
  "*nonempty* ⟷ *lang_aut* ≠ {}"

lemma *nonempty_alt*:
  "*nonempty* ⟷ (∃ *p*. ∃ *q* ∈ *finals*. ∃ *w*. (*Init p*, *w*, *q*) ∈ *trans_star*)"
  ⟨*proof*⟩

typedef 'a *mark_state* = "{(*Q* :: 'a *set*, *I*). *I* ⊆ *Q*}"
  ⟨*proof*⟩
setup-lifting *type_definition_mark_state*
lift-definition *get_visited* :: "'a *mark_state* ⇒ 'a *set*" **is** *fst* ⟨*proof*⟩
lift-definition *get_next* :: "'a *mark_state* ⇒ 'a *set*" **is** *snd* ⟨*proof*⟩
lift-definition *make_mark_state* :: "'a *set* ⇒ 'a *set* ⇒ 'a *mark_state*" **is** "λ*Q J*. (*Q* ∪ *J*, *J*)" ⟨*proof*⟩
lemma *get_next_get_visited*: "*get_next ms* ⊆ *get_visited ms*"
  ⟨*proof*⟩
lemma *get_next_set_next*[*simp*]: "*get_next* (*make_mark_state Q J*) = *J*"
  ⟨*proof*⟩
lemma *get_visited_set_next*[*simp*]: "*get_visited* (*make_mark_state Q J*) = *Q* ∪ *J*"
  ⟨*proof*⟩

function *mark* **where**
  "*mark ms* ⟷
    (*let Q* = *get_visited ms*; *I* = *get_next ms in*
    if I ∩ *finals* ≠ {} *then True*
    else let J = (⋃(*q*,*w*,*q'*)∈*transition_relation*. *if q* ∈ *I* ∧ *q'* ∉ *Q then* {*q'*} *else* {}) *in*
      if J = {} *then False else mark* (*make_mark_state Q J*))"
  ⟨*proof*⟩
termination ⟨*proof*⟩

declare *mark.simps*[*simp del*]

lemma *trapped_transitions*: "(*p*, *w*, *q*) ∈ *trans_star* ⟹
  ∀ *p* ∈ *Q*. (∀ γ *q*. (*p*, γ, *q*) ∈ *transition_relation* ⟶ *q* ∈ *Q*) ⟹
  p ∈ *Q* ⟹ *q* ∈ *Q*"
  ⟨*proof*⟩

lemma *mark_complete*: "(*p*, *w*, *q*) ∈ *trans_star* ⟹ (*get_visited ms* − *get_next ms*) ∩ *finals* = {} ⟹
  ∀ *p* ∈ *get_visited ms* − *get_next ms*. ∀ *q* γ. (*p*, γ, *q*) ∈ *transition_relation* ⟶ *q* ∈ *get_visited ms* ⟹

p ∈ get_visited ms ⟹ q ∈ finals ⟹ mark ms"
⟨*proof*⟩

lemma *mark_sound*: *"mark ms ⟹ (∃ p ∈ get_next ms. ∃ q ∈ finals. ∃ w. (p, w, q) ∈ trans_star)"*
⟨*proof*⟩

lemma *nonempty_code*[*code*]: *"nonempty = mark (make_mark_state {} (set (map Init Enum.enum)))"*
⟨*proof*⟩

end

2.2 Intersection P-Automaton locale

locale *Intersection_P_Automaton* =
A1: *P_Automaton ts1 Init finals1* +
A2: *P_Automaton ts2 Init finals2*
for *ts1* :: *"('state :: finite, 'label) transition set"*
and *Init* :: *"'ctr_loc :: enum ⇒ 'state"*
and *finals1* :: *"'state set"*
and *ts2* :: *"('state, 'label) transition set"*
and *finals2* :: *"'state set"*
begin

sublocale *pa*: *P_Automaton "inters ts1 ts2" "(λp. (Init p, Init p))" "inters_finals finals1 finals2"*
⟨*proof*⟩

definition *accepts_aut_inters* **where**
"accepts_aut_inters p w = pa.accepts_aut p w"

definition *lang_aut_inters* :: *"('ctr_loc * 'label list) set"* **where**
"lang_aut_inters = {(p,w). accepts_aut_inters p w}"

lemma *trans_star_inter*:
assumes *"(p1, w, p2) ∈ A1.trans_star"*
assumes *"(q1, w, q2) ∈ A2.trans_star"*
shows *"((p1,q1), w :: 'label list, (p2,q2)) ∈ pa.trans_star"*
⟨*proof*⟩

lemma *inters_trans_star1*:
assumes *"(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star"*
shows *"(fst p1q2, w, fst p2q2) ∈ A1.trans_star"*
⟨*proof*⟩

lemma *inters_trans_star*:
assumes *"(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star"*
shows *"(snd p1q2, w, snd p2q2) ∈ A2.trans_star"*
⟨*proof*⟩

lemma *inters_trans_star_iff*:
"((p1,q2), w :: 'label list, (p2,q2)) ∈ pa.trans_star ⟷ (p1, w, p2) ∈ A1.trans_star ∧ (q2, w, q2) ∈ A2.trans_star"
⟨*proof*⟩

lemma *inters_accept_iff*: *"accepts_aut_inters p w ⟷ A1.accepts_aut p w ∧ A2.accepts_aut p w"*
⟨*proof*⟩

lemma *lang_aut_alt*:
"pa.lang_aut = {(p, w). (p, w) ∈ lang_aut_inters}"
⟨*proof*⟩

lemma *inters_lang*: *"lang_aut_inters = A1.lang_aut ∩ A2.lang_aut"*
⟨*proof*⟩

end

3 Automata with epsilon

3.1 P-Automaton with epsilon locale

locale *P_Automaton_ε = LTS_ε transition_relation* **for** *transition_relation* :: *"('state::finite, 'label option) transition set" +*
  fixes *finals* :: *"'state set"* **and** *Init* :: *"'ctr_loc :: enum ⇒ 'state"*
begin

definition *accepts_aut_ε* :: *"'ctr_loc ⇒ 'label list ⇒ bool"* **where**
  "accepts_aut_ε ≡ λp w. (∃ q ∈ finals. (Init p, w, q) ∈ trans_star_ε)"

definition *lang_aut_ε* :: *"('ctr_loc * 'label list) set"* **where**
  "lang_aut_ε = {(p,w). accepts_aut_ε p w}"

definition *nonempty_ε* **where**
  "nonempty_ε ⟷ lang_aut_ε ≠ {}"

end

3.2 Intersection P-Automaton with epsilon locale

locale *Intersection_P_Automaton_ε =*
  A1: *P_Automaton_ε ts1 finals1 Init +*
  A2: *P_Automaton_ε ts2 finals2 Init*
  for *ts1* :: *"('state :: finite, 'label option) transition set"*
    and *finals1* :: *"'state set"*
    and *Init* :: *"'ctr_loc :: enum ⇒ 'state"*
    and *ts2* :: *"('state, 'label option) transition set"*
    and *finals2* :: *"'state set"*
begin

abbreviation *ε* :: *"'label option"* **where**
  "ε == None"

sublocale *pa*: *P_Automaton_ε "inters_ε ts1 ts2" "inters_finals finals1 finals2" "(λp. (Init p, Init p))"*
  ⟨*proof*⟩

definition *accepts_aut_inters_ε* **where**
  "accepts_aut_inters_ε p w = pa.accepts_aut_ε p w"

definition *lang_aut_inters_ε* :: *"('ctr_loc * 'label list) set"* **where**
  "lang_aut_inters_ε = {(p,w). accepts_aut_inters_ε p w}"

lemma *trans_star_trans_star_ε_inter*:
  assumes *"LTS_ε.ε_exp w1 w"*
  assumes *"LTS_ε.ε_exp w2 w"*
  assumes *"(p1, w1, p2) ∈ A1.trans_star"*
  assumes *"(q1, w2, q2) ∈ A2.trans_star"*
  shows *"((p1,q1), w :: 'label list, (p2,q2)) ∈ pa.trans_star_ε"*
  ⟨*proof*⟩

lemma *trans_star_ε_inter*:
  assumes *"(p1, w :: 'label list, p2) ∈ A1.trans_star_ε"*
  assumes *"(q1, w, q2) ∈ A2.trans_star_ε"*
  shows *"((p1, q1), w, (p2, q2)) ∈ pa.trans_star_ε"*
⟨*proof*⟩

lemma *inters_trans_star_ε1*:
  assumes *"(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star_ε"*

shows "(*fst p1q2*, *w*, *fst p2q2*) ∈ *A1.trans_star_ε*"
⟨*proof*⟩

lemma *inters_trans_star_ε*:
assumes "(*p1q2*, *w* :: '*label list*, *p2q2*) ∈ *pa.trans_star_ε*"
shows "(*snd p1q2*, *w*, *snd p2q2*) ∈ *A2.trans_star_ε*"
⟨*proof*⟩

lemma *inters_trans_star_ε_iff*:
"((*p1*,*q2*), *w* :: '*label list*, (*p2*,*q2*)) ∈ *pa.trans_star_ε* ⟷
(*p1*, *w*, *p2*) ∈ *A1.trans_star_ε* ∧ (*q2*, *w*, *q2*) ∈ *A2.trans_star_ε*"
⟨*proof*⟩

lemma *inters_ε_accept_ε_iff*:
"*accepts_aut_inters_ε p w* ⟷ *A1.accepts_aut_ε p w* ∧ *A2.accepts_aut_ε p w*"
⟨*proof*⟩

lemma *inters_ε_lang_ε*: "*lang_aut_inters_ε* = *A1.lang_aut_ε* ∩ *A2.lang_aut_ε*"
⟨*proof*⟩

end

end
theory *PDS* **imports** "*P_Automata*" "*HOL−Library.While_Combinator*" **begin**

4 PDS

datatype '*label operation* = *pop* | *swap* '*label* | *push* '*label* '*label*
type-synonym ('*ctr_loc*, '*label*) *rule* = "('*ctr_loc* × '*label*) × ('*ctr_loc* × '*label operation*)"
type-synonym ('*ctr_loc*, '*label*) *conf* = "'*ctr_loc* × '*label list*"

We define push down systems.

locale *PDS* =
fixes Δ :: "('*ctr_loc*, '*label*::*finite*) *rule set*"

begin

primrec *lbl* :: "'*label operation* ⇒ '*label list*" **where**
"*lbl pop* = []"
| "*lbl* (*swap* γ) = [γ]"
| "*lbl* (*push* γ γ′) = [γ, γ′]"

definition *is_rule* :: "'*ctr_loc* × '*label* ⇒ '*ctr_loc* × '*label operation* ⇒ *bool*" (**infix** "↪" *80*) **where**
"*pγ* ↪ *p′w* ≡ (*pγ*,*p′w*) ∈ Δ"

inductive-set *transition_rel* :: "(('*ctr_loc*, '*label*) *conf* × *unit* × ('*ctr_loc*, '*label*) *conf*) *set*" **where**
"(*p*, γ) ↪ (*p′*, *w*) ⟹
((*p*, γ#*w′*), (), (*p′*, (*lbl w*)@*w′*)) ∈ *transition_rel*"

interpretation *LTS transition_rel* ⟨*proof*⟩

notation *step_relp* (**infix** "⇒" *80*)
notation *step_starp* (**infix** "⇒*" *80*)

lemma *step_relp_def2*:
"(*p*, γ*w′*) ⇒ (*p′*,*ww′*) ⟷ (∃γ *w′ w*. γ*w′* = γ#*w′* ∧ *ww′* = (*lbl w*)@*w′* ∧ (*p*, γ) ↪ (*p′*, *w*))"
⟨*proof*⟩

end

5 PDS with P automata

type-synonym (*'ctr_loc*, *'label*) *sat_rule* = "(*'ctr_loc*, *'label*) *transition set* ⇒ (*'ctr_loc*, *'label*) *transition set* ⇒ *bool*"

datatype (*'ctr_loc*, *'noninit*, *'label*) *state* =
  is_Init: *Init* (*the_Ctr_Loc*: *'ctr_loc*)
  | *is_Noninit*: *Noninit* (*the_St*: *'noninit*)
  | *is_Isolated*: *Isolated* (*the_Ctr_Loc*: *'ctr_loc*) (*the_Label*: *'label*)

lemma *finitely_many_states*:
  assumes "*finite* (*UNIV* :: *'ctr_loc set*)"
  assumes "*finite* (*UNIV* :: *'noninit set*)"
  assumes "*finite* (*UNIV* :: *'label set*)"
  shows "*finite* (*UNIV* :: (*'ctr_loc*, *'noninit*, *'label*) *state set*)"
⟨*proof*⟩

instantiation *state* :: (*finite*, *finite*, *finite*) *finite* **begin**

instance ⟨*proof*⟩

end

locale *PDS_with_P_automata* = *PDS* Δ
  for Δ :: "(*'ctr_loc*::*enum*, *'label*::*finite*) *rule set*"
    +
  fixes *final_inits* :: "(*'ctr_loc*::*enum*) *set*"
  fixes *final_noninits* :: "(*'noninit*::*finite*) *set*"
begin

definition *finals* :: "(*'ctr_loc*, *'noninit*::*finite*, *'label*) *state set*" **where**
  "*finals* = *Init* ' *final_inits* ∪ *Noninit* ' *final_noninits*"

lemma *F_not_Ext*: "¬(∃ *f*∈*finals*. *is_Isolated f*)"
  ⟨*proof*⟩

definition *inits* :: "(*'ctr_loc*, *'noninit*, *'label*) *state set*" **where**
  "*inits* = {*q*. *is_Init q*}"

lemma *inits_code*[*code*]: "*inits* = *set* (*map Init Enum.enum*)"
  ⟨*proof*⟩

definition *noninits* :: "(*'ctr_loc*, *'noninit*, *'label*) *state set*" **where**
  "*noninits* = {*q*. *is_Noninit q*}"

definition *isols* :: "(*'ctr_loc*, *'noninit*, *'label*) *state set*" **where**
  "*isols* = {*q*. *is_Isolated q*}"

sublocale *LTS transition_rel* ⟨*proof*⟩
notation *step_relp* (**infix** "⇒" *80*)
notation *step_starp* (**infix** "⇒* " *80*)

definition *accepts* :: "((*'ctr_loc*, *'noninit*, *'label*) *state*, *'label*) *transition set* ⇒ (*'ctr_loc*, *'label*) *conf* ⇒ *bool*" **where**
  "*accepts ts* ≡ λ(*p*,*w*). (∃ *q* ∈ *finals*. (*Init p*,*w*,*q*) ∈ *LTS.trans_star ts*)"

lemma *accepts_accepts_aut*: "*accepts ts* (*p*, *w*) ⟷ *P_Automaton.accepts_aut ts Init finals p w*"
  ⟨*proof*⟩

definition *accepts_ε* :: "((*'ctr_loc*, *'noninit*, *'label*) *state*, *'label option*) *transition set* ⇒ (*'ctr_loc*, *'label*) *conf* ⇒ *bool*" **where**
  "*accepts_ε ts* ≡ λ(*p*,*w*). (∃ *q* ∈ *finals*. (*Init p*,*w*,*q*) ∈ *LTS_ε.trans_star_ε ts*)"

abbreviation ε :: *"'label option"* **where**
"ε == None"

lemma *accepts_mono*[*mono*]: *"mono accepts"*
⟨*proof*⟩

lemma *accepts_cons*: *"(Init p, γ, Init p') $\in$ ts $\Longrightarrow$ accepts ts (p', w) $\Longrightarrow$ accepts ts (p, γ # w)"*
⟨*proof*⟩

definition *lang* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ ('ctr_loc, 'label) conf set"* **where**
"lang ts = {c. accepts ts c}"

lemma *lang_lang_aut*: *"lang ts = (λ(s,w). (s, w)) ' (P_Automaton.lang_aut ts Init finals)"*
⟨*proof*⟩

lemma *lang_aut_lang*: *"P_Automaton.lang_aut ts Init finals = lang ts"*
⟨*proof*⟩

definition *lang_ε* :: *"(('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ ('ctr_loc, 'label) conf set"* **where**
"lang_ε ts = {c. accepts_ε ts c}"

5.1 Saturations

definition *saturated* :: *"('c, 'l) sat_rule ⇒ ('c, 'l) transition set ⇒ bool"* **where**
"saturated rule ts ⟷ (∄ ts'. rule ts ts')"

definition *saturation* :: *"('c, 'l) sat_rule ⇒ ('c, 'l) transition set ⇒ ('c, 'l) transition set ⇒ bool"* **where**
*"saturation rule ts ts' ⟷ rule** ts ts' ∧ saturated rule ts'"*

lemma *no_infinite*:
assumes *"⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"*
assumes *"∀ i :: nat. rule (tts i) (tts (Suc i))"*
shows *"False"*
⟨*proof*⟩

lemma *saturation_termination*:
assumes *"⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"*
shows *"¬(∃ tts. (∀ i :: nat. rule (tts i) (tts (Suc i))))"*
⟨*proof*⟩

lemma *saturation_exi*:
assumes *"⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"*
shows *"∃ ts'. saturation rule ts ts'"*
⟨*proof*⟩

5.2 Saturation rules

inductive *pre_star_rule* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ bool"* **where**
add_trans: *"(p, γ) ↪ (p', w) ⟹ (Init p', lbl w, q) ∈ LTS.trans_star ts ⟹*
(Init p, γ, q) ∉ ts ⟹ pre_star_rule ts (ts ∪ {(Init p, γ, q)})"

definition *pre_star1* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state, 'label) transition set"* **where**
"pre_star1 ts =
(⋃((p, γ), (p', w)) ∈ Δ. ⋃ q ∈ LTS.reach ts (Init p') (lbl w). {(Init p, γ, q)})"

lemma *pre_star1_mono*: *"mono pre_star1"*
⟨*proof*⟩

lemma *pre_star_rule_pre_star1*:
assumes *"X ⊆ pre_star1 ts"*

shows *"pre_star_rule** ts (ts ∪ X)"*
⟨*proof*⟩

lemma *pre_star_rule_pre_star1s*: *"pre_star_rule** ts (((λs. s ∪ pre_star1 s) ^^ k) ts)"*
⟨*proof*⟩

definition *"pre_star_loop = while_option (λs. s ∪ pre_star1 s ≠ s) (λs. s ∪ pre_star1 s)"*
definition *"pre_star_exec = the o pre_star_loop"*
definition *"pre_star_exec_check A = (if inits ⊆ LTS.srcs A then pre_star_loop A else None)"*

definition *"accept_pre_star_exec_check A c = (if inits ⊆ LTS.srcs A then Some (accepts (pre_star_exec A) c) else None)"*

lemma *while_option_finite_subset_Some*: **fixes** *C* :: *"'a set"*
assumes *"mono f"* **and** *"!!X. X ⊆ C ⟹ f X ⊆ C"* **and** *"finite C"* **and** *X*: *"X ⊆ C" "X ⊆ f X"*
shows *"∃ P. while_option (λA. f A ≠ A) f X = Some P"*
⟨*proof*⟩

lemma *pre_star_exec_terminates*: *"∃ t. pre_star_loop s = Some t"*
⟨*proof*⟩

lemma *pre_star_exec_code*[*code*]:
"pre_star_exec s = (let s' = pre_star1 s in if s' ⊆ s then s else pre_star_exec (s ∪ s'))"
⟨*proof*⟩

lemma *saturation_pre_star_exec*: *"saturation pre_star_rule ts (pre_star_exec ts)"*
⟨*proof*⟩

inductive *post_star_rules* :: *"(('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ (('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ bool"* **where**
add_trans_pop:
"(p, γ) ↪ (p', pop) ⟹
(Init p, [γ], q) ∈ LTS_ε.trans_star_ε ts ⟹
(Init p', ε, q) ∉ ts ⟹
post_star_rules ts (ts ∪ {(Init p', ε, q)})"
| *add_trans_swap*:
"(p, γ) ↪ (p', swap γ') ⟹
(Init p, [γ], q) ∈ LTS_ε.trans_star_ε ts ⟹
(Init p', Some γ', q) ∉ ts ⟹
post_star_rules ts (ts ∪ {(Init p', Some γ', q)})"
| *add_trans_push_1*:
"(p, γ) ↪ (p', push γ' γ'') ⟹
(Init p, [γ], q) ∈ LTS_ε.trans_star_ε ts ⟹
(Init p', Some γ', Isolated p' γ') ∉ ts ⟹
post_star_rules ts (ts ∪ {(Init p', Some γ', Isolated p' γ')})"
| *add_trans_push_2*:
"(p, γ) ↪ (p', push γ' γ'') ⟹
(Init p, [γ], q) ∈ LTS_ε.trans_star_ε ts ⟹
(Isolated p' γ', Some γ'', q) ∉ ts ⟹
(Init p', Some γ', Isolated p' γ') ∈ ts ⟹
post_star_rules ts (ts ∪ {(Isolated p' γ', Some γ'', q)})"

lemma *pre_star_rule_mono*:
"pre_star_rule ts ts' ⟹ ts ⊂ ts'"
⟨*proof*⟩

lemma *post_star_rules_mono*:
"post_star_rules ts ts' ⟹ ts ⊂ ts'"
⟨*proof*⟩

lemma *pre_star_rule_card_Suc*: *"pre_star_rule ts ts' ⟹ card ts' = Suc (card ts)"*
⟨*proof*⟩

lemma *post_star_rules_card_Suc*: "*post_star_rules ts ts'* $\Longrightarrow$ *card ts'* = *Suc* (*card ts*)"
⟨*proof*⟩

lemma *pre_star_saturation_termination*:
"¬(∃ *tts*. (∀ *i* :: *nat*. *pre_star_rule* (*tts i*) (*tts* (*Suc i*))))"
⟨*proof*⟩

lemma *post_star_saturation_termination*:
"¬(∃ *tts*. (∀ *i* :: *nat*. *post_star_rules* (*tts i*) (*tts* (*Suc i*))))"
⟨*proof*⟩

lemma *pre_star_saturation_exi*:
shows "∃ *ts'*. *saturation pre_star_rule ts ts'*"
⟨*proof*⟩

lemma *post_star_saturation_exi*:
shows "∃ *ts'*. *saturation post_star_rules ts ts'*"
⟨*proof*⟩

lemma *pre_star_rule_incr*: "*pre_star_rule A B* $\Longrightarrow$ *A* ⊆ *B*"
⟨*proof*⟩

lemma *post_star_rules_incr*: "*post_star_rules A B* $\Longrightarrow$ *A* ⊆ *B*"
⟨*proof*⟩

lemma *saturation_rtranclp_pre_star_rule_incr*: "*pre_star_rule*** *A B* $\Longrightarrow$ *A* ⊆ *B*"
⟨*proof*⟩

lemma *saturation_rtranclp_post_star_rule_incr*: "*post_star_rules*** *A B* $\Longrightarrow$ *A* ⊆ *B*"
⟨*proof*⟩

lemma *pre_star'_incr_trans_star*:
"*pre_star_rule*** *A A'* $\Longrightarrow$ *LTS.trans_star A* ⊆ *LTS.trans_star A'*"
⟨*proof*⟩

lemma *post_star'_incr_trans_star*:
"*post_star_rules*** *A A'* $\Longrightarrow$ *LTS.trans_star A* ⊆ *LTS.trans_star A'*"
⟨*proof*⟩

lemma *post_star'_incr_trans_star_ε*:
"*post_star_rules*** *A A'* $\Longrightarrow$ *LTS_ε.trans_star_ε A* ⊆ *LTS_ε.trans_star_ε A'*"
⟨*proof*⟩

lemma *pre_star_lim'_incr_trans_star*:
"*saturation pre_star_rule A A'* $\Longrightarrow$ *LTS.trans_star A* ⊆ *LTS.trans_star A'*"
⟨*proof*⟩

lemma *post_star_lim'_incr_trans_star*:
"*saturation post_star_rules A A'* $\Longrightarrow$ *LTS.trans_star A* ⊆ *LTS.trans_star A'*"
⟨*proof*⟩

lemma *post_star_lim'_incr_trans_star_ε*:
"*saturation post_star_rules A A'* $\Longrightarrow$ *LTS_ε.trans_star_ε A* ⊆ *LTS_ε.trans_star_ε A'*"
⟨*proof*⟩

5.3 Pre* lemmas

lemma *inits_srcs_iff_Ctr_Loc_srcs*:
"*inits* ⊆ *LTS.srcs A* ⟷ (∄ *q* γ *q'*. (*q*, γ, *Init q'*) ∈ *A*)"
⟨*proof*⟩

lemma *lemma_3_1*:
assumes "*p'w* ⇒* *pv*"

assumes *"pv ∈ lang A"*
assumes *"saturation pre_star_rule A A'"*
shows *"accepts A' p'w"*
⟨*proof*⟩

lemma *word_into_init_empty_states*:
fixes *A* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set"*
assumes *"(p, w, ss, Init q) ∈ LTS.trans_star_states A"*
assumes *"inits ⊆ LTS.srcs A"*
shows *"w = [] ∧ p = Init q ∧ ss=[p]"*
⟨*proof*⟩

lemma *word_into_init_empty*:
fixes *A* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set"*
assumes *"(p, w, Init q) ∈ LTS.trans_star A"*
assumes *"inits ⊆ LTS.srcs A"*
shows *"w = [] ∧ p = Init q"*
⟨*proof*⟩

lemma *step_relp_append_aux*:
assumes *"pu ⇒* p1y"*
shows *"(fst pu, snd pu @ v) ⇒* (fst p1y, snd p1y @ v)"*
⟨*proof*⟩

lemma *step_relp_append*:
assumes *"(p, u) ⇒* (p1, y)"*
shows *"(p, u @ v) ⇒* (p1, y @ v)"*
⟨*proof*⟩

lemma *step_relp_append_empty*:
assumes *"(p, u) ⇒* (p1, [])"*
shows *"(p, u @ v) ⇒* (p1, v)"*
⟨*proof*⟩

lemma *lemma_3_2_a'*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"pre_star_rule** A A'"*
assumes *"(Init p, w, q) ∈ LTS.trans_star A'"*
shows *"∃ p' w'. (Init p', w', q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p', w')"*
⟨*proof*⟩

lemma *lemma_3_2_a*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"saturation pre_star_rule A A'"*
assumes *"(Init p, w, q) ∈ LTS.trans_star A'"*
shows *"∃ p' w'. (Init p', w', q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p', w')"*
⟨*proof*⟩
theorem *pre_star_rule_subset_pre_star_lang*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"pre_star_rule** A A'"*
shows *"{c. accepts A' c} ⊆ pre_star (lang A)"*
⟨*proof*⟩
theorem *pre_star_rule_accepts_correct*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"saturation pre_star_rule A A'"*
shows *"{c. accepts A' c} = pre_star (lang A)"*
⟨*proof*⟩
theorem *pre_star_rule_correct*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"saturation pre_star_rule A A'"*
shows *"lang A' = pre_star (lang A)"*
⟨*proof*⟩

theorem *pre_star_exec_accepts_correct*:
assumes "*inits* ⊆ *LTS.srcs A*"
shows "{*c*. *accepts* (*pre_star_exec A*) *c*} = *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *pre_star_exec_lang_correct*:
assumes "*inits* ⊆ *LTS.srcs A*"
shows "*lang* (*pre_star_exec A*) = *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *pre_star_exec_check_accepts_correct*:
assumes "*pre_star_exec_check A* ≠ *None*"
shows "{*c*. *accepts* (*the* (*pre_star_exec_check A*)) *c*} = *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *pre_star_exec_check_correct*:
assumes "*pre_star_exec_check A* ≠ *None*"
shows "*lang* (*the* (*pre_star_exec_check A*)) = *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *accept_pre_star_exec_correct_True*:
assumes "*inits* ⊆ *LTS.srcs A*"
assumes "*accepts* (*pre_star_exec A*) *c*"
shows "*c* ∈ *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *accept_pre_star_exec_correct_False*:
assumes "*inits* ⊆ *LTS.srcs A*"
assumes "¬*accepts* (*pre_star_exec A*) *c*"
shows "*c* ∉ *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *accept_pre_star_exec_correct_Some_True*:
assumes "*accept_pre_star_exec_check A c* = *Some True*"
shows "*c* ∈ *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *accept_pre_star_exec_correct_Some_False*:
assumes "*accept_pre_star_exec_check A c* = *Some False*"
shows "*c* ∉ *pre_star* (*lang A*)"
⟨*proof*⟩

theorem *accept_pre_star_exec_correct_None*:
assumes "*accept_pre_star_exec_check A c* = *None*"
shows "¬*inits* ⊆ *LTS.srcs A*"
⟨*proof*⟩

5.4 Post* lemmas

lemma *lemma_3_3′*:
assumes "*pv* ⇒* *p′w*"
and "(*fst pv*, *snd pv*) ∈ *lang_ε A*"
and "*saturation post_star_rules A A′*"
shows "*accepts_ε A′* (*fst p′w*, *snd p′w*)"
⟨*proof*⟩

lemma *lemma_3_3*:
assumes "(*p*,*v*) ⇒* (*p′*,*w*)"
and "(*p*, *v*) ∈ *lang_ε A*"
and "*saturation post_star_rules A A′*"
shows "*accepts_ε A′* (*p′*, *w*)"
⟨*proof*⟩

lemma *init_only_hd*:
assumes "(*ss*, *w*) ∈ *LTS.path_with_word A*"
assumes "*inits* ⊆ *LTS.srcs A*"
assumes "*count* (*transitions_of* (*ss*, *w*)) *t* > *0*"
assumes "*t* = (*Init p1*, γ, *q1*)"
shows "*hd* (*transition_list* (*ss*, *w*)) = *t* ∧ *count* (*transitions_of* (*ss*, *w*)) *t* = *1*"
⟨*proof*⟩

lemma *no_edge_to_Ctr_Loc_avoid_Ctr_Loc*:
assumes "(*p*, *w*, *qq*) ∈ *LTS.trans_star Aiminus1*"
assumes "*w* ≠ []"
assumes "*inits* ⊆ *LTS.srcs Aiminus1*"
shows "*qq* ∉ *inits*"
⟨*proof*⟩

lemma *no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε*:
assumes "(*p*, [γ], *qq*) ∈ *LTS_ε.trans_star_ε Aiminus1*"
assumes "*inits* ⊆ *LTS.srcs Aiminus1*"
shows "*qq* ∉ *inits*"
⟨*proof*⟩

lemma *no_edge_to_Ctr_Loc_post_star_rules′*:
assumes "*post_star_rules*** *A Ai*"
assumes "∄ *q* γ *q′*. (*q*, γ, *Init q′*) ∈ *A*"
shows "∄ *q* γ *q′*. (*q*, γ, *Init q′*) ∈ *Ai*"
⟨*proof*⟩

lemma *no_edge_to_Ctr_Loc_post_star_rules*:
assumes "*post_star_rules*** *A Ai*"
assumes "*inits* ⊆ *LTS.srcs A*"
shows "*inits* ⊆ *LTS.srcs Ai*"
⟨*proof*⟩

lemma *source_and_sink_isolated*:
assumes "*N* ⊆ *LTS.srcs A*"
assumes "*N* ⊆ *LTS.sinks A*"
shows "∀ *p* γ *q*. (*p*, γ, *q*) ∈ *A* ⟶ *p* ∉ *N* ∧ *q* ∉ *N*"
⟨*proof*⟩

lemma *post_star_rules_Isolated_source_invariant′*:
assumes "*post_star_rules*** *A A′*"
assumes "*isols* ⊆ *LTS.isolated A*"
assumes "(*Init p′*, *Some* γ′, *Isolated p′* γ′) ∉ *A′*"
shows "∄ *p* γ. (*p*, γ, *Isolated p′* γ′) ∈ *A′*"
⟨*proof*⟩

lemma *post_star_rules_Isolated_source_invariant*:
assumes "*post_star_rules*** *A A′*"
assumes "*isols* ⊆ *LTS.isolated A*"
assumes "(*Init p′*, *Some* γ′, *Isolated p′* γ′) ∉ *A′*"
shows "*Isolated p′* γ′ ∈ *LTS.srcs A′*"
⟨*proof*⟩

lemma *post_star_rules_Isolated_sink_invariant′*:
assumes "*post_star_rules*** *A A′*"
assumes "*isols* ⊆ *LTS.isolated A*"
assumes "(*Init p′*, *Some* γ′, *Isolated p′* γ′) ∉ *A′*"
shows "∄ *p* γ. (*Isolated p′* γ′, γ, *p*) ∈ *A′*"
⟨*proof*⟩

lemma *post_star_rules_Isolated_sink_invariant*:
assumes "*post_star_rules*** *A A′*"
assumes "*isols* ⊆ *LTS.isolated A*"

assumes *"(Init p', Some γ', Isolated p' γ') ∉ A'"*
shows *"Isolated p' γ' ∈ LTS.sinks A'"*
⟨*proof*⟩
lemma *rtranclp_post_star_rules_constains_successors_states*:
assumes *"post_star_rules** A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
assumes *"(Init p, w, ss, q) ∈ LTS.trans_star_states A'"*
shows *"(¬is_Isolated q ⟶ (∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε w))) ∧
(is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w))"*
⟨*proof*⟩
lemma *rtranclp_post_star_rules_constains_successors*:
assumes *"post_star_rules** A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
assumes *"(Init p, w, q) ∈ LTS.trans_star A'"*
shows *"(¬is_Isolated q ⟶ (∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε w))) ∧
(is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w))"*
⟨*proof*⟩
lemma *post_star_rules_saturation_constains_successors*:
assumes *"saturation post_star_rules A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
assumes *"(Init p, w, q) ∈ LTS.trans_star A'"*
shows *"(¬is_Isolated q ⟶ (∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε w))) ∧
(is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w))"*
⟨*proof*⟩
theorem *post_star_rules_subset_post_star_lang*:
assumes *"post_star_rules** A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"{c. accepts_ε A' c} ⊆ post_star (lang_ε A)"*
⟨*proof*⟩
theorem *post_star_rules_accepts_ε_correct*:
assumes *"saturation post_star_rules A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"{c. accepts_ε A' c} = post_star (lang_ε A)"*
⟨*proof*⟩
theorem *post_star_rules_correct*:
assumes *"saturation post_star_rules A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"lang_ε A' = post_star (lang_ε A)"*
⟨*proof*⟩

end

5.5 Intersection Automata

definition *accepts_inters* :: *"(('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state) set ⇒ ('ctr_loc, 'label) conf ⇒ bool"* **where**
"accepts_inters ts finals ≡ λ(p,w). (∃ qq ∈ finals. ((Init p, Init p),w,qq) ∈ LTS.trans_star ts)"

lemma *accepts_inters_accepts_aut_inters*:
assumes *"ts12 = inters ts1 ts2"*
assumes *"finals12 = inters_finals finals1 finals2"*
shows *"accepts_inters ts12 finals12 (p,w) ⟷
Intersection_P_Automaton.accepts_aut_inters ts1 Init finals1 ts2
finals2 p w"*
⟨*proof*⟩

definition *lang_inters* :: "(('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state) set ⇒ ('ctr_loc, 'label) conf set" **where**
"lang_inters ts finals = {c. accepts_inters ts finals c}"

lemma *lang_inters_lang_aut_inters*:
assumes "ts12 = inters ts1 ts2"
assumes "finals12 = inters_finals finals1 finals2"
shows "(λ(p,w). (p, w)) ‘ lang_inters ts12 finals12 =
Intersection_P_Automaton.lang_aut_inters ts1 Init finals1 ts2 finals2"
⟨*proof*⟩

lemma *inters_accept_iff*:
assumes "ts12 = inters ts1 ts2"
assumes "finals12 = inters_finals (PDS_with_P_automata.finals final_initss1 final_noninits1)
(PDS_with_P_automata.finals final_initss2 final_noninits2)"
shows
"accepts_inters ts12 finals12 (p,w) ⟷
PDS_with_P_automata.accepts final_initss1 final_noninits1 ts1 (p,w) ∧
PDS_with_P_automata.accepts final_initss2 final_noninits2 ts2 (p,w)"
⟨*proof*⟩

lemma *inters_lang*:
assumes "ts12 = inters ts1 ts2"
assumes "finals12 =
inters_finals (PDS_with_P_automata.finals final_initss1 final_noninits1)
(PDS_with_P_automata.finals final_initss2 final_noninits2)"
shows "lang_inters ts12 finals12 =
PDS_with_P_automata.lang final_initss1 final_noninits1 ts1 ∩
PDS_with_P_automata.lang final_initss2 final_noninits2 ts2"
⟨*proof*⟩

5.6 Intersection epsilon-Automata

context *PDS_with_P_automata* **begin**

interpretation *LTS transition_rel* ⟨*proof*⟩
notation *step_relp* (**infix** "⇒" 80)
notation *step_starp* (**infix** "⇒* " 80)

definition *accepts_ε_inters* :: "(('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ ('ctr_loc, 'label) conf ⇒ bool" **where**
"accepts_ε_inters ts ≡ λ(p,w). (∃ q1 ∈ finals. ∃ q2 ∈ finals. ((Init p, Init p),w,(q1,q2)) ∈ LTS_ε.trans_star_ε ts)"

definition *lang_ε_inters* :: "(('ctr_loc, 'noninit, 'label) state * ('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ ('ctr_loc, 'label) conf set" **where**
"lang_ε_inters ts = {c. accepts_ε_inters ts c}"

lemma *trans_star_trans_star_ε_inter*:
assumes "LTS_ε.ε_exp w1 w"
assumes "LTS_ε.ε_exp w2 w"
assumes "(p1, w1, p2) ∈ LTS.trans_star ts1"
assumes "(q1, w2, q2) ∈ LTS.trans_star ts2"
shows "((p1,q1), w :: 'label list, (p2,q2)) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
⟨*proof*⟩

lemma *trans_star_ε_inter*:
assumes "(p1, w :: 'label list, p2) ∈ LTS_ε.trans_star_ε ts1"
assumes "(q1, w, q2) ∈ LTS_ε.trans_star_ε ts2"
shows "((p1, q1), w, (p2, q2)) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
⟨*proof*⟩

lemma *inters_trans_star_ε1*:

assumes "(*p1q2*, *w* :: *'label list*, *p2q2*) ∈ *LTS_ε.trans_star_ε* (*inters_ε ts1 ts2*)"
shows "(*fst p1q2*, *w*, *fst p2q2*) ∈ *LTS_ε.trans_star_ε ts1*"
⟨*proof*⟩

lemma *inters_trans_star_ε*:
assumes "(*p1q2*, *w* :: *'label list*, *p2q2*) ∈ *LTS_ε.trans_star_ε* (*inters_ε ts1 ts2*)"
shows "(*snd p1q2*, *w*, *snd p2q2*) ∈ *LTS_ε.trans_star_ε ts2*"
⟨*proof*⟩

lemma *inters_trans_star_ε_iff*:
"((*p1*,*q2*), *w* :: *'label list*, (*p2*,*q2*)) ∈ *LTS_ε.trans_star_ε* (*inters_ε ts1 ts2*) ⟷
(*p1*, *w*, *p2*) ∈ *LTS_ε.trans_star_ε ts1* ∧ (*q2*, *w*, *q2*) ∈ *LTS_ε.trans_star_ε ts2*"
⟨*proof*⟩

lemma *inters_ε_accept_ε_iff*:
"*accepts_ε_inters* (*inters_ε ts1 ts2*) *c* ⟷ *accepts_ε ts1 c* ∧ *accepts_ε ts2 c*"
⟨*proof*⟩

lemma *inters_ε_lang_ε*: "*lang_ε_inters* (*inters_ε ts1 ts2*) = *lang_ε ts1* ∩ *lang_ε ts2*"
⟨*proof*⟩

5.7 Dual search

lemma *dual1*:
"*post_star* (*lang_ε A1*) ∩ *pre_star* (*lang A2*) = {*c*. ∃ *c1* ∈ *lang_ε A1*. ∃ *c2* ∈ *lang A2*. *c1* ⇒* *c* ∧ *c* ⇒* *c2*}"
⟨*proof*⟩

lemma *dual2*:
"*post_star* (*lang_ε A1*) ∩ *pre_star* (*lang A2*) ≠ {} ⟷ (∃ *c1* ∈ *lang_ε A1*. ∃ *c2* ∈ *lang A2*. *c1* ⇒* *c2*)"
⟨*proof*⟩

lemma *LTS_ε_of_LTS_Some*: "(*p*, *Some* γ, *q'*) ∈ *LTS_ε_of_LTS A2'* ⟷ (*p*, γ, *q'*) ∈ *A2'*"
⟨*proof*⟩

lemma *LTS_ε_of_LTS_None*: "(*p*, *None*, *q'*) ∉ *LTS_ε_of_LTS A2'*"
⟨*proof*⟩

lemma *trans_star_ε_LTS_ε_of_LTS_trans_star*:
assumes "(*p*,*w*,*q*) ∈ *LTS_ε.trans_star_ε* (*LTS_ε_of_LTS A2'*)"
shows "(*p*,*w*,*q*) ∈ *LTS.trans_star A2'*"
⟨*proof*⟩

lemma *trans_star_trans_star_ε_LTS_ε_of_LTS*:
assumes "(*p*,*w*,*q*) ∈ *LTS.trans_star A2'*"
shows "(*p*,*w*,*q*) ∈ *LTS_ε.trans_star_ε* (*LTS_ε_of_LTS A2'*)"
⟨*proof*⟩

lemma *accepts_ε_LTS_ε_of_LTS_iff_accepts*: "*accepts_ε* (*LTS_ε_of_LTS A2'*) (*p*, *w*) ⟷ *accepts A2'* (*p*, *w*)"
⟨*proof*⟩

lemma *lang_ε_LTS_ε_of_LTS_is_lang*: "*lang_ε* (*LTS_ε_of_LTS A2'*) = *lang A2'*"
⟨*proof*⟩

theorem *dual_star_correct_early_termination*:
assumes "*inits* ⊆ *LTS.srcs A1*"
assumes "*inits* ⊆ *LTS.srcs A2*"
assumes "*isols* ⊆ *LTS.isolated A1*"
assumes "*isols* ⊆ *LTS.isolated A2*"
assumes "*post_star_rules*** *A1 A1'*"
assumes "*pre_star_rule*** *A2 A2'*"
assumes "*lang_ε_inters* (*inters_ε A1'* (*LTS_ε_of_LTS A2'*)) ≠ {}"
shows "∃ *c1* ∈ *lang_ε A1*. ∃ *c2* ∈ *lang A2*. *c1* ⇒* *c2*"
⟨*proof*⟩

theorem *dual_star_correct_saturation*:
assumes *"inits ⊆ LTS.srcs A1"*
assumes *"inits ⊆ LTS.srcs A2"*
assumes *"isols ⊆ LTS.isolated A1"*
assumes *"isols ⊆ LTS.isolated A2"*
assumes *"saturation post_star_rules A1 A1'"*
assumes *"saturation pre_star_rule A2 A2'"*
shows *"lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {} ⟷ (∃ c1 ∈ lang_ε A1. ∃ c2 ∈ lang A2. c1 ⇒* c2)"*
⟨*proof*⟩

theorem *dual_star_correct_early_termination_configs*:
assumes *"inits ⊆ LTS.srcs A1"*
assumes *"inits ⊆ LTS.srcs A2"*
assumes *"isols ⊆ LTS.isolated A1"*
assumes *"isols ⊆ LTS.isolated A2"*
assumes *"lang_ε A1 = {c1}"*
assumes *"lang A2 = {c2}"*
assumes *"post_star_rules** A1 A1'"*
assumes *"pre_star_rule** A2 A2'"*
assumes *"lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {}"*
shows *"c1 ⇒* c2"*
⟨*proof*⟩

theorem *dual_star_correct_saturation_configs*:
assumes *"inits ⊆ LTS.srcs A1"*
assumes *"inits ⊆ LTS.srcs A2"*
assumes *"isols ⊆ LTS.isolated A1"*
assumes *"isols ⊆ LTS.isolated A2"*
assumes *"lang_ε A1 = {c1}"*
assumes *"lang A2 = {c2}"*
assumes *"saturation post_star_rules A1 A1'"*
assumes *"saturation pre_star_rule A2 A2'"*
shows *"lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {} ⟷ c1 ⇒* c2"*
⟨*proof*⟩

end

end
theory *PDS_Code*
imports *PDS "Deriving.Derive"*
begin

global-interpretation *pds*: *PDS_with_P_automata Δ F_ctr_loc F_ctr_loc_st*
for *Δ* :: *"('ctr_loc::{enum, linorder}, 'label::{finite, linorder}) rule set"*
and *F_ctr_loc* :: *"('ctr_loc) set"*
and *F_ctr_loc_st* :: *"('state::finite) set"*
defines *pre_star = "PDS_with_P_automata.pre_star_exec Δ"*
and *pre_star_check = "PDS_with_P_automata.pre_star_exec_check Δ"*
and *inits = "PDS_with_P_automata.inits"*
and *finals = "PDS_with_P_automata.finals F_ctr_loc F_ctr_loc_st"*
and *accepts = "PDS_with_P_automata.accepts F_ctr_loc F_ctr_loc_st"*
and *language = "PDS_with_P_automata.lang F_ctr_loc F_ctr_loc_st"*
and *step_starp = "rtranclp (LTS.step_relp (PDS.transition_rel Δ))"*
and *accepts_pre_star_check = "PDS_with_P_automata.accept_pre_star_exec_check Δ F_ctr_loc F_ctr_loc_st"*
⟨*proof*⟩

global-interpretation *inter*: *Intersection_P_Automaton*
initial_automaton Init "finals initial_F_ctr_loc initial_F_ctr_loc_st"
"pre_star Δ final_automaton" "finals final_F_ctr_loc final_F_ctr_loc_st"
for *Δ* :: *"('ctr_loc::{enum, linorder}, 'label::{finite, linorder}) rule set"*
and *initial_automaton* :: *"(('ctr_loc, 'state::finite, 'label) state, 'label) transition set"*

```
  and initial_F_ctr_loc :: "'ctr_loc set"
  and initial_F_ctr_loc_st :: "'state set"
  and final_automaton :: "(('ctr_loc, 'state, 'label) state, 'label) transition set"
  and final_F_ctr_loc :: "'ctr_loc set"
  and final_F_ctr_loc_st :: "'state set"
  defines nonempty_inter = "P_Automaton.nonempty
    (inters initial_automaton (pre_star Δ final_automaton))
    ((λp. (Init p, Init p)))
    (inters_finals (finals initial_F_ctr_loc initial_F_ctr_loc_st)
                   (finals final_F_ctr_loc final_F_ctr_loc_st))"
  ⟨proof⟩

definition "check Δ I IF IF_st F FF FF_st =
  (if pds.inits ⊆ LTS.srcs F then Some (nonempty_inter Δ I IF IF_st F FF FF_st) else None)"

lemma check_None: "check Δ I IF IF_st F FF FF_st = None ⟷ ¬ (inits ⊆ LTS.srcs F)"
  ⟨proof⟩

lemma check_Some: "check Δ I IF IF_st F FF FF_st = Some b ⟷
  (inits ⊆ LTS.srcs F ∧ b = (∃p w p' w'.
     (p, w) ∈ language IF IF_st I ∧
     (p', w') ∈ language FF FF_st F ∧
     step_starp Δ (p, w) (p', w')))"
  ⟨proof⟩

declare P_Automaton.mark.simps[code]

export-code check checking SML

end
```

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