

Ptolemy's Theorem

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Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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1 Ptolemy's Theorem

```
theory Ptolemys-Theorem
imports
  HOL-Analysis.Multivariate-Analysis
begin
```

1.1 Preliminaries

1.1.1 Additions to Rat theory

```
hide-const (open) normalize
```

1.1.2 Additions to Transcendental theory

Lemmas about *arcsin* and *arccos* commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between -1 and 1 . As the argumentation for $-1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term $\cos (\arccos y)$ is already provided in the Isabelle distribution with name *cos-arccos-abs*. Here, we further provide the analogue on *arcsin* for rewriting $\sin (\arcsin y)$.

lemma *sin-arcsin-abs*: $|y| \leq 1 \implies \sin (\arcsin y) = y$
 $\langle \text{proof} \rangle$

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

lemma *arccos-lbound-abs* [simp]:
 $|y| \leq 1 \implies 0 \leq \arccos y$
 $\langle \text{proof} \rangle$

lemma *arccos-ubound-abs* [simp]:
 $|y| \leq 1 \implies \arccos y \leq \pi$
 $\langle \text{proof} \rangle$

As we choose angles to be between 0 between $2 * \pi$, we need some lemmas to reason about the sign of $\sin x$ for angles x .

lemma *sin-ge-zero-iff*:
assumes $0 \leq x < 2 * \pi$
shows $0 \leq \sin x \longleftrightarrow x \leq \pi$
 $\langle \text{proof} \rangle$

lemma *sin-less-zero-iff*:
assumes $0 \leq x < 2 * \pi$
shows $\sin x < 0 \longleftrightarrow \pi < x$
 $\langle \text{proof} \rangle$

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [simp]:
assumes $i \neq j$
shows $\text{axis } i \ x \ \$ \ j = 0$
 $\langle \text{proof} \rangle$

lemma *norm-axis*:
fixes $x :: \text{real}$
shows $\text{norm } (\text{axis } i \ x) = \text{abs } x$
 $\langle \text{proof} \rangle$

lemma *norm-eq-on-real-2-vec*:

fixes $x :: \text{real}^2$

shows $\text{norm } x = \text{sqrt } ((x \$ 1)^2 + (x \$ 2)^2)$

$\langle \text{proof} \rangle$

lemma *dist-eq-on-real-2-vec*:

fixes $a \ b :: \text{real}^2$

shows $\text{dist } a \ b = \text{sqrt } ((a \$ 1 - b \$ 1)^2 + (a \$ 2 - b \$ 2)^2)$

$\langle \text{proof} \rangle$

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* $:: \text{real} \Rightarrow \text{real}^2$

where

$\text{of-radiant } \omega = \text{axis } 1 (\cos \omega) + \text{axis } 2 (\sin \omega)$

definition *normalize* $:: \text{real}^2 \Rightarrow \text{real}^2$

where

$\text{normalize } p = (\text{if } p = 0 \text{ then axis } 1 \ 1 \text{ else } (1 / \text{norm } p) *_{\text{R}} p)$

definition *radiant-of* $:: \text{real}^2 \Rightarrow \text{real}$

where

$\text{radiant-of } p = (\text{THE } \omega. 0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{of-radiant } \omega = \text{normalize } p)$

The vector *of-radiant* ω is the vector with length 1 and angle ω to the first axis. We normalize vectors to length 1 keeping their orientation with the *normalize* function. Conversely, *radiant-of* p is the angle of vector p to the first axis, where we choose *radiant-of* to return angles between 0 and $2 * \pi$, following the usual high-school convention. With these definitions, we can express the main result $\text{norm } p *_{\text{R}} \text{of-radiant } (\text{radiant-of } p) = p$. Note that the main result holds for any definition of *radiant-of* 0. So, we choose to define *normalize* 0 and *radiant-of* 0, such that *radiant-of* 0 = 0.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [simp]:

$\text{of-radiant } \omega \$ 1 = \cos \omega$

$\langle \text{proof} \rangle$

lemma *nth-of-radiant-2* [simp]:

$\text{of-radiant } \omega \$ 2 = \sin \omega$

$\langle \text{proof} \rangle$

lemma *norm-of-radiant*:

$\text{norm } (\text{of-radiant } \omega) = 1$

$\langle \text{proof} \rangle$

lemma *of-radiant-plus-2pi*:
 $\text{of-radiant } (\omega + 2 * \pi) = \text{of-radiant } \omega$
 $\langle \text{proof} \rangle$

lemma *of-radiant-minus-2pi*:
 $\text{of-radiant } (\omega - 2 * \pi) = \text{of-radiant } \omega$
 $\langle \text{proof} \rangle$

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:
 $\text{norm } p *_R \text{ normalize } p = p$
 $\langle \text{proof} \rangle$

lemma *norm-normalize*:
 $\text{norm } (\text{normalize } p) = 1$
 $\langle \text{proof} \rangle$

lemma *nth-normalize [simp]*:
 $|\text{normalize } p \$ i| \leq 1$
 $\langle \text{proof} \rangle$

lemma *normalize-square*:
 $(\text{normalize } p \$ 1)^2 + (\text{normalize } p \$ 2)^2 = 1$
 $\langle \text{proof} \rangle$

lemma *nth-normalize-ge-zero-iff*:
 $0 \leq \text{normalize } p \$ i \longleftrightarrow 0 \leq p \$ i$
 $\langle \text{proof} \rangle$

lemma *nth-normalize-less-zero-iff*:
 $\text{normalize } p \$ i < 0 \longleftrightarrow p \$ i < 0$
 $\langle \text{proof} \rangle$

lemma *normalize-boundary-iff*:
 $|\text{normalize } p \$ 1| = 1 \longleftrightarrow p \$ 2 = 0$
 $\langle \text{proof} \rangle$

lemma *between-normalize-if-distant-from-0*:
assumes $\text{norm } p \geq 1$
shows $\text{between } (0, p) (\text{normalize } p)$
 $\langle \text{proof} \rangle$

lemma *between-normalize-if-near-0*:
assumes $\text{norm } p \leq 1$
shows $\text{between } (0, \text{normalize } p) p$
 $\langle \text{proof} \rangle$

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:

$0 \leq \text{radiant-of } p \wedge \text{radiant-of } p < 2 * \pi \wedge \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p$
 $\langle \text{proof} \rangle$

lemma *radiant-of-bounds* [simp]:

$0 \leq \text{radiant-of } p \text{ radiant-of } p < 2 * \pi$
 $\langle \text{proof} \rangle$

lemma *radiant-of-weak-ubound* [simp]:

$\text{radiant-of } p \leq 2 * \pi$
 $\langle \text{proof} \rangle$

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:

$\text{norm } p *_R \text{of-radiant } (\text{radiant-of } p) = p$
 $\langle \text{proof} \rangle$

lemma *relative-polar-form-eq*:

$Q + \text{dist } P \ Q *_R \text{of-radiant } (\text{radiant-of } (P - Q)) = P$
 $\langle \text{proof} \rangle$

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:

assumes $0 \leq \text{radius}$ $0 \leq \alpha$ $\alpha \leq \beta$ $\beta \leq 2 * \pi$
shows $\text{dist } (\text{center} + \text{radius} *_R \text{of-radiant } \alpha) (\text{center} + \text{radius} *_R \text{of-radiant } \beta)$
 $= 2 * \text{radius} * \sin ((\beta - \alpha) / 2)$
(is ?lhs = ?rhs)
 $\langle \text{proof} \rangle$

theorem *ptolemy-trigonometric*:

fixes $\omega_1 \ \omega_2 \ \omega_3 :: \text{real}$
shows $\sin (\omega_1 + \omega_2) * \sin (\omega_2 + \omega_3) = \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin (\omega_1 + \omega_2 + \omega_3)$
 $\langle \text{proof} \rangle$

theorem *ptolemy*:

fixes $A \ B \ C \ D \ \text{center} :: \text{real} \wedge 2$
assumes $\text{dist center } A = \text{radius}$ **and** $\text{dist center } B = \text{radius}$
assumes $\text{dist center } C = \text{radius}$ **and** $\text{dist center } D = \text{radius}$
assumes *ordering-of-points*:
 $\text{radiant-of } (A - \text{center}) \leq \text{radiant-of } (B - \text{center})$
 $\text{radiant-of } (B - \text{center}) \leq \text{radiant-of } (C - \text{center})$
 $\text{radiant-of } (C - \text{center}) \leq \text{radiant-of } (D - \text{center})$
shows $\text{dist } A \ C * \text{dist } B \ D = \text{dist } A \ B * \text{dist } C \ D + \text{dist } A \ D * \text{dist } B \ C$
 $\langle \text{proof} \rangle$

end

References

- [1] J. Harrison. Ptolemy's theorem. <https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml>.
- [2] F. Wiedijk. Formalizing 100 theorems. <http://www.cs.ru.nl/~freek/100/>.
- [3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].