Pseudo-hoops

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Abstract

Pseudo-hoops are algebraic structures introduced in [1, 2] by B. Bosbach under the name of complementary semigroups. This is a formalization of the paper [4]. Following [4] we prove some properties of pseudo-hoops and we define the basic concepts of filter and normal filter. The lattice of normal filters is isomorphic with the lattice of congruences of a pseudo-hoop. We also study some important classes of pseudo-hoops. Bounded Wajsberg pseudo-hoops are equivalent to pseudo-Wajsberg algebras and bounded basic pseudo-hoops are equivalent to pseudo-BL algebras. Some examples of pseudo-hoops are given in the last section of the formalization.

Contents

1 Overview

Section 2 introduces some operations and their infix syntax. Section 3 and 4 introduces some facts about residuated and complemented monoids. Section

5 introduces the pseudo-hoops and some of their properties. Section 6 introduces filters and normal filters and proves that the lattice of normal filters and the lattice of congruences are isomorphic. Following [3], section 7 introduces pseudo-Waisberg algebras and some of their properties. In Section 8 we investigate some classes of pseudo-hoops. Finally section 9 presents some examples of pseudo-hoops and normal filters.

2 Operations

theory *Operations*
imports *Main*
begin

class *left-imp* =
  fixes *imp-l* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $l\rightarrow$ *65*)

class *right-imp* =
  fixes *imp-r* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $r\rightarrow$ *65*)

class *left-uminus* =
  fixes *uminus-l* :: $'a$ => $'a$ ($-l$ - [*81*] *80*)

class *right-uminus* =
  fixes *uminus-r* :: $'a$ => $'a$ ($-r$ - [*81*] *80*)

end

3 Left Complemented Monoid

theory *LeftComplementedMonoid*
  imports *Operations LatticeProperties.Lattice-Prop*
begin

class *right-pordered-monoid-mult* = *order* + *monoid-mult* +
  assumes *mult-right-mono*: $a \le b \Longrightarrow a * c \le b * c$

class *one-greatest* = *one* + *ord* +
  assumes *one-greatest* [*simp*]: $a \le 1$

class *integral-right-pordered-monoid-mult* = *right-pordered-monoid-mult* + *one-greatest*

class *left-residuated* = *ord* + *times* + *left-imp* +
  assumes *left-residual*: $(x * a \le b) = (x \le a\ l\rightarrow b)$

class *left-residuated-pordered-monoid* = *integral-right-pordered-monoid-mult* + *left-residuated*

class *left-inf* = *inf* + *times* + *left-imp* +
  assumes *inf-l-def*: $(a \sqcap b) = (a\ l\rightarrow b) * a$

class *left-complemented-monoid* = *left-residuated-pordered-monoid* + *left-inf* +
  assumes *right-divisibility*: $(a \leq b) = (\exists\ c\ .\ a = c * b)$
begin
lemma *lcm-D*: $a\ l\!\!\rightarrow\ a = 1$
  ⟨*proof*⟩

subclass *semilattice-inf*
    ⟨*proof*⟩

  lemma *left-one-inf* [*simp*]: $1 \sqcap a = a$
    ⟨*proof*⟩

  lemma *left-one-impl* [*simp*]: $1\ l\!\!\rightarrow\ a = a$

    ⟨*proof*⟩

  lemma *lcm-A*: $(a\ l\!\!\rightarrow\ b) * a = (b\ l\!\!\rightarrow\ a) * b$
    ⟨*proof*⟩

  lemma *lcm-B*: $((a * b)\ l\!\!\rightarrow\ c) = (a\ l\!\!\rightarrow\ (b\ l\!\!\rightarrow\ c))$
    ⟨*proof*⟩

  lemma *lcm-C*: $(a \leq b) = ((a\ l\!\!\rightarrow\ b) = 1)$

    ⟨*proof*⟩

  end

class *less-def* = *ord* +
  assumes *less-def*: $(a < b) = ((a \leq b) \land \neg\ (b \leq a))$

class *one-times* = *one* + *times* +
  assumes *one-left* [*simp*]: $1 * a = a$
  and *one-right* [*simp*]: $a * 1 = a$

class *left-complemented-monoid-algebra* = *left-imp* + *one-times* + *left-inf* + *less-def*
+
  assumes *left-impl-one* [*simp*]: $a\ l\!\!\rightarrow\ a = 1$
  and *left-impl-times*: $(a\ l\!\!\rightarrow\ b) * a = (b\ l\!\!\rightarrow\ a) * b$
  and *left-impl-ded*: $((a * b)\ l\!\!\rightarrow\ c) = (a\ l\!\!\rightarrow\ (b\ l\!\!\rightarrow\ c))$
  and *left-lesseq*: $(a \leq b) = ((a\ l\!\!\rightarrow\ b) = 1)$
begin
  lemma *A*: $(1\ l\!\!\rightarrow\ a)\ l\!\!\rightarrow\ 1 = 1$
    ⟨*proof*⟩

  subclass *order*
    ⟨*proof*⟩

lemma *B*: $(1\ l{\rightarrow}\ a) \le 1$
⟨*proof*⟩

lemma *C*: $a \le (1\ l{\rightarrow}\ a)$
⟨*proof*⟩

lemma *D*: $a \le 1$
⟨*proof*⟩

lemma *less-eq*: $(a \le b) = (\exists\ c\ .\ a = (c * b))$

⟨*proof*⟩

lemma *F*: $(a * b) * c\ l{\rightarrow}\ z = a * (b * c)\ l{\rightarrow}\ z$
⟨*proof*⟩

lemma *associativity*: $(a * b) * c = a * (b * c)$

⟨*proof*⟩

lemma *H*: $a * b \le b$
⟨*proof*⟩

lemma *I*: $a * b\ l{\rightarrow}\ b = 1$
⟨*proof*⟩

lemma *K*: $a \le b \Longrightarrow a * c \le b * c$
⟨*proof*⟩

lemma *L*: $(x * a \le b) = (x \le a\ l{\rightarrow}\ b)$
⟨*proof*⟩

subclass *left-complemented-monoid*
⟨*proof*⟩
end

lemma (**in** *left-complemented-monoid*) *left-complemented-monoid*:
class.left-complemented-monoid-algebra $(*)$ *inf* $(l{\rightarrow})$ $(\le)$ $(<)$ *1*
⟨*proof*⟩

end

4 Right Complemented Monoid

theory *RightComplementedMonoid*

imports *LeftComplementedMonoid*
begin

class *left-pordered-monoid-mult* = *order* + *monoid-mult* +
  assumes *mult-left-mono*: $a \leq b \Longrightarrow c * a \leq c * b$

class *integral-left-pordered-monoid-mult* = *left-pordered-monoid-mult* + *one-greatest*

class *right-residuated* = *ord* + *times* + *right-imp* +
  assumes *right-residual*: $(a * x \leq b) = (x \leq a\ r\!\rightarrow\ b)$

class *right-residuated-pordered-monoid* = *integral-left-pordered-monoid-mult* + *right-residuated*

class *right-inf* = *inf* + *times* + *right-imp* +
  assumes *inf-r-def*: $(a \sqcap b) = a * (a\ r\!\rightarrow\ b)$

class *right-complemented-monoid* = *right-residuated-pordered-monoid* + *right-inf*
+
  assumes *left-divisibility*: $(a \leq b) = (\exists\ c\ .\ a = b * c)$

sublocale *right-complemented-monoid* < *dual*: *left-complemented-monoid* $\lambda\ a\ b\ .$
$b * a\ (\sqcap)\ (r\!\rightarrow)\ 1\ (\leq)\ (<)$
  ⟨*proof*⟩

context *right-complemented-monoid* **begin**
lemma *rcm-D*: $a\ r\!\rightarrow\ a = 1$
  ⟨*proof*⟩

subclass *semilattice-inf*
    ⟨*proof*⟩

lemma *right-semilattice-inf*: *class.semilattice-inf inf* $(\leq)\ (<)$
  ⟨*proof*⟩

  lemma *right-one-inf* [*simp*]: $1 \sqcap a = a$
    ⟨*proof*⟩

  lemma *right-one-impl* [*simp*]: $1\ r\!\rightarrow\ a = a$
    ⟨*proof*⟩

  lemma *rcm-A*: $a * (a\ r\!\rightarrow\ b) = b * (b\ r\!\rightarrow\ a)$
    ⟨*proof*⟩

  lemma *rcm-B*: $((b * a)\ r\!\rightarrow\ c) = (a\ r\!\rightarrow\ (b\ r\!\rightarrow\ c))$
    ⟨*proof*⟩

  lemma *rcm-C*: $(a \leq b) = ((a\ r\!\rightarrow\ b) = 1)$
    ⟨*proof*⟩
  end

class *right-complemented-monoid-nole-algebra* = *right-imp* + *one-times* + *right-inf* + *less-def* +
  assumes *right-impl-one* [*simp*]: $a\ r\!\rightarrow\ a = 1$
  and *right-impl-times*: $a * (a\ r\!\rightarrow\ b) = b * (b\ r\!\rightarrow\ a)$
  and *right-impl-ded*: $((a * b)\ r\!\rightarrow\ c) = (b\ r\!\rightarrow\ (a\ r\!\rightarrow\ c))$

class *right-complemented-monoid-algebra* = *right-complemented-monoid-nole-algebra* +
  assumes *right-lesseq*: $(a \le b) = ((a\ r\!\rightarrow\ b) = 1)$
begin
end

sublocale *right-complemented-monoid-algebra* < *dual-algebra*: *left-complemented-monoid-algebra* $\lambda\ a\ b\ .\ b * a$ *inf* $(r\!\rightarrow)$ $(\le)$ $(<)$ *1*
  ⟨*proof*⟩

context *right-complemented-monoid-algebra* **begin**

subclass *right-complemented-monoid*
  ⟨*proof*⟩
end

lemma (**in** *right-complemented-monoid*) *right-complemented-monoid*: *class.right-complemented-monoid-algebr*
$(\le)$ $(<)$ *1* $(*)$ *inf* $(r\!\rightarrow)$
  ⟨*proof*⟩

end

5 Pseudo-Hoops

theory *PseudoHoops*
imports *RightComplementedMonoid*
begin

lemma *drop-assumption*:
  $p \Longrightarrow$ *True*
  ⟨*proof*⟩

class *pseudo-hoop-algebra* = *left-complemented-monoid-algebra* + *right-complemented-monoid-nole-algebra* +
  assumes *left-right-impl-times*: $(a\ l\!\rightarrow\ b) * a = a * (a\ r\!\rightarrow\ b)$
begin
  definition
    inf-rr $a\ b\ = a * (a\ r\!\rightarrow\ b)$

  definition

lesseq-r a b = (*a r*→ *b* = *1*)

definition
less-r a b = (*lesseq-r a b* ∧ ¬ *lesseq-r b a*)
end

context *pseudo-hoop-algebra* **begin**

lemma *right-complemented-monoid-algebra*: *class.right-complemented-monoid-algebra lesseq-r less-r 1* (∗) *inf-rr* (*r*→)

⟨*proof*⟩

lemma *inf-rr-inf* [*simp*]: *inf-rr* = (⊓)
⟨*proof*⟩

lemma *lesseq-lesseq-r*: *lesseq-r a b* = (*a* ≤ *b*)
⟨*proof*⟩
lemma [*simp*]: *lesseq-r* = (≤)
⟨*proof*⟩

lemma [*simp*]: *less-r* = (<)
⟨*proof*⟩

subclass *right-complemented-monoid-algebra*
⟨*proof*⟩
end

sublocale *pseudo-hoop-algebra* <
pseudo-hoop-dual: *pseudo-hoop-algebra* λ *a b* . *b* ∗ *a* (⊓) (*r*→) (≤) (<) *1* (*l*→)
⟨*proof*⟩

context *pseudo-hoop-algebra* **begin**
lemma *commutative-ps*: (∀ *a b* . *a* ∗ *b* = *b* ∗ *a*) = ((*l*→) = (*r*→))
⟨*proof*⟩

lemma *lemma-2-4-5*: *a l*→ *b* ≤ (*c l*→ *a*) *l*→ (*c l*→ *b*)
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-4-6*: *a r*→ *b* ≤ (*c r*→ *a*) *r*→ (*c r*→ *b*)
⟨*proof*⟩

primrec
imp-power-l:: $'a$ => $nat \Rightarrow 'a \Rightarrow 'a$ ((-) $l-$(-)$\rightarrow$ (-) [65,0,65] 65) **where**
$a\ l-0\rightarrow\ b = b$ |
$a\ l-(Suc\ n)\rightarrow\ b = (a\ l\rightarrow\ (a\ l-n\rightarrow\ b))$

primrec
imp-power-r:: $'a$ => $nat \Rightarrow 'a \Rightarrow 'a$ ((-) $r-$(-)$\rightarrow$ (-) [65,0,65] 65) **where**
$a\ r-0\rightarrow\ b = b$ |
$a\ r-(Suc\ n)\rightarrow\ b = (a\ r\rightarrow\ (a\ r-n\rightarrow\ b))$

lemma *lemma-2-4-7-a*: $a\ l-n\rightarrow\ b = a\ \hat{}\ n\ l\rightarrow\ b$
⟨*proof*⟩

lemma *lemma-2-4-7-b*: $a\ r-n\rightarrow\ b = a\ \hat{}\ n\ r\rightarrow\ b$
⟨*proof*⟩

lemma *lemma-2-5-8-a* [*simp*]: $a * b \leq a$
⟨*proof*⟩

lemma *lemma-2-5-8-b* [*simp*]: $a * b \leq b$
⟨*proof*⟩

lemma *lemma-2-5-9-a*: $a \leq b\ l\rightarrow\ a$
⟨*proof*⟩

lemma *lemma-2-5-9-b*: $a \leq b\ r\rightarrow\ a$
⟨*proof*⟩

lemma *lemma-2-5-11*: $a * b \leq a \sqcap b$
⟨*proof*⟩

lemma *lemma-2-5-12-a*: $a \leq b \Longrightarrow\ c\ l\rightarrow\ a \leq c\ l\rightarrow\ b$
⟨*proof*⟩

lemma *lemma-2-5-13-a*: $a \leq b \Longrightarrow\ b\ l\rightarrow\ c \leq a\ l\rightarrow\ c$
⟨*proof*⟩

lemma *lemma-2-5-14*: $(b\ l\rightarrow\ c) * (a\ l\rightarrow\ b) \leq a\ l\rightarrow\ c$
⟨*proof*⟩

lemma *lemma-2-5-16*: $(a\ l\rightarrow\ b) \leq\ (b\ l\rightarrow\ c)\ r\rightarrow\ (a\ l\rightarrow\ c)$
⟨*proof*⟩

lemma *lemma-2-5-18*: $(a\ l\rightarrow\ b) \leq\ a * c\ l\rightarrow\ b * c$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-5-12-b*: $a \leq b \Longrightarrow\ c\ r{\rightarrow}\ a \leq c\ r{\rightarrow}\ b$
⟨*proof*⟩

lemma *lemma-2-5-13-b*: $a \leq b \Longrightarrow\ b\ r{\rightarrow}\ c \leq a\ r{\rightarrow}\ c$
⟨*proof*⟩

lemma *lemma-2-5-15*: $(a\ r{\rightarrow}\ b) * (b\ r{\rightarrow}\ c) \leq a\ r{\rightarrow}\ c$
⟨*proof*⟩

lemma *lemma-2-5-17*: $(a\ r{\rightarrow}\ b) \leq\ (b\ r{\rightarrow}\ c)\ l{\rightarrow}\ (a\ r{\rightarrow}\ c)$
⟨*proof*⟩

lemma *lemma-2-5-19*: $(a\ r{\rightarrow}\ b) \leq\ c * a\ r{\rightarrow}\ c * b$
⟨*proof*⟩

definition
$\mathit{lower\text{-}bound}\ A = \{a\ .\ \forall\ x \in A\ .\ a \leq x\}$

definition
$\mathit{infimum}\ A = \{a \in \mathit{lower\text{-}bound}\ A\ .\ (\forall\ x \in \mathit{lower\text{-}bound}\ A\ .\ x \leq a)\}$

lemma *infimum-unique*: $(\mathit{infimum}\ A = \{x\}) = (x \in \mathit{infimum}\ A)$
⟨*proof*⟩

lemma *lemma-2-6-20*:
$a \in \mathit{infimum}\ A \Longrightarrow b\ l{\rightarrow}\ a \in \mathit{infimum}\ (((l{\rightarrow})\ b)\text{`}A)$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-6-21*:
$a \in \mathit{infimum}\ A \Longrightarrow b\ r{\rightarrow}\ a \in \mathit{infimum}\ (((r{\rightarrow})\ b)\text{`}A)$
⟨*proof*⟩

lemma *infimum-pair*: $a \in \mathit{infimum}\ \{x\ .\ x = b \lor x = c\} = (a = b \sqcap c)$
⟨*proof*⟩

lemma *lemma-2-6-20-a*:
$a\ l{\rightarrow}\ (b \sqcap c) = (a\ l{\rightarrow}\ b) \sqcap (a\ l{\rightarrow}\ c)$
⟨*proof*⟩
end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-6-21-a*:
$a\ r\!\!\rightarrow (b \sqcap c) = (a\ r\!\!\rightarrow b) \sqcap (a\ r\!\!\rightarrow c)$
⟨*proof*⟩

lemma *mult-mono*: $a \leq b \Longrightarrow c \leq d \Longrightarrow a * c \leq b * d$
⟨*proof*⟩

lemma *lemma-2-7-22*: $(a\ l\!\!\rightarrow b) * (c\ l\!\!\rightarrow d) \leq (a \sqcap c)\ l\!\!\rightarrow (b \sqcap d)$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**
lemma *lemma-2-7-23*: $(a\ r\!\!\rightarrow b) * (c\ r\!\!\rightarrow d) \leq (a \sqcap c)\ r\!\!\rightarrow (b \sqcap d)$
⟨*proof*⟩

definition
upper-bound $A = \{a\ .\ \forall\ x \in A\ .\ x \leq a\}$

definition
supremum $A = \{a \in$ *upper-bound* $A\ .\ (\forall\ x \in$ *upper-bound* $A\ .\ a \leq x)\}$

lemma *supremum-unique*:
$a \in$ *supremum* $A \Longrightarrow b \in$ *supremum* $A \Longrightarrow a = b$
⟨*proof*⟩

lemma *lemma-2-8-i*:
$a \in$ *supremum* $A \Longrightarrow a\ l\!\!\rightarrow b \in$ *infimum* $((\lambda\ x\ .\ x\ l\!\!\rightarrow b)`A)$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-8-i1*:
$a \in$ *supremum* $A \Longrightarrow a\ r\!\!\rightarrow b \in$ *infimum* $((\lambda\ x\ .\ x\ r\!\!\rightarrow b)`A)$
⟨*proof*⟩

definition
times-set :: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set$ (**infixl** $**$ *70*) **where**
$(A ** B) = \{a\ .\ \exists\ x \in A\ .\ \exists\ y \in B\ .\ a = x * y\}$

lemma *times-set-assoc*: $A ** (B ** C) = (A ** B) ** C$
⟨*proof*⟩

primrec *power-set* :: $'a\ set \Rightarrow nat \Rightarrow 'a\ set$ (**infixr** $*\hat{}$ *80*) **where**
power-set-0: $(A *\hat{}\ 0) = \{1\}$

| *power-set-Suc*: $(A *\hat{}\ (Suc\ n)) = (A ** (A *\hat{}\ n))$

lemma *infimum-singleton* [*simp*]: *infimum* $\{a\} = \{a\}$
⟨*proof*⟩

lemma *lemma-2-8-ii*:
$a \in supremum\ A \Longrightarrow (a\ \hat{}\ n)\ l{\rightarrow}\ b \in infimum\ ((\lambda\ x\ .\ x\ l{\rightarrow}\ b)`(A *\hat{}\ n))$
⟨*proof*⟩

lemma *power-set-Suc2*:
$A *\hat{}\ (Suc\ n) = A *\hat{}\ n ** A$
⟨*proof*⟩

lemma *power-set-add*:
$A *\hat{}\ (n + m) = (A *\hat{}\ n) ** (A *\hat{}\ m)$
⟨*proof*⟩
end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-8-ii1*:
$a \in supremum\ A \Longrightarrow (a\ \hat{}\ n)\ r{\rightarrow}\ b \in infimum\ ((\lambda\ x\ .\ x\ r{\rightarrow}\ b)`(A *\hat{}\ n))$
⟨*proof*⟩

lemma *lemma-2-9-i*:
$b \in supremum\ A \Longrightarrow a * b \in supremum\ ((*)\ a\ `\ A)$
⟨*proof*⟩

lemma *lemma-2-9-i1*:
$b \in supremum\ A \Longrightarrow b * a \in supremum\ ((\lambda\ x\ .\ x * a)\ `\ A)$
⟨*proof*⟩

lemma *lemma-2-9-ii*:
$b \in supremum\ A \Longrightarrow a \sqcap b \in supremum\ ((\sqcap)\ a\ `\ A)$
⟨*proof*⟩

lemma *lemma-2-10-24*:
$a \leq (a\ l{\rightarrow}\ b)\ r{\rightarrow}\ b$
⟨*proof*⟩

lemma *lemma-2-10-25*:
$a \leq (a\ l{\rightarrow}\ b)\ r{\rightarrow}\ a$
⟨*proof*⟩
end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-26*:
  $a \leq (a\ r\rightarrow\ b)\ l\rightarrow\ b$
  ⟨*proof*⟩

lemma *lemma-2-10-27*:
  $a \leq (a\ r\rightarrow\ b)\ l\rightarrow\ a$
  ⟨*proof*⟩

lemma *lemma-2-10-28*:
  $b\ l\rightarrow\ ((a\ l\rightarrow\ b)\ r\rightarrow\ a) = b\ l\rightarrow\ a$
  ⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-29*:
  $b\ r\rightarrow\ ((a\ r\rightarrow\ b)\ l\rightarrow\ a) = b\ r\rightarrow\ a$
  ⟨*proof*⟩

lemma *lemma-2-10-30*:
  $((b\ l\rightarrow\ a)\ r\rightarrow\ a)\ l\rightarrow\ a = b\ l\rightarrow\ a$
  ⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-31*:
  $((b\ r\rightarrow\ a)\ l\rightarrow\ a)\ r\rightarrow\ a = b\ r\rightarrow\ a$
  ⟨*proof*⟩

lemma *lemma-2-10-32*:
  $(((b\ l\rightarrow\ a)\ r\rightarrow\ a)\ l\rightarrow\ b)\ l\rightarrow\ (b\ l\rightarrow\ a) = b\ l\rightarrow\ a$
  ⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-33*:
  $(((b\ r\rightarrow\ a)\ l\rightarrow\ a)\ r\rightarrow\ b)\ r\rightarrow\ (b\ r\rightarrow\ a) = b\ r\rightarrow\ a$
  ⟨*proof*⟩
end

class *pseudo-hoop-sup-algebra* = *pseudo-hoop-algebra* + *sup* +
  assumes

sup-comute: $a \sqcup b = b \sqcup a$
and *sup-le* [*simp*]: $a \leq a \sqcup b$
and *le-sup-equiv*: $(a \leq b) = (a \sqcup b = b)$
begin
lemma *sup-le-2* [*simp*]:
$b \leq a \sqcup b$
⟨*proof*⟩

lemma *le-sup-equiv-r*:
$(a \sqcup b = b) = (a \leq b)$
⟨*proof*⟩

lemma *sup-idemp* [*simp*]:
$a \sqcup a = a$
⟨*proof*⟩
end

class *pseudo-hoop-sup1-algebra* = *pseudo-hoop-algebra* + *sup* +
assumes
sup-def: $a \sqcup b = ((a\ l{\rightarrow}\ b)\ r{\rightarrow}\ b) \sqcap ((b\ l{\rightarrow}\ a)\ r{\rightarrow}\ a)$
begin

lemma *sup-comute1*: $a \sqcup b = b \sqcup a$
⟨*proof*⟩

lemma *sup-le1* [*simp*]: $a \leq a \sqcup b$
⟨*proof*⟩

lemma *le-sup-equiv1*: $(a \leq b) = (a \sqcup b = b)$
⟨*proof*⟩

subclass *pseudo-hoop-sup-algebra*
⟨*proof*⟩
end

class *pseudo-hoop-sup2-algebra* = *pseudo-hoop-algebra* + *sup* +
assumes
sup-2-def: $a \sqcup b = ((a\ r{\rightarrow}\ b)\ l{\rightarrow}\ b) \sqcap ((b\ r{\rightarrow}\ a)\ l{\rightarrow}\ a)$

context *pseudo-hoop-sup1-algebra* **begin end**

sublocale *pseudo-hoop-sup2-algebra* < *sup1-dual*: *pseudo-hoop-sup1-algebra* $(\sqcup)$ λ $a\ b\ .\ b * a$ $(\sqcap)$ $(r{\rightarrow})$ $(\leq)$ $(<)$ 1 $(l{\rightarrow})$
⟨*proof*⟩

context *pseudo-hoop-sup2-algebra* **begin**

lemma *sup-comute-2*: $a \sqcup b = b \sqcup a$

⟨*proof*⟩

lemma *sup-le2* [*simp*]: $a \leq a \sqcup b$
⟨*proof*⟩

lemma *le-sup-equiv2*: $(a \leq b) = (a \sqcup b = b)$
⟨*proof*⟩

subclass *pseudo-hoop-sup-algebra*
⟨*proof*⟩

end

class *pseudo-hoop-lattice-a* = *pseudo-hoop-sup-algebra* +
assumes *sup-inf-le-distr*: $a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap (a \sqcup c)$
begin
lemma *sup-lower-upper-bound* [*simp*]:
$a \leq c \Longrightarrow b \leq c \Longrightarrow a \sqcup b \leq c$
⟨*proof*⟩
end

sublocale *pseudo-hoop-lattice-a* < *lattice* $(\sqcap)$ $(\leq)$ $(<)$ $(\sqcup)$
⟨*proof*⟩

class *pseudo-hoop-lattice-b* = *pseudo-hoop-sup-algebra* +
assumes *le-sup-cong*: $a \leq b \Longrightarrow a \sqcup c \leq b \sqcup c$

begin
lemma *sup-lower-upper-bound-b* [*simp*]:
$a \leq c \Longrightarrow b \leq c \Longrightarrow a \sqcup b \leq c$
⟨*proof*⟩

lemma *sup-inf-le-distr-b*:
$a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap (a \sqcup c)$
⟨*proof*⟩
end

context *pseudo-hoop-lattice-a* **begin end**

sublocale *pseudo-hoop-lattice-b* < *pseudo-hoop-lattice-a* $(\sqcup)$ $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(\leq)$ $(<)$ *1* $(r\rightarrow)$
⟨*proof*⟩

class *pseudo-hoop-lattice* = *pseudo-hoop-sup-algebra* +
assumes *sup-assoc-1*: $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$
begin
lemma *le-sup-cong-c*:
$a \leq b \Longrightarrow a \sqcup c \leq b \sqcup c$
⟨*proof*⟩

end

sublocale *pseudo-hoop-lattice* < *pseudo-hoop-lattice-b* (⊔) (∗) (⊓) (*l*→) (≤) (<) *1* (*r*→)
⟨*proof*⟩

sublocale *pseudo-hoop-lattice* < *semilattice-sup* (⊔) (≤) (<)
⟨*proof*⟩

sublocale *pseudo-hoop-lattice* < *lattice* (⊓) (≤) (<) (⊔)
⟨*proof*⟩

lemma (**in** *pseudo-hoop-lattice-a*) *supremum-pair* [*simp*]:
supremum {*a*, *b*} = {*a* ⊔ *b*}
⟨*proof*⟩

sublocale *pseudo-hoop-lattice* < *distrib-lattice* (⊓) (≤) (<) (⊔)
⟨*proof*⟩

class *bounded-semilattice-inf-top* = *semilattice-inf* + *order-top*
begin
lemma *inf-eq-top-iff* [*simp*]:
(*inf x y* = *top*) = (*x* = *top* ∧ *y* = *top*)
⟨*proof*⟩
end

sublocale *pseudo-hoop-algebra* < *bounded-semilattice-inf-top* (⊓) (≤) (<) *1*
⟨*proof*⟩

definition (**in** *pseudo-hoop-algebra*)
sup1::′*a* ⇒ ′*a* ⇒ ′*a* (**infixl** ⊔*1* *70*) **where**
a ⊔*1* *b* = ((*a* *l*→ *b*) *r*→ *b*) ⊓ ((*b* *l*→ *a*) *r*→ *a*)

sublocale *pseudo-hoop-algebra* < *sup1*: *pseudo-hoop-sup1-algebra* (⊔*1*) (∗) (⊓) (*l*→) (≤) (<) *1* (*r*→)
⟨*proof*⟩

definition (**in** *pseudo-hoop-algebra*)
sup2::′*a* ⇒ ′*a* ⇒ ′*a* (**infixl** ⊔*2* *70*) **where**
a ⊔*2* *b* = ((*a* *r*→ *b*) *l*→ *b*) ⊓ ((*b* *r*→ *a*) *l*→ *a*)

sublocale *pseudo-hoop-algebra* < *sup2*: *pseudo-hoop-sup2-algebra* (⊔*2*) (∗) (⊓) (*l*→) (≤) (<) *1* (*r*→)
⟨*proof*⟩

context *pseudo-hoop-algebra*
begin
lemma *lemma-2-15-i*:
$1 \in supremum\ \{a,\ b\} \Longrightarrow a * b = a \sqcap b$
⟨*proof*⟩

lemma *lemma-2-15-ii*:
$1 \in supremum\ \{a,\ b\} \Longrightarrow a \le c \Longrightarrow b \le d \Longrightarrow 1 \in supremum\ \{c,\ d\}$
⟨*proof*⟩

lemma *sup-union*:
$a \in supremum\ A \Longrightarrow b \in supremum\ B \Longrightarrow supremum\ \{a,\ b\} = supremum\ (A \cup B)$
⟨*proof*⟩

lemma *sup-singleton* [*simp*]: $a \in supremum\ \{a\}$
⟨*proof*⟩

lemma *sup-union-singleton*: $a \in supremum\ X \Longrightarrow supremum\ \{a,\ b\} = supremum\ (X \cup \{b\})$
⟨*proof*⟩

lemma *sup-le-union* [*simp*]: $a \le b \Longrightarrow supremum\ (A \cup \{a,\ b\}) = supremum\ (A \cup \{b\})$
⟨*proof*⟩

lemma *sup-sup-union*: $a \in supremum\ A \Longrightarrow b \in supremum\ (B \cup \{a\}) \Longrightarrow b \in supremum\ (A \cup B)$
⟨*proof*⟩

end

lemma [*simp*]:
$n \le 2\ \hat{}\ n$
⟨*proof*⟩

context *pseudo-hoop-algebra*
begin

lemma *sup-le-union-2*:
$a \le b \Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow supremum\ A = supremum\ ((A - \{a\}) \cup \{b\})$
⟨*proof*⟩

lemma *lemma-2-15-iii-0*:
$1 \in supremum\ \{a,\ b\} \Longrightarrow 1 \in supremum\ \{a\ \hat{}\ 2,\ b\ \hat{}\ 2\}$
⟨*proof*⟩

lemma [*simp*]: $m \leq n \Longrightarrow a\ \hat{}\ n \leq a\ \hat{}\ m$
⟨*proof*⟩

lemma [*simp*]: $a\ \hat{}\ (2\ \hat{}\ n) \leq a\ \hat{}\ n$
⟨*proof*⟩

lemma *lemma-2-15-iii-1*: $1 \in supremum\ \{a,\ b\} \Longrightarrow 1 \in supremum\ \{a\ \hat{}\ (2\ \hat{}\ n),\ b\ \hat{}\ (2\ \hat{}\ n)\}$
⟨*proof*⟩

lemma *lemma-2-15-iii*:
$1 \in supremum\ \{a,\ b\} \Longrightarrow 1 \in supremum\ \{a\ \hat{}\ n,\ b\ \hat{}\ n\}$
⟨*proof*⟩
end

end

6 Filters and Congruences

theory *PseudoHoopFilters*
imports *PseudoHoops*
begin

context *pseudo-hoop-algebra*
begin
definition
filters $= \{F\ .\ F \neq \{\} \wedge (\forall\ a\ b\ .\ a \in F \wedge b \in F \longrightarrow a * b \in F) \wedge (\forall\ a\ b\ .\ a \in F \wedge a \leq b \longrightarrow b \in F)\}$

definition
properfilters $= \{F\ .\ F \in filters \wedge F \neq UNIV\}$

definition
maximalfilters $= \{F\ .\ F \in filters \wedge (\forall\ A\ .\ A \in filters \wedge F \subseteq A \longrightarrow A = F \vee A = UNIV)\}$

definition
ultrafilters $= properfilters \cap maximalfilters$

lemma *filter-i*: $F \in filters \Longrightarrow a \in F \Longrightarrow b \in F \Longrightarrow a * b \in F$
⟨*proof*⟩

lemma *filter-ii*: $F \in filters \Longrightarrow a \in F \Longrightarrow a \leq b \Longrightarrow b \in F$

⟨*proof*⟩

lemma *filter-iii* [*simp*]: $F \in \mathit{filters} \Longrightarrow \mathit{1} \in F$
⟨*proof*⟩

lemma *filter-left-impl*:
$(F \in \mathit{filters}) = ((\mathit{1} \in F) \wedge (\forall\ a\ b\ .\ a \in F \wedge a\ l\!\!\rightarrow b \in F \longrightarrow b \in F))$
⟨*proof*⟩

lemma *filter-right-impl*:
$(F \in \mathit{filters}) = ((\mathit{1} \in F) \wedge (\forall\ a\ b\ .\ a \in F \wedge a\ r\!\!\rightarrow b \in F \longrightarrow b \in F))$
⟨*proof*⟩

lemma [*simp*]: $A \subseteq \mathit{filters} \Longrightarrow \bigcap\ A \in \mathit{filters}$
⟨*proof*⟩

definition
$\mathit{filterof}\ X = \bigcap\ \{F\ .\ F \in \mathit{filters} \wedge X \subseteq F\}$

lemma [*simp*]: $\mathit{filterof}\ X \in \mathit{filters}$
⟨*proof*⟩

lemma *times-le-mono* [*simp*]: $x \leq y \Longrightarrow u \leq v \Longrightarrow x * u \leq y * v$
⟨*proof*⟩

lemma *prop-3-2-i*:
$\mathit{filterof}\ X = \{a\ .\ \exists\ n\ x\ .\ x \in X *\hat{}\ n \wedge x \leq a\}$
⟨*proof*⟩

lemma *ultrafilter-union*:
$\mathit{ultrafilters} = \{F\ .\ F \in \mathit{filters} \wedge F \neq \mathit{UNIV} \wedge (\forall\ x\ .\ x \notin F \longrightarrow \mathit{filterof}\ (F \cup \{x\}) = \mathit{UNIV})\}$
⟨*proof*⟩

lemma *filterof-sub*: $F \in \mathit{filters} \Longrightarrow X \subseteq F \Longrightarrow \mathit{filterof}\ X \subseteq F$
⟨*proof*⟩

lemma *filterof-elem* [*simp*]: $x \in X \Longrightarrow x \in\ \mathit{filterof}\ X$
⟨*proof*⟩

lemma [*simp*]: $\mathit{filterof}\ X \in \mathit{filters}$
⟨*proof*⟩

lemma *singleton-power* [*simp*]: $\{a\} *\hat{}\ n = \{b\ .\ b = a\ \hat{}\ n\}$
⟨*proof*⟩

lemma *power-pair*: $x \in \{a,\ b\} *\hat{}\ n \Longrightarrow \exists\ i\ j\ .\ i + j = n \wedge x \leq a\ \hat{}\ i \wedge x \leq b\ \hat{}$

j
⟨*proof*⟩

lemma *filterof-supremum*:
$c \in supremum\ \{a,\ b\} \Longrightarrow filterof\ \{c\} = filterof\ \{a\} \cap filterof\ \{b\}$
⟨*proof*⟩

definition $d1\ a\ b = (a\ l{\rightarrow}\ b) * (b\ l{\rightarrow}\ a)$
definition $d2\ a\ b = (a\ r{\rightarrow}\ b) * (b\ r{\rightarrow}\ a)$

definition $d3\ a\ b = d1\ b\ a$
definition $d4\ a\ b = d2\ b\ a$

lemma [*simp*]: $(a * b = 1) = (a = 1 \wedge b = 1)$
⟨*proof*⟩

lemma *lemma-3-5-i-1*: $(d1\ a\ b = 1) = (a = b)$
⟨*proof*⟩

lemma *lemma-3-5-i-2*: $(d2\ a\ b = 1) = (a = b)$
⟨*proof*⟩

lemma *lemma-3-5-i-3*: $(d3\ a\ b = 1) = (a = b)$
⟨*proof*⟩

lemma *lemma-3-5-i-4*: $(d4\ a\ b = 1) = (a = b)$
⟨*proof*⟩

lemma *lemma-3-5-ii-1* [*simp*]: $d1\ a\ a = 1$
⟨*proof*⟩

lemma *lemma-3-5-ii-2* [*simp*]: $d2\ a\ a = 1$
⟨*proof*⟩

lemma *lemma-3-5-ii-3* [*simp*]: $d3\ a\ a = 1$
⟨*proof*⟩

lemma *lemma-3-5-ii-4* [*simp*]: $d4\ a\ a = 1$
⟨*proof*⟩

lemma [*simp*]: $(a\ l{\rightarrow}\ 1) = 1$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma [*simp*]: $(a\ r\rightarrow\ 1) = 1$
⟨*proof*⟩

lemma *lemma-3-5-iii-1* [*simp*]: $d1\ a\ 1 = a$
⟨*proof*⟩

lemma *lemma-3-5-iii-2* [*simp*]: $d2\ a\ 1 = a$
⟨*proof*⟩

lemma *lemma-3-5-iii-3* [*simp*]: $d3\ a\ 1 = a$
⟨*proof*⟩

lemma *lemma-3-5-iii-4* [*simp*]: $d4\ a\ 1 = a$
⟨*proof*⟩

lemma *lemma-3-5-iv-1*: $(d1\ b\ c) * (d1\ a\ b) * (d1\ b\ c) \leq d1\ a\ c$
⟨*proof*⟩

lemma *lemma-3-5-iv-2*: $(d2\ a\ b) * (d2\ b\ c) * (d2\ a\ b) \leq d2\ a\ c$
⟨*proof*⟩

lemma *lemma-3-5-iv-3*: $(d3\ a\ b) * (d3\ b\ c) * (d3\ a\ b) \leq d3\ a\ c$
⟨*proof*⟩

lemma *lemma-3-5-iv-4*: $(d4\ b\ c) * (d4\ a\ b) * (d4\ b\ c) \leq d4\ a\ c$
⟨*proof*⟩

definition
$cong\text{-}r\ F\ a\ b \equiv d1\ a\ b \in F$

definition
$cong\text{-}l\ F\ a\ b \equiv d2\ a\ b \in F$

lemma *cong-r-filter*: $F \in filters \Longrightarrow (cong\text{-}r\ F\ a\ b) = (a\ l\rightarrow\ b \in F \wedge b\ l\rightarrow\ a \in F)$
⟨*proof*⟩

lemma *cong-r-symmetric*: $F \in filters \Longrightarrow (cong\text{-}r\ F\ a\ b) = (cong\text{-}r\ F\ b\ a)$
⟨*proof*⟩

lemma *cong-r-d3*: $F \in filters \Longrightarrow (cong\text{-}r\ F\ a\ b) = (d3\ a\ b \in F)$
⟨*proof*⟩

lemma *cong-l-filter*: $F \in filters \Longrightarrow (cong\text{-}l\ F\ a\ b) = (a\ r\rightarrow\ b \in F \wedge b\ r\rightarrow\ a \in F)$
⟨*proof*⟩

lemma *cong-l-symmetric*: $F \in filters \Longrightarrow (cong\text{-}l\ F\ a\ b) = (cong\text{-}l\ F\ b\ a)$
⟨*proof*⟩

lemma *cong-l-d4*: $F \in filters \Longrightarrow (cong\text{-}l\ F\ a\ b) = (d4\ a\ b \in F)$
⟨*proof*⟩

lemma *cong-r-reflexive*: $F \in filters \Longrightarrow cong\text{-}r\ F\ a\ a$
⟨*proof*⟩

lemma *cong-r-transitive*: $F \in filters \Longrightarrow cong\text{-}r\ F\ a\ b \Longrightarrow cong\text{-}r\ F\ b\ c \Longrightarrow cong\text{-}r\ F\ a\ c$
⟨*proof*⟩

lemma *cong-l-reflexive*: $F \in filters \Longrightarrow cong\text{-}l\ F\ a\ a$
⟨*proof*⟩

lemma *cong-l-transitive*: $F \in filters \Longrightarrow cong\text{-}l\ F\ a\ b \Longrightarrow cong\text{-}l\ F\ b\ c \Longrightarrow cong\text{-}l\ F\ a\ c$
⟨*proof*⟩

lemma *lemma-3-7-i*: $F \in filters \Longrightarrow F = \{a\ .\ cong\text{-}r\ F\ a\ 1\}$
⟨*proof*⟩

lemma *lemma-3-7-ii*: $F \in filters \Longrightarrow F = \{a\ .\ cong\text{-}l\ F\ a\ 1\}$
⟨*proof*⟩

lemma *lemma-3-8-i*: $F \in filters \Longrightarrow (cong\text{-}r\ F\ a\ b) = (\exists\ x\ y\ .\ x \in F \wedge y \in F \wedge x * a = y * b)$
⟨*proof*⟩

lemma *lemma-3-8-ii*: $F \in filters \Longrightarrow (cong\text{-}l\ F\ a\ b) = (\exists\ x\ y\ .\ x \in F \wedge y \in F \wedge a * x = b * y)$
⟨*proof*⟩

lemma *lemma-3-9-i*: $F \in filters \Longrightarrow cong\text{-}r\ F\ a\ b \Longrightarrow cong\text{-}r\ F\ c\ d \Longrightarrow (a\ l{\rightarrow}\ c \in F) = (b\ l{\rightarrow}\ d \in F)$
⟨*proof*⟩

lemma *lemma-3-9-ii*: $F \in filters \Longrightarrow cong\text{-}l\ F\ a\ b \Longrightarrow cong\text{-}l\ F\ c\ d \Longrightarrow (a\ r{\rightarrow}\ c \in F) = (b\ r{\rightarrow}\ d \in F)$
⟨*proof*⟩

definition
$normalfilters = \{F\ .\ F \in filters \wedge (\forall\ a\ b\ .\ (a\ l{\rightarrow}\ b \in F) = (a\ r{\rightarrow}\ b \in F))\}$

lemma *normalfilter-i*:

$H \in normalfilters \Longrightarrow a\ l\rightarrow\ b \in H \Longrightarrow a\ r\rightarrow\ b \in H$
⟨*proof*⟩

lemma *normalfilter-ii*:
$H \in normalfilters \Longrightarrow a\ r\rightarrow\ b \in H \Longrightarrow a\ l\rightarrow\ b \in H$
⟨*proof*⟩

lemma [*simp*]: $H \in normalfilters \Longrightarrow H \in filters$
⟨*proof*⟩

lemma *lemma-3-10-i-ii*:
$H \in normalfilters \Longrightarrow \{a\} ** H = H ** \{a\}$
⟨*proof*⟩

lemma *lemma-3-10-ii-iii*:
$H \in filters \Longrightarrow (\forall\ a\ .\ \{a\} ** H = H ** \{a\}) \Longrightarrow cong\text{-}r\ H = cong\text{-}l\ H$
⟨*proof*⟩

lemma *lemma-3-10-i-iii*:
$H \in normalfilters \Longrightarrow cong\text{-}r\ H = cong\text{-}l\ H$
⟨*proof*⟩

lemma *order-impl-l* [*simp*]: $a \leq b \Longrightarrow a\ l\rightarrow\ b = 1$
⟨*proof*⟩

end

context *pseudo-hoop-algebra* **begin**

lemma *impl-l-d1*: $(a\ l\rightarrow\ b) = d1\ a\ (a \sqcap b)$
⟨*proof*⟩

lemma *impl-r-d2*: $(a\ r\rightarrow\ b) = d2\ a\ (a \sqcap b)$
⟨*proof*⟩

lemma *lemma-3-10-iii-i*:
$H \in filters \Longrightarrow cong\text{-}r\ H = cong\text{-}l\ H \Longrightarrow H \in normalfilters$
⟨*proof*⟩

lemma *lemma-3-10-ii-i*:
$H \in filters \Longrightarrow (\forall\ a\ .\ \{a\} ** H = H ** \{a\}) \Longrightarrow H \in normalfilters$
⟨*proof*⟩

definition
$allpowers\ x\ n = \{y\ .\ \exists\ i.\ i < n \wedge y = x\ \hat{}\ i\}$

lemma *times-set-in*: $a \in A \Longrightarrow b \in B \Longrightarrow c = a * b \Longrightarrow c \in A ** B$
⟨*proof*⟩

lemma *power-set-power*: $a \in A \Longrightarrow a\ \hat{}\ n \in A *\hat{}\ n$
⟨*proof*⟩

lemma *normal-filter-union*: $H \in normalfilters \Longrightarrow (H \cup \{x\}) *\hat{}\ n = (H ** (allpowers\ x\ n)) \cup \{x\ \hat{}\ n\}$
⟨*proof*⟩

lemma *lemma-3-11-i*: $H \in normalfilters \Longrightarrow filterof\ (H \cup \{x\}) = \{a\ .\ \exists\ n\ h\ .\ h \in H \wedge h * x\ \hat{}\ n \leq a\}$
⟨*proof*⟩

lemma *lemma-3-11-ii*: $H \in normalfilters \Longrightarrow filterof\ (H \cup \{x\}) = \{a\ .\ \exists\ n\ h\ .\ h \in H \wedge (x\ \hat{}\ n) * h \leq a\}$
⟨*proof*⟩

lemma *lemma-3-12-i-ii*:
$H \in normalfilters \Longrightarrow H \in ultrafilters \Longrightarrow x \notin H \Longrightarrow (\exists\ n\ .\ x\ \hat{}\ n\ l{\rightarrow}\ a \in H)$
⟨*proof*⟩

lemma *lemma-3-12-ii-i*:
$H \in normalfilters \Longrightarrow H \in properfilters \Longrightarrow (\forall\ x\ a\ .\ x \notin H \longrightarrow (\exists\ n\ .\ x\ \hat{}\ n\ l{\rightarrow}\ a \in H)) \Longrightarrow H \in maximalfilters$
⟨*proof*⟩

lemma *lemma-3-12-i-iii*:
$H \in normalfilters \Longrightarrow H \in ultrafilters \Longrightarrow x \notin H \Longrightarrow (\exists\ n\ .\ x\ \hat{}\ n\ r{\rightarrow}\ a \in H)$
⟨*proof*⟩

lemma *lemma-3-12-iii-i*:
$H \in normalfilters \Longrightarrow H \in properfilters \Longrightarrow (\forall\ x\ a\ .\ x \notin H \longrightarrow (\exists\ n\ .\ x\ \hat{}\ n\ r{\rightarrow}\ a \in H)) \Longrightarrow H \in maximalfilters$
⟨*proof*⟩

definition
$cong\ H = (\lambda\ x\ y\ .\ cong\text{-}l\ H\ x\ y \wedge cong\text{-}r\ H\ x\ y)$

definition
$congruences = \{R\ .\ equivp\ R \wedge (\forall\ a\ b\ c\ d\ .\ R\ a\ b \wedge R\ c\ d \longrightarrow R\ (a * c)\ (b * d) \wedge\ R\ (a\ l{\rightarrow}\ c)\ (b\ l{\rightarrow}\ d) \wedge\ R\ (a\ r{\rightarrow}\ c)\ (b\ r{\rightarrow}\ d))\}$

lemma *cong-l*: $H \in normalfilters \Longrightarrow cong\ H = cong\text{-}l\ H$
⟨*proof*⟩

lemma *cong-r*: $H \in normalfilters \Longrightarrow cong\ H = cong\text{-}r\ H$
⟨*proof*⟩

lemma *cong-equiv*: $H \in normalfilters \Longrightarrow equivp\ (cong\ H)$
⟨*proof*⟩

lemma *cong-trans*: $H \in normalfilters \Longrightarrow cong\ H\ x\ y \Longrightarrow cong\ H\ y\ z \Longrightarrow cong\ H\ x\ z$
⟨*proof*⟩

lemma *lemma-3-13* [*simp*]:
$H \in normalfilters \Longrightarrow cong\ H \in congruences$
⟨*proof*⟩

lemma *cong-times*: $R \in congruences \Longrightarrow R\ a\ b \Longrightarrow R\ c\ d \Longrightarrow R\ (a * c)\ (b * d)$
⟨*proof*⟩

lemma *cong-impl-l*: $R \in congruences \Longrightarrow R\ a\ b \Longrightarrow R\ c\ d \Longrightarrow R\ (a\ l{\rightarrow}\ c)\ (b\ l{\rightarrow}\ d)$
⟨*proof*⟩

lemma *cong-impl-r*: $R \in congruences \Longrightarrow R\ a\ b \Longrightarrow R\ c\ d \Longrightarrow R\ (a\ r{\rightarrow}\ c)\ (b\ r{\rightarrow}\ d)$
⟨*proof*⟩

lemma *cong-refl* [*simp*]: $R \in congruences \Longrightarrow R\ a\ a$
⟨*proof*⟩

lemma *cong-trans-a*: $R \in congruences \Longrightarrow R\ a\ b \Longrightarrow R\ b\ c \Longrightarrow R\ a\ c$
⟨*proof*⟩

lemma *cong-sym*: $R \in congruences \Longrightarrow R\ a\ b \Longrightarrow R\ b\ a$
⟨*proof*⟩

definition
$normalfilter\ R = \{a\ .\ R\ a\ 1\}$

lemma *lemma-3-14* [*simp*]:
$R \in congruences \Longrightarrow (normalfilter\ R) \in normalfilters$
⟨*proof*⟩

lemma *lemma-3-15-i*:
$H \in normalfilters \Longrightarrow normalfilter\ (cong\ H) = H$
⟨*proof*⟩

lemma *lemma-3-15-ii*:
$R \in congruences \Longrightarrow cong\ (normalfilter\ R) = R$
⟨*proof*⟩

lemma *lemma-3-15-iii*: $H1 \in normalfilters \Longrightarrow H2 \in normalfilters \Longrightarrow (H1 \subseteq H2) = (cong\ H1 \leq cong\ H2)$
⟨*proof*⟩

definition
$p\ x\ y\ z = ((x\ l\rightarrow\ y)\ r\rightarrow\ z) \sqcap ((z\ l\rightarrow\ y)\ r\rightarrow\ x)$

lemma *lemma-3-16-i*: $p\ x\ x\ y = y \wedge p\ x\ y\ y = x$
⟨*proof*⟩

definition $M\ x\ y\ z = ((y\ l\rightarrow\ x)\ r\rightarrow\ x) \sqcap ((z\ l\rightarrow\ y)\ r\rightarrow\ y) \sqcap ((x\ l\rightarrow\ z)\ r\rightarrow\ z)$

lemma $M\ x\ x\ y = x \wedge M\ x\ y\ x = x \wedge M\ y\ x\ x = x$
⟨*proof*⟩
end

end

7 Pseudo Waisberg Algebra

theory *PseudoWaisbergAlgebra*
imports *Operations*
begin

class *impl-lr-algebra* = *one* + *left-imp* + *right-imp* +
assumes *W1a* [*simp*]: $1\ l\rightarrow\ a = a$
and *W1b* [*simp*]: $1\ r\rightarrow\ a = a$

and *W2a*: $(a\ l\rightarrow\ b)\ r\rightarrow\ b = (b\ l\rightarrow\ a)\ r\rightarrow\ a$
and *W2b*: $(b\ l\rightarrow\ a)\ r\rightarrow\ a = (b\ r\rightarrow\ a)\ l\rightarrow\ a$
and *W2c*: $(b\ r\rightarrow\ a)\ l\rightarrow\ a = (a\ r\rightarrow\ b)\ l\rightarrow\ b$

and *W3a*: $(a\ l\rightarrow\ b)\ l\rightarrow\ ((b\ l\rightarrow\ c)\ r\rightarrow\ (a\ l\rightarrow\ c)) = 1$
and *W3b*: $(a\ r\rightarrow\ b)\ r\rightarrow\ ((b\ r\rightarrow\ c)\ l\rightarrow\ (a\ r\rightarrow\ c)) = 1$

begin

lemma *P1-a* [*simp*]: $x\ l\rightarrow\ x = 1$
⟨*proof*⟩

lemma *P1-b* [*simp*]: $x\ r\rightarrow\ x = 1$
⟨*proof*⟩

lemma *P2-a*: $x\ l\rightarrow\ y = 1 \Longrightarrow y\ l\rightarrow\ x = 1 \Longrightarrow x = y$
⟨*proof*⟩

lemma *P2-b*: $x\ r\rightarrow\ y = 1 \Longrightarrow y\ r\rightarrow\ x = 1 \Longrightarrow x = y$
⟨*proof*⟩

lemma *P2-c*: $x\ l\!\!\rightarrow\ y = 1 \Longrightarrow y\ r\!\!\rightarrow\ x = 1 \Longrightarrow x = y$
⟨*proof*⟩

lemma *P3-a*: $(x\ l\!\!\rightarrow\ 1)\ r\!\!\rightarrow\ 1 = 1$
⟨*proof*⟩

lemma *P3-b*: $(x\ r\!\!\rightarrow\ 1)\ l\!\!\rightarrow\ 1 = 1$
⟨*proof*⟩

lemma *P4-a* [*simp*]: $x\ l\!\!\rightarrow\ 1 = 1$
⟨*proof*⟩

lemma *P4-b* [*simp*]: $x\ r\!\!\rightarrow\ 1 = 1$
⟨*proof*⟩

lemma *P5-a*: $x\ l\!\!\rightarrow\ y = 1 \Longrightarrow y\ l\!\!\rightarrow\ z = 1 \Longrightarrow x\ l\!\!\rightarrow\ z = 1$
⟨*proof*⟩

lemma *P5-b*: $x\ r\!\!\rightarrow\ y = 1 \Longrightarrow y\ r\!\!\rightarrow\ z = 1 \Longrightarrow x\ r\!\!\rightarrow\ z = 1$
⟨*proof*⟩

lemma *P6-a*: $x\ l\!\!\rightarrow\ (y\ r\!\!\rightarrow\ x) = 1$
⟨*proof*⟩

lemma *P6-b*: $x\ r\!\!\rightarrow\ (y\ l\!\!\rightarrow\ x) = 1$
⟨*proof*⟩

lemma *P7*: $(x\ l\!\!\rightarrow\ (y\ r\!\!\rightarrow\ z) = 1) = (y\ r\!\!\rightarrow\ (x\ l\!\!\rightarrow\ z) = 1)$
⟨*proof*⟩

lemma *P8-a*: $(x\ l\!\!\rightarrow\ y)\ r\!\!\rightarrow\ ((z\ l\!\!\rightarrow\ x)\ l\!\!\rightarrow\ (z\ l\!\!\rightarrow\ y)) = 1$
⟨*proof*⟩

lemma *P8-b*: $(x\ r\!\!\rightarrow\ y)\ l\!\!\rightarrow\ ((z\ r\!\!\rightarrow\ x)\ r\!\!\rightarrow\ (z\ r\!\!\rightarrow\ y)) = 1$
⟨*proof*⟩

lemma *P9*: $x\ l\!\!\rightarrow\ (y\ r\!\!\rightarrow\ z) = y\ r\!\!\rightarrow\ (x\ l\!\!\rightarrow\ z)$
⟨*proof*⟩

definition
lesseq-a $a\ b = (a\ l\!\!\rightarrow\ b = 1)$

definition
less-a $a\ b =$ (*lesseq-a* $a\ b \wedge \neg$ *lesseq-a* $b\ a$)

definition
lesseq-b $a\ b = (a\ r\!\!\rightarrow\ b = 1)$

definition
less-b a b = (*lesseq-b a b* ∧ ¬ *lesseq-b b a*)

definition
sup-a a b = (*a l→ b*) *r→ b*

end

sublocale *impl-lr-algebra* < *order-a*:*order lesseq-a less-a*
⟨*proof*⟩

sublocale *impl-lr-algebra* < *order-b*:*order lesseq-b less-b*
⟨*proof*⟩

sublocale *impl-lr-algebra* < *slattice-a*:*semilattice-sup sup-a lesseq-a less-a*
⟨*proof*⟩

sublocale *impl-lr-algebra* < *slattice-b*:*semilattice-sup sup-a lesseq-b less-b*
⟨*proof*⟩

context *impl-lr-algebra*
begin
lemma *lesseq-a-b*: *lesseq-b* = *lesseq-a*
⟨*proof*⟩

lemma *P10*: (*a l→ b* = *1*) = (*a r→ b* = *1*)
⟨*proof*⟩
end

class *one-ord* = *one* + *ord*

class *impl-lr-ord-algebra* = *impl-lr-algebra* + *one-ord* +
assumes
order: *a* ≤ *b* = (*a l→ b* = *1*)
and
strict: *a* < *b* = (*a* ≤ *b* ∧ ¬ *b* ≤ *a*)
begin
lemma *order-l*: (*a* ≤ *b*) = (*a l→ b* = *1*)
⟨*proof*⟩

lemma *order-r*: (*a* ≤ *b*) = (*a r→ b* = *1*)
⟨*proof*⟩

lemma *P11-a*: *a* ≤ *b l→ a*
⟨*proof*⟩

lemma *P11-b*: *a* ≤ *b r→ a*
⟨*proof*⟩

lemma *P12*: $(a \le b\ l\rightarrow\ c) = (b \le a\ r\rightarrow\ c)$
  ⟨*proof*⟩

lemma *P13-a*: $a \le b \Longrightarrow b\ l\rightarrow\ c \le a\ l\rightarrow\ c$
  ⟨*proof*⟩

lemma *P13-b*: $a \le b \Longrightarrow b\ r\rightarrow\ c \le a\ r\rightarrow\ c$
  ⟨*proof*⟩

lemma *P14-a*: $a \le b \Longrightarrow c\ l\rightarrow\ a \le c\ l\rightarrow\ b$
  ⟨*proof*⟩

lemma *P14-b*: $a \le b \Longrightarrow c\ r\rightarrow\ a \le c\ r\rightarrow\ b$
  ⟨*proof*⟩

subclass *order*
  ⟨*proof*⟩

end

class *one-zero-uminus* = *one* + *zero* + *left-uminus* + *right-uminus*

class *impl-neg-lr-algebra* = *impl-lr-ord-algebra* + *one-zero-uminus* +
  assumes
      W4: $-l\ 1 = -r\ 1$
  and *W5a*: $(-l\ a\ r\rightarrow\ -l\ b)\ l\rightarrow\ (b\ l\rightarrow\ a) = 1$
  and *W5b*: $(-r\ a\ l\rightarrow\ -r\ b)\ l\rightarrow\ (b\ r\rightarrow\ a) = 1$
  and *zero-def*: $0 = -l\ 1$
begin

lemma *zero-r-def*: $0 = -r\ 1$
  ⟨*proof*⟩

lemma *C1-a* [*simp*]: $(-l\ x\ r\rightarrow\ 0)\ l\rightarrow\ x = 1$
  ⟨*proof*⟩

lemma *C1-b* [*simp*]: $(-r\ x\ l\rightarrow\ 0)\ r\rightarrow\ x = 1$
  ⟨*proof*⟩

lemma *C2-b* [*simp*]: $0\ r\rightarrow\ x = 1$
  ⟨*proof*⟩

lemma *C2-a* [*simp*]: $0\ l\rightarrow\ x = 1$
  ⟨*proof*⟩

lemma *C3-a*: $x\ l\rightarrow\ 0 = -l\ x$
  ⟨*proof*⟩

lemma *C3-b*: $x\ r\!\rightarrow\ 0 = -r\ x$
⟨*proof*⟩

lemma *C4-a* [*simp*]: $-r\ (-l\ x) = x$
⟨*proof*⟩

lemma *C4-b* [*simp*]: $-l\ (-r\ x) = x$
⟨*proof*⟩

lemma *C5-a*: $-r\ x\ l\!\rightarrow\ -r\ y = y\ r\!\rightarrow\ x$
⟨*proof*⟩

lemma *C5-b*: $-l\ x\ r\!\rightarrow\ -l\ y = y\ l\!\rightarrow\ x$
⟨*proof*⟩

lemma *C6*: $-r\ x\ l\!\rightarrow\ y = -l\ y\ r\!\rightarrow\ x$
⟨*proof*⟩

lemma *C7-a*: $(x \le y) = (-l\ y \le -l\ x)$
⟨*proof*⟩

lemma *C7-b*: $(x \le y) = (-r\ y \le -r\ x)$
⟨*proof*⟩

end

class *pseudo-wajsberg-algebra* = *impl-neg-lr-algebra* +
assumes
W6: $-r\ (a\ l\!\rightarrow\ -l\ b) = -l\ (b\ r\!\rightarrow\ -r\ a)$
begin
definition
mult $a\ b = -r\ (a\ l\!\rightarrow\ -l\ b)$

definition
inf-a $a\ b = -l\ (a\ r\!\rightarrow\ -r\ (a\ l\!\rightarrow\ b))$

definition
inf-b $a\ b = -r\ (b\ l\!\rightarrow\ -l\ (b\ r\!\rightarrow\ a))$

end

sublocale *pseudo-wajsberg-algebra* < *slattice-inf-a*:*semilattice-inf inf-a* (≤) (<)
⟨*proof*⟩

sublocale *pseudo-wajsberg-algebra* < *slattice-inf-b*:*semilattice-inf inf-b* (≤) (<)
⟨*proof*⟩

context *pseudo-wajsberg-algebra*
begin
lemma *inf-a-b*: *inf-a* = *inf-b*
⟨*proof*⟩

end
end

8 Some Classes of Pseudo-Hoops

theory *SpecialPseudoHoops*
imports *PseudoHoopFilters PseudoWaisbergAlgebra*
begin

class *cancel-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *mult-cancel-left*: $a * b = a * c \Longrightarrow b = c$
and *mult-cancel-right*: $b * a = c * a \Longrightarrow b = c$
begin
lemma *cancel-left-a*: $b\ l{\rightarrow}\ (a * b) = a$
⟨*proof*⟩

lemma *cancel-right-a*: $b\ r{\rightarrow}\ (b * a) = a$
⟨*proof*⟩

end

class *cancel-pseudo-hoop-algebra-2* = *pseudo-hoop-algebra* +
assumes *cancel-left*: $b\ l{\rightarrow}\ (a * b) = a$
and *cancel-right*: $b\ r{\rightarrow}\ (b * a) = a$

begin
subclass *cancel-pseudo-hoop-algebra*
⟨*proof*⟩

end

context *cancel-pseudo-hoop-algebra*
begin

lemma *lemma-4-2-i*: $a\ l{\rightarrow}\ b = (a * c)\ l{\rightarrow}\ (b * c)$
⟨*proof*⟩

lemma *lemma-4-2-ii*: $a\ r{\rightarrow}\ b = (c * a)\ r{\rightarrow}\ (c * b)$
⟨*proof*⟩

lemma *lemma-4-2-iii*: $(a * c \leq b * c) = (a \leq b)$
⟨*proof*⟩

lemma *lemma-4-2-iv*: $(c * a \leq c * b) = (a \leq b)$
⟨*proof*⟩

end

class *wajsberg-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *wajsberg1*: $(a\ l{\rightarrow}\ b)\ r{\rightarrow}\ b = (b\ l{\rightarrow}\ a)\ r{\rightarrow}\ a$
and *wajsberg2*: $(a\ r{\rightarrow}\ b)\ l{\rightarrow}\ b = (b\ r{\rightarrow}\ a)\ l{\rightarrow}\ a$

context *wajsberg-pseudo-hoop-algebra*
begin

lemma *lemma-4-3-i-a*: $a \sqcup 1\ b = (a\ l{\rightarrow}\ b)\ r{\rightarrow}\ b$
⟨*proof*⟩

lemma *lemma-4-3-i-b*: $a \sqcup 1\ b = (b\ l{\rightarrow}\ a)\ r{\rightarrow}\ a$
⟨*proof*⟩

lemma *lemma-4-3-ii-a*: $a \sqcup 2\ b = (a\ r{\rightarrow}\ b)\ l{\rightarrow}\ b$
⟨*proof*⟩

lemma *lemma-4-3-ii-b*: $a \sqcup 2\ b = (b\ r{\rightarrow}\ a)\ l{\rightarrow}\ a$
⟨*proof*⟩
end

sublocale *wajsberg-pseudo-hoop-algebra* < *lattice1*:*pseudo-hoop-lattice-b* $(\sqcup 1)$ $(*)$
$(\sqcap)$ $(l{\rightarrow})$ $(\leq)$ $(<)$ *1* $(r{\rightarrow})$
⟨*proof*⟩

class *zero-one* = *zero* + *one*

class *bounded-wajsberg-pseudo-hoop-algebra* = *zero-one* + *wajsberg-pseudo-hoop-algebra*
+
assumes *zero-smallest* [*simp*]: $0 \leq a$
begin
end

sublocale *wajsberg-pseudo-hoop-algebra* < *lattice2*:*pseudo-hoop-lattice-b* $(\sqcup 2)$ $(*)$
$(\sqcap)$ $(l{\rightarrow})$ $(\leq)$ $(<)$ *1* $(r{\rightarrow})$
⟨*proof*⟩

lemma (**in** *wajsberg-pseudo-hoop-algebra*) *sup1-eq-sup2*: $(\sqcup 1) = (\sqcup 2)$
⟨*proof*⟩

context *bounded-wajsberg-pseudo-hoop-algebra*
begin
definition
negl a = a l→ 0

definition
negr a = a r→ 0

lemma [*simp*]: *0 l→ a = 1*
⟨*proof*⟩

lemma [*simp*]: *0 r→ a = 1*
⟨*proof*⟩
end

sublocale *bounded-wajsberg-pseudo-hoop-algebra* < *wajsberg*: *pseudo-wajsberg-algebra 1 (l→) (r→) (≤) (<) 0 negl negr*
⟨*proof*⟩

context *pseudo-wajsberg-algebra*
begin
lemma *class.bounded-wajsberg-pseudo-hoop-algebra mult inf-a (l→) (≤) (<) 1 (r→) (0::'a)*
⟨*proof*⟩

end

class *basic-pseudo-hoop-algebra = pseudo-hoop-algebra +*
assumes *B1*: *(a l→ b) l→ c ≤ ((b l→ a) l→ c) l→ c*
and *B2*: *(a r→ b) r→ c ≤ ((b r→ a) r→ c) r→ c*
begin
lemma *lemma-4-5-i*: *(a l→ b) ⊔1 (b l→ a) = 1*
⟨*proof*⟩

lemma *lemma-4-5-ii*: *(a r→ b) ⊔2 (b r→ a) = 1*
⟨*proof*⟩

lemma *lemma-4-5-iii*: *a l→ b = (a ⊔1 b) l→ b*
⟨*proof*⟩

lemma *lemma-4-5-iv*: *a r→ b = (a ⊔2 b) r→ b*
⟨*proof*⟩

lemma *lemma-4-5-v*: *(a ⊔1 b) l→ c = (a l→ c) ⊓ (b l→ c)*
⟨*proof*⟩

lemma *lemma-4-5-vi*: $(a \sqcup 2\ b)\ r\!\rightarrow\ c = (a\ r\!\rightarrow\ c) \sqcap (b\ r\!\rightarrow\ c)$
⟨*proof*⟩

lemma *lemma-4-5-a*: $a \le c \Longrightarrow b \le c \Longrightarrow a \sqcup 1\ b \le c$
⟨*proof*⟩

lemma *lemma-4-5-b*: $a \le c \Longrightarrow b \le c \Longrightarrow a \sqcup 2\ b \le c$
⟨*proof*⟩

lemma *lemma-4-5*: $a \sqcup 1\ b = a \sqcup 2\ b$
⟨*proof*⟩
end

sublocale *basic-pseudo-hoop-algebra* < *basic-lattice*:*lattice* $(\sqcap)$ $(\le)$ $(<)$ $(\sqcup 1)$
⟨*proof*⟩

context *pseudo-hoop-lattice* **begin end**

sublocale *basic-pseudo-hoop-algebra* < *pseudo-hoop-lattice* $(\sqcup 1)$ $(*)$ $(\sqcap)$ $(l\!\rightarrow)$ $(\le)$ $(<)$ 1 $(r\!\rightarrow)$
⟨*proof*⟩

class *sup-assoc-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *sup1-assoc*: $a \sqcup 1\ (b \sqcup 1\ c) = (a \sqcup 1\ b) \sqcup 1\ c$
and *sup2-assoc*: $a \sqcup 2\ (b \sqcup 2\ c) = (a \sqcup 2\ b) \sqcup 2\ c$

sublocale *sup-assoc-pseudo-hoop-algebra* < *sup1-lattice*: *pseudo-hoop-lattice* $(\sqcup 1)$ $(*)$ $(\sqcap)$ $(l\!\rightarrow)$ $(\le)$ $(<)$ 1 $(r\!\rightarrow)$
⟨*proof*⟩

sublocale *sup-assoc-pseudo-hoop-algebra* < *sup2-lattice*: *pseudo-hoop-lattice* $(\sqcup 2)$ $(*)$ $(\sqcap)$ $(l\!\rightarrow)$ $(\le)$ $(<)$ 1 $(r\!\rightarrow)$
⟨*proof*⟩

class *sup-assoc-pseudo-hoop-algebra-1* = *sup-assoc-pseudo-hoop-algebra* +
assumes *union-impl*: $(a\ l\!\rightarrow\ b) \sqcup 1\ (b\ l\!\rightarrow\ a) = 1$
and *union-impr*: $(a\ r\!\rightarrow\ b) \sqcup 1\ (b\ r\!\rightarrow\ a) = 1$

lemma (**in** *pseudo-hoop-algebra*) [*simp*]: *infimum* $\{a,\ b\} = \{a \sqcap b\}$
⟨*proof*⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-impl-inf*:
$(a \sqcup b)\ l\!\rightarrow\ c = (a\ l\!\rightarrow\ c) \sqcap (b\ l\!\rightarrow c)$
⟨*proof*⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-impr-inf*:

$(a \sqcup b)\ r\rightarrow\ c = (a\ r\rightarrow\ c) \sqcap (b\ r\rightarrow c)$
⟨*proof*⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-times*:
$a * (b \sqcup c) = (a * b) \sqcup (a * c)$
⟨*proof*⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-times-right*:
$(b \sqcup c) * a = (b * a) \sqcup (c * a)$
⟨*proof*⟩

context *basic-pseudo-hoop-algebra* **begin end**

sublocale *sup-assoc-pseudo-hoop-algebra-1* < *basic-1*: *basic-pseudo-hoop-algebra* $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(\leq)$ $(<)$ 1 $(r\rightarrow)$
⟨*proof*⟩

context *basic-pseudo-hoop-algebra*
begin

lemma *lemma-4-8-i*: $a * (b \sqcap c) = (a * b) \sqcap (a * c)$
⟨*proof*⟩

lemma *lemma-4-8-ii*: $(b \sqcap c) * a = (b * a) \sqcap (c * a)$
⟨*proof*⟩

lemma *lemma-4-8-iii*: $(a\ l\rightarrow\ b)\ l\rightarrow\ (b\ l\rightarrow\ a) = b\ l\rightarrow\ a$
⟨*proof*⟩

lemma *lemma-4-8-iv*: $(a\ r\rightarrow\ b)\ r\rightarrow\ (b\ r\rightarrow\ a) = b\ r\rightarrow\ a$
⟨*proof*⟩

end

context *wajsberg-pseudo-hoop-algebra*
begin
subclass *sup-assoc-pseudo-hoop-algebra-1*
⟨*proof*⟩
end

class *bounded-basic-pseudo-hoop-algebra* = *zero-one* + *basic-pseudo-hoop-algebra* +
assumes *zero-smallest* [*simp*]: $0 \leq a$
begin
end

class *inf-a* =
fixes *inf-a* :: $'a => 'a => 'a$ (**infixl** $\sqcap1\ 65$)

class *pseudo-bl-algebra* = *zero* + *sup* + *inf* + *monoid-mult* + *ord* + *left-imp* + *right-imp* +
  assumes *bounded-lattice*: *class.bounded-lattice inf* (≤) (<) *sup 0 1*
  and *left-residual-bl*: $(x * a \leq b) = (x \leq a\ l\!\rightarrow\ b)$
  and *right-residual-bl*: $(a * x \leq b) = (x \leq a\ r\!\rightarrow\ b)$
  and *inf-l-bl-def*: $a \sqcap b = (a\ l\!\rightarrow\ b) * a$
  and *inf-r-bl-def*: $a \sqcap b = a * (a\ r\!\rightarrow\ b)$
  and *impl-sup-bl*: $(a\ l\!\rightarrow\ b) \sqcup (b\ l\!\rightarrow\ a) = 1$
  and *impr-sup-bl*: $(a\ r\!\rightarrow\ b) \sqcup (b\ r\!\rightarrow\ a) = 1$
begin
end

context *pseudo-bl-algebra* **begin end**

sublocale *bounded-basic-pseudo-hoop-algebra* < *basic*:*pseudo-bl-algebra 1* (∗) *0* (⊓) (⊔*1*) (*l*→) (*r*→) (≤) (<)
  ⟨*proof*⟩

sublocale *pseudo-bl-algebra* < *bounded-lattice*: *bounded-lattice inf* (≤) (<) *sup 0 1*
  ⟨*proof*⟩

context *pseudo-bl-algebra*
begin
  lemma *impl-one-bl* [*simp*]: $a\ l\!\rightarrow\ a = 1$
    ⟨*proof*⟩

  lemma *impr-one-bl* [*simp*]: $a\ r\!\rightarrow\ a = 1$
    ⟨*proof*⟩

  lemma *impl-ded-bl*: $((a * b)\ l\!\rightarrow\ c) = (a\ l\!\rightarrow\ (b\ l\!\rightarrow\ c))$
    ⟨*proof*⟩

  lemma *impr-ded-bl*: $(b * a\ r\!\rightarrow\ c) = (a\ r\!\rightarrow\ (b\ r\!\rightarrow\ c))$
    ⟨*proof*⟩

  lemma *lesseq-impl-bl*: $(a \leq b) = (a\ l\!\rightarrow\ b = 1)$
    ⟨*proof*⟩

end

context *pseudo-bl-algebra*
begin
subclass *pseudo-hoop-lattice*
  ⟨*proof*⟩

subclass *bounded-basic-pseudo-hoop-algebra*
⟨*proof*⟩

end

class *product-pseudo-hoop-algebra* = *basic-pseudo-hoop-algebra* +
assumes *P1*: $b\ l{\rightarrow}\ b * b \leq (a \sqcap (a\ l{\rightarrow}\ b))\ l{\rightarrow}\ b$
and *P2*: $b\ r{\rightarrow}\ b * b \leq (a \sqcap (a\ r{\rightarrow}\ b))\ r{\rightarrow}\ b$
and *P3*: $((a\ l{\rightarrow}\ b)\ l{\rightarrow}\ b) * (c * a\ l{\rightarrow}\ d * a) * (c * b\ l{\rightarrow}\ d * b) \leq c\ l{\rightarrow}\ d$
and *P4*: $((a\ r{\rightarrow}\ b)\ r{\rightarrow}\ b) * (a * c\ r{\rightarrow}\ a * d) * (b * c\ r{\rightarrow}\ b * d) \leq c\ r{\rightarrow}\ d$

class *cancel-basic-pseudo-hoop-algebra* = *basic-pseudo-hoop-algebra* + *cancel-pseudo-hoop-algebra*
begin
subclass *product-pseudo-hoop-algebra*
⟨*proof*⟩

end

class *simple-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *simple*: *normalfilters* ∩ *properfilters* = {{*1*}}

class *proper* = *one* +
assumes *proper*: $\exists\ a\ .\ a \neq 1$

class *strong-simple-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *strong-simple*: *properfilters* = {{*1*}}
begin

subclass *proper*
⟨*proof*⟩

lemma *lemma-4-12-i-ii*: $a \neq 1 \Longrightarrow$ *filterof*({*a*}) = *UNIV*
⟨*proof*⟩

lemma *lemma-4-12-i-iii*: $a \neq 1 \Longrightarrow (\exists\ n\ .\ a\ \hat{}\ n \leq b)$
⟨*proof*⟩

lemma *lemma-4-12-i-iv*: $a \neq 1 \Longrightarrow (\exists\ n\ .\ a\ l{-}n{\rightarrow}\ b = 1)$
⟨*proof*⟩

lemma *lemma-4-12-i-v*: $a \neq 1 \Longrightarrow (\exists\ n\ .\ a\ r{-}n{\rightarrow}\ b = 1)$
⟨*proof*⟩

end

lemma (**in** *pseudo-hoop-algebra*) [*simp*]: {*1*} ∈ *filters*
⟨*proof*⟩

class *strong-simple-pseudo-hoop-algebra-a* = *pseudo-hoop-algebra* + *proper* +

assumes *strong-simple-a*: $a \neq 1 \Longrightarrow$ *filterof*$(\{a\}) = $ *UNIV*
begin
subclass *strong-simple-pseudo-hoop-algebra*
⟨*proof*⟩
end

sublocale *strong-simple-pseudo-hoop-algebra* < *strong-simple-pseudo-hoop-algebra-a*
⟨*proof*⟩

lemma (**in** *pseudo-hoop-algebra*) *power-impl*: $b\ l\!\to\ a = a \Longrightarrow b\ \hat{}\ n\ l\!\to\ a = a$
⟨*proof*⟩

lemma (**in** *pseudo-hoop-algebra*) *power-impr*: $b\ r\!\to\ a = a \Longrightarrow b\ \hat{}\ n\ r\!\to\ a = a$
⟨*proof*⟩

context *strong-simple-pseudo-hoop-algebra*
begin

lemma *lemma-4-13-i*: $b\ l\!\to\ a = a \Longrightarrow a = 1 \lor b = 1$
⟨*proof*⟩

lemma *lemma-4-13-ii*: $b\ r\!\to\ a = a \Longrightarrow a = 1 \lor b = 1$
⟨*proof*⟩
end

class *basic-pseudo-hoop-algebra-A* = *basic-pseudo-hoop-algebra* +
assumes *left-impl-one*: $b\ l\!\to\ a = a \Longrightarrow a = 1 \lor b = 1$
and *right-impl-one*: $b\ r\!\to\ a = a \Longrightarrow a = 1 \lor b = 1$
begin
subclass *linorder*
⟨*proof*⟩

lemma [*simp*]: $(a\ l\!\to\ b)\ r\!\to\ b \leq (b\ l\!\to\ a)\ r\!\to\ a$
⟨*proof*⟩

end

context *basic-pseudo-hoop-algebra-A* **begin**

lemma [*simp*]: $(a\ r\!\to\ b)\ l\!\to\ b \leq (b\ r\!\to\ a)\ l\!\to\ a$
⟨*proof*⟩

subclass *wajsberg-pseudo-hoop-algebra*
⟨*proof*⟩

end

class *strong-simple-basic-pseudo-hoop-algebra* = *strong-simple-pseudo-hoop-algebra* + *basic-pseudo-hoop-algebra*

begin
subclass *basic-pseudo-hoop-algebra-A*
  ⟨*proof*⟩

subclass *wajsberg-pseudo-hoop-algebra*
  ⟨*proof*⟩

end

end

9 Examples of Pseudo-Hoops

theory *Examples*
imports *SpecialPseudoHoops LatticeProperties.Lattice-Ordered-Group*
begin

declare *add-uminus-conv-diff* [*simp del*] *right-minus* [*simp*]
lemmas *diff-minus* = *diff-conv-add-uminus*

context *lgroup*
begin
lemma (**in** *lgroup*) *less-eq-inf-2*: $(x \leq y) = (\mathit{inf}\ y\ x = x)$
  ⟨*proof*⟩
end

class *lgroup-with-const* = *lgroup* +
  fixes $u::'a$
  assumes [*simp*]: $0 \leq u$

definition $G = \{a::'a::\mathit{lgroup\text{-}with\text{-}const}.\ (0 \leq a \wedge a \leq u)\}$
typedef (**overloaded**) $'a\ G = G::'a::\mathit{lgroup\text{-}with\text{-}const}\ \mathit{set}$
⟨*proof*⟩

instantiation G :: (*lgroup-with-const*) *bounded-wajsberg-pseudo-hoop-algebra*
begin

definition
  times-def: $a * b \equiv \mathit{Abs\text{-}G}\ (\mathit{sup}\ (\mathit{Rep\text{-}G}\ a - u + \mathit{Rep\text{-}G}\ b)\ 0)$

lemma [*simp*]: $\mathit{sup}\ (\mathit{Rep\text{-}G}\ a - u + \mathit{Rep\text{-}G}\ b)\ 0 \in G$
  ⟨*proof*⟩

definition
  impl-def: $a\ l\!\rightarrow b \equiv \mathit{Abs\text{-}G}\ ((\mathit{Rep\text{-}G}\ b - \mathit{Rep\text{-}G}\ a + u) \sqcap u)$

lemma [*simp*]: $inf\ (Rep\text{-}G\ (b::'a\ G) - Rep\text{-}G\ a + u)\ u \in G$
⟨*proof*⟩

definition
impr-def: $a\ r{\rightarrow}\ b \equiv Abs\text{-}G\ (inf\ (u - Rep\text{-}G\ a + Rep\text{-}G\ b)\ u)$

lemma [*simp*]: $inf\ (u - Rep\text{-}G\ a + Rep\text{-}G\ b)\ u \in G$
⟨*proof*⟩

definition
one-def: $1 \equiv Abs\text{-}G\ u$

definition
zero-def: $0 \equiv Abs\text{-}G\ 0$

definition
order-def: $((a::'a\ G) \leq b) \equiv (a\ l{\rightarrow}\ b = 1)$

definition
strict-order-def: $(a::'a\ G) < b \equiv (a \leq b \wedge \neg\ b \leq a)$

definition
inf-def: $(a::'a\ G) \sqcap b = ((a\ l{\rightarrow}\ b) * a)$

lemma [*simp*]: $(u::'a) \in G$
⟨*proof*⟩

lemma [*simp*]: $(1::'a\ G) * a = a$
⟨*proof*⟩

lemma [*simp*]: $a * (1::'a\ G) = a$
⟨*proof*⟩

lemma [*simp*]: $a\ l{\rightarrow}\ a = (1::'a\ G)$
⟨*proof*⟩

lemma [*simp*]: $a\ r{\rightarrow}\ a = (1::'a\ G)$
⟨*proof*⟩

lemma [*simp*]: $a \in G \Longrightarrow Rep\text{-}G\ (Abs\text{-}G\ a) = a$
⟨*proof*⟩

lemma *inf-def-1*: $((a::'a\ G)\ l{\rightarrow}\ b) * a = Abs\text{-}G\ (inf\ (Rep\text{-}G\ a)\ (Rep\text{-}G\ b))$
⟨*proof*⟩

lemma *inf-def-2*: $(a::'a\ G) * (a\ r{\rightarrow}\ b) = Abs\text{-}G\ (inf\ (Rep\text{-}G\ a)\ (Rep\text{-}G\ b))$
⟨*proof*⟩

lemma *Rep-G-order*: $(a \leq b) = (Rep\text{-}G\ a \leq Rep\text{-}G\ b)$

⟨*proof*⟩

lemma *ded-left*: $((a::'a\ G) * b)\ l\!\rightarrow\ c = a\ l\!\rightarrow\ b\ l\!\rightarrow\ c$
⟨*proof*⟩

lemma *ded-right*: $((a::'a\ G) * b)\ r\!\rightarrow\ c = b\ r\!\rightarrow\ a\ r\!\rightarrow\ c$
⟨*proof*⟩

lemma [*simp*]: $0 \in G$
⟨*proof*⟩

lemma [*simp*]: $0 \leq (a::'a\ G)$
⟨*proof*⟩

lemma *lemma-W1*: $((a::'a\ G)\ l\!\rightarrow\ b)\ r\!\rightarrow\ b = (b\ l\!\rightarrow\ a)\ r\!\rightarrow\ a$
⟨*proof*⟩

lemma *lemma-W2*: $((a::'a\ G)\ r\!\rightarrow\ b)\ l\!\rightarrow\ b = (b\ r\!\rightarrow\ a)\ l\!\rightarrow\ a$
⟨*proof*⟩

instance ⟨*proof*⟩

end

context *order*
begin
definition
closed-interval::$'a \Rightarrow 'a \Rightarrow 'a$ *set* (|[- , -]| [0, 0] 900) **where**
closed-interval $a\ b = \{c\ .\ a \leq c \wedge c \leq b\}$

definition
convex $= \{A\ .\ \forall\ a\ b\ .\ a \in A \wedge b \in A \longrightarrow |[a,\ b]| \subseteq A\}$

end

context *group-add*
begin
definition
subgroup $= \{A\ .\ 0 \in A \wedge (\forall\ a\ b\ .\ a \in A \wedge b \in A \longrightarrow a + b \in A \wedge -a \in A)\}$

lemma [*simp*]: $A \in$ *subgroup* $\Longrightarrow 0 \in A$
⟨*proof*⟩

lemma [*simp*]: $A \in$ *subgroup* $\Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow a + b \in A$
⟨*proof*⟩

lemma *minus-subgroup*: $A \in subgroup \Longrightarrow -a \in A \Longrightarrow a \in A$
⟨*proof*⟩

definition
add-set:: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set$ (**infixl** +++ *70*) **where**
$add\text{-}set\ A\ B = \{c\ .\ \exists\ a \in A\ .\exists\ b \in B\ .\ c = a + b\}$

definition
$normal = \{A\ .\ (\forall\ a\ .\ A +++ \{a\} = \{a\} +++ A)\}$
end

context *lgroup*
begin
definition
$lsubgroup = \{A\ .\ A \in subgroup \wedge (\forall\ a\ b\ .\ a \in A \wedge b \in A \longrightarrow inf\ a\ b \in A \wedge sup\ a\ b \in A)\}$

lemma *inf-lsubgroup*:
$A \in lsubgroup \Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow inf\ a\ b \in A$
⟨*proof*⟩

lemma *sup-lsubgroup*:
$A \in lsubgroup \Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow sup\ a\ b \in A$
⟨*proof*⟩
end

definition
$F\ K = \{a::\ 'a\ G\ .\ (u::'a::lgroup\text{-}with\text{-}const) - Rep\text{-}G\ a \in K\}$

lemma *F-def2*: $K \in normal \Longrightarrow F\ K = \{a::\ 'a\ G\ .\ - Rep\text{-}G\ a + (u::'a::lgroup\text{-}with\text{-}const) \in K\}$
⟨*proof*⟩

context *lgroup* **begin**
lemma *sup-assoc-lgroup*: $a \sqcup b \sqcup c = a \sqcup (b \sqcup c)$
⟨*proof*⟩

end

lemma *normal-1*: $K \in normal \Longrightarrow K \in convex \Longrightarrow K \in lsubgroup \Longrightarrow x \in \{a\} ** F\ K \Longrightarrow x \in F\ K ** \{a\}$
⟨*proof*⟩

lemma *normal-2*: $K \in normal \Longrightarrow K \in convex \Longrightarrow K \in lsubgroup \Longrightarrow x \in F\ K ** \{a\} \Longrightarrow x \in \{a\} ** F\ K$
⟨*proof*⟩

lemma $K \in \textit{normal} \Longrightarrow K \in \textit{convex} \Longrightarrow K \in \textit{lsubgroup} \Longrightarrow F\ K \in \textit{normalfilters}$
  ⟨*proof*⟩

definition $N = \{a::'a::lgroup.\ a \leq 0\}$
typedef (**overloaded**) $('a::lgroup)\ N = N :: 'a::lgroup\ set$
⟨*proof*⟩

class *cancel-product-pseudo-hoop-algebra* = *cancel-pseudo-hoop-algebra* + *product-pseudo-hoop-algebra*

context *group-add*
begin
subclass *cancel-semigroup-add*
⟨*proof*⟩

end

instantiation N :: (*lgroup*) *pseudo-hoop-algebra*
begin

definition
  times-N-def: $a * b \equiv \textit{Abs-N}\ (\textit{Rep-N}\ a + \ \textit{Rep-N}\ b)$

lemma [*simp*]: $\textit{Rep-N}\ a + \textit{Rep-N}\ b \in N$
  ⟨*proof*⟩

definition
  impl-N-def: $a\ l\!\rightarrow\ b \equiv \textit{Abs-N}\ (\textit{inf}\ (\textit{Rep-N}\ b - \textit{Rep-N}\ a)\ 0)$

definition
  inf-N-def: $(a::\ 'a\ N) \sqcap b = (a\ l\!\rightarrow\ b) * a$

lemma [*simp*]: $\textit{inf}\ (\textit{Rep-N}\ b - \textit{Rep-N}\ a)\ 0 \in N$
  ⟨*proof*⟩

definition
  impr-N-def: $a\ r\!\rightarrow\ b \equiv \textit{Abs-N}\ (\textit{inf}\ (-\ \textit{Rep-N}\ a + \textit{Rep-N}\ b)\ 0)$

lemma [*simp*]: $\textit{inf}\ (-\ \textit{Rep-N}\ a + \textit{Rep-N}\ b)\ 0 \in N$
  ⟨*proof*⟩

definition
  one-N-def: $1 \equiv \textit{Abs-N}\ 0$

lemma [*simp*]: $0 \in N$
  ⟨*proof*⟩

definition

order-N-def: $((a::'a\ N) \leq b) \equiv (a\ l\rightarrow\ b = 1)$

definition
strict-order-N-def: $(a::'a\ N) < b \equiv (a \leq b \wedge \neg\ b \leq a)$

lemma *order-Rep-N*:
$((a::'a\ N) \leq b) = (Rep\text{-}N\ a \leq Rep\text{-}N\ b)$
⟨*proof*⟩

lemma *order-Abs-N*:
$a \in N \Longrightarrow b \in N \Longrightarrow (a \leq b) = (Abs\text{-}N\ a \leq Abs\text{-}N\ b)$
⟨*proof*⟩

lemma [*simp*]: $(1::'a\ N) * a = a$
⟨*proof*⟩

lemma [*simp*]: $a * (1::'a\ N) = a$
⟨*proof*⟩

lemma [*simp*]: $a\ l\rightarrow\ a = (1::'a\ N)$
⟨*proof*⟩

lemma [*simp*]: $a\ r\rightarrow\ a = (1::'a\ N)$
⟨*proof*⟩

lemma *impl-times*: $(a\ l\rightarrow\ b) * a = (b\ l\rightarrow\ a) * (b::'a\ N)$
⟨*proof*⟩

lemma *impr-times*: $a * (a\ r\rightarrow\ b) = (b::'a\ N) * (b\ r\rightarrow\ a)$
⟨*proof*⟩

lemma *impr-impl-times*: $(a\ l\rightarrow\ b) * a = (a::'a\ N) * (a\ r\rightarrow\ b)$
⟨*proof*⟩

lemma *impl-ded*: $(a::'a\ N) * b\ l\rightarrow\ c = a\ l\rightarrow\ b\ l\rightarrow\ c$
⟨*proof*⟩

lemma *impr-ded*: $(a::'a\ N) * b\ r\rightarrow\ c = b\ r\rightarrow\ a\ r\rightarrow\ c$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *Rep-N-inf*: $Rep\text{-}N\ ((a::'a::lgroup\ N) \sqcap b) = (Rep\text{-}N\ a) \sqcap (Rep\text{-}N\ b)$
⟨*proof*⟩

context *lgroup* **begin**

lemma *sup-inf-distrib2-lgroup*: $(b \sqcap c) \sqcup a = (b \sqcup a) \sqcap (c \sqcup a)$
⟨*proof*⟩

lemma *inf-sup-distrib2-lgroup*: $(b \sqcup c) \sqcap a = (b \sqcap a) \sqcup (c \sqcap a)$
⟨*proof*⟩
end

instantiation N :: (*lgroup*) *cancel-product-pseudo-hoop-algebra*
begin

lemma *cancel-times-left*: $(a::'a\ N) * b = a * c \Longrightarrow b = c$
⟨*proof*⟩

lemma *cancel-times-right*: $b * (a::'a\ N) = c * a \Longrightarrow b = c$
⟨*proof*⟩

lemma *prod-1*: $((a::'a\ N)\ l\!\rightarrow b)\ l\!\rightarrow c \le ((b\ l\!\rightarrow a)\ l\!\rightarrow c)\ l\!\rightarrow c$
⟨*proof*⟩

lemma *prod-2*: $((a::'a\ N)\ r\!\rightarrow b)\ r\!\rightarrow c \le ((b\ r\!\rightarrow a)\ r\!\rightarrow c)\ r\!\rightarrow c$
⟨*proof*⟩

lemma *prod-3*: $(b::'a\ N)\ l\!\rightarrow b * b \le a \sqcap (a\ l\!\rightarrow b)\ l\!\rightarrow b$
⟨*proof*⟩

lemma *prod-4*: $(b::'a\ N)\ r\!\rightarrow b * b \le a \sqcap (a\ r\!\rightarrow b)\ r\!\rightarrow b$
⟨*proof*⟩

lemma *prod-5*: $(((a::'a\ N)\ l\!\rightarrow b)\ l\!\rightarrow b) * (c * a\ l\!\rightarrow f * a) * (c * b\ l\!\rightarrow f * b) \le c\ l\!\rightarrow f$
⟨*proof*⟩

lemma *prod-6*: $(((a::'a\ N)\ r\!\rightarrow b)\ r\!\rightarrow b) * (a * c\ r\!\rightarrow a * f) * (b * c\ r\!\rightarrow b * f) \le c\ r\!\rightarrow f$
⟨*proof*⟩

instance
⟨*proof*⟩

end

definition *OrdSum* =
$\{x.\ (\exists\, a::'a::pseudo\text{-}hoop\text{-}algebra.\ x = (a,\ 1::'b::pseudo\text{-}hoop\text{-}algebra)) \lor (\exists\, b::'b.\ x = (1::'a,\ b))\}$

typedef (**overloaded**) $('a,\ 'b)$ *OrdSum* = *OrdSum* :: $('a$::*pseudo-hoop-algebra* $\times$ $'b$::*pseudo-hoop-algebra*) *set*
⟨*proof*⟩

lemma [*simp*]: $(1,\ b) \in$ *OrdSum*
⟨*proof*⟩

lemma [*simp*]: $(a,\ 1) \in$ *OrdSum*
⟨*proof*⟩

definition
first x = *fst* (*Rep-OrdSum* x)

definition
second x = *snd* (*Rep-OrdSum* x)

lemma *if-unfold-left*: ((*if* a *then* b *else* c) = d) = (($a \longrightarrow b = d$) $\wedge$ ($\neg\ a \longrightarrow c = d$))
⟨*proof*⟩

lemma *if-unfold-right*: (d = (*if* a *then* b *else* c)) = (($a \longrightarrow d = b$) $\wedge$ ($\neg\ a \longrightarrow d = c$))
⟨*proof*⟩

lemma *fst-snd-eq*: *fst* $a = x \Longrightarrow$ *snd* $a = y \Longrightarrow (x,\ y) = a$
⟨*proof*⟩

instantiation *OrdSum* :: (*pseudo-hoop-algebra*, *pseudo-hoop-algebra*) *pseudo-hoop-algebra*
begin

definition
times-OrdSum-def: $a * b \equiv$ (
if second $a = 1 \wedge$ *second* $b = 1$ *then*
Abs-OrdSum (*first* a $*$ *first* b, 1)
else if first $a = 1 \wedge$ *first* $b = 1$ *then*
Abs-OrdSum (1, *second* a $*$ *second* b)
else if first $a = 1 \wedge$ *second* $b = 1$ *then*
b
else
a)

definition
one-OrdSum-def: $1 \equiv$ *Abs-OrdSum* $(1,\ 1)$

definition
impl-OrdSum-def: a $l\rightarrow$ $b \equiv$
(*if second* $a = 1 \wedge$ *second* $b = 1$ *then*
Abs-OrdSum (*first* a $l\rightarrow$ *first* b, 1)

else if first a = *1* ∧ *first b* = *1 then*
Abs-OrdSum (*1*, *second a l*→ *second b*)
else if first a = *1* ∧ *second b* = *1 then*
b
else
1)

definition
impr-OrdSum-def: *a r*→ *b* ≡
(*if second a* = *1* ∧ *second b* = *1 then*
Abs-OrdSum (*first a r*→ *first b*, *1*)
else if first a = *1* ∧ *first b* = *1 then*
Abs-OrdSum (*1*, *second a r*→ *second b*)
else if first a = *1* ∧ *second b* = *1 then*
b
else
1)

definition
order-OrdSum-def: ((*a*::(′*a*, ′*b*) *OrdSum*) ≤ *b*) ≡ (*a l*→ *b* = *1*)

definition
inf-OrdSum-def: (*a*::(′*a*, ′*b*) *OrdSum*) ⊓ *b* = (*a l*→ *b*) ∗ *a*

definition
strict-order-OrdSum-def: (*a*::(′*a*, ′*b*) *OrdSum*) < *b* ≡ (*a* ≤ *b* ∧ ¬ *b* ≤ *a*)

lemma *OrdSum-first* [*simp*]: (*a*, *1*) ∈ *OrdSum*
⟨*proof*⟩

lemma *OrdSum-second* [*simp*]: (*1*, *b*) ∈ *OrdSum*
⟨*proof*⟩

lemma *Rep-OrdSum-eq*: *Rep-OrdSum x* = *Rep-OrdSum y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *Abs-OrdSum-eq*: *x* ∈ *OrdSum* ⟹ *y* ∈ *OrdSum* ⟹ *Abs-OrdSum x* = *Abs-OrdSum y* ⟹ *x* = *y*
⟨*proof*⟩

lemma [*simp*]: *fst* (*Rep-OrdSum a*) ≠ *1* ⟹ (*snd* (*Rep-OrdSum a*) ≠ *1* = *False*)
⟨*proof*⟩

lemma *fst-not-one-snd*: *fst* (*Rep-OrdSum a*) ≠ *1* ⟹ (*snd* (*Rep-OrdSum a*) = *1*)
⟨*proof*⟩

lemma *snd-not-one-fst*: *snd* (*Rep-OrdSum a*) ≠ *1* ⟹ (*fst* (*Rep-OrdSum a*) = *1*)
⟨*proof*⟩

lemma *fst-not-one-simp* [*simp*]: *fst* (*Rep-OrdSum* *c*) $\neq$ *1* $\Longrightarrow$ *Abs-OrdSum* (*fst* (*Rep-OrdSum* *c*), *1*) = *c*
⟨*proof*⟩

lemma *snd-not-one-simp* [*simp*]: *snd* (*Rep-OrdSum* *c*) $\neq$ *1* $\Longrightarrow$ *Abs-OrdSum* (*1*, *snd* (*Rep-OrdSum* *c*)) = *c*
⟨*proof*⟩

lemma *A*: **fixes** *a b*::(′*a*, ′*b*) *OrdSum* **shows** (*a l*→ *b*) ∗ *a* = *a* ∗ (*a r*→ *b*)
⟨*proof*⟩

instance
⟨*proof*⟩

definition
Second = {*x* . ∃ *b* . *x* = *Abs-OrdSum*(*1*::′*a*, *b*::′*b*)}

end

lemma *Second* ∈ *normalfilters*
⟨*proof*⟩

class *linear-pseudo-hoop-algebra* = *pseudo-hoop-algebra* + *linorder*

instantiation *OrdSum* :: (*linear-pseudo-hoop-algebra*, *linear-pseudo-hoop-algebra*) *linear-pseudo-hoop-algebra*
begin
instance
⟨*proof*⟩
end

instantiation *bool*:: *pseudo-hoop-algebra*
begin
definition *impl-bool-def*:
a l→ *b* = (*a* ⟶ *b*)

definition *impr-bool-def*:
a r→ *b* = (*a* ⟶ *b*)

definition *one-bool-def*:
1 = *True*

definition *times-bool-def*:
a ∗ *b* = (*a* ∧ *b*)

lemma *inf-bool-def*: (*a* :: *bool*) ⊓ *b* = (*a l*→ *b*) ∗ *a*
⟨*proof*⟩

instance
  ⟨*proof*⟩

end

context *cancel-pseudo-hoop-algebra* **begin end**

lemma $\neg$ *class.cancel-pseudo-hoop-algebra* $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(\leq)$ $(<)$ $(1{::}\ bool)$ $(r\rightarrow)$
  ⟨*proof*⟩

context *pseudo-hoop-algebra* **begin**
lemma *classorder*: *class.order* $(\leq)$ $(<)$
  ⟨*proof*⟩
end

lemma *impl-OrdSum-first*: *Abs-OrdSum* $(x,\ 1)$ $l\rightarrow$ *Abs-OrdSum* $(y,\ 1) =$ *Abs-OrdSum* $(x\ l\rightarrow\ y,\ 1)$
  ⟨*proof*⟩

lemma *impl-OrdSum-second*: *Abs-OrdSum* $(1,\ x)$ $l\rightarrow$ *Abs-OrdSum* $(1,\ y) =$ *Abs-OrdSum* $(1,\ x\ l\rightarrow\ y)$
  ⟨*proof*⟩

lemma *impl-OrdSum-first-second*: $x \neq 1 \Longrightarrow$ *Abs-OrdSum* $(x,\ 1)$ $l\rightarrow$ *Abs-OrdSum* $(1,\ y) = 1$
  ⟨*proof*⟩

lemma *Abs-OrdSum-bijective*: $x \in$ *OrdSum* $\Longrightarrow y \in$ *OrdSum* $\Longrightarrow$ (*Abs-OrdSum* x $=$ *Abs-OrdSum* $y) = (x = y)$
  ⟨*proof*⟩

context *pseudo-hoop-algebra* **begin end**

context *linear-pseudo-hoop-algebra* **begin end**
context *basic-pseudo-hoop-algebra* **begin end**

lemma *class.pseudo-hoop-algebra* $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(\leq)$ $(<)$ $(1{::}'a{::}pseudo\text{-}hoop\text{-}algebra)$ $(r\rightarrow)$
  $\Longrightarrow \neg$ (*class.linear-pseudo-hoop-algebra* $(\leq)$ $(<)$ $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(1{::}'a)$ $(r\rightarrow))$
  $\Longrightarrow \neg$ *class.basic-pseudo-hoop-algebra* $(*)$ $(\sqcap)$ $(l\rightarrow)$ $(\leq)$ $(<)$ $(1{::}('a,\ bool)$ *OrdSum*) $(r\rightarrow)$
  ⟨*proof*⟩

end

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