

Pseudo-hoops

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Abstract

Pseudo-hoops are algebraic structures introduced in [1, 2] by B. Bosbach under the name of complementary semigroups. This is a formalization of the paper [4]. Following [4] we prove some properties of pseudo-hoops and we define the basic concepts of filter and normal filter. The lattice of normal filters is isomorphic with the lattice of congruences of a pseudo-hoop. We also study some important classes of pseudo-hoops. Bounded Wajsberg pseudo-hoops are equivalent to pseudo-Wajsberg algebras and bounded basic pseudo-hoops are equivalent to pseudo-BL algebras. Some examples of pseudo-hoops are given in the last section of the formalization.

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1 Overview

Section 2 introduces some operations and their infix syntax. Section 3 and 4 introduces some facts about residuated and complemented monoids. Section

5 introduces the pseudo-hoops and some of their properties. Section 6 introduces filters and normal filters and proves that the lattice of normal filters and the lattice of congruences are isomorphic. Following [3], section 7 introduces pseudo-Waisberg algebras and some of their properties. In Section 8 we investigate some classes of pseudo-hoops. Finally section 9 presents some examples of pseudo-hoops and normal filters.

2 Operations

```

theory Operations
imports Main
begin

class left-imp =
  fixes imp-l :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr l  $\rightarrow$  65)

class right-imp =
  fixes imp-r :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr r  $\rightarrow$  65)

class left-uminus =
  fixes uminus-l :: 'a  $\Rightarrow$  'a (-l - [81] 80)

class right-uminus =
  fixes uminus-r :: 'a  $\Rightarrow$  'a (-r - [81] 80)

end

```

3 Left Complemented Monoid

```

theory LeftComplementedMonoid
  imports Operations LatticeProperties.Lattice-Prop
begin

class right-pordered-monoid-mult = order + monoid-mult +
  assumes mult-right-mono:  $a \leq b \implies a * c \leq b * c$ 

class one-greatest = one + ord +
  assumes one-greatest [simp]:  $a \leq 1$ 

class integral-right-pordered-monoid-mult = right-pordered-monoid-mult + one-greatest

class left-residuated = ord + times + left-imp +
  assumes left-residual:  $(x * a \leq b) = (x \leq a \mathit{l} \rightarrow b)$ 

class left-residuated-pordered-monoid = integral-right-pordered-monoid-mult + left-residuated

class left-inf = inf + times + left-imp +
  assumes inf-l-def:  $(a \sqcap b) = (a \mathit{l} \rightarrow b) * a$ 

```

```

class left-complemented-monoid = left-residuated-pordered-monoid + left-inf +
  assumes right-divisibility:  $(a \leq b) = (\exists c . a = c * b)$ 
begin
lemma lcm-D:  $a \text{ l}\rightarrow a = 1$ 
   $\langle \text{proof} \rangle$ 

subclass semilattice-inf
   $\langle \text{proof} \rangle$ 

lemma left-one-inf [simp]:  $1 \sqcap a = a$ 
   $\langle \text{proof} \rangle$ 

lemma left-one-impl [simp]:  $1 \text{ l}\rightarrow a = a$ 
   $\langle \text{proof} \rangle$ 

lemma lcm-A:  $(a \text{ l}\rightarrow b) * a = (b \text{ l}\rightarrow a) * b$ 
   $\langle \text{proof} \rangle$ 

lemma lcm-B:  $((a * b) \text{ l}\rightarrow c) = (a \text{ l}\rightarrow (b \text{ l}\rightarrow c))$ 
   $\langle \text{proof} \rangle$ 

lemma lcm-C:  $(a \leq b) = ((a \text{ l}\rightarrow b) = 1)$ 
   $\langle \text{proof} \rangle$ 

end

class less-def = ord +
  assumes less-def:  $(a < b) = ((a \leq b) \wedge \neg (b \leq a))$ 

class one-times = one + times +
  assumes one-left [simp]:  $1 * a = a$ 
  and one-right [simp]:  $a * 1 = a$ 

class left-complemented-monoid-algebra = left-imp + one-times + left-inf + less-def
+
  assumes left-impl-one [simp]:  $a \text{ l}\rightarrow a = 1$ 
  and left-impl-times:  $(a \text{ l}\rightarrow b) * a = (b \text{ l}\rightarrow a) * b$ 
  and left-impl-ded:  $((a * b) \text{ l}\rightarrow c) = (a \text{ l}\rightarrow (b \text{ l}\rightarrow c))$ 
  and left-lesseq:  $(a \leq b) = ((a \text{ l}\rightarrow b) = 1)$ 
begin
lemma A:  $(1 \text{ l}\rightarrow a) \text{ l}\rightarrow 1 = 1$ 
   $\langle \text{proof} \rangle$ 

subclass order
   $\langle \text{proof} \rangle$ 

```

lemma B: $(1 \text{ l}\rightarrow a) \leq 1$

<proof>

lemma C: $a \leq (1 \text{ l}\rightarrow a)$

<proof>

lemma D: $a \leq 1$

<proof>

lemma less-eq: $(a \leq b) = (\exists c . a = (c * b))$

<proof>

lemma F: $(a * b) * c \text{ l}\rightarrow z = a * (b * c) \text{ l}\rightarrow z$

<proof>

lemma associativity: $(a * b) * c = a * (b * c)$

<proof>

lemma H: $a * b \leq b$

<proof>

lemma I: $a * b \text{ l}\rightarrow b = 1$

<proof>

lemma K: $a \leq b \implies a * c \leq b * c$

<proof>

lemma L: $(x * a \leq b) = (x \leq a \text{ l}\rightarrow b)$

<proof>

subclass left-complemented-monoid

<proof>

end

lemma (in left-complemented-monoid) left-complemented-monoid:

class.left-complemented-monoid-algebra () inf (l→) (≤) (<) 1*

<proof>

end

4 Right Complemented Monoid

theory RightComplementedMonoid

```

imports LeftComplementedMonoid
begin

class left-pordered-monoid-mult = order + monoid-mult +
  assumes mult-left-mono:  $a \leq b \implies c * a \leq c * b$ 

class integral-left-pordered-monoid-mult = left-pordered-monoid-mult + one-greatest

class right-residuated = ord + times + right-imp +
  assumes right-residual:  $(a * x \leq b) = (x \leq a \text{ r}\rightarrow b)$ 

class right-residuated-pordered-monoid = integral-left-pordered-monoid-mult + right-residuated

class right-inf = inf + times + right-imp +
  assumes inf-r-def:  $(a \sqcap b) = a * (a \text{ r}\rightarrow b)$ 

class right-complemented-monoid = right-residuated-pordered-monoid + right-inf
+
  assumes left-divisibility:  $(a \leq b) = (\exists c . a = b * c)$ 

sublocale right-complemented-monoid < dual: left-complemented-monoid  $\lambda a b .$ 
 $b * a (\sqcap) (\text{r}\rightarrow) 1 (\leq) (<)$ 
   $\langle \text{proof} \rangle$ 

context right-complemented-monoid begin
lemma rcm-D:  $a \text{ r}\rightarrow a = 1$ 
   $\langle \text{proof} \rangle$ 

subclass semilattice-inf
   $\langle \text{proof} \rangle$ 

lemma right-semilattice-inf: class.semilattice-inf inf  $(\leq) (<)$ 
   $\langle \text{proof} \rangle$ 

lemma right-one-inf [simp]:  $1 \sqcap a = a$ 
   $\langle \text{proof} \rangle$ 

lemma right-one-impl [simp]:  $1 \text{ r}\rightarrow a = a$ 
   $\langle \text{proof} \rangle$ 

lemma rcm-A:  $a * (a \text{ r}\rightarrow b) = b * (b \text{ r}\rightarrow a)$ 
   $\langle \text{proof} \rangle$ 

lemma rcm-B:  $((b * a) \text{ r}\rightarrow c) = (a \text{ r}\rightarrow (b \text{ r}\rightarrow c))$ 
   $\langle \text{proof} \rangle$ 

lemma rcm-C:  $(a \leq b) = ((a \text{ r}\rightarrow b) = 1)$ 
   $\langle \text{proof} \rangle$ 
end

```

```

class right-complemented-monoid-nole-algebra = right-imp + one-times + right-inf
+ less-def +
  assumes right-impl-one [simp]:  $a \ r \rightarrow \ a = 1$ 
  and right-impl-times:  $a * (a \ r \rightarrow \ b) = b * (b \ r \rightarrow \ a)$ 
  and right-impl-ded:  $((a * b) \ r \rightarrow \ c) = (b \ r \rightarrow \ (a \ r \rightarrow \ c))$ 

```

```

class right-complemented-monoid-algebra = right-complemented-monoid-nole-algebra
+
  assumes right-lesseq:  $(a \leq b) = ((a \ r \rightarrow \ b) = 1)$ 
begin
end

```

```

sublocale right-complemented-monoid-algebra < dual-algebra: left-complemented-monoid-algebra
 $\lambda \ a \ b . \ b * a \ \text{inf} \ (r \rightarrow) \ (\leq) \ (<) \ 1$ 
  <proof>

```

```

context right-complemented-monoid-algebra begin

```

```

subclass right-complemented-monoid
  <proof>
end

```

```

lemma (in right-complemented-monoid) right-complemented-monoid: class.right-complemented-monoid-algebra
 $(\leq) \ (<) \ 1 \ (*) \ \text{inf} \ (r \rightarrow)$ 
  <proof>

```

```

end

```

5 Pseudo-Hoops

```

theory PseudoHoops
imports RightComplementedMonoid
begin

```

```

lemma drop-assumption:
   $p \implies \ \text{True}$ 
  <proof>

```

```

class pseudo-hoop-algebra = left-complemented-monoid-algebra + right-complemented-monoid-nole-algebra
+
  assumes left-right-impl-times:  $(a \ l \rightarrow \ b) * a = a * (a \ r \rightarrow \ b)$ 
begin
  definition
     $\text{inf-rr} \ a \ b = a * (a \ r \rightarrow \ b)$ 

  definition

```

$lesseq-r\ a\ b = (a\ r\rightarrow\ b = 1)$

definition

$less-r\ a\ b = (lesseq-r\ a\ b \wedge \neg\ lesseq-r\ b\ a)$

end

context *pseudo-hoop-algebra* **begin**

lemma *right-complemented-monoid-algebra*: *class.right-complemented-monoid-algebra*
 $lesseq-r\ less-r\ 1\ (*)\ inf-rr\ (r\rightarrow)$

$\langle proof \rangle$

lemma *inf-rr-inf* [*simp*]: $inf-rr = (\sqcap)$

$\langle proof \rangle$

lemma *lesseq-lesseq-r*: $lesseq-r\ a\ b = (a \leq b)$

$\langle proof \rangle$

lemma [*simp*]: $lesseq-r = (\leq)$

$\langle proof \rangle$

lemma [*simp*]: $less-r = (<)$

$\langle proof \rangle$

subclass *right-complemented-monoid-algebra*

$\langle proof \rangle$

end

sublocale *pseudo-hoop-algebra* $<$

pseudo-hoop-dual: *pseudo-hoop-algebra* $\lambda\ a\ b . b * a\ (\sqcap)\ (r\rightarrow)\ (\leq)\ (<)\ 1\ (l\rightarrow)$

$\langle proof \rangle$

context *pseudo-hoop-algebra* **begin**

lemma *commutative-ps*: $(\forall\ a\ b . a * b = b * a) = ((l\rightarrow) = (r\rightarrow))$

$\langle proof \rangle$

lemma *lemma-2-4-5*: $a\ l\rightarrow\ b \leq (c\ l\rightarrow\ a)\ l\rightarrow\ (c\ l\rightarrow\ b)$

$\langle proof \rangle$

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-4-6*: $a\ r\rightarrow\ b \leq (c\ r\rightarrow\ a)\ r\rightarrow\ (c\ r\rightarrow\ b)$

$\langle proof \rangle$

primrec

imp-power-l:: 'a => nat => 'a => 'a ((-) l-(-)→ (-) [65,0,65] 65) **where**
a l-0→ b = b |
a l-(Suc n)→ b = (a l→ (a l-n→ b))

primrec

imp-power-r:: 'a => nat => 'a => 'a ((-) r-(-)→ (-) [65,0,65] 65) **where**
a r-0→ b = b |
a r-(Suc n)→ b = (a r→ (a r-n→ b))

lemma *lemma-2-4-7-a*: a l-n→ b = a ^ n l→ b
{proof}

lemma *lemma-2-4-7-b*: a r-n→ b = a ^ n r→ b
{proof}

lemma *lemma-2-5-8-a* [*simp*]: a * b ≤ a
{proof}

lemma *lemma-2-5-8-b* [*simp*]: a * b ≤ b
{proof}

lemma *lemma-2-5-9-a*: a ≤ b l→ a
{proof}

lemma *lemma-2-5-9-b*: a ≤ b r→ a
{proof}

lemma *lemma-2-5-11*: a * b ≤ a □ b
{proof}

lemma *lemma-2-5-12-a*: a ≤ b ⇒ c l→ a ≤ c l→ b
{proof}

lemma *lemma-2-5-13-a*: a ≤ b ⇒ b l→ c ≤ a l→ c
{proof}

lemma *lemma-2-5-14*: (b l→ c) * (a l→ b) ≤ a l→ c
{proof}

lemma *lemma-2-5-16*: (a l→ b) ≤ (b l→ c) r→ (a l→ c)
{proof}

lemma *lemma-2-5-18*: (a l→ b) ≤ a * c l→ b * c
{proof}

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-5-12-b*: $a \leq b \implies c \ r \rightarrow a \leq c \ r \rightarrow b$
<proof>

lemma *lemma-2-5-13-b*: $a \leq b \implies b \ r \rightarrow c \leq a \ r \rightarrow c$
<proof>

lemma *lemma-2-5-15*: $(a \ r \rightarrow b) * (b \ r \rightarrow c) \leq a \ r \rightarrow c$
<proof>

lemma *lemma-2-5-17*: $(a \ r \rightarrow b) \leq (b \ r \rightarrow c) \ l \rightarrow (a \ r \rightarrow c)$
<proof>

lemma *lemma-2-5-19*: $(a \ r \rightarrow b) \leq c * a \ r \rightarrow c * b$
<proof>

definition

lower-bound $A = \{a . \forall x \in A . a \leq x\}$

definition

infimum $A = \{a \in \text{lower-bound } A . (\forall x \in \text{lower-bound } A . x \leq a)\}$

lemma *infimum-unique*: $(\text{infimum } A = \{x\}) = (x \in \text{infimum } A)$
<proof>

lemma *lemma-2-6-20*:

$a \in \text{infimum } A \implies b \ l \rightarrow a \in \text{infimum } (((l \rightarrow) b) 'A)$
<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-6-21*:

$a \in \text{infimum } A \implies b \ r \rightarrow a \in \text{infimum } (((r \rightarrow) b) 'A)$
<proof>

lemma *infimum-pair*: $a \in \text{infimum } \{x . x = b \vee x = c\} = (a = b \sqcap c)$
<proof>

lemma *lemma-2-6-20-a*:

$a \ l \rightarrow (b \sqcap c) = (a \ l \rightarrow b) \sqcap (a \ l \rightarrow c)$
<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-6-21-a*:

$$a \rightarrow (b \sqcap c) = (a \rightarrow b) \sqcap (a \rightarrow c)$$

<proof>

lemma *mult-mono*: $a \leq b \implies c \leq d \implies a * c \leq b * d$

<proof>

lemma *lemma-2-7-22*: $(a \rightarrow b) * (c \rightarrow d) \leq (a \sqcap c) \rightarrow (b \sqcap d)$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-7-23*: $(a \rightarrow b) * (c \rightarrow d) \leq (a \sqcap c) \rightarrow (b \sqcap d)$

<proof>

definition

$$\text{upper-bound } A = \{a . \forall x \in A . x \leq a\}$$

definition

$$\text{supremum } A = \{a \in \text{upper-bound } A . (\forall x \in \text{upper-bound } A . a \leq x)\}$$

lemma *supremum-unique*:

$$a \in \text{supremum } A \implies b \in \text{supremum } A \implies a = b$$

<proof>

lemma *lemma-2-8-i*:

$$a \in \text{supremum } A \implies a \rightarrow b \in \text{infimum } ((\lambda x . x \rightarrow b)'A)$$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-8-i1*:

$$a \in \text{supremum } A \implies a \rightarrow b \in \text{infimum } ((\lambda x . x \rightarrow b)'A)$$

<proof>

definition

times-set :: 'a set \Rightarrow 'a set \Rightarrow 'a set (**infixl** ** 70) **where**

$$(A ** B) = \{a . \exists x \in A . \exists y \in B . a = x * y\}$$

lemma *times-set-assoc*: $A ** (B ** C) = (A ** B) ** C$

<proof>

primrec *power-set* :: 'a set \Rightarrow nat \Rightarrow 'a set (**infixr** *[^] 80) **where**

$$\text{power-set-0}: (A *[^] 0) = \{1\}$$

| *power-set-Suc*: $(A *^\wedge (\text{Suc } n)) = (A ** (A *^\wedge n))$

lemma *infimum-singleton* [*simp*]: $\text{infimum } \{a\} = \{a\}$
<proof>

lemma *lemma-2-8-ii*:
 $a \in \text{supremum } A \implies (a \wedge n) \text{ l} \rightarrow b \in \text{infimum } ((\lambda x . x \text{ l} \rightarrow b) \text{ ` } (A *^\wedge n))$
<proof>

lemma *power-set-Suc2*:
 $A *^\wedge (\text{Suc } n) = A *^\wedge n ** A$
<proof>

lemma *power-set-add*:
 $A *^\wedge (n + m) = (A *^\wedge n) ** (A *^\wedge m)$
<proof>
end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-8-ii1*:
 $a \in \text{supremum } A \implies (a \wedge n) \text{ r} \rightarrow b \in \text{infimum } ((\lambda x . x \text{ r} \rightarrow b) \text{ ` } (A *^\wedge n))$
<proof>

lemma *lemma-2-9-i*:
 $b \in \text{supremum } A \implies a * b \in \text{supremum } ((* \text{ ` } a \text{ ` } A)$
<proof>

lemma *lemma-2-9-i1*:
 $b \in \text{supremum } A \implies b * a \in \text{supremum } ((\lambda x . x * a) \text{ ` } A)$
<proof>

lemma *lemma-2-9-ii*:
 $b \in \text{supremum } A \implies a \sqcap b \in \text{supremum } ((\sqcap) \text{ ` } a \text{ ` } A)$
<proof>

lemma *lemma-2-10-24*:
 $a \leq (a \text{ l} \rightarrow b) \text{ r} \rightarrow b$
<proof>

lemma *lemma-2-10-25*:
 $a \leq (a \text{ l} \rightarrow b) \text{ r} \rightarrow a$
<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-26*:

$$a \leq (a \ r \rightarrow \ b) \ l \rightarrow \ b$$

<proof>

lemma *lemma-2-10-27*:

$$a \leq (a \ r \rightarrow \ b) \ l \rightarrow \ a$$

<proof>

lemma *lemma-2-10-28*:

$$b \ l \rightarrow \ ((a \ l \rightarrow \ b) \ r \rightarrow \ a) = b \ l \rightarrow \ a$$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-29*:

$$b \ r \rightarrow \ ((a \ r \rightarrow \ b) \ l \rightarrow \ a) = b \ r \rightarrow \ a$$

<proof>

lemma *lemma-2-10-30*:

$$((b \ l \rightarrow \ a) \ r \rightarrow \ a) \ l \rightarrow \ a = b \ l \rightarrow \ a$$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-31*:

$$((b \ r \rightarrow \ a) \ l \rightarrow \ a) \ r \rightarrow \ a = b \ r \rightarrow \ a$$

<proof>

lemma *lemma-2-10-32*:

$$(((b \ l \rightarrow \ a) \ r \rightarrow \ a) \ l \rightarrow \ b) \ l \rightarrow \ (b \ l \rightarrow \ a) = b \ l \rightarrow \ a$$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *lemma-2-10-33*:

$$(((b \ r \rightarrow \ a) \ l \rightarrow \ a) \ r \rightarrow \ b) \ r \rightarrow \ (b \ r \rightarrow \ a) = b \ r \rightarrow \ a$$

<proof>

end

class *pseudo-hoop-sup-algebra* = *pseudo-hoop-algebra* + *sup* +
assumes

```

    sup-comute:  $a \sqcup b = b \sqcup a$ 
    and sup-le [simp]:  $a \leq a \sqcup b$ 
    and le-sup-equiv:  $(a \leq b) = (a \sqcup b = b)$ 
begin
  lemma sup-le-2 [simp]:
     $b \leq a \sqcup b$ 
    <proof>

  lemma le-sup-equiv-r:
     $(a \sqcup b = b) = (a \leq b)$ 
    <proof>

  lemma sup-idemp [simp]:
     $a \sqcup a = a$ 
    <proof>
end

class pseudo-hoop-sup1-algebra = pseudo-hoop-algebra + sup +
  assumes
    sup-def:  $a \sqcup b = ((a \text{ l}\rightarrow b) \text{ r}\rightarrow b) \sqcap ((b \text{ l}\rightarrow a) \text{ r}\rightarrow a)$ 
begin

  lemma sup-comute1:  $a \sqcup b = b \sqcup a$ 
    <proof>

  lemma sup-le1 [simp]:  $a \leq a \sqcup b$ 
    <proof>

  lemma le-sup-equiv1:  $(a \leq b) = (a \sqcup b = b)$ 
    <proof>

  subclass pseudo-hoop-sup-algebra
    <proof>
end

class pseudo-hoop-sup2-algebra = pseudo-hoop-algebra + sup +
  assumes
    sup-2-def:  $a \sqcup b = ((a \text{ r}\rightarrow b) \text{ l}\rightarrow b) \sqcap ((b \text{ r}\rightarrow a) \text{ l}\rightarrow a)$ 

context pseudo-hoop-sup1-algebra begin end

sublocale pseudo-hoop-sup2-algebra < sup1-dual: pseudo-hoop-sup1-algebra ( $\sqcup$ )  $\lambda$ 
  a b . b * a ( $\sqcap$ ) ( $\text{r}\rightarrow$ ) ( $\leq$ ) ( $<$ ) 1 ( $\text{l}\rightarrow$ )
  <proof>

context pseudo-hoop-sup2-algebra begin

  lemma sup-comute-2:  $a \sqcup b = b \sqcup a$ 

```

```

    <proof>

lemma sup-le2 [simp]:  $a \leq a \sqcup b$ 
  <proof>

lemma le-sup-equiv2:  $(a \leq b) = (a \sqcup b = b)$ 
  <proof>

subclass pseudo-hoop-sup-algebra
  <proof>

end

class pseudo-hoop-lattice-a = pseudo-hoop-sup-algebra +
  assumes sup-inf-le-distr:  $a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap (a \sqcup c)$ 
begin
  lemma sup-lower-upper-bound [simp]:
     $a \leq c \implies b \leq c \implies a \sqcup b \leq c$ 
    <proof>
end

sublocale pseudo-hoop-lattice-a < lattice ( $\sqcap$ ) ( $\leq$ ) ( $<$ ) ( $\sqcup$ )
  <proof>

class pseudo-hoop-lattice-b = pseudo-hoop-sup-algebra +
  assumes le-sup-cong:  $a \leq b \implies a \sqcup c \leq b \sqcup c$ 

begin
  lemma sup-lower-upper-bound-b [simp]:
     $a \leq c \implies b \leq c \implies a \sqcup b \leq c$ 
    <proof>

  lemma sup-inf-le-distr-b:
     $a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap (a \sqcup c)$ 
    <proof>
end

context pseudo-hoop-lattice-a begin end

sublocale pseudo-hoop-lattice-b < pseudo-hoop-lattice-a ( $\sqcup$ ) ( $*$ ) ( $\sqcap$ ) ( $l \rightarrow$ ) ( $\leq$ ) ( $<$ )
  1 ( $r \rightarrow$ )
  <proof>

class pseudo-hoop-lattice = pseudo-hoop-sup-algebra +
  assumes sup-assoc-1:  $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$ 
begin
  lemma le-sup-cong-c:
     $a \leq b \implies a \sqcup c \leq b \sqcup c$ 
    <proof>

```

end

sublocale *pseudo-hoop-lattice* < *pseudo-hoop-lattice-b* (\sqcup) (*) (\sqcap) ($l \rightarrow$) (\leq) (<)
1 ($r \rightarrow$)
<proof>

sublocale *pseudo-hoop-lattice* < *semilattice-sup* (\sqcup) (\leq) (<)
<proof>

sublocale *pseudo-hoop-lattice* < *lattice* (\sqcap) (\leq) (<) (\sqcup)
<proof>

lemma (in *pseudo-hoop-lattice-a*) *supremum-pair* [simp]:
 $\text{supremum } \{a, b\} = \{a \sqcup b\}$
<proof>

sublocale *pseudo-hoop-lattice* < *distrib-lattice* (\sqcap) (\leq) (<) (\sqcup)
<proof>

class *bounded-semilattice-inf-top* = *semilattice-inf* + *order-top*
begin

lemma *inf-eq-top-iff* [simp]:
 $(\text{inf } x \ y = \text{top}) = (x = \text{top} \wedge y = \text{top})$
<proof>

end

sublocale *pseudo-hoop-algebra* < *bounded-semilattice-inf-top* (\sqcap) (\leq) (<) 1
<proof>

definition (in *pseudo-hoop-algebra*)
 $\text{sup1}::'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup 1$ 70) **where**
 $a \sqcup 1 b = ((a \ l \rightarrow b) \ r \rightarrow b) \sqcap ((b \ l \rightarrow a) \ r \rightarrow a)$

sublocale *pseudo-hoop-algebra* < *sup1: pseudo-hoop-sup1-algebra* ($\sqcup 1$) (*) (\sqcap)
($l \rightarrow$) (\leq) (<) 1 ($r \rightarrow$)
<proof>

definition (in *pseudo-hoop-algebra*)
 $\text{sup2}::'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup 2$ 70) **where**
 $a \sqcup 2 b = ((a \ r \rightarrow b) \ l \rightarrow b) \sqcap ((b \ r \rightarrow a) \ l \rightarrow a)$

sublocale *pseudo-hoop-algebra* < *sup2: pseudo-hoop-sup2-algebra* ($\sqcup 2$) (*) (\sqcap)
($l \rightarrow$) (\leq) (<) 1 ($r \rightarrow$)
<proof>

context *pseudo-hoop-algebra*

begin

lemma *lemma-2-15-i*:

$1 \in \text{supremum } \{a, b\} \implies a * b = a \sqcap b$
<proof>

lemma *lemma-2-15-ii*:

$1 \in \text{supremum } \{a, b\} \implies a \leq c \implies b \leq d \implies 1 \in \text{supremum } \{c, d\}$
<proof>

lemma *sup-union*:

$a \in \text{supremum } A \implies b \in \text{supremum } B \implies \text{supremum } \{a, b\} = \text{supremum } (A \cup B)$
<proof>

lemma *sup-singleton [simp]*: $a \in \text{supremum } \{a\}$

<proof>

lemma *sup-union-singleton*: $a \in \text{supremum } X \implies \text{supremum } \{a, b\} = \text{supremum } (X \cup \{b\})$

<proof>

lemma *sup-le-union [simp]*: $a \leq b \implies \text{supremum } (A \cup \{a, b\}) = \text{supremum } (A \cup \{b\})$

<proof>

lemma *sup-sup-union*: $a \in \text{supremum } A \implies b \in \text{supremum } (B \cup \{a\}) \implies b \in \text{supremum } (A \cup B)$

<proof>

end

lemma *[simp]*:

$n \leq 2^{\wedge} n$
<proof>

context *pseudo-hoop-algebra*

begin

lemma *sup-le-union-2*:

$a \leq b \implies a \in A \implies b \in A \implies \text{supremum } A = \text{supremum } ((A - \{a\}) \cup \{b\})$
<proof>

lemma *lemma-2-15-iii-0*:
 $1 \in \text{supremum } \{a, b\} \implies 1 \in \text{supremum } \{a \wedge 2, b \wedge 2\}$
 ⟨proof⟩

lemma [*simp*]: $m \leq n \implies a \wedge n \leq a \wedge m$
 ⟨proof⟩

lemma [*simp*]: $a \wedge (2 \wedge n) \leq a \wedge n$
 ⟨proof⟩

lemma *lemma-2-15-iii-1*: $1 \in \text{supremum } \{a, b\} \implies 1 \in \text{supremum } \{a \wedge (2 \wedge n), b \wedge (2 \wedge n)\}$
 ⟨proof⟩

lemma *lemma-2-15-iii*:
 $1 \in \text{supremum } \{a, b\} \implies 1 \in \text{supremum } \{a \wedge n, b \wedge n\}$
 ⟨proof⟩

end

end

6 Filters and Congruences

theory *PseudoHoopFilters*
imports *PseudoHoops*
begin

context *pseudo-hoop-algebra*
begin

definition
 $\text{filters} = \{F . F \neq \{\}\} \wedge (\forall a b . a \in F \wedge b \in F \longrightarrow a * b \in F) \wedge (\forall a b . a \in F \wedge a \leq b \longrightarrow b \in F)$

definition
 $\text{properfilters} = \{F . F \in \text{filters} \wedge F \neq \text{UNIV}\}$

definition
 $\text{maximalfilters} = \{F . F \in \text{filters} \wedge (\forall A . A \in \text{filters} \wedge F \subseteq A \longrightarrow A = F \vee A = \text{UNIV})\}$

definition
 $\text{ultrafilters} = \text{properfilters} \cap \text{maximalfilters}$

lemma *filter-i*: $F \in \text{filters} \implies a \in F \implies b \in F \implies a * b \in F$
 ⟨proof⟩

lemma *filter-ii*: $F \in \text{filters} \implies a \in F \implies a \leq b \implies b \in F$

$\langle proof \rangle$

lemma *filter-iii* [*simp*]: $F \in filters \implies 1 \in F$
 $\langle proof \rangle$

lemma *filter-left-impl*:
 $(F \in filters) = ((1 \in F) \wedge (\forall a b . a \in F \wedge a l \rightarrow b \in F \longrightarrow b \in F))$
 $\langle proof \rangle$

lemma *filter-right-impl*:
 $(F \in filters) = ((1 \in F) \wedge (\forall a b . a \in F \wedge a r \rightarrow b \in F \longrightarrow b \in F))$
 $\langle proof \rangle$

lemma [*simp*]: $A \subseteq filters \implies \bigcap A \in filters$
 $\langle proof \rangle$

definition
 $filterof X = \bigcap \{F . F \in filters \wedge X \subseteq F\}$

lemma [*simp*]: $filterof X \in filters$
 $\langle proof \rangle$

lemma *times-le-mono* [*simp*]: $x \leq y \implies u \leq v \implies x * u \leq y * v$
 $\langle proof \rangle$

lemma *prop-3-2-i*:
 $filterof X = \{a . \exists n x . x \in X * ^n \wedge x \leq a\}$
 $\langle proof \rangle$

lemma *ultrafilter-union*:
 $ultrafilters = \{F . F \in filters \wedge F \neq UNIV \wedge (\forall x . x \notin F \longrightarrow filterof (F \cup \{x\}) = UNIV)\}$
 $\langle proof \rangle$

lemma *filterof-sub*: $F \in filters \implies X \subseteq F \implies filterof X \subseteq F$
 $\langle proof \rangle$

lemma *filterof-elem* [*simp*]: $x \in X \implies x \in filterof X$
 $\langle proof \rangle$

lemma [*simp*]: $filterof X \in filters$
 $\langle proof \rangle$

lemma *singleton-power* [*simp*]: $\{a\} * ^n = \{b . b = a ^n\}$
 $\langle proof \rangle$

lemma *power-pair*: $x \in \{a, b\} * ^n \implies \exists i j . i + j = n \wedge x \leq a ^i \wedge x \leq b ^j$

<proof>

lemma *filterof-supremum*:

$c \in \text{supremum } \{a, b\} \implies \text{filterof } \{c\} = \text{filterof } \{a\} \cap \text{filterof } \{b\}$
<proof>

definition $d1$ a $b = (a \text{ l} \rightarrow b) * (b \text{ l} \rightarrow a)$

definition $d2$ a $b = (a \text{ r} \rightarrow b) * (b \text{ r} \rightarrow a)$

definition $d3$ a $b = d1$ b a

definition $d4$ a $b = d2$ b a

lemma [*simp*]: $(a * b = 1) = (a = 1 \wedge b = 1)$
<proof>

lemma *lemma-3-5-i-1*: $(d1$ a $b = 1) = (a = b)$
<proof>

lemma *lemma-3-5-i-2*: $(d2$ a $b = 1) = (a = b)$
<proof>

lemma *lemma-3-5-i-3*: $(d3$ a $b = 1) = (a = b)$
<proof>

lemma *lemma-3-5-i-4*: $(d4$ a $b = 1) = (a = b)$
<proof>

lemma *lemma-3-5-ii-1* [*simp*]: $d1$ a $a = 1$
<proof>

lemma *lemma-3-5-ii-2* [*simp*]: $d2$ a $a = 1$
<proof>

lemma *lemma-3-5-ii-3* [*simp*]: $d3$ a $a = 1$
<proof>

lemma *lemma-3-5-ii-4* [*simp*]: $d4$ a $a = 1$
<proof>

lemma [*simp*]: $(a \text{ l} \rightarrow 1) = 1$
<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma [*simp*]: $(a \text{ r} \rightarrow 1) = 1$

<proof>

lemma *lemma-3-5-iii-1 [simp]:* $d1\ a\ 1 = a$
<proof>

lemma *lemma-3-5-iii-2 [simp]:* $d2\ a\ 1 = a$
<proof>

lemma *lemma-3-5-iii-3 [simp]:* $d3\ a\ 1 = a$
<proof>

lemma *lemma-3-5-iii-4 [simp]:* $d4\ a\ 1 = a$
<proof>

lemma *lemma-3-5-iv-1:* $(d1\ b\ c) * (d1\ a\ b) * (d1\ b\ c) \leq d1\ a\ c$
<proof>

lemma *lemma-3-5-iv-2:* $(d2\ a\ b) * (d2\ b\ c) * (d2\ a\ b) \leq d2\ a\ c$
<proof>

lemma *lemma-3-5-iv-3:* $(d3\ a\ b) * (d3\ b\ c) * (d3\ a\ b) \leq d3\ a\ c$
<proof>

lemma *lemma-3-5-iv-4:* $(d4\ b\ c) * (d4\ a\ b) * (d4\ b\ c) \leq d4\ a\ c$
<proof>

definition

cong-r $F\ a\ b \equiv d1\ a\ b \in F$

definition

cong-l $F\ a\ b \equiv d2\ a\ b \in F$

lemma *cong-r-filter:* $F \in \text{filters} \implies (\text{cong-r}\ F\ a\ b) = (a\ l\rightarrow\ b \in F \wedge b\ l\rightarrow\ a \in F)$
<proof>

lemma *cong-r-symmetric:* $F \in \text{filters} \implies (\text{cong-r}\ F\ a\ b) = (\text{cong-r}\ F\ b\ a)$
<proof>

lemma *cong-r-d3:* $F \in \text{filters} \implies (\text{cong-r}\ F\ a\ b) = (d3\ a\ b \in F)$
<proof>

lemma *cong-l-filter:* $F \in \text{filters} \implies (\text{cong-l}\ F\ a\ b) = (a\ r\rightarrow\ b \in F \wedge b\ r\rightarrow\ a \in F)$
<proof>

lemma *cong-l-symmetric:* $F \in \text{filters} \implies (\text{cong-l}\ F\ a\ b) = (\text{cong-l}\ F\ b\ a)$

<proof>

lemma *cong-l-d4*: $F \in \text{filters} \implies (\text{cong-l } F \ a \ b) = (d4 \ a \ b \in F)$
<proof>

lemma *cong-r-reflexive*: $F \in \text{filters} \implies \text{cong-r } F \ a \ a$
<proof>

lemma *cong-r-transitive*: $F \in \text{filters} \implies \text{cong-r } F \ a \ b \implies \text{cong-r } F \ b \ c \implies \text{cong-r } F \ a \ c$
<proof>

lemma *cong-l-reflexive*: $F \in \text{filters} \implies \text{cong-l } F \ a \ a$
<proof>

lemma *cong-l-transitive*: $F \in \text{filters} \implies \text{cong-l } F \ a \ b \implies \text{cong-l } F \ b \ c \implies \text{cong-l } F \ a \ c$
<proof>

lemma *lemma-3-7-i*: $F \in \text{filters} \implies F = \{a . \text{cong-r } F \ a \ 1\}$
<proof>

lemma *lemma-3-7-ii*: $F \in \text{filters} \implies F = \{a . \text{cong-l } F \ a \ 1\}$
<proof>

lemma *lemma-3-8-i*: $F \in \text{filters} \implies (\text{cong-r } F \ a \ b) = (\exists \ x \ y . x \in F \wedge y \in F \wedge x * a = y * b)$
<proof>

lemma *lemma-3-8-ii*: $F \in \text{filters} \implies (\text{cong-l } F \ a \ b) = (\exists \ x \ y . x \in F \wedge y \in F \wedge a * x = b * y)$
<proof>

lemma *lemma-3-9-i*: $F \in \text{filters} \implies \text{cong-r } F \ a \ b \implies \text{cong-r } F \ c \ d \implies (a \ l \rightarrow c \in F) = (b \ l \rightarrow d \in F)$
<proof>

lemma *lemma-3-9-ii*: $F \in \text{filters} \implies \text{cong-l } F \ a \ b \implies \text{cong-l } F \ c \ d \implies (a \ r \rightarrow c \in F) = (b \ r \rightarrow d \in F)$
<proof>

definition

normalfilters = $\{F . F \in \text{filters} \wedge (\forall \ a \ b . (a \ l \rightarrow b \in F) = (a \ r \rightarrow b \in F))\}$

lemma *normalfilter-i*:

$H \in \text{normalfilters} \implies a \ l \rightarrow b \in H \implies a \ r \rightarrow b \in H$
<proof>

lemma *normalfilter-ii*:

$H \in \text{normalfilters} \implies a \ r \rightarrow b \in H \implies a \ l \rightarrow b \in H$
<proof>

lemma [*simp*]: $H \in \text{normalfilters} \implies H \in \text{filters}$

<proof>

lemma *lemma-3-10-i-ii*:

$H \in \text{normalfilters} \implies \{a\} ** H = H ** \{a\}$
<proof>

lemma *lemma-3-10-ii-iii*:

$H \in \text{filters} \implies (\forall a . \{a\} ** H = H ** \{a\}) \implies \text{cong-r } H = \text{cong-l } H$
<proof>

lemma *lemma-3-10-i-iii*:

$H \in \text{normalfilters} \implies \text{cong-r } H = \text{cong-l } H$
<proof>

lemma *order-impl-l* [*simp*]: $a \leq b \implies a \ l \rightarrow b = 1$

<proof>

end

context *pseudo-hoop-algebra* **begin**

lemma *impl-l-d1*: $(a \ l \rightarrow b) = d1 \ a \ (a \sqcap b)$

<proof>

lemma *impl-r-d2*: $(a \ r \rightarrow b) = d2 \ a \ (a \sqcap b)$

<proof>

lemma *lemma-3-10-iii-i*:

$H \in \text{filters} \implies \text{cong-r } H = \text{cong-l } H \implies H \in \text{normalfilters}$
<proof>

lemma *lemma-3-10-ii-i*:

$H \in \text{filters} \implies (\forall a . \{a\} ** H = H ** \{a\}) \implies H \in \text{normalfilters}$
<proof>

definition

allpowers $x \ n = \{y . \exists i . i < n \wedge y = x \wedge i\}$

lemma *times-set-in*: $a \in A \implies b \in B \implies c = a * b \implies c \in A ** B$

<proof>

lemma *power-set-power*: $a \in A \implies a \wedge n \in A * \wedge n$
 ⟨proof⟩

lemma *normal-filter-union*: $H \in \text{normalfilters} \implies (H \cup \{x\}) * \wedge n = (H ** (\text{allpowers } x \ n)) \cup \{x \wedge n\}$
 ⟨proof⟩

lemma *lemma-3-11-i*: $H \in \text{normalfilters} \implies \text{filterof } (H \cup \{x\}) = \{a . \exists n \ h . h \in H \wedge h * x \wedge n \leq a\}$
 ⟨proof⟩

lemma *lemma-3-11-ii*: $H \in \text{normalfilters} \implies \text{filterof } (H \cup \{x\}) = \{a . \exists n \ h . h \in H \wedge (x \wedge n) * h \leq a\}$
 ⟨proof⟩

lemma *lemma-3-12-i-ii*:
 $H \in \text{normalfilters} \implies H \in \text{ultrafilters} \implies x \notin H \implies (\exists n . x \wedge n \ l \rightarrow a \in H)$
 ⟨proof⟩

lemma *lemma-3-12-ii-i*:
 $H \in \text{normalfilters} \implies H \in \text{properfilters} \implies (\forall x \ a . x \notin H \longrightarrow (\exists n . x \wedge n \ l \rightarrow a \in H)) \implies H \in \text{maximalfilters}$
 ⟨proof⟩

lemma *lemma-3-12-i-iii*:
 $H \in \text{normalfilters} \implies H \in \text{ultrafilters} \implies x \notin H \implies (\exists n . x \wedge n \ r \rightarrow a \in H)$
 ⟨proof⟩

lemma *lemma-3-12-iii-i*:
 $H \in \text{normalfilters} \implies H \in \text{properfilters} \implies (\forall x \ a . x \notin H \longrightarrow (\exists n . x \wedge n \ r \rightarrow a \in H)) \implies H \in \text{maximalfilters}$
 ⟨proof⟩

definition
 $\text{cong } H = (\lambda x \ y . \text{cong-l } H \ x \ y \wedge \text{cong-r } H \ x \ y)$

definition
 $\text{congruences} = \{R . \text{equivp } R \wedge (\forall a \ b \ c \ d . R \ a \ b \wedge R \ c \ d \longrightarrow R \ (a * c) \ (b * d)) \wedge R \ (a \ l \rightarrow c) \ (b \ l \rightarrow d) \wedge R \ (a \ r \rightarrow c) \ (b \ r \rightarrow d)\}$

lemma *cong-l*: $H \in \text{normalfilters} \implies \text{cong } H = \text{cong-l } H$
 ⟨proof⟩

lemma *cong-r*: $H \in \text{normalfilters} \implies \text{cong } H = \text{cong-r } H$
 ⟨proof⟩

lemma *cong-equiv*: $H \in \text{normalfilters} \implies \text{equiv} (\text{cong } H)$
<proof>

lemma *cong-trans*: $H \in \text{normalfilters} \implies \text{cong } H \ x \ y \implies \text{cong } H \ y \ z \implies \text{cong } H \ x \ z$
<proof>

lemma *lemma-3-13* [*simp*]:
 $H \in \text{normalfilters} \implies \text{cong } H \in \text{congruences}$
<proof>

lemma *cong-times*: $R \in \text{congruences} \implies R \ a \ b \implies R \ c \ d \implies R \ (a * c) \ (b * d)$
<proof>

lemma *cong-impl-l*: $R \in \text{congruences} \implies R \ a \ b \implies R \ c \ d \implies R \ (a \ l \rightarrow c) \ (b \ l \rightarrow d)$
<proof>

lemma *cong-impl-r*: $R \in \text{congruences} \implies R \ a \ b \implies R \ c \ d \implies R \ (a \ r \rightarrow c) \ (b \ r \rightarrow d)$
<proof>

lemma *cong-refl* [*simp*]: $R \in \text{congruences} \implies R \ a \ a$
<proof>

lemma *cong-trans-a*: $R \in \text{congruences} \implies R \ a \ b \implies R \ b \ c \implies R \ a \ c$
<proof>

lemma *cong-sym*: $R \in \text{congruences} \implies R \ a \ b \implies R \ b \ a$
<proof>

definition
normalfilter $R = \{a \ . \ R \ a \ 1\}$

lemma *lemma-3-14* [*simp*]:
 $R \in \text{congruences} \implies (\text{normalfilter } R) \in \text{normalfilters}$
<proof>

lemma *lemma-3-15-i*:
 $H \in \text{normalfilters} \implies \text{normalfilter} (\text{cong } H) = H$
<proof>

lemma *lemma-3-15-ii*:
 $R \in \text{congruences} \implies \text{cong} (\text{normalfilter } R) = R$
<proof>

lemma *lemma-3-15-iii*: $H1 \in \text{normalfilters} \implies H2 \in \text{normalfilters} \implies (H1 \subseteq H2) = (\text{cong } H1 \leq \text{cong } H2)$

<proof>

definition

$$p\ x\ y\ z = ((x\ l\rightarrow\ y)\ r\rightarrow\ z) \sqcap ((z\ l\rightarrow\ y)\ r\rightarrow\ x)$$

lemma *lemma-3-16-i*: $p\ x\ x\ y = y \wedge p\ x\ y\ y = x$

<proof>

definition $M\ x\ y\ z = ((y\ l\rightarrow\ x)\ r\rightarrow\ x) \sqcap ((z\ l\rightarrow\ y)\ r\rightarrow\ y) \sqcap ((x\ l\rightarrow\ z)\ r\rightarrow\ z)$

lemma $M\ x\ x\ y = x \wedge M\ x\ y\ x = x \wedge M\ y\ x\ x = x$

<proof>

end

end

7 Pseudo Waisberg Algebra

theory *PseudoWaisbergAlgebra*

imports *Operations*

begin

class *impl-lr-algebra* = *one* + *left-imp* + *right-imp* +

assumes *W1a* [*simp*]: $1\ l\rightarrow\ a = a$

and *W1b* [*simp*]: $1\ r\rightarrow\ a = a$

and *W2a*: $(a\ l\rightarrow\ b)\ r\rightarrow\ b = (b\ l\rightarrow\ a)\ r\rightarrow\ a$

and *W2b*: $(b\ l\rightarrow\ a)\ r\rightarrow\ a = (b\ r\rightarrow\ a)\ l\rightarrow\ a$

and *W2c*: $(b\ r\rightarrow\ a)\ l\rightarrow\ a = (a\ r\rightarrow\ b)\ l\rightarrow\ b$

and *W3a*: $(a\ l\rightarrow\ b)\ l\rightarrow\ ((b\ l\rightarrow\ c)\ r\rightarrow\ (a\ l\rightarrow\ c)) = 1$

and *W3b*: $(a\ r\rightarrow\ b)\ r\rightarrow\ ((b\ r\rightarrow\ c)\ l\rightarrow\ (a\ r\rightarrow\ c)) = 1$

begin

lemma *P1-a* [*simp*]: $x\ l\rightarrow\ x = 1$

<proof>

lemma *P1-b* [*simp*]: $x\ r\rightarrow\ x = 1$

<proof>

lemma *P2-a*: $x\ l\rightarrow\ y = 1 \implies y\ l\rightarrow\ x = 1 \implies x = y$

<proof>

lemma *P2-b*: $x\ r\rightarrow\ y = 1 \implies y\ r\rightarrow\ x = 1 \implies x = y$

<proof>

lemma *P2-c*: $x\ l\rightarrow\ y = 1 \implies y\ r\rightarrow\ x = 1 \implies x = y$

<proof>

lemma P3-a: $(x \text{ l} \rightarrow 1) \text{ r} \rightarrow 1 = 1$
<proof>

lemma P3-b: $(x \text{ r} \rightarrow 1) \text{ l} \rightarrow 1 = 1$
<proof>

lemma P4-a [simp]: $x \text{ l} \rightarrow 1 = 1$
<proof>

lemma P4-b [simp]: $x \text{ r} \rightarrow 1 = 1$
<proof>

lemma P5-a: $x \text{ l} \rightarrow y = 1 \implies y \text{ l} \rightarrow z = 1 \implies x \text{ l} \rightarrow z = 1$
<proof>

lemma P5-b: $x \text{ r} \rightarrow y = 1 \implies y \text{ r} \rightarrow z = 1 \implies x \text{ r} \rightarrow z = 1$
<proof>

lemma P6-a: $x \text{ l} \rightarrow (y \text{ r} \rightarrow x) = 1$
<proof>

lemma P6-b: $x \text{ r} \rightarrow (y \text{ l} \rightarrow x) = 1$
<proof>

lemma P7: $(x \text{ l} \rightarrow (y \text{ r} \rightarrow z) = 1) = (y \text{ r} \rightarrow (x \text{ l} \rightarrow z) = 1)$
<proof>

lemma P8-a: $(x \text{ l} \rightarrow y) \text{ r} \rightarrow ((z \text{ l} \rightarrow x) \text{ l} \rightarrow (z \text{ l} \rightarrow y)) = 1$
<proof>

lemma P8-b: $(x \text{ r} \rightarrow y) \text{ l} \rightarrow ((z \text{ r} \rightarrow x) \text{ r} \rightarrow (z \text{ r} \rightarrow y)) = 1$
<proof>

lemma P9: $x \text{ l} \rightarrow (y \text{ r} \rightarrow z) = y \text{ r} \rightarrow (x \text{ l} \rightarrow z)$
<proof>

definition
 $\text{lesseq-a } a \ b = (a \text{ l} \rightarrow b = 1)$

definition
 $\text{less-a } a \ b = (\text{lesseq-a } a \ b \wedge \neg \text{lesseq-a } b \ a)$

definition
 $\text{lesseq-b } a \ b = (a \text{ r} \rightarrow b = 1)$

definition
 $\text{less-b } a \ b = (\text{lesseq-b } a \ b \wedge \neg \text{lesseq-b } b \ a)$

definition

$sup-a\ a\ b = (a\ l\rightarrow\ b)\ r\rightarrow\ b$

end

sublocale *impl-lr-algebra* < *order-a:order lesseq-a less-a*

<proof>

sublocale *impl-lr-algebra* < *order-b:order lesseq-b less-b*

<proof>

sublocale *impl-lr-algebra* < *slattice-a:semilattice-sup sup-a lesseq-a less-a*

<proof>

sublocale *impl-lr-algebra* < *slattice-b:semilattice-sup sup-a lesseq-b less-b*

<proof>

context *impl-lr-algebra*

begin

lemma *lesseq-a-b: lesseq-b = lesseq-a*

<proof>

lemma *P10: (a l→ b = 1) = (a r→ b = 1)*

<proof>

end

class *one-ord = one + ord*

class *impl-lr-ord-algebra = impl-lr-algebra + one-ord +*

assumes

order: a ≤ b = (a l→ b = 1)

and

strict: a < b = (a ≤ b ∧ ¬ b ≤ a)

begin

lemma *order-l: (a ≤ b) = (a l→ b = 1)*

<proof>

lemma *order-r: (a ≤ b) = (a r→ b = 1)*

<proof>

lemma *P11-a: a ≤ b l→ a*

<proof>

lemma *P11-b: a ≤ b r→ a*

<proof>

lemma *P12: (a ≤ b l→ c) = (b ≤ a r→ c)*

<proof>

lemma *P13-a*: $a \leq b \implies b \text{ l} \rightarrow c \leq a \text{ l} \rightarrow c$
<proof>

lemma *P13-b*: $a \leq b \implies b \text{ r} \rightarrow c \leq a \text{ r} \rightarrow c$
<proof>

lemma *P14-a*: $a \leq b \implies c \text{ l} \rightarrow a \leq c \text{ l} \rightarrow b$
<proof>

lemma *P14-b*: $a \leq b \implies c \text{ r} \rightarrow a \leq c \text{ r} \rightarrow b$
<proof>

subclass *order*
<proof>

end

class *one-zero-uminus* = *one* + *zero* + *left-uminus* + *right-uminus*

class *impl-neg-lr-algebra* = *impl-lr-ord-algebra* + *one-zero-uminus* +
assumes

W4: $-l \ 1 = -r \ 1$

and *W5a*: $(-l \ a \ \text{r} \rightarrow -l \ b) \ \text{l} \rightarrow (b \ \text{l} \rightarrow a) = 1$

and *W5b*: $(-r \ a \ \text{l} \rightarrow -r \ b) \ \text{r} \rightarrow (b \ \text{r} \rightarrow a) = 1$

and *zero-def*: $0 = -l \ 1$

begin

lemma *zero-r-def*: $0 = -r \ 1$
<proof>

lemma *C1-a* [*simp*]: $(-l \ x \ \text{r} \rightarrow 0) \ \text{l} \rightarrow x = 1$
<proof>

lemma *C1-b* [*simp*]: $(-r \ x \ \text{l} \rightarrow 0) \ \text{r} \rightarrow x = 1$
<proof>

lemma *C2-b* [*simp*]: $0 \ \text{r} \rightarrow x = 1$
<proof>

lemma *C2-a* [*simp*]: $0 \ \text{l} \rightarrow x = 1$
<proof>

lemma *C3-a*: $x \ \text{l} \rightarrow 0 = -l \ x$
<proof>

lemma *C3-b*: $x \ \text{r} \rightarrow 0 = -r \ x$
<proof>

lemma *C4-a [simp]*: $-r (-l x) = x$
<proof>

lemma *C4-b [simp]*: $-l (-r x) = x$
<proof>

lemma *C5-a*: $-r x l \rightarrow -r y = y r \rightarrow x$
<proof>

lemma *C5-b*: $-l x r \rightarrow -l y = y l \rightarrow x$
<proof>

lemma *C6*: $-r x l \rightarrow y = -l y r \rightarrow x$
<proof>

lemma *C7-a*: $(x \leq y) = (-l y \leq -l x)$
<proof>

lemma *C7-b*: $(x \leq y) = (-r y \leq -r x)$
<proof>

end

class *pseudo-wajsberg-algebra* = *impl-neg-lr-algebra* +
assumes

W6: $-r (a l \rightarrow -l b) = -l (b r \rightarrow -r a)$

begin

definition

mult $a b = -r (a l \rightarrow -l b)$

definition

inf-a $a b = -l (a r \rightarrow -r (a l \rightarrow b))$

definition

inf-b $a b = -r (b l \rightarrow -l (b r \rightarrow a))$

end

sublocale *pseudo-wajsberg-algebra* < *slattice-inf-a:semilattice-inf inf-a* (\leq) ($<$)
<proof>

sublocale *pseudo-wajsberg-algebra* < *slattice-inf-b:semilattice-inf inf-b* (\leq) ($<$)
<proof>

context *pseudo-wajsberg-algebra*

begin

lemma *inf-a-b*: $\text{inf-}a = \text{inf-}b$
<proof>

end
end

8 Some Classes of Pseudo-Hoops

theory *SpecialPseudoHoops*
imports *PseudoHoopFilters PseudoWaisbergAlgebra*
begin

class *cancel-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
 assumes *mult-cancel-left*: $a * b = a * c \implies b = c$
 and *mult-cancel-right*: $b * a = c * a \implies b = c$

begin

lemma *cancel-left-a*: $b \text{ l} \rightarrow (a * b) = a$
<proof>

lemma *cancel-right-a*: $b \text{ r} \rightarrow (b * a) = a$
<proof>

end

class *cancel-pseudo-hoop-algebra-2* = *pseudo-hoop-algebra* +
 assumes *cancel-left*: $b \text{ l} \rightarrow (a * b) = a$
 and *cancel-right*: $b \text{ r} \rightarrow (b * a) = a$

begin

subclass *cancel-pseudo-hoop-algebra*
<proof>

end

context *cancel-pseudo-hoop-algebra*
begin

lemma *lemma-4-2-i*: $a \text{ l} \rightarrow b = (a * c) \text{ l} \rightarrow (b * c)$
<proof>

lemma *lemma-4-2-ii*: $a \text{ r} \rightarrow b = (c * a) \text{ r} \rightarrow (c * b)$
<proof>

lemma *lemma-4-2-iii*: $(a * c \leq b * c) = (a \leq b)$
<proof>

lemma *lemma-4-2-iv*: $(c * a \leq c * b) = (a \leq b)$

```

    <proof>

end

class wajsberg-pseudo-hoop-algebra = pseudo-hoop-algebra +
  assumes wajsberg1: (a l→ b) r→ b = (b l→ a) r→ a
  and wajsberg2: (a r→ b) l→ b = (b r→ a) l→ a

context wajsberg-pseudo-hoop-algebra
begin

lemma lemma-4-3-i-a: a ⊔1 b = (a l→ b) r→ b
  <proof>

lemma lemma-4-3-i-b: a ⊔1 b = (b l→ a) r→ a
  <proof>

lemma lemma-4-3-ii-a: a ⊔2 b = (a r→ b) l→ b
  <proof>

lemma lemma-4-3-ii-b: a ⊔2 b = (b r→ a) l→ a
  <proof>
end

sublocale wajsberg-pseudo-hoop-algebra < lattice1:pseudo-hoop-lattice-b (⊔1) (*)
  (⊔) (l→) (≤) (<) 1 (r→)
  <proof>

class zero-one = zero + one

class bounded-wajsberg-pseudo-hoop-algebra = zero-one + wajsberg-pseudo-hoop-algebra
  +
  assumes zero-smallest [simp]: 0 ≤ a
begin
end

sublocale wajsberg-pseudo-hoop-algebra < lattice2:pseudo-hoop-lattice-b (⊔2) (*)
  (⊔) (l→) (≤) (<) 1 (r→)
  <proof>

lemma (in wajsberg-pseudo-hoop-algebra) sup1-eq-sup2: (⊔1) = (⊔2)
  <proof>

context bounded-wajsberg-pseudo-hoop-algebra

```

```

begin
definition
  negl a = a l→ 0

definition
  negr a = a r→ 0

lemma [simp]: 0 l→ a = 1
  ⟨proof⟩

lemma [simp]: 0 r→ a = 1
  ⟨proof⟩
end

sublocale bounded-wajsberg-pseudo-hoop-algebra < wajsberg: pseudo-wajsberg-algebra
1 (l→) (r→) (≤) (<) 0 negl negr
  ⟨proof⟩

context pseudo-wajsberg-algebra
begin
  lemma class.bounded-wajsberg-pseudo-hoop-algebra mult inf-a (l→) (≤) (<) 1
    (r→) (0::'a)
    ⟨proof⟩
end

class basic-pseudo-hoop-algebra = pseudo-hoop-algebra +
  assumes B1: (a l→ b) l→ c ≤ ((b l→ a) l→ c) l→ c
  and B2: (a r→ b) r→ c ≤ ((b r→ a) r→ c) r→ c
begin
  lemma lemma-4-5-i: (a l→ b) ⊔1 (b l→ a) = 1
    ⟨proof⟩

  lemma lemma-4-5-ii: (a r→ b) ⊔2 (b r→ a) = 1
    ⟨proof⟩

  lemma lemma-4-5-iii: a l→ b = (a ⊔1 b) l→ b
    ⟨proof⟩

  lemma lemma-4-5-iv: a r→ b = (a ⊔2 b) r→ b
    ⟨proof⟩

  lemma lemma-4-5-v: (a ⊔1 b) l→ c = (a l→ c) ⊓ (b l→ c)
    ⟨proof⟩
end

```


lemma *lemma-4-5-vi*: $(a \sqcup 2 b) r \rightarrow c = (a r \rightarrow c) \sqcap (b r \rightarrow c)$
 ⟨proof⟩

lemma *lemma-4-5-a*: $a \leq c \implies b \leq c \implies a \sqcup 1 b \leq c$
 ⟨proof⟩

lemma *lemma-4-5-b*: $a \leq c \implies b \leq c \implies a \sqcup 2 b \leq c$
 ⟨proof⟩

lemma *lemma-4-5*: $a \sqcup 1 b = a \sqcup 2 b$
 ⟨proof⟩
end

sublocale *basic-pseudo-hoop-algebra* < *basic-lattice:lattice* (\sqcap) (\leq) ($<$) ($\sqcup 1$)
 ⟨proof⟩

context *pseudo-hoop-lattice* **begin end**

sublocale *basic-pseudo-hoop-algebra* < *pseudo-hoop-lattice* ($\sqcup 1$) ($*$) (\sqcap) ($l \rightarrow$) (\leq)
 ($<$) 1 ($r \rightarrow$)
 ⟨proof⟩

class *sup-assoc-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *sup1-assoc*: $a \sqcup 1 (b \sqcup 1 c) = (a \sqcup 1 b) \sqcup 1 c$
and *sup2-assoc*: $a \sqcup 2 (b \sqcup 2 c) = (a \sqcup 2 b) \sqcup 2 c$

sublocale *sup-assoc-pseudo-hoop-algebra* < *sup1-lattice: pseudo-hoop-lattice* ($\sqcup 1$)
 ($*$) (\sqcap) ($l \rightarrow$) (\leq) ($<$) 1 ($r \rightarrow$)
 ⟨proof⟩

sublocale *sup-assoc-pseudo-hoop-algebra* < *sup2-lattice: pseudo-hoop-lattice* ($\sqcup 2$)
 ($*$) (\sqcap) ($l \rightarrow$) (\leq) ($<$) 1 ($r \rightarrow$)
 ⟨proof⟩

class *sup-assoc-pseudo-hoop-algebra-1* = *sup-assoc-pseudo-hoop-algebra* +
assumes *union-impl*: $(a l \rightarrow b) \sqcup 1 (b l \rightarrow a) = 1$
and *union-impr*: $(a r \rightarrow b) \sqcup 1 (b r \rightarrow a) = 1$

lemma (**in** *pseudo-hoop-algebra*) [*simp*]: $\text{infimum } \{a, b\} = \{a \sqcap b\}$
 ⟨proof⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-impl-inf*:
 $(a \sqcup b) l \rightarrow c = (a l \rightarrow c) \sqcap (b l \rightarrow c)$
 ⟨proof⟩

lemma (**in** *pseudo-hoop-lattice*) *sup-impr-inf*:
 $(a \sqcup b) r \rightarrow c = (a r \rightarrow c) \sqcap (b r \rightarrow c)$
 ⟨proof⟩

lemma (in *pseudo-hoop-lattice*) *sup-times*:

$$a * (b \sqcup c) = (a * b) \sqcup (a * c)$$

<proof>

lemma (in *pseudo-hoop-lattice*) *sup-times-right*:

$$(b \sqcup c) * a = (b * a) \sqcup (c * a)$$

<proof>

context *basic-pseudo-hoop-algebra* **begin end**

sublocale *sup-assoc-pseudo-hoop-algebra-1* < *basic-1*: *basic-pseudo-hoop-algebra*

$$(*) (\sqcap) (l\rightarrow) (\leq) (<) 1 (r\rightarrow)$$

<proof>

context *basic-pseudo-hoop-algebra*

begin

lemma *lemma-4-8-i*: $a * (b \sqcap c) = (a * b) \sqcap (a * c)$

<proof>

lemma *lemma-4-8-ii*: $(b \sqcap c) * a = (b * a) \sqcap (c * a)$

<proof>

lemma *lemma-4-8-iii*: $(a \ l\rightarrow \ b) \ l\rightarrow \ (b \ l\rightarrow \ a) = b \ l\rightarrow \ a$

<proof>

lemma *lemma-4-8-iv*: $(a \ r\rightarrow \ b) \ r\rightarrow \ (b \ r\rightarrow \ a) = b \ r\rightarrow \ a$

<proof>

end

context *wajsberg-pseudo-hoop-algebra*

begin

subclass *sup-assoc-pseudo-hoop-algebra-1*

<proof>

end

class *bounded-basic-pseudo-hoop-algebra* = *zero-one* + *basic-pseudo-hoop-algebra*

+

assumes *zero-smallest* [*simp*]: $0 \leq a$

class *inf-a* =

fixes *inf-a* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcap 1$ 65)

class *pseudo-bl-algebra* = *zero* + *sup* + *inf* + *monoid-mult* + *ord* + *left-imp* + *right-imp* +

assumes *bounded-lattice*: *class.bounded-lattice* *inf* (\leq) ($<$) *sup* 0 1

and *left-residual-bl*: $(x * a \leq b) = (x \leq a \rightarrow b)$
and *right-residual-bl*: $(a * x \leq b) = (x \leq a \rightarrow b)$
and *inf-l-bl-def*: $a \sqcap b = (a \rightarrow b) * a$
and *inf-r-bl-def*: $a \sqcap b = a * (a \rightarrow b)$
and *impl-sup-bl*: $(a \rightarrow b) \sqcup (b \rightarrow a) = 1$
and *impr-sup-bl*: $(a \rightarrow b) \sqcup (b \rightarrow a) = 1$

sublocale *bounded-basic-pseudo-hoop-algebra* < *basic: pseudo-bl-algebra* 1 (*) 0
 (\sqcap) $(\sqcup 1)$ (\rightarrow) (\rightarrow) (\leq) $(<)$
 <proof>

sublocale *pseudo-bl-algebra* < *bounded-lattice: bounded-lattice* inf (\leq) $(<)$ sup 0
 1
 <proof>

context *pseudo-bl-algebra*

begin

lemma *impl-one-bl* [*simp*]: $a \rightarrow a = 1$
 <proof>

lemma *impr-one-bl* [*simp*]: $a \rightarrow a = 1$
 <proof>

lemma *impl-ded-bl*: $((a * b) \rightarrow c) = (a \rightarrow (b \rightarrow c))$
 <proof>

lemma *impr-ded-bl*: $(b * a \rightarrow c) = (a \rightarrow (b \rightarrow c))$
 <proof>

lemma *lesseq-impl-bl*: $(a \leq b) = (a \rightarrow b = 1)$
 <proof>

end

context *pseudo-bl-algebra*

begin

subclass *pseudo-hoop-lattice*
 <proof>

subclass *bounded-basic-pseudo-hoop-algebra*
 <proof>

end

class *product-pseudo-hoop-algebra* = *basic-pseudo-hoop-algebra* +

assumes *P1*: $b \rightarrow b * b \leq (a \sqcap (a \rightarrow b)) \rightarrow b$

and *P2*: $b \rightarrow b * b \leq (a \sqcap (a \rightarrow b)) \rightarrow b$

and *P3*: $((a \rightarrow b) \rightarrow b) * (c * a \rightarrow d * a) * (c * b \rightarrow d * b) \leq c \rightarrow d$

and P_4 : $((a \rightarrow b) \rightarrow b) * (a * c \rightarrow a * d) * (b * c \rightarrow b * d) \leq c \rightarrow d$

class *cancel-basic-pseudo-hoop-algebra* = *basic-pseudo-hoop-algebra* + *cancel-pseudo-hoop-algebra*
begin
subclass *product-pseudo-hoop-algebra*
 ⟨*proof*⟩

end

class *simple-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *simple*: $\text{normalfilters} \cap \text{properfilters} = \{\{1\}\}$

class *proper* = *one* +
assumes *proper*: $\exists a . a \neq 1$

class *strong-simple-pseudo-hoop-algebra* = *pseudo-hoop-algebra* +
assumes *strong-simple*: $\text{properfilters} = \{\{1\}\}$
begin

subclass *proper*
 ⟨*proof*⟩

lemma *lemma-4-12-i-ii*: $a \neq 1 \implies \text{filterof}(\{a\}) = \text{UNIV}$
 ⟨*proof*⟩

lemma *lemma-4-12-i-iii*: $a \neq 1 \implies (\exists n . a \wedge^n \leq b)$
 ⟨*proof*⟩

lemma *lemma-4-12-i-iv*: $a \neq 1 \implies (\exists n . a \multimap^n \rightarrow b = 1)$
 ⟨*proof*⟩

lemma *lemma-4-12-i-v*: $a \neq 1 \implies (\exists n . a \multimap^n \rightarrow b = 1)$
 ⟨*proof*⟩

end

lemma (**in** *pseudo-hoop-algebra*) [*simp*]: $\{1\} \in \text{filters}$
 ⟨*proof*⟩

class *strong-simple-pseudo-hoop-algebra-a* = *pseudo-hoop-algebra* + *proper* +
assumes *strong-simple-a*: $a \neq 1 \implies \text{filterof}(\{a\}) = \text{UNIV}$
begin
subclass *strong-simple-pseudo-hoop-algebra*
 ⟨*proof*⟩

end

sublocale *strong-simple-pseudo-hoop-algebra* < *strong-simple-pseudo-hoop-algebra-a*
 ⟨*proof*⟩

lemma (in *pseudo-hoop-algebra*) *power-impl*: $b \multimap a = a \implies b \wedge^n \multimap a = a$
 ⟨*proof*⟩

lemma (in *pseudo-hoop-algebra*) *power-impr*: $b \multimap a = a \implies b \wedge^n \multimap a = a$
 ⟨*proof*⟩

context *strong-simple-pseudo-hoop-algebra*
begin

lemma *lemma-4-13-i*: $b \multimap a = a \implies a = 1 \vee b = 1$
 ⟨*proof*⟩

lemma *lemma-4-13-ii*: $b \multimap a = a \implies a = 1 \vee b = 1$
 ⟨*proof*⟩

end

class *basic-pseudo-hoop-algebra-A* = *basic-pseudo-hoop-algebra* +
assumes *left-impl-one*: $b \multimap a = a \implies a = 1 \vee b = 1$
and *right-impl-one*: $b \multimap a = a \implies a = 1 \vee b = 1$
begin
subclass *linorder*
 ⟨*proof*⟩

lemma [*simp*]: $(a \multimap b) \multimap b \leq (b \multimap a) \multimap a$
 ⟨*proof*⟩

end

context *basic-pseudo-hoop-algebra-A* **begin**

lemma [*simp*]: $(a \multimap b) \multimap b \leq (b \multimap a) \multimap a$
 ⟨*proof*⟩

subclass *wajsberg-pseudo-hoop-algebra*
 ⟨*proof*⟩

end

class *strong-simple-basic-pseudo-hoop-algebra* = *strong-simple-pseudo-hoop-algebra*
 + *basic-pseudo-hoop-algebra*
begin
subclass *basic-pseudo-hoop-algebra-A*
 ⟨*proof*⟩

subclass *wajsberg-pseudo-hoop-algebra*
 ⟨*proof*⟩

end

end

9 Examples of Pseudo-Hoops

theory *Examples*

imports *SpecialPseudoHoops LatticeProperties.Lattice-Ordered-Group*
begin

declare *add-uminus-conv-diff* [*simp del*] *right-minus* [*simp*]

lemmas *diff-minus = diff-conv-add-uminus*

context *lgroup*

begin

lemma (**in** *lgroup*) *less-eq-inf-2*: $(x \leq y) = (\text{inf } y \ x = x)$

<proof>

end

class *lgroup-with-const* = *lgroup* +

fixes *u::'a*

assumes [*simp*]: $0 \leq u$

definition $G = \{a::'a::\text{lgroup-with-const}. (0 \leq a \wedge a \leq u)\}$

typedef (**overloaded**) *'a G = G::'a::lgroup-with-const set*

<proof>

instantiation $G :: (\text{lgroup-with-const}) \text{ bounded-wajsberg-pseudo-hoop-algebra}$

begin

definition

times-def: $a * b \equiv \text{Abs-}G (\text{sup } (\text{Rep-}G \ a - u + \text{Rep-}G \ b) \ 0)$

lemma [*simp*]: $\text{sup } (\text{Rep-}G \ a - u + \text{Rep-}G \ b) \ 0 \in G$

<proof>

definition

impl-def: $a \mapsto b \equiv \text{Abs-}G ((\text{Rep-}G \ b - \text{Rep-}G \ a + u) \sqcap u)$

lemma [*simp*]: $\text{inf } (\text{Rep-}G \ (b::'a \ G) - \text{Rep-}G \ a + u) \ u \in G$

<proof>

definition

impr-def: $a \mapsto b \equiv \text{Abs-}G (\text{inf } (u - \text{Rep-}G \ a + \text{Rep-}G \ b) \ u)$

lemma [*simp*]: $\text{inf } (u - \text{Rep-}G \ a + \text{Rep-}G \ b) \ u \in G$

<proof>

definition

one-def: $1 \equiv \text{Abs-G } u$

definition

zero-def: $0 \equiv \text{Abs-G } 0$

definition

order-def: $((a::'a \ G) \leq b) \equiv (a \ l \rightarrow b = 1)$

definition

strict-order-def: $(a::'a \ G) < b \equiv (a \leq b \wedge \neg b \leq a)$

definition

inf-def: $(a::'a \ G) \sqcap b = ((a \ l \rightarrow b) * a)$

lemma [*simp*]: $(u::'a) \in G$

<proof>

lemma [*simp*]: $(1::'a \ G) * a = a$

<proof>

lemma [*simp*]: $a * (1::'a \ G) = a$

<proof>

lemma [*simp*]: $a \ l \rightarrow a = (1::'a \ G)$

<proof>

lemma [*simp*]: $a \ r \rightarrow a = (1::'a \ G)$

<proof>

lemma [*simp*]: $a \in G \implies \text{Rep-G } (\text{Abs-G } a) = a$

<proof>

lemma *inf-def-1*: $((a::'a \ G) \ l \rightarrow b) * a = \text{Abs-G } (\text{inf } (\text{Rep-G } a) (\text{Rep-G } b))$

<proof>

lemma *inf-def-2*: $(a::'a \ G) * (a \ r \rightarrow b) = \text{Abs-G } (\text{inf } (\text{Rep-G } a) (\text{Rep-G } b))$

<proof>

lemma *Rep-G-order*: $(a \leq b) = (\text{Rep-G } a \leq \text{Rep-G } b)$

<proof>

lemma *ded-left*: $((a::'a \ G) * b) \ l \rightarrow c = a \ l \rightarrow b \ l \rightarrow c$

<proof>

lemma *ded-right*: $((a::'a \ G) * b) \ r \rightarrow c = b \ r \rightarrow a \ r \rightarrow c$

<proof>

lemma [*simp*]: $0 \in G$
<proof>

lemma [*simp*]: $0 \leq (a::'a G)$
<proof>

lemma *lemma-W1*: $((a::'a G) l \rightarrow b) r \rightarrow b = (b l \rightarrow a) r \rightarrow a$
<proof>

lemma *lemma-W2*: $((a::'a G) r \rightarrow b) l \rightarrow b = (b r \rightarrow a) l \rightarrow a$
<proof>

instance *<proof>*

end

context *order*

begin

definition

closed-interval:: $'a \Rightarrow 'a \Rightarrow 'a$ set ($|| \text{ - , - } || [0, 0]$ 900) **where**
closed-interval $a b = \{c . a \leq c \wedge c \leq b\}$

definition

convex = $\{A . \forall a b . a \in A \wedge b \in A \longrightarrow |[a, b]| \subseteq A\}$

end

context *group-add*

begin

definition

subgroup = $\{A . 0 \in A \wedge (\forall a b . a \in A \wedge b \in A \longrightarrow a + b \in A \wedge -a \in A)\}$

lemma [*simp*]: $A \in \text{subgroup} \Longrightarrow 0 \in A$
<proof>

lemma [*simp*]: $A \in \text{subgroup} \Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow a + b \in A$
<proof>

lemma *minus-subgroup*: $A \in \text{subgroup} \Longrightarrow -a \in A \Longrightarrow a \in A$
<proof>

definition

add-set:: $'a$ set $\Rightarrow 'a$ set $\Rightarrow 'a$ set (**infixl** +++ 70) **where**
add-set $A B = \{c . \exists a \in A . \exists b \in B . c = a + b\}$

definition

$normal = \{A . (\forall a . A \text{ +++ } \{a\} = \{a\} \text{ +++ } A)\}$
end

context *lgroup*

begin

definition

$lsubgroup = \{A . A \in subgroup \wedge (\forall a b . a \in A \wedge b \in A \longrightarrow inf\ a\ b \in A \wedge sup\ a\ b \in A)\}$

lemma *inf-lsubgroup*:

$A \in lsubgroup \implies a \in A \implies b \in A \implies inf\ a\ b \in A$
 ⟨*proof*⟩

lemma *sup-lsubgroup*:

$A \in lsubgroup \implies a \in A \implies b \in A \implies sup\ a\ b \in A$
 ⟨*proof*⟩

end

definition

$F\ K = \{a::'a\ G . (u::'a::lgroup-with-const) - Rep-G\ a \in K\}$

lemma *F-def2*: $K \in normal \implies F\ K = \{a::'a\ G . - Rep-G\ a + (u::'a::lgroup-with-const) \in K\}$

⟨*proof*⟩

context *lgroup* **begin**

lemma *sup-assoc-lgroup*: $a \sqcup b \sqcup c = a \sqcup (b \sqcup c)$

⟨*proof*⟩

end

lemma *normal-1*: $K \in normal \implies K \in convex \implies K \in lsubgroup \implies x \in \{a\}$

$**\ F\ K \implies x \in F\ K **\ \{a\}$

⟨*proof*⟩

lemma *normal-2*: $K \in normal \implies K \in convex \implies K \in lsubgroup \implies x \in F\ K$

$**\ \{a\} \implies x \in \{a\} **\ F\ K$

⟨*proof*⟩

lemma $K \in normal \implies K \in convex \implies K \in lsubgroup \implies F\ K \in normalfilters$

⟨*proof*⟩

definition $N = \{a::'a::lgroup . a \leq 0\}$

typedef (**overloaded**) (*'a::lgroup*) $N = N :: 'a::lgroup\ set$

⟨*proof*⟩

class *cancel-product-pseudo-hoop-algebra* = *cancel-pseudo-hoop-algebra* + *product-pseudo-hoop-algebra*

context *group-add*
begin
subclass *cancel-semigroup-add*
 $\langle proof \rangle$
end

instantiation $N :: (lgroup) \textit{pseudo-hoop-algebra}$
begin

definition
times-N-def: $a * b \equiv Abs-N (Rep-N a + Rep-N b)$

lemma [*simp*]: $Rep-N a + Rep-N b \in N$
 $\langle proof \rangle$

definition
impl-N-def: $a l \rightarrow b \equiv Abs-N (inf (Rep-N b - Rep-N a) 0)$

definition
inf-N-def: $(a :: 'a N) \sqcap b = (a l \rightarrow b) * a$

lemma [*simp*]: $inf (Rep-N b - Rep-N a) 0 \in N$
 $\langle proof \rangle$

definition
impr-N-def: $a r \rightarrow b \equiv Abs-N (inf (- Rep-N a + Rep-N b) 0)$

lemma [*simp*]: $inf (- Rep-N a + Rep-N b) 0 \in N$
 $\langle proof \rangle$

definition
one-N-def: $1 \equiv Abs-N 0$

lemma [*simp*]: $0 \in N$
 $\langle proof \rangle$

definition
order-N-def: $((a :: 'a N) \leq b) \equiv (a l \rightarrow b = 1)$

definition
strict-order-N-def: $(a :: 'a N) < b \equiv (a \leq b \wedge \neg b \leq a)$

lemma *order-Rep-N*:
 $((a :: 'a N) \leq b) = (Rep-N a \leq Rep-N b)$
 $\langle proof \rangle$

lemma *order-Abs-N*:

$$a \in N \implies b \in N \implies (a \leq b) = (\text{Abs-}N\ a \leq \text{Abs-}N\ b)$$

<proof>

lemma [*simp*]: $(1::'a\ N) * a = a$
<proof>

lemma [*simp*]: $a * (1::'a\ N) = a$
<proof>

lemma [*simp*]: $a \text{ l}\rightarrow a = (1::'a\ N)$
<proof>

lemma [*simp*]: $a \text{ r}\rightarrow a = (1::'a\ N)$
<proof>

lemma *impl-times*: $(a \text{ l}\rightarrow b) * a = (b \text{ l}\rightarrow a) * (b::'a\ N)$
<proof>

lemma *impr-times*: $a * (a \text{ r}\rightarrow b) = (b::'a\ N) * (b \text{ r}\rightarrow a)$
<proof>

lemma *impr-impl-times*: $(a \text{ l}\rightarrow b) * a = (a::'a\ N) * (a \text{ r}\rightarrow b)$
<proof>

lemma *impl-ded*: $(a::'a\ N) * b \text{ l}\rightarrow c = a \text{ l}\rightarrow b \text{ l}\rightarrow c$
<proof>

lemma *impr-ded*: $(a::'a\ N) * b \text{ r}\rightarrow c = b \text{ r}\rightarrow a \text{ r}\rightarrow c$
<proof>

instance *<proof>*

end

lemma *Rep-N-inf*: $\text{Rep-}N\ ((a::'a::\text{lgroup}\ N) \sqcap b) = (\text{Rep-}N\ a) \sqcap (\text{Rep-}N\ b)$
<proof>

context *lgroup* **begin**

lemma *sup-inf-distrib2-lgroup*: $(b \sqcap c) \sqcup a = (b \sqcup a) \sqcap (c \sqcup a)$
<proof>

lemma *inf-sup-distrib2-lgroup*: $(b \sqcup c) \sqcap a = (b \sqcap a) \sqcup (c \sqcap a)$
<proof>

end

instantiation $N :: (\text{lgroup}) \text{cancel-product-pseudo-hoop-algebra}$
begin

lemma *cancel-times-left*: $(a::'a N) * b = a * c \implies b = c$
 $\langle \text{proof} \rangle$

lemma *cancel-times-right*: $b * (a::'a N) = c * a \implies b = c$
 $\langle \text{proof} \rangle$

lemma *prod-1*: $((a::'a N) \text{l} \rightarrow b) \text{l} \rightarrow c \leq ((b \text{l} \rightarrow a) \text{l} \rightarrow c) \text{l} \rightarrow c$
 $\langle \text{proof} \rangle$

lemma *prod-2*: $((a::'a N) \text{r} \rightarrow b) \text{r} \rightarrow c \leq ((b \text{r} \rightarrow a) \text{r} \rightarrow c) \text{r} \rightarrow c$
 $\langle \text{proof} \rangle$

lemma *prod-3*: $(b::'a N) \text{l} \rightarrow b * b \leq a \sqcap (a \text{l} \rightarrow b) \text{l} \rightarrow b$
 $\langle \text{proof} \rangle$

lemma *prod-4*: $(b::'a N) \text{r} \rightarrow b * b \leq a \sqcap (a \text{r} \rightarrow b) \text{r} \rightarrow b$
 $\langle \text{proof} \rangle$

lemma *prod-5*: $((a::'a N) \text{l} \rightarrow b) \text{l} \rightarrow b * (c * a \text{l} \rightarrow f * a) * (c * b \text{l} \rightarrow f * b) \leq c \text{l} \rightarrow f$
 $\langle \text{proof} \rangle$

lemma *prod-6*: $((a::'a N) \text{r} \rightarrow b) \text{r} \rightarrow b * (a * c \text{r} \rightarrow a * f) * (b * c \text{r} \rightarrow b * f) \leq c \text{r} \rightarrow f$
 $\langle \text{proof} \rangle$

instance
 $\langle \text{proof} \rangle$

end

definition *OrdSum* =

$\{x. (\exists a::'a::\text{pseudo-hoop-algebra}. x = (a, 1::'b::\text{pseudo-hoop-algebra})) \vee (\exists b::'b. x = (1::'a, b))\}$

typedef (**overloaded**) $('a, 'b) \text{OrdSum} = \text{OrdSum} :: ('a::\text{pseudo-hoop-algebra} \times 'b::\text{pseudo-hoop-algebra}) \text{set}$
 $\langle \text{proof} \rangle$

lemma [*simp*]: $(1, b) \in \text{OrdSum}$
 $\langle \text{proof} \rangle$

lemma [*simp*]: $(a, 1) \in \text{OrdSum}$
 $\langle \text{proof} \rangle$

definition

first $x = \text{fst } (\text{Rep-OrdSum } x)$

definition

second $x = \text{snd } (\text{Rep-OrdSum } x)$

lemma *if-unfold-left*: $((\text{if } a \text{ then } b \text{ else } c) = d) = ((a \longrightarrow b = d) \wedge (\neg a \longrightarrow c = d))$

<proof>

lemma *if-unfold-right*: $(d = (\text{if } a \text{ then } b \text{ else } c)) = ((a \longrightarrow d = b) \wedge (\neg a \longrightarrow d = c))$

<proof>

lemma *fst-snd-eq*: $\text{fst } a = x \implies \text{snd } a = y \implies (x, y) = a$

<proof>

instantiation *OrdSum* :: $(\text{pseudo-hoop-algebra}, \text{pseudo-hoop-algebra}) \text{ pseudo-hoop-algebra}$
begin

definition

times-OrdSum-def: $a * b \equiv$ (
 if *second* $a = 1 \wedge$ *second* $b = 1$ then
 Abs-OrdSum (*first* $a *$ *first* $b, 1$)
 else if *first* $a = 1 \wedge$ *first* $b = 1$ then
 Abs-OrdSum ($1, \text{second } a * \text{second } b$)
 else if *first* $a = 1 \wedge$ *second* $b = 1$ then
 b
 else
 a)

definition

one-OrdSum-def: $1 \equiv \text{Abs-OrdSum } (1, 1)$

definition

impl-OrdSum-def: $a \text{ l} \rightarrow b \equiv$ (
 if *second* $a = 1 \wedge$ *second* $b = 1$ then
 Abs-OrdSum (*first* $a \text{ l} \rightarrow$ *first* $b, 1$)
 else if *first* $a = 1 \wedge$ *first* $b = 1$ then
 Abs-OrdSum ($1, \text{second } a \text{ l} \rightarrow \text{second } b$)
 else if *first* $a = 1 \wedge$ *second* $b = 1$ then
 b
 else
 1)

definition

impr-OrdSum-def: $a \text{ r} \rightarrow b \equiv$

(if second $a = 1 \wedge$ second $b = 1$ then
 Abs-OrdSum (first $a \rightarrow$ first b , 1)
 else if first $a = 1 \wedge$ first $b = 1$ then
 Abs-OrdSum (1, second $a \rightarrow$ second b)
 else if first $a = 1 \wedge$ second $b = 1$ then
 b
 else
 1)

definition

order-OrdSum-def: $((a::('a, 'b) \text{ OrdSum}) \leq b) \equiv (a \rightarrow b = 1)$

definition

inf-OrdSum-def: $(a::('a, 'b) \text{ OrdSum}) \sqcap b = (a \rightarrow b) * a$

definition

strict-order-OrdSum-def: $(a::('a, 'b) \text{ OrdSum}) < b \equiv (a \leq b \wedge \neg b \leq a)$

lemma OrdSum-first [simp]: $(a, 1) \in \text{OrdSum}$

<proof>

lemma OrdSum-second [simp]: $(1, b) \in \text{OrdSum}$

<proof>

lemma Rep-OrdSum-eq: $\text{Rep-OrdSum } x = \text{Rep-OrdSum } y \implies x = y$

<proof>

lemma Abs-OrdSum-eq: $x \in \text{OrdSum} \implies y \in \text{OrdSum} \implies \text{Abs-OrdSum } x = \text{Abs-OrdSum } y \implies x = y$

<proof>

lemma [simp]: $\text{fst } (\text{Rep-OrdSum } a) \neq 1 \implies (\text{snd } (\text{Rep-OrdSum } a) \neq 1 = \text{False})$

<proof>

lemma fst-not-one-snd: $\text{fst } (\text{Rep-OrdSum } a) \neq 1 \implies (\text{snd } (\text{Rep-OrdSum } a) = 1)$

<proof>

lemma snd-not-one-fst: $\text{snd } (\text{Rep-OrdSum } a) \neq 1 \implies (\text{fst } (\text{Rep-OrdSum } a) = 1)$

<proof>

lemma fst-not-one-simp [simp]: $\text{fst } (\text{Rep-OrdSum } c) \neq 1 \implies \text{Abs-OrdSum } (\text{fst } (\text{Rep-OrdSum } c), 1) = c$

<proof>

lemma snd-not-one-simp [simp]: $\text{snd } (\text{Rep-OrdSum } c) \neq 1 \implies \text{Abs-OrdSum } (1, \text{snd } (\text{Rep-OrdSum } c)) = c$

<proof>

lemma *A: fixes a b::('a, 'b) OrdSum shows (a l→ b) * a = a * (a r→ b)*
⟨*proof*⟩

instance
⟨*proof*⟩

definition
 $Second = \{x . \exists b . x = Abs-OrdSum(1::'a, b::'b)\}$

end

lemma *Second ∈ normalfilters*
⟨*proof*⟩

class *linear-pseudo-hoop-algebra = pseudo-hoop-algebra + linorder*

instantiation *OrdSum :: (linear-pseudo-hoop-algebra, linear-pseudo-hoop-algebra)*
linear-pseudo-hoop-algebra

begin
instance
⟨*proof*⟩
end

instantiation *bool:: pseudo-hoop-algebra*
begin

definition *impl-bool-def:*
 $a \text{ l} \rightarrow b = (a \longrightarrow b)$

definition *impr-bool-def:*
 $a \text{ r} \rightarrow b = (a \longrightarrow b)$

definition *one-bool-def:*
 $1 = True$

definition *times-bool-def:*
 $a * b = (a \wedge b)$

lemma *inf-bool-def: (a :: bool) □ b = (a l→ b) * a*
⟨*proof*⟩

instance
⟨*proof*⟩

end

context *cancel-pseudo-hoop-algebra begin end*

lemma \neg *class.cancel-pseudo-hoop-algebra* (*) (\sqcap) ($l \rightarrow$) (\leq) ($<$) ($1 :: \text{bool}$) ($r \rightarrow$)
 $\langle \text{proof} \rangle$

context *pseudo-hoop-algebra* **begin**

lemma *classorder: class.order* (\leq) ($<$)
 $\langle \text{proof} \rangle$

end

lemma *impl-OrdSum-first: Abs-OrdSum* ($x, 1$) $l \rightarrow$ *Abs-OrdSum* ($y, 1$) = *Abs-OrdSum*
 ($x l \rightarrow y, 1$)
 $\langle \text{proof} \rangle$

lemma *impl-OrdSum-second: Abs-OrdSum* ($1, x$) $l \rightarrow$ *Abs-OrdSum* ($1, y$) = *Abs-OrdSum*
 ($1, x l \rightarrow y$)
 $\langle \text{proof} \rangle$

lemma *impl-OrdSum-first-second: $x \neq 1 \implies$* *Abs-OrdSum* ($x, 1$) $l \rightarrow$ *Abs-OrdSum*
 ($1, y$) = 1
 $\langle \text{proof} \rangle$

lemma *Abs-OrdSum-bijective: $x \in \text{OrdSum} \implies y \in \text{OrdSum} \implies$* (*Abs-OrdSum* x
 = *Abs-OrdSum* y) = ($x = y$)
 $\langle \text{proof} \rangle$

context *pseudo-hoop-algebra* **begin end**

context *linear-pseudo-hoop-algebra* **begin end**

context *basic-pseudo-hoop-algebra* **begin end**

lemma *class.pseudo-hoop-algebra* (*) (\sqcap) ($l \rightarrow$) (\leq) ($<$) ($1 :: 'a :: \text{pseudo-hoop-algebra}$)
 ($r \rightarrow$)
 $\implies \neg$ (*class.linear-pseudo-hoop-algebra* (\leq) ($<$) (*) (\sqcap) ($l \rightarrow$) ($1 :: 'a$) ($r \rightarrow$))
 $\implies \neg$ *class.basic-pseudo-hoop-algebra* (*) (\sqcap) ($l \rightarrow$) (\leq) ($<$) ($1 :: ('a, \text{bool})$
OrdSum) ($r \rightarrow$)
 $\langle \text{proof} \rangle$

end

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