Propositional Resolution and Prime Implicates Generation

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Abstract

We provide formal proofs in Isabelle-HOL (using mostly structured Isar proofs) of the soundness and completeness of the Resolution rule in propositional logic. The completeness proofs take into account the usual redundancy elimination rules (namely tautology elimination and subsumption), and several refinements of the Resolution rule are considered: ordered resolution (with selection functions), positive and negative resolution, semantic resolution and unit resolution (the latter refinement is complete only for clause sets that are Horn-renamable). We also define a concrete procedure for computing saturated sets and establish its soundness and completeness. The clause sets are not assumed to be finite, so that the results can be applied to formulas obtained by grounding sets of first-order clauses (however, a total ordering among atoms is assumed to be given).

Next, we show that the unrestricted Resolution rule is deductive-complete, in the sense that it is able to generate all (prime) implicates of any set of propositional clauses (i.e., all entailment-minimal, non-valid, clausal consequences of the considered set). The generation of prime implicates is an important problem, with many applications in artificial intelligence and verification (for abductive reasoning, knowledge compilation, diagnosis, debugging etc.). We also show that implicates can be computed in an incremental way, by fixing an ordering among all the atoms and resolving upon these atoms one by one in the considered order (with no backtracking). This feature is critical for the efficient computation of prime implicates. Building on these results, we provide a procedure for computing such implicates and establish its soundness and completeness.

Contents

1 Syntax of Propositional Clausal Logic

We define the usual syntactic notions of clausal propositional logic. The set of atoms may be arbitrary (even uncountable), but a well-founded total order is assumed to be given.

theory *Propositional-Resolution*

imports *Main*

begin

locale *propositional-atoms* =
  fixes *atom-ordering* :: $('at \times 'at)$ *set*
  assumes
    atom-ordering-wf :(*wf atom-ordering*)
  and *atom-ordering-total*:($\forall x\ y.\ (x \neq y \longrightarrow ((x,y) \in$ *atom-ordering* $\vee (y,x) \in$ *atom-ordering*)))

and *atom-ordering-trans*: $\forall\, x\ y\ z.\ (x,y) \in$ *atom-ordering* $\longrightarrow (y,z) \in$ *atom-ordering* $\longrightarrow (x,z) \in$ *atom-ordering*
and *atom-ordering-irrefl*: $\forall\, x\ y.\ (x,y) \in$ *atom-ordering* $\longrightarrow (y,x) \notin$ *atom-ordering*
begin

Literals are defined as usual and clauses and formulas are considered as sets. Clause sets are not assumed to be finite (so that the results can be applied to sets of clauses obtained by grounding first-order clauses).

datatype $'a$ *Literal* = *Pos* $'a$ | *Neg* $'a$

definition *atoms* = $\{\ x{::}'at.\ True\ \}$

fun *atom* :: $'a$ *Literal* $\Rightarrow$ $'a$
where
(*atom* (*Pos* A)) = A |
(*atom* (*Neg* A)) = A

fun *complement* :: $'a$ *Literal* $\Rightarrow$ $'a$ *Literal*
where
(*complement* (*Pos* A)) = (*Neg* A) |
(*complement* (*Neg* A)) = (*Pos* A)

lemma *atom-property* : $A = (atom\ L) \Longrightarrow (L = (Pos\ A) \vee L = (Neg\ A))$
$\langle proof \rangle$

fun *positive* :: $'at$ *Literal* $\Rightarrow$ *bool*
where
(*positive* (*Pos* A)) = *True* |
(*positive* (*Neg* A)) = *False*

fun *negative* :: $'at$ *Literal* $\Rightarrow$ *bool*
where
(*negative* (*Pos* A)) = *False* |
(*negative* (*Neg* A)) = *True*

type-synonym $'a$ *Clause* = $'a$ *Literal set*

type-synonym $'a$ *Formula* = $'a$ *Clause set*

Note that the clauses are not assumed to be finite (some of the properties below hold for infinite clauses).

The following functions return the set of atoms occurring in a clause or formula.

fun *atoms-clause* :: $'at$ *Clause* $\Rightarrow$ $'at$ *set*
where *atoms-clause* $C = \{\ A.\ \exists\, L.\ L \in C \wedge A = atom(L)\ \}$

fun *atoms-formula* :: $'at$ *Formula* $\Rightarrow$ $'at$ *set*
where *atoms-formula* $S = \{\ A.\ \exists\, C.\ C \in S \wedge A \in$ *atoms-clause*$(C)\ \}$

lemma *atoms-formula-subset*: $S1 \subseteq S2 \Longrightarrow$ *atoms-formula* $S1 \subseteq$ *atoms-formula* $S2$
⟨*proof*⟩

lemma *atoms-formula-union*: *atoms-formula* $(S1 \cup S2) =$ *atoms-formula* $S1 \cup$ *atoms-formula* $S2$
⟨*proof*⟩

The following predicate is useful to state that every clause in a set fulfills some property.

definition *all-fulfill* :: $('at\ Clause \Rightarrow bool) \Rightarrow 'at\ Formula \Rightarrow bool$
  where *all-fulfill* $P\ S = (\forall C.\ (C \in S \longrightarrow (P\ C)))$

The order on atoms induces a (non total) order among literals:

fun *literal-ordering* :: $'at\ Literal \Rightarrow 'at\ Literal \Rightarrow bool$
where
  (*literal-ordering* $L1\ L2$) = ((*atom* $L1$,*atom* $L2$) $\in$ *atom-ordering*)

lemma *literal-ordering-trans* :
  assumes *literal-ordering* $A\ B$
  assumes *literal-ordering* $B\ C$
  shows *literal-ordering* $A\ C$
⟨*proof*⟩

definition *strictly-maximal-literal* :: $'at\ Clause \Rightarrow 'at\ Literal \Rightarrow bool$
where
  (*strictly-maximal-literal* $S\ A$) $\equiv (A \in S) \land (\forall B.\ (\ B \in S \land A \neq B) \longrightarrow$ (*literal-ordering* $B\ A$))

2 Semantics

We define the notions of interpretation, satisfiability and entailment and establish some basic properties.

type-synonym $'a\ Interpretation = 'a\ set$

fun *validate-literal* :: $'at\ Interpretation \Rightarrow 'at\ Literal \Rightarrow bool$ (**infix** ‹⊨› *65*)
  where
  (*validate-literal* I (*Pos* A)) $= (A \in I)$ |
  (*validate-literal* I (*Neg* A)) $= (A \notin I)$

fun *validate-clause* :: $'at\ Interpretation \Rightarrow 'at\ Clause \Rightarrow bool$ (**infix** ‹⊨› *65*)
  where
  (*validate-clause* $I\ C$) = $(\exists L.\ (L \in C) \land$ (*validate-literal* $I\ L$))

fun *validate-formula* :: $'at\ Interpretation \Rightarrow 'at\ Formula \Rightarrow bool$ (**infix** ‹⊨› *65*)
  where

$(validate\text{-}formula\ I\ S) = (\forall\, C.\ (C \in S \longrightarrow (validate\text{-}clause\ I\ C)))$

definition *satisfiable* :: $'at\ Formula \Rightarrow bool$
where
$(satisfiable\ S) \equiv (\exists\, I.\ (validate\text{-}formula\ I\ S))$

We define the usual notions of entailment between clauses and formulas.

definition *entails* :: $'at\ Formula \Rightarrow\ 'at\ Clause \Rightarrow bool$
where
$(entails\ S\ C) \equiv (\forall\, I.\ (validate\text{-}formula\ I\ S) \longrightarrow (validate\text{-}clause\ I\ C))$

lemma *entails-member*:
assumes $C \in S$
shows *entails S C*
⟨*proof*⟩

definition *entails-formula* :: $'at\ Formula \Rightarrow\ 'at\ Formula \Rightarrow bool$
where $(entails\text{-}formula\ S1\ S2) = (\forall\, C \in S2.\ (entails\ S1\ C))$

definition *equivalent* :: $'at\ Formula \Rightarrow\ 'at\ Formula \Rightarrow bool$
where $(equivalent\ S1\ S2) = (entails\text{-}formula\ S1\ S2 \wedge entails\text{-}formula\ S2\ S1)$

lemma *equivalent-symmetric*: $equivalent\ S1\ S2 \Longrightarrow equivalent\ S2\ S1$
⟨*proof*⟩

lemma *entailment-implies-validity*:
assumes *entails-formula S1 S2*
assumes *validate-formula I S1*
shows *validate-formula I S2*
⟨*proof*⟩

lemma *validity-implies-entailment*:
assumes $\forall\, I.\ validate\text{-}formula\ I\ S1 \longrightarrow validate\text{-}formula\ I\ S2$
shows *entails-formula S1 S2*
⟨*proof*⟩

lemma *entails-transitive*:
assumes *entails-formula S1 S2*
assumes *entails-formula S2 S3*
shows *entails-formula S1 S3*
⟨*proof*⟩

lemma *equivalent-transitive*:
assumes *equivalent S1 S2*
assumes *equivalent S2 S3*
shows *equivalent S1 S3*
⟨*proof*⟩

lemma *entailment-subset* :

assumes $S2 \subseteq S1$
shows *entails-formula S1 S2*
$\langle proof \rangle$

lemma *entailed-formula-entails-implicates*:
assumes *entails-formula S1 S2*
assumes *entails S2 C*
shows *entails S1 C*
$\langle proof \rangle$

3 Inference Rules

We first define an abstract notion of a binary inference rule.

type-synonym $'a\ BinaryRule = 'a\ Clause \Rightarrow 'a\ Clause \Rightarrow 'a\ Clause \Rightarrow bool$

definition *less-restrictive* :: $'at\ BinaryRule \Rightarrow 'at\ BinaryRule \Rightarrow bool$
where
$(less\text{-}restrictive\ R1\ R2) = (\forall\ P1\ P2\ C.\ (R2\ P1\ P2\ C) \longrightarrow ((R1\ P1\ P2\ C) \vee (R1\ P2\ P1\ C)))$

The following functions allow to generate all the clauses that are deducible from a given clause set (in one step).

fun *all-deducible-clauses*:: $'at\ BinaryRule \Rightarrow 'at\ Formula \Rightarrow 'at\ Formula$
where $all\text{-}deducible\text{-}clauses\ R\ S = \{\ C.\ \exists P1\ P2.\ P1 \in S \wedge P2 \in S \wedge (R\ P1\ P2\ C)\ \}$

fun *add-all-deducible-clauses*:: $'at\ BinaryRule \Rightarrow 'at\ Formula \Rightarrow 'at\ Formula$
where $add\text{-}all\text{-}deducible\text{-}clauses\ R\ S = (S \cup all\text{-}deducible\text{-}clauses\ R\ S)$

definition *derived-clauses-are-finite* :: $'at\ BinaryRule \Rightarrow bool$
where $derived\text{-}clauses\text{-}are\text{-}finite\ R =$
$(\forall\ P1\ P2\ C.\ (finite\ P1 \longrightarrow finite\ P2 \longrightarrow (R\ P1\ P2\ C) \longrightarrow finite\ C))$

lemma *less-restrictive-and-finite* :
assumes *less-restrictive R1 R2*
assumes *derived-clauses-are-finite R1*
shows *derived-clauses-are-finite R2*
$\langle proof \rangle$

We then define the unrestricted resolution rule and usual resolution refinements.

3.1 Unrestricted Resolution

definition *resolvent* :: $'at\ BinaryRule$
where
$(resolvent\ P1\ P2\ C) \equiv$

$(\exists A. ((Pos\ A) \in P1 \wedge (Neg\ A) \in P2 \wedge (C = (\ (P1 - \{\ Pos\ A\}) \cup (\ P2 - \{$
$Neg\ A\ \})))))$

For technical convience, we now introduce a slightly extended definition in which resolution upon a literal not occurring in the premises is allowed (the obtained resolvent is then redundant with the premises). If the atom is fixed then this version of the resolution rule can be turned into a total function.

fun *resolvent-upon* :: *'at Clause* ⇒ *'at Clause* ⇒ *'at* ⇒ *'at Clause*
where
(*resolvent-upon P1 P2 A*) =
((*P1* − { *Pos A*}) ∪ (*P2* − { *Neg A* }))

lemma *resolvent-upon-is-resolvent* :
assumes *Pos A* ∈ *P1*
assumes *Neg A* ∈ *P2*
shows *resolvent P1 P2* (*resolvent-upon P1 P2 A*)
⟨*proof*⟩

lemma *resolvent-is-resolvent-upon* :
assumes *resolvent P1 P2 C*
shows ∃ *A*. *C* = *resolvent-upon P1 P2 A*
⟨*proof*⟩

lemma *resolvent-is-finite* :
shows *derived-clauses-are-finite resolvent*
⟨*proof*⟩

In the next subsections we introduce various resolution refinements and show that they are more restrictive than unrestricted resolution.

3.2 Ordered Resolution

In the first refinement, resolution is only allowed on maximal literals.

definition *ordered-resolvent* :: *'at Clause* ⇒ *'at Clause* ⇒ *'at Clause* ⇒ *bool*
where
(*ordered-resolvent P1 P2 C*) ≡
(∃ *A*. ((*C* = ((*P1* − { *Pos A*}) ∪ (*P2* − { *Neg A* })))
∧ (*strictly-maximal-literal P1* (*Pos A*)) ∧ (*strictly-maximal-literal P2* (*Neg*
A))))

We now show that the maximal literal of the resolvent is always smaller than those of the premises.

lemma *resolution-and-max-literal* :
assumes *R* = *resolvent-upon P1 P2 A*
assumes *strictly-maximal-literal P1* (*Pos A*)
assumes *strictly-maximal-literal P2* (*Neg A*)
assumes *strictly-maximal-literal R M*

shows $(atom\ M,\ A) \in atom\text{-}ordering$
$\langle proof \rangle$

3.3 Ordered Resolution with Selection

In the next restriction strategy, some negative literals are selected with highest priority for applying the resolution rule, regardless of the ordering. Relaxed ordering restrictions also apply.

definition $(selected\text{-}part\ Sel\ C) = \{\ L.\ L \in C \land (\exists\, A \in Sel.\ L = (Neg\ A))\ \}$

definition $ordered\text{-}sel\text{-}resolvent :: {'}at\ set \Rightarrow {'}at\ Clause \Rightarrow {'}at\ Clause \Rightarrow {'}at\ Clause \Rightarrow bool$
where
$(ordered\text{-}sel\text{-}resolvent\ Sel\ P1\ P2\ C) \equiv$
$(\exists\, A.\ ((C = (\ (P1 - \{\ Pos\ A\}) \cup (\ P2 - \{\ Neg\ A\ \})))$
$\land\ (strictly\text{-}maximal\text{-}literal\ P1\ (Pos\ A)) \land ((selected\text{-}part\ Sel\ P1) = \{\}) \land$
$(\ ((strictly\text{-}maximal\text{-}literal\ P2\ (Neg\ A)) \land (selected\text{-}part\ Sel\ P2) = \{\})$
$\lor\ (strictly\text{-}maximal\text{-}literal\ (selected\text{-}part\ Sel\ P2)\ (Neg\ A)))))$

lemma $ordered\text{-}resolvent\text{-}is\text{-}resolvent : less\text{-}restrictive\ resolvent\ ordered\text{-}resolvent$
$\langle proof \rangle$

The next lemma states that ordered resolution with selection coincides with ordered resolution if the selected part is empty.

lemma $ordered\text{-}sel\text{-}resolvent\text{-}is\text{-}ordered\text{-}resolvent$:
assumes $ordered\text{-}resolvent\ P1\ P2\ C$
assumes $selected\text{-}part\ Sel\ P1 = \{\}$
assumes $selected\text{-}part\ Sel\ P2 = \{\}$
shows $ordered\text{-}sel\text{-}resolvent\ Sel\ P1\ P2\ C$
$\langle proof \rangle$

lemma $ordered\text{-}resolvent\text{-}upon\text{-}is\text{-}resolvent$:
assumes $strictly\text{-}maximal\text{-}literal\ P1\ (Pos\ A)$
assumes $strictly\text{-}maximal\text{-}literal\ P2\ (Neg\ A)$
shows $ordered\text{-}resolvent\ P1\ P2\ (resolvent\text{-}upon\ P1\ P2\ A)$
$\langle proof \rangle$

3.4 Semantic Resolution

In this strategy, resolution is applied only if one parent is false in some (fixed) interpretation. Note that ordering restrictions still apply, although they are relaxed.

definition $validated\text{-}part :: {'}at\ set \Rightarrow {'}at\ Clause \Rightarrow {'}at\ Clause$
where $(validated\text{-}part\ I\ C) = \{\ L.\ L \in C \land (validate\text{-}literal\ I\ L)\ \}$

definition $ordered\text{-}model\text{-}resolvent ::$
${'}at\ Interpretation \Rightarrow {'}at\ Clause \Rightarrow {'}at\ Clause \Rightarrow {'}at\ Clause \Rightarrow bool$

where
$(ordered\text{-}model\text{-}resolvent\ I\ P1\ P2\ C) =$
$(\exists L.\ (C = (P1 - \{\ L\ \} \cup (P2 - \{\ complement\ L\ \}))) \wedge$
$((validated\text{-}part\ I\ P1) = \{\}\ \ \wedge (strictly\text{-}maximal\text{-}literal\ P1\ L))$
$\wedge\ (strictly\text{-}maximal\text{-}literal\ (validated\text{-}part\ I\ P2)\ (complement\ L)))$

lemma *ordered-model-resolvent-is-resolvent* : *less-restrictive resolvent* (*ordered-model-resolvent* *I*)
⟨*proof*⟩

3.5 Unit Resolution

Resolution is applied only if one parent is unit (this restriction is incomplete).

definition *Unit* :: *'at Clause* ⇒ *bool*
where $(Unit\ C) = ((card\ C) = 1)$

definition *unit-resolvent* :: *'at BinaryRule*
where $(unit\text{-}resolvent\ P1\ P2\ C) =\ \ ((\exists L.\ (C = (\ (P1 - \{\ L\}) \cup (\ P2 - \{$
$complement\ L\ \})))$
$\wedge\ L \in P1 \wedge (complement\ L) \in P2) \wedge Unit\ P1)$

lemma *unit-resolvent-is-resolvent* : *less-restrictive resolvent unit-resolvent*
⟨*proof*⟩

3.6 Positive and Negative Resolution

Resolution is applied only if one parent is positive (resp. negative). Again, relaxed ordering restrictions apply.

definition *positive-part* :: *'at Clause* ⇒ *'at Clause*
where
$(positive\text{-}part\ C) = \{\ L.\ (\exists A.\ L = Pos\ A) \wedge L \in C\ \}$

definition *negative-part* :: *'at Clause* ⇒ *'at Clause*
where
$(negative\text{-}part\ C) = \{\ L.\ (\exists A.\ L = Neg\ A) \wedge L \in C\ \}$

lemma *decomposition-clause-pos-neg* :
$C = (negative\text{-}part\ C) \cup (positive\text{-}part\ C)$
⟨*proof*⟩

definition *ordered-positive-resolvent* :: *'at Clause* ⇒ *'at Clause* ⇒ *'at Clause* ⇒ *bool*
where
$(ordered\text{-}positive\text{-}resolvent\ P1\ P2\ C) =$
$(\exists L.\ (C = (P1 - \{\ L\ \} \cup (P2 - \{\ complement\ L\ \}))) \wedge$
$((negative\text{-}part\ P1) = \{\}\ \ \wedge (strictly\text{-}maximal\text{-}literal\ P1\ L))$
$\wedge\ (strictly\text{-}maximal\text{-}literal\ (negative\text{-}part\ P2)\ (complement\ L)))$

definition *ordered-negative-resolvent* :: $'at$ *Clause* $\Rightarrow$ $'at$ *Clause* $\Rightarrow$ $'at$ *Clause* $\Rightarrow$ *bool*
where
$(ordered\text{-}negative\text{-}resolvent\ P1\ P2\ C) =$
$(\exists L.\ (C = (P1 - \{\ L\ \} \cup (P2 - \{\ complement\ L\ \}))) \wedge$
$((positive\text{-}part\ P1) = \{\}\ \wedge (strictly\text{-}maximal\text{-}literal\ P1\ L))$
$\wedge (strictly\text{-}maximal\text{-}literal\ (positive\text{-}part\ P2)\ (complement\ L)))$

lemma *positive-resolvent-is-resolvent* : *less-restrictive resolvent ordered-positive-resolvent*
⟨*proof*⟩

lemma *negative-resolvent-is-resolvent* : *less-restrictive resolvent ordered-negative-resolvent*
⟨*proof*⟩

4 Redundancy Elimination Rules

We define the usual redundancy elimination rules.

definition *tautology* :: $'a$ *Clause* $\Rightarrow$ *bool*
where
$(tautology\ C) \equiv (\exists\ A.\ (Pos\ A \in C \wedge Neg\ A \in C))$

definition *subsumes* :: $'a$ *Clause* $\Rightarrow$ $'a$ *Clause* $\Rightarrow$ *bool*
where
$(subsumes\ C\ D)\ \equiv (C \subseteq D)$

definition *redundant* :: $'a$ *Clause* $\Rightarrow$ $'a$ *Formula* $\Rightarrow$ *bool*
where
$redundant\ C\ S = ((tautology\ C) \vee (\exists D.\ (D \in S \wedge subsumes\ D\ C)))$

definition *strictly-redundant* :: $'a$ *Clause* $\Rightarrow$ $'a$ *Formula* $\Rightarrow$ *bool*
where
$strictly\text{-}redundant\ C\ S = ((tautology\ C) \vee (\exists D.\ (D \in S \wedge (D \subset C))))$

definition *simplify* :: $'at$ *Formula* $\Rightarrow$ $'at$ *Formula*
where
$simplify\ S = \{\ C.\ C \in S \wedge \neg strictly\text{-}redundant\ C\ S\ \}$

We first establish some basic syntactic properties.

lemma *tautology-monotonous* : $(tautology\ C) \Longrightarrow (C \subseteq D) \Longrightarrow (tautology\ D)$
⟨*proof*⟩

lemma *simplify-involutive*:
shows $simplify\ (simplify\ S) = (simplify\ S)$
⟨*proof*⟩

lemma *simplify-finite*:
assumes *all-fulfill finite S*
shows *all-fulfill finite* $(simplify\ S)$

⟨*proof*⟩

lemma *atoms-formula-simplify*:
 shows *atoms-formula* (*simplify S*) ⊆ *atoms-formula S*
⟨*proof*⟩

lemma *subsumption-preserves-redundancy* :
 assumes *redundant C S*
 assumes *subsumes C D*
 shows *redundant D S*
⟨*proof*⟩

lemma *subsumption-and-max-literal* :
 assumes *subsumes C1 C2*
 assumes *strictly-maximal-literal C1 L1*
 assumes *strictly-maximal-literal C2 L2*
 assumes *A1* = *atom L1*
 assumes *A2* = *atom L2*
 shows (*A1* = *A2*) ∨ (*A1*,*A2*) ∈ *atom-ordering*
⟨*proof*⟩

lemma *superset-preserves-redundancy*:
 assumes *redundant C S*
 assumes $S \subseteq S'$
 shows *redundant C* S'
⟨*proof*⟩

lemma *superset-preserves-strict-redundancy*:
 assumes *strictly-redundant C S*
 assumes *S* ⊆ *SS*
 shows *strictly-redundant C SS*
⟨*proof*⟩

The following lemmas relate the above notions with that of semantic entailment and thus establish the soundness of redundancy elimination rules.

lemma *tautologies-are-valid* :
 assumes *tautology C*
 shows *validate-clause I C*
⟨*proof*⟩

lemma *subsumption-and-semantics* :
 assumes *subsumes C D*
 assumes *validate-clause I C*
 shows *validate-clause I D*
⟨*proof*⟩

lemma *redundancy-and-semantics* :
 assumes *redundant C S*
 assumes *validate-formula I S*

shows *validate-clause I C*
⟨*proof*⟩

lemma *redundancy-implies-entailment*:
assumes *redundant C S*
shows *entails S C*
⟨*proof*⟩

lemma *simplify-and-membership* :
assumes *all-fulfill finite S*
assumes $T = simplify\ S$
assumes $C \in S$
shows *redundant C T*
⟨*proof*⟩

lemma *simplify-preserves-redundancy*:
assumes *all-fulfill finite S*
assumes *redundant C S*
shows *redundant C* (*simplify S*)
⟨*proof*⟩

lemma *simplify-preserves-strict-redundancy*:
assumes *all-fulfill finite S*
assumes *strictly-redundant C S*
shows *strictly-redundant C* (*simplify S*)
⟨*proof*⟩

lemma *simplify-preserves-semantic* :
assumes $T = simplify\ S$
assumes *all-fulfill finite S*
shows *validate-formula I S* $\longleftrightarrow$ *validate-formula I T*
⟨*proof*⟩

lemma *simplify-preserves-equivalence* :
assumes $T = simplify\ S$
assumes *all-fulfill finite S*
shows *equivalent S T*
⟨*proof*⟩

After simplification, the formula contains no strictly redundant clause:

definition *non-redundant* :: *'at Formula* $\Rightarrow$ *bool*
where *non-redundant S* = ($\forall$ *C*. ($C \in S \longrightarrow \neg$*strictly-redundant C S*))

lemma *simplify-non-redundant*:
shows *non-redundant* (*simplify S*)
⟨*proof*⟩

lemma *deducible-clause-preserve-redundancy*:
assumes *redundant C S*

shows *redundant C* (*add-all-deducible-clauses R S*)
⟨*proof*⟩

5 Renaming

A renaming is a function changing the sign of some literals. We show that this operation preserves most of the previous syntactic and semantic notions.

definition *rename-literal* :: *'at set* ⇒ *'at Literal* ⇒ *'at Literal*
where *rename-literal A L* = (*if* ((*atom L*) ∈ *A*) *then* (*complement L*) *else L*)

definition *rename-clause* :: *'at set* ⇒ *'at Clause* ⇒ *'at Clause*
where *rename-clause A C* = {*L*. ∃ *LL*. *LL* ∈ *C* ∧ *L* = (*rename-literal A LL*)}

definition *rename-formula* :: *'at set* ⇒ *'at Formula* ⇒ *'at Formula*
where *rename-formula A S* = {*C*. ∃ *CC*. *CC* ∈ *S* ∧ *C* = (*rename-clause A CC*)}

lemma *inverse-renaming* : (*rename-literal A* (*rename-literal A L*)) = *L*
⟨*proof*⟩

lemma *inverse-clause-renaming* : (*rename-clause A* (*rename-clause A L*)) = *L*
⟨*proof*⟩

lemma *inverse-formula-renaming* : *rename-formula A* (*rename-formula A L*) = *L*
⟨*proof*⟩

lemma *renaming-preserves-cardinality* :
card (*rename-clause A C*) = *card C*
⟨*proof*⟩

lemma *renaming-preserves-literal-order* :
assumes *literal-ordering L1 L2*
shows *literal-ordering* (*rename-literal A L1*) (*rename-literal A L2*)
⟨*proof*⟩

lemma *inverse-renaming-preserves-literal-order* :
assumes *literal-ordering* (*rename-literal A L1*) (*rename-literal A L2*)
shows *literal-ordering L1 L2*
⟨*proof*⟩

lemma *renaming-is-injective*:
assumes *rename-literal A L1* = *rename-literal A L2*
shows *L1* = *L2*
⟨*proof*⟩

lemma *renaming-preserves-strictly-maximal-literal* :
assumes *strictly-maximal-literal C L*
shows *strictly-maximal-literal* (*rename-clause A C*) (*rename-literal A L*)
⟨*proof*⟩

lemma *renaming-and-selected-part* :
  selected-part UNIV C = *rename-clause Sel* (*validated-part Sel* (*rename-clause Sel C*))
⟨*proof*⟩

lemma *renaming-preserves-tautology*:
  assumes *tautology C*
  shows *tautology* (*rename-clause Sel C*)
⟨*proof*⟩

lemma *rename-union* : *rename-clause Sel* ($C \cup D$) = *rename-clause Sel C* $\cup$ *rename-clause Sel D*
⟨*proof*⟩

lemma *renaming-set-minus-subset* :
  rename-clause Sel ($C - \{ L \}$) $\subseteq$ *rename-clause Sel C* $-$ {*rename-literal Sel L* }
⟨*proof*⟩

lemma *renaming-set-minus* : *rename-clause Sel* ($C - \{ L \}$)
  = (*rename-clause Sel C*) $-$ {*rename-literal Sel L* }
⟨*proof*⟩

definition *rename-interpretation* :: *'at set* $\Rightarrow$ *'at Interpretation* $\Rightarrow$ *'at Interpretation*
where
  rename-interpretation Sel I = { *A*. ($A \in I \wedge A \notin$ *Sel*) } $\cup$ { *A*. ($A \notin I \wedge A \in$ *Sel*) }

lemma *renaming-preserves-semantic* :
  assumes *validate-literal I L*
  shows *validate-literal* (*rename-interpretation Sel I*) (*rename-literal Sel L*)
⟨*proof*⟩

lemma *renaming-preserves-satisfiability*:
  assumes *satisfiable S*
  shows *satisfiable* (*rename-formula Sel S*)
⟨*proof*⟩

lemma *renaming-preserves-subsumption*:
  assumes *subsumes C D*
  shows *subsumes* (*rename-clause Sel C*) (*rename-clause Sel D*)
⟨*proof*⟩

6 Soundness

In this section we prove that all the rules introduced in the previous section are sound. We first introduce an abstract notion of soundness.

definition *Sound* :: *'at BinaryRule* ⇒ *bool*
where
  (*Sound Rule*) ≡ ∀ *I P1 P2 C*. (*Rule P1 P2 C* ⟶ (*validate-clause I P1*) ⟶ (*validate-clause I P2*)
    ⟶ (*validate-clause I C*))

lemma *soundness-and-entailment* :
  assumes *Sound Rule*
  assumes *Rule P1 P2 C*
  assumes *P1* ∈ *S*
  assumes *P2* ∈ *S*
  shows *entails S C*
⟨*proof*⟩

lemma *all-deducible-sound*:
  assumes *Sound R*
  shows *entails-formula S* (*all-deducible-clauses R S*)
⟨*proof*⟩

lemma *add-all-deducible-sound*:
  assumes *Sound R*
  shows *entails-formula S* (*add-all-deducible-clauses R S*)
⟨*proof*⟩

If a rule is more restrictive than a sound rule then it is necessarily sound.

lemma *less-restrictive-correct*:
  assumes *less-restrictive R1 R2*
  assumes *Sound R1*
  shows *Sound R2*
⟨*proof*⟩

We finally establish usual concrete soundness results.

theorem *resolution-is-correct*:
  (*Sound resolvent*)
⟨*proof*⟩

theorem *ordered-resolution-correct* : *Sound ordered-resolvent*
⟨*proof*⟩

theorem *ordered-model-resolution-correct* : *Sound* (*ordered-model-resolvent I*)
⟨*proof*⟩

theorem *ordered-positive-resolution-correct* : *Sound ordered-positive-resolvent*
⟨*proof*⟩

theorem *ordered-negative-resolution-correct* : *Sound ordered-negative-resolvent*
⟨*proof*⟩

theorem *unit-resolvent-correct* : *Sound unit-resolvent*

$\langle proof \rangle$

7 Refutational Completeness

In this section we establish the refutational completeness of the previous inference rules (under adequate restrictions for the unit resolution rule). Completeness is proven w.r.t. redundancy elimination rules, i.e., we show that every saturated unsatisfiable clause set contains the empty clause.

We first introduce an abstract notion of saturation.

definition *saturated-binary-rule* :: *'a BinaryRule* ⇒ *'a Formula* ⇒ *bool*
where
(*saturated-binary-rule Rule S*) ≡ (∀ *P1 P2 C*. (((*P1* ∈ *S*) ∧ (*P2* ∈ *S*) ∧ (*Rule P1 P2 C*)))
⟶ *redundant C S*)

definition *Complete* :: *'at BinaryRule* ⇒ *bool*
where
(*Complete Rule*) = (∀ *S*. ((*saturated-binary-rule Rule S*) ⟶ (*all-fulfill finite S*)
⟶ ({} ∉ *S*) ⟶ *satisfiable S*))

If a set of clauses is saturated under some rule then it is necessarily saturated under more restrictive rules, which entails that if a rule is less restrictive than a complete rule then it is also complete.

lemma *less-restrictive-saturated*:
assumes *less-restrictive R1 R2*
assumes *saturated-binary-rule R1 S*
shows *saturated-binary-rule R2 S*
$\langle proof \rangle$

lemma *less-restrictive-complete*:
assumes *less-restrictive R1 R2*
assumes *Complete R2*
shows *Complete R1*
$\langle proof \rangle$

7.1 Ordered Resolution

We define a function associating every set of clauses S with a "canonic" interpretation constructed from S. If S is saturated under ordered resolution and does not contain the empty clause then the interpretation is a model of S. The interpretation is defined by mean of an auxiliary function that maps every atom to a function indicating whether the atom occurs in the interpretation corresponding to a given clause set. The auxiliary function is defined by induction on the set of atoms.

function *canonic-int-fun-ordered* :: *'at* ⇒ (*'at Formula* ⇒ *bool*)

where
($canonic\text{-}int\text{-}fun\text{-}ordered\ A$) =
($\lambda S.$ ($\exists$ $C.$ ($C \in S$) $\wedge$ ($strictly\text{-}maximal\text{-}literal\ C$ ($Pos\ A$))
$\wedge$ ($\forall$ $B.$ ($Pos\ B \in C \longrightarrow (B, A) \in atom\text{-}ordering \longrightarrow (\neg(canonic\text{-}int\text{-}fun\text{-}ordered$
$B)\ S)))$
$\wedge$ ($\forall$ $B.$ ($Neg\ B \in C \longrightarrow (B, A) \in atom\text{-}ordering \longrightarrow ((canonic\text{-}int\text{-}fun\text{-}ordered$
$B)\ S)))))$
⟨*proof*⟩
termination ⟨*proof*⟩

definition *canonic-int-ordered* :: *'at Formula* ⇒ *'at Interpretation*
where
($canonic\text{-}int\text{-}ordered\ S$) = $\{\ A.\ ((canonic\text{-}int\text{-}fun\text{-}ordered\ A)\ S)\ \}$

We first prove that the canonic interpretation validates every clause having a positive strictly maximal literal

lemma *int-validate-cl-with-pos-max* :
assumes *strictly-maximal-literal C* (*Pos A*)
assumes $C \in S$
shows *validate-clause* (*canonic-int-ordered S*) *C*
⟨*proof*⟩

lemma *strictly-maximal-literal-exists* :

$\forall C.$ (((*finite C*) $\wedge$ (*card C*) $= n \wedge n \neq 0 \wedge$ ($\neg$ (*tautology C*))))
$\longrightarrow$ ($\exists A.$ (*strictly-maximal-literal C A*)) (**is** *?P n*)

⟨*proof*⟩

We then deduce that all clauses are validated.

lemma *canonic-int-validates-all-clauses* :
assumes *saturated-binary-rule ordered-resolvent S*
assumes *all-fulfill finite S*
assumes $\{\} \notin S$
assumes $C \in S$
shows *validate-clause* (*canonic-int-ordered S*) *C*
⟨*proof*⟩

theorem *ordered-resolution-is-complete* :
Complete ordered-resolvent
⟨*proof*⟩

7.2 Ordered Resolution with Selection

We now consider the case where some negative literals are considered with highest priority. The proof reuses the canonic interpretation defined in the previous section. The interpretation is constructed using only clauses with no selected literals. By the previous result, all such clauses must be satisfied.

We then show that the property carries over to the clauses with non empty selected part.

definition *empty-selected-part Sel S* = { *C*. *C* ∈ *S* ∧ (*selected-part Sel C*) = {} }

lemma *saturated-ordered-sel-res-empty-sel* :
 assumes *saturated-binary-rule* (*ordered-sel-resolvent Sel*) *S*
 shows *saturated-binary-rule ordered-resolvent* (*empty-selected-part Sel S*)
⟨*proof*⟩

definition *ordered-sel-resolvent-upon* :: *'at set* ⇒ *'at Clause* ⇒ *'at Clause* ⇒ *'at Clause* ⇒ *'at* ⇒ *bool*
 where
 ordered-sel-resolvent-upon Sel P1 P2 C A ≡
  (((*C* = ((*P1* − { *Pos A*}) ∪ (*P2* − { *Neg A* })))
   ∧ (*strictly-maximal-literal P1* (*Pos A*)) ∧ ((*selected-part Sel P1*) = {})
   ∧ (((*strictly-maximal-literal P2* (*Neg A*)) ∧ (*selected-part Sel P2*) = {})
    ∨ (*strictly-maximal-literal* (*selected-part Sel P2*) (*Neg A*)))))

lemma *ordered-sel-resolvent-upon-is-resolvent*:
 assumes *ordered-sel-resolvent-upon Sel P1 P2 C A*
 shows *ordered-sel-resolvent Sel P1 P2 C*
⟨*proof*⟩

lemma *resolution-decreases-selected-part*:
 assumes *ordered-sel-resolvent-upon Sel P1 P2 C A*
 assumes *Neg A* ∈ *P2*
 assumes *finite P1*
 assumes *finite P2*
 assumes *card* (*selected-part Sel P2*) = *Suc n*
 shows *card* (*selected-part Sel C*) = *n*
⟨*proof*⟩

lemma *canonic-int-validates-all-clauses-sel* :
 assumes *saturated-binary-rule* (*ordered-sel-resolvent Sel*) *S*
 assumes *all-fulfill finite S*
 assumes {} ∉ *S*
 assumes *C* ∈ *S*
 shows *validate-clause* (*canonic-int-ordered* (*empty-selected-part Sel S*)) *C*
⟨*proof*⟩

theorem *ordered-resolution-is-complete-ordered-sel* :
 Complete (*ordered-sel-resolvent Sel*)
⟨*proof*⟩

7.3 Semantic Resolution

We show that under some particular renaming, model resolution simulates ordered resolution where all negative literals are selected, which immediately entails the refutational completeness of model resolution.

lemma *ordered-res-with-selection-is-model-res* :
assumes *ordered-sel-resolvent UNIV P1 P2 C*
shows *ordered-model-resolvent Sel* (*rename-clause Sel P1*) (*rename-clause Sel P2*)
(*rename-clause Sel C*)
⟨*proof*⟩

theorem *ordered-resolution-is-complete-model-resolution*:
Complete (*ordered-model-resolvent Sel*)
⟨*proof*⟩

7.4 Positive and Negative Resolution

We show that positive and negative resolution simulate model resolution with some specific interpretation. Then completeness follows from previous results.

lemma *empty-interpretation-validate* :
validate-literal {} $L = (\exists A. (L = Neg\ A))$
⟨*proof*⟩

lemma *universal-interpretation-validate* :
validate-literal UNIV $L = (\exists A. (L = Pos\ A))$
⟨*proof*⟩

lemma *negative-part-lemma*:
(*negative-part C*) = (*validated-part* {} *C*)
⟨*proof*⟩

lemma *positive-part-lemma*:
(*positive-part C*) = (*validated-part UNIV C*)
⟨*proof*⟩

lemma *negative-resolvent-is-model-res*:
less-restrictive ordered-negative-resolvent (*ordered-model-resolvent UNIV*)
⟨*proof*⟩

lemma *positive-resolvent-is-model-res*:
less-restrictive ordered-positive-resolvent (*ordered-model-resolvent* {})
⟨*proof*⟩

theorem *ordered-positive-resolvent-is-complete* : *Complete ordered-positive-resolvent*
⟨*proof*⟩

theorem *ordered-negative-resolvent-is-complete*: *Complete ordered-negative-resolvent*
⟨*proof*⟩

7.5 Unit Resolution and Horn Renamable Clauses

Unit resolution is complete if the considered clause set can be transformed into a Horn clause set by renaming. This result is proven by showing that unit resolution simulates semantic resolution for Horn-renamable clauses (for some specific interpretation).

definition *Horn* :: *'at Clause* ⇒ *bool*
  where (*Horn C*) = ((*card* (*positive-part C*)) ≤ *1*)

definition *Horn-renamable-formula* :: *'at Formula* ⇒ *bool*
  where *Horn-renamable-formula S* = (∃ *I*. (*all-fulfill Horn* (*rename-formula I S*)))

theorem *unit-resolvent-complete-for-Horn-renamable-set*:
  assumes *saturated-binary-rule unit-resolvent S*
  assumes *all-fulfill finite S*
  assumes {} ∉ *S*
  assumes *Horn-renamable-formula S*
  shows *satisfiable S*
⟨*proof*⟩

8 Computation of Saturated Clause Sets

We now provide a concrete (rather straightforward) procedure for computing saturated clause sets. Starting from the initial set, we define a sequence of clause sets, where each set is obtained from the previous one by applying the resolution rule in a systematic way, followed by redundancy elimination rules. The algorithm is generic, in the sense that it applies to any binary inference rule.

fun *inferred-clause-sets* :: *'at BinaryRule* ⇒ *'at Formula* ⇒ *nat* ⇒ *'at Formula*
 where
  (*inferred-clause-sets R S 0*) = (*simplify S*) |
  (*inferred-clause-sets R S* (*Suc N*)) =
    (*simplify* (*add-all-deducible-clauses R* (*inferred-clause-sets R S N*)))

The saturated set is constructed by considering the set of persistent clauses, i.e., the clauses that are generated and never deleted.

fun *saturation* :: *'at BinaryRule* ⇒ *'at Formula* ⇒ *'at Formula*
 where
  saturation R S = { *C*. ∃ *N*. (∀ *M*. (*M* ≥ *N* ⟶ *C* ∈ *inferred-clause-sets R S M*))
}

We prove that all inference rules yield finite clauses.

theorem *ordered-resolvent-is-finite* : *derived-clauses-are-finite ordered-resolvent*
⟨*proof*⟩

theorem *model-resolvent-is-finite* : *derived-clauses-are-finite* (*ordered-model-resolvent* I)
⟨*proof*⟩

theorem *positive-resolvent-is-finite* : *derived-clauses-are-finite ordered-positive-resolvent*
⟨*proof*⟩

theorem *negative-resolvent-is-finite* : *derived-clauses-are-finite ordered-negative-resolvent*
⟨*proof*⟩

theorem *unit-resolvent-is-finite* : *derived-clauses-are-finite unit-resolvent*
⟨*proof*⟩

lemma *all-deducible-clauses-are-finite*:
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows *all-fulfill finite* (*all-deducible-clauses R S*)
⟨*proof*⟩

This entails that all the clauses occurring in the sets in the sequence are finite.

lemma *all-inferred-clause-sets-are-finite*:
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows *all-fulfill finite* (*inferred-clause-sets R S N*)
⟨*proof*⟩

lemma *add-all-deducible-clauses-finite*:
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows *all-fulfill finite* (*add-all-deducible-clauses R* (*inferred-clause-sets R S N*))
⟨*proof*⟩

We show that the set of redundant clauses can only increase.

lemma *sequence-of-inferred-clause-sets-is-monotonous*:
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows $\forall$ *C*. *redundant C* (*inferred-clause-sets R S N*)
$\longrightarrow$ *redundant C* (*inferred-clause-sets R S* (N+M::*nat*))

⟨*proof*⟩

We show that non-persistent clauses are strictly redundant in some element of the sequence.

lemma *non-persistent-clauses-are-redundant*:
assumes $D \in$ *inferred-clause-sets R S N*
assumes $D \notin$ *saturation R S*
assumes *all-fulfill finite S*

assumes *derived-clauses-are-finite R*
shows $\exists M$. *strictly-redundant D* (*inferred-clause-sets R S M*)
⟨*proof*⟩

This entails that the clauses that are redundant in some set in the sequence are also redundant in the set of persistent clauses.

lemma *persistent-clauses-subsume-redundant-clauses*:
assumes *redundant C* (*inferred-clause-sets R S N*)
assumes *all-fulfill finite S*
assumes *derived-clauses-are-finite R*
assumes *finite C*
shows *redundant C* (*saturation R S*)
⟨*proof*⟩

We deduce that the set of persistent clauses is saturated.

theorem *persistent-clauses-are-saturated*:
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows *saturated-binary-rule R* (*saturation R S*)
⟨*proof*⟩

Finally, we show that the computed saturated set is equivalent to the initial formula.

theorem *saturation-is-correct*:
assumes *Sound R*
assumes *derived-clauses-are-finite R*
assumes *all-fulfill finite S*
shows *equivalent S* (*saturation R S*)
⟨*proof*⟩

end

end

9 Prime Implicates Generation

We show that the unrestricted resolution rule is deductive complete, i.e. that it is able to generate all (prime) implicates of any given clause set.

theory *Prime-Implicates*

imports *Propositional-Resolution*

begin

context *propositional-atoms*

begin

9.1 Implicates and Prime Implicates

We first introduce the definitions of implicates and prime implicates.

definition *implicates* :: $'at$ *Formula* $\Rightarrow$ $'at$ *Formula*
where *implicates* $S = \{ \ C. \ entails \ S \ C \ \}$

definition *prime-implicates* :: $'at$ *Formula* $\Rightarrow$ $'at$ *Formula*
where *prime-implicates* $S =$ *simplify* (*implicates* S)

9.2 Generation of Prime Implicates

We introduce a function simplifying a given clause set by evaluating some literals to false. We show that this partial evaluation operation preserves saturatedness and that if the considered set of literals is an implicate of the initial clause set then the partial evaluation yields a clause set that is unsatisfiable. Then the proof follows from refutational completeness: since the partially evaluated set is unsatisfiable and saturated it must contain the empty clause, and therefore the initial clause set necessarily contains a clause subsuming the implicate.

fun *partial-evaluation* :: $'a$ *Formula* $\Rightarrow$ $'a$ *Literal set* $\Rightarrow$ $'a$ *Formula*
where
(*partial-evaluation* S C) $= \{ \ E. \ \exists D. \ D \in S \land E = D - C \land \neg(\exists L. \ (L \in C) \land$ (*complement* $L) \in D)\}$

lemma *partial-evaluation-is-saturated* :
assumes *saturated-binary-rule resolvent* S
shows *saturated-binary-rule ordered-resolvent* (*partial-evaluation* S C)
⟨*proof*⟩

lemma *evaluation-wrt-implicate-is-unsat* :
assumes *entails* S C
assumes ¬*tautology* C
shows ¬*satisfiable* (*partial-evaluation* S C)
⟨*proof*⟩

lemma *entailment-and-implicates*:
assumes *entails-formula S1 S2*
shows *implicates S2* $\subseteq$ *implicates S1*
⟨*proof*⟩

lemma *equivalence-and-implicates*:
assumes *equivalent S1 S2*
shows *implicates S1* $=$ *implicates S2*
⟨*proof*⟩

lemma *equivalence-and-prime-implicates*:
assumes *equivalent S1 S2*
shows *prime-implicates S1* $=$ *prime-implicates S2*

⟨*proof*⟩

lemma *unrestricted-resolution-is-deductive-complete* :
  assumes *saturated-binary-rule resolvent S*
  assumes *all-fulfill finite S*
  assumes $C \in$ *implicates S*
  shows *redundant C S*
⟨*proof*⟩

lemma *prime-implicates-generation-correct* :
  assumes *saturated-binary-rule resolvent S*
  assumes *non-redundant S*
  assumes *all-fulfill finite S*
  shows $S \subseteq$ *prime-implicates S*
⟨*proof*⟩

theorem *prime-implicates-of-saturated-sets*:
  assumes *saturated-binary-rule resolvent S*
  assumes *all-fulfill finite S*
  assumes *non-redundant S*
  shows $S =$ *prime-implicates S*
⟨*proof*⟩

9.3 Incremental Prime Implicates Computation

We show that it is possible to compute the set of prime implicates incrementally i.e., to fix an ordering among atoms, and to compute the set of resolvents upon each atom one by one, without backtracking (in the sense that if the resolvents upon a given atom are generated at some step i then no resolvents upon the same atom are generated at step $i < j$. This feature is critical in practice for the efficiency of prime implicates generation algorithms.

We first introduce a function computing all resolvents upon a given atom.

definition *all-resolvents-upon* :: *'at Formula* $\Rightarrow$ *'at* $\Rightarrow$ *'at Formula*
 where (*all-resolvents-upon S A*) = { *C*. $\exists$ *P1 P2*. *P1* $\in$ *S* $\wedge$ *P2* $\in$ *S* $\wedge$ *C* = (*resolvent-upon P1 P2 A*) }

lemma *resolvent-upon-correct*:
  assumes *P1* $\in$ *S*
  assumes *P2* $\in$ *S*
  assumes *C* = *resolvent-upon P1 P2 A*
  shows *entails S C*
⟨*proof*⟩

lemma *all-resolvents-upon-is-finite*:
  assumes *all-fulfill finite S*
  shows *all-fulfill finite* (*S* $\cup$ (*all-resolvents-upon S A*))

$\langle proof \rangle$

lemma *atoms-formula-resolvents*:
shows *atoms-formula* (*all-resolvents-upon S A*) $\subseteq$ *atoms-formula S*
$\langle proof \rangle$

We define a partial saturation predicate that is restricted to a specific atom.

definition *partial-saturation* :: *'at Formula* $\Rightarrow$ *'at* $\Rightarrow$ *'at Formula* $\Rightarrow$ *bool*
where
(*partial-saturation S A R*) = ($\forall$ *P1 P2*. (*P1* $\in$ *S* $\longrightarrow$ *P2* $\in$ *S*
$\longrightarrow$(*redundant* (*resolvent-upon P1 P2 A*) *R*)))

We show that the resolvent of two redundant clauses in a partially saturated set is itself redundant.

lemma *resolvent-upon-and-partial-saturation* :
assumes *redundant P1 S*
assumes *redundant P2 S*
assumes *partial-saturation S A* (*S* $\cup$ *R*)
assumes *C* = *resolvent-upon P1 P2 A*
shows *redundant C* (*S* $\cup$ *R*)
$\langle proof \rangle$

We show that if R is a set of resolvents of a set of clauses S then the same holds for $S \cup R$. For the clauses in S, the premises are identical to the resolvent and the inference is thus redundant (this trick is useful to simplify proofs).

definition *in-all-resolvents-upon*:: *'at Formula* $\Rightarrow$ *'at* $\Rightarrow$ *'at Clause* $\Rightarrow$ *bool*
where
in-all-resolvents-upon S A C = ($\exists$ *P1 P2*. (*P1* $\in$ *S* $\wedge$ *P2* $\in$ *S* $\wedge$ *C* = *resolvent-upon P1 P2 A*))

lemma *every-clause-is-a-resolvent*:
assumes *all-fulfill* (*in-all-resolvents-upon S A*) *R*
assumes *all-fulfill* (λx. $\neg$(*tautology x*)) *S*
assumes *P1* $\in$ *S* $\cup$ *R*
shows *in-all-resolvents-upon S A P1*
$\langle proof \rangle$

We show that if a formula is partially saturated then it stays so when new resolvents are added in the set.

lemma *partial-saturation-is-preserved* :
assumes *partial-saturation S E1 S*
assumes *partial-saturation S E2* (*S* $\cup$ *R*)
assumes *all-fulfill* (λx. $\neg$(*tautology x*)) *S*
assumes *all-fulfill* (*in-all-resolvents-upon S E2*) *R*
shows *partial-saturation* (*S* $\cup$ *R*) *E1* (*S* $\cup$ *R*)
$\langle proof \rangle$

The next lemma shows that the clauses inferred by applying the resolution rule upon a given atom contain no occurrence of this atom, unless the inference is redundant.

lemma *resolvents-do-not-contain-atom* :
  assumes $\neg$ *tautology P1*
  assumes $\neg$ *tautology P2*
  assumes C = *resolvent-upon P1 P2 E2*
  assumes $\neg$ *subsumes P1 C*
  assumes $\neg$ *subsumes P2 C*
  shows $(Neg\ E2) \notin C \wedge (Pos\ E2) \notin C$
$\langle proof \rangle$

The next lemma shows that partial saturation can be ensured by computing all (non-redundant) resolvents upon the considered atom.

lemma *ensures-partial-saturation* :
  assumes *partial-saturation* $S\ E2\ (S \cup R)$
  assumes *all-fulfill* $(\lambda x.\ \neg(tautology\ x))\ S$
  assumes *all-fulfill* $(in\text{-}all\text{-}resolvents\text{-}upon\ S\ E2)\ R$
  assumes *all-fulfill* $(\lambda x.\ (\neg redundant\ x\ S))\ R$
  shows *partial-saturation* $(S \cup R)\ E2\ (S \cup R)$
$\langle proof \rangle$

lemma *resolvents-preserve-equivalence*:
  shows *equivalent* $S\ (S \cup (all\text{-}resolvents\text{-}upon\ S\ A))$
$\langle proof \rangle$

Given a sequence of atoms, we define a sequence of clauses obtained by resolving upon each atom successively. Simplification rules are applied at each iteration step.

fun *resolvents-sequence* :: $(nat \Rightarrow 'at) \Rightarrow 'at\ Formula \Rightarrow nat \Rightarrow 'at\ Formula$
 where
  $(resolvents\text{-}sequence\ A\ S\ 0) = (simplify\ S)\ |$
  $(resolvents\text{-}sequence\ A\ S\ (Suc\ N)) =$
    $(simplify\ ((resolvents\text{-}sequence\ A\ S\ N)$
      $\cup\ (all\text{-}resolvents\text{-}upon\ (resolvents\text{-}sequence\ A\ S\ N)\ (A\ N))))$

The following lemma states that partial saturation is preserved by simplification.

lemma *redundancy-implies-partial-saturation*:
  assumes *partial-saturation S1 A S1*
  assumes $S2 \subseteq S1$
  assumes *all-fulfill* $(\lambda x.\ redundant\ x\ S2)\ S1$
  shows *partial-saturation S2 A S2*
$\langle proof \rangle$

The next theorem finally states that the implicate generation algorithm is sound and complete in the sense that the final clause set in the sequence is exactly the set of prime implicates of the considered clause set.

theorem *incremental-prime-implication-generation*:
  assumes *atoms-formula* $S = \{ X. \exists I{::}nat.\ I < N \wedge X = (A\ I) \}$
  assumes *all-fulfill finite* S
  shows (*prime-implicates* S) = (*resolvents-sequence* $A\ S\ N$)
⟨*proof*⟩

end
end