

Probabilistic Noninterference

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June 17, 2024

Abstract

We formalize a probabilistic noninterference for a multi-threaded language with uniform scheduling, where probabilistic behaviour comes from both the scheduler and the individual threads. We define notions probabilistic noninterference in two variants: resumption-based and trace-based. For the resumption-based notions, we prove compositionality w.r.t. the language constructs and establish sound type-system-like syntactic criteria.

This is a formalization of the mathematical development presented in the papers [1, 2]. It is the probabilistic variant of the [Possibilistic Noninterference AFP](#) entry.

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1 Simple While Language with probabilistic choice and parallel execution

```
theory Language-Semantics
imports Interface
begin
```

1.1 Preliminaries

Trivia

```
declare zero-le-mult-iff[simp]
declare split-mult-pos-le[simp]
declare zero-le-divide-iff[simp]
```

```
lemma in-minus-Un[simp]:
assumes  $i \in I$ 
shows  $I - \{i\} \text{ Un } \{i\} = I$  and  $\{i\} \text{ Un } (I - \{i\}) = I$ 
apply(metis Un-commute assms insert-Diff-single insert-absorb insert-is-Un)
by (metis assms insert-Diff-single insert-absorb insert-is-Un)
```

```
lemma less-plus-cases[case-names Left Right]:
assumes
*:  $(i::nat) < n1 \implies phi$  and
**:  $\bigwedge i2. i = n1 + i2 \implies phi$ 
shows phi
proof(cases  $i < n1$ )
case True
thus ?thesis using * by simp
next
case False hence  $n1 \leq i$  by simp
then obtain i2 where  $i = n1 + i2$  by (metis le-iff-add)
thus ?thesis using ** by blast
```

qed

lemma *less-plus-elim*[*elim!*, *consumes 1*, *case-names Left Right*]:
assumes $i: (i:: \text{nat}) < n1 + n2$ **and**
*****: $i < n1 \implies \text{phi}$ **and**
******: $\bigwedge i2. \llbracket i2 < n2; i = n1 + i2 \rrbracket \implies \text{phi}$
shows *phi*
apply(*rule less-plus-cases*[*of i n1*])
using *assms* **by** *auto*

lemma *nth-append-singl*[*simp*]:
 $i < \text{length } al \implies (al @ [a]) ! i = al ! i$
by (*auto simp add: nth-append*)

lemma *take-append-singl*[*simp*]:
assumes $n < \text{length } al$ **shows** $\text{take } n (al @ [a]) = \text{take } n al$
using *assms* **by** (*induct al rule: rev-induct*) *auto*

lemma *length-unique-prefix*:
 $al1 \leq al \implies al2 \leq al \implies \text{length } al1 = \text{length } al2 \implies al1 = al2$
by (*metis not-equal-is-parallel parallelE prefix-same-cases less-eq-list-def*)

take:

lemma *take-length*[*simp*]:
 $\text{take } (\text{length } al) al = al$
using *take-all* **by** *auto*

lemma *take-le*:
assumes $n < \text{length } al$
shows $\text{take } n al @ [al ! n] \leq al$
by (*metis assms take-Suc-conv-app-nth take-is-prefix less-eq-list-def*)

lemma *list-less-iff-prefix*: $a < b \iff \text{strict-prefix } a b$
by (*metis le-less less-eq-list-def less-irrefl prefix-order.le-less prefix-order.less-irrefl*)

lemma *take-lt*:
 $n < \text{length } al \implies \text{take } n al < al$
unfolding *list-less-iff-prefix*
using *prefix-order.le-less*[*of take n al al*]
by (*simp add: take-is-prefix*)

lemma *le-take*:
assumes $n1 \leq n2$
shows $\text{take } n1 al \leq \text{take } n2 al$
using *assms* **proof**(*induct al arbitrary: n1 n2*)
 case (*Cons a al*)
 thus *?case*
 by (*cases n1 n2 rule: nat.exhaust[case-product nat.exhaust]*) *auto*
qed *auto*

lemma *inj-take*:
assumes $n1 \leq \text{length } al$ **and** $n2 \leq \text{length } al$
shows $\text{take } n1 \text{ } al = \text{take } n2 \text{ } al \longleftrightarrow n1 = n2$
proof –
 {**assume** $\text{take } n1 \text{ } al = \text{take } n2 \text{ } al$
 hence $n1 = n2$
 using *assms* **proof**(*induct al arbitrary: n1 n2*)
 case (*Cons a al*)
 thus ?*case*
 by (*cases n1 n2 rule: nat.exhaust[case-product nat.exhaust]*) *auto*
 qed *auto*
 }
thus ?*thesis* **by** *auto*
qed

lemma *lt-take*: $n1 < n2 \implies n2 \leq \text{length } al \implies \text{take } n1 \text{ } al < \text{take } n2 \text{ } al$
by (*metis inj-take le-less-trans le-take not-less-iff-gr-or-eq order.not-eq-order-implies-strict order.strict-implies-order*)

lsum:

definition *lsum* :: $('a \Rightarrow \text{nat}) \Rightarrow 'a \text{ list} \Rightarrow \text{nat}$ **where**
lsum *f* *al* $\equiv \text{sum-list } (\text{map } f \text{ } al)$

lemma *lsum-simps[simp]*:
lsum *f* [] = 0
lsum *f* (*al* @ [*a*]) = *lsum* *f* *al* + *f* *a*
unfolding *lsum-def* **by** *auto*

lemma *lsum-append*:
lsum *f* (*al* @ *bl*) = *lsum* *f* *al* + *lsum* *f* *bl*
unfolding *lsum-def* **by** *auto*

lemma *lsum-cong[fundef-cong]*:
assumes $\bigwedge a. a \in \text{set } al \implies f \ a = g \ a$
shows $\text{lsum } f \ al = \text{lsum } g \ al$
using *assms* **unfolding** *lsum-def* **by** (*metis map-eq-conv*)

lemma *lsum-gt-0[simp]*:
assumes $al \neq []$ **and** $\bigwedge a. a \in \text{set } al \implies 0 < f \ a$
shows $0 < \text{lsum } f \ al$
using *assms* **by** (*induct rule: rev-induct*) *auto*

lemma *lsum-mono[simp]*:
assumes $al \leq bl$
shows $\text{lsum } f \ al \leq \text{lsum } f \ bl$
proof –
 obtain *cl* **where** *bl*: $bl = al @ cl$ **using** *assms* **unfolding** *prefix-def less-eq-list-def*
by *blast*

thus *?thesis* **unfolding** *bl lsum-append* **by** *simp*
qed

lemma *lsum-mono2*[*simp*]:
assumes $f: \bigwedge b. b \in \text{set } bl \implies f b > 0$ **and** *le*: $al < bl$
shows $lsum f al < lsum f bl$
proof –

obtain *cl* **where** *bl*: $bl = al @ cl$ **and** *cl*: $cl \neq []$
 using *assms* **unfolding** *list-less-iff-prefix prefix-def strict-prefix-def* **by** *blast*
 have $lsum f al < lsum f al + lsum f cl$
 using *f cl* **unfolding** *bl* **by** *simp*
 also have $\dots = lsum f bl$ **unfolding** *bl lsum-append* **by** *simp*
 finally show *?thesis* .
qed

lemma *lsum-take*[*simp*]:
 $lsum f (\text{take } n \text{ } al) \leq lsum f al$
by (*metis lsum-mono take-is-prefix less-eq-list-def*)

lemma *less-lsum-nchotomy*:
assumes $f: \bigwedge a. a \in \text{set } al \implies 0 < f a$
and *i*: $(i::nat) < lsum f al$
shows $\exists n j. n < \text{length } al \wedge j < f (al ! n) \wedge i = lsum f (\text{take } n \text{ } al) + j$
using *assms* **proof**(*induct* rule: *rev-induct*)
 case (*snoc a al*)
 hence $i < lsum f al + f a$ **and**
 pos: $0 < f a \wedge a'. a' \in \text{set } al \implies 0 < f a'$ **by** *auto*
 from *i* **show** *?case*
 proof(*cases* rule: *less-plus-elim*)
 case *Left*
 then obtain *n j* **where** $n < \text{length } al \wedge j < f (al ! n) \wedge i = lsum f (\text{take } n \text{ } al) + j$
 using *pos snoc* **by** *auto*
 thus *?thesis*
 apply – **apply**(rule *exI*[*of - n*]) **apply**(rule *exI*[*of - j*]) **by** *auto*
 next
 case (*Right j*)
 thus *?thesis*
 apply – **apply**(rule *exI*[*of - length al*]) **apply**(rule *exI*[*of - j*]) **by** *simp*
 qed
qed *auto*

lemma *less-lsum-unique*:
assumes $\bigwedge a. a \in \text{set } al \implies (0::nat) < f a$
and $n1 < \text{length } al \wedge j1 < f (al ! n1)$
and $n2 < \text{length } al \wedge j2 < f (al ! n2)$
and $lsum f (\text{take } n1 \text{ } al) + j1 = lsum f (\text{take } n2 \text{ } al) + j2$
shows $n1 = n2 \wedge j1 = j2$
using *assms* **proof**(*induct* *al arbitrary*: *n1 n2 j1 j2* rule: *rev-induct*)

```

case (snoc a al)
hence pos: 0 < f a  $\wedge$  a'. a'  $\in$  set al  $\implies$  0 < f a'
and n1: n1 < length al + 1 and n2: n2 < length al + 1 by auto
from n1 show ?case
proof(cases rule: less-plus-elim)
  case Left note n1 = Left
  hence j1: j1 < f (al ! n1) using snoc by auto
  obtain al' where al: al = (take n1 al) @ ((al ! n1) # al')
  using n1 by (metis append-take-drop-id Cons-nth-drop-Suc)
  have j1 < lsum f ((al ! n1) # al') using pos j1 unfolding lsum-def by simp
  hence lsum f (take n1 al) + j1 < lsum f (take n1 al) + lsum f ((al ! n1) #
al') by simp
  also have ... = lsum f al unfolding lsum-append[THEN sym] using al by
simp
  finally have lsum1: lsum f (take n1 al) + j1 < lsum f al .
  from n2 show ?thesis
proof(cases rule: less-plus-elim)
  case Left note n2 = Left
  hence j2: j2 < f (al ! n2) using snoc by auto
  show ?thesis apply(rule snoc(1)) using snoc using pos n1 j1 n2 j2 by auto
next
  case Right
  hence n2: n2 = length al by simp
  hence j2: j2 < f a using snoc by simp
  have lsum f (take n1 al) + j1 = lsum f al + j2 using n1 n2 snoc by simp
  hence False using lsum1 by auto
  thus ?thesis by simp
qed
next
case Right
hence n1: n1 = length al by simp
hence j1: j1 < f a using snoc by simp
from n2 show ?thesis
proof(cases rule: less-plus-elim)
  case Left note n2 = Left
  hence j2: j2 < f (al ! n2) using snoc by auto
  obtain al' where al: al = (take n2 al) @ ((al ! n2) # al')
  using n2 by (metis append-take-drop-id Cons-nth-drop-Suc)
  have j2 < lsum f ((al ! n2) # al') using pos j2 unfolding lsum-def by simp
  hence lsum f (take n2 al) + j2 < lsum f (take n2 al) + lsum f ((al ! n2) #
al') by simp
  also have ... = lsum f al unfolding lsum-append[THEN sym] using al by
simp
  finally have lsum2: lsum f (take n2 al) + j2 < lsum f al .
  have lsum f al + j1 = lsum f (take n2 al) + j2 using n1 n2 snoc by simp
  hence False using lsum2 by auto
  thus ?thesis by simp
next
  case Right

```

hence $n2: n2 = \text{length } al$ **by** *simp*
have $j1 = j2$ **using** $n1\ n2$ *snoc* **by** *simp*
thus *?thesis* **using** $n1\ n2$ **by** *simp*
qed
qed
qed *auto*

definition *locate-pred* **where**
 $\text{locate-pred } f\ al\ (i::nat)\ n-j \equiv$
 $\text{fst } n-j < \text{length } al \wedge$
 $\text{snd } n-j < f\ (al\ !\ (\text{fst } n-j)) \wedge$
 $i = \text{lsum } f\ (\text{take } (\text{fst } n-j)\ al) + (\text{snd } n-j)$

definition *locate* **where**
 $\text{locate } f\ al\ i \equiv \text{SOME } n-j. \text{locate-pred } f\ al\ i\ n-j$

definition *locate1* **where** $\text{locate1 } f\ al\ i \equiv \text{fst } (\text{locate } f\ al\ i)$

definition *locate2* **where** $\text{locate2 } f\ al\ i \equiv \text{snd } (\text{locate } f\ al\ i)$

lemma *locate-pred-ex*:
assumes $\bigwedge a. a \in \text{set } al \implies 0 < f\ a$
and $i < \text{lsum } f\ al$
shows $\exists n-j. \text{locate-pred } f\ al\ i\ n-j$
using *assms less-lsum-nchotomy* **unfolding** *locate-pred-def* **by** *force*

lemma *locate-pred-unique*:
assumes $\bigwedge a. a \in \text{set } al \implies 0 < f\ a$
and $\text{locate-pred } f\ al\ i\ n1-j1\ \text{locate-pred } f\ al\ i\ n2-j2$
shows $n1-j1 = n2-j2$
using *assms less-lsum-unique* **unfolding** *locate-pred-def*
apply(*cases n1-j1, cases n2-j2*) **apply** *simp* **by** *blast*

lemma *locate-locate-pred*:
assumes $\bigwedge a. a \in \text{set } al \implies (0::nat) < f\ a$
and $i < \text{lsum } f\ al$
shows $\text{locate-pred } f\ al\ i\ (\text{locate } f\ al\ i)$
proof–
obtain $n-j$ **where** $\text{locate-pred } f\ al\ i\ n-j$
using *assms locate-pred-ex*[*of al f i*] **by** *auto*
thus *?thesis* **unfolding** *locate-def* **apply**(*intro someI*[*of locate-pred f al i*]) **by**
blast
qed

lemma *locate-locate-pred-unique*:
assumes $\bigwedge a. a \in \text{set } al \implies (0::nat) < f\ a$
and $\text{locate-pred } f\ al\ i\ n-j$
shows $n-j = \text{locate } f\ al\ i$
unfolding *locate-def* **apply**(*rule sym, rule some-equality*)
using *assms locate-locate-pred* **apply** *force*

using *assms locate-pred-unique* **by** *blast*

lemma *locate*:

assumes $\bigwedge a. a \in \text{set } al \implies 0 < f a$

and $i < \text{lsum } f al$

shows $\text{locate1 } f al i < \text{length } al \wedge$

$\text{locate2 } f al i < f (al ! (\text{locate1 } f al i)) \wedge$

$i = \text{lsum } f (\text{take } (\text{locate1 } f al i) al) + (\text{locate2 } f al i)$

using *assms locate-locate-pred*

unfolding *locate1-def locate2-def locate-pred-def* **by** *auto*

lemma *locate-unique*:

assumes $\bigwedge a. a \in \text{set } al \implies 0 < f a$

and $n < \text{length } al$ **and** $j < f (al ! n)$ **and** $i = \text{lsum } f (\text{take } n al) + j$

shows $n = \text{locate1 } f al i \wedge j = \text{locate2 } f al i$

proof –

define $n\text{-}j$ **where** $n\text{-}j = (n, j)$

have *locate-pred* $f al i n\text{-}j$ **using** *assms unfolding n-j-def locate-pred-def* **by** *auto*

hence $n\text{-}j = \text{locate } f al i$ **using** *assms locate-locate-pred-unique* **by** *blast*

thus *?thesis* **unfolding** *n-j-def locate1-def locate2-def* **by** (*metis fst-conv n-j-def snd-conv*)

qed

sum:

lemma *sum-2[simp]*:

$\text{sum } f \{..< 2\} = f 0 + f (\text{Suc } 0)$

proof –

have $\{..< 2\} = \{0, \text{Suc } 0\}$ **by** *auto*

thus *?thesis* **by** *force*

qed

lemma *inj-Plus[simp]*:

$\text{inj } ((+) (a::\text{nat}))$

unfolding *inj-on-def* **by** *auto*

lemma *inj-on-Plus[simp]*:

$\text{inj-on } ((+) (a::\text{nat})) A$

unfolding *inj-on-def* **by** *auto*

lemma *Plus-int[simp]*:

fixes $a :: \text{nat}$

shows $(+) b ' \{..< a\} = \{b ..< b + a\}$

proof *safe*

fix $x::\text{nat}$ **assume** $x \in \{b..< b + a\}$

hence $b \leq x$ **and** $x: x < b + a$ **by** *auto*

then obtain y **where** $xb: x = b + y$ **by** (*metis le-iff-add*)

hence $y < a$ **using** x **by** *simp*

thus $x \in (+) b ' \{..< a\}$ **using** xb **by** *auto*

qed *auto*

lemma *sum-minus[simp]*:
fixes $a :: nat$
shows $sum\ f\ \{a ..< a + b\} = sum\ (\%x.\ f\ (a + x))\ \{..< b\}$
using *sum.reindex[of (+) a {..< b} f]* **by** *auto*

lemma *sum-Un-introL*:
assumes $A1 = B1\ Un\ C1$ **and** $x = x1 + x2$
finite A1 **and**
 $B1\ Int\ C1 = \{\}$ **and**
 $sum\ f1\ B1 = x1$ **and** $sum\ f1\ C1 = x2$
shows $sum\ f1\ A1 = x$
by (*metis assms finite-Un sum.union-disjoint*)

lemma *sum-Un-intro*:
assumes $A1 = B1\ Un\ C1$ **and** $A2 = B2\ Un\ C2$ **and**
finite A1 **and** *finite A2* **and**
 $B1\ Int\ C1 = \{\}$ **and** $B2\ Int\ C2 = \{\}$ **and**
 $sum\ f1\ B1 = sum\ f2\ B2$ **and** $sum\ f1\ C1 = sum\ f2\ C2$
shows $sum\ f1\ A1 = sum\ f2\ A2$
by (*metis assms finite-Un sum.union-disjoint*)

lemma *sum-UN-introL*:
assumes $A1: A1 = (UN\ n : N.\ B1\ n)$ **and** $a2: a2 = sum\ b2\ N$ **and**
fin: finite N $\wedge n. n \in N \implies finite\ (B1\ n)$ **and**
int: $\wedge m\ n. \{m, n\} \subseteq N \wedge m \neq n \implies B1\ m \cap B1\ n = \{\}$ **and**
ss: $\wedge n. n \in N \implies sum\ f1\ (B1\ n) = b2\ n$
shows $sum\ f1\ A1 = a2$ (**is** $?L = a2$)
proof –
 have $?L = sum\ (\%n.\ sum\ f1\ (B1\ n))\ N$
 unfolding $A1$ **using** *sum.UNION-disjoint[of N B1 f1]* *fin int* **by** *simp*
 also have $... = sum\ b2\ N$ **using** *ss fin* **by** *auto*
 also have $... = a2$ **using** $a2$ **by** *simp*
 finally show $?thesis$.
qed

lemma *sum-UN-intro*:
assumes $A1: A1 = (UN\ n : N.\ B1\ n)$ **and** $A2: A2 = (UN\ n : N.\ B2\ n)$ **and**
fin: finite N $\wedge n. n \in N \implies finite\ (B1\ n) \wedge finite\ (B2\ n)$ **and**
int: $\wedge m\ n. \{m, n\} \subseteq N \wedge m \neq n \implies B1\ m \cap B1\ n = \{\} \wedge m\ n. \{m, n\} \subseteq N$
 $\implies B2\ m \cap B2\ n = \{\}$ **and**
ss: $\wedge n. n \in N \implies sum\ f1\ (B1\ n) = sum\ f2\ (B2\ n)$
shows $sum\ f1\ A1 = sum\ f2\ A2$ (**is** $?L = ?R$)
proof –
 have $?L = sum\ (\%n.\ sum\ f1\ (B1\ n))\ N$
 unfolding $A1$ **using** *sum.UNION-disjoint[of N B1 f1]* *fin int* **by** *simp*
 also have $... = sum\ (\%n.\ sum\ f2\ (B2\ n))\ N$ **using** *ss fin sum.mono-neutral-left*
by *auto*
 also have $... = ?R$

unfolding $A2$ **using** $sum.UNION-disjoint[of\ N\ B2\ f2]$ $fin\ int$ **by** $simp$
finally show $?thesis$.
qed

lemma $sum-Minus-intro$:

fixes $f1 :: 'a1 \Rightarrow real$ **and** $f2 :: 'a2 \Rightarrow real$
assumes $B1 = A1 - \{a1\}$ **and** $B2 = A2 - \{a2\}$ **and**
 $a1 : A1$ **and** $a2 : A2$ **and** $finite\ A1$ **and** $finite\ A2$
 $sum\ f1\ A1 = sum\ f2\ A2$ **and** $f1\ a1 = f2\ a2$
shows $sum\ f1\ B1 = sum\ f2\ B2$

proof –

have $1: A1 = B1\ Un\ \{a1\}$ **and** $2: A2 = B2\ Un\ \{a2\}$
using $assms$ **by** $blast+$
from $assms$ **have** $a1 \notin B1$ **by** $simp$
with 1 $\langle finite\ A1 \rangle$ **have** $sum\ f1\ A1 = sum\ f1\ B1 + f1\ a1$
by $simp$
hence $3: sum\ f1\ B1 = sum\ f1\ A1 - f1\ a1$ **by** $simp$
from $assms$ **have** $a2 \notin B2$ **by** $simp$
with 2 $\langle finite\ A2 \rangle$ **have** $sum\ f2\ A2 = sum\ f2\ B2 + f2\ a2$
by $simp$
hence $sum\ f2\ B2 = sum\ f2\ A2 - f2\ a2$ **by** $simp$
thus $?thesis$ **using** 3 $assms$ **by** $simp$

qed

lemma $sum-singl-intro$:

assumes $b = f\ a$
and $finite\ A$ **and** $a \in A$
and $\bigwedge a'. \llbracket a' \in A; a' \neq a \rrbracket \implies f\ a' = 0$
shows $sum\ f\ A = b$

proof –

define B **where** $B = A - \{a\}$
have $A = B\ Un\ \{a\}$ **unfolding** $B-def$ **using** $assms$ **by** $blast$
moreover **have** $B\ Int\ \{a\} = \{\}$ **unfolding** $B-def$ **by** $blast$
ultimately **have** $sum\ f\ A = sum\ f\ B + sum\ f\ \{a\}$
using $assms\ sum.union-disjoint$ **by** $blast$
moreover **have** $\forall b \in B. f\ b = 0$ **using** $assms$ **unfolding** $B-def$ **by** $auto$
ultimately **show** $?thesis$ **using** $assms$ **by** $auto$

qed

lemma $sum-all0-intro$:

assumes $b = 0$
and $\bigwedge a. a \in A \implies f\ a = 0$
shows $sum\ f\ A = b$
using $assms$ **by** $simp$

lemma $sum-1$:

assumes $I: finite\ I$ **and** $ss: sum\ f\ I = 1$ **and** $i: i \in I - I1$ **and** $I1: I1 \subseteq I$
and $f: \bigwedge i. i \in I \implies (0::real) \leq f\ i$
shows $f\ i \leq 1 - sum\ f\ I1$

proof–
have $\text{sum } f \ I = \text{sum } f \ (\{i\} \ \text{Un} \ (I - \{i\}))$ **using** i
by (*metis DiffE insert-Diff-single insert-absorb insert-is-Un*)
also have $\dots = \text{sum } f \ \{i\} + \text{sum } f \ (I - \{i\})$
apply(*rule sum.union-disjoint*) **using** I **by** *auto*
finally have $1 = f \ i + \text{sum } f \ (I - \{i\})$ **unfolding** *ss[THEN sym]* **by** *simp*
moreover have $\text{sum } f \ (I - \{i\}) \geq \text{sum } f \ I1$
apply(*rule sum-mono2*) **using** *assms* **by** *auto*
ultimately have $1 \geq f \ i + \text{sum } f \ I1$ **by** *auto*
thus *?thesis* **by** *auto*
qed

1.2 Syntax

datatype (*'test, 'atom, 'choice*) *cmd* =
Done
| *Atm 'atom*
| *Seq ('test, 'atom, 'choice) cmd ('test, 'atom, 'choice) cmd (- ;; - [60, 61] 60)*
| *While 'test ('test, 'atom, 'choice) cmd*
| *Ch 'choice ('test, 'atom, 'choice) cmd ('test, 'atom, 'choice) cmd*
| *Par ('test, 'atom, 'choice) cmd list*
| *ParT ('test, 'atom, 'choice) cmd list*

fun *noWhile* **where**
noWhile Done \longleftrightarrow *True*
| *noWhile (Atm atm)* \longleftrightarrow *True*
| *noWhile (c1 ;; c2)* \longleftrightarrow *noWhile c1* \wedge *noWhile c2*
| *noWhile (While tst c)* \longleftrightarrow *False*
| *noWhile (Ch ch c1 c2)* \longleftrightarrow *noWhile c1* \wedge *noWhile c2*
| *noWhile (Par cl)* \longleftrightarrow $(\forall c \in \text{set } cl. \text{noWhile } c)$
| *noWhile (ParT cl)* \longleftrightarrow $(\forall c \in \text{set } cl. \text{noWhile } c)$

fun *finished* **where**
finished Done \longleftrightarrow *True*
| *finished (Atm atm)* \longleftrightarrow *False*
| *finished (c1 ;; c2)* \longleftrightarrow *False*
| *finished (While tst c)* \longleftrightarrow *False*
| *finished (Ch ch c1 c2)* \longleftrightarrow *False*
| *finished (Par cl)* \longleftrightarrow $(\forall c \in \text{set } cl. \text{finished } c)$
| *finished (ParT cl)* \longleftrightarrow $(\forall c \in \text{set } cl. \text{finished } c)$

definition *noWhileL* **where**
noWhileL cl $\equiv \forall c \in \text{set } cl. \text{noWhile } c$

lemma *fin-Par-noWhileL[simp]*:
noWhile (Par cl) \longleftrightarrow *noWhileL cl*
unfolding *noWhileL-def* **by** *simp*

```

lemma fin-ParT-noWhileL[simp]:
noWhile (ParT cl)  $\longleftrightarrow$  noWhileL cl
unfolding noWhileL-def by simp

declare noWhile.simps(6) [simp del]
declare noWhile.simps(7) [simp del]

lemma noWhileL-intro[intro]:
assumes  $\bigwedge c. c \in \text{set } cl \implies \text{noWhile } c$ 
shows noWhileL cl
using assms unfolding noWhileL-def by auto

lemma noWhileL-fin[simp]:
assumes noWhileL cl and  $c \in \text{set } cl$ 
shows noWhile c
using assms unfolding noWhileL-def by simp

lemma noWhileL-update[simp]:
assumes cl: noWhileL cl and c': noWhile c'
shows noWhileL (cl[n := c'])
proof(cases n < length cl)
  case True
  show ?thesis
  unfolding noWhileL-def proof safe
    fix c assume  $c \in \text{set } (cl[n := c'])$ 
    hence  $c \in \text{insert } c' (\text{set } cl)$  using set-update-subset-insert by fastforce
    thus noWhile c using assms by (cases c = c') auto
  qed
qed (insert cl, auto)

definition finishedL where
finishedL cl  $\equiv \forall c \in \text{set } cl. \text{finished } c$ 

lemma finished-Par-finishedL[simp]:
finished (Par cl)  $\longleftrightarrow$  finishedL cl
unfolding finishedL-def by simp

lemma finished-ParT-finishedL[simp]:
finished (ParT cl)  $\longleftrightarrow$  finishedL cl
unfolding finishedL-def by simp

declare finished.simps(6) [simp del]
declare finished.simps(7) [simp del]

lemma finishedL-intro[intro]:
assumes  $\bigwedge c. c \in \text{set } cl \implies \text{finished } c$ 
shows finishedL cl
using assms unfolding finishedL-def by auto

```

lemma *finishedL-finished[simp]*:
assumes *finishedL cl and c ∈ set cl*
shows *finished c*
using *assms unfolding finishedL-def by simp*

lemma *finishedL-update[simp]*:
assumes *cl: finishedL cl and c': finished c'*
shows *finishedL (cl[n := c'])*
proof(*cases n < length cl*)
 case *True*
 show *?thesis*
 unfolding *finishedL-def proof safe*
 fix *c assume c ∈ set (cl[n := c'])*
 hence *c ∈ insert c' (set cl) using set-update-subset-insert by fastforce*
 thus *finished c using assms by (cases c = c') auto*
qed
qed (*insert cl, auto*)

lemma *finished-fin[simp]*:
finished c ⇒ noWhile c
by(*induct c*) *auto*

lemma *finishedL-noWhileL[simp]*:
finishedL cl ⇒ noWhileL cl
unfolding *finishedL-def noWhileL-def by auto*

locale *PL =*
 fixes
 aval :: 'atom ⇒ 'state ⇒ 'state and
 tval :: 'test ⇒ 'state ⇒ bool and
 cval :: 'choice ⇒ 'state ⇒ real
 assumes
 properCh: ∧ ch s. 0 ≤ cval ch s ∧ cval ch s ≤ 1
begin

lemma [*simp*]: *(n::nat) < N ⇒ 0 ≤ 1 / N by auto*

lemma [*simp*]: *(n::nat) < N ⇒ 1 / N ≤ 1 by auto*

lemma [*simp*]: *(n::nat) < N ⇒ 0 ≤ 1 - 1 / N by (simp add: divide-simps)*

lemma *sum-equal: 0 < (N::nat) ⇒ sum (λ n. 1/N) {..< N} = 1*
unfolding *sum-constant by auto*

fun *proper where*
 proper Done ↔ True
 | *proper (Atm x) ↔ True*
 | *proper (Seq c1 c2) ↔ proper c1 ∧ proper c2*

| $\text{proper} (\text{While } \text{tst } c) \longleftrightarrow \text{proper } c$
 | $\text{proper} (\text{Ch } \text{ch } c1 \ c2) \longleftrightarrow \text{proper } c1 \wedge \text{proper } c2$
 | $\text{proper} (\text{Par } cl) \longleftrightarrow cl \neq [] \wedge (\forall c \in \text{set } cl. \text{proper } c)$
 | $\text{proper} (\text{ParT } cl) \longleftrightarrow cl \neq [] \wedge (\forall c \in \text{set } cl. \text{proper } c)$

definition *properL* **where**
 $\text{properL } cl \equiv cl \neq [] \wedge (\forall c \in \text{set } cl. \text{proper } c)$

lemma *proper-Par-properL[simp]*:
 $\text{proper} (\text{Par } cl) \longleftrightarrow \text{properL } cl$
unfolding *properL-def* **by** *simp*

lemma *proper-ParT-properL[simp]*:
 $\text{proper} (\text{ParT } cl) \longleftrightarrow \text{properL } cl$
unfolding *properL-def* **by** *simp*

declare *proper.simps(6)* [*simp del*]
declare *proper.simps(7)* [*simp del*]

lemma *properL-intro[intro]*:
 $\llbracket cl \neq []; \bigwedge c. c \in \text{set } cl \implies \text{proper } c \rrbracket \implies \text{properL } cl$
unfolding *properL-def* **by** *auto*

lemma *properL-notEmp[simp]*: $\text{properL } cl \implies cl \neq []$
unfolding *properL-def* **by** *simp*

lemma *properL-proper[simp]*:
 $\llbracket \text{properL } cl; c \in \text{set } cl \rrbracket \implies \text{proper } c$
unfolding *properL-def* **by** *simp*

lemma *properL-update[simp]*:
assumes *cl: properL cl and c': proper c'*
shows $\text{properL } (cl[n := c'])$
proof(*cases n < length cl*)
 case *True*
 show *?thesis*
 unfolding *properL-def* **proof** *safe*
 fix *c* **assume** $c \in \text{set } (cl[n := c'])$
 hence $c \in \text{insert } c' (\text{set } cl)$ **using** *set-update-subset-insert* **by** *fastforce*
 thus $\text{proper } c$ **using** *assms* **by** (*cases c = c'*) *auto*
 qed (*insert cl, auto*)
qed (*insert cl, auto*)

lemma *proper-induct[consumes 1, case-names Done Atm Seq While Ch Par ParT]*:
assumes **: proper c*
and *Done: phi Done*
and *Atm: $\bigwedge atm. phi (Atm atm)$*
and *Seq: $\bigwedge c1 \ c2. \llbracket phi \ c1; phi \ c2 \rrbracket \implies phi (c1 ;; c2)$*
and *While: $\bigwedge \text{tst } c. phi \ c \implies phi (\text{While } \text{tst } c)$*

and $Ch: \bigwedge ch\ c1\ c2. \llbracket \text{phi } c1; \text{phi } c2 \rrbracket \implies \text{phi } (Ch\ ch\ c1\ c2)$
and $Par: \bigwedge cl. \llbracket \text{properL } cl; \bigwedge c. c \in \text{set } cl \implies \text{phi } c \rrbracket \implies \text{phi } (Par\ cl)$
and $ParT: \bigwedge cl. \llbracket \text{properL } cl; \bigwedge c. c \in \text{set } cl \implies \text{phi } c \rrbracket \implies \text{phi } (ParT\ cl)$
shows $\text{phi } c$
using * **apply**($\text{induct } c$)
using assms **unfolding** properL-def **by** auto

1.2.1 Operational Small-Step Semantics

definition $\text{theFT } cl \equiv \{n. n < \text{length } cl \wedge \text{finished } (cl!n)\}$

definition $\text{theNFT } cl \equiv \{n. n < \text{length } cl \wedge \neg \text{finished } (cl!n)\}$

lemma $\text{finite-theFT}[simp]: \text{finite } (\text{theFT } cl)$
unfolding theFT-def **by** simp

lemma $\text{theFT-length}[simp]: n \in \text{theFT } cl \implies n < \text{length } cl$
unfolding theFT-def **by** simp

lemma $\text{theFT-finished}[simp]: n \in \text{theFT } cl \implies \text{finished } (cl!n)$
unfolding theFT-def **by** simp

lemma $\text{finite-theNFT}[simp]: \text{finite } (\text{theNFT } cl)$
unfolding theNFT-def **by** simp

lemma $\text{theNFT-length}[simp]: n \in \text{theNFT } cl \implies n < \text{length } cl$
unfolding theNFT-def **by** simp

lemma $\text{theNFT-notFinished}[simp]: n \in \text{theNFT } cl \implies \neg \text{finished } (cl!n)$
unfolding theNFT-def **by** simp

lemma $\text{theFT-Int-theNFT}[simp]:$
 $\text{theFT } cl \text{ Int } \text{theNFT } cl = \{\}$ **and** $\text{theNFT } cl \text{ Int } \text{theFT } cl = \{\}$
unfolding theFT-def theNFT-def **by** auto

lemma $\text{theFT-Un-theNFT}[simp]:$
 $\text{theFT } cl \text{ Un } \text{theNFT } cl = \{.. < \text{length } cl\}$ **and**
 $\text{theNFT } cl \text{ Un } \text{theFT } cl = \{.. < \text{length } cl\}$
unfolding theFT-def theNFT-def **by** auto

lemma $\text{in-theFT-theNFT}[simp]:$
assumes $n1 \in \text{theFT } cl$ **and** $n2 \in \text{theNFT } cl$
shows $n1 \neq n2$ **and** $n2 \neq n1$
using assms theFT-Int-theNFT **by** blast+

definition $\text{WtFT } cl \equiv \text{sum } (\lambda (n::\text{nat}). 1/(\text{length } cl)) (\text{theFT } cl)$

definition $WtNFT\ cl \equiv \text{sum } (\lambda\ (n::\text{nat}).\ 1/(\text{length}\ cl))\ (\text{theNFT}\ cl)$

lemma $WtFT\text{-}WtNFT[simp]$:

assumes $0 < \text{length}\ cl$

shows $WtFT\ cl + WtNFT\ cl = 1$ (**is** $?A = 1$)

proof –

let $?w = \lambda\ n.\ 1 / \text{length}\ cl$ **let** $?L = \text{theFT}\ cl$ **let** $?Lnot = \text{theNFT}\ cl$

have $1: \{.. < \text{length}\ cl\} = ?L\ \text{Un}\ ?Lnot$ **by** *auto*

have $?A = \text{sum}\ ?w\ ?L + \text{sum}\ ?w\ ?Lnot$ **unfolding** $WtFT\text{-}def\ WtNFT\text{-}def$ **by** *simp*

also have $... = \text{sum}\ ?w\ \{.. < \text{length}\ cl\}$ **unfolding** 1

apply(*rule sum.union-disjoint[THEN sym]*) **by** *auto*

also have $... = 1$ **unfolding** $\text{sum-equal}[OF\ \text{assms}]$ **by** *auto*

finally show $?thesis$.

qed

lemma $WtNFT\text{-}1\text{-}WtFT$: $0 < \text{length}\ cl \implies WtNFT\ cl = 1 - WtFT\ cl$

by (*simp add: algebra-simps*)

lemma $WtNFT\text{-}WtFT\text{-}1[simp]$:

assumes $0 < \text{length}\ cl$ **and** $WtFT\ cl \neq 1$

shows $WtNFT\ cl / (1 - WtFT\ cl) = 1$ (**is** $?A / ?B = 1$)

proof –

have $A: ?A = ?B$ **using** *assms WtNFT-1-WtFT* **by** *auto*

show $?thesis$ **unfolding** A **using** *assms* **by** *auto*

qed

lemma $WtFT\text{-}ge\text{-}0[simp]$: $WtFT\ cl \geq 0$

unfolding $WtFT\text{-}def$ **by** (*rule sum-nonneg*) *simp*

lemma $WtFT\text{-}le\text{-}1[simp]$: $WtFT\ cl \leq 1$ (**is** $?L \leq 1$)

proof –

let $?N = \text{length}\ cl$

have $?L \leq \text{sum } (\lambda\ n::\text{nat}.\ 1/?N)\ \{.. < ?N\}$

unfolding $WtFT\text{-}def$ **apply**(*rule sum-mono2*) **by** *auto*

also have $... \leq 1$

by (*metis div-by-0 le-cases neq0-conv not-one-le-zero of-nat-0 sum.neutral sum-equal*)

finally show $?thesis$.

qed

lemma $le\text{-}1\text{-}WtFT[simp]$: $0 \leq 1 - WtFT\ cl$ (**is** $0 \leq ?R$)

proof –

have $a: -1 \leq - WtFT\ cl$ **by** *simp*

have $(0 :: \text{real}) = 1 + (-1)$ **by** *simp*

also have $1 + (-1) \leq 1 + (- WtFT\ cl)$ **using** a **by** *arith*

also have $... = ?R$ **by** *simp*

finally show $?thesis$.

qed

lemma *WtFT-lt-1*[simp]: $WtFT\ cl \neq 1 \implies WtFT\ cl < 1$
using *WtFT-le-1* **by** (*auto simp add: le-less*)

lemma *lt-1-WtFT*[simp]: $WtFT\ cl \neq 1 \implies 0 < 1 - WtFT\ cl$
using *le-1-WtFT* **by** (*metis le-1-WtFT eq-iff-diff-eq-0 less-eq-real-def*)

lemma *notFinished-WtFT*[simp]:
assumes $n < length\ cl$ **and** $\neg finished\ (cl\ !\ n)$
shows $1 / length\ cl \leq 1 - WtFT\ cl$
proof –
 have $0 < length\ cl$ **using** *assms* **by** *auto*
 thus *?thesis* **unfolding** *WtFT-def* **apply**(*intro sum-1*[of $\{.. < length\ cl\}$])
 using *assms* **by** *auto*
qed

fun *brn* :: ('test, 'atom, 'choice) *cmd* \Rightarrow *nat* **where**
 brn Done = 1
 | *brn (Atm atm)* = 1
 | *brn (c1 ;; c2)* = *brn c1*
 | *brn (While tst c)* = 1
 | *brn (Ch ch c1 c2)* = 2
 | *brn (Par cl)* = *lsum brn cl*
 | *brn (ParT cl)* = *lsum brn cl*

lemma *brn-gt-0*: *proper c* $\implies 0 < brn\ c$
by (*induct rule: proper-induct*) *auto*

lemma *brn-gt-0-L*: $\llbracket properL\ cl; c \in set\ cl \rrbracket \implies 0 < brn\ c$
by (*metis brn-gt-0 properL-def*)

definition *locateT* $\equiv locate1\ brn$ **definition** *locateI* $\equiv locate2\ brn$

definition *brnL cl n* $\equiv lsum\ brn\ (take\ n\ cl)$

lemma *brnL-lsum*: *brnL cl (length cl)* = *lsum brn cl*
unfolding *brnL-def* **by** *simp*

lemma *brnL-unique*:
assumes *properL cl* **and** $n1 < length\ cl \wedge j1 < brn\ (cl\ !\ n1)$
and $n2 < length\ cl \wedge j2 < brn\ (cl\ !\ n2)$ **and** $brnL\ cl\ n1 + j1 = brnL\ cl\ n2 + j2$
shows $n1 = n2 \wedge j1 = j2$
apply (*rule less-lsum-unique*) **using** *assms brn-gt-0* **unfolding** *brnL-def properL-def* **by** *auto*

lemma *brn-Par-simp*[simp]: *brn (Par cl)* = *brnL cl (length cl)*
unfolding *brnL-lsum* **by** *simp*

lemma *brn-ParT-simp*[simp]: $\text{brn} (\text{ParT } cl) = \text{brnL } cl (\text{length } cl)$
unfolding *brnL-lsum* **by** *simp*

declare *brn.simps(6)*[simp del] **declare** *brn.simps(7)*[simp del]

lemma *brnL-0*[simp]: $\text{brnL } cl \ 0 = 0$
unfolding *brnL-def* **by** *auto*

lemma *brnL-Suc*[simp]: $n < \text{length } cl \implies \text{brnL } cl (\text{Suc } n) = \text{brnL } cl \ n + \text{brn} (cl \ ! \ n)$
unfolding *brnL-def* **using** *take-Suc-conv-app-nth*[of *n cl*] **by** *simp*

lemma *brnL-mono*[simp]: $n1 \leq n2 \implies \text{brnL } cl \ n1 \leq \text{brnL } cl \ n2$
using *le-take*[of *n1 n2 cl*] **unfolding** *brnL-def* **by** *simp*

lemma *brnL-mono2*[simp]:
assumes *p*: *properL cl* **and** *n*: $n1 < n2$ **and** *l*: $n2 \leq \text{length } cl$
shows $\text{brnL } cl \ n1 < \text{brnL } cl \ n2$ (**is** $?L < ?R$)
proof–
 have $1: \bigwedge c. c \in \text{set } (\text{take } n2 \ cl) \implies 0 < \text{brn } c$
 using *p* **by** (*metis brn-gt-0 in-set-takeD properL-proper*)
 have $\text{take } n1 \ cl < \text{take } n2 \ cl$ **using** *n l lt-take* **by** *auto*
 hence $\text{lsum } \text{brn} (\text{take } n1 \ cl) < \text{lsum } \text{brn} (\text{take } n2 \ cl)$
 using *lsum-mono2*[of *take n2 cl %c. brn c take n1 cl*] 1 **by** *simp*
 thus *?thesis* **unfolding** *brnL-def* .

qed

lemma *brn-index*[simp]:
assumes *n*: $n < \text{length } cl$ **and** *i*: $i < \text{brn} (cl \ ! \ n)$
shows $\text{brnL } cl \ n + i < \text{brnL } cl (\text{length } cl)$ (**is** $?L < ?R$)
proof–
 have $?L < \text{brnL } cl (\text{Suc } n)$ **using** *assms* **by** *simp*
 also have $\dots \leq ?R$
 using *n brnL-mono*[of *Suc n length cl cl*] **by** *simp*
 finally show *?thesis* .

qed

lemma *brnL-gt-0*[simp]: $\llbracket \text{properL } cl; 0 < n \rrbracket \implies 0 < \text{brnL } cl \ n$
by (*metis properL-def brnL-mono brnL-mono2 le-0-eq length-greater-0-conv nat-le-linear neq0-conv*)

lemma *locateTI*:
assumes *properL cl* **and** $ii < \text{brn} (\text{Par } cl)$
shows
 $\text{locateT } cl \ ii < \text{length } cl \wedge$
 $\text{locateI } cl \ ii < \text{brn} (cl \ ! \ (\text{locateT } cl \ ii)) \wedge$
 $ii = \text{brnL } cl (\text{locateT } cl \ ii) + \text{locateI } cl \ ii$
using *assms locate*[of *cl brn ii*] *brn-gt-0*

unfolding *locateT-def locateI-def brnL-def*
unfolding *brnL-lsum[THEN sym]* **by** *auto*

lemma *locateTI-unique*:
assumes *properL cl* **and** $n < \text{length } cl$
and $i < \text{brn } (cl ! n)$ **and** $ii = \text{brnL } cl \ n + i$
shows $n = \text{locateT } cl \ ii \wedge i = \text{locateI } cl \ ii$
using *assms locate-unique[of cl brn] brn-gt-0*
unfolding *locateT-def locateI-def brnL-def*
unfolding *brnL-lsum[THEN sym]* **by** *auto*

definition *pickFT-pred* **where** $\text{pickFT-pred } cl \ n \equiv n < \text{length } cl \wedge \text{finished } (cl ! n)$
definition *pickFT* **where** $\text{pickFT } cl \equiv \text{SOME } n. \text{pickFT-pred } cl \ n$

lemma *pickFT-pred*:
assumes $WtFT \ cl = 1$ **shows** $\exists n. \text{pickFT-pred } cl \ n$
proof(*rule ccontr, unfold not-ex*)
assume $\forall n. \neg \text{pickFT-pred } cl \ n$
hence $\bigwedge n. n < \text{length } cl \implies \neg \text{finished } (cl ! n)$
unfolding *pickFT-pred-def* **by** *auto*
hence $\text{theFT } cl = \{\}$ **unfolding** *theFT-def* **by** *auto*
hence $WtFT \ cl = 0$ **unfolding** *WtFT-def* **by** *simp*
thus *False* **using** *assms* **by** *simp*
qed

lemma *pickFT-pred-pickFT*: $WtFT \ cl = 1 \implies \text{pickFT-pred } cl \ (\text{pickFT } cl)$
unfolding *pickFT-def* **by**(*auto intro: someI-ex pickFT-pred*)

lemma *pickFT-length[simp]*: $WtFT \ cl = 1 \implies \text{pickFT } cl < \text{length } cl$
using *pickFT-pred-pickFT* **unfolding** *pickFT-pred-def* **by** *auto*

lemma *pickFT-finished[simp]*: $WtFT \ cl = 1 \implies \text{finished } (cl ! (\text{pickFT } cl))$
using *pickFT-pred-pickFT* **unfolding** *pickFT-pred-def* **by** *auto*

lemma *pickFT-theFT[simp]*: $WtFT \ cl = 1 \implies \text{pickFT } cl \in \text{theFT } cl$
unfolding *theFT-def* **by** *auto*

fun *wt-cont-eff* **where**
wt-cont-eff Done s i = (1, Done, s)
|
wt-cont-eff (Atm atm) s i = (1, Done, aval atm s)
|
wt-cont-eff (c1 ;; c2) s i =
(case wt-cont-eff c1 s i of
(x, c1', s') =>
if finished c1' then (x, c2, s') else (x, c1' ;; c2, s'))
|

```

wt-cont-eff (While tst c) s i =
  (if tval tst s
   then (1, c ;; (While tst c), s)
   else (1, Done, s))
|
wt-cont-eff (Ch ch c1 c2) s i =
  (if i = 0 then cval ch s else 1 - cval ch s,
   if i = 0 then c1 else c2,
   s)
|
wt-cont-eff (Par cl) s ii =
  (if cl ! (locateT cl ii) ∈ set cl then
   (case wt-cont-eff
    (cl ! (locateT cl ii))
    s
    (locateI cl ii) of
    (w, c', s') ⇒
    ((1 / (length cl)) * w,
     Par (cl [(locateT cl ii) := c'],
          s'))
    else undefined)
  |
wt-cont-eff (ParT cl) s ii =
  (if cl ! (locateT cl ii) ∈ set cl
   then
    (case wt-cont-eff
     (cl ! (locateT cl ii))
     s
     (locateI cl ii) of
     (w, c', s') ⇒
     (if WtFT cl = 1
      then (if locateT cl ii = pickFT cl ∧ locateI cl ii = 0
            then 1
            else 0)
      else if finished (cl ! (locateT cl ii))
            then 0
            else (1 / (length cl))
                  / (1 - WtFT cl)
                  * w,
      ParT (cl [(locateT cl ii) := c'],
            s'))
     else undefined)
  )

```

definition *wt where* $wt\ c\ s\ i = fst\ (wt\text{-cont-eff}\ c\ s\ i)$

definition *cont where* $cont\ c\ s\ i = fst\ (snd\ (wt\text{-cont-eff}\ c\ s\ i))$

definition *eff where* $eff\ c\ s\ i = snd\ (snd\ (wt\text{-cont-eff}\ c\ s\ i))$

lemma *wt-Done*[simp]: *wt Done s i = 1*
unfolding *wt-def* **by** *simp*

lemma *wt-Atm*[simp]: *wt (Atm atm) s i = 1*
unfolding *wt-def* **by** *simp*

lemma *wt-Seq*[simp]:
wt (c1 ;; c2) s = wt c1 s

proof –
 {**fix** *i* **have** *wt (c1 ;; c2) s i = wt c1 s i*
proof(*cases wt-cont-eff c1 s i*)
case (*fields - c1' -*)
thus *?thesis* **unfolding** *wt-def* **by**(*cases c1', auto*)
qed
 }
thus *?thesis* **by** *auto*
qed

lemma *wt-While*[simp]: *wt (While tst c) s i = 1*
unfolding *wt-def* **by** *simp*

lemma *wt-Ch-L*[simp]: *wt (Ch ch c1 c2) s 0 = cval ch s*
unfolding *wt-def* **by** *simp*

lemma *wt-Ch-R*[simp]: *wt (Ch ch c1 c2) s (Suc n) = 1 - cval ch s*
unfolding *wt-def* **by** *simp*

lemma *cont-Done*[simp]: *cont Done s i = Done*
unfolding *cont-def* **by** *simp*

lemma *cont-Atm*[simp]: *cont (Atm atm) s i = Done*
unfolding *cont-def* **by** *simp*

lemma *cont-Seq-finished*[simp]: *finished (cont c1 s i) \implies cont (c1 ;; c2) s i = c2*
unfolding *cont-def* **by**(*cases wt-cont-eff c1 s i*) *auto*

lemma *cont-Seq-notFinished*[simp]:
assumes \neg *finished (cont c1 s i)*
shows *cont (c1 ;; c2) s i = (cont c1 s i) ;; c2*
proof(*cases wt-cont-eff c1 s i*)
case (*fields - c1' -*)
thus *?thesis* **using** *assms* **unfolding** *cont-def* **by**(*cases c1'*) *auto*
qed

lemma *cont-Seq-not-eq-finished*[simp]: \neg *finished c2 \implies \neg finished (cont (Seq c1 c2) s i)*
by (*cases finished (cont c1 s i)*) *auto*

lemma *cont-While-False*[simp]: $tval\ tst\ s = False \implies cont\ (While\ tst\ c)\ s\ i = Done$

unfolding *cont-def* **by** *simp*

lemma *cont-While-True*[simp]: $tval\ tst\ s = True \implies cont\ (While\ tst\ c)\ s\ i = c\ ;\ ;$
(*While\ tst\ c*)

unfolding *cont-def* **by** *simp*

lemma *cont-Ch-L*[simp]: $cont\ (Ch\ ch\ c1\ c2)\ s\ 0 = c1$

unfolding *cont-def* **by** *simp*

lemma *cont-Ch-R*[simp]: $cont\ (Ch\ ch\ c1\ c2)\ s\ (Suc\ n) = c2$

unfolding *cont-def* **by** *simp*

lemma *eff-Done*[simp]: $eff\ Done\ s\ i = s$

unfolding *eff-def* **by** *simp*

lemma *eff-Atm*[simp]: $eff\ (Atm\ atm)\ s\ i = aval\ atm\ s$

unfolding *eff-def* **by** *simp*

lemma *eff-Seq*[simp]: $eff\ (c1\ ;\ ;\ c2)\ s = eff\ c1\ s$

proof –

{fix *i* **have** $eff\ (c1\ ;\ ;\ c2)\ s\ i = eff\ c1\ s\ i$

proof(*cases\ wt-cont-eff\ c1\ s\ i*)

case (*fields - c1' -*)

thus *?thesis*

unfolding *eff-def* **by**(*cases\ c1'*) *auto*

qed

}

thus *?thesis* **by** *auto*

qed

lemma *eff-While*[simp]: $eff\ (While\ tst\ c)\ s\ i = s$

unfolding *eff-def* **by** *simp*

lemma *eff-Ch*[simp]: $eff\ (Ch\ ch\ c1\ c2)\ s\ i = s$

unfolding *eff-def* **by** *simp*

lemma *brnL-nchotomy*:

assumes *properL\ cl* **and** $ii < brnL\ cl\ (length\ cl)$

shows $\exists\ n\ i.\ n < length\ cl \wedge i < brn\ (cl\ !\ n) \wedge ii = brnL\ cl\ n + i$

unfolding *brnL-def* **apply**(*rule\ less-lsum-nchotomy*) **using** *assms\ brn-gt-0*

unfolding *brnL-lsum*[*THEN\ sym*] **by** *auto*

corollary *brnL-cases*[*consumes\ 2, case-names\ Local, elim*]:

assumes *properL\ cl* **and** $ii < brnL\ cl\ (length\ cl)$ **and**

$\bigwedge n i. \llbracket n < \text{length } cl; i < \text{brn } (cl ! n); ii = \text{brnL } cl \ n + i \rrbracket \implies phi$
shows *phi*
using *assms brnL-nchotomy* **by** *blast*

lemma *wt-cont-eff-Par[simp]*:
assumes *p: properL cl*
and *n: n < length cl* **and** *i: i < brn (cl ! n)*
shows
 $wt (Par \ cl) \ s \ (\text{brnL } cl \ n + i) =$
 $1 / (\text{length } cl) * wt (cl ! n) \ s \ i$
(is *?wL = ?wR***)**

$cont (Par \ cl) \ s \ (\text{brnL } cl \ n + i) =$
 $Par (cl \ [n := cont (cl ! n) \ s \ i])$
(is *?mL = ?mR***)**

$eff (Par \ cl) \ s \ (\text{brnL } cl \ n + i) =$
 $eff (cl ! n) \ s \ i$
(is *?eL = ?eR***)**

proof –

define *ii* **where** $ii = \text{brnL } cl \ n + i$
define *n1* **where** $n1 = \text{locateT } cl \ ii$
define *i1* **where** $i1 = \text{locateI } cl \ ii$
have *n-i: n = n1 i = i1* **using** *p* **unfolding** *n1-def i1-def*
using *ii-def locateTI-unique n i* **by** *auto*
have *lsum1: ii < brnL cl (length cl)* **unfolding** *ii-def* **using** *n i* **by** *simp*
hence $n1 < \text{length } cl$
unfolding *n1-def* **using** *i p locateTI[of cl ii]* **by** *simp*
hence *set: cl ! n1 ∈ set cl* **by** *simp*

have $?wL = 1 / (\text{length } cl) * wt (cl ! n1) \ s \ i1$
unfolding *ii-def[THEN sym]*
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using *set* **unfolding** *n1-def i1-def* **unfolding** *wt-def* **by** *simp*
also have $\dots = ?wR$ **unfolding** *n-i* **by** *simp*
finally show $?wL = ?wR$.

have $?mL = Par (cl \ [n1 := cont (cl ! n1) \ s \ i1])$
unfolding *ii-def[THEN sym]*
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using *set* **unfolding** *n1-def i1-def* **unfolding** *cont-def* **by** *simp*
also have $\dots = ?mR$ **unfolding** *n-i* **by** *simp*
finally show $?mL = ?mR$.

have $?eL = eff (cl ! n1) \ s \ i1$
unfolding *ii-def[THEN sym]*
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using *set* **unfolding** *n1-def i1-def* **unfolding** *eff-def* **by** *simp*
also have $eff (cl ! n1) \ s \ i1 = ?eR$ **unfolding** *n-i* **by** *simp*

finally show $?eL = ?eR$.
qed

lemma *cont-eff-ParT[simp]*:
assumes p : *properL cl*
and n : $n < \text{length } cl$ **and** i : $i < \text{brn } (cl ! n)$
shows
 $\text{cont } (ParT \ cl) \ s \ (\text{brnL } \ cl \ n \ + \ i) =$
 $\text{ParT } (cl \ [n := \text{cont } (cl ! n) \ s \ i])$
(is $?mL = ?mR$ **)**

$\text{eff } (ParT \ cl) \ s \ (\text{brnL } \ cl \ n \ + \ i) =$
 $\text{eff } (cl ! n) \ s \ i$
(is $?eL = ?eR$ **)**

proof –

define ii **where** $ii = \text{brnL } \ cl \ n \ + \ i$
define $n1$ **where** $n1 = \text{locateT } \ cl \ ii$
define $i1$ **where** $i1 = \text{locateI } \ cl \ ii$
have $n-i$: $n = n1 \ i = i1$ **using** p **unfolding** $n1\text{-def } i1\text{-def}$
using $ii\text{-def } \text{locateTI}\text{-unique } n \ i$ **by** *auto*
have $lsum1$: $ii < \text{brnL } \ cl \ (\text{length } \ cl)$ **unfolding** $ii\text{-def}$ **using** $n \ i$ **by** *simp*
hence $n1 < \text{length } \ cl$
unfolding $n1\text{-def}$ **using** $i \ p \ \text{locateTI}[of \ cl \ ii]$ **by** *simp*
hence $set: \ cl ! \ n1 \in set \ cl$ **by** *simp*

have $?mL = \text{ParT } (cl \ [n1 := \text{cont } (cl ! n1) \ s \ i1])$
unfolding $ii\text{-def}[THEN \ sym]$
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using set **unfolding** $n1\text{-def } i1\text{-def}$ **unfolding** $cont\text{-def}$ **by** *simp*
also have $\dots = ?mR$ **unfolding** $n-i$ **by** *simp*
finally show $?mL = ?mR$.

have $?eL = \text{eff } (cl ! n1) \ s \ i1$
unfolding $ii\text{-def}[THEN \ sym]$
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using set **unfolding** $n1\text{-def } i1\text{-def}$ **unfolding** $eff\text{-def}$ **by** *simp*
also have $\text{eff } (cl ! n1) \ s \ i1 = ?eR$ **unfolding** $n-i$ **by** *simp*
finally show $?eL = ?eR$.

qed

lemma *wt-ParT-WtFT-pickFT-0[simp]*:
assumes p : *properL cl* **and** $WtFT$: $WtFT \ cl = 1$
shows $wt \ (ParT \ cl) \ s \ (\text{brnL } \ cl \ (\text{pickFT } \ cl)) = 1$
(is $?wL = 1$ **)**

proof –

define ii **where** $ii = \text{brnL } \ cl \ (\text{pickFT } \ cl)$
define $n1$ **where** $n1 = \text{locateT } \ cl \ ii$
define $i1$ **where** $i1 = \text{locateI } \ cl \ ii$
have ni : $\text{pickFT } \ cl < \text{length } \ cl \ 0 < \text{brn } (cl ! (\text{pickFT } \ cl))$

using *WtFT p brn-gt-0* **by** *auto*
hence *n-i: pickFT cl = n1 0 = i1* **using** *p* **unfolding** *n1-def i1-def*
using *ii-def locateTI-unique[of cl pickFT cl 0 ii]* **by** *auto*
have *lsum1: ii < brnL cl (length cl)* **unfolding** *ii-def* **using** *ni*
by (*metis add.comm-neutral brn-index*)
hence *n1 < length cl*
unfolding *n1-def* **using** *ni p locateTI[of cl ii]* **by** *simp*
hence *set: cl ! n1 ∈ set cl* **by** *simp*

have *n1i1: n1 = pickFT cl ∧ i1 = 0* **using** *WtFT ni* **unfolding** *n-i* **by** *auto*
show *?wL = 1*
unfolding *ii-def[THEN sym]*
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using *WtFT n1i1 set* **unfolding** *n1-def i1-def* **unfolding** *wt-def* **by** *simp*
qed

lemma *wt-ParT-WtFT-notPickFT-0[simp]:*
assumes *p: properL cl* **and** *n: n < length cl* **and** *i: i < brn (cl ! n)*
and *WtFT: WtFT cl = 1* **and** *ni: n = pickFT cl → i ≠ 0*
shows *wt (ParT cl) s (brnL cl n + i) = 0* (**is** *?wL = 0*)
proof –

define *ii* **where** *ii = brnL cl n + i*
define *n1* **where** *n1 = locateT cl ii*
define *i1* **where** *i1 = locateI cl ii*
have *n-i: n = n1 i = i1* **using** *p* **unfolding** *n1-def i1-def*
using *ii-def locateTI-unique n i* **by** *auto*
have *lsum1: ii < brnL cl (length cl)* **unfolding** *ii-def* **using** *n i* **by** *simp*
hence *n1 < length cl*
unfolding *n1-def* **using** *i p locateTI[of cl ii]* **by** *simp*
hence *set: cl ! n1 ∈ set cl* **by** *simp*

have *n1i1: n1 ≠ pickFT cl ∨ i1 ≠ 0* **using** *WtFT ni* **unfolding** *n-i* **by** *auto*
show *?wL = 0*
unfolding *ii-def[THEN sym]*
apply (*cases wt-cont-eff (cl ! n1) s i1*)
using *WtFT n1i1 set* **unfolding** *n1-def i1-def* **unfolding** *wt-def* **by** *force*
qed

lemma *wt-ParT-notWtFT-finished[simp]:*
assumes *p: properL cl* **and** *n: n < length cl* **and** *i: i < brn (cl ! n)*
and *WtFT: WtFT cl ≠ 1* **and** *f: finished (cl ! n)*
shows *wt (ParT cl) s (brnL cl n + i) = 0* (**is** *?wL = 0*)
proof –

define *ii* **where** *ii = brnL cl n + i*
define *n1* **where** *n1 = locateT cl ii*
define *i1* **where** *i1 = locateI cl ii*
have *n-i: n = n1 i = i1* **using** *p* **unfolding** *n1-def i1-def*
using *ii-def locateTI-unique n i* **by** *auto*
have *lsum1: ii < brnL cl (length cl)* **unfolding** *ii-def* **using** *n i* **by** *simp*

hence $n1 < \text{length } cl$
unfolding $n1\text{-def}$ **using** $i\ p\ \text{locateTI}[of\ cl\ ii]$ **by** simp
hence $set: cl ! n1 \in set\ cl$ **by** simp

have $f: \text{finished } (cl ! n1)$ **using** f **unfolding** $n\text{-}i$ **by** auto
show $?wL = 0$
unfolding $ii\text{-def}[THEN\ \text{sym}]$
apply $(\text{cases } wt\text{-cont-eff } (cl ! n1)\ s\ i1)$
using $WtFT\ f\ set$ **unfolding** $n1\text{-def } i1\text{-def}$ **unfolding** $wt\text{-def}$ **by** simp
qed

lemma $wt\text{-cont-eff-ParT-notWtFT-notFinished}[\text{simp}]$:
assumes $p: \text{properL } cl$ **and** $n: n < \text{length } cl$ **and** $i: i < \text{brn } (cl ! n)$
and $WtFT: WtFT\ cl \neq 1$ **and** $nf: \neg \text{finished } (cl ! n)$
shows $wt\ (ParT\ cl)\ s\ (\text{brnL } cl\ n + i) =$
 $(1 / (\text{length } cl)) / (1 - WtFT\ cl) * wt\ (cl ! n)\ s\ i$ **(is** $?wL = ?wR)$
proof –

define ii **where** $ii = \text{brnL } cl\ n + i$
define $n1$ **where** $n1 = \text{locateT } cl\ ii$
define $i1$ **where** $i1 = \text{locateI } cl\ ii$
have $n\text{-}i: n = n1\ i = i1$ **using** p **unfolding** $n1\text{-def } i1\text{-def}$
using $ii\text{-def } \text{locateTI-unique } n\ i$ **by** auto
have $lsum1: ii < \text{brnL } cl\ (\text{length } cl)$ **unfolding** $ii\text{-def}$ **using** $n\ i$ **by** simp
hence $n1 < \text{length } cl$ **unfolding** $n1\text{-def}$ **using** $i\ p\ \text{locateTI}[of\ cl\ ii]$ **by** simp
hence $set: cl ! n1 \in set\ cl$ **by** simp

have $nf: \neg \text{finished } (cl ! n1)$ **using** nf **unfolding** $n\text{-}i$ **by** auto
have $?wL = (1 / (\text{length } cl)) / (1 - WtFT\ cl) * wt\ (cl ! n1)\ s\ i1$
unfolding $ii\text{-def}[THEN\ \text{sym}]$
apply $(\text{cases } wt\text{-cont-eff } (cl ! n1)\ s\ i1)$
using $WtFT\ nf\ set$ **unfolding** $n1\text{-def } i1\text{-def}$ **unfolding** $wt\text{-def}$ **by** simp
also **have** $\dots = ?wR$ **unfolding** $n\text{-}i$ **by** simp
finally **show** $?wL = ?wR$.
qed

lemma $wt\text{-ge-}0[\text{simp}]$:
assumes $\text{proper } c$ **and** $i < \text{brn } c$
shows $0 \leq wt\ c\ s\ i$
using assms **proof** $(\text{induct } c\ \text{arbitrary};\ i\ s\ \text{rule: proper-induct})$
case $(Ch\ ch\ c1\ c2)$
thus $?case$
using properCh **by** $(\text{cases } i)$ $(\text{auto } \text{simp: algebra-simps})$
next
case $(Par\ cl\ ii)$
have $\text{properL } cl$ **and** $ii < \text{brnL } cl\ (\text{length } cl)$ **using** Par **by** auto
thus $?case$
apply $(\text{cases rule: brnL-cases})$
using Par **by** simp

```

next
  case (ParT cl ii)
  have properL cl and ii < brnL cl (length cl) using ParT by auto
  thus ?case
  proof(cases rule: brnL-cases)
    case (Local n i)
    show ?thesis
    proof (cases WtFT cl = 1)
      case True
      thus ?thesis using Local ParT by (cases n = pickFT cl ∧ i = 0) auto
    next
      case False
      thus ?thesis using Local ParT by (cases finished (cl ! n)) auto
    qed
  qed
qed auto

lemma wt-le-1[simp]:
  assumes proper c and i < brn c
  shows wt c s i ≤ 1
  using assms proof (induct c arbitrary: i s rule: proper-induct)
    case (Ch ch c1 c2)
    thus ?case
    using properCh by (cases i) auto
  next
  case (Par cl ii)
  hence properL cl and ii < brnL cl (length cl) by auto
  thus ?case
  apply (cases rule: brnL-cases) apply simp
  using Par apply auto
  by (metis add-increasing2 diff-is-0-eq gr0-conv-Suc less-imp-diff-less less-or-eq-imp-le
  nth-mem of-nat-0-le-iff of-nat-Suc)
  next
  case (ParT cl ii)
  have properL cl and ii < brnL cl (length cl) using ParT by auto
  thus ?case
  proof(cases rule: brnL-cases)
    case (Local n i)
    show ?thesis
    proof (cases WtFT cl = 1)
      case True
      thus ?thesis using Local ParT by (cases n = pickFT cl ∧ i = 0, auto)
    next
      case False note sch = False
      thus ?thesis using Local ParT proof (cases finished (cl ! n))
        case False note cln = False
        let ?L1 = 1 / (length cl) let ?L2 = wt (cl ! n) s i
        let ?R = WtFT cl
        have 0 ≤ ?L1 and 0 ≤ ?L2 using ParT Local by auto
    qed
  qed

```

moreover have $?L2 \leq 1$ **using** *ParT Local* **by** *auto*
ultimately have $?L1 * ?L2 \leq ?L1$ **by** (*metis mult-right-le-one-le*)
also have $?L1 \leq 1 - ?R$ **using** *ParT Local cln* **by** *auto*
finally have $?L1 * ?L2 \leq 1 - ?R$.
thus *?thesis* **using** *Local ParT cln sch*
by (*auto simp: pos-divide-le-eq mult.commute*)
qed (*insert sch ParT Local, auto*)
qed
qed
qed *auto*

abbreviation *fromPlus* ($(1\{..\<+-\})$) **where**
 $\{a ..\<+ b\} \equiv \{a ..\< a + b\}$

lemma *brnL-UN*:
assumes *properL cl*
shows $\{..\< \text{brnL } cl \text{ (length } cl)\} = (\bigcup n < \text{length } cl. \{\text{brnL } cl \ n ..\<+ \text{brn } (cl!n)\})$
(is $?L = (\bigcup n < \text{length } cl. ?R \ n)$ **)**
proof *safe*
fix *ii* **assume** *ii: ii < brnL cl (length cl)*
from *assms ii* **show** $ii \in (\bigcup n < \text{length } cl. ?R \ n)$
proof(*cases rule: brnL-cases*)
case (*Local n i*)
hence $ii \in ?R \ n$ **by** *simp*
thus *?thesis* **using** *Local* **by** *force*
qed
qed *auto*

lemma *brnL-Int-lt*:
assumes $n12: n1 < n2$ **and** $n2: n2 < \text{length } cl$
shows $\{\text{brnL } cl \ n1 ..\<+ \text{brn } (cl!n1)\} \cap \{\text{brnL } cl \ n2 ..\<+ \text{brn } (cl!n2)\} = \{\}$
proof–
have $\text{Suc } n1 \leq n2$ **using** *assms* **by** *simp*
hence $\text{brnL } cl \ (\text{Suc } n1) \leq \text{brnL } cl \ n2$ **by** *simp*
thus *?thesis* **using** *assms* **by** *simp*
qed

lemma *brnL-Int*:
assumes $n1 \neq n2$ **and** $n1 < \text{length } cl$ **and** $n2 < \text{length } cl$
shows $\{\text{brnL } cl \ n1 ..\<+ \text{brn } (cl!n1)\} \cap \{\text{brnL } cl \ n2 ..\<+ \text{brn } (cl!n2)\} = \{\}$
proof(*cases n1 < n2*)
case *True* **thus** *?thesis* **using** *assms brnL-Int-lt* **by** *auto*
next
case *False*
hence $n2 < n1$ **using** *assms* **by** *simp*
thus *?thesis* **using** *brnL-Int-lt assms* **by** *fastforce*
qed

lemma *sum-wt-Par-sub*[simp]:
assumes *cl*: properL *cl* **and** *n*: $n < \text{length } cl$ **and** *I*: $I \subseteq \{..
shows $\text{sum } (wt \ (Par \ cl) \ s) \ ((+) \ (brnL \ cl \ n) \ 'I) =$
 $1 \ /(\text{length } cl) * \text{sum } (wt \ (cl \ ! \ n) \ s) \ I \ (\text{is } ?L = ?wSch * ?R)$
proof –
have $?L = \text{sum } (\%i. ?wSch * wt \ (cl \ ! \ n) \ s \ i) \ I$
apply(rule *sum.reindex-cong*[of (+) (brnL cl n)]) **using** *assms* **by** *auto*
also have ... = $?wSch * ?R$
unfolding *sum-distrib-left* **by** *simp*
finally show *?thesis* .
qed$

lemma *sum-wt-Par*[simp]:
assumes *cl*: properL *cl* **and** *n*: $n < \text{length } cl$
shows $\text{sum } (wt \ (Par \ cl) \ s) \ \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} =$
 $1 \ /(\text{length } cl) * \text{sum } (wt \ (cl \ ! \ n) \ s) \ \{..
using *assms* **by** (*simp add: sum-distrib-left*)$

lemma *sum-wt-ParT-sub-WtFT-pickFT-0*[simp]:
assumes *cl*: properL *cl* **and** *nf*: *WtFT* *cl* = 1
and *I*: $I \subseteq \{..
shows $\text{sum } (wt \ (ParT \ cl) \ s) \ ((+) \ (brnL \ cl \ (pickFT \ cl)) \ 'I) = 1 \ (\text{is } ?L = 1)$
proof –
let $?n = pickFT \ cl$
let $?w = \%i. \text{if } i = 0 \ \text{then } (1::real) \ \text{else } 0$
have *n*: $?n < \text{length } cl$ **using** *nf* **by** *simp*
have *0*: $I = \{0\} \cup (I - \{0\})$ **using** *I* **by** *auto*
have *finI*: *finite* *I* **using** *I* **by** (*metis finite-nat-iff-bounded*)
have $?L = \text{sum } ?w \ I$
proof (rule *sum.reindex-cong* [of plus (brnL cl ?n)])
fix *i* **assume** *i*: $i \in I$
have $i < brn \ (cl \ ! \ ?n)$ **using** *i* *I* **by** *auto*
note *i* = *this* *i*
show $wt \ (ParT \ cl) \ s \ (brnL \ cl \ ?n + i) = ?w \ i$
using *nf* *n* *i* *cl* **by** (*cases* $i = 0$) *auto*
qed (*insert* *assms*, *auto*)
also have ... = $\text{sum } ?w \ (\{0\} \cup (I - \{0\}))$ **using** *0* **by** *auto*
also have ... = $\text{sum } ?w \ \{0::real\} + \text{sum } ?w \ (I - \{0\})$
using *sum.union-disjoint*[of $\{0\}$ $I - \{0\}$ $?w$] *finI* **by** *auto*
also have ... = 1 **by** *simp*
finally show *?thesis* .
qed$

lemma *sum-wt-ParT-sub-WtFT-pickFT-0-2*[simp]:
assumes *cl*: properL *cl* **and** *nf*: *WtFT* *cl* = 1
and *II*: $II \subseteq \{..
shows $\text{sum } (wt \ (ParT \ cl) \ s) \ II = 1 \ (\text{is } ?L = 1)$
proof –
let $?n = pickFT \ cl$$

```

let ?w = %ii. if ii = brnL cl (pickFT cl) then (1::real) else 0
have n: ?n < length cl using nf by simp
have 0: II = {brnL cl ?n} Un (II - {brnL cl ?n}) using II by auto
have finI: finite II using II by (metis finite-nat-iff-bounded)
have ?L = sum ?w II
proof(rule sum.cong)
  fix ii assume ii: ii ∈ II
  hence ii: ii < brnL cl (length cl) using II by auto
  from cl ii show wt (ParT cl) s ii = ?w ii
  proof(cases rule: brnL-cases)
    case (Local n i)
    show ?thesis
    proof(cases ii = brnL cl (pickFT cl))
      case True
      have n = pickFT cl ∧ i = 0
      apply(intro brnL-unique[of cl]) using Local cl nf brn-gt-0 unfolding True
    by auto
    thus ?thesis using cl nf True by simp
  next
  case False
  hence n = pickFT cl → i ≠ 0 unfolding Local by auto
  thus ?thesis using Local ii nf cl False by auto
  qed
  qed
  qed simp
  also have ... = sum ?w ({brnL cl ?n} Un (II - {brnL cl ?n})) using 0 by simp
  also have ... = sum ?w {brnL cl ?n} + sum ?w (II - {brnL cl ?n})
  apply(rule sum.union-disjoint) using finI by auto
  also have ... = 1 by simp
  finally show ?thesis .
qed

```

```

lemma sum-wt-ParT-sub-WtFT-notPickFT-0[simp]:
  assumes cl: properL cl and nf: WtFT cl = 1 and n: n < length cl
  and I: I ⊆ {..  

  shows sum (wt (ParT cl) s) ((+) (brnL cl n) ' I) = 0 (is ?L = 0)
  proof-
  have ?L = sum (%i. 0) I
  proof (rule sum.reindex-cong [of plus (brnL cl n)])
    fix i assume i: i ∈ I
    hence n = pickFT cl → i ≠ 0 using nI by metis
    moreover have i < brn (cl ! n) using i I by auto
    ultimately show wt (ParT cl) s (brnL cl n + i) = 0
    using nf n cl by simp
  qed (insert asms, auto)
  also have ... = 0 by simp
  finally show ?thesis .
qed

```

lemma *sum-wt-ParT-sub-notWtFT-finished*[simp]:
assumes *cl*: properL *cl* **and** *nf*: WtFT *cl* $\neq 1$
and *n*: $n < \text{length } cl$ **and** *cln*: finished (*cl*!*n*) **and** *I*: $I \subseteq \{..< \text{brn } (cl ! n)\}$
shows $\text{sum } (wt \text{ (ParT } cl) s) ((+) (\text{brnL } cl \ n) \ 'I) = 0$ (**is** $?L = 0$)
proof –
 have $?L = \text{sum } (\%i. 0) I$
 apply(rule *sum.reindex-cong*[of (+) (*brnL* *cl* *n*))] **using** *assms* **by** *auto*
 also **have** ... = 0 **by** *simp*
 finally **show** *?thesis* .
qed

lemma *sum-wt-ParT-sub-notWtFT-notFinished*[simp]:
assumes *cl*: properL *cl* **and** *nf*: WtFT *cl* $\neq 1$ **and** *n*: $n < \text{length } cl$
and *cln*: \neg finished (*cl*!*n*) **and** *I*: $I \subseteq \{..< \text{brn } (cl ! n)\}$
shows
 $\text{sum } (wt \text{ (ParT } cl) s) ((+) (\text{brnL } cl \ n) \ 'I) =$
 $(1 / (\text{length } cl)) / (1 - \text{WtFT } cl) * \text{sum } (wt \text{ (cl ! } n) s) I$
(**is** $?L = ?w / (1 - ?wF) * ?R$)
proof –
 have $?L = \text{sum } (\%i. ?w / (1 - ?wF) * wt \text{ (cl ! } n) s \ i) I$
 apply(rule *sum.reindex-cong*[of (+) (*brnL* *cl* *n*)])
 using *assms* **by** *auto*
 also **have** ... = $?w / (1 - ?wF) * ?R$
 unfolding *sum-distrib-left* **by** *simp*
 finally **show** *?thesis* .
qed

lemma *sum-wt-ParT-WtFT-pickFT-0*[simp]:
assumes *cl*: properL *cl* **and** *nf*: WtFT *cl* = 1
shows $\text{sum } (wt \text{ (ParT } cl) s) \{ \text{brnL } cl \ (\text{pickFT } cl) \ ..<+ \text{brn } (cl ! (\text{pickFT } cl)) \} =$
1
proof –
 let $?n = \text{pickFT } cl$
 have 1: $\{ \text{brnL } cl \ ?n \ ..<+ \text{brn } (cl ! ?n) \} =$
 $(+) (\text{brnL } cl \ ?n) \ ' \{ ..< \text{brn } (cl ! ?n) \}$ **by** *simp*
 show *?thesis* **unfolding** 1
 apply(rule *sum-wt-ParT-sub-WtFT-pickFT-0*)
 using *assms* **apply** *simp-all*
 by (*metis* *brn-gt-0-L* *nth-mem* *pickFT-length*)
qed

lemma *sum-wt-ParT-WtFT-notPickFT-0*[simp]:
assumes *cl*: properL *cl* **and** *nf*: WtFT *cl* = 1 **and** *n*: $n < \text{length } cl$ $n \neq \text{pickFT } cl$
shows $\text{sum } (wt \text{ (ParT } cl) s) \{ \text{brnL } cl \ n \ ..<+ \text{brn } (cl ! n) \} = 0$
proof –
 have 1: $\{ \text{brnL } cl \ n \ ..<+ \text{brn } (cl ! n) \} = (+) (\text{brnL } cl \ n) \ ' \{ ..< \text{brn } (cl ! n) \}$ **by**
simp
 show *?thesis* **unfolding** 1 **apply**(rule *sum-wt-ParT-sub-WtFT-notPickFT-0*)

using *assms* by *auto*
qed

lemma *sum-wt-ParT-notWtFT-finished*[*simp*]:
assumes *cl*: *properL cl* **and** *WtFT cl* $\neq 1$
and *n*: $n < \text{length } cl$ **and** *cln*: *finished (cl!n)*
shows $\text{sum } (wt \ (ParT \ cl) \ s) \ \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} = 0$
proof –
 have $1: \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} = (+) \ (brnL \ cl \ n) \ ' \ \{..< \ brn \ (cl!n)\}$ **by**
 simp
 show *?thesis* **unfolding** 1
 apply(*rule sum-wt-ParT-sub-notWtFT-finished*) **using** *assms* **by** *auto*
qed

lemma *sum-wt-ParT-notWtFT-notFinished*[*simp*]:
assumes *cl*: *properL cl* **and** *nf*: *WtFT cl* $\neq 1$
and *n*: $n < \text{length } cl$ **and** *cln*: $\neg \text{finished } (cl!n)$
shows
 $\text{sum } (wt \ (ParT \ cl) \ s) \ \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} =$
 $(1 / (\text{length } cl)) / (1 - WtFT \ cl) * \ \text{sum } (wt \ (cl \ ! \ n) \ s) \ \{..< \ brn \ (cl \ ! \ n)\}$
proof –
 have $1: \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} = (+) \ (brnL \ cl \ n) \ ' \ \{..< \ brn \ (cl!n)\}$ **by**
 simp
 show *?thesis* **unfolding** 1 **apply**(*rule sum-wt-ParT-sub-notWtFT-notFinished*)
using *assms* **by** *auto*
qed

lemma *sum-wt*[*simp*]:
assumes *proper c*
shows $\text{sum } (wt \ c \ s) \ \{..< \ brn \ c\} = 1$
using *assms* **proof** (*induct c arbitrary: s rule: proper-induct*)
 case (*Par cl*)
 let $?w = \lambda \ n. \ 1 / (\text{length } cl) * \ \text{sum } (wt \ (cl \ ! \ n) \ s) \ \{..< \ brn \ (cl \ ! \ n)\}$
 show *?case*
 proof (*rule sum-UN-introL [of - %n. \{brnL cl n ..<+ brn (cl!n)\} \{..< length cl\}*
 - *?w]*)
 have $1 = \text{sum } (\lambda \ n. \ 1 / (\text{length } cl)) \ \{..< \ \text{length } cl\}$
 using *Par* **by** *simp*
 also **have** $\dots = \text{sum } ?w \ \{..< \ \text{length } cl\}$ **using** *Par* **by** *simp*
 finally **show** $1 = \text{sum } ?w \ \{..< \ \text{length } cl\} .$
next
 fix *m n* **assume** $\{m, n\} \subseteq \{..< \ \text{length } cl\} \wedge m \neq n$
 thus
 $\{brnL \ cl \ m \ ..<+ \ brn \ (cl!m)\} \cap \{brnL \ cl \ n \ ..<+ \ brn \ (cl!n)\} = \{\}$
 using *brnL-Int* **by** *auto*
 qed(*insert Par brnL-UN sum-wt-Par, auto*)
next
 case (*ParT cl*)


```

let ?v = 1/(length cl) let ?wtF = WtFT cl let ?wtNF = WtNFT cl
let ?w = λn.
if ?wtF = 1
  then
    (if n = pickFT cl then 1 else 0)
  else
    (if finished (cl!n)
      then 0
      else ?v / (1 - ?wtF) *
        sum (wt (cl ! n) s) {..  
brn (cl ! n)})
define w where w = ?w
have w: ∧ n. ?wtF ≠ 1 ∧ n < length cl ∧ ¬ finished (cl!n)
⇒ w n = ?v / (1 - ?wtF)
unfolding w-def using ParT by simp
show ?case
proof(cases WtFT cl = 1)
  case True
    with ParT show ?thesis by simp
  next
    case False note nf = False
    show ?thesis
    proof (rule sum-UN-introL [of - %n. {brnL cl n ..<+ brn (cl!n)} {..  
length
cl} - w])
      show 1 = sum w {..  
length cl}
      proof(cases ?wtF = 1)
        case True note sch = True
          let ?n = pickFT cl
          let ?L = {?n} let ?Lnot = {..  
length cl} - {?n}
          have ?n < length cl using ParT True by auto
          hence {..  
length cl} = ?L Un ?Lnot by auto
          hence sum w {..  
length cl} = sum w (?L Un ?Lnot) by simp
          also have ... = sum w ?L + sum w ?Lnot
          apply(rule sum.union-disjoint) by auto
          also have ... = 1 unfolding w-def using sch by simp
          finally show ?thesis by simp
        next
          case False note sch = False
          let ?L = theFT cl let ?Lnot = theNFT cl
          have 1: {..  
length cl} = ?L Un ?Lnot by auto
          have sum w {..  
length cl} = sum w ?L + sum w ?Lnot
          unfolding 1 apply(rule sum.union-disjoint) by auto
          also have ... = sum w ?Lnot unfolding w-def using sch by simp
          also have ... = sum (%n. ?v / (1 - ?wtF)) ?Lnot
          apply(rule sum.cong) using w sch by auto
          also have ... = sum (%n. ?v) ?Lnot / (1 - ?wtF)
          unfolding sum-divide-distrib by simp
          also have ... = ?wtNF / (1 - ?wtF) unfolding WtNFT-def by simp
          also have ... = 1 using nf ParT by simp
          finally show ?thesis by simp

```

```

qed
next
fix n assume n: n ∈ {.. $\text{length } cl$ }
show  $\text{sum } (wt (ParT cl) s) \{brnL cl n..<+brn (cl ! n)\} = w n$ 
proof-
  have  $(\sum i < brn (cl ! n). ?v * wt (cl ! n) s i / (1 - ?wtF)) =$ 
     $?v * (\sum i < brn (cl ! n). wt (cl ! n) s i) / (1 - ?wtF)$ 
  unfolding sum-distrib-left sum-divide-distrib by simp
  also have ... =  $?v / (1 - ?wtF)$  using ParT n by simp
  finally have  $(\sum i < brn (cl ! n). ?v * wt (cl ! n) s i / (1 - ?wtF)) =$ 
     $?v / (1 - ?wtF)$  .
  thus ?thesis unfolding w-def using n nf ParT by simp
qed
qed(insert ParT brnL-UN brnL-Int sum-wt-Par, auto)
qed
qed auto

lemma proper-cont[simp]:
assumes proper c and  $i < brn c$ 
shows proper (cont c s i)
using assms proof(induct c arbitrary: i s rule: cmd.induct)
  case (Ch ch c1 c2)
  thus ?case by (cases i) auto
next
  case (Seq c1 c2) thus ?case
  by (cases finished (cont c1 s i)) auto
next
  case (While tst c) thus ?case
  by (cases tval tst s) auto
next
  case (Par cl ii)
  hence properL cl and  $ii < brnL cl (\text{length } cl)$  by auto
  thus ?case
  using Par by (cases rule: brnL-cases) auto
next
  case (ParT cl ii)
  have properL cl and  $ii < brnL cl (\text{length } cl)$  using ParT by auto
  thus ?case apply (cases rule: brnL-cases) using ParT by auto
qed auto

lemma sum-subset-le-1[simp]:
assumes *: proper c and **:  $I \subseteq \{.. $\text{brn } c$ \}$ 
shows  $\text{sum } (wt c s) I \leq 1$ 
proof-
  define J where  $J = \{.. $\text{brn } c$ \}$ 
  have  $I \subseteq J$  and finite J using ** unfolding J-def by auto
  moreover have  $\forall j \in J. wt c s j \geq 0$  unfolding J-def using * by simp
  ultimately have  $\text{sum } (wt c s) I \leq \text{sum } (wt c s) J$ 
  using sum-mono2[of J I wt c s] by auto

```

also have $\dots = 1$ using * unfolding J -def by simp
 finally show $\text{sum } (wt \ c \ s) \ I \leq 1$ unfolding J -def by simp
 qed

lemma sum-le-1 [simp]:
 assumes *: proper c and **: $i < \text{brn } c$
 shows $\text{sum } (wt \ c \ s) \ \{..i\} \leq 1$
 proof –
 have $\{..i\} \subseteq \{..< \text{brn } c\}$ using ** by auto
 thus ?thesis using assms sum-subset-le-1 [of $c \ \{..i\} \ s$] by blast
 qed

1.2.2 Operations on configurations

definition $\text{cont-eff } cf \ b = \text{snd } (wt\text{-cont-eff } (fst \ cf) \ (\text{snd } cf) \ b)$

lemma cont-eff : $\text{cont-eff } cf \ b = (\text{cont } (fst \ cf) \ (\text{snd } cf) \ b, \text{eff } (fst \ cf) \ (\text{snd } cf) \ b)$
 unfolding $\text{cont-eff-def } \text{cont-def } \text{eff-def}$ by simp

end

end

2 Resumption-Based Noninterference

theory Resumption-Based
 imports $\text{Language-Semantics}$
 begin

type-synonym $'a \ \text{rel} = ('a \times 'a) \ \text{set}$

2.1 Preliminaries

lemma int-emp [simp]:
 assumes $i > 0$
 shows $\{..<i\} \neq \{\}$
 by (metis assms $\text{emptyE lessThan-iff}$)

lemma inj-on-inv-into [simp]:
 assumes $\text{inj-on } F \ P$
 shows $\text{inv-into } P \ F \ ' (F \ ' P) = P$
 using assms by auto

lemma inj-on-inv-into2 [simp]:
 $\text{inj-on } (\text{inv-into } P \ F) \ (F \ ' P)$
 by (metis $\text{Hilbert-Choice.inj-on-inv-into subset-refl}$)

lemma *refl-gfp*:
assumes 1: *mono Retr* **and** 2: \bigwedge *theta*. *refl theta* \implies *refl (Retr theta)*
shows *refl (gfp Retr)*
proof –
 define *bis* **where** *bis* = *gfp Retr*
 define *th* **where** *th* = *Id Un bis*
 have *bis* \subseteq *Retr bis*
 using 1 **unfolding** *bis-def* **by** (*metis gfp-unfold subset-refl*)
 also have $\dots \subseteq$ *Retr th* **using** 1 **unfolding** *mono-def th-def* **by** *auto*
 finally have *bis* \subseteq *Retr th* .
 moreover
 { **have** *refl th* **unfolding** *th-def* **by** (*metis Un-commute refl-reflcl*)
 hence *refl (Retr th)* **using** 2 **by** *simp*
 }
 ultimately have *Id* \subseteq *Retr th* **unfolding** *th-def refl-on-def* **by** *auto*
 hence *Id* \subseteq *bis* **using** 1 **coinduct** **unfolding** *th-def bis-def* **by** *blast*
 thus ?*thesis* **unfolding** *bis-def refl-on-def* **by** *auto*
qed

lemma *sym-gfp*:
assumes 1: *mono Retr* **and** 2: \bigwedge *theta*. *sym theta* \implies *sym (Retr theta)*
shows *sym (gfp Retr)*
proof –
 define *bis* **where** *bis* = *gfp Retr*
 define *th* **where** *th* = *bis* $\hat{-}1$ *Un bis*
 have *bis* \subseteq *Retr bis*
 using 1 **unfolding** *bis-def* **by** (*metis gfp-unfold subset-refl*)
 also have $\dots \subseteq$ *Retr th* **using** 1 **unfolding** *mono-def th-def* **by** *auto*
 finally have *bis* \subseteq *Retr th* .
 moreover
 { **have** *sym th* **unfolding** *th-def* **by** (*metis Un-commute sym-Un-converse*)
 hence *sym (Retr th)* **using** 2 **by** *simp*
 }
 ultimately have *bis* $\hat{-}1$ \subseteq *Retr th*
 by (*metis Un-absorb2 Un-commute Un-upper1 converse-Un sym-conv-converse-eq*)
 hence *bis* $\hat{-}1$ \subseteq *bis* **using** 1 **coinduct**[of *Retr bis* $\hat{-}1$] **unfolding** *th-def bis-def*
by *blast*
 thus ?*thesis* **unfolding** *bis-def sym-def* **by** *blast*
qed

lemma *trancl-trans[simp]*:
assumes *trans R*
shows $P \hat{+} \subseteq R \iff P \subseteq R$
proof –
 { **assume** $P \subseteq R$
 hence $P \hat{+} \subseteq R \hat{+}$ **using** *trancl-mono* **by** *auto*
 also have $R \hat{+} = R$ **using** *assms trans-trancl* **by** *auto*
 finally have $P \hat{+} \subseteq R$.
 }

```

}
thus ?thesis by auto
qed

```

lemma *trans-gfp*:

assumes 1: *mono Retr* **and** 2: $\bigwedge theta. trans\ theta \implies trans (Retr\ theta)$

shows $trans (gfp\ Retr)$

proof –

define *bis* **where** $bis = gfp\ Retr$

define *th* **where** $th = bis\ \hat{+}$

have $bis \subseteq Retr\ bis$

using 1 **unfolding** *bis-def* **by** (*metis gfp-unfold subset-refl*)

also have $\dots \subseteq Retr\ th$ **using** 1 **unfolding** *mono-def th-def*
by (*metis trancl-trans order-refl trans-trancl*)

finally have $bis \subseteq Retr\ th$.

moreover

{**have** $trans\ th$ **unfolding** *th-def* **by** (*metis th-def trans-trancl*)

hence $trans (Retr\ th)$ **using** 2 **by** *simp*

}

ultimately have $bis\ \hat{+} \subseteq Retr\ th$ **by** *simp*

hence $bis\ \hat{+} \subseteq bis$ **using** 1 *coinduct* **unfolding** *th-def bis-def*

by (*metis bis-def gfp-upperbound th-def*)

thus ?thesis **unfolding** *bis-def trans-def* **by** *auto*

qed

lemma *O-subset-trans*:

assumes $r\ O\ r \subseteq r$

shows $trans\ r$

using *assms* **unfolding** *trans-def* **by** *blast*

lemma *trancl-imp-trans*:

assumes $r\ \hat{+} \subseteq r$

shows $trans\ r$

by (*metis Int-absorb1 Int-commute trancl-trans assms subset-refl trans-trancl*)

lemma *sym-trans-gfp*:

assumes 1: *mono Retr* **and** 2: $\bigwedge theta. sym\ theta \wedge trans\ theta \implies sym (Retr\ theta) \wedge trans (Retr\ theta)$

shows $sym (gfp\ Retr) \wedge trans (gfp\ Retr)$

proof –

define *bis* **where** $bis = gfp\ Retr$

define *th* **where** $th = (bis\ Un\ bis\ \hat{-}1)\ \hat{+}$

have $bis \subseteq Retr\ bis$

using 1 **unfolding** *bis-def* **by** (*metis gfp-unfold subset-refl*)

also have $\dots \subseteq Retr\ th$ **using** 1 **unfolding** *mono-def th-def*

by (*metis inf-sup-absorb le-iff-inf sup-aci(2) trancl-unfold*)

finally have $bis \subseteq Retr\ th$.

hence $(bis\ Un\ bis\ \hat{-}1)\ \hat{+} \subseteq ((Retr\ th)\ Un\ (Retr\ th)\ \hat{-}1)\ \hat{+}$ **by** *auto*

moreover

```

{have sym th unfolding th-def by (metis sym-Un-converse sym-trancl)
 moreover have trans th unfolding th-def by (metis th-def trans-trancl)
 ultimately have sym (Retr th)  $\wedge$  trans (Retr th) using 2 by simp
 hence ((Retr th) Un (Retr th)  $\hat{-}1$ )  $\hat{+} \subseteq$  Retr th
 by (metis Un-absorb subset-refl sym-conv-converse-eq trancl-id)
}
ultimately have (bis Un bis  $\hat{-}1$ )  $\hat{+} \subseteq$  Retr th by blast
hence (bis Un bis  $\hat{-}1$ )  $\hat{+} \subseteq$  bis using 1 coinduct unfolding th-def bis-def
by (metis bis-def gfp-upperbound th-def)
hence bis  $\hat{-}1 \subseteq$  bis and bis  $\hat{+} \subseteq$  bis
apply (metis equalityI gfp-upperbound le-supI1 subset-refl sym-Un-converse sym-conv-converse-eq
th-def trancl-id trancl-imp-trans)
by (metis Un-absorb  $\langle (bis \cup bis^{-1})^+ \subseteq bis \rangle$  less-supI1 psubset-eq sym-Un-converse
sym-conv-converse-eq sym-trancl trancl-id trancl-imp-trans)
thus ?thesis unfolding bis-def sym-def using trancl-imp-trans by auto
qed

```

2.2 Infrastructure for partitions

definition *part* where

```

part J P  $\equiv$ 
  Union P = J  $\wedge$ 
  ( $\forall J1 J2. J1 \in P \wedge J2 \in P \wedge J1 \neq J2 \implies J1 \cap J2 = \{\}$ )

```

inductive-set *gen*

for $P :: 'a \text{ set set}$ and $I :: 'a \text{ set}$ where

```

incl[simp]:  $i \in I \implies i \in \text{gen } P \ I$ 
|ext[simp]:  $\llbracket J \in P; j0 \in J; j0 \in \text{gen } P \ I; j \in J \rrbracket \implies j \in \text{gen } P \ I$ 

```

definition *partGen* where

```

partGen P  $\equiv$  {gen P I | I . I  $\in$  P}

```

definition *finer* where

```

finer P Q  $\equiv$ 
  ( $\forall J \in Q. J = \text{Union } \{I \in P . I \subseteq J\}$ )  $\wedge$ 
  (P  $\neq \{\}$   $\implies$  Q  $\neq \{\}$ )

```

definition *partJoin* where

```

partJoin P Q  $\equiv$  partGen (P  $\cup$  Q)

```

definition *compat* where

```

compat I theta f  $\equiv$   $\forall i j. \{i, j\} \subseteq I \wedge i \neq j \implies (f \ i, f \ j) \in \text{theta}$ 

```

definition *partCompat* **where**
partCompat *P* *theta* *f* \equiv
 $\forall I \in P. \text{compat } I \text{ theta } f$

definition *lift* **where**
lift *P* *F* *II* $\equiv \text{Union } \{F \ I \mid I . I \in P \wedge I \subseteq II\}$

part:

lemma *part-emp*[*simp*]:
part *J* (*insert* $\{\}$ *P*) = *part* *J* *P*
unfolding *part-def* **by** *auto*

lemma *finite-part*[*simp*]:
assumes *finite* *I* **and** *part* *I* *P*
shows *finite* *P*
using *assms* *finite-UnionD* **unfolding** *part-def* **by** *auto*

lemma *part-sum*:
assumes *P*: *part* $\{..<n::\text{nat}\}$ *P*
shows $(\sum i<n. f \ i) = (\sum p \in P. \sum i \in p. f \ i)$
proof (*subst* *sum.Union-disjoint* [*symmetric*, *simplified*])
show $\forall p \in P. \text{finite } p$
proof
fix *p* **assume** $p \in P$
with *P* **have** $p \subseteq \{0..<n\}$ **by** (*auto* *simp*: *part-def*)
then show *finite* *p* **by** (*rule* *finite-subset*) *simp*
qed
show $\forall A \in P. \forall B \in P. A \neq B \longrightarrow A \cap B = \{\}$
using *P* **by** (*auto* *simp*: *part-def*)
show $\text{sum } f \ \{..<n\} = \text{sum } f \ (\bigcup P)$
using *P* **by** (*auto* *simp*: *part-def* *atLeast0LessThan*)
qed

lemma *part-Un*[*simp*]:
assumes *part* *I1* *P1* **and** *part* *I2* *P2* **and** *I1* *Int* *I2* = $\{\}$
shows *part* (*I1* *Un* *I2*) (*P1* *Un* *P2*)
using *assms* **unfolding** *part-def*
by (*metis* *Union-Un-distrib* *Union-disjoint* *inf-aci*(1) *mem-simps*(3))

lemma *part-Un-singl*[*simp*]:
assumes *part* *K* *P* **and** $\bigwedge I. I \in P \implies I0 \text{ Int } I = \{\}$
shows *part* (*I0* *Un* *K*) ($\{I0\}$ *Un* *P*)
using *assms* **unfolding** *part-def*
by (*metis* *complete-lattice-class.Sup-insert* *Int-commute* *insert-iff* *insert-is-Un*)

lemma *part-Un-singl2*:
assumes *K01* = *I0* *Un* *K1*
and *part* *K1* *P* **and** $\bigwedge I. I \in P \implies I0 \text{ Int } I = \{\}$

shows *part K01* ($\{I0\}$ *Un P*)
using *assms part-Un-singl* **by** *blast*

lemma *part-UN*:
assumes $\bigwedge n. n \in N \implies \text{part } (I \ n) \ (P \ n)$
and $\bigwedge n1 \ n2. \{n1, n2\} \subseteq N \wedge n1 \neq n2 \implies I \ n1 \cap I \ n2 = \{\}$
shows $\text{part } (UN \ n : N. I \ n) \ (UN \ n : N. P \ n)$
using *assms unfolding part-def* **apply** *auto*
apply (*metis UnionE*)
apply (*metis Union-upper disjoint-iff-not-equal insert-absorb insert-subset*)
by (*metis UnionI disjoint-iff-not-equal*)

gen:

lemma *incl-gen[simp]*:
 $I \subseteq \text{gen } P \ I$
by *auto*

lemma *gen-incl-Un*:
 $\text{gen } P \ I \subseteq I \cup (Union \ P)$
proof
fix *j* **assume** $j \in \text{gen } P \ I$
thus $j \in I \cup \bigcup P$ **apply** *induct* **by** *blast+*
qed

lemma *gen-incl*:
assumes $I \in P$
shows $\text{gen } P \ I \subseteq Union \ P$
using *assms gen-incl-Un[of P I]* **by** *blast*

lemma *finite-gen*:
assumes *finite P* **and** $\bigwedge J. J \in P \implies \text{finite } J$ **and** *finite I*
shows *finite (gen P I)*
by (*metis assms finite-Union gen-incl-Un infinite-Un infinite-super*)

lemma *subset-gen[simp]*:
assumes $J \in P$ **and** $\text{gen } P \ I \cap J \neq \{\}$
shows $J \subseteq \text{gen } P \ I$
using *assms gen.ext[of J P - I]* **by** *blast*

lemma *gen-subset-gen[simp]*:
assumes $J \in P$ **and** $\text{gen } P \ I \cap J \neq \{\}$
shows $\text{gen } P \ J \subseteq \text{gen } P \ I$
proof–
have *J*: $J \subseteq \text{gen } P \ I$ **using** *assms* **by** *auto*
show *?thesis* **proof**
fix *i* **assume** $i \in \text{gen } P \ J$
thus $i \in \text{gen } P \ I$
proof *induct*
case (*ext J' j0 j*)


```

    thus ?case
    using gen.ext[of J' P j0 I j] by blast
  qed (insert J, auto)
qed
qed

lemma gen-mono[simp]:
  assumes  $I \subseteq J$ 
  shows  $gen\ P\ I \subseteq gen\ P\ J$ 
  proof
    fix i assume  $i \in gen\ P\ I$  thus  $i \in gen\ P\ J$ 
    proof induct
      case (ext I' j0 j)
      thus ?case
      using gen.ext[of I' P j0 J j] by blast
    qed (insert assms, auto)
  qed

lemma gen-idem[simp]:
   $gen\ P\ (gen\ P\ I) = gen\ P\ I$ 
  proof-
    define J where  $J = gen\ P\ I$ 
    have  $I \subseteq J$  unfolding J-def by auto
    hence  $gen\ P\ I \subseteq gen\ P\ J$  by simp
    moreover have  $gen\ P\ J \subseteq gen\ P\ I$ 
    proof
      fix i assume  $i \in gen\ P\ J$ 
      thus  $i \in gen\ P\ I$ 
      proof induct
        case (ext J' j0 j)
        thus ?case
        using gen.ext[of J' P j0 I j] by blast
      qed (unfold J-def, auto)
    qed
    ultimately show ?thesis unfolding J-def by auto
  qed

lemma gen-nchotomy:
  assumes  $J \in P$ 
  shows  $J \subseteq gen\ P\ I \vee gen\ P\ I \cap J = \{\}$ 
  using assms subset-gen[of J P I] by blast

lemma gen-Union:
  assumes  $I \in P$ 
  shows  $gen\ P\ I = Union\ \{J \in P . J \subseteq gen\ P\ I\}$ 
  proof safe
    fix i assume  $i \in gen\ P\ I$ 
    then obtain J where  $J: J \in P\ i \in J$  using assms gen-incl by blast
    hence  $J \subseteq gen\ P\ I$  using assms i gen-nchotomy by auto
  qed

```

thus $i \in \bigcup \{J \in P. J \subseteq \text{gen } P I\}$ using J by *auto*
qed *auto*

lemma *subset-gen2*:

assumes $\ast: \{I, J\} \subseteq P$ **and** $\ast\ast: \text{gen } P I \cap \text{gen } P J \neq \{\}$

shows $I \subseteq \text{gen } P J$

proof –

{**fix** $i0\ i$ **assume** $i0: i0 \in I \wedge i0 \notin \text{gen } P J$

assume $i \in \text{gen } P I$

hence $i \notin \text{gen } P J$

proof *induct*

case (*incl* i)

thus $?case$ using $i0$ *gen-nchotomy*[of $I\ P\ J$] \ast by *blast*

next

case (*ext* $I'\ j0\ j$)

thus $?case$

using *gen.ext*[of $I'\ P\ j0\ J\ j$] *gen-nchotomy*[of $I'\ P\ J$] by *blast*

qed

}

thus $?thesis$ using *assms* by *auto*

qed

lemma *gen-subset-gen2*[*simp*]:

assumes $\{I, J\} \subseteq P$ **and** $\text{gen } P I \cap \text{gen } P J \neq \{\}$

shows $\text{gen } P I \subseteq \text{gen } P J$

proof

fix i **assume** $i \in \text{gen } P I$

thus $i \in \text{gen } P J$

proof *induct*

case (*incl* i)

thus $?case$

using *assms* *subset-gen2* by *auto*

next

case (*ext* $I'\ j0\ j$)

thus $?case$

using *gen.ext*[of $I'\ P\ j0\ J\ j$] by *blast*

qed

qed

lemma *gen-eq-gen*:

assumes $\{I, J\} \subseteq P$ **and** $\text{gen } P I \cap \text{gen } P J \neq \{\}$

shows $\text{gen } P I = \text{gen } P J$

using *assms* *gen-subset-gen2*[of $I\ J\ P$] *gen-subset-gen2*[of $J\ I\ P$] by *blast*

lemma *gen-empty*[*simp*]:

$\text{gen } P \{\} = \{\}$

proof –

{**fix** j **assume** $j \in \text{gen } P \{\}$ **hence** *False*

apply *induct* by *auto*

```

}
thus ?thesis by auto
qed

```

```

lemma gen-empty2[simp]:
gen {} I = I
proof -
  {fix j assume j ∈ gen {} I hence j ∈ I
  apply induct by auto
  }
thus ?thesis by auto
qed

```

```

lemma emp-gen[simp]:
assumes gen P I = {}
shows I = {}
by (metis all-not-in-conv assms gen.incl)

```

partGen:

```

lemma partGen-ex:
assumes I ∈ P
shows ∃ J ∈ partGen P. I ⊆ J
using assms unfolding partGen-def
apply(intro best[of - gen P I]) by auto

```

```

lemma ex-partGen:
assumes J ∈ partGen P and j: j ∈ J
shows ∃ I ∈ P. j ∈ I
proof -
  obtain I0 where I0: I0 ∈ P and J: J = gen P I0
  using assms unfolding partGen-def by auto
  thus ?thesis using j gen-incl[of I0 P] by auto
qed

```

```

lemma Union-partGen: ∪ (partGen P) = ∪ P
using ex-partGen[of - P] partGen-ex[of - P] by fastforce

```

```

lemma Int-partGen:
assumes *: {I,J} ⊆ partGen P and **: I ∩ J ≠ {}
shows I = J
proof -
  obtain I0 where I0: I0 ∈ P and I: I = gen P I0
  using assms unfolding partGen-def by auto
  obtain J0 where J0: J0 ∈ P and J: J = gen P J0
  using assms unfolding partGen-def by auto
  show ?thesis using gen-eq-gen[of I0 J0 P] I0 J0 ** unfolding I J by blast
qed

```

```

lemma part-partGen:

```

part (Union P) (partGen P)
unfolding *part-def* **apply**(*intro conjI allI impI*)
apply (*metis Union-partGen*)
using *Int-partGen* **by** *blast*

lemma *finite-partGen[simp]*:
assumes *finite P*
shows *finite (partGen P)*
using *assms* **unfolding** *partGen-def* **by** *auto*

lemma *emp-partGen[simp]*:
assumes $\{\} \notin P$
shows $\{\} \notin \text{partGen } P$
using *assms* **unfolding** *partGen-def* **using** *emp-gen[of P]* **by** *blast*

finer:

lemma *finer-partGen*:
finer P (partGen P)
unfolding *finer-def partGen-def* **using** *gen-Union* **by** *auto*

lemma *finer-nchotomy*:
assumes *P: part I0 P and Q: part I0 Q and PQ: finer P Q*
and *I: I ∈ P and II: II ∈ Q*
shows $I \subseteq II \vee (I \cap II = \{\})$
proof(*cases I ∩ II = {\}*)
 case False
 then obtain *i* **where** $i \in I \wedge i \in II$ **by** *auto*
 then obtain *I'* **where** $i \in I'$ **and** $I': I' \in P \wedge I' \subseteq II$
 using *PQ II* **unfolding** *finer-def* **by** *blast*
 hence $I \text{ Int } I' \neq \{\}$ **using** *i* **by** *blast*
 hence $I = I'$ **using** $I I' P$ **unfolding** *part-def* **by** *auto*
 hence $I \subseteq II$ **using** *I'* **by** *simp*
 thus *?thesis* **by** *auto*
qed *auto*

lemma *finer-ex*:
assumes *P: part I0 P and Q: part I0 Q and PQ: finer P Q*
and *I: I ∈ P*
shows $\exists II. II \in Q \wedge I \subseteq II$
proof(*cases I = {\}*)
 case True
 have $Q \neq \{\}$ **using** *I PQ* **unfolding** *finer-def* **by** *auto*
 then obtain *JJ* **where** $JJ \in Q$ **by** *auto*
 with *True* **show** *?thesis* **by** *blast*
next
 case False
 then obtain *i* **where** $i \in I$ **by** *auto*
 hence $i \in I0$ **using** $I P$ **unfolding** *part-def* **by** *auto*
 then obtain *II* **where** $II \in Q$ **and** $i \in II$ **using** *Q* **unfolding** *part-def* **by**

auto
hence $I \text{ Int } II \neq \{\}$ **using** i **by** *auto*
thus *?thesis* **using** *assms* I II *finer-nchotomy*[of $I0$ P Q I II] **by** *auto*
qed

partJoin:

lemma *partJoin-commute*:
 $\text{partJoin } P \ Q = \text{partJoin } Q \ P$
unfolding *partJoin-def* *partGen-def*
using *Un-commute* **by** *metis*

lemma *Union-partJoin-L*:
 $\text{Union } P \subseteq \text{Union } (\text{partJoin } P \ Q)$
unfolding *partJoin-def* *partGen-def* **by** *auto*

lemma *Union-partJoin-R*:
 $\text{Union } Q \subseteq \text{Union } (\text{partJoin } P \ Q)$
unfolding *partJoin-def* *partGen-def* **by** *auto*

lemma *part-partJoin[simp]*:
assumes *part* I P **and** *part* I Q
shows *part* I $(\text{partJoin } P \ Q)$
proof–
have 1 : $\text{Union } (P \ \text{Un } Q) = I$
using *assms* **unfolding** *part-def* **by** *auto*
show *?thesis* **using** *part-partGen*[of P $\text{Un } Q$]
unfolding 1 *partJoin-def* **by** *auto*
qed

lemma *finer-partJoin-L[simp]*:
assumes $*$: *part* I P **and** $**$: *part* I Q
shows *finer* P $(\text{partJoin } P \ Q)$
proof–
have 1 : *part* I $(\text{partJoin } P \ Q)$ **using** *assms* **by** *simp*
{fix J j **assume** J : $J \in \text{partJoin } P \ Q$ **and** j : $j \in J$
hence $J \subseteq I$ **using** 1 **by** (*metis* *Union-upper* *part-def*)
with j **have** $j \in I$ **by** *auto*
then obtain J' **where** jJ' : $j \in J'$ **and** J' : $J' \in P$
using $*$ **unfolding** *part-def* **by** *auto*
hence $J \cap J' \neq \{\}$ **using** j **by** *auto*
moreover obtain $J0$ **where** $J = \text{gen } (P \ \text{Un } Q)$ $J0$
and $J0 \in P \ \text{Un } Q$
using J **unfolding** *partJoin-def* *partGen-def* **by** *blast*
ultimately have $J' \subseteq J$
using J' *gen-nchotomy*[of J' $P \ \text{Un } Q$ $J0$] **by** *blast*
hence $j \in \bigcup \{J' \in P. J' \subseteq J\}$ **using** $J' jJ'$ **by** *blast*
}
thus *?thesis* **unfolding** *finer-def*
unfolding *partJoin-def* *partGen-def* **by** *blast*

qed

lemma *finer-partJoin-R[simp]*:
assumes *: *part I P* and **: *part I Q*
shows *finer Q (partJoin P Q)*
using *assms finer-partJoin-L[of I Q P] partJoin-commute[of P Q]* by *auto*

lemma *finer-emp[simp]*:
assumes *finer {} Q*
shows $Q \subseteq \{ \{ \} \}$
using *assms unfolding finer-def* by *auto*

compat:

lemma *part-emp-R[simp]*:
 $part I \{ \} \longleftrightarrow I = \{ \}$
unfolding *part-def* by *auto*

lemma *part-emp-L[simp]*:
 $part \{ \} P \implies P \subseteq \{ \{ \} \}$
unfolding *part-def* by *auto*

lemma *finite-partJoin[simp]*:
assumes *finite P* and *finite Q*
shows *finite (partJoin P Q)*
using *assms unfolding partJoin-def* by *auto*

lemma *emp-partJoin[simp]*:
assumes $\{ \} \notin P$ and $\{ \} \notin Q$
shows $\{ \} \notin partJoin P Q$
using *assms unfolding partJoin-def* by *auto*

partCompat:

lemma *partCompat-Un[simp]*:
 $partCompat (P Un Q) theta f \longleftrightarrow$
 $partCompat P theta f \wedge partCompat Q theta f$
unfolding *partCompat-def* by *auto*

lemma *partCompat-gen-aux*:
assumes *theta: sym theta trans theta*
and *fP: partCompat P theta f* and *I: I ∈ P*
and *i: i ∈ I* and *j: j ∈ gen P I* and *ij: i ≠ j*
shows $(f i, f j) \in theta$
using *j ij proof induct*
 case (incl j)
 thus ?case
 using *fP I i unfolding partCompat-def compat-def* by *blast*
next
 case (ext J j0 j)
 show ?case

```

proof(cases i = j0)
  case False note case-i = False
  hence 1: (f i, f j0) ∈ theta using ext by blast
  show ?thesis
  proof(cases j = j0)
    case True
    thus ?thesis using case-i 1 by simp
  next
    case False
    hence (f j, f j0) ∈ theta using ⟨j0 ∈ J⟩ ⟨j ∈ J⟩ ⟨J ∈ P⟩
    using fP unfolding partCompat-def compat-def by auto
    hence (f j0, f j) ∈ theta using theta unfolding sym-def by simp
    thus ?thesis using 1 theta unfolding trans-def by blast
  qed
next
  case True note case-i = True
  hence j0 ≠ j using ⟨i ≠ j⟩ by auto
  hence (f j0, f j) ∈ theta using ⟨j0 ∈ J⟩ ⟨j ∈ J⟩ ⟨J ∈ P⟩
  using fP unfolding partCompat-def compat-def by auto
  thus ?thesis unfolding case-i .
qed
qed

```

lemma partCompat-gen:

assumes theta: sym theta trans theta

and fP: partCompat P theta f **and** I: I ∈ P

shows compat (gen P I) theta f

unfolding compat-def **proof** clarify

fix i j **assume** ij: {i, j} ⊆ gen P I i ≠ j

show (f i, f j) ∈ theta

proof(cases i ∈ I)

case True **note** i = True

show ?thesis

proof(cases j ∈ I)

case True

thus ?thesis **using** i ij I fP **unfolding** partCompat-def compat-def **by** blast

next

case False

hence i ≠ j **using** i **by** auto

thus ?thesis **using** assms partCompat-gen-aux i ij **by** auto

qed

next

case False **note** i = False

show ?thesis

proof(cases j ∈ I)

case True

hence j ≠ i **using** i **by** auto

hence (f j, f i) ∈ theta **using** assms partCompat-gen-aux[of theta P f I j i]

True ij **by** auto

```

thus ?thesis using theta unfolding sym-def by auto
next
case False note j = False
show ?thesis
proof(cases I = {})
  case True
  hence False using ij by simp
  thus ?thesis by simp
next
case False
then obtain i0 where i0: i0 ∈ I by auto
hence i0-not: i0 ∉ {i,j} using i j by auto
have (f i0, f i) ∈ theta
using assms i0 i0-not ij partCompat-gen-aux[of theta P f I i0 i] by blast
hence (f i, f i0) ∈ theta using theta unfolding sym-def by auto
moreover have (f i0, f j) ∈ theta
using assms i0 i0-not ij partCompat-gen-aux[of theta P f I i0 j] by blast
ultimately show ?thesis using theta unfolding trans-def by blast
qed
qed
qed
qed

```

lemma partCompat-partGen:
assumes sym theta **and** trans theta
and partCompat P theta f
shows partCompat (partGen P) theta f
unfolding partCompat-def partGen-def
using assms partCompat-gen[of theta P f] **by** auto

lemma partCompat-partJoin[simp]:
assumes sym theta **and** trans theta
and partCompat P theta f **and** partCompat Q theta f
shows partCompat (partJoin P Q) theta f
by (metis assms partCompat-Un partCompat-partGen partJoin-def)

lift:

lemma inj-on-lift:
assumes P: part I0 P **and** Q: part I0 Q **and** PQ: finer P Q
and F: inj-on F P **and** FP: part J0 (F ' P) **and** emp: {} ∉ F ' P
shows inj-on (lift P F) Q
unfolding inj-on-def **proof** clarify
fix II' **assume** II: II ∈ Q **and** II': II' ∈ Q **and** eq: lift P F II = lift P F II'
have 1: II = Union {I ∈ P . I ⊆ II} **using** PQ II **unfolding** finer-def **by** auto
have 2: II' = Union {I ∈ P . I ⊆ II'} **using** PQ II' **unfolding** finer-def **by** auto
 {**fix** I
assume I: I ∈ P I ⊆ II
hence F I ⊆ lift P F II **unfolding** lift-def[abs-def] **by** blast

hence $\exists: F I \subseteq \text{lift } P F II'$ **unfolding eq .**
 have $F I \neq \{\}$ **using emp I FP by auto**
 then obtain j where $j: j \in F I$ **by auto**
 with \exists obtain I' where $I': I' \in P \wedge I' \subseteq II'$ **and $j \in F I'$ unfolding lift-def**
 [abs-def] **by auto**
 hence $F I \text{ Int } F I' \neq \{\}$ **using j by auto**
 hence $F I = F I'$ **using FP I I' unfolding part-def by auto**
 hence $I = I'$ **using F I I' unfolding inj-on-def by auto**
 hence $I \subseteq II'$ **using I' by auto**
 }
 hence $a: II \subseteq II'$ **using 1 2 by blast**

{fix I
 assume $I: I \in P \ I \subseteq II'$
 hence $F I \subseteq \text{lift } P F II'$ **unfolding lift-def [abs-def] by blast**
 hence $\exists: F I \subseteq \text{lift } P F II$ **unfolding eq .**
 have $F I \neq \{\}$ **using emp I FP by auto**
 then obtain j where $j: j \in F I$ **by auto**
 with \exists obtain I' where $I': I' \in P \wedge I' \subseteq II$ **and $j \in F I'$ unfolding lift-def**
 [abs-def] **by auto**
 hence $F I \text{ Int } F I' \neq \{\}$ **using j by auto**
 hence $F I = F I'$ **using FP I I' unfolding part-def by auto**
 hence $I = I'$ **using F I I' unfolding inj-on-def by auto**
 hence $I \subseteq II$ **using I' by auto**
 }
 hence $II' \subseteq II$ **using 1 2 by blast**
 with a show $II = II'$ **by auto**

qed

lemma part-lift:

assumes P : part I0 P and Q : part I0 Q and PQ : finer P Q

and F : inj-on F P and FP : part J0 (F ' P) and emp: $\{\} \notin P \ \{\} \notin F ' P$

shows part J0 (lift P F ' Q)

unfolding part-def proof (intro conjI allI impI)

show $\bigcup (\text{lift } P F ' Q) = J0$

proof safe

fix j II assume $j \in \text{lift } P F II$ and $II: II \in Q$

then obtain I where $j \in F I$ and $I \in P$ and $I \subseteq II$

unfolding lift-def by auto

thus $j \in J0$ using FP unfolding part-def by auto

next

fix j assume $j \in J0$

then obtain J where $J: J \in F ' P$ and $j: j \in J$ using FP unfolding part-def

by auto

define I where $I = \text{inv-into } P F J$

have $j: j \in F I$ unfolding I-def using $j J F$ by auto

have $I: I \in P$ unfolding I-def using $F J$ by auto

obtain II where $I \subseteq II$ and $II \in Q$ using $P Q PQ I$ finer-ex[of I0 P Q I]

by auto

```

    thus  $j \in \bigcup (\text{lift } P \ F \ ' \ Q)$  unfolding lift-def [abs-def] using  $j \ I$  by auto
  qed
next
fix  $JJ1 \ JJ2$  assume  $JJ12: JJ1 \in \text{lift } P \ F \ ' \ Q \wedge JJ2 \in \text{lift } P \ F \ ' \ Q \wedge JJ1 \neq JJ2$ 
then obtain  $II1 \ II2$  where  $II12: \{II1, II2\} \subseteq Q$  and  $JJ1: JJ1 = \text{lift } P \ F \ II1$ 
and  $JJ2: JJ2 = \text{lift } P \ F \ II2$  by auto
have  $II1 \neq II2$  using  $JJ12$  unfolding  $JJ1 \ JJ2$  using  $II12$  assms
using inj-on-lift[of  $I0 \ P \ Q \ F$ ] by auto
hence  $\_4: II1 \ \text{Int} \ II2 = \{\}$  using  $II12 \ Q$  unfolding part-def by auto
show  $JJ1 \cap JJ2 = \{\}$ 
proof(rule ccontr)
  assume  $JJ1 \cap JJ2 \neq \{\}$ 
  then obtain  $j$  where  $j: j \in JJ1 \ j \in JJ2$  by auto
  from  $j$  obtain  $I1$  where  $j \in F \ I1$  and  $I1: I1 \in P \ I1 \subseteq II1$ 
  unfolding  $JJ1$  lift-def [abs-def] by auto
  moreover from  $j$  obtain  $I2$  where  $j \in F \ I2$  and  $I2: I2 \in P \ I2 \subseteq II2$ 
  unfolding  $JJ2$  lift-def [abs-def] by auto
  ultimately have  $F \ I1 \ \text{Int} \ F \ I2 \neq \{\}$  by blast
  hence  $F \ I1 = F \ I2$  using  $I1 \ I2 \ FP$  unfolding part-def by blast
  hence  $I1 = I2$  using  $I1 \ I2 \ F$  unfolding inj-on-def by auto
  moreover have  $I1 \neq \{\}$  using  $I1 \ \text{emp}$  by auto
  ultimately have  $II1 \ \text{Int} \ II2 \neq \{\}$  using  $I1 \ I2$  by auto
  thus False using  $\_4$  by simp
  qed
qed

```

lemma *finer-lift*:

```

assumes finer  $P \ Q$ 
shows finer  $(F \ ' \ P) (\text{lift } P \ F \ ' \ Q)$ 
unfolding finer-def proof (intro conjI ballI impI)
  fix  $JJ$  assume  $JJ: JJ \in \text{lift } P \ F \ ' \ Q$ 
  show  $JJ = \bigcup \{J \in F \ ' \ P. J \subseteq JJ\}$ 
  proof safe
    fix  $j$  assume  $j: j \in JJ$ 
    obtain  $II$  where  $II: II \in Q$  and  $JJ: JJ = \text{lift } P \ F \ II$  using  $JJ$  by auto
    then obtain  $I$  where  $j: j \in F \ I$  and  $I: I \in P \wedge I \subseteq II$  and  $F \ I \subseteq JJ$ 
    using  $j$  unfolding lift-def [abs-def] by auto
    thus  $j \in \bigcup \{J \in F \ ' \ P. J \subseteq JJ\}$  using  $I \ j$  by auto
  qed auto
next
assume  $F \ ' \ P \neq \{\}$ 
thus  $\text{lift } P \ F \ ' \ Q \neq \{\}$ 
using assms unfolding lift-def [abs-def] finer-def by simp
qed

```

2.3 Basic setting for bisimilarity

locale *PL-Indis* =
PL aval tval cval

for $aval :: 'atom \Rightarrow 'state \Rightarrow 'state$ **and**
 $tval :: 'test \Rightarrow 'state \Rightarrow bool$ **and**
 $cval :: 'choice \Rightarrow 'state \Rightarrow real +$
fixes
 $indis :: 'state rel$
assumes
 $equiv-indis: equiv UNIV indis$

context $PL-Indis$
begin

no-notation $eqpoll$ (**infixl** ≈ 50)

abbreviation $indisAbbrev$ (**infix** ≈ 50)
where $s1 \approx s2 \equiv (s1, s2) \in indis$

lemma $refl-indis: refl indis$
and $trans-indis: trans indis$
and $sym-indis: sym indis$
using $equiv-indis$ **unfolding** $equiv-def$ **by** $auto$

lemma $indis-refl[intro]: s \approx s$
using $refl-indis$ **unfolding** $refl-on-def$ **by** $simp$

lemma $indis-trans[trans]: \llbracket s \approx s'; s' \approx s'' \rrbracket \Longrightarrow s \approx s''$
using $trans-indis$ **unfolding** $trans-def$ **by** $blast$

lemma $indis-sym[sym]: s \approx s' \Longrightarrow s' \approx s$
using $sym-indis$ **unfolding** $sym-def$ **by** $blast$

2.4 Discreetness

coinductive $discr$ **where**

$intro:$
 $(\bigwedge s i. i < brn\ c \longrightarrow s \approx eff\ c\ s\ i \wedge discr\ (cont\ c\ s\ i))$
 $\Longrightarrow discr\ c$

definition $discrL$ **where**

$discrL\ cl \equiv \forall c \in set\ cl. discr\ c$

lemma $discrL-intro[intro]:$
assumes $\bigwedge c. c \in set\ cl \Longrightarrow discr\ c$
shows $discrL\ cl$
using $assms$ **unfolding** $discrL-def$ **by** $auto$

lemma $discrL-discr[simp]:$
assumes $discrL\ cl$ **and** $c \in set\ cl$

shows *discr c*
using *assms unfolding discrL-def by simp*

lemma *discrL-update[simp]*:
assumes *cl: discrL cl and c': discr c'*
shows *discrL (cl[n := c'])*
proof(*cases n < length cl*)
 case *True*
 show *?thesis*
 unfolding *discrL-def proof safe*
 fix *c assume c ∈ set (cl[n := c'])*
 hence *c ∈ insert c' (set cl) using set-update-subset-insert by fastforce*
 thus *discr c using assms by (cases c = c') auto*
qed
qed (*insert cl, auto*)

Coinduction for discreteness:

lemma *discr-coind[consumes 1, case-names Hyp, induct pred: discr]*:
assumes **: phi c and*
*** : $\bigwedge c s i. \llbracket \text{phi } c ; i < \text{brn } c \rrbracket$*
 $\implies s \approx \text{eff } c s i \wedge (\text{phi } (\text{cont } c s i) \vee \text{discr } (\text{cont } c s i))$
shows *discr c*
using ** apply(induct rule: discr.coinduct) using ** by auto*

lemma *discr-raw-coind[consumes 1, case-names Hyp]*:
assumes **: phi c and*
*** : $\bigwedge c s i. \llbracket i < \text{brn } c ; \text{phi } c \rrbracket \implies s \approx \text{eff } c s i \wedge \text{phi } (\text{cont } c s i)$*
shows *discr c*
using ** apply(induct) using ** by blast*

Discreteness versus transition:

lemma *discr-cont[simp]*:
assumes **: discr c and **: i < brn c*
shows *discr (cont c s i)*
using ** apply(cases rule: discr.cases) using ** by blast*

lemma *discr-eff-indis[simp]*:
assumes **: discr c and **: i < brn c*
shows $s \approx \text{eff } c s i$
using ** apply(cases rule: discr.cases) using ** by blast*

2.5 Self-isomorphism

coinductive *siso* where

intro:

$\llbracket \bigwedge s i. i < \text{brn } c \implies \text{siso } (\text{cont } c s i);$

$\bigwedge s t i.$

$i < \text{brn } c \wedge s \approx t \implies$

$\text{eff } c s i \approx \text{eff } c t i \wedge \text{wt } c s i = \text{wt } c t i \wedge \text{cont } c s i = \text{cont } c t i \rrbracket$

$\implies \text{siso } c$

definition *sisoL* **where**
sisoL cl $\equiv \forall c \in \text{set } cl. \text{siso } c$

lemma *sisoL-intro*[*intro*]:
assumes $\bigwedge c. c \in \text{set } cl \implies \text{siso } c$
shows *sisoL cl*
using *assms unfolding sisoL-def by auto*

lemma *sisoL-siso*[*simp*]:
assumes *sisoL cl* **and** $c \in \text{set } cl$
shows *siso c*
using *assms unfolding sisoL-def by simp*

lemma *sisoL-update*[*simp*]:
assumes *cl: sisoL cl* **and** *c': siso c'*
shows *sisoL (cl[n := c'])*
proof(*cases n < length cl*)
 case *True*
 show *?thesis*
 unfolding *sisoL-def proof safe*
 fix *c* **assume** $c \in \text{set } (cl[n := c'])$
 hence $c \in \text{insert } c' (\text{set } cl)$ **using** *set-update-subset-insert by fastforce*
 thus *siso c* **using** *assms by (cases c = c') auto*
 qed
qed (*insert cl, auto*)

Coinduction for self-isomorphism:

lemma *siso-coind*[*consumes 1, case-names Obs Cont, induct pred: siso*]:
assumes $*$: *phi c* **and**
 $**$: $\bigwedge c s t i. \llbracket i < \text{brn } c; \text{phi } c; s \approx t \rrbracket \implies$
 $\text{eff } c s i \approx \text{eff } c t i \wedge \text{wt } c s i = \text{wt } c t i \wedge \text{cont } c s i = \text{cont } c t i$ **and**
 $***$: $\bigwedge c s i. \llbracket i < \text{brn } c; \text{phi } c \rrbracket \implies \text{phi } (\text{cont } c s i) \vee \text{siso } (\text{cont } c s i)$
shows *siso c*
using $*$ **apply**(*induct rule: siso.coinduct*) **using** $**$ $***$ **by** *auto*

lemma *siso-raw-coind*[*consumes 1, case-names Obs Cont*]:
assumes $*$: *phi c* **and**
 $***$: $\bigwedge c s t i. \llbracket i < \text{brn } c; \text{phi } c; s \approx t \rrbracket \implies$
 $\text{eff } c s i \approx \text{eff } c t i \wedge \text{wt } c s i = \text{wt } c t i \wedge \text{cont } c s i = \text{cont } c t i$ **and**
 $**$: $\bigwedge c s i. \llbracket i < \text{brn } c; \text{phi } c \rrbracket \implies \text{phi } (\text{cont } c s i)$
shows *siso c*
using $*$ **apply** *induct using* $**$ $***$ **by** *blast+*

Self-Isomorphism versus transition:

lemma *siso-cont*[*simp*]:
assumes $*$: *siso c* **and** $**$: $i < \text{brn } c$
shows *siso (cont c s i)*

using * apply(cases rule: siso.cases) using ** by blast

lemma *siso-cont-indis*[simp]:

assumes *: *siso c* **and** **: $s \approx t \ i < \text{brn } c$

shows $\text{eff } c \ s \ i \approx \text{eff } c \ t \ i \wedge \text{wt } c \ s \ i = \text{wt } c \ t \ i \wedge \text{cont } c \ s \ i = \text{cont } c \ t \ i$

using * apply(cases rule: siso.cases) using ** by blast

2.6 Notions of bisimilarity

Matchers

definition *mC-C-part* **where**

mC-C-part $c \ d \ P \ F \equiv$

$\{\} \notin P \wedge \{\} \notin F \ ' \ P \wedge$

$\text{part } \{.. < \text{brn } c\} P \wedge \text{part } \{.. < \text{brn } d\} (F \ ' \ P)$

definition *mC-C-wt* **where**

mC-C-wt $c \ d \ s \ t \ P \ F \equiv \forall I \in P. \text{sum } (\text{wt } c \ s) \ I = \text{sum } (\text{wt } d \ t) \ (F \ I)$

definition *mC-C-eff-cont* **where**

mC-C-eff-cont $\text{theta } c \ d \ s \ t \ P \ F \equiv$

$\forall I \ i \ j.$

$I \in P \wedge i \in I \wedge j \in F \ I \longrightarrow$

$\text{eff } c \ s \ i \approx \text{eff } d \ t \ j \wedge (\text{cont } c \ s \ i, \text{cont } d \ t \ j) \in \text{theta}$

definition *mC-C* **where**

mC-C $\text{theta } c \ d \ s \ t \ P \ F \equiv$

$\text{mC-C-part } c \ d \ P \ F \wedge \text{inj-on } F \ P \wedge \text{mC-C-wt } c \ d \ s \ t \ P \ F \wedge \text{mC-C-eff-cont } \text{theta } c \ d \ s \ t \ P \ F$

definition *matchC-C* **where**

matchC-C $\text{theta } c \ d \equiv \forall s \ t. s \approx t \longrightarrow (\exists P \ F. \text{mC-C } \text{theta } c \ d \ s \ t \ P \ F)$

definition *mC-ZOC-part* **where**

mC-ZOC-part $c \ d \ s \ t \ I0 \ P \ F \equiv$

$\{\} \notin P - \{I0\} \wedge \{\} \notin F \ ' \ (P - \{I0\}) \wedge I0 \in P \wedge$

$\text{part } \{.. < \text{brn } c\} P \wedge \text{part } \{.. < \text{brn } d\} (F \ ' \ P)$

definition *mC-ZOC-wt* **where**

mC-ZOC-wt $c \ d \ s \ t \ I0 \ P \ F \equiv$

$\text{sum } (\text{wt } c \ s) \ I0 < 1 \wedge \text{sum } (\text{wt } d \ t) \ (F \ I0) < 1 \longrightarrow$

$(\forall I \in P - \{I0\}.$

$\text{sum } (\text{wt } c \ s) \ I / (1 - \text{sum } (\text{wt } c \ s) \ I0) =$

$\text{sum } (\text{wt } d \ t) \ (F \ I) / (1 - \text{sum } (\text{wt } d \ t) \ (F \ I0)))$

definition *mC-ZOC-eff-cont0* **where**

mC-ZOC-eff-cont0 $\text{theta } c \ d \ s \ t \ I0 \ F \equiv$

$(\forall i \in I0. s \approx \text{eff } c \ s \ i \wedge (\text{cont } c \ s \ i, d) \in \text{theta}) \wedge$

$(\forall j \in F I0. t \approx \text{eff } d t j \wedge (c, \text{cont } d t j) \in \text{theta})$

definition *mC-ZOC-eff-cont* **where**
mC-ZOC-eff-cont $\text{theta } c d s t I0 P F \equiv$
 $\forall I i j.$
 $I \in P - \{I0\} \wedge i \in I \wedge j \in F I \longrightarrow$
 $\text{eff } c s i \approx \text{eff } d t j \wedge$
 $(\text{cont } c s i, \text{cont } d t j) \in \text{theta}$

definition *mC-ZOC* **where**
mC-ZOC $\text{theta } c d s t I0 P F \equiv$
mC-ZOC-part $c d s t I0 P F \wedge$
inj-on $F P \wedge$
mC-ZOC-wt $c d s t I0 P F \wedge$
mC-ZOC-eff-cont0 $\text{theta } c d s t I0 F \wedge$
mC-ZOC-eff-cont $\text{theta } c d s t I0 P F$

definition *matchC-LC* **where**
matchC-LC $\text{theta } c d \equiv$
 $\forall s t. s \approx t \longrightarrow (\exists I0 P F. \text{mC-ZOC } \text{theta } c d s t I0 P F)$

lemmas *m-defs* = *mC-C-def* *mC-ZOC-def*

lemmas *m-defsAll* =
mC-C-def *mC-C-part-def* *mC-C-wt-def* *mC-C-eff-cont-def*
mC-ZOC-def *mC-ZOC-part-def* *mC-ZOC-wt-def* *mC-ZOC-eff-cont0-def* *mC-ZOC-eff-cont-def*

lemmas *match-defs* =
matchC-C-def *matchC-LC-def*

lemma *mC-C-mono*:
assumes *mC-C* $\text{theta } c d s t P F$ **and** $\text{theta} \subseteq \text{theta}'$
shows *mC-C* $\text{theta}' c d s t P F$
using *assms* **unfolding** *m-defsAll* **by** *fastforce+*

lemma *matchC-C-mono*:
assumes *matchC-C* $\text{theta } c d$ **and** $\text{theta} \subseteq \text{theta}'$
shows *matchC-C* $\text{theta}' c d$
using *assms* *mC-C-mono* **unfolding** *matchC-C-def* **by** *blast*

lemma *mC-ZOC-mono*:
assumes *mC-ZOC* $\text{theta } c d s t I0 P F$ **and** $\text{theta} \subseteq \text{theta}'$
shows *mC-ZOC* $\text{theta}' c d s t I0 P F$
using *assms* **unfolding** *m-defsAll* *subset-eq* **by** *auto*

lemma *matchC-LC-mono*:
assumes *matchC-LC* $\text{theta } c d$ **and** $\text{theta} \subseteq \text{theta}'$
shows *matchC-LC* $\text{theta}' c d$
using *assms* *mC-ZOC-mono* **unfolding** *matchC-LC-def*

by *metis*

lemma *Int-not-in-eq-emp*:

$P \cap \{I. I \notin P\} = \{\}$

by *blast*

lemma *mC-C-mC-ZOC*:

assumes *mC-C theta c d s t P F*

shows *mC-ZOC theta c d s t {} (P Un { {} }) (%I. if I ∈ P then F I else {})*

(**is** *mC-ZOC theta c d s t ?I0 ?Q ?G*)

unfolding *mC-ZOC-def* **proof**(*intro conjI*)

show *mC-ZOC-part c d s t ?I0 ?Q ?G*

unfolding *mC-ZOC-part-def* **using** *assms* **unfolding** *mC-C-def mC-C-part-def*

by (*auto simp add: Int-not-in-eq-emp*)

next

show *inj-on ?G ?Q*

unfolding *inj-on-def* **proof** *clarify*

fix *I1 I2* **assume** *I12: I1 ∈ ?Q I2 ∈ ?Q*

and *G: ?G I1 = ?G I2*

show *I1 = I2*

proof(*cases I1 ∈ P*)

case *True*

hence *I2: I2 ∈ P* **apply**(*rule-tac ccontr*)

using *G assms* **unfolding** *mC-C-def mC-C-part-def* **by** *auto*

with *True G* **have** *F I1 = F I2* **by** *simp*

thus *?thesis* **using** *True I2 I12 assms* **unfolding** *mC-C-def inj-on-def* **by**

blast

next

case *False* **note** *case1 = False*

hence *I1: I1 = {}* **using** *I12* **by** *blast*

show *?thesis*

proof(*cases I2 ∈ P*)

case *False*

hence *I2 = {}* **using** *I12* **by** *blast*

thus *?thesis* **using** *I1* **by** *blast*

next

case *True*

hence *I1 ∈ P* **apply**(*rule-tac ccontr*)

using *G assms* **unfolding** *mC-C-def mC-C-part-def* **by** *auto*

thus *?thesis* **using** *case1* **by** *simp*

qed

qed

qed

qed(*insert assms, unfold m-defsAll, fastforce+*)

lemma *matchC-C-matchC-LC*:

assumes *matchC-C theta c d*

shows *matchC-LC theta c d*

using *assms mC-C-mC-ZOC* **unfolding** *match-defs* **by** *blast*

Retracts:

definition *Sretr* **where**

Sretr *theta* \equiv
 $\{(c, d). \text{matchC-C } \textit{theta } c \ d\}$

definition *ZOretr* **where**

ZOretr *theta* \equiv
 $\{(c,d). \text{matchC-LC } \textit{theta } c \ d\}$

lemmas *Retr-defs* =

Sretr-def
ZOretr-def

lemma *mono-Retr*:

mono Sretr
mono ZOretr

unfolding *mono-def Retr-defs*

by (*auto simp add: matchC-C-mono matchC-LC-mono*)

lemma *Retr-incl*:

Sretr theta \subseteq *ZOretr theta*

unfolding *Retr-defs*

using *matchC-C-matchC-LC* **by** *blast+*

The associated bisimilarity relations:

definition *Sbis* **where** *Sbis* \equiv *gfp Sretr*

definition *ZObis* **where** *ZObis* \equiv *gfp ZOretr*

abbreviation *Sbis-abbrev* (**infix** $\approx s$ 55) **where** $c \approx s \ d \equiv (c, d) : \textit{Sbis}$

abbreviation *ZObis-abbrev* (**infix** ≈ 01 55) **where** $c \approx 01 \ d \equiv (c, d) : \textit{ZObis}$

lemmas *bis-defs* = *Sbis-def ZObis-def*

lemma *bis-incl*:

Sbis \leq *ZObis*

unfolding *bis-defs*

using *Retr-incl gfp-mono* **by** *blast+*

lemma *bis-imp[simp]*:

$\bigwedge c1 \ c2. c1 \approx s \ c2 \implies c1 \approx 01 \ c2$

using *bis-incl* **unfolding** *bis-defs* **by** *auto*

lemma *Sbis-prefix*:

Sbis \leq *Sretr Sbis*

unfolding *Sbis-def*

using *def-gfp-unfold mono-Retr(1)* **by** *blast*

lemma *Sbis-fix*:

Sretr Sbis = Sbis

unfolding *Sbis-def*

by (*metis def-gfp-unfold mono-Retr(1)*)

lemma *Sbis-mC-C*:

assumes $c \approx s$ d **and** $s \approx t$

shows $\exists P F. mC-C\ Sbis\ c\ d\ s\ t\ P\ F$

using *assms Sbis-prefix* **unfolding** *Sretr-def matchC-C-def* **by** *auto*

lemma *Sbis-coind*:

assumes $\theta \leq Sretr\ (\theta \ Un\ Sbis)$

shows $\theta \leq Sbis$

using *assms* **unfolding** *Sbis-def*

by (*metis Sbis-def assms def-coinduct mono-Retr(1)*)

lemma *Sbis-raw-coind*:

assumes $\theta \leq Sretr\ \theta$

shows $\theta \leq Sbis$

proof–

have $Sretr\ \theta \subseteq Sretr\ (\theta \ Un\ Sbis)$

by (*metis Un-upper1 monoD mono-Retr(1)*)

hence $\theta \subseteq Sretr\ (\theta \ Un\ Sbis)$ **using** *assms* **by** *blast*

thus *?thesis* **using** *Sbis-coind* **by** *blast*

qed

lemma *mC-C-sym*:

assumes *mC-C* $\theta \ c\ d\ s\ t\ P\ F$

shows *mC-C* $(\theta \ \hat{-} 1)\ d\ c\ t\ s\ (F \ ' P)\ (inv-into\ P\ F)$

unfolding *mC-C-def* **proof** (*intro conjI*)

show *mC-C-eff-cont* $(\theta^{-1})\ d\ c\ t\ s\ (F \ ' P)\ (inv-into\ P\ F)$

unfolding *mC-C-eff-cont-def* **proof** *clarify*

fix $i\ j\ I$

assume $I: I \in P$ **and** $j: j \in F\ I$ **and** $i \in inv-into\ P\ F\ (F\ I)$

hence $i \in I$ **using** *assms* **unfolding** *mC-C-def* **by** *auto*

hence $eff\ c\ s\ i \approx eff\ d\ t\ j \wedge (cont\ c\ s\ i, cont\ d\ t\ j) \in \theta$

using *assms* $I\ j$ **unfolding** *mC-C-def mC-C-eff-cont-def* **by** *auto*

thus $eff\ d\ t\ j \approx eff\ c\ s\ i \wedge (cont\ d\ t\ j, cont\ c\ s\ i) \in \theta^{-1}$

by (*metis converseI indis-sym*)

qed

qed(*insert assms, unfold m-defsAll, auto*)

lemma *matchC-C-sym*:

assumes *matchC-C* $\theta \ c\ d$

shows *matchC-C* $(\theta \ \hat{-} 1)\ d\ c$

unfolding *matchC-C-def* **proof** *clarify*
fix $t\ s$
assume $t \approx s$ **hence** $s \approx t$ **by** (*metis indis-sym*)
then obtain $P\ F$ **where** *mC-C* $\theta\ c\ d\ s\ t\ P\ F$
using *assms* **unfolding** *matchC-C-def* **by** *blast*
hence *mC-C* $(\theta^{-1})\ d\ c\ t\ s\ (F\ 'P)$ (*inv-into P F*)
using *mC-C-sym* **by** *auto*
thus $\exists Q\ G.$ *mC-C* $(\theta^{-1})\ d\ c\ t\ s\ Q\ G$ **by** *blast*
qed

lemma *Sretr-sym*:
assumes *sym* θ
shows *sym* (*Sretr* θ)
unfolding *sym-def* **proof** *clarify*
fix $c\ d$ **assume** $(c, d) \in \text{Sretr } \theta$
hence $(d, c) \in \text{Sretr } (\theta^{-1})$
unfolding *Sretr-def* **using** *matchC-C-sym* **by** *simp*
thus $(d, c) \in \text{Sretr } \theta$
using *assms* **by** (*metis sym-conv-converse-eq*)
qed

lemma *sym-Sbis*: *sym* *Sbis*
by (*metis Sbis-def Sretr-sym mono-Retr(1) sym-gfp*)

lemma *Sbis-sym*: $c \approx s\ d \implies d \approx s\ c$
using *sym-Sbis* **unfolding** *sym-def* **by** *blast*

lemma *mC-C-trans*:
assumes $*$: *mC-C* $\theta_1\ c\ d\ s\ t\ P\ F$ **and** $*$: *mC-C* $\theta_2\ d\ e\ t\ u\ (F\ 'P)\ G$
shows *mC-C* $(\theta_1\ O\ \theta_2)\ c\ e\ s\ u\ P\ (G\ o\ F)$
unfolding *mC-C-def* **proof** (*intro conjI*)
show *mC-C-part* $c\ e\ P\ (G\ o\ F)$
using *assms* **unfolding** *mC-C-def mC-C-part-def* **by** (*auto simp add: image-comp*)
next
show *inj-on* $(G\ o\ F)\ P$
using *assms* **unfolding** *mC-C-def* **by** (*auto simp add: comp-inj-on*)
next
show *mC-C-eff-cont* $(\theta_1\ O\ \theta_2)\ c\ e\ s\ u\ P\ (G\ o\ F)$
unfolding *mC-C-eff-cont-def* **proof** *clarify*
fix $I\ i\ k$
assume $I: I \in P$ **and** $i: i \in I$ **and** $k: k \in (G\ o\ F)\ I$
have $F\ I \neq \{\}$ **using** $*$ *I* **unfolding** *mC-C-def mC-C-part-def* **by** *auto*
then obtain j **where** $j: j \in F\ I$ **by** *auto*
have *eff* $c\ s\ i \approx \text{eff } d\ t\ j \wedge (\text{cont } c\ s\ i, \text{cont } d\ t\ j) \in \theta_1$
using $I\ i\ j\ *$ **unfolding** *mC-C-def mC-C-eff-cont-def* **by** *auto*
moreover

```

have eff d t j ≈ eff e u k ∧ (cont d t j, cont e u k) ∈ theta2
using I j k ** unfolding mC-C-def mC-C-eff-cont-def by auto
ultimately
show eff c s i ≈ eff e u k ∧ (cont c s i, cont e u k) ∈ theta1 O theta2
using indis-trans by blast
qed
qed(insert assms, unfold m-defsAll, auto)

lemma mC-C-finer:
assumes *: mC-C theta c d s t P F
and theta: trans theta
and Q: finer P Q finite Q {} ∉ Q part {..

```

```

proof clarify
  fix  $I J$  assume  $I \in S$  and  $J \in S$  and diff:  $I \neq J$ 
  hence  $IJ: \{I, J\} \subseteq P$  unfolding S-def by simp
  hence  $F I \neq F J$  using * diff unfolding mC-C-def inj-on-def by auto
  thus  $F I \cap F J = \{\}$  using * IJ unfolding mC-C-def mC-C-part-def part-def
by auto
qed
have  $\text{sum } (wt \ c \ s) \ II = \text{sum } (\text{sum } (wt \ c \ s)) \ S$ 
unfolding II using S SS sum.UNION-disjoint[of S id wt c s] by simp
also have  $\dots = \text{sum } (\% \ I. \ \text{sum } (wt \ d \ t) \ (F \ I)) \ S$ 
apply(rule sum.cong)
using S apply force
  unfolding S-def using * unfolding mC-C-def mC-C-part-def mC-C-wt-def
by auto
also have  $\dots = \text{sum } (wt \ d \ t) \ (UN \ I : S . F \ I)$ 
unfolding lift-def using S sum.UNION-disjoint[of S F wt d t] S SS SSS by
simp
also have  $(UN \ I : S . F \ I) = \text{lift } P \ F \ II$  unfolding lift-def S-def by auto
finally show  $\text{sum } (wt \ c \ s) \ II = \text{sum } (wt \ d \ t) \ (\text{lift } P \ F \ II)$  .
qed
next
show mC-C-eff-cont theta c d s t Q (lift P F)
unfolding mC-C-eff-cont-def proof clarify
  fix  $II \ i \ j$ 
  assume  $II: II \in Q$  and  $i: i \in II$  and  $j \in \text{lift } P \ F \ II$ 
  then obtain  $I$  where  $j: j \in F \ I$  and  $I: I \in P \wedge I \subseteq II$  unfolding lift-def
by auto
hence  $I \neq \{\}$  using * unfolding mC-C-def mC-C-part-def by auto
then obtain  $i'$  where  $i': i' \in I$  by auto
hence  $1: \text{eff } c \ s \ i' \approx \text{eff } d \ t \ j \wedge (\text{cont } c \ s \ i', \text{cont } d \ t \ j) \in \text{theta}$ 
using *  $j \ I$  unfolding mC-C-def mC-C-eff-cont-def by auto
show  $\text{eff } c \ s \ i \approx \text{eff } d \ t \ j \wedge (\text{cont } c \ s \ i, \text{cont } d \ t \ j) \in \text{theta}$ 
proof(cases i' = i)
  case True show ?thesis using  $1$  unfolding True .
next
  case False
  moreover have  $i' \in II$  using  $I \ i'$  by auto
  ultimately have  $\text{eff } c \ s \ i \approx \text{eff } c \ s \ i' \wedge (\text{cont } c \ s \ i, \text{cont } c \ s \ i') \in \text{theta}$ 
using  $i \ II \ c$  unfolding partCompat-def compat-def by auto
thus ?thesis using  $1$  indis-trans theta unfolding trans-def by blast
qed
qed
qed

lemma mC-C-partCompat-eff:
assumes *: mC-C theta c d s t P F
shows partCompat P indis (eff c s)
unfolding partCompat-def compat-def proof clarify
  fix  $i \ i'$  assume  $I: I \in P$  and  $ii': \{i, i'\} \subseteq I \ i \neq i'$ 

```

hence $F I \neq \{\}$ **using** * **unfolding** $m\text{-defsAll}$ **by** blast
 then **obtain** j where $j: j \in F I$ **by** auto
 from $ii' j I$ **have** 1: $\text{eff } c s i \approx \text{eff } d t j$
using * **unfolding** $m\text{-defsAll}$ **by** blast
 from $ii' j I$ **have** 2: $\text{eff } c s i' \approx \text{eff } d t j$
using * **unfolding** $m\text{-defsAll}$ **by** blast
 from 1 2 **show** $\text{eff } c s i \approx \text{eff } c s i'$
using $\text{indis-trans indis-sym}$ **by** blast
qed

lemma $mC\text{-}C\text{-partCompat-cont}$:
assumes *: $mC\text{-}C$ θ $c d s t P F$
and θ : $\text{sym } \theta$ $\text{trans } \theta$
shows $\text{partCompat } P \theta$ ($\text{cont } c s$)
unfolding $\text{partCompat-def compat-def}$ **proof** clarify
fix $I i i'$ **assume** $I: I \in P$ **and** $ii': \{i, i'\} \subseteq I i \neq i'$
 hence $F I \neq \{\}$ **using** * **unfolding** $m\text{-defsAll}$ **by** blast
 then **obtain** j where $j: j \in F I$ **by** auto
 from $ii' j I$ **have** 1: $(\text{cont } c s i, \text{cont } d t j) \in \theta$
using * **unfolding** $m\text{-defsAll}$ **by** blast
 from $ii' j I$ **have** 2: $(\text{cont } c s i', \text{cont } d t j) \in \theta$
using * **unfolding** $m\text{-defsAll}$ **by** blast
 from 1 2 **show** $(\text{cont } c s i, \text{cont } c s i') \in \theta$
using θ **unfolding** trans-def sym-def **by** blast
qed

lemma $\text{matchC-C-sym-trans}$:
assumes *: $\text{matchC-C } \theta$ $c1 c$ **and** **: $\text{matchC-C } \theta$ $c c2$
and θ : $\text{sym } \theta$ $\text{trans } \theta$
shows $\text{matchC-C } \theta$ $c1 c2$
unfolding matchC-C-def **proof** clarify
fix $s1 s2$ **assume** $s1s2: s1 \approx s2$
define s where $s = s1$
have $s: s \approx s1 \wedge s \approx s2$ **unfolding** $s\text{-def}$ **using** $s1s2$ **by** auto
have θ : $\theta^{-1} = \theta$ $\theta \circ \theta \subseteq \theta$ **using** θ
by ($\text{metis sym-conv-converse-eq trans-O-subset}$)
hence *: $\text{matchC-C } \theta$ $c c1$ **by** ($\text{metis } * \text{matchC-C-sym}$)
from s * **obtain** $P1 F1$ where $m1: mC\text{-}C \theta c c1 s s1 P1 F1$
unfolding matchC-C-def **by** blast
from s ** **obtain** $P2 F2$ where $m2: mC\text{-}C \theta c c2 s s2 P2 F2$
unfolding matchC-C-def **by** blast
define P where $P = \text{partJoin } P1 P2$

have P :
 $\text{finer } P1 P \wedge \text{finer } P2 P \wedge$
 $\text{finite } P \wedge \{\} \notin P \wedge \text{part } \{.. < \text{brn } c\} P$
using $m1 m2 \text{finer-partJoin-L finite-partJoin emp-partJoin part-partJoin finite-part}$ [of
 $\{.. < \text{brn } c\} P$]
unfolding $P\text{-def } mC\text{-}C\text{-def } mC\text{-}C\text{-part-def}$ **by** force

```

have mC-C theta c c1 s s1 P (lift P1 F1)
proof(rule mC-C-finer)
  show partCompat P indis (eff c s)
  unfolding P-def apply(rule partCompat-partJoin)
  using m1 m2 sym-indis trans-indis mC-C-partCompat-eff by auto
next
  show partCompat P theta (cont c s)
  unfolding P-def apply(rule partCompat-partJoin)
  using m1 m2 theta mC-C-partCompat-cont by auto
qed(insert P m1 theta, auto)
hence A: mC-C theta c1 c c2 s s1 s (lift P1 F1 ‘ P) (inv-into P (lift P1 F1))
using mC-C-sym[of theta c c1 s s1 P lift P1 F1] unfolding th by blast

have B: mC-C theta c c2 s s2 P (lift P2 F2)
proof(rule mC-C-finer)
  show partCompat P indis (eff c s)
  unfolding P-def apply(rule partCompat-partJoin)
  using m1 m2 sym-indis trans-indis mC-C-partCompat-eff by auto
next
  show partCompat P theta (cont c s)
  unfolding P-def apply(rule partCompat-partJoin)
  using m1 m2 theta mC-C-partCompat-cont by auto
qed(insert P m2 theta, auto)

have inj-on (lift P1 F1) P
apply(rule inj-on-lift) using m1 m2 P unfolding m-defsAll by auto
hence inv-into P (lift P1 F1) ‘ lift P1 F1 ‘ P = P
by (metis inj-on-inv-into)
hence mC-C (theta O theta) c1 c2 s1 s2 (lift P1 F1 ‘ P) (lift P2 F2 o (inv-into
P (lift P1 F1)))
apply – apply(rule mC-C-trans[of theta c1 c s1 s - - theta c2 s2])
using A B by auto
hence mC-C theta c1 c2 s1 s2 (lift P1 F1 ‘ P) (lift P2 F2 o (inv-into P (lift P1
F1)))
using th mC-C-mono by blast
thus  $\exists R H. mC-C \theta c1 c2 s1 s2 R H$  by blast
qed

lemma Sretr-sym-trans:
assumes sym theta  $\wedge$  trans theta
shows trans (Sretr theta)
unfolding trans-def proof clarify
  fix c d e assume (c, d)  $\in$  Sretr theta and (d, e)  $\in$  Sretr theta
  thus (c, e)  $\in$  Sretr theta
  unfolding Sretr-def using assms matchC-C-sym-trans by blast
qed

lemma trans-Sbis: trans Sbis

```

by (metis *Sbis-def Sretr-sym Sretr-sym-trans mono-Retr sym-trans-gfp*)

lemma *Sbis-trans*: $\llbracket c \approx s d; d \approx s e \rrbracket \implies c \approx s e$
using *trans-Sbis unfolding trans-def* by *blast*

lemma *ZObis-prefix*:
 $ZObis \leq ZOretr ZObis$
unfolding *ZObis-def*
using *def-gfp-unfold mono-Retr(2)* by *blast*

lemma *ZObis-fix*:
 $ZOretr ZObis = ZObis$
unfolding *ZObis-def*
by (metis *def-gfp-unfold mono-Retr(2)*)

lemma *ZObis-mC-ZOC*:
assumes $c \approx 01 d$ and $s \approx t$
shows $\exists I0 P F. mC-ZOC ZObis c d s t I0 P F$
using *assms ZObis-prefix unfolding ZOretr-def matchC-LC-def* by *blast*

lemma *ZObis-coind*:
assumes $theta \leq ZOretr (theta Un ZObis)$
shows $theta \leq ZObis$
using *assms unfolding ZObis-def*
by (metis *ZObis-def assms def-coinduct mono-Retr(2)*)

lemma *ZObis-raw-coind*:
assumes $theta \leq ZOretr theta$
shows $theta \leq ZObis$
proof–
 have $ZOretr theta \subseteq ZOretr (theta Un ZObis)$
 by (metis *Un-upper1 monoD mono-Retr*)
 hence $theta \subseteq ZOretr (theta Un ZObis)$ using *assms* by *blast*
 thus *?thesis* using *ZObis-coind* by *blast*
qed

lemma *mC-ZOC-sym*:
assumes $theta: sym\ theta$ and $*: mC-ZOC\ theta\ c\ d\ s\ t\ I0\ P\ F$
shows $mC-ZOC\ theta\ d\ c\ t\ s\ (F\ I0)\ (F\ 'P)\ (inv-into\ P\ F)$
unfolding *mC-ZOC-def* **proof** (*intro conjI*)
 show $mC-ZOC-part\ d\ c\ t\ s\ (F\ I0)\ (F\ 'P)\ (inv-into\ P\ F)$
 unfolding *mC-ZOC-part-def* **proof** (*intro conjI*)
 have $0: inj-on\ F\ P\ I0 \in P$ using $*$ unfolding *mC-ZOC-def mC-ZOC-part-def*
 by *blast+*

have $\text{inv-into } P F' (F' P - \{F I0\}) = \text{inv-into } P F' (F' (P - \{I0\}))$
using $0 \text{ inj-on-image-set-diff[of } F P P \{I0\}, OF - \text{Set.Diff-subset]}$ **by** *simp*
also have $\dots = P - \{I0\}$ **using** 0 **by** (*metis Diff-subset inv-into-image-cancel*)
finally have $\text{inv-into } P F' (F' P - \{F I0\}) = P - \{I0\}$.
thus $\{\} \notin \text{inv-into } P F' (F' P - \{F I0\})$
using * **unfolding** *mC-ZOC-def mC-ZOC-part-def* **by** *simp*

have $\text{part } \{..<\text{brn } c\} P$ **using** * **unfolding** *mC-ZOC-def mC-ZOC-part-def* **by**
blast
thus $\text{part } \{..<\text{brn } c\} (\text{inv-into } P F' F' P)$ **using** 0 **by** *auto*
qed(*insert *, unfold mC-ZOC-def mC-ZOC-part-def, blast+*)
next
have $0: \text{inj-on } F P I0 \in P$ **using** * **unfolding** *mC-ZOC-def mC-ZOC-part-def*
by *blast+*
show *mC-ZOC-wt d c t s (F I0) (F' P) (inv-into P F)*
unfolding *mC-ZOC-wt-def* **proof**(*intro conjI ballI impI*)
fix J **assume** $J \in F' P - \{F I0\}$ **and** $le-1: \text{sum } (wt d t) (F I0) < 1 \wedge \text{sum}$
 $(wt c s) (\text{inv-into } P F (F I0)) < 1$
then obtain I **where** $I: I \in P - \{I0\}$ **and** $J: J = F I$
by (*metis image-iff member-remove remove-def*)
have $2: \text{inv-into } P F J = I$ **unfolding** J **using** $0 I$ **by** *simp*
have $3: \text{inv-into } P F (F I0) = I0$ **using** 0 **by** *simp*
show
 $\text{sum } (wt d t) J / (1 - \text{sum } (wt d t) (F I0)) =$
 $\text{sum } (wt c s) (\text{inv-into } P F J) / (1 - \text{sum } (wt c s) (\text{inv-into } P F (F I0)))$
unfolding $2 \ 3$ **unfolding** J
using * $I \ le-1$ **unfolding** *mC-ZOC-def mC-ZOC-wt-def* **by** (*metis 3 J*)
qed
next
show *mC-ZOC-eff-cont theta d c t s (F I0) (F' P) (inv-into P F)*
unfolding *mC-ZOC-eff-cont-def* **proof**(*intro allI impI, elim conjE*)
fix $i \ j \ J$
assume $J \in F' P - \{F I0\}$ **and** $j: j \in J$ **and** $i: i \in \text{inv-into } P F J$
then obtain I **where** $J: J = F I$ **and** $I: I \in P - \{I0\}$
by (*metis image-iff member-remove remove-def*)
hence $i \in I$ **using** *assms i* **unfolding** *mC-ZOC-def* **by** *auto*
hence $\text{eff } c \ s \ i \approx \text{eff } d \ t \ j \wedge (\text{cont } c \ s \ i, \text{cont } d \ t \ j) \in \text{theta}$
using *assms I j* **unfolding** *mC-ZOC-def mC-ZOC-eff-cont-def J* **by** *auto*
thus $\text{eff } d \ t \ j \approx \text{eff } c \ s \ i \wedge (\text{cont } d \ t \ j, \text{cont } c \ s \ i) \in \text{theta}$
by (*metis assms indis-sym sym-def*)
qed
qed(*insert assms, unfold sym-def m-defsAll, auto*)

lemma *matchC-LC-sym:*
assumes *: *sym theta* **and** *matchC-LC theta c d*
shows *matchC-LC theta d c*
unfolding *matchC-LC-def* **proof** *clarify*
fix $t \ s$
assume $t \approx s$ **hence** $s \approx t$ **by** (*metis indis-sym*)

then obtain $I0 P F$ **where** $mC\text{-}ZOC\ theta\ c\ d\ s\ t\ I0\ P\ F$
using $assms\ unfolding\ matchC\text{-}LC\text{-}def$ **by** $blast$
hence $mC\text{-}ZOC\ theta\ d\ c\ t\ s\ (F\ I0)\ (F\ 'P)\ (inv\text{-}into\ P\ F)$
using $mC\text{-}ZOC\text{-}sym\ *$ **by** $auto$
thus $\exists I0\ Q\ G.\ mC\text{-}ZOC\ theta\ d\ c\ t\ s\ I0\ Q\ G$ **by** $blast$
qed

lemma $ZOretr\text{-}sym$:
assumes $sym\ theta$
shows $sym\ (ZOretr\ theta)$
unfolding $sym\text{-}def$ **proof** $clarify$
fix $c\ d$ **assume** $(c, d) \in ZOretr\ theta$
hence $(d, c) \in ZOretr\ theta$
unfolding $ZOretr\text{-}def$ **using** $assms\ matchC\text{-}LC\text{-}sym$ **by** $simp$
thus $(d, c) \in ZOretr\ theta$
using $assms$ **by** $(metis\ sym\text{-}conv\text{-}converse\text{-}eq)$
qed

lemma $sym\text{-}ZObis$: $sym\ ZObis$
by $(metis\ ZObis\text{-}def\ ZOretr\text{-}sym\ mono\text{-}Retr\ sym\text{-}gfp)$

lemma $ZObis\text{-}sym$: $c \approx_{01} d \implies d \approx_{01} c$
using $sym\text{-}ZObis\ unfolding\ sym\text{-}def$ **by** $blast$

2.7 List versions of the bisimilarities

definition $SbisL$ **where**
 $SbisL\ cl\ dl \equiv$
 $length\ cl = length\ dl \wedge (\forall n < length\ cl.\ cl\ !\ n \approx_s\ dl\ !\ n)$

lemma $SbisL\text{-}intro[intr0]$:
assumes $length\ cl = length\ dl$ **and**
 $\bigwedge n.\ [n < length\ cl; n < length\ dl] \implies cl\ !\ n \approx_s\ dl\ !\ n$
shows $SbisL\ cl\ dl$
using $assms\ unfolding\ SbisL\text{-}def$ **by** $auto$

lemma $SbisL\text{-}length[simp]$:
assumes $SbisL\ cl\ dl$
shows $length\ cl = length\ dl$
using $assms\ unfolding\ SbisL\text{-}def$ **by** $simp$

lemma $SbisL\text{-}Sbis[simp]$:
assumes $SbisL\ cl\ dl$ **and** $n < length\ cl \vee n < length\ dl$
shows $cl\ !\ n \approx_s\ dl\ !\ n$
using $assms\ unfolding\ SbisL\text{-}def$ **by** $simp$

lemma $SbisL\text{-}update[simp]$:
assumes $cldl: SbisL\ cl\ dl$ **and** $c'd': c' \approx_s\ d'$
shows $SbisL\ (cl\ [n := c']\ (dl\ [n := d']\ (is\ SbisL\ ?cl'\ ?dl'))$

```

proof(cases n < length cl)
  case True
  show ?thesis proof
    fix m assume m: m < length ?cl' m < length ?dl'
    thus ?cl' ! m ≈s ?dl' ! m using assms by (cases m = n) auto
  qed (insert cldl, simp)
qed (insert cldl, simp)

```

```

lemma SbisL-update-L[simp]:
assumes SbisL cl dl and c' ≈s dl!n
shows SbisL (cl[n := c']) dl
proof –
  let ?d' = dl!n
  have SbisL (cl[n := c']) (dl[n := ?d'])
  apply(rule SbisL-update) using assms by auto
  thus ?thesis by simp
qed

```

```

lemma SbisL-update-R[simp]:
assumes SbisL cl dl and cl!n ≈s d'
shows SbisL cl (dl[n := d'])
proof –
  let ?c' = cl!n
  have SbisL (cl[n := ?c']) (dl[n := d'])
  apply(rule SbisL-update) using assms by auto
  thus ?thesis by simp
qed

```

definition ZObisL **where**
ZObisL cl dl ≡
length cl = length dl ∧ (∀ n < length cl. cl ! n ≈01 dl ! n)

```

lemma ZObisL-intro[intro]:
assumes length cl = length dl and
  ∧ n. [n < length cl; n < length dl] ⇒ cl ! n ≈01 dl ! n
shows ZObisL cl dl
using assms unfolding ZObisL-def by auto

```

```

lemma ZObisL-length[simp]:
assumes ZObisL cl dl
shows length cl = length dl
using assms unfolding ZObisL-def by simp

```

```

lemma ZObisL-ZObis[simp]:
assumes ZObisL cl dl and n < length cl ∨ n < length dl
shows cl ! n ≈01 dl ! n
using assms unfolding ZObisL-def by simp

```

lemma *ZObisL-update[simp]*:
assumes *cddl*: *ZObisL cl dl* **and** *c'd'*: *c' ≈01 d'*
shows *ZObisL (cl [n := c[∧]]) (dl [n := d[∧]]) (is ZObisL ?cl' ?dl')*
proof(*cases n < length cl*)
 case *True*
 show *?thesis* **proof**
 fix *m* **assume** *m*: *m < length ?cl' m < length ?dl'*
 thus *?cl' ! m ≈01 ?dl' ! m* **using** *assms* **by** (*cases m = n*) *auto*
 qed (*insert cddl, simp*)
qed (*insert cddl, simp*)

lemma *ZObisL-update-L[simp]*:
assumes *ZObisL cl dl* **and** *c' ≈01 dl!n*
shows *ZObisL (cl[n := c[∧]]) dl*
proof –
 let *?d' = dl!n*
 have *ZObisL (cl[n := c[∧]]) (dl[n := ?d[∧]])*
 apply(*rule ZObisL-update*) **using** *assms* **by** *auto*
 thus *?thesis* **by** *simp*
qed

lemma *ZObisL-update-R[simp]*:
assumes *ZObisL cl dl* **and** *cl!n ≈01 d'*
shows *ZObisL cl (dl[n := d[∧]])*
proof –
 let *?c' = cl!n*
 have *ZObisL (cl[n := ?c[∧]]) (dl[n := d[∧]])*
 apply(*rule ZObisL-update*) **using** *assms* **by** *auto*
 thus *?thesis* **by** *simp*
qed

2.8 Discreetness for configurations

coinductive *discrCf* **where**

intro:

$(\bigwedge i. i < \text{brn} (\text{fst } cf) \longrightarrow$
 $\quad \text{snd } cf \approx \text{snd} (\text{cont-eff } cf \ i) \wedge \text{discrCf} (\text{cont-eff } cf \ i))$
 $\implies \text{discrCf } cf$

Coinduction for discrness:

lemma *discrCf-coind[consumes 1, case-names Hyp, induct pred: discr]*:

assumes ***: *phi cf* **and**

****: $\bigwedge cf \ i.$

$\llbracket i < \text{brn} (\text{fst } cf); \text{phi } cf \rrbracket \implies$

$\text{snd } cf \approx \text{snd} (\text{cont-eff } cf \ i) \wedge (\text{phi} (\text{cont-eff } cf \ i) \vee \text{discrCf} (\text{cont-eff } cf \ i))$

shows *discrCf cf*

using *** **apply**(*induct rule: discrCf.coinduct*) **using** **** **by** *auto*

lemma *discrCf-raw-coind*[*consumes 1, case-names Hyp*]:
assumes *: *phi cf* **and**
 **: $\bigwedge cf\ i. \llbracket i < brn\ (fst\ cf);\ phi\ cf \rrbracket \implies$
 $snd\ cf \approx snd\ (cont\text{-}eff\ cf\ i) \wedge phi\ (cont\text{-}eff\ cf\ i)$
shows *discrCf cf*
using * **apply**(*induct*) **using** ** **by** *blast*

Discreetness versus transition:

lemma *discrCf-cont*[*simp*]:
assumes *: *discrCf cf* **and** **: $i < brn\ (fst\ cf)$
shows *discrCf (cont-eff cf i)*
using * **apply**(*cases rule: discrCf.cases*) **using** ** **by** *blast*

lemma *discrCf-eff-indis*[*simp*]:
assumes *: *discrCf cf* **and** **: $i < brn\ (fst\ cf)$
shows $snd\ cf \approx snd\ (cont\text{-}eff\ cf\ i)$
using * **apply**(*cases rule: discrCf.cases*) **using** ** **by** *blast*

lemma *discr-discrCf*:
assumes *discr c*
shows *discrCf (c, s)*
proof –
 { **fix** *cf* :: (*'test, 'atom, 'choice*) *cmd* \times *'state*
 assume *discr (fst cf)*
 hence *discrCf cf*
 apply (*induct*)
 using *discr-eff-indis discr-cont unfolding eff-def cont-def cont-eff-def* **by** *auto*
 }
thus *?thesis* **using** *assms* **by** *auto*
qed

lemma *ZObis-pres-discrCfL*:
assumes $fst\ cf \approx_{01}\ fst\ df$ **and** $snd\ cf \approx snd\ df$ **and** *discrCf df*
shows *discrCf cf*
proof –
have $\exists df. fst\ cf \approx_{01}\ fst\ df \wedge snd\ cf \approx snd\ df \wedge discrCf\ df$ **using** *assms* **by**
blast
thus *?thesis*
proof (*induct rule: discrCf-raw-coind*)
case (*Hyp cf i*)
then obtain *df* **where** $i: i < brn\ (fst\ cf)$ **and**
 $df: discrCf\ df$ **and** $cf\text{-}df: fst\ cf \approx_{01}\ fst\ df\ snd\ cf \approx snd\ df$ **by** *blast*
then obtain *I0 P F* **where**
 $match: mC\text{-}ZOC\ ZObis\ (fst\ cf)\ (fst\ df)\ (snd\ cf)\ (snd\ df)\ I0\ P\ F$
using *ZObis-mC-ZOC* **by** *blast*
hence $\bigcup P = \{..<brn\ (fst\ cf)\}$
using *i* **unfolding** *mC-ZOC-def mC-ZOC-part-def part-def* **by** *simp*
then obtain *I* **where** $I: I \in P$ **and** $i: i \in I$ **using** *i* **by** *auto*
show *?case*

```

proof(cases I = I0)
  case False
  then obtain j where j: j ∈ F I using match I False unfolding m-defsAll
by blast
  hence j < brn (fst df) using I match
  unfolding mC-ZOC-def mC-ZOC-part-def part-def apply simp by blast
  hence md: discrCf (cont-eff df j) and s: snd df ≈ snd (cont-eff df j)
  using df discrCf-cont[of df j] discrCf-eff-indis[of df j] by auto
  have 0: snd (cont-eff cf i) ≈ snd (cont-eff df j) and
  md2: fst (cont-eff cf i) ≈01 fst (cont-eff df j)
  using I i j match False unfolding mC-ZOC-def mC-ZOC-eff-cont-def
  unfolding eff-def cont-def cont-eff-def by auto
  hence snd cf ≈ snd (cont-eff cf i) using s indis-sym indis-trans cf-df by
blast
  thus ?thesis using md md2 0 by blast
next
  case True
  hence snd cf ≈ snd (cont-eff cf i) ∧ fst (cont-eff cf i) ≈01 fst df
  using match i ZObis-sym unfolding mC-ZOC-def mC-ZOC-eff-cont0-def
  unfolding cont-eff-def cont-def eff-def by blast
  moreover have snd (cont-eff cf i) ≈ snd df
  using match i indis-sym indis-trans cf-df unfolding mC-ZOC-def mC-ZOC-eff-cont0-def
  unfolding cont-eff-def cont-def eff-def True by blast
  ultimately show ?thesis using df cf-df by blast
  qed
  qed
qed

corollary ZObis-pres-discrCfR:
assumes discrCf cf and fst cf ≈01 fst df and snd cf ≈ snd df
shows discrCf df
using assms ZObis-pres-discrCfL ZObis-sym indis-sym by blast

```

end

end

3 Trace-Based Noninterference

```

theory Trace-Based
  imports Resumption-Based
begin

```

3.1 Preliminaries

lemma dist-sum:

fixes $f :: 'a \Rightarrow \text{real}$ **and** $g :: 'a \Rightarrow \text{real}$
assumes $\bigwedge i. i \in I \implies \text{dist } (f \ i) \ (g \ i) \leq e \ i$
shows $\text{dist } (\sum i \in I. f \ i) \ (\sum i \in I. g \ i) \leq (\sum i \in I. e \ i)$
proof *cases*
assume *finite I from this assms show ?thesis*
proof *induct*
case *(insert i I)*
then have $\text{dist } (\sum i \in \text{insert } i \ I. f \ i) \ (\sum i \in \text{insert } i \ I. g \ i) \leq \text{dist } (f \ i) \ (g \ i) +$
 $\text{dist } (\sum i \in I. f \ i) \ (\sum i \in I. g \ i)$
by *(simp add: dist-triangle-add)*
also have $\dots \leq e \ i + (\sum i \in I. e \ i)$
using *insert by (intro add-mono) auto*
finally show *?case using insert by simp*
qed *simp*
qed *simp*

lemma *dist-mult[simp]*: $\text{dist } (x * y) \ (x * z) = |x| * \text{dist } y \ z$
unfolding *dist-real-def abs-mult[symmetric] right-diff-distrib ..*

lemma *dist-divide[simp]*: $\text{dist } (y / r) \ (z / r) = \text{dist } y \ z / |r|$
unfolding *dist-real-def abs-divide[symmetric] diff-divide-distrib ..*

lemma *dist-weighted-sum*:

fixes $f :: 'a \Rightarrow \text{real}$ **and** $g :: 'b \Rightarrow \text{real}$
assumes *eps*: $\bigwedge i \ j. i \in I \implies j \in J \implies w \ i \neq 0 \implies v \ j \neq 0 \implies \text{dist } (f \ i) \ (g \ j) \leq d \ i + e \ j$
and *pos*: $\bigwedge i. i \in I \implies 0 \leq w \ i \ \bigwedge j. j \in J \implies 0 \leq v \ j$
and *sum*: $(\sum i \in I. w \ i) = 1 \ (\sum j \in J. v \ j) = 1$
shows $\text{dist } (\sum i \in I. w \ i * f \ i) \ (\sum j \in J. v \ j * g \ j) \leq (\sum i \in I. w \ i * d \ i) + (\sum j \in J. v \ j * e \ j)$
proof *-*
let $?W \ h = (\sum (i,j) \in I \times J. (w \ i * v \ j) * h \ (i,j))$
{ **fix** $g :: 'b \Rightarrow \text{real}$
have $(\sum j \in J. v \ j * g \ j) = (\sum i \in I. w \ i) * (\sum j \in J. v \ j * g \ j)$
using *sum by simp*
also have $\dots = ?W \ (g \circ \text{snd})$
by *(simp add: ac-simps sum-product sum.cartesian-product)*
finally have $(\sum j \in J. v \ j * g \ j) = ?W \ (g \circ \text{snd}) \ . \}$
moreover
{ **fix** $f :: 'a \Rightarrow \text{real}$
have $(\sum i \in I. w \ i * f \ i) = (\sum i \in I. w \ i * f \ i) * (\sum j \in J. v \ j)$
using *sum by simp*
also have $\dots = ?W \ (f \circ \text{fst})$
unfolding *sum-product sum.cartesian-product by (simp add: ac-simps)*
finally have $(\sum i \in I. w \ i * f \ i) = ?W \ (f \circ \text{fst}) \ . \}$
moreover
{ **have** $\text{dist } (?W \ (f \circ \text{fst})) \ (?W \ (g \circ \text{snd})) \leq ?W \ (\lambda(i,j). d \ i + e \ j)$
using *eps pos*
by *(intro dist-sum)*

```

      (auto simp: mult-le-cancel-left zero-less-mult-iff mult-le-0-iff not-le[symmetric])
    also have ... = ?W (d o fst) + ?W (e o snd)
      by (auto simp add: sum.distrib[symmetric] field-simps intro!: sum.cong)
    finally have dist (?W (f o fst)) (?W (g o snd)) ≤ ?W (d o fst) + ?W (e o snd)
  by simp }
  ultimately show ?thesis by simp
qed

```

```

lemma field-abs-le-zero-epsilon:
  fixes x :: 'a::{\linordered-field}
  assumes epsilon:  $\bigwedge e. 0 < e \implies |x| \leq e$ 
  shows  $|x| = 0$ 
proof (rule antisym)
  show  $|x| \leq 0$ 
  proof (rule field-le-epsilon)
    fix e :: 'a assume  $0 < e$  then show  $|x| \leq 0 + e$  by simp fact
  qed
qed simp

```

```

lemma nat-nat-induct[case-names less]:
  fixes P :: nat  $\Rightarrow$  nat  $\Rightarrow$  bool
  assumes less:  $\bigwedge n m. (\bigwedge j k. j + k < n + m \implies P j k) \implies P n m$ 
  shows P n m
proof -
  define N where  $N \equiv n + m$ 
  then show ?thesis
  proof (induct N arbitrary: n m rule: nat-less-induct)
    case 1 show P n m
      apply (rule less)
      apply (rule 1[rule-format])
      apply auto
      done
  qed
qed

```

```

lemma part-insert:
  assumes part A P assumes  $X \cap A = \{\}$ 
  shows part (A  $\cup$  X) (insert X P)
  using assms by (auto simp: part-def)

```

```

lemma part-insert-subset:
  assumes X: part (A - X) P  $X \subseteq A$ 
  shows part A (insert X P)
  using X part-insert[of A - X P X] by (simp add: Un-absorb2)

```

```

lemma part-is-subset:
  part S P  $\implies p \in P \implies p \subseteq S$ 
  unfolding part-def by (metis Union-upper)

```


lemma *dist-nonneg-bounded*:

fixes $l\ u\ x\ y :: \text{real}$
assumes $l \leq x\ x \leq u\ l \leq y\ y \leq u$
shows $\text{dist } x\ y \leq u - l$
using *assms* **unfolding** *dist-real-def* **by** *arith*

lemma *integrable-count-space-finite-support*:

fixes $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$
shows $\text{finite } \{x \in X. f\ x \neq 0\} \implies \text{integrable } (\text{count-space } X)\ f$
by (*auto simp: nn-integral-count-space integrable-iff-bounded*)

lemma *lebesgue-integral-point-measure*:

fixes $g :: - \Rightarrow \text{real}$
assumes $f: \text{finite } \{a \in A. 0 < f\ a \wedge g\ a \neq 0\}$
shows $\text{integral}^L (\text{point-measure } A\ f)\ g = (\sum a | a \in A \wedge 0 < f\ a \wedge g\ a \neq 0. f\ a * g\ a)$

proof –

have *eq*: $\{a \in A. \max\ 0\ (f\ a) \neq 0 \wedge g\ a \neq 0\} = \{a \in A. 0 < f\ a \wedge g\ a \neq 0\}$
by *auto*
have $*$: $\text{ennreal } (f\ x) = \text{ennreal } (\max\ 0\ (f\ x))$ **for** x
by (*cases* $0 \leq f\ x$) (*auto simp: max.absorb1 ennreal-neg*)

show *?thesis*

unfolding *point-measure-def* $*$

using f

apply (*subst integral-real-density*)

apply (*auto simp add: integral-real-density lebesgue-integral-count-space-finite-support*

eq

intro!: *sum.cong*)

done

qed

lemma (*in finite-measure*) *finite-measure-dist*:

assumes *AE*: $AE\ x\ \text{in } M. x \notin C \longrightarrow (x \in A \longleftrightarrow x \in B)$

assumes *sets*: $A \in \text{sets } M\ B \in \text{sets } M\ C \in \text{sets } M$

shows $\text{dist } (\text{measure } M\ A)\ (\text{measure } M\ B) \leq \text{measure } M\ C$

proof –

{ **have** $\text{measure } M\ A \leq \text{measure } M\ (B \cup C)$

using *AE sets* **by** (*auto intro: finite-measure-mono-AE*)

also have $\dots \leq \text{measure } M\ B + \text{measure } M\ C$

using *sets* **by** (*auto intro: finite-measure-subadditive*)

finally have $A: \text{measure } M\ A \leq \text{measure } M\ B + \text{measure } M\ C . }$

moreover

{ **have** $\text{measure } M\ B \leq \text{measure } M\ (A \cup C)$

using *AE sets* **by** (*auto intro: finite-measure-mono-AE*)

also have $\dots \leq \text{measure } M\ A + \text{measure } M\ C$

using *sets* **by** (*auto intro: finite-measure-subadditive*)

finally have $B: \text{measure } M\ B \leq \text{measure } M\ A + \text{measure } M\ C . }$

ultimately show *?thesis*
by (*simp add: dist-real-def*)
qed

lemma (*in prob-space*) *prob-dist*:
assumes *AE*: $AE\ x\ in\ M.\ \neg\ C\ x\ \longrightarrow\ (A\ x\ \longleftrightarrow\ B\ x)$
assumes *sets*: *Measurable.pred M A Measurable.pred M B Measurable.pred M C*
shows $dist\ \mathcal{P}(x\ in\ M.\ A\ x)\ \mathcal{P}(x\ in\ M.\ B\ x)\ \leq\ \mathcal{P}(x\ in\ M.\ C\ x)$
using *assms* **by** (*intro finite-measure-dist*) *auto*

lemma *Least-eq-0-iff*: $(\exists\ i::nat.\ P\ i)\ \Longrightarrow\ (LEAST\ i.\ P\ i) = 0\ \longleftrightarrow\ P\ 0$
by (*metis LeastI-ex neq0-conv not-less-Least*)

lemma *case-nat-comp-Suc[*simp*]*: $case\ nat\ x\ f\ \circ\ Suc = f$
by *auto*

lemma *sum-eq-0-iff*:
fixes $f :: - \Rightarrow 'a :: \{comm-monoid-add, ordered-ab-group-add\}$
shows $finite\ A\ \Longrightarrow\ (\bigwedge i. i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow (\sum_{i \in A} f\ i) = 0 \longleftrightarrow (\forall i \in A. f\ i = 0)$
apply *rule*
apply (*blast intro: sum-nonneg-0*)
apply *simp*
done

lemma *sum-less-0-iff*:
fixes $f :: - \Rightarrow 'a :: \{comm-monoid-add, ordered-ab-group-add\}$
shows $finite\ A\ \Longrightarrow\ (\bigwedge i. i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow 0 < (\sum_{i \in A} f\ i) \longleftrightarrow (\exists i \in A. 0 < f\ i)$
using *sum-nonneg[of A f] sum-eq-0-iff[of A f]* **by** (*simp add: less-le*)

context *PL-Indis*
begin

declare *emp-gen[*simp del*]*

interpretation *pmf-as-function* .

lift-definition *wt-pmf* :: $('test, 'atom, 'choice)\ cmd \times 'state \Rightarrow nat\ pmf\ is$
 $\lambda(c, s)\ i.\ if\ proper\ c\ then\ if\ i < brn\ c\ then\ wt\ c\ s\ i\ else\ 0\ else\ if\ i = 0\ then\ 1\ else\ 0$

proof

let $?f = \lambda(c, s)\ i.\ if\ proper\ c\ then\ if\ i < brn\ c\ then\ wt\ c\ s\ i\ else\ 0\ else\ if\ i = 0\ then\ 1\ else\ 0$
fix *cf* **show** $\forall i. 0 \leq ?f\ cf\ i$
by (*auto split: prod.split*)
show $(\int^+ i. ?f\ cf\ i\ \partial count-space\ UNIV) = 1$
by (*subst nn-integral-count-space'[where A=if proper (fst cf) then {..<brn (fst cf)} else {0} for n]*)

(*auto split: prod.split*)
qed

definition *trans* :: ('test, 'atom, 'choice) cmd × 'state ⇒ (('test, 'atom, 'choice) cmd × 'state) pmf **where**
trans cf = *map-pmf* (λ*i*. *if proper* (*fst cf*) *then cont-eff cf i* *else cf*) (*wt-pmf cf*)

sublocale *T?*: *MC-syntax trans* .

abbreviation *G cf* ≡ *set-pmf* (*trans cf*)

lemma *set-pmf-map*: *set-pmf* (*map-pmf f M*) = *f* ‘ *set-pmf M*
using *pmf-set-map[of f]* **by** (*simp add: comp-def fun-eq-iff*)

lemma *set-pmf-wt*:
set-pmf (*wt-pmf cf*) = (*if proper* (*fst cf*) *then* {*i*. *i* < *brn* (*fst cf*) ∧ *0* < *wt* (*fst cf*) (*snd cf*) *i*} *else* {*0*})
by (*auto simp: set-eq-iff set-pmf-iff wt-pmf.rep-eq split: prod.split*) (*metis less-le wt-ge-0*)

lemma *G-eq*:
G cf = (*if proper* (*fst cf*) *then* {*cont-eff cf i* | *i*. *i* < *brn* (*fst cf*) ∧ *0* < *wt* (*fst cf*) (*snd cf*) *i*} *else* {*cf*})
by (*auto simp add: trans-def set-pmf-map set-pmf-wt*)

lemma *discrCf-G*: *discrCf cf* ⇒ *cf' ∈ G cf* ⇒ *discrCf cf'*
using *discrCf-cont[of cf]* **by** (*auto simp: G-eq split: if-split-asm*)

lemma *measurable-discrCf[measurable]*: *Measurable.pred* (*count-space UNIV*) *discrCf*
by *auto*

lemma *measurable-indis[measurable]*: *Measurable.pred* (*count-space UNIV*) (λ*x*. *snd x* ≈ *c*)
by *auto*

lemma *integral-trans*:
proper (*fst cf*) ⇒
($\int x. f x \partial \text{trans } cf$) = ($\sum i < \text{brn } (fst \text{ } cf). \text{wt } (fst \text{ } cf) (\text{snd } cf) i * f (\text{cont-eff } cf \text{ } i)$)
unfolding *trans-def map-pmf-rep-eq*
apply (*simp add: integral-distr*)
apply (*subst integral-measure-pmf[of {..
apply (*auto simp: set-pmf-wt mult-ac wt-pmf.rep-eq split: prod.split*)
done*

3.2 Quasi strong termination traces

abbreviation *qsend* ≡ *sfirst* (*holds discrCf*)

lemma *qsend-eq-0-iff*: $qsend\ cfT = 0 \iff discrCf\ (shd\ cfT)$
by (*simp add: sfirst.simps[of - cfT]*)

lemma *qsend-eq-0[simp]*: $discrCf\ cf \implies qsend\ (cf\ \#\#\ \omega) = 0$
by (*simp add: qsend-eq-0-iff*)

lemma *alw-discrCf*: $enabled\ cf\ \omega \implies discrCf\ cf \implies alw\ (holds\ discrCf)\ \omega$
by (*coinduction arbitrary: cf \omega*)
(*auto simp add: HLD-iff elim: enabled.cases intro: discrCf-G simp del: split-paired-Ex*)

lemma *alw-discrCf-indis'*:
 $enabled\ cf\ \omega \implies discrCf\ cf \implies snd\ cf \approx t \implies alw\ (holds\ (\lambda cf'.\ snd\ cf' \approx t))\ \omega$
proof (*coinduction arbitrary: cf \omega*)
case *alw with* *discrCf-eff-indis[of cf]* **show** *?case*
by (*auto simp add: HLD-iff enabled.simps[of - \omega] G-eq*)
simp del: split-paired-Ex discrCf-eff-indis
intro!: exI[of - shd \omega]
split: if-split-asm
(*blast intro: indis-trans indis-sym*)+

qed

lemma *alw-discrCf-indis*:
 $enabled\ cf\ \omega \implies discrCf\ cf \implies alw\ (holds\ (\lambda cf'.\ snd\ cf' \approx snd\ cf))\ (cf\ \#\#\ \omega)$
using *alw-discrCf-indis'[of cf \omega, OF - - indis-refl]*
by (*simp add: alw.simps[of - cf \#\#\ \omega] indis-refl*)

lemma *enabled-sdrop*: $enabled\ cf\ \omega \implies enabled\ ((cf\ \#\#\ \omega)\ \#\ n)\ (sdrop\ n\ \omega)$
proof (*induction n arbitrary: cf \omega*)
case (*Suc n*) **from** *Suc.IH[of shd \omega stl \omega] Suc.prem* **show** *?case*
unfolding *enabled.simps[of cf]* **by** *simp*

qed *simp*

lemma *sfirst-eq-eSuc*: $sfirst\ P\ \omega = eSuc\ n \iff (\neg\ P\ \omega) \wedge sfirst\ P\ (stl\ \omega) = n$
by (*subst sfirst.simps*) *auto*

lemma *qsend-snth*: $qsend\ \omega = enat\ n \implies discrCf\ (\omega\ \#\ n)$
by (*induction n arbitrary: \omega*)
(*simp-all add: eSuc-enat[symmetric] enat-0 sfirst-eq-0 sfirst-eq-eSuc*)

lemma *indis-iff*: $a \approx d \implies b \approx d \implies a \approx c \iff b \approx c$
by (*metis indis-trans indis-sym indis-refl*)

lemma *enabled-qsend-indis*:
assumes $enabled\ cf\ \omega\ qsend\ (cf\ \#\#\ \omega) \leq n\ qsend\ (cf\ \#\#\ \omega) \leq m$
shows $snd\ ((cf\ \#\#\ \omega)\ \#\ n) \approx t \iff snd\ ((cf\ \#\#\ \omega)\ \#\ m) \approx t$
proof –
from *assms* **obtain** *N :: nat* **where** $N: qsend\ (cf\ \#\#\ \omega) = N$
by (*cases qsend (cf \#\#\ \omega)*) *auto*

note $discr-N = qsend\text{-}snth[OF\ this]$ **and** $sd = enabled\text{-}sdrop[OF\ assms(1),\ of\ N]$
have $\bigwedge \omega. \omega !! N \##\ sdrop\ N\ (stl\ \omega) = sdrop\ N\ \omega$
by $(induct\ N)\ auto$
from $this[of\ cf\ \##\ \omega]$ **have** $eq: (cf\ \##\ \omega) !! N \##\ sdrop\ N\ \omega = sdrop\ N\ (cf\ \##\ \omega)$
by $simp$

from $discr-N\ alw\text{-}discrCf\text{-}indis[OF\ sd\ -]$ $assms(2,3)$ **show** $?thesis$
by $(simp\ add: N\ alw\text{-}iff\text{-}sdrop\ le\text{-}iff\text{-}add[where\ 'a=nat]\ eq)$
 $(metis\ indis\text{-}iff)$

qed

lemma $enabled\text{-}imp\text{-}UNTIL\text{-}alw\text{-}discrCf$:

$enabled\ (shd\ \omega)\ (stl\ \omega) \implies (not\ (holds\ discrCf)\ until\ (alw\ (holds\ discrCf)))\ \omega$

proof $(coinduction\ arbitrary: \omega)$

case $UNTIL$ **then** **show** $?case$

using $alw\text{-}discrCf[of\ shd\ \omega\ stl\ \omega]$

by $(cases\ discrCf\ (shd\ \omega))$

$(simp\text{-}all\ add: enabled.\text{simps}[of\ shd\ \omega]\ alw.\text{simps}[of\ -\ \omega])$

qed

lemma $less\text{-}qsend\text{-}iff\text{-}not\text{-}discrCf$:

$enabled\ cf\ bT \implies n < qsend\ (cf\ \##\ bT) \longleftrightarrow \neg\ discrCf\ ((cf\ \##\ bT) !! n)$

using $enabled\text{-}imp\text{-}UNTIL\text{-}alw\text{-}discrCf[THEN\ less\text{-}sfirst\text{-}iff,\ of\ cf\ \##\ bT]$

by $(simp\ add: holds.\text{simps}[abs\text{-}def])$

3.3 Terminating configurations

definition $qstermCf\ cf \longleftrightarrow (\forall\ cfT. enabled\ cf\ cfT \longrightarrow qsend\ (cf\ \##\ cfT) < \infty)$

lemma $qstermCf\text{-}E$:

$qstermCf\ cf \implies cf' \in G\ cf \implies qstermCf\ cf'$

apply $(auto\ simp: qstermCf\text{-}def)$

apply $(erule\ tac\ x=cf' \##\ cfT\ in\ allE)$

apply $(auto\ simp: sfirst\text{-}Stream\ intro: enat\text{-}0\ discrCf\text{-}G\ split: if\text{-}split\text{-}asm\ intro: enabled.\text{intros})$

done

abbreviation $eff\text{-}at\ cf\ bT\ n \equiv snd\ ((cf\ \##\ bT) !! n)$

abbreviation $cont\text{-}at\ cf\ bT\ n \equiv fst\ ((cf\ \##\ bT) !! n)$

definition $amSec\ c \longleftrightarrow (\forall\ s1\ s2\ n\ t. s1 \approx s2 \longrightarrow$

$\mathcal{P}(bT\ in\ T.T\ (c, s1). eff\text{-}at\ (c, s1)\ bT\ n \approx t) =$

$\mathcal{P}(bT\ in\ T.T\ (c, s2). eff\text{-}at\ (c, s2)\ bT\ n \approx t))$

definition $eSec\ c \longleftrightarrow (\forall\ s1\ s2\ t. s1 \approx s2 \longrightarrow$

$\mathcal{P}(bT\ in\ T.T\ (c, s1). \exists n. qsend\ ((c, s1) \##\ bT) = n \wedge eff\text{-}at\ (c, s1)\ bT\ n \approx t) =$

$\mathcal{P}(bT\ in\ T.T\ (c, s2). \exists n. qsend\ ((c, s2) \##\ bT) = n \wedge eff\text{-}at\ (c, s2)\ bT\ n \approx t))$

t))

definition $aeT\ c \longleftrightarrow (\forall s. AE\ bT\ in\ T.T\ (c,s).\ qsend\ ((c,s)\ \#\# \ bT) < \infty)$

lemma *dist-Ps-upper-bound:*

fixes $cf1\ cf2 :: ('test, 'atom, 'choice)\ cmd \times 'state$ **and** $s :: 'state$ **and** S

defines $S\ cf\ bT \equiv \exists n. qsend\ (cf\ \#\# \ bT) = n \wedge eff-at\ cf\ bT\ n \approx s$

defines $Ps\ cf \equiv \mathcal{P}(bT\ in\ T.T\ cf.\ S\ cf\ bT)$

defines $N\ cf\ n\ bT \equiv \neg\ discrCf\ ((cf\ \#\# \ bT)\ \#\#\ n)$

defines $Pn\ cf\ n \equiv \mathcal{P}(bT\ in\ T.T\ cf.\ N\ cf\ n\ bT)$

assumes *bisim: proper (fst cf1) proper (fst cf2) fst cf1 \approx_{01} fst cf2 snd cf1 \approx snd cf2*

shows $dist\ (Ps\ cf1)\ (Ps\ cf2) \leq Pn\ cf1\ n + Pn\ cf2\ m$

using *bisim proof (induct n m arbitrary: cf1 cf2 rule: nat-nat-induct)*

case (*less n m*)

note $\langle proper\ (fst\ cf1) \rangle [simp, intro]$

note $\langle proper\ (fst\ cf2) \rangle [simp, intro]$

define W **where** $W\ c = sum\ (wt\ (fst\ c)\ (snd\ c))$ **for** c

from $ZObis-mC-ZOC[OF\ \langle fst\ cf1\ \approx_{01}\ fst\ cf2 \rangle\ \langle snd\ cf1\ \approx\ snd\ cf2 \rangle]$

obtain $I0\ P\ F$ **where** $mC: mC-ZOC\ ZObis\ (fst\ cf1)\ (fst\ cf2)\ (snd\ cf1)\ (snd\ cf2)\ I0\ P\ F$ **by** *blast*

then **have** $P: \{\} \notin P - \{I0\}$ *part* $\{..<brn\ (fst\ cf1)\}$ P **and** $I0 \in P$

and $FP: \{\} \notin F(P - \{I0\})$ *part* $\{..<brn\ (fst\ cf2)\}$ $(F\ P)$ *inj-on* $F\ P$

and $eff: \bigwedge q\ i\ j. q \in P \implies q \neq I0 \implies i \in q \implies j \in F\ q \implies eff\ (fst\ cf1)\ (snd\ cf1)\ i \approx eff\ (fst\ cf2)\ (snd\ cf2)\ j$

and $cont: \bigwedge q\ i\ j. q \in P \implies q \neq I0 \implies i \in q \implies j \in F\ q \implies cont\ (fst\ cf1)\ (snd\ cf1)\ i \approx_{01}\ cont\ (fst\ cf2)\ (snd\ cf2)\ j$

and $wt:$

$\bigwedge I. I \in P - \{I0\} \implies W\ cf1\ I0 < 1 \implies W\ cf2\ (F\ I0) < 1 \implies$

$W\ cf1\ I / (1 - W\ cf1\ I0) = W\ cf2\ (F\ I) / (1 - W\ cf2\ (F\ I0))$

and $I0:$

$\bigwedge i. i \in I0 \implies snd\ cf1 \approx eff\ (fst\ cf1)\ (snd\ cf1)\ i$

$\bigwedge i. i \in I0 \implies cont\ (fst\ cf1)\ (snd\ cf1)\ i \approx_{01}\ fst\ cf2$

and $FI0:$

$\bigwedge i. i \in F\ I0 \implies snd\ cf2 \approx eff\ (fst\ cf2)\ (snd\ cf2)\ i$

$\bigwedge i. i \in F\ I0 \implies fst\ cf1 \approx_{01}\ cont\ (fst\ cf2)\ (snd\ cf2)\ i$

unfolding $mC-ZOC-def\ mC-ZOC-part-def\ mC-ZOC-eff-cont-def\ mC-ZOC-eff-cont0-def\ mC-ZOC-wt-def\ W-def$

by *simp-all*

have *finite* P *inj-on* $F\ (P - \{I0\})$ **and** FP' : *finite* $(F\ P)$ $F\ I0 \in F\ P$

using *finite-part[OF - P(2)] finite-part[OF - FP(2)] $\langle I0 \in P \rangle$ $\langle inj-on\ F\ P \rangle$*

by (*auto intro: inj-on-diff*)

{ **fix** $I\ i$ **assume** $I \in P\ i \in I$

with P **have** $0 \leq wt\ (fst\ cf1)\ (snd\ cf1)\ i$

by (*auto dest: part-is-subset intro!: wt-ge-0*) }

note $w1-nneg[intro] = this$

```

{ fix I i assume I ∈ P i ∈ F I
  with FP have 0 ≤ wt (fst cf2) (snd cf2) i
  by (auto dest: part-is-subset intro!: wt-ge-0) }
note wt2-nneg[intro] = this

{ fix I assume I ∈ P
  then have 0 ≤ W cf1 I
  unfolding W-def by (auto intro!: sum-nonneg) }
note W1-nneg[intro] = this

{ fix I assume I ∈ P
  then have 0 ≤ W cf2 (F I)
  unfolding W-def by (auto intro!: sum-nonneg) }
note W2-nneg[intro] = this

show ?case
proof cases
  { fix n cf1 cf2 assume *: fst cf1 ≈01 fst cf2 snd cf1 ≈ snd cf2
    have dist (Ps cf1) (Ps cf2) ≤ Pn cf1 0
    proof cases
      assume cf1: discrCf cf1
      moreover
      note cf2 = ZObis-pres-discrCfR[OF cf1 *]
      from cf1 cf2 have S cf1 = (λbT. snd cf1 ≈ s) S cf2 = (λbT. snd cf2 ≈ s)
      unfolding S-def[abs-def] by (auto simp: enat-0[symmetric])
      moreover from ⟨snd cf1 ≈ snd cf2⟩ have snd cf1 ≈ s ↔ snd cf2 ≈ s
      by (blast intro: indis-sym indis-trans)
      ultimately show ?thesis
      using T.prob-space by (cases snd cf2 ≈ s) (simp-all add: Ps-def Pn-def
measure-nonneg)
    next
      assume cf1: ¬ discrCf cf1
      then have Pn cf1 0 = 1
      using T.prob-space by (simp add: Pn-def N-def)
      moreover have dist (Ps cf1) (Ps cf2) ≤ 1
      using dist-nonneg-bounded[where u=1 and l=0 and x=Ps cf1 and y=Ps
cf2]
      by (auto simp add: dist-real-def Ps-def measure-nonneg split: abs-split)
      ultimately show ?thesis by simp
    qed }
  note base-case = this

assume n = 0 ∨ m = 0
then show ?thesis
proof
  assume n = 0
  moreover
  with T.prob-space ⟨fst cf1 ≈01 fst cf2⟩ ⟨snd cf1 ≈ snd cf2⟩

```

```

have dist (Ps cf1) (Ps cf2) ≤ Pn cf1 0
  by (intro base-case) (auto simp: Ps-def Pn-def)
moreover have 0 ≤ Pn cf2 m
  by (simp add: Pn-def measure-nonneg)
ultimately show ?thesis
  by simp
next
assume m = 0
moreover
with T.prob-space ⟨fst cf1 ≈01 fst cf2⟩ ⟨snd cf1 ≈ snd cf2⟩
have dist (Ps cf2) (Ps cf1) ≤ Pn cf2 0
  by (intro base-case) (auto simp: Ps-def Pn-def intro: indis-sym ZObis-sym)
moreover have 0 ≤ Pn cf1 n
  by (simp add: Pn-def measure-nonneg)
ultimately show ?thesis
  by (simp add: dist-commute)
qed
next
assume ¬ (n = 0 ∨ m = 0)
then have n ≠ 0 m ≠ 0 by auto
then obtain n' m' where nm: n = Suc n' m = Suc m'
  by (metis nat.exhaust)

define ps pn
  where ps cf I = (∑ b∈I. wt (fst cf) (snd cf) b / W cf I * Ps (cont-eff cf b))
  and pn cf I n = (∑ b∈I. wt (fst cf) (snd cf) b / W cf I * Pn (cont-eff cf
b) n)
  for cf I n

{ fix I assume I ∈ P I ≠ I0 and W: W cf1 I ≠ 0 W cf2 (F I) ≠ 0
then have dist (ps cf1 I) (ps cf2 (F I)) ≤ pn cf1 I n' + pn cf2 (F I) m'
  unfolding ps-def pn-def
proof (intro dist-weighted-sum)
  fix i j assume ij: i ∈ I j ∈ F I
  assume wt (fst cf1) (snd cf1) i / W cf1 I ≠ 0
  and wt (fst cf2) (snd cf2) j / W cf2 (F I) ≠ 0
  from ⟨I ∈ P⟩ ij P(2) FP(2) have br: i < brn (fst cf1) j < brn (fst cf2)
  by (auto dest: part-is-subset)
  show dist (Ps (cont-eff cf1 i)) (Ps (cont-eff cf2 j)) ≤
    Pn (cont-eff cf1 i) n' + Pn (cont-eff cf2 j) m'
  proof (rule less.hyps)
    show n' + m' < n + m using nm by simp
    show proper (fst (cont-eff cf1 i)) proper (fst (cont-eff cf2 j))
      using br less.prem by (auto simp: cont-eff)
    show fst (cont-eff cf1 i) ≈01 fst (cont-eff cf2 j)
      snd (cont-eff cf1 i) ≈ snd (cont-eff cf2 j)
      using cont[OF ⟨I ∈ P⟩ ⟨I ≠ I0⟩ ij] eff[OF ⟨I ∈ P⟩ ⟨I ≠ I0⟩ ij] by
(auto simp: cont-eff)
  qed
}

```



```

next
  show  $(\sum_{b \in F I}. wt (fst cf2) (snd cf2) b / W cf2 (F I)) = 1$ 
     $(\sum_{b \in I}. wt (fst cf1) (snd cf1) b / W cf1 I) = 1$ 
  using  $W$  by (auto simp: sum-divide-distrib[symmetric] sum-nonneg W-def)
qed auto }
note  $dist-n'-m' = this$ 

{ fix  $I$  assume  $I \in P I \neq I0$  and  $W: W cf1 I = 0 \longleftrightarrow W cf2 (F I) = 0$ 
  have  $dist (ps cf1 I) (ps cf2 (F I)) \leq pn cf1 I n' + pn cf2 (F I) m'$ 
  proof cases
    assume  $W cf2 (F I) = 0$ 
    then have  $W cf2 (F I) = 0 W cf1 I = 0$  by (auto simp: W)
    then show ?thesis by (simp add: ps-def pn-def)
  next
    assume  $W cf2 (F I) \neq 0$ 
    then have  $W cf1 I \neq 0 W cf2 (F I) \neq 0$  by (auto simp: W)
    from  $dist-n'-m'[OF \langle I \in P \rangle \langle I \neq I0 \rangle this]$  show ?thesis .
  qed }
note  $dist-n'-m'-W-iff = this$ 

{ fix  $I j$  assume  $W: W cf2 (F I0) \neq 0$ 
  from  $\langle I0 \in P \rangle$  have  $dist (\sum_{i \in \{()\}}. 1 * Ps cf1) (ps cf2 (F I0)) \leq (\sum_{i \in \{()\}}. 1 * Pn cf1 n) + pn cf2 (F I0) m'$ 
  unfolding ps-def pn-def
  proof (intro dist-weighted-sum)
    fix  $j$  assume  $j \in F I0$ 
    with  $FP(2) \langle I0 \in P \rangle$  have  $br: j < brn (fst cf2)$ 
    by (auto dest: part-is-subset)
    show  $dist (Ps cf1) (Ps (cont-eff cf2 j)) \leq Pn cf1 n + Pn (cont-eff cf2 j) m'$ 
    proof (rule less.hyps)
      show  $n + m' < n + m$  using  $nm$  by simp
      show  $proper (fst cf1) proper (fst (cont-eff cf2 j))$ 
        using  $br$  by (auto simp: cont-eff)
      show  $fst cf1 \approx_{01} fst (cont-eff cf2 j)$ 
         $snd cf1 \approx snd (cont-eff cf2 j)$ 
        using  $FIO[OF \langle j \in F I0 \rangle \langle snd cf1 \approx snd cf2 \rangle]$ 
        by (auto simp: cont-eff intro: indis-trans)
    qed
  next
    show  $(\sum_{b \in F I0}. wt (fst cf2) (snd cf2) b / W cf2 (F I0)) = 1$ 
    using  $W \langle I0 \in P \rangle$  by (auto simp: sum-divide-distrib[symmetric] sum-nonneg W-def)
  qed auto
  then have  $dist (Ps cf1) (ps cf2 (F I0)) \leq Pn cf1 n + pn cf2 (F I0) m'$ 
  by simp }
note  $dist-n-m' = this$ 

{ fix  $I j$  assume  $W: W cf1 I0 \neq 0$ 
  from  $\langle I0 \in P \rangle$  have  $dist (ps cf1 I0) (\sum_{i \in \{()\}}. 1 * Ps cf2) \leq pn cf1 I0 n'$ 

```

```

+ (∑ i∈{()}. 1 * Pn cf2 m)
  unfolding ps-def pn-def
  proof (intro dist-weighted-sum)
  fix i assume i ∈ I0
  with P(2) ⟨I0 ∈ P⟩ have br: i < brn (fst cf1)
  by (auto dest: part-is-subset)
  show dist (Ps (cont-eff cf1 i)) (Ps cf2) ≤ Pn (cont-eff cf1 i) n' + Pn cf2 m
  proof (rule less.hyps)
  show n' + m < n + m using nm by simp
  show proper (fst (cont-eff cf1 i)) proper (fst cf2)
  using br less.premis by (auto simp: cont-eff)
  show fst (cont-eff cf1 i) ≈01 fst cf2
  snd (cont-eff cf1 i) ≈ snd cf2
  using I0[OF ⟨i ∈ I0⟩] ⟨snd cf1 ≈ snd cf2⟩
  by (auto simp: cont-eff intro: indis-trans indis-sym)
  qed
next
show (∑ b∈I0. wt (fst cf1) (snd cf1) b / W cf1 I0) = 1
using W ⟨I0 ∈ P⟩ by (auto simp: sum-divide-distrib[symmetric] sum-nonneg
W-def)
qed auto
then have dist (ps cf1 I0) (Ps cf2) ≤ pn cf1 I0 n' + Pn cf2 m
by simp }
note dist-n'-m = this

have S-measurable[measurable]: ∧cf. Measurable.pred T.S (S cf)
unfolding S-def[abs-def]
by measurable

{ fix cf' cf assume cf[simp]: proper (fst cf) and cf': cf' ∈ G cf
let ?S = λbT n. qsend bT = enat n ∧ snd (bT !! n) ≈ s
{ fix bT n assume *: ?S (cf ## cf' ## bT) n have S cf' bT
proof (cases n)
case 0 with * cf' show ?thesis
by (auto simp: S-def enat-0 sfirst-Stream G-eq split: if-split-asm intro!:
exI[of - 0])
(blast intro: indis-trans indis-sym discrCf-eff-indis)
next
case (Suc n) with * cf' show S cf' bT
by (auto simp: eSuc-enat[symmetric] sfirst-Stream S-def split: if-split-asm)
qed }
moreover
{ fix bT n assume ?S (cf' ## bT) n with cf' have ?S (cf ## cf' ##
bT) (if discrCf cf then 0 else Suc n)
by (auto simp: eSuc-enat[symmetric] enat-0[symmetric] sfirst-Stream G-eq
split: if-split-asm)
(blast intro: indis-trans indis-sym discrCf-eff-indis)
then have S cf (cf' ## bT)
by (auto simp: S-def) }
}

```

ultimately have $AE\ bT\ in\ T.T\ cf'.\ S\ cf\ (cf'\ \#\#\ bT) = S\ cf'\ bT$
by (*auto simp: S-def*) }
note $S\text{-sets} = this$

{ **fix** $cf :: ('test, 'atom, 'choice)\ cmd \times 'state\ \mathbf{and}\ P\ IO\ S\ n\ st$
assume $cf[simp]:\ proper\ (fst\ cf)\ \mathbf{and}\ P:$ $part\ \{..<brn\ (fst\ cf)\}\ P\ \mathbf{and}\ finite$
 $P\ IO \in P$

{ **fix** $I\ i\ \mathbf{assume}\ I \in P\ i \in I$
with $P\ \mathbf{have}\ 0 \leq wt\ (fst\ cf)\ (snd\ cf)\ i$
by (*auto dest: part-is-subset intro!: wt-ge-0*) }
note $wt\text{-nneg}[simp] = this$

assume $S\text{-measurable}[measurable]:\ \bigwedge cf\ n.\ Measurable.pred\ T.S\ (\lambda bT.\ S\ cf\ n$
 $bT)$
and $S\text{-next}: \bigwedge cf\ cf'\ n.\ proper\ (fst\ cf) \implies cf' \in G\ cf \implies$
 $AE\ bT\ in\ T.T\ cf'.\ S\ cf\ (Suc\ n)\ (cf'\ \#\#\ bT) = S\ cf'\ n\ bT$
have $finite\text{-}I: \bigwedge I.\ I \in P \implies finite\ I$
using $finite\text{-subset}[OF\ part\text{-is}\text{-subset}[OF\ P]]$ **by** *blast*
let $?P = \lambda I.\ \sum i \in I.\ wt\ (fst\ cf)\ (snd\ cf)\ i * \mathcal{P}(bT\ in\ T.T\ (cont\text{-}eff\ cf\ i).\ S$
 $(cont\text{-}eff\ cf\ i)\ n\ bT)$
let $?P' = \lambda I.\ W\ cf\ I * (\sum i \in I.\ wt\ (fst\ cf)\ (snd\ cf)\ i / W\ cf\ I * \mathcal{P}(bT\ in$
 $T.T\ (cont\text{-}eff\ cf\ i).\ S\ (cont\text{-}eff\ cf\ i)\ n\ bT)$
have $\mathcal{P}(bT\ in\ T.T\ cf.\ S\ cf\ (Suc\ n)\ bT) = (\int cf'.\ \mathcal{P}(bT\ in\ T.T\ cf'.\ S\ cf'\ n$
 $bT)\ \partial trans\ cf)$
apply (*subst T.prob-T[OF S-measurable]*)
apply (*intro integral-cong-AE*)
apply (*auto simp: AE-measure-pmf-iff intro!: T.prob-eq-AE S-next simp del:*
 $space\text{-}T)$
apply *auto*
done
also have $\dots = (\sum I \in P.\ ?P\ I)$
unfolding $integral\text{-}trans[OF\ cf]$ **by** (*simp add: part-sum[OF P]*)
also have $\dots = (\sum I \in P.\ ?P'\ I)$
using $finite\text{-}I$
by (*auto intro!: sum.cong simp add: sum-distrib-left sum-nonneg-eq-0-iff*
 $W\text{-}def)$
also have $\dots = ?P'\ IO + (\sum I \in P - \{IO\}.\ ?P'\ I)$
unfolding $sum.remove[OF\ \langle finite\ P \rangle \langle IO \in P \rangle]$ **..**
finally have $\mathcal{P}(bT\ in\ T.T\ cf.\ S\ cf\ (Suc\ n)\ bT) = \dots . }$
note $P\text{-}split = this$

have $Ps1: Ps\ cf1 = W\ cf1\ IO * ps\ cf1\ IO + (\sum I \in P - \{IO\}.\ W\ cf1\ I * ps\ cf1$
 $I)$
unfolding $Ps\text{-}def\ ps\text{-}def$ **using** $P(2)\ \langle finite\ P \rangle \langle IO \in P \rangle$ **by** (*intro P-split*
 $S\text{-}sets)$ *simp-all*

have $Ps\ cf2 = W\ cf2\ (F\ IO) * ps\ cf2\ (F\ IO) + (\sum I \in F'P - \{F\ IO\}.\ W\ cf2\ I *$
 $ps\ cf2\ I)$

unfolding $Ps\text{-def } ps\text{-def}$ **using** $FP(2)$ $\langle finite P \rangle \langle I0 \in P \rangle$ **by** $(intro P\text{-split } S\text{-sets}) simp\text{-all}$

moreover have $F\text{-diff}$: $F \text{ ` } P - \{F I0\} = F \text{ ` } (P - \{I0\})$

by $(auto simp: \langle inj\text{-on } F P \rangle [THEN inj\text{-on}\text{-eq}\text{-iff}] \langle I0 \in P \rangle)$

ultimately have $Ps2$: $Ps\ cf2 = W\ cf2 (F I0) * ps\ cf2 (F I0) + (\sum I \in P - \{I0\}. W\ cf2 (F I) * ps\ cf2 (F I))$

by $(simp\ add: sum.reindex \langle inj\text{-on } F (P - \{I0\}) \rangle)$

have $Pn1$: $Pn\ cf1\ n = W\ cf1\ I0 * pn\ cf1\ I0\ n' + (\sum I \in P - \{I0\}. W\ cf1\ I * pn\ cf1\ I\ n')$

unfolding $Pn\text{-def } pn\text{-def } nm$ **using** $P(2)$ $\langle finite P \rangle \langle I0 \in P \rangle$ **by** $(intro P\text{-split}) (simp\text{-all } add: N\text{-def})$

have $Pn\ cf2\ m = W\ cf2 (F I0) * pn\ cf2 (F I0)\ m' + (\sum I \in F'P - \{F I0\}. W\ cf2\ I * pn\ cf2\ I\ m')$

unfolding $Pn\text{-def } pn\text{-def } nm$ **using** $FP(2)$ $\langle finite P \rangle \langle I0 \in P \rangle$ **by** $(intro P\text{-split}) (simp\text{-all } add: N\text{-def})$

with $F\text{-diff}$ **have** $Pn2$: $Pn\ cf2\ m = W\ cf2 (F I0) * pn\ cf2 (F I0)\ m' + (\sum I \in P - \{I0\}. W\ cf2 (F I) * pn\ cf2 (F I)\ m')$

by $(simp\ add: sum.reindex \langle inj\text{-on } F (P - \{I0\}) \rangle)$

show $?thesis$

proof $cases$

assume $W\ cf1\ I0 = 1 \vee W\ cf2 (F I0) = 1$

then show $?thesis$

proof

assume $*$: $W\ cf1\ I0 = 1$

then have $W\ cf1\ I0 = W\ cf1 \{..$

also have $... = W\ cf1\ I0 + (\sum I \in P - \{I0\}. W\ cf1\ I)$

unfolding $\langle part \{..$

unfolding $sum.remove[OF \langle finite P \rangle \langle I0 \in P \rangle] ..$

finally have $(\sum I \in P - \{I0\}. W\ cf1\ I) = 0$ **by** $simp$

then have $\forall I \in P - \{I0\}. W\ cf1\ I = 0$

using $\langle finite P \rangle$ **by** $(subst (asm) sum\text{-nonneg}\text{-eq}\text{-0}\text{-iff}) auto$

then have $Ps\ cf1 = ps\ cf1\ I0\ Pn\ cf1\ n = pn\ cf1\ I0\ n'$

unfolding $Ps1\ Pn1$ ***** **by** $simp\text{-all}$

moreover note $dist\text{-}n'\text{-}m$ $*$

ultimately show $?thesis$ **by** $simp$

next

assume $*$: $W\ cf2 (F I0) = 1$

then have $W\ cf2 (F I0) = W\ cf2 \{..$

also have $... = W\ cf2 (F I0) + (\sum I \in F'P - \{F I0\}. W\ cf2\ I)$

unfolding $FP(2)[THEN part\text{-sum}] W\text{-def}$

unfolding $sum.remove[OF FP] ..$

finally have $(\sum I \in F'P - \{F I0\}. W\ cf2\ I) = 0$ **by** $simp$

then have $\forall I \in F'P - \{F I0\}. W\ cf2\ I = 0$

using $\langle finite P \rangle$ **by** $(subst (asm) sum\text{-nonneg}\text{-eq}\text{-0}\text{-iff}) auto$

then have $Ps\ cf2 = ps\ cf2 (F I0)\ Pn\ cf2\ m = pn\ cf2 (F I0)\ m'$

unfolding $Ps2\ Pn2$ ***** **by** $(simp\text{-all } add: F\text{-diff})$

```

    moreover note dist-n-m' *
    ultimately show ?thesis by simp
qed
next
assume W-neq1:  $\neg (W\ cf1\ I0 = 1 \vee W\ cf2\ (F\ I0) = 1)$ 
moreover
{ fix cf :: ('test, 'atom, 'choice) cmd  $\times$  'state and I P
  assume proper (fst cf) part {..brn (fst cf)} P I  $\in$  P
  then have  $W\ cf\ I \leq W\ cf\ \{..brn (fst cf)\}$ 
    unfolding W-def
    by (intro sum-mono2 part-is-subset) auto
  then have  $W\ cf\ I \leq 1$  using  $\langle$ proper (fst cf) $\rangle$  by (simp add: W-def) }
ultimately have wt-less1:  $W\ cf1\ I0 < 1 \wedge W\ cf2\ (F\ I0) < 1$ 
  using FP( $\varrho$ ) FP'( $\varrho$ ) P( $\varrho$ )  $\langle I0 \in P \rangle$ 
  unfolding le-less by blast+

{ fix I assume *:  $I \in P - \{I0\}$ 
  have  $W\ cf1\ I = 0 \iff W\ cf2\ (F\ I) = 0$ 
    using wt[OF * wt-less1] wt-less1 by auto
  with * have  $dist\ (ps\ cf1\ I)\ (ps\ cf2\ (F\ I)) \leq pn\ cf1\ I\ n' + pn\ cf2\ (F\ I)\ m'$ 
    by (intro dist-n'-m'-W-iff) auto }
note dist-eps = this

{ fix a b c d :: real
  have  $dist\ a\ b = dist\ ((a - c) + c)\ ((b - d) + d)$  by simp
  also have  $\dots \leq dist\ (a - c)\ (b - d) + dist\ c\ d$ 
    using dist-triangle-add .
  finally have  $dist\ a\ b \leq dist\ (a - c)\ (b - d) + dist\ c\ d$  . }
note dist-triangle-diff = this

have dist-diff-diff:  $\bigwedge a\ b\ c\ d::real. dist\ (a - b)\ (c - d) \leq dist\ a\ b + dist\ c\ d$ 
  unfolding dist-real-def by auto

let ?v0 =  $W\ cf1\ I0$  and ?w0 =  $W\ cf2\ (F\ I0)$ 
let ?v1 =  $1 - ?v0$  and ?w1 =  $1 - ?w0$ 
let ?wQ =  $(Ps\ cf2 - ?w0 * ps\ cf2\ (F\ I0)) / ?w1$ 
let ?wP =  $(Ps\ cf1 - ?v0 * ps\ cf1\ I0) / ?v1$ 
let ?D =  $(?w0 * ?v1 * Ps\ cf1 + ?w1 * ?v0 * ps\ cf1\ I0)$ 
let ?E =  $(?v0 * ?w1 * Ps\ cf2 + ?v1 * ?w0 * ps\ cf2\ (F\ I0))$ 

have w0v0-less1:  $?w0 * ?v0 < 1 * 1$ 
  using wt-less1  $\langle I0 \in P \rangle$  by (intro mult-strict-mono) auto
then have neg-w0v0-nonneg:  $0 \leq 1 - ?w0 * ?v0$  by simp

let ?e1 =  $(\sum I \in P - \{I0\}. W\ cf1\ I / ?v1 * pn\ cf1\ I\ n') +$ 
   $(\sum I \in P - \{I0\}. W\ cf2\ (F\ I) / ?w1 * pn\ cf2\ (F\ I)\ m')$ 
have  $dist\ ((1 - ?w0 * ?v0) * Ps\ cf1)\ ((1 - ?w0 * ?v0) * Ps\ cf2) \leq$ 
   $dist\ ((1 - ?w0 * ?v0) * Ps\ cf1 - ?D)\ ((1 - ?w0 * ?v0) * Ps\ cf2 - ?E)$ 
+ dist ?D ?E

```

by (*rule dist-triangle-diff*)
also have $\dots \leq ?v1 * ?w1 * ?e1 + (?v1 * ?w0 * (Pn\ cf1\ n + pn\ cf2\ (F\ I0)\ m') + ?w1 * ?v0 * (pn\ cf1\ I0\ n' + Pn\ cf2\ m))$
proof (*rule add-mono*)
{ **have** $?wP = (\sum_{I \in P - \{I0\}} W\ cf1\ I * ps\ cf1\ I) / ?v1$
unfolding *Ps1* **by** (*simp add: field-simps*)
also have $\dots = (\sum_{I \in P - \{I0\}} W\ cf1\ I / ?v1 * ps\ cf1\ I)$
by (*subst sum-divide-distrib*) *simp*
finally have $?wP = (\sum_{I \in P - \{I0\}} W\ cf1\ I / ?v1 * ps\ cf1\ I) .$ }
moreover
{ **have** $?wQ = (\sum_{I \in P - \{I0\}} W\ cf2\ (F\ I) * ps\ cf2\ (F\ I)) / ?w1$
using *Ps2* **by** (*simp add: field-simps*)
also have $\dots = (\sum_{I \in P - \{I0\}} W\ cf2\ (F\ I) / ?w1 * ps\ cf2\ (F\ I))$
by (*subst sum-divide-distrib*) *simp*
also have $\dots = (\sum_{I \in P - \{I0\}} W\ cf1\ I / ?v1 * ps\ cf2\ (F\ I))$
using *wt[OF - wt-less1]* **by** *simp*
finally have $?wQ = (\sum_{I \in P - \{I0\}} W\ cf1\ I / ?v1 * ps\ cf2\ (F\ I)) .$ }
ultimately
have $dist\ ?wP\ ?wQ \leq (\sum_{I \in P - \{I0\}} W\ cf1\ I / ?v1 * (pn\ cf1\ I\ n' + pn\ cf2\ (F\ I)\ m'))$
using *wt-less1 dist-eps*
by (*simp, intro dist-sum*)
(*simp add: sum-nonneg divide-le-cancel mult-le-cancel-left not-le[symmetric]*)
W1-nneg)
also have $\dots = ?e1$
unfolding *sum.distrib[symmetric]* **using** *wt[OF - wt-less1]*
by (*simp add: field-simps add-divide-distrib*)
finally have $dist\ (?v1 * ?w1 * ?wP)\ (?v1 * ?w1 * ?wQ) \leq ?v1 * ?w1 * ?e1$
using *wt-less1 unfolding dist-mult* **by** *simp*
also {
have $?v1 * ?w1 * ?wP = ?w1 * (?v0 * Ps\ cf1 + ?v1 * Ps\ cf1) - ?w1 * ?v0 * ps\ cf1\ I0$
using *wt-less1 unfolding divide-eq-eq* **by** (*simp add: field-simps*)
also have $\dots = (1 - ?w0 * ?v0) * Ps\ cf1 - ?D$
by (*simp add: field-simps*)
finally have $?v1 * ?w1 * ?wP = (1 - ?w0 * ?v0) * Ps\ cf1 - ?D .$ }
also {
have $?v1 * ?w1 * ?wQ = ?v1 * (?w0 * Ps\ cf2 + ?w1 * Ps\ cf2) - ?v1 * ?w0 * (ps\ cf2\ (F\ I0))$
using *wt-less1 unfolding divide-eq-eq* **by** (*simp add: field-simps*)
also have $\dots = (1 - ?w0 * ?v0) * Ps\ cf2 - ?E$
by (*simp add: field-simps*)
finally have $?v1 * ?w1 * ?wQ = (1 - ?w0 * ?v0) * Ps\ cf2 - ?E .$ }
finally show $dist\ ((1 - ?w0 * ?v0) * Ps\ cf1 - ?D)\ ((1 - ?w0 * ?v0) * Ps\ cf2 - ?E) \leq ?v1 * ?w1 * ?e1 .$
next
have $dist\ ?D\ ?E = dist\ (?v1 * ?w0 * Ps\ cf1 - ?v1 * ?w0 * ps\ cf2\ (F\ I0))$

$(?w1 * ?v0 * Ps\ cf2 - ?w1 * ?v0 * ps\ cf1\ I0)$
unfolding *dist-real-def* **by** (*simp add: ac-simps*)
also have $\dots \leq dist\ (?v1 * ?w0 * Ps\ cf1)\ (?v1 * ?w0 * ps\ cf2\ (F\ I0)) +$
 $dist\ (?w1 * ?v0 * Ps\ cf2)\ (?w1 * ?v0 * ps\ cf1\ I0)$
using *dist-diff-diff* .
also have $\dots \leq ?v1 * ?w0 * (Pn\ cf1\ n + pn\ cf2\ (F\ I0)\ m') + ?w1 * ?v0$
 $* (pn\ cf1\ I0\ n' + Pn\ cf2\ m)$
proof (*rule add-mono*)
show $dist\ (?v1 * ?w0 * Ps\ cf1)\ (?v1 * ?w0 * ps\ cf2\ (F\ I0)) \leq ?v1 * ?w0$
 $* (Pn\ cf1\ n + pn\ cf2\ (F\ I0)\ m')$
using *wt-less1 dist-n-m' <I0 ∈ P>*
by (*simp add: sum-nonneg mult-le-cancel-left not-le[symmetric] mult-le-0-iff*
W2-nneg)
show $dist\ (?w1 * ?v0 * Ps\ cf2)\ (?w1 * ?v0 * ps\ cf1\ I0) \leq ?w1 * ?v0 *$
 $(pn\ cf1\ I0\ n' + Pn\ cf2\ m)$
using *wt-less1 dist-n'-m <I0 ∈ P>*
by (*subst dist-commute*)
(simp add: sum-nonneg mult-le-cancel-left not-le[symmetric] mult-le-0-iff
W1-nneg)
qed
finally show $dist\ ?D\ ?E \leq ?v1 * ?w0 * (Pn\ cf1\ n + pn\ cf2\ (F\ I0)\ m') +$
 $?w1 * ?v0 * (pn\ cf1\ I0\ n' + Pn\ cf2\ m)$.
qed
also have $\dots = ?w1 * (\sum I \in P - \{I0\}. W\ cf1\ I * pn\ cf1\ I\ n') + ?v1 *$
 $(\sum I \in P - \{I0\}. W\ cf2\ (F\ I) * pn\ cf2\ (F\ I)\ m') +$
 $?v1 * ?w0 * (Pn\ cf1\ n + pn\ cf2\ (F\ I0)\ m') + ?w1 * ?v0 * (pn\ cf1\ I0\ n' +$
 $Pn\ cf2\ m)$
using *W-neq1* **by** (*simp add: sum-divide-distrib[symmetric] add-divide-eq-iff*
divide-add-eq-iff)
also have $\dots = (1 - ?w0 * ?v0) * (Pn\ cf1\ n + Pn\ cf2\ m)$
unfolding *Pn1 Pn2* **by** (*simp add: field-simps*)
finally show $dist\ (Ps\ cf1)\ (Ps\ cf2) \leq Pn\ cf1\ n + Pn\ cf2\ m$
using *neg-w0v0-nonneg w0v0-less1* **by** (*simp add: mult-le-cancel-left*)
qed
qed
qed

lemma *AE-T-max-qsend-time:*

fixes *cf* **and** $e :: real$ **assumes** *AE: AE bT in T.T cf. qsend (cf ## bT) < ∞*
 $0 < e$

shows $\exists N. \mathcal{P}(bT\ in\ T.T\ cf. \neg\ discrCf\ ((cf\ ##\ bT)\ !!\ N)) < e$

proof –

from *AE-T-max-sfirst[OF - AE]* **obtain** $N :: nat$

where $\mathcal{P}(bT\ in\ T.T\ cf. N < qsend\ (cf\ ##\ bT)) < e$

by *auto*

also have $\mathcal{P}(bT\ in\ T.T\ cf. N < qsend\ (cf\ ##\ bT)) = \mathcal{P}(bT\ in\ T.T\ cf. \neg\$
 $discrCf\ ((cf\ ##\ bT)\ !!\ N))$

using *less-qsend-iff-not-discrCf[of cf] AE-T-enabled[of cf]*

by (*intro T.prob-eq-AE*) *auto*

finally show *?thesis ..*
qed

lemma *Ps-eq*:

fixes *cf1 cf2 s* and *S*

defines $S\ cf\ bT \equiv \exists n. \text{qsend}\ (cf\ \#\# \ bT) = n \wedge \text{eff-at}\ cf\ bT\ n \approx s$

defines $Ps\ cf \equiv \mathcal{P}(bT\ \text{in}\ T.T\ cf. S\ cf\ bT)$

assumes *qsterm1*: $AE\ bT\ \text{in}\ T.T\ cf1. \text{qsend}\ (cf1\ \#\# \ bT) < \infty$

assumes *qsterm2*: $AE\ bT\ \text{in}\ T.T\ cf2. \text{qsend}\ (cf2\ \#\# \ bT) < \infty$

and *bisim*: *proper* (*fst cf1*) *proper* (*fst cf2*) *fst cf1* \approx_{01} *fst cf2* *snd cf1* \approx *snd cf2*

shows $Ps\ cf1 = Ps\ cf2$

proof –

let $?nT = \lambda cf\ n\ bT. \neg \text{discrCf}\ ((cf\ \#\# \ bT)\ \#\ n)$

let $?PnT = \lambda cf\ n. \mathcal{P}(bT\ \text{in}\ T.T\ cf. ?nT\ cf\ n\ bT)$

have $\text{dist}\ (Ps\ cf1)\ (Ps\ cf2) = 0$

unfolding *dist-real-def*

proof (*rule field-abs-le-zero-epsilon*)

fix *e* :: *real* assume $0 < e$

then have $0 < e / 2$ by *auto*

from *AE-T-max-qsend-time*[*OF qsterm1 this*] *AE-T-max-qsend-time*[*OF qsterm2*

this]

obtain *n m* where $?PnT\ cf1\ n < e / 2$ $?PnT\ cf2\ m < e / 2$ by *auto*

moreover have $\text{dist}\ (Ps\ cf1)\ (Ps\ cf2) \leq ?PnT\ cf1\ n + ?PnT\ cf2\ m$

unfolding *Ps-def S-def* using *bisim* by (*rule dist-Ps-upper-bound*)

ultimately show $|Ps\ cf1 - Ps\ cf2| \leq e$

unfolding *dist-real-def* by *auto*

qed

then show $Ps\ cf1 = Ps\ cf2$ by *auto*

qed

lemma *siso-trace*:

assumes *siso* $c\ s \approx t$ *enabled* (*c*, *t*) *cfT*

shows *siso* (*cont-at* (*c*, *s*) *cfT* *n*)

and *cont-at* (*c*, *s*) *cfT* *n* = *cont-at* (*c*, *t*) *cfT* *n*

and *eff-at* (*c*, *s*) *cfT* *n* \approx *eff-at* (*c*, *t*) *cfT* *n*

using *assms*

proof (*induction n arbitrary: c s t cfT*)

case (*Suc n*) case 1

with *Suc*(1)[*of fst* (*shd cfT*) *snd* (*shd cfT*) *snd* (*shd cfT*) *stl cfT*] show *?case*

by (*auto simp add: enabled.simps*[*of - cfT*] *G-eq cont-eff indis-refl split: if-split-asm*)

qed *auto*

lemma *Sbis-trace*:

assumes *proper* (*fst cf1*) *proper* (*fst cf2*) *fst cf1* \approx_s *fst cf2* *snd cf1* \approx *snd cf2*

shows $\mathcal{P}(cfT\ \text{in}\ T.T\ cf1. \text{eff-at}\ cf1\ cfT\ n \approx s') = \mathcal{P}(cfT\ \text{in}\ T.T\ cf2. \text{eff-at}\ cf2\ cfT\ n \approx s')$

(is $?P\ cf1\ n = ?P\ cf2\ n$)

using *assms* proof (*induct n arbitrary: cf1 cf2*)


```

case 0
show ?case
proof cases
  assume  $\text{snd } cf1 \approx s'$ 
  with  $\langle \text{snd } cf1 \approx \text{snd } cf2 \rangle \langle \text{fst } cf1 \approx_s \text{fst } cf2 \rangle$  have  $\text{snd } cf1 \approx s' \text{snd } cf2 \approx s'$ 
    by (metis indis-trans indis-sym)+
  then show ?case
    using T.prob-space by simp
next
  assume  $\neg \text{snd } cf1 \approx s'$ 
  with  $\langle \text{snd } cf1 \approx \text{snd } cf2 \rangle \langle \text{fst } cf1 \approx_s \text{fst } cf2 \rangle$  have  $\neg \text{snd } cf1 \approx s' \wedge \neg \text{snd } cf2$ 
 $\approx s'$ 
    by (metis indis-trans indis-sym)
  then show ?case
    by auto
qed
next
case (Suc n)
note  $\langle \text{proper } (fst\ cf1) \rangle [simp] \langle \text{proper } (fst\ cf2) \rangle [simp]$ 

from Sbis-mC-C  $\langle \text{fst } cf1 \approx_s \text{fst } cf2 \rangle \langle \text{snd } cf1 \approx \text{snd } cf2 \rangle$ 
obtain P F where mP: mC-C Sbis (fst cf1) (fst cf2) (snd cf1) (snd cf2) P F
  by blast
then have
  P: part  $\{..<brn\ (fst\ cf1)\} P \{\} \notin P$  and
  FP: part  $\{..<brn\ (fst\ cf2)\} (F'P) \{\} \notin F' P$  inj-on F P and
  W:  $\bigwedge I. I \in P \implies \text{sum } (wt\ (fst\ cf1)\ (snd\ cf1))\ I = \text{sum } (wt\ (fst\ cf2)\ (snd\ cf2))\ (F\ I)$  and
  eff:  $\bigwedge I\ i\ j. I \in P \implies i \in I \implies j \in F\ I \implies$ 
    eff (fst cf1) (snd cf1) i  $\approx$  eff (fst cf2) (snd cf2) j and
  cont:  $\bigwedge I\ i\ j. I \in P \implies i \in I \implies j \in F\ I \implies$ 
    cont (fst cf1) (snd cf1) i  $\approx_s$  cont (fst cf2) (snd cf2) j
  unfolding mC-C-def mC-C-eff-cont-def mC-C-part-def mC-C-wt-def by metis+
  { fix cf1 :: ('test, 'atom, 'choice) cmd  $\times$  'state and P assume cf[simp]: proper
  (fst cf1) and P: part  $\{..<brn\ (fst\ cf1)\} P$ 
  have ?P cf1 (Suc n) =  $(\int cf'. ?P\ cf'\ n\ \partial\text{trans } cf1)$ 
    by (subst T.prob-T) auto
  also have ... =  $(\sum b<brn\ (fst\ cf1). wt\ (fst\ cf1)\ (snd\ cf1)\ b * ?P\ (cont\text{-eff } cf1\ b)\ n)$ 
    unfolding integral-trans[OF cf] ..
  also have ... =  $(\sum I \in P. \sum b \in I. wt\ (fst\ cf1)\ (snd\ cf1)\ b * ?P\ (cont\text{-eff } cf1\ b)\ n)$ 
    unfolding part-sum[OF P] ..
  finally have ?P cf1 (Suc n) = ... . }
note split = this

{ fix I i assume I  $\in$  P i  $\in$  I
  with  $\langle \text{proper } (fst\ cf1) \rangle$  have i < brn (fst cf1)
    using part-is-subset[OF P(1)  $\langle I \in P \rangle$ ] by auto }

```

```

note  $brn\text{-}cf[simp] = this$ 

{ fix  $I\ i$  assume  $I \in P\ i \in F\ I$ 
  with  $\langle proper\ (fst\ cf2) \rangle$  have  $i < brn\ (fst\ cf2)$ 
  using  $part\text{-}is\text{-}subset[OF\ FP(1),\ of\ F\ I]$  by  $auto$  }
note  $brn\text{-}cf2[simp] = this$ 

{ fix  $I$  assume  $I \in P$ 
  with  $\langle \{\} \notin P \rangle$  obtain  $i$  where  $i \in I$  by  $(metis\ all\text{-}not\text{-}in\text{-}conv)$ 
  from  $\langle I \in P \rangle\ FP$  have  $F\ I \neq \{\}$   $F\ I \subseteq \{..<brn\ (fst\ cf2)\}$ 
  by  $(auto\ simp:\ part\text{-}is\text{-}subset)$ 
  then obtain  $j$  where  $j < brn\ (fst\ cf2)\ j \in F\ I$  by  $auto$ 
  { fix  $b$  assume  $b \in F\ I$ 
    then have  $?P\ (cont\text{-}eff\ cf1\ i)\ n = ?P\ (cont\text{-}eff\ cf2\ b)\ n$ 
    using  $\langle I \in P \rangle\ \langle i \in I \rangle\ cont\ eff$ 
    by  $(intro\ Suc)\ (auto\ simp\ add:\ cont\text{-}eff)$  }
  note  $cont\text{-}d\text{-}const = this[symmetric]$ 
  { fix  $a$  assume  $a \in I$ 
    with  $\langle I \in P \rangle\ \langle i \in I \rangle\ \langle j \in F\ I \rangle\ cont\ eff$ 
    have  $?P\ (cont\text{-}eff\ cf1\ i)\ n = ?P\ (cont\text{-}eff\ cf2\ j)\ n \wedge$ 
     $?P\ (cont\text{-}eff\ cf1\ a)\ n = ?P\ (cont\text{-}eff\ cf2\ j)\ n$ 
    by  $(intro\ conjI\ Suc)\ (auto\ simp\ add:\ cont\text{-}eff)$ 
    then have  $?P\ (cont\text{-}eff\ cf1\ a)\ n = ?P\ (cont\text{-}eff\ cf1\ i)\ n$  by  $simp$  }
  then have  $(\sum b \in I. wt\ (fst\ cf1)\ (snd\ cf1)\ b * ?P\ (cont\text{-}eff\ cf1\ b)\ n) =$ 
 $(\sum b \in I. wt\ (fst\ cf1)\ (snd\ cf1)\ b) * ?P\ (cont\text{-}eff\ cf1\ i)\ n$ 
  by  $(simp\ add:\ sum\text{-}distrib\text{-}right)$ 
  also have  $\dots = (\sum b \in F\ I. wt\ (fst\ cf2)\ (snd\ cf2)\ b) * ?P\ (cont\text{-}eff\ cf1\ i)\ n$ 
  using  $W\ \langle I \in P \rangle$  by  $auto$ 
  also have  $\dots = (\sum b \in F\ I. wt\ (fst\ cf2)\ (snd\ cf2)\ b) * ?P\ (cont\text{-}eff\ cf2\ b)\ n$ 
  using  $cont\text{-}d\text{-}const$  by  $(auto\ simp\ add:\ sum\text{-}distrib\text{-}right)$ 
  finally have  $(\sum b \in I. wt\ (fst\ cf1)\ (snd\ cf1)\ b * ?P\ (cont\text{-}eff\ cf1\ b)\ n) = \dots$  }
  note  $sum\text{-}eq = this$ 

  have  $?P\ cf1\ (Suc\ n) = (\sum I \in P. \sum b \in I. wt\ (fst\ cf1)\ (snd\ cf1)\ b * ?P\ (cont\text{-}eff\ cf1\ b)\ n)$ 
  using  $\langle proper\ (fst\ cf1) \rangle\ P(1)$  by  $(rule\ split)$ 
  also have  $\dots = (\sum I \in P. \sum b \in F\ I. wt\ (fst\ cf2)\ (snd\ cf2)\ b * ?P\ (cont\text{-}eff\ cf2\ b)\ n)$ 
  using  $sum\text{-}eq$  by  $simp$ 
  also have  $\dots = (\sum I \in F'P. \sum b \in I. wt\ (fst\ cf2)\ (snd\ cf2)\ b * ?P\ (cont\text{-}eff\ cf2\ b)\ n)$ 
  using  $\langle inj\text{-}on\ F\ P \rangle$  by  $(simp\ add:\ sum.\text{reindex})$ 
  also have  $\dots = ?P\ cf2\ (Suc\ n)$ 
  using  $\langle proper\ (fst\ cf2) \rangle\ FP(1)$  by  $(rule\ split[symmetric])$ 
  finally show  $?case$  .
qed

```

3.4 Final Theorems

theorem *ZObis-eSec*: $\llbracket \text{proper } c; c \approx_{01} c; \text{aeT } c \rrbracket \implies \text{eSec } c$
 by (*auto simp*: *aeT-def eSec-def intro!*: *Ps-eq[simplified]*)

theorem *Sbis-amSec*: $\llbracket \text{proper } c; c \approx_s c \rrbracket \implies \text{amSec } c$
 by (*auto simp*: *amSec-def intro!*: *Sbis-trace[simplified]*)

theorem *amSec-eSec*:

assumes [*simp*]: *proper c* and *aeT c amSec c* shows *eSec c*

proof (*unfold eSec-def, intro allI impI*)

fix *s1 s2 t* assume *s1 ≈ s2*

let $?T = \lambda s. bT. \exists n. \text{qsend } ((c, s) \#\# bT) = n \wedge \text{eff-at } (c, s) bT n \approx t$

let $?P = \lambda s. \mathcal{P}(bT \text{ in } T.T (c, s). ?T s bT)$

have $\text{dist } (?P s1) (?P s2) = 0$

unfolding *dist-real-def*

proof (*rule field-abs-le-zero-epsilon*)

fix *e :: real* assume $0 < e$

then have $0 < e / 2$ by *simp*

let $?N = \lambda s n bT. \neg \text{discrCf } (((c, s) \#\# bT) !! n)$

from *AE-T-max-qsend-time*[*OF - <0 < e / 2>, of (c,s1)*]

obtain *N1* where $N1: \mathcal{P}(bT \text{ in } T.T (c, s1). ?N s1 N1 bT) < e / 2$

using $\langle \text{aeT } c \rangle$ unfolding *aeT-def* by *auto*

from *AE-T-max-qsend-time*[*OF - <0 < e / 2>, of (c,s2)*]

obtain *N2* where $N2: \mathcal{P}(bT \text{ in } T.T (c, s2). ?N s2 N2 bT) < e / 2$

using $\langle \text{aeT } c \rangle$ unfolding *aeT-def* by *auto*

define *N* where $N = \max N1 N2$

let $?Tn = \lambda n s bT. \text{eff-at } (c, s) bT n \approx t$

have $\text{dist } \mathcal{P}(bT \text{ in } T.T (c, s1). ?T s1 bT) \mathcal{P}(bT \text{ in } T.T (c, s1). ?Tn N s1 bT)$

\leq

$\mathcal{P}(bT \text{ in } T.T (c, s1). ?N s1 N1 bT)$

using $\langle \text{aeT } c \rangle$ [*unfolded aeT-def, rule-format*] *AE-T-enabled AE-space*

proof (*intro T.prob-dist, eventually-elim, intro impI*)

fix *bT* assume *bT: enabled (c,s1) bT* and $\neg \neg \text{discrCf } (((c, s1) \#\# bT) !!$

N1)

with *bT* have $\text{qsend } ((c, s1) \#\# bT) \leq N1$

using *less-qsend-iff-not-discrCf*[*of (c,s1) bT N1*] by *simp*

then show $?T s1 bT \longleftrightarrow ?Tn N s1 bT$

using *bT*

by (*cases qsend ((c, s1) \#\# bT)*)

(*auto intro!*: *enabled-qsend-indis del: iffI simp: N-def*)

qed *measurable*

moreover

have $\text{dist } \mathcal{P}(bT \text{ in } T.T (c, s2). ?T s2 bT) \mathcal{P}(bT \text{ in } T.T (c, s2). ?Tn N s2 bT)$

\leq

$\mathcal{P}(bT \text{ in } T.T (c, s2). ?N s2 N2 bT)$

using $\langle \text{aeT } c \rangle$ [*unfolded aeT-def, rule-format*] *AE-T-enabled AE-space*

```

proof (intro T.prob-dist, eventually-elim, intro impI)
  fix bT assume bT: enabled (c,s2) bT  $\neg \neg$  discrCf (((c,s2) ## bT) !! N2)
  with bT have qsend ((c,s2) ## bT)  $\leq$  N2
    using less-qsend-iff-not-discrCf[of (c,s2) bT N2] by simp
  then show ?T s2 bT  $\longleftrightarrow$  ?Tn N s2 bT
    using bT
    by (cases qsend ((c, s2) ## bT))
      (auto intro!: enabled-qsend-indis del: iffI simp: N-def)
  qed measurable
  ultimately have dist  $\mathcal{P}(bT \text{ in } T.T (c, s1). ?T s1 bT) \mathcal{P}(bT \text{ in } T.T (c, s1). ?Tn N s1 bT) +$ 
    dist  $\mathcal{P}(bT \text{ in } T.T (c, s2). ?T s2 bT) \mathcal{P}(bT \text{ in } T.T (c, s1). ?Tn N s1 bT) \leq e$ 
    using <amSec c>[unfolded amSec-def, rule-format, OF <s1  $\approx$  s2>, of N t]
    using N1 N2 by simp
  from dist-triangle-le[OF this]
  show  $|\?P s1 - \?P s2| \leq e$  by (simp add: dist-real-def)
  qed
  then show ?P s1 = ?P s2
    by simp
qed
end
end

```

4 Compositionality of Resumption-Based Noninterference

```

theory Compositionality
imports Resumption-Based
begin

```

```

context PL-Indis
begin

```

4.1 Compatibility and discreteness of atoms, tests and choices

```

definition compatAtm where
  compatAtm atm  $\equiv$ 
    ALL s t. s  $\approx$  t  $\longrightarrow$  aval atm s  $\approx$  aval atm t

```

```

definition presAtm where
  presAtm atm  $\equiv$ 
    ALL s. s  $\approx$  aval atm s

```

```

definition compatTst where

```

compatTst tst \equiv
 $\text{ALL } s \ t. \ s \approx t \longrightarrow \text{tval } \text{tst } s = \text{tval } \text{tst } t$

lemma *discrAt-compatAt[simp]*:
assumes *presAtm atm*
shows *compatAtm atm*
using *assms unfolding compatAtm-def*
by (*metis presAtm-def indis-sym indis-trans*)

definition *compatCh* **where**
 $\text{compatCh } ch \equiv \forall \ s \ t. \ s \approx t \longrightarrow \text{cval } ch \ s = \text{cval } ch \ t$

lemma *compatCh-cval[simp]*:
assumes *compatCh ch* **and** $s \approx t$
shows $\text{cval } ch \ s = \text{cval } ch \ t$
using *assms unfolding compatCh-def* **by** *auto*

4.2 Compositionality of self-isomorphism

Self-Isomorphism versus language constructs:

lemma *siso-Done[simp]*:
siso Done
proof –
 {**fix** $c :: ('test, 'atom, 'choice) \text{cmd}$
 assume $c = \text{Done}$ **hence** *siso c*
 apply *induct* **by** *auto*
 }
 thus *?thesis* **by** *blast*
qed

lemma *siso-Atm[simp]*:
 $\text{siso } (\text{Atm } atm) = \text{compatAtm } atm$
proof –
 {**fix** $c :: ('test, 'atom, 'choice) \text{cmd}$
 assume $\exists \ atm. \ c = \text{Atm } atm \wedge \text{compatAtm } atm$
 hence *siso c*
 apply *induct*
 apply (*metis compatAtm-def eff-Atm cont-Atm wt-Atm*)
 by (*metis cont-Atm siso-Done*)
 }
 moreover **have** $\text{siso } (\text{Atm } atm) \Longrightarrow \text{compatAtm } atm$ **unfolding** *compatAtm-def*
 by (*metis brn.simps eff-Atm less-Suc-eq mult-less-cancel1 nat-mult-1 siso-cont-indis*)
 ultimately show *?thesis* **by** *blast*
qed

lemma *siso-Seq[simp]*:
assumes $*$: *siso c1* **and** $**$: *siso c2*
shows $\text{siso } (c1 ;; c2)$
proof –

```

{fix c :: ('test, 'atom, 'choice) cmd
assume  $\exists c1 c2. c = c1 ;; c2 \wedge \text{siso } c1 \wedge \text{siso } c2$ 
hence siso c
proof induct
  case (Obs c s t i)
  then obtain c1 c2 where  $i < \text{brn } c1$ 
  and  $c = c1 ;; c2$  and siso c1  $\wedge$  siso c2
  and  $s \approx t$  by auto
  thus ?case by (cases finished (cont c1 s i)) auto
next
  case (Cont c s i)
  then obtain c1 c2 where  $i < \text{brn } c1$  and
   $c = c1 ;; c2 \wedge \text{siso } c1 \wedge \text{siso } c2$  by fastforce
  thus ?case by (cases finished (cont c1 s i), auto)
qed
}
thus ?thesis using assms by blast
qed

```

lemma *siso-While[simp]*:

assumes *compatTst tst* and *siso c*

shows *siso (While tst c)*

proof—

```

{fix c :: ('test, 'atom, 'choice) cmd
assume
( $\exists \text{tst } d. \text{compatTst } \text{tst} \wedge c = \text{While } \text{tst } d \wedge \text{siso } d$ )  $\vee$ 
( $\exists \text{tst } d1 d. \text{compatTst } \text{tst} \wedge c = d1 ;; (\text{While } \text{tst } d) \wedge \text{siso } d1 \wedge \text{siso } d$ )
hence siso c
proof induct
  case (Obs c s t i)
  hence  $i: i < \text{brn } c$  and  $st: s \approx t$  by auto
  from Obs show ?case
  proof (elim disjE exE conjE)
    fix tst d
    assume compatTst tst and  $c = \text{While } \text{tst } d$  and siso d
    thus ?thesis using  $i$  st unfolding compatTst-def
    by (cases tval tst s, simp-all)
  next
    fix tst d1 d
    assume compatTst tst and  $c = d1 ;; \text{While } \text{tst } d$ 
    and siso d1 and siso d
    thus ?thesis
    using  $i$  st unfolding compatTst-def
    apply (cases tval tst s, simp-all)
    by (cases finished (cont d1 s i), simp-all)
  qed
next
  case (Cont c s i)
  hence  $i: i < \text{brn } c$  by simp

```

```

from Cont show ?case
proof(elim disjE exE conjE)
  fix tst d
  assume compatTst tst and c = While tst d and siso d
  thus ?thesis by (cases tval tst s, simp-all)
next
  fix tst d1 d
  assume compatTst tst and c = d1 ;; While tst d and siso d1 and siso d
  thus ?thesis using i unfolding compatTst-def
  apply (cases tval tst s, simp-all)
  by (cases finished (cont d1 s i), simp-all)+
qed
qed
}
thus ?thesis using assms by blast
qed

```

```

lemma siso-Ch[simp]:
assumes compatCh ch
and *: siso c1 and **: siso c2
shows siso (Ch ch c1 c2)
proof–
  {fix c :: ('test, 'atom, 'choice) cmd
  assume  $\exists$  ch c1 c2. compatCh ch  $\wedge$  c = Ch ch c1 c2  $\wedge$  siso c1  $\wedge$  siso c2
  hence siso c
  proof induct
    case (Obs c s t i)
    then obtain ch c1 c2 where  $i < 2$ 
    and compatCh ch and c = Ch ch c1 c2 and siso c1  $\wedge$  siso c2
    and  $s \approx t$  by fastforce
    thus ?case by (cases i) auto
  next
    case (Cont c s i)
    then obtain ch c1 c2 where  $i < 2$  and
    compatCh ch  $\wedge$  c = Ch ch c1 c2  $\wedge$  siso c1  $\wedge$  siso c2 by fastforce
    thus ?case by (cases i) auto
  qed
}
thus ?thesis using assms by blast
qed

```

```

lemma siso-Par[simp]:
assumes properL cl and sisoL cl
shows siso (Par cl)
proof–
  {fix c :: ('test, 'atom, 'choice) cmd
  assume  $\exists$  cl. c = Par cl  $\wedge$  properL cl  $\wedge$  sisoL cl
  hence siso c
  proof induct

```

```

    case (Obs c s t ii)
    then obtain cl where ii: ii < brnL cl (length cl)
    and cl: properL cl
    and c: c = Par cl and siso: sisoL cl
    and st: s ≈ t by auto
    let ?N = length cl
    from cl ii show ?case
    apply(cases rule: brnL-cases)
    using siso st cl unfolding c by fastforce
next
  case (Cont c s ii)
  then obtain cl where ii: ii < brnL cl (length cl)
  and cl: properL cl
  and c: c = Par cl and sisoL: sisoL cl
  by auto
  from cl ii show ?case
  apply (cases rule: brnL-cases)
  using cl sisoL unfolding c by auto
qed
}
thus ?thesis using assms by blast
qed

```

```

lemma siso-ParT[simp]:
  assumes properL cl and sisoL cl
  shows siso (ParT cl)
proof-
  {fix c :: ('test, 'atom, 'choice) cmd
  assume ∃ cl. c = ParT cl ∧ properL cl ∧ sisoL cl
  hence siso c
  proof induct
    case (Obs c s t ii)
    then obtain cl where ii: ii < brnL cl (length cl)
    and cl: properL cl
    and c: c = ParT cl and siso: sisoL cl
    and st: s ≈ t by auto
    let ?N = length cl
    from cl ii show ?case proof (cases rule: brnL-cases)
      case (Local n i)
      show ?thesis (is ?eff ∧ ?wt ∧ ?mv)
      proof-
        have eff-mv: ?eff ∧ ?mv using Local siso cl st unfolding c by force
        have wt: ?wt
        proof(cases WtFT cl = 1)
          case True
          thus ?thesis unfolding c using Local cl st siso True
          by (cases n = pickFT cl ∧ i = 0) auto
        next
          case False

```



```

      thus ?thesis unfolding c using Local cl st siso False
      by (cases finished (cl!n)) auto
    qed
  from eff-mv wt show ?thesis by simp
  qed
  qed
  next
  case (Cont c s ii)
  then obtain cl where ii: ii < brnL cl (length cl)
  and cl: properL cl
  and c: c = ParT cl and siso: sisoL cl
  by auto
  from cl ii show ?case apply (cases rule: brnL-cases)
  using siso cl unfolding c by force
  qed
}
thus ?thesis using assms by blast
qed

```

Self-isomorphism implies strong bisimilarity:

```

lemma bij-betw-emp[simp]:
  bij-betw f {} {}
  unfolding bij-betw-def by auto

```

```

lemma part-full[simp]:
  part I {I}
  unfolding part-def by auto

```

```

definition singlPart where
  singlPart I ≡ {{i} | i . i ∈ I}

```

```

lemma part-singlPart[simp]:
  part I (singlPart I)
  unfolding part-def singlPart-def by auto

```

```

lemma singlPart-inj-on[simp]:
  inj-on (image f) (singlPart I) = inj-on f I
  unfolding inj-on-def singlPart-def
  apply auto
  by (metis image-insert insertI1 insert-absorb insert-code singleton-inject)

```

```

lemma singlPart-surj[simp]:
  (image f) ` (singlPart I) = (singlPart J) ↔ f ` I = J
  unfolding inj-on-def singlPart-def apply auto by blast

```

```

lemma singlPart-bij-betw[simp]:
  bij-betw (image f) (singlPart I) (singlPart J) = bij-betw f I J
  unfolding bij-betw-def by auto

```

lemma *singlPart-finite1*:
assumes *finite (singlPart I)*
shows *finite (I::'a set)*
proof –
 define *u* **where** $u\ i = \{i\}$ **for** $i :: 'a$
 have $u\ 'I \subseteq \text{singlPart } I$ **unfolding** *u-def singlPart-def* **by** *auto*
 moreover **have** *inj-on u I* **unfolding** *u-def inj-on-def* **by** *auto*
 ultimately show *?thesis* **using** *assms*
 by (*metis <inj-on u I> finite-imageD infinite-super*)
qed

lemma *singlPart-finite[simp]*:
finite (singlPart I) = finite I
using *singlPart-finite1[of I]* **unfolding** *singlPart-def* **by** *auto*

lemma *emp-notIn-singlPart[simp]*:
 $\{\} \notin \text{singlPart } I$
unfolding *singlPart-def* **by** *auto*

lemma *Sbis-coinduct[consumes 1, case-names step, coinduct set]*:
 $R\ c\ d \implies$
 $(\bigwedge c\ d\ s\ t. R\ c\ d \implies s \approx t \implies$
 $\exists P\ F. mC\text{-}C\text{-part } c\ d\ P\ F \wedge \text{inj-on } F\ P \wedge mC\text{-}C\text{-wt } c\ d\ s\ t\ P\ F \wedge$
 $(\forall I \in P. \forall i \in I. \forall j \in F\ I.$
 $\text{eff } c\ s\ i \approx \text{eff } d\ t\ j \wedge (R\ (\text{cont } c\ s\ i)\ (\text{cont } d\ t\ j) \vee (\text{cont } c\ s\ i, \text{cont } d$
 $t\ j) \in \text{Sbis}))$
 $\implies (c, d) \in \text{Sbis}$
using *Sbis-coind[of {(x, y). R x y}]*
unfolding *Sretr-def matchC-C-def mC-C-def mC-C-eff-cont-def*
apply (*simp add: subset-eq Ball-def*)
apply *metis*
done

lemma *siso-Sbis[simp]*: *siso c $\implies c \approx s\ c$*
proof (*coinduction arbitrary: c*)
 case (*step s t c*) **with** *siso-cont-indis[of c s t] part-singlPart[of {..
brn c}]* **show**
 ?*case*
 by (*intro exI[of - singlPart {..
brn c}] exI[of - id]*)
 (*auto simp add: mC-C-part-def mC-C-wt-def singlPart-def*)
qed

4.3 Discreteness versus language constructs:

lemma *discr-Done[simp]*: *discr Done*
by *coinduction auto*

lemma *discr-Atm-presAtm[simp]*: *discr (Atm atm) = presAtm atm*
proof –
 have *presAtm atm \implies discr (Atm atm)*

by (coinduction arbitrary: atm) (auto simp: presAtm-def)
 moreover have $\text{discr } (\text{Atm atm}) \implies \text{presAtm atm}$
 unfolding presAtm-def
 by (metis One-nat-def brn.simps(2) discr.simps eff-Atm lessI)
 ultimately show ?thesis by blast
 qed

lemma *discr-Seq[simp]*:
 $\text{discr } c1 \implies \text{discr } c2 \implies \text{discr } (c1 ;; c2)$
 by (coinduction arbitrary: c1 c2)
 (simp, metis cont-Seq-finished discr-cont cont-Seq-notFinished)

lemma *discr-While[simp]*: **assumes** $\text{discr } c$ **shows** $\text{discr } (\text{While } \text{tst } c)$
proof –
 {fix c :: ('test, 'atom, 'choice) cmd
 assume
 ($\exists \text{tst } d. c = \text{While } \text{tst } d \wedge \text{discr } d$) \vee
 ($\exists \text{tst } d1 \text{ } d. c = d1 ;; (\text{While } \text{tst } d) \wedge \text{discr } d1 \wedge \text{discr } d$)
 hence $\text{discr } c$
 apply induct apply safe
 apply (metis eff-While indis-refl)
 apply (metis cont-While-False discr-Done cont-While-True)
 apply (metis eff-Seq discr.simps brn.simps)
 by (metis cont-Seq-finished discr.simps cont-Seq-notFinished brn.simps)
 }
 thus ?thesis using assms by blast
 qed

lemma *discr-Ch[simp]*: $\text{discr } c1 \implies \text{discr } c2 \implies \text{discr } (\text{Ch } \text{ch } c1 \text{ } c2)$
 by coinduction (simp, metis indis-refl less-2-cases cont-Ch-L cont-Ch-R)

lemma *discr-Par[simp]*: $\text{properL } cl \implies \text{discrL } cl \implies \text{discr } (\text{Par } cl)$
proof (coinduction arbitrary: cl, clarsimp)
 fix cl ii s
 assume *: $\text{properL } cl \text{ } ii < \text{brnL } cl \text{ } (\text{length } cl)$ and $\text{discrL } cl$
from * **show** $s \approx \text{eff } (\text{Par } cl) \text{ } s \text{ } ii \wedge$
 ($(\exists cl'. \text{cont } (\text{Par } cl) \text{ } s \text{ } ii = \text{Par } cl' \wedge \text{properL } cl' \wedge \text{discrL } cl') \vee \text{discr } (\text{cont } (\text{Par } cl) \text{ } s \text{ } ii)$)
proof (cases rule: brnL-cases)
 case (Local n i)
 with $\langle \text{discrL } cl \rangle$ **have** $s \approx \text{eff } (cl ! n) \text{ } s \text{ } i$ by simp
 thus ?thesis
 using Local $\langle \text{discrL } cl \rangle \langle \text{properL } cl \rangle$ by auto
 qed
 qed

lemma *discr-ParT[simp]*: $\text{properL } cl \implies \text{discrL } cl \implies \text{discr } (\text{ParT } cl)$
proof (coinduction arbitrary: cl, clarsimp)
 fix p cl s ii **assume** $\text{properL } cl \text{ } ii < \text{brnL } cl \text{ } (\text{length } cl)$ $\text{discrL } cl$

then show $s \approx \text{eff } (\text{ParT } cl) s ii \wedge$
 $((\exists cl'. \text{cont } (\text{ParT } cl) s ii = \text{ParT } cl' \wedge \text{properL } cl' \wedge \text{discrL } cl') \vee \text{discr } (\text{cont } (\text{ParT } cl) s ii))$
proof (*cases rule: brnL-cases*)
case (*Local n i*)
have $s \approx \text{eff } (cl ! n) s i$ **using** *Local* $\langle \text{discrL } cl \rangle$ **by** *simp*
thus *?thesis* **using** *Local* $\langle \text{discrL } cl \rangle$ $\langle \text{properL } cl \rangle$ **by** *simp*
qed
qed

lemma *discr-finished[simp]*: $\text{proper } c \implies \text{finished } c \implies \text{discr } c$
by (*induct c rule: proper-induct*) (*auto simp: discrL-intro*)

4.4 Strong bisimilarity versus language constructs

lemma *Sbis-pres-discr-L*:
 $c \approx_s d \implies \text{discr } d \implies \text{discr } c$
proof (*coinduction arbitrary: d c, clarsimp*)
fix $c d s i$
assume $d: c \approx_s d$ **discr** d **and** $i: i < \text{brn } c$
then obtain $P F$ **where**
 $\text{match: } mC\text{-}C \text{ Sbis } c d s s P F$
using *Sbis-mC-C[of c d s s]* **by** *blast*
hence $\bigcup P = \{.. < \text{brn } c\}$
using i **unfolding** *mC-C-def mC-C-part-def part-def* **by** *simp*
then obtain I **where** $I: I \in P$ **and** $i: i \in I$ **using** i **by** *auto*
obtain j **where** $j: j \in F I$
using $\text{match } I$ **unfolding** *mC-C-def mC-C-part-def* **by** *blast*
hence $j < \text{brn } d$ **using** $I \text{ match}$
unfolding *mC-C-def mC-C-part-def part-def* **apply** *simp* **by** *blast*
hence $md: \text{discr } (\text{cont } d s j)$ **and** $s: s \approx \text{eff } d s j$
using $d \text{ discr-cont}[of d j s]$ *discr-eff-indis[of d j s]* **by** *auto*
have $\text{eff } c s i \approx \text{eff } d s j$ **and** $md2: \text{cont } c s i \approx_s \text{cont } d s j$
using $I i j \text{ match}$ **unfolding** *mC-C-def mC-C-eff-cont-def* **by** *auto*
hence $s \approx \text{eff } c s i$ **using** $s \text{ indis-sym indis-trans}$ **by** *blast*
thus $s \approx \text{eff } c s i \wedge ((\exists d. \text{cont } c s i \approx_s d \wedge \text{discr } d) \vee \text{discr } (\text{cont } c s i))$
using $md \text{ md2}$ **by** *blast*
qed

lemma *Sbis-pres-discr-R*:
assumes $\text{discr } c$ **and** $c \approx_s d$
shows $\text{discr } d$
using *assms Sbis-pres-discr-L Sbis-sym* **by** *blast*

lemma *Sbis-finished-discr-L*:
assumes $c \approx_s d$ **and** $\text{proper } d$ **and** $\text{finished } d$
shows $\text{discr } c$
using *assms Sbis-pres-discr-L* **by** *auto*

lemma *Sbis-finished-discr-R*:
assumes *proper c and finished c and $c \approx s d$*
shows *discr d*
using *assms Sbis-pres-discr-R[of c d]* **by** *auto*

definition *thetaSD* **where**
 $thetaSD \equiv \{(c, d) \mid c d . proper\ c \wedge proper\ d \wedge discr\ c \wedge discr\ d\}$

lemma *thetaSD-Sretr*:
 $thetaSD \subseteq Sretr\ thetaSD$
unfolding *Sretr-def matchC-C-def* **proof** *safe*
fix *c d s t*
assume *c-d: (c, d) ∈ thetaSD and st: s ≈ t*
hence *p: proper c ∧ proper d* **unfolding** *thetaSD-def* **by** *auto*
let $?P = \{\{..<brn\ c\}\}$
let $?F = \% I. \{\{..<brn\ d\}\}$
have $0: \{\{..<brn\ c\}\} \neq \{\} \ \ \ \ \ \{\{..<brn\ d\}\} \neq \{\}$
using *p int-emp brn-gt-0* **by** *blast+*
show $\exists P\ F. mC-C\ thetaSD\ c\ d\ s\ t\ P\ F$
apply $-$
apply(*rule exI[of - ?P]*) **apply**(*rule exI[of - ?F]*)
unfolding *mC-C-def* **proof**(*intro conjI*)
show *mC-C-part c d ?P ?F*
unfolding *mC-C-part-def* **using** 0 **unfolding** *part-def* **by** *auto*
next
show *inj-on ?F ?P* **unfolding** *inj-on-def* **by** *simp*
next
show *mC-C-wt c d s t ?P ?F* **unfolding** *mC-C-wt-def* **using** *p* **by** *auto*
next
show *mC-C-eff-cont thetaSD c d s t ?P ?F*
unfolding *mC-C-eff-cont-def* **proof** *clarify*
fix *I i j*
assume *i: i < brn c and j: j < brn d*
hence $s \approx eff\ c\ s\ i$ **using** *c-d* **unfolding** *thetaSD-def* **by** *simp*
moreover **have** $t \approx eff\ d\ t\ j$ **using** *i j c-d* **unfolding** *thetaSD-def* **by** *simp*
ultimately **have** $eff\ c\ s\ i \approx eff\ d\ t\ j$ **using** *st indis-sym indis-trans* **by** *blast*
thus $eff\ c\ s\ i \approx eff\ d\ t\ j \wedge (cont\ c\ s\ i, cont\ d\ t\ j) \in thetaSD$
using *c-d i j* **unfolding** *thetaSD-def* **by** *auto*
qed
qed
qed

lemma *thetaSD-Sbis*:
 $thetaSD \subseteq Sbis$
using *Sbis-raw-coind thetaSD-Sretr* **by** *blast*

theorem *discr-Sbis[simp]*:

assumes *proper c and proper d and discr c and discr d*
shows $c \approx_s d$
using *assms thetaSD-Sbis unfolding thetaSD-def by auto*

definition *thetaSDone* **where**
 $thetaSDone \equiv \{(Done, Done)\}$

lemma *thetaSDone-Sretr*:
 $thetaSDone \subseteq Sretr\ thetaSDone$
unfolding *Sretr-def matchC-C-def thetaSDone-def* **proof** *safe*
fix $s\ t$ **assume** $st: s \approx t$
let $?P = \{\{0\}\}$ **let** $?F = id$
show $\exists P\ F. mC-C\ \{(Done, Done)\}\ Done\ Done\ s\ t\ P\ F$
apply(*intro exI[of - ?P]*) **apply**(*intro exI[of - ?F]*)
unfolding *m-defsAll part-def* **using** st **by** *auto*
qed

lemma *thetaSDone-Sbis*:
 $thetaSDone \subseteq Sbis$
using *Sbis-raw-coind thetaSDone-Sretr* **by** *blast*

theorem *Done-Sbis[simp]*:
 $Done \approx_s Done$
using *thetaSDone-Sbis unfolding thetaSDone-def* **by** *auto*

definition *thetaSAtm* **where**
 $thetaSAtm\ atm \equiv$
 $\{(Atm\ atm, Atm\ atm), (Done, Done)\}$

lemma *thetaSAtm-Sretr*:
assumes *compatAtm atm*
shows $thetaSAtm\ atm \subseteq Sretr\ (thetaSAtm\ atm)$
unfolding *Sretr-def matchC-C-def thetaSAtm-def* **proof** *safe*
fix $s\ t$ **assume** $st: s \approx t$
let $?P = \{\{0\}\}$ **let** $?F = id$
show $\exists P\ F. mC-C\ \{(Atm\ atm, Atm\ atm), (Done, Done)\}\ Done\ Done\ s\ t\ P\ F$
apply(*intro exI[of - ?P]*) **apply**(*intro exI[of - ?F]*)
unfolding *m-defsAll part-def* **using** st **by** *auto*
next
fix $s\ t$ **assume** $st: s \approx t$
let $?P = \{\{0\}\}$ **let** $?F = id$
show $\exists P\ F. mC-C\ \{(Atm\ atm, Atm\ atm), (Done, Done)\}\ (Atm\ atm)\ (Atm\ atm)$
 $s\ t\ P\ F$
apply(*intro exI[of - ?P]*) **apply**(*intro exI[of - ?F]*)
unfolding *m-defsAll part-def* **using** st **assms** *unfolding compatAtm-def* **by** *auto*
qed

lemma *thetaSAtm-Sbis*:
assumes *compatAtm atm*
shows *thetaSAtm atm* \subseteq *Sbis*
using *assms Sbis-raw-coind thetaSAtm-Sretr* **by** *blast*

theorem *Atm-Sbis[simp]*:
assumes *compatAtm atm*
shows *Atm atm* \approx_s *Atm atm*
using *assms thetaSAtm-Sbis* **unfolding** *thetaSAtm-def* **by** *auto*

definition *thetaSSeqI* **where**
thetaSSeqI \equiv
 $\{(e ;; c, e ;; d) \mid e c d . \text{siso } e \wedge c \approx_s d\}$

lemma *thetaSSeqI-Sretr*:
thetaSSeqI \subseteq *Sretr* (*thetaSSeqI Un Sbis*)
unfolding *Sretr-def matchC-C-def* **proof** *safe*
fix *c d s t*
assume *c-d*: $(c, d) \in \text{thetaSSeqI}$ **and** *st*: $s \approx t$
then obtain *e c1 d1* **where** *e*: *siso e* **and** *c1d1*: $c1 \approx_s d1$
and *c*: $c = e ;; c1$ **and** *d*: $d = e ;; d1$
unfolding *thetaSSeqI-def* **by** *auto*
let *?P* = $\{\{i\} \mid i . i < \text{brn } e\}$
let *?F* = $\%I . I$
show $\exists P F . mC-C$ (*thetaSSeqI Un Sbis*) *c d s t P F*
apply(*rule exI[of - ?P]*) **apply**(*rule exI[of - ?F]*)
unfolding *mC-C-def* **proof** (*intro conjI*)
show *mC-C-part c d ?P ?F*
unfolding *mC-C-part-def* **proof** (*intro conjI*)
show *part* $\{..<\text{brn } c\}$ *?P*
unfolding *part-def* **proof** *safe*
fix *i* **assume** $i < \text{brn } c$
thus $i \in \bigcup ?P$ **using** *c e st* *siso-cont-indis[of e s t]* **by** *auto*
qed (*unfold c, simp*)

thus *part* $\{..<\text{brn } d\}$ (*?F ' ?P*) **unfolding** *c d* **by** *auto*
qed *auto*

next
show *mC-C-eff-cont* (*thetaSSeqI Un Sbis*) *c d s t ?P ?F*
unfolding *mC-C-eff-cont-def* **proof** (*intro impI allI, elim conjE*)
fix *I i j*
assume *I*: $I \in ?P$ **and** *i*: $i \in I$ **and** *j*: $j \in I$
show $\text{eff } c s i \approx \text{eff } d t j \wedge (\text{cont } c s i, \text{cont } d t j) \in \text{thetaSSeqI} \cup \text{Sbis}$
proof(*cases* $I = \{\}$)
case *True* **thus** *?thesis* **using** *i* **by** *simp*
next
case *False*

```

    then obtain  $i'$  where  $j \in ?F \{i'\}$  and  $i' < \text{brn } e$ 
    using  $I j$  by auto
    thus ?thesis
    using  $st \ c\text{-}d \ e \ i \ j \ I$  unfolding  $c \ d \ \text{thetaSSeqI}\text{-}def$ 
    by (cases finished (cont  $e \ s \ i'$ )) auto
  qed
qed
qed (insert  $st \ c\text{-}d \ c$ , unfold  $m\text{-}defsAll \ \text{thetaSSeqI}\text{-}def \ \text{part}\text{-}def$ , auto)
qed

```

```

lemma  $\text{thetaSSeqI}\text{-}Sbis$ :
 $\text{thetaSSeqI} \subseteq Sbis$ 
using  $Sbis\text{-}coind \ \text{thetaSSeqI}\text{-}Sretr$  by blast

```

```

theorem  $Seq\text{-}siso\text{-}Sbis[simp]$ :
assumes  $siso \ e$  and  $c2 \approx_s d2$ 
shows  $e ;; c2 \approx_s e ;; d2$ 
using  $assms \ \text{thetaSSeqI}\text{-}Sbis$  unfolding  $\text{thetaSSeqI}\text{-}def$  by auto

```

```

definition  $\text{thetaSSeqD}$  where
 $\text{thetaSSeqD} \equiv$ 
 $\{(c1 ;; c2, d1 ;; d2) \mid$ 
 $c1 \ c2 \ d1 \ d2.$ 
 $\text{proper } c1 \wedge \text{proper } d1 \wedge \text{proper } c2 \wedge \text{proper } d2 \wedge$ 
 $\text{discr } c2 \wedge \text{discr } d2 \wedge$ 
 $c1 \approx_s d1\}$ 

```

```

lemma  $\text{thetaSSeqD}\text{-}Sretr$ :
 $\text{thetaSSeqD} \subseteq Sretr (\text{thetaSSeqD} \cup Sbis)$ 
unfolding  $Sretr\text{-}def \ \text{matchC}\text{-}C\text{-}def$  proof safe
  fix  $c \ d \ s \ t$ 
  assume  $c\text{-}d: (c, d) \in \text{thetaSSeqD}$  and  $st: s \approx t$ 
  then obtain  $c1 \ c2 \ d1 \ d2$  where
     $c1d1: \text{proper } c1 \ \text{proper } d1 \ c1 \approx_s d1$  and
     $c2d2: \text{proper } c2 \ \text{proper } d2 \ \text{discr } c2 \ \text{discr } d2$ 
    and  $c: c = c1 ;; c2$  and  $d: d = d1 ;; d2$ 
  unfolding  $\text{thetaSSeqD}\text{-}def$  by auto
  from  $c1d1 \ st$  obtain  $P \ F$ 
  where  $match: mC\text{-}C \ Sbis \ c1 \ d1 \ s \ t \ P \ F$ 
  using  $Sbis\text{-}mC\text{-}C$  by blast
  have  $P: \bigcup P = \{..<\text{brn } c1\}$  and  $FP: \bigcup (F \text{ ' } P) = \{..<\text{brn } d1\}$ 
  using  $match$  unfolding  $mC\text{-}C\text{-}def \ mC\text{-}C\text{-}part\text{-}def \ \text{part}\text{-}def$  by metis+
  show  $\exists P \ F. mC\text{-}C (\text{thetaSSeqD} \cup Sbis) \ c \ d \ s \ t \ P \ F$ 
  apply (rule  $exI[of \ - \ P]$ ) apply (rule  $exI[of \ - \ F]$ )
  unfolding  $mC\text{-}C\text{-}def$  proof (intro  $conjI$ )
    show  $mC\text{-}C\text{-}eff\text{-}cont (\text{thetaSSeqD} \cup Sbis) \ c \ d \ s \ t \ P \ F$ 
    unfolding  $mC\text{-}C\text{-}eff\text{-}cont\text{-}def$  proof (intro  $allI \ \text{impI}, \ \text{elim } conjE$ )
      fix  $i \ j \ I$  assume  $I : I \in P$  and  $i: i \in I$  and  $j: j \in F \ I$ 

```



```

let ?c1' = cont c1 s i let ?d1' = cont d1 t j
let ?s' = eff c1 s i let ?t' = eff d1 t j
have i < brn c1 using i I P by blast note i = this i
have j < brn d1 using j I FP by blast note j = this j
have c1'd1': ?c1' ≈s ?d1'
proper ?c1' proper ?d1'
using c1d1 i j I match unfolding c mC-C-def mC-C-eff-cont-def by auto
show eff c s i ≈ eff d t j ∧ (cont c s i, cont d t j) ∈ thetaSSeqD ∪ Sbis
(is ?eff ∧ ?cont) proof
  show ?eff using match I i j unfolding c d m-defsAll by simp
next
show ?cont
proof(cases finished ?c1')
  case True note c1' = True
  hence csi: cont c s i = c2 using i match unfolding c m-defsAll by simp
  show ?thesis
  proof(cases finished ?d1')
    case True
    hence cont d t j = d2 using j match unfolding d m-defsAll by simp
    thus ?thesis using csi c2d2 by simp
  next
  case False
  hence dtj: cont d t j = ?d1' ;; d2
  using j match unfolding d m-defsAll by simp
  have discr ?d1' using c1'd1' c1' Sbis-finished-discr-R by blast
  thus ?thesis using c1'd1' c2d2 unfolding csi dtj by simp
qed
next
case False note Done-c = False
hence csi: cont c s i = ?c1' ;; c2
using i match unfolding c m-defsAll by simp
show ?thesis
proof(cases finished (cont d1 t j))
  case True note d1' = True
  hence dtj: cont d t j = d2 using j match unfolding d m-defsAll by
simp
  have discr ?c1' using c1'd1' d1' Sbis-finished-discr-L by blast
  thus ?thesis using c1'd1' c2d2 unfolding csi dtj by simp
next
case False
hence dtj: cont d t j = ?d1' ;; d2 using j match unfolding d m-defsAll
by simp
  thus ?thesis unfolding csi dtj thetaSSeqD-def
  using c1'd1' c2d2 by blast
qed
qed
qed
qed
qed(insert match, unfold m-defsAll c d, auto)

```

qed

lemma *thetaSSeqD-Sbis*:
thetaSSeqD \subseteq *Sbis*
using *Sbis-coind thetaSSeqD-Sretr* by *blast*

theorem *Seq-Sbis[simp]*:
assumes *proper c1* and *proper d1* and *proper c2* and *proper d2*
and *c1 \approx_s d1* and *discr c2* and *discr d2*
shows *c1* ;; *c2 \approx_s d1* ;; *d2*
using *assms thetaSSeqD-Sbis* unfolding *thetaSSeqD-def* by *auto*

definition *thetaSCh* where
thetaSCh ch c1 c2 d1 d2 \equiv $\{(Ch\ ch\ c1\ c2, Ch\ ch\ d1\ d2)\}$

lemma *thetaSCh-Sretr*:
assumes *compatCh ch* and *c1 \approx_s d1* and *c2 \approx_s d2*
shows *thetaSCh ch c1 c2 d1 d2* \subseteq
 Sretr (thetaSCh ch c1 c2 d1 d2 \cup Sbis)
(is *?th* \subseteq *Sretr (?th \cup Sbis)*)
unfolding *Sretr-def matchC-C-def* proof safe
 fix *c d s t*
 assume *c-d: (c, d) \in ?th* and *st: s \approx t*
 hence *c: c = Ch ch c1 c2 brn c = 2*
 and *d: d = Ch ch d1 d2 brn d = 2*
 unfolding *thetaSCh-def* by *auto*
 let *?P = $\{\{0\}, \{1\}\}$*
 let *?F = $\%_0 I. I$*
 show $\exists P\ F. mC-C\ (?th\ Un\ Sbis)\ c\ d\ s\ t\ P\ F$
 apply(*rule exI[of - ?P]*) apply(*rule exI[of - ?F]*)
 using *assms st c-d c* unfolding *m-defsAll thetaSCh-def part-def* by *auto*
qed

lemma *thetaSCh-Sbis*:
assumes *compatCh ch* and *c1 \approx_s d1* and *c2 \approx_s d2*
shows *thetaSCh ch c1 c2 d1 d2* \subseteq *Sbis*
using *Sbis-coind thetaSCh-Sretr[OF assms]* by *blast*

theorem *Ch-iso-Sbis[simp]*:
assumes *compatCh ch* and *c1 \approx_s d1* and *c2 \approx_s d2*
shows *Ch ch c1 c2 \approx_s Ch ch d1 d2*
using *thetaSCh-Sbis[OF assms]* unfolding *thetaSCh-def* by *auto*

definition *shift* where

$shift\ cl\ n \equiv image\ (\%i.\ brnL\ cl\ n + i)$

definition *back where*

$back\ cl\ n \equiv image\ (\%i.\ ii - brnL\ cl\ n)$

lemma *emp-shift[simp]:*

$shift\ cl\ n\ I = \{\} \longleftrightarrow I = \{\}$

unfolding *shift-def by auto*

lemma *emp-shift-rev[simp]:*

$\{\} = shift\ cl\ n\ I \longleftrightarrow I = \{\}$

unfolding *shift-def by auto*

lemma *emp-back[simp]:*

$back\ cl\ n\ II = \{\} \longleftrightarrow II = \{\}$

unfolding *back-def by force*

lemma *emp-back-rev[simp]:*

$\{\} = back\ cl\ n\ II \longleftrightarrow II = \{\}$

unfolding *back-def by force*

lemma *in-shift[simp]:*

$brnL\ cl\ n + i \in shift\ cl\ n\ I \longleftrightarrow i \in I$

unfolding *shift-def by auto*

lemma *in-back[simp]:*

$ii \in II \implies ii - brnL\ cl\ n \in back\ cl\ n\ II$

unfolding *back-def by auto*

lemma *in-back2[simp]:*

assumes $ii > brnL\ cl\ n$ **and** $II \subseteq \{brnL\ cl\ n ..<+ brn\ (cl!n)\}$

shows $ii - brnL\ cl\ n \in back\ cl\ n\ II \longleftrightarrow ii \in II$ (**is** ?L \longleftrightarrow ?R)

using *assms unfolding back-def by force*

lemma *shift[simp]:*

assumes $I \subseteq \{..
brn\ (cl!n)\}$

shows $shift\ cl\ n\ I \subseteq \{brnL\ cl\ n ..<+ brn\ (cl!n)\}$

using *assms unfolding shift-def by auto*

lemma *shift2[simp]:*

assumes $I \subseteq \{..
brn\ (cl!n)\}$

and $ii \in shift\ cl\ n\ I$

shows $brnL\ cl\ n \leq ii \wedge ii < brnL\ cl\ n + brn\ (cl!n)$

using *assms unfolding shift-def by auto*

lemma *shift3[simp]:*

assumes $n: n < length\ cl$ **and** $I: I \subseteq \{..
brn\ (cl!n)\}$

and $ii: ii \in shift\ cl\ n\ I$

shows $ii < brnL\ cl\ (length\ cl)$

proof–

have $ii < brnL\ cl\ n + brn\ (cl!n)$ **using** $I\ ii$ **by** $simp$
also have $\dots \leq brnL\ cl\ (length\ cl)$ **using** n
by $(metis\ brnL-Suc\ brnL-mono\ Suc-leI)$
finally show $?thesis$.

qed

lemma $back[simp]$:

assumes $II \subseteq \{brnL\ cl\ n \dots < +\ brn\ (cl!n)\}$
shows $back\ cl\ n\ II \subseteq \{\dots < brn\ (cl!n)\}$
using $assms$ **unfolding** $back-def$ **by** $force$

lemma $back2[simp]$:

assumes $II \subseteq \{brnL\ cl\ n \dots < +\ brn\ (cl!n)\}$
and $i \in back\ cl\ n\ II$
shows $i < brn\ (cl!n)$
using $assms$ **unfolding** $back-def$ **by** $force$

lemma $shift-inj[simp]$:

$shift\ cl\ n\ I1 = shift\ cl\ n\ I2 \longleftrightarrow I1 = I2$
unfolding $shift-def$ **by** $force$

lemma $shift-mono[simp]$:

$shift\ cl\ n\ I1 \subseteq shift\ cl\ n\ I2 \longleftrightarrow I1 \subseteq I2$
unfolding $shift-def$ **by** $auto$

lemma $shift-Int[simp]$:

$shift\ cl\ n\ I1 \cap shift\ cl\ n\ I2 = \{\} \longleftrightarrow I1 \cap I2 = \{\}$
unfolding $shift-def$ **by** $force$

lemma $inj-shift$: $inj\ (shift\ cl\ n)$

unfolding $inj-on-def$ **by** $simp$

lemma $inj-on-shift$: $inj-on\ (shift\ cl\ n)\ K$

using $inj-shift$ **unfolding** $inj-on-def$ **by** $simp$

lemma $back-shift[simp]$:

$back\ cl\ n\ (shift\ cl\ n\ I) = I$
unfolding $back-def\ shift-def$ **by** $force$

lemma $shift-back[simp]$:

assumes $II \subseteq \{brnL\ cl\ n \dots < +\ brn\ (cl!n)\}$
shows $shift\ cl\ n\ (back\ cl\ n\ II) = II$
using $assms$ **unfolding** $shift-def\ back-def\ atLeastLessThan-def$ **by** $force$

lemma $back-inj[simp]$:

assumes $III: III \subseteq \{brnL\ cl\ n \dots < +\ brn\ (cl!n)\}$
and $II2: II2 \subseteq \{brnL\ cl\ n \dots < +\ brn\ (cl!n)\}$
shows $back\ cl\ n\ III = back\ cl\ n\ II2 \longleftrightarrow III = II2$ (**is** $?L = ?R \longleftrightarrow III = II2$)

proof
 have $III = \text{shift } cl \ n \ ?L$ **using** $II1$ **by** simp
 also **assume** $?L = ?R$
 also **have** $\text{shift } cl \ n \ ?R = II2$ **using** $II2$ **by** simp
finally show $III = II2$.
qed *auto*

lemma *back-mono*[simp]:
assumes $III \subseteq \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\}$
and $II2 \subseteq \{\text{brnL } cl \ n \ ..< \text{brnL } cl \ n + \text{brn } (cl!n)\}$
shows $\text{back } cl \ n \ III \subseteq \text{back } cl \ n \ II2 \longleftrightarrow III \subseteq II2$
(is $?L \subseteq ?R \longleftrightarrow III \subseteq II2$ **)**
proof –
 have $?L \subseteq ?R \longleftrightarrow \text{shift } cl \ n \ ?L \subseteq \text{shift } cl \ n \ ?R$ **by** simp
 also **have** $\dots \longleftrightarrow III \subseteq II2$ **using** *assms* **by** simp
finally show *thesis* .
qed

lemma *back-Int*[simp]:
assumes $III \subseteq \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\}$
and $II2 \subseteq \{\text{brnL } cl \ n \ ..< \text{brnL } cl \ n + \text{brn } (cl!n)\}$
shows $\text{back } cl \ n \ III \cap \text{back } cl \ n \ II2 = \{\} \longleftrightarrow III \cap II2 = \{\}$
(is $?L \cap ?R = \{\} \longleftrightarrow III \cap II2 = \{\}$ **)**
proof –
 have $?L \cap ?R = \{\} \longleftrightarrow \text{shift } cl \ n \ ?L \cap \text{shift } cl \ n \ ?R = \{\}$ **by** simp
 also **have** $\dots \longleftrightarrow III \cap II2 = \{\}$ **using** *assms* **by** simp
finally show *thesis* .
qed

lemma *inj-on-back*:
inj-on ($\text{back } cl \ n$) ($\text{Pow } \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\}$)
unfolding *inj-on-def* **by** simp

lemma *shift-surj*:
assumes $II \subseteq \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\}$
shows $\exists I. I \subseteq \{..< \text{brn } (cl!n)\} \wedge \text{shift } cl \ n \ I = II$
apply(*intro exI*[*of - back } cl \ n \ II*]) **using** *assms* **by** simp

lemma *back-surj*:
assumes $I \subseteq \{..< \text{brn } (cl!n)\}$
shows $\exists II. II \subseteq \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\} \wedge \text{back } cl \ n \ II = I$
apply(*intro exI*[*of - shift } cl \ n \ I*]) **using** *assms* **by** simp

lemma *shift-part*[simp]:
assumes $\text{part } \{..< \text{brn } (cl!n)\} P$
shows $\text{part } \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\} (\text{shift } cl \ n \ ' P)$
unfolding *part-def* **proof**(*intro conjI allI impI*)
show $\bigcup (\text{shift } cl \ n \ ' P) = \{\text{brnL } cl \ n \ ..<+ \text{brn } (cl!n)\}$
proof *safe*

fix $ii\ I$ **assume** $ii: ii \in \text{shift } cl\ n\ I$ **and** $I \in P$
hence $I \subseteq \{..
thus $ii \in \{brnL\ cl\ n\ ..<+ brn\ (cl!n)\}$ **using** ii **by** *simp*
next
fix ii **assume** $ii\text{-in}: ii \in \{brnL\ cl\ n..<+brn\ (cl!\ n)\}$
define i **where** $i = ii - brnL\ cl\ n$
have $ii: ii = brnL\ cl\ n + i$ **unfolding** $i\text{-def}$ **using** $ii\text{-in}$ **by** *force*
have $i \in \{..
then obtain I **where** $i: i \in I$ **and** $I: I \in P$
using *assms* **unfolding** *part-def* **by** *blast*
thus $ii \in \cup (\text{shift } cl\ n\ 'P)$ **unfolding** ii **by** *force*
qed
qed(*insert assms, unfold part-def, force*)$$

lemma *part-brn-disj1*:
assumes $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..
and $n1: n1 < \text{length } cl$ **and** $n2: n2 < \text{length } cl$
and $II1: II1 \in \text{shift } cl\ n1\ '(P\ n1)$ **and** $II2: II2 \in \text{shift } cl\ n2\ '(P\ n2)$ **and** $d: n1 \neq n2$
shows $II1 \cap II2 = \{\}$
proof–
let $?N = \text{length } cl$
obtain $I1\ I2$ **where** $I1: I1 \in P\ n1$ **and** $I2: I2 \in P\ n2$
and $III1: III1 = \text{shift } cl\ n1\ I1$ **and** $III2: III2 = \text{shift } cl\ n2\ I2$
using $III1\ III2$ **by** *auto*
have $I1 \subseteq \{..
using $n1\ I1\ n2\ I2\ P$ **unfolding** *part-def* **by** *blast+*
hence $III1 \subseteq \{brnL\ cl\ n1\ ..<+ brn\ (cl!n1)\}$ **and** $III2 \subseteq \{brnL\ cl\ n2\ ..<+ brn\ (cl!n2)\}$
unfolding $III1\ III2$ **by** *auto*
thus $?thesis$ **using** $n1\ n2\ d\ brnL\text{-Int}$ **by** *blast*
qed$$

lemma *part-brn-disj2*:
assumes $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..
and $n1: n1 < \text{length } cl$ **and** $n2: n2 < \text{length } cl$ **and** $d: n1 \neq n2$
shows $\text{shift } cl\ n1\ '(P\ n1) \cap \text{shift } cl\ n2\ '(P\ n2) = \{\}$ (**is** $?L \cap ?R = \{\}$)
proof–
{fix II **assume** $II: II \in ?L \cap ?R$
hence $II = \{\}$ **using** *part-brn-disj1*[*of cl P n1 n2 II II*] *assms* **by** *blast*
hence $\{\} \in ?L$ **using** II **by** *blast*
then obtain I **where** $I: I \in P\ n1$ **and** $II: \{\} = \text{shift } cl\ n1\ I$ **by** *blast*
hence $I = \{\}$ **by** *simp*
hence *False* **using** $n1\ P\ I$ **by** *blast*
}
thus $?thesis$ **by** *blast*
qed$

lemma *part-brn-disj3*:

assumes $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n)$
and $n1: n1 < \text{length } cl$ **and** $n2: n2 < \text{length } cl$
and $I1: I1 \in P \ n1$ **and** $I2: I2 \in P \ n2$ **and** $d: n1 \neq n2$
shows $\text{shift } cl \ n1 \ I1 \cap \text{shift } cl \ n2 \ I2 = \{\}$
apply(*rule part-brn-disj1*)
using *assms* **by** *auto*

lemma *sum-wt-Par-sub-shift[simp]*:
assumes $cl: \text{properL } cl$ **and** $n: n < \text{length } cl$ **and**
 $I: I \subseteq \{..< \text{brn } (cl!n)\}$
shows
 $\text{sum } (wt \ (Par \ cl) \ s) \ (\text{shift } cl \ n \ I) =$
 $1 / (\text{length } cl) * \text{sum } (wt \ (cl!n) \ s) \ I$
using *assms* *sum-wt-Par-sub* **unfolding** *shift-def* **by** *simp*

lemma *sum-wt-ParT-sub-WtFT-pickFT-0-shift[simp]*:
assumes $cl: \text{properL } cl$ **and** $nf: WtFT \ cl = 1$
and $I: I \subseteq \{..< \text{brn } (cl! (\text{pickFT } cl))\}$ $0 \in I$
shows
 $\text{sum } (wt \ (ParT \ cl) \ s) \ (\text{shift } cl \ (\text{pickFT } cl) \ I) = 1$
using *assms* *sum-wt-ParT-sub-WtFT-pickFT-0*
unfolding *shift-def* **by** *simp*

lemma *sum-wt-ParT-sub-WtFT-notPickFT-0-shift[simp]*:
assumes $cl: \text{properL } cl$ **and** $nf: WtFT \ cl = 1$ **and** $n: n < \text{length } cl$
and $I: I \subseteq \{..< \text{brn } (cl!n)\}$ **and** $nI: n = \text{pickFT } cl \longrightarrow 0 \notin I$
shows $\text{sum } (wt \ (ParT \ cl) \ s) \ (\text{shift } cl \ n \ I) = 0$
using *assms* *sum-wt-ParT-sub-WtFT-notPickFT-0* **unfolding** *shift-def* **by** *simp*

lemma *sum-wt-ParT-sub-notWtFT-finished-shift[simp]*:
assumes $cl: \text{properL } cl$ **and** $nf: WtFT \ cl \neq 1$ **and** $n: n < \text{length } cl$ **and** $cln:$
 $\text{finished } (cl!n)$
and $I: I \subseteq \{..< \text{brn } (cl!n)\}$
shows $\text{sum } (wt \ (ParT \ cl) \ s) \ (\text{shift } cl \ n \ I) = 0$
using *assms* *sum-wt-ParT-sub-notWtFT-finished*
unfolding *shift-def* **by** *simp*

lemma *sum-wt-ParT-sub-notWtFT-notFinished-shift[simp]*:
assumes $cl: \text{properL } cl$ **and** $nf: WtFT \ cl \neq 1$
and $n: n < \text{length } cl$ **and** $cln: \neg \text{finished } (cl!n)$
and $I: I \subseteq \{..< \text{brn } (cl!n)\}$
shows
 $\text{sum } (wt \ (ParT \ cl) \ s) \ (\text{shift } cl \ n \ I) =$
 $(1 / (\text{length } cl)) / (1 - WtFT \ cl) * \text{sum } (wt \ (cl!n) \ s) \ I$
using *assms* *sum-wt-ParT-sub-notWtFT-notFinished*
unfolding *shift-def* **by** *simp*

definition *UNpart where*

$UNpart\ cl\ P \equiv \bigcup n < length\ cl.\ shift\ cl\ n\ ' (P\ n)$

lemma *UNpart-cases*[*elim, consumes 1, case-names Local*]:

assumes $II \in UNpart\ cl\ P$ **and**

$\bigwedge n\ I.\ \llbracket n < length\ cl; I \in P\ n; II = shift\ cl\ n\ I \rrbracket \implies phi$

shows *phi*

using *assms unfolding UNpart-def by auto*

lemma *emp-UNpart*:

assumes $\bigwedge n.\ n < length\ cl \implies \{\} \notin P\ n$

shows $\{\} \notin UNpart\ cl\ P$

using *assms unfolding UNpart-def by auto*

lemma *part-UNpart*:

assumes *cl: properL cl and*

$P: \bigwedge n.\ n < length\ cl \implies part\ \{.. < brn\ (cl!n)\} (P\ n)$

shows $part\ \{.. < brnL\ cl\ (length\ cl)\} (UNpart\ cl\ P)$

(**is part** *?J ?Q*)

proof –

let $?N = length\ cl$

have $J: ?J = (\bigcup n \in \{.. < ?N\}.\ \{brnL\ cl\ n\ .. < +\ brn\ (cl!n)\})$

using *cl brnL-UN by auto*

have $Q: ?Q = (\bigcup n \in \{.. < ?N\}.\ shift\ cl\ n\ ' (P\ n))$

unfolding *UNpart-def by auto*

show *?thesis unfolding J Q apply(rule part-UN)*

using *P brnL-Int by auto*

qed

definition *pickT-pred where*

$pickT-pred\ cl\ P\ II\ n \equiv n < length\ cl \wedge II \in shift\ cl\ n\ ' (P\ n)$

definition *pickT where*

$pickT\ cl\ P\ II \equiv SOME\ n.\ pickT-pred\ cl\ P\ II\ n$

lemma *pickT-pred*:

assumes $II \in UNpart\ cl\ P$

shows $\exists n.\ pickT-pred\ cl\ P\ II\ n$

using *assms unfolding UNpart-def pickT-pred-def by auto*

lemma *pickT-pred-unique*:

assumes $P: \bigwedge n.\ n < length\ cl \implies part\ \{.. < brn\ (cl!n)\} (P\ n) \wedge \{\} \notin P\ n$

and *1: pickT-pred cl P II n1 and 2: pickT-pred cl P II n2*

shows $n1 = n2$

proof –

{assume $n1 \neq n2$

hence $shift\ cl\ n1\ ' (P\ n1) \cap shift\ cl\ n2\ ' (P\ n2) = \{\}$


```

    using assms part-brn-disj2 unfolding pickT-pred-def by blast
    hence False using 1 2 unfolding pickT-pred-def by blast
  }
  thus ?thesis by auto
qed

```

```

lemma pickT-pred-pickT:
  assumes II ∈ UNpart cl P
  shows pickT-pred cl P II (pickT cl P II)
  unfolding pickT-def apply(rule someI-ex)
  using assms pickT-pred by auto

```

```

lemma pickT-pred-pickT-unique:
  assumes P:  $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P n) \wedge \{\} \notin P n$ 
  and pickT-pred cl P II n
  shows n = pickT cl P II
  unfolding pickT-def apply(rule sym, rule some-equality)
  using assms pickT-pred-unique[of cl P II] by auto

```

```

lemma pickT-length[simp]:
  assumes II ∈ UNpart cl P
  shows pickT cl P II < length cl
  using assms pickT-pred-pickT unfolding pickT-pred-def by auto

```

```

lemma pickT-shift[simp]:
  assumes II ∈ UNpart cl P
  shows II ∈ shift cl (pickT cl P II) ‘ (P (pickT cl P II))
  using assms pickT-pred-pickT unfolding pickT-pred-def by auto

```

```

lemma pickT-unique:
  assumes P:  $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P n) \wedge \{\} \notin P n$ 
  and n < length cl and II ∈ shift cl n ‘ (P n)
  shows n = pickT cl P II
  using assms pickT-pred-pickT-unique unfolding pickT-pred-def by auto

```

```

definition UNlift where
  UNlift cl dl P F II  $\equiv$ 
    shift dl (pickT cl P II) (F (pickT cl P II) (back cl (pickT cl P II) II))

```

```

lemma UNlift-shift[simp]:
  assumes P:  $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P n) \wedge \{\} \notin P n$ 
  and n: n < length cl and I: I ∈ P n
  shows UNlift cl dl P F (shift cl n I) = shift dl n (F n I)
  proof –

```

```

    let ?N = length cl
    define II where II = shift cl n I
    have II: shift cl n I = II using II-def by simp
    have n: n = pickT cl P II apply(rule pickT-unique)
    using assms unfolding II-def by auto

```

have $\text{back } cl \ n \ II = I$ **unfolding** $II\text{-def}$ **by** simp
hence $\text{shift } dl \ n \ (F \ n \ (\text{back } cl \ n \ II)) = \text{shift } dl \ n \ (F \ n \ I)$ **by** simp
thus $?thesis$ **unfolding** $UN\text{lift-def } II \ n[\text{THEN } sym]$.
qed

lemma $UN\text{lift-inj-on}$:

assumes l : $\text{length } cl = \text{length } dl$

and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$

and FP : $\bigwedge n. n < \text{length } dl \implies \text{part } \{..< \text{brn } (dl!n)\} (F \ n \ ' (P \ n)) \wedge \{\} \notin F \ n \ ' (P \ n)$

and F : $\bigwedge n. n < \text{length } cl \implies \text{inj-on } (F \ n) (P \ n)$

shows $\text{inj-on } (UN\text{lift } cl \ dl \ P \ F) (UN\text{part } cl \ P)$ (**is** $\text{inj-on } ?G \ ?Q$)

unfolding inj-on-def **proof** clarify

fix $III \ II2$

assume $III1$: $III1 \in ?Q$ **and** $III2$: $III2 \in ?Q$ **and** G : $?G \ III1 = ?G \ III2$

from $III1$ **show** $III1 = III2$

proof($\text{cases rule: } UN\text{part-cases}$)

case ($\text{Local } n1 \ I1$)

hence $n1$: $n1 < \text{length } cl \ n1 < \text{length } dl$ **and** $I1$: $I1 \in P \ n1$

and $III1$: $III1 = \text{shift } cl \ n1 \ I1$ **using** l **by** auto

hence $G1\text{-def}$: $?G \ III1 = \text{shift } dl \ n1 \ (F \ n1 \ I1)$ **using** P **by** simp

have $Pn1$: $\text{part } \{..< \text{brn } (dl!n1)\} (F \ n1 \ ' (P \ n1)) \{\} \notin F \ n1 \ ' (P \ n1)$

using $n1 \ FP$ **by** auto

have $F1\text{-in}$: $F \ n1 \ I1 \in F \ n1 \ ' (P \ n1)$ **using** $I1$ **by** simp

hence $Fn1I1$: $F \ n1 \ I1 \neq \{\} \ F \ n1 \ I1 \subseteq \{..< \text{brn } (dl!n1)\}$

using $Pn1$ **by** (blast , unfold part-def , blast)

hence $G1$: $?G \ III1 \neq \{\} \ ?G \ III1 \subseteq \{\text{brnL } dl \ n1 \ ..<+ \text{brn } (dl!n1)\}$

unfolding $G1\text{-def}$ **by** simp-all

from $III2$ **show** $?thesis$

proof($\text{cases rule: } UN\text{part-cases}$)

case ($\text{Local } n2 \ I2$)

hence $n2$: $n2 < \text{length } cl \ n2 < \text{length } dl$ **and** $I2$: $I2 \in P \ n2$

and $III2$: $III2 = \text{shift } cl \ n2 \ I2$ **using** l **by** auto

hence $G2\text{-def}$: $?G \ III2 = \text{shift } dl \ n2 \ (F \ n2 \ I2)$ **using** P **by** simp

have $Pn2$: $\text{part } \{..< \text{brn } (dl!n2)\} (F \ n2 \ ' (P \ n2)) \{\} \notin F \ n2 \ ' (P \ n2)$

using $n2 \ FP$ **by** auto

have $F2\text{-in}$: $F \ n2 \ I2 \in F \ n2 \ ' (P \ n2)$ **using** $I2$ **by** simp

hence $Fn2I2$: $F \ n2 \ I2 \neq \{\} \ F \ n2 \ I2 \subseteq \{..< \text{brn } (dl!n2)\}$

using $Pn2$ **by** (blast , unfold part-def , blast)

hence $G2$: $?G \ III2 \neq \{\} \ ?G \ III2 \subseteq \{\text{brnL } dl \ n2 \ ..<+ \text{brn } (dl!n2)\}$

unfolding $G2\text{-def}$ **by** simp-all

have $n12$: $n1 = n2$ **using** $n1 \ n2 \ G1 \ G2 \ G \ \text{brnL-Int}$ **by** blast

have $F \ n1 \ I1 = F \ n2 \ I2$ **using** G **unfolding** $G1\text{-def}$ $G2\text{-def}$ $n12$ **by** simp

hence $I1 = I2$ **using** $I1 \ I2 \ n1 \ F$ **unfolding** $n12 \ \text{inj-on-def}$ **by** simp

thus $?thesis$ **unfolding** $III1 \ III2 \ n12$ **by** simp

qed

qed

qed

lemma UNlift-UNpart:
assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
shows $(\text{UNlift } cl \ dl \ P \ F) \ ' (\text{UNpart } cl \ P) = \text{UNpart } dl \ (\%n. F \ n \ ' (P \ n))$ (**is** $?G \ ' ?Q = ?R$)
proof *safe*
 fix II **assume** II : $II \in ?Q$
 thus $?G \ II \in ?R$
 proof(*cases rule: UNpart-cases*)
 case (*Local* $n \ I$)
 hence n : $n < \text{length } cl \ n < \text{length } dl$ **and** I : $I \in P \ n$
 and II : $II = \text{shift } cl \ n \ I$ **using** l **by** *auto*
 hence G : $?G \ II = \text{shift } dl \ n \ (F \ n \ I)$ **using** P **by** *simp*
 show *?thesis* **using** $n \ I$ **unfolding** $G \ \text{UNpart-def}$ **by** *auto*
 qed
 next
 fix JJ **assume** JJ : $JJ \in ?R$
 thus $JJ \in ?G \ ' ?Q$
 proof(*cases rule: UNpart-cases*)
 case (*Local* $n \ J$)
 hence n : $n < \text{length } cl \ n < \text{length } dl$ **and** J : $J \in F \ n \ ' (P \ n)$
 and JJ : $JJ = \text{shift } dl \ n \ J$ **using** l **by** *auto*
 then obtain I **where** I : $I \in P \ n$ **and** $J = F \ n \ I$ **by** *auto*
 hence $JJ = \text{shift } dl \ n \ (F \ n \ I)$ **using** JJ **by** *simp*
 also have $\dots = \text{UNlift } cl \ dl \ P \ F \ (\text{shift } cl \ n \ I)$ **using** $n \ I \ P$ **by** *simp*
 finally have JJ : $JJ = \text{UNlift } cl \ dl \ P \ F \ (\text{shift } cl \ n \ I)$.
 show *?thesis* **using** $n \ I$ **unfolding** $JJ \ \text{UNpart-def}$ **by** *auto*
 qed
 qed

lemma emp-UNlift-UNpart:
assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and FP : $\bigwedge n. n < \text{length } dl \implies \{\} \notin F \ n \ ' (P \ n)$
shows $\{\} \notin (\text{UNlift } cl \ dl \ P \ F) \ ' (\text{UNpart } cl \ P)$ (**is** $\{\} \notin ?R$)
proof–
 have R : $?R = \text{UNpart } dl \ (\%n. F \ n \ ' (P \ n))$
 apply(*rule UNlift-UNpart*) **using** *assms* **by** *auto*
 show *?thesis* **unfolding** R **apply**(*rule emp-UNpart*) **using** FP **by** *simp*
qed

lemma part-UNlift-UNpart:
assumes l : $\text{length } cl = \text{length } dl$ **and** dl : *properL* dl
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and FP : $\bigwedge n. n < \text{length } dl \implies \text{part } \{.. < \text{brn } (dl!n)\} (F \ n \ ' (P \ n))$
shows $\text{part } \{.. < \text{brnL } dl \ (\text{length } dl)\} ((\text{UNlift } cl \ dl \ P \ F) \ ' (\text{UNpart } cl \ P))$ (**is** *part* $?C \ ?R$)
proof–

have $R: ?R = UNpart\ dl\ (\%n.\ F\ n\ ' (P\ n))$
apply(rule *UNlift-UNpart*) **using** *assms* **by** *auto*
show *?thesis unfolding R apply(rule part-UNpart) using dl FP by auto*
qed

lemma *ss-wt-Par-UNlift:*

assumes $l: length\ cl = length\ dl$

and $cldl: properL\ cl\ properL\ dl$ **and** $II: II \in UNpart\ cl\ P$

and $P: \bigwedge n. n < length\ cl \implies part\ \{..< brn\ (cl!n)\}\ (P\ n) \wedge \{\} \notin P\ n$

and $FP: \bigwedge n. n < length\ dl \implies part\ \{..< brn\ (dl!n)\}\ (F\ n\ ' (P\ n))$

and $sw:$

$\bigwedge n\ I. \llbracket n < length\ cl; I \in P\ n \rrbracket \implies$

$sum\ (wt\ (cl!\ n)\ s)\ I =$

$sum\ (wt\ (dl!\ n)\ t)\ (F\ n\ I)$

and $st: s \approx t$

shows

$sum\ (wt\ (Par\ cl)\ s)\ II =$

$sum\ (wt\ (Par\ dl)\ t)\ (UNlift\ cl\ dl\ P\ F\ II)$ (**is** $?L = ?R$)

proof –

let $?N = length\ cl$

let $?p = \%n.\ 1 / ?N$ **let** $?q = \%n.\ 1 / (length\ dl)$

let $?ss = \%n.\ s$ **let** $?tt = \%n.\ t$

have $sstt: \bigwedge n. n < ?N \implies ?ss\ n \approx ?tt\ n$ **using** st **by** *auto*

have $pq: \bigwedge n. n < ?N \implies ?p\ n = ?q\ n$ **and** $sstt: \bigwedge n. n < ?N \implies ?ss\ n \approx ?tt$

n

using *assms l* **by** *auto*

from II **show** *?thesis*

proof(*cases rule: UNpart-cases*)

case (*Local n I*)

hence $n: n < ?N\ n < length\ dl$ **and** $I: I \in P\ n$

and $II: II = shift\ cl\ n\ I$ **using** l **by** *auto*

have $I\text{-sub}: I \subseteq \{..< brn\ (cl!n)\}$ **using** $n\ I\ P$ **unfolding** *part-def* **by** *blast*

hence $FnI\text{-sub}: F\ n\ I \subseteq \{..< brn\ (dl!n)\}$ **using** $n\ I\ FP$ **unfolding** *part-def* **by**

blast

have $?L = (?p\ n) * sum\ (wt\ (cl!\ n)\ (?ss\ n))\ I$

unfolding II **using** $n\ cldl\ I\text{-sub}$ **by** *simp*

also have $... = (?q\ n) * sum\ (wt\ (dl!\ n)\ (?tt\ n))\ (F\ n\ I)$

using $n\ pq$ **apply** *simp* **using** $I\ sw[of\ n\ I]$ **unfolding** l **by** *auto*

also have $... = ?R$

unfolding II **using** $l\ cldl\ n\ FnI\text{-sub}\ P\ I$ **by** *simp*

finally show *?thesis* .

qed

qed

definition *thetaSPar* **where**

$thetaSPar \equiv$

$\{(Par\ cl, Par\ dl) \mid$

$cl\ dl.\ properL\ cl \wedge properL\ dl \wedge SbisL\ cl\ dl\}$

lemma *cont-eff-Par-UNlift*:

assumes l : $length\ cl = length\ dl$

and $cldl$: $properL\ cl\ properL\ dl\ SbisL\ cl\ dl$

and II : $II \in UNpart\ cl\ P$ **and** ii : $ii \in II$ **and** jj : $jj \in UNlift\ cl\ dl\ P\ F\ II$

and P : $\bigwedge n. n < length\ cl \implies part\ \{..< brn\ (cl!n)\}\ (P\ n) \wedge \{\}\ \notin P\ n$

and FP : $\bigwedge n. n < length\ dl \implies part\ \{..< brn\ (dl!n)\}\ (F\ n\ ' (P\ n))$

and *eff-cont*:

$\bigwedge n\ I\ i\ j. \llbracket n < length\ cl; I \in P\ n; i \in I; j \in F\ n\ I \rrbracket \implies$

$eff\ (cl!n)\ s\ i \approx eff\ (dl!n)\ t\ j \wedge$

$cont\ (cl!n)\ s\ i \approx_s cont\ (dl!n)\ t\ j$

and st : $s \approx t$

shows

$eff\ (Par\ cl)\ s\ ii \approx eff\ (Par\ dl)\ t\ jj \wedge$

$(cont\ (Par\ cl)\ s\ ii, cont\ (Par\ dl)\ t\ jj) \in thetaSPar$

(**is** $?eff \wedge ?cont$)

proof –

let $?N = length\ cl$

let $?p = \%n. 1/?N$ **let** $?q = \%n. 1/(length\ dl)$

let $?ss = \%n. s$ **let** $?tt = \%n. t$

have $sstt$: $\bigwedge n. n < ?N \implies ?ss\ n \approx ?tt\ n$

using $st\ l$ **by** *auto*

have pq : $\bigwedge n. n < ?N \implies ?p\ n = ?q\ n$ **and** $sstt$: $\bigwedge n. n < ?N \implies ?ss\ n \approx ?tt\ n$

n

using $assms\ l$ **by** *auto*

from II **show** $?thesis$

proof(*cases rule: UNpart-cases*)

case (*Local n I*)

hence n : $n < length\ cl\ n < length\ dl$ **and** I : $I \in P\ n$

and II : $II = shift\ cl\ n\ I$ **using** l **by** *auto*

from $ii\ II$ **obtain** i **where** i : $i \in I$ **and** ii : $ii = brnL\ cl\ n + i$

unfolding *shift-def* **by** *auto*

have $i < brn\ (cl!n)$ **using** $i\ I\ n\ P$ **unfolding** *part-def* **by** *blast* **note** $i = this\ i$

have jj : $jj \in shift\ dl\ n\ (F\ n\ I)$ **using** $jj\ P\ n\ I$ **unfolding** II **by** *simp*

from $jj\ II$ **obtain** j **where** j : $j \in F\ n\ I$ **and** jj : $jj = brnL\ dl\ n + j$

unfolding *shift-def* **by** *auto*

have $j < brn\ (dl!n)$ **using** $j\ I\ n\ FP$ **unfolding** *part-def* **by** *blast* **note** $j = this$

j

show $?thesis$

proof

have $eff\ (cl!n)\ (?ss\ n)\ i \approx eff\ (dl!n)\ (?tt\ n)\ j$

using $n\ I\ i\ j$ *eff-cont* **by** *blast*

thus $?eff$ **unfolding** $ii\ jj$ **using** $st\ cldl\ n\ i\ j$ **by** *simp*

next

have $cont\ (cl!n)\ (?ss\ n)\ i \approx_s cont\ (dl!n)\ (?tt\ n)\ j$

using $n\ I\ i\ j$ *eff-cont* **by** *blast*

thus $?cont$ **unfolding** $ii\ jj$ *thetaSPar-def* **using** $n\ i\ j\ l\ cldl$ **by** *simp*

qed

qed
qed

lemma *thetaSPar-Sretr*: $\text{thetaSPar} \subseteq \text{Sretr} (\text{thetaSPar})$

unfolding *Sretr-def matchC-C-def* **proof** *safe*

fix $c\ d\ s\ t$

assume $c\text{-}d$: $(c, d) \in \text{thetaSPar}$ **and** st : $s \approx t$

then obtain $cl\ dl$ **where**

c : $c = \text{Par } cl$ **and** d : $d = \text{Par } dl$ **and**

$cldl$: $\text{properL } cl\ \text{properL } dl\ \text{SbisL } cl\ dl$

unfolding *thetaSPar-def* **by** *blast*

let $?N = \text{length } cl$

let $?ss = \%n. s$ **let** $?tt = \%n. t$

have N : $?N = \text{length } dl$ **using** $cldl$ **by** *simp*

have $sstt$: $\bigwedge n. n < ?N \implies ?ss\ n \approx ?tt\ n$

using $st\ N$ **by** *auto*

let $?phi = \%n\ \text{PFn. } mC\text{-}C\ \text{Sbis } (cl\ !\ n)\ (dl\ !\ n)\ (?ss\ n)\ (?tt\ n)\ (\text{fst } \text{PFn})\ (\text{snd } \text{PFn})$

{fix n **assume** n : $n < ?N$

hence $cl\ !\ n \approx_s dl\ !\ n$ **using** $cldl$ **by** *auto*

hence $\exists\ \text{PFn. } ?phi\ n\ \text{PFn}$ **using** $n\ \text{Sbis-}mC\text{-}C\ sstt$ **by** *fastforce*

}

then obtain PF **where** phi : $\bigwedge n. n < ?N \implies ?phi\ n\ (PF\ n)$

using *bchoice*[of $\{.. < ?N\}\ ?phi$] **by** *blast*

define $P\ F$ **where** $P = \text{fst } o\ PF$ **and** $F = \text{snd } o\ PF$

have m : $\bigwedge n. n < ?N \implies mC\text{-}C\ \text{Sbis } (cl\ !\ n)\ (dl\ !\ n)\ (?ss\ n)\ (?tt\ n)\ (P\ n)\ (F\ n)$

using phi **unfolding** $P\text{-def } F\text{-def}$ **by** *auto*

have $brn\text{-}c$: $brn\ c = brnL\ cl\ ?N$ **unfolding** c **by** *simp*

have $brn\text{-}d$: $brn\ d = brnL\ dl\ (\text{length } dl)$ **unfolding** d **by** *simp*

have P : $\bigwedge n. n < ?N \implies \text{part } \{.. < brn\ (cl\ !\ n)\}\ (P\ n) \wedge \{\}\ \notin (P\ n)$

using m **unfolding** $m\text{-defsAll } part\text{-def}$ **by** *auto*

have FP : $\bigwedge n. n < \text{length } dl \implies \text{part } \{.. < brn\ (dl\ !\ n)\}\ (F\ n\ ' (P\ n)) \wedge \{\}\ \notin F\ n\ ' (P\ n)$

using $m\ N$ **unfolding** $m\text{-defsAll } part\text{-def}$ **by** *auto*

have F : $\bigwedge n. n < ?N \implies \text{inj-on } (F\ n)\ (P\ n)$ **using** m **unfolding** $m\text{-defsAll}$ **by** *auto*

have sw : $\bigwedge n\ I. \llbracket n < \text{length } cl; I \in P\ n \rrbracket \implies$

$\text{sum } (wt\ (cl\ !\ n)\ (?ss\ n))\ I = \text{sum } (wt\ (dl\ !\ n)\ (?tt\ n))\ (F\ n\ I)$

using m **unfolding** $mC\text{-}C\text{-def } mC\text{-}C\text{-wt-def}$ **by** *auto*

have eff-cont : $\bigwedge n\ I\ i\ j. \llbracket n < \text{length } cl; I \in P\ n; i \in I; j \in F\ n\ I \rrbracket \implies$

$\text{eff } (cl!n)\ (?ss\ n)\ i \approx \text{eff } (dl!n)\ (?tt\ n)\ j \wedge$

$\text{cont } (cl!n)\ (?ss\ n)\ i \approx_s \text{cont } (dl!n)\ (?tt\ n)\ j$

using m **unfolding** $mC\text{-}C\text{-def } mC\text{-}C\text{-eff-cont-def}$ **by** *auto*

define $Q\ G$ **where** $Q = \text{UNpart } cl\ P$ **and** $G = \text{UNlift } cl\ dl\ P\ F$

note $\text{defi} = Q\text{-def } G\text{-def } brn\text{-}c\ brn\text{-}d$

show $\exists\ Q\ G. mC\text{-}C\ (\text{thetaSPar})\ c\ d\ s\ t\ Q\ G$

```

apply(rule exI[of - Q]) apply(rule exI[of - G])
unfolding mC-C-def proof (intro conjI)
  show mC-C-part c d Q unfolding mC-C-part-def proof(intro conjI)
    show {}  $\notin$  Q unfolding defi apply(rule emp-UNpart) using P by simp
    show {}  $\notin$  G ' Q unfolding defi apply(rule emp-UNlift-UNpart) using N
  P FP by auto
  show part {..  

  unfolding defi apply(rule part-UNpart) using cddl P by auto
  show part {..  

  unfolding defi apply(rule part-UNlift-UNpart) using N cddl P FP by auto
qed
next
show inj-on G Q
unfolding defi apply(rule UNlift-inj-on) using N P FP F by auto
next
show mC-C-wt c d s t Q G
unfolding mC-C-wt-def defi proof clarify
  fix I assume I  $\in$  UNpart cl P
  thus sum (wt c s) I = sum (wt d t) (UNlift cl dl P F I)
  unfolding c d apply(intro ss-wt-Par-UNlift)
  using N cddl P FP sw st by auto
qed
next
show mC-C-eff-cont (thetaSPar) c d s t Q G
unfolding mC-C-eff-cont-def proof clarify
  fix II ii jj assume II: II  $\in$  Q and ii: ii  $\in$  II and jj: jj  $\in$  G II
  thus eff c s ii  $\approx$  eff d t jj  $\wedge$  (cont c s ii, cont d t jj)  $\in$  thetaSPar
  unfolding defi c d apply(intro cont-eff-Par-UNlift)
  using N cddl P FP eff-cont st by blast+
qed
qed
qed

```

lemma thetaSPar-Sbis: thetaSPar \subseteq Sbis
using Sbis-raw-coind thetaSPar-Sretr **by** blast

theorem Par-Sbis[simp]:
assumes properL cl **and** properL dl SbisL cl dl
shows Par cl \approx_s Par dl
using assms thetaSPar-Sbis **unfolding** thetaSPar-def **by** blast

4.5 01-bisimilarity versus language constructs

lemma ZObis-pres-discr-L: c \approx_{01} d \implies discr d \implies discr c
proof (coinduction arbitrary: d c, clarsimp)
fix s i c d **assume** i: i < brn c **and** d: discr d **and** c-d: c \approx_{01} d
then obtain I0 P F **where**
 match: mC-ZOC ZObis c d s s I0 P F
using ZObis-mC-ZOC[of c d s s] **by** blast

hence $\bigcup P = \{..<brn\ c\}$
 using *i* **unfolding** *mC-ZOC-def mC-ZOC-part-def part-def* **by** *simp*
 then obtain *I* where *I*: $I \in P$ and *i*: $i \in I$ **using** *i* **by** *auto*
 show $s \approx \text{eff } c\ s\ i \wedge ((\exists d. \text{cont } c\ s\ i \approx 01\ d \wedge \text{discr } d) \vee \text{discr } (\text{cont } c\ s\ i))$
proof(*cases* $I = I0$)
 case *False*
 then obtain *j* where *j*: $j \in F\ I$
 using *match* *I False* **unfolding** *mC-ZOC-def mC-ZOC-part-def* **by** *blast*
 hence $j < brn\ d$ **using** *I match*
unfolding *mC-ZOC-def mC-ZOC-part-def part-def* **apply** *simp* **by** *blast*
 hence *md*: $\text{discr } (\text{cont } d\ s\ j)$ and *s*: $s \approx \text{eff } d\ s\ j$
 using *d discr-cont[of d j s] discr-eff-indis[of d j s]* **by** *auto*
 have $\text{eff } c\ s\ i \approx \text{eff } d\ s\ j$ and *md2*: $\text{cont } c\ s\ i \approx 01\ \text{cont } d\ s\ j$
 using *I i j match False* **unfolding** *mC-ZOC-def mC-ZOC-eff-cont-def* **by**
auto
 hence $s \approx \text{eff } c\ s\ i$ **using** *s indis-sym indis-trans* **by** *blast*
 thus *?thesis* **using** *md md2* **by** *blast*
 next
 case *True*
 hence $s \approx \text{eff } c\ s\ i \wedge \text{cont } c\ s\ i \approx 01\ d$
 using *match i ZObis-sym* **unfolding** *mC-ZOC-def mC-ZOC-eff-cont0-def* **by**
blast
 thus *?thesis* **using** *d* **by** *blast*
qed
qed

theorem *ZObis-pres-discr-R*:
assumes *discr c* and $c \approx 01\ d$
shows *discr d*
using *assms ZObis-pres-discr-L ZObis-sym* **by** *blast*

theorem *ZObis-finished-discr-L*:
assumes $c \approx 01\ d$ and *proper d* and *finished d*
shows *discr c*
using *assms ZObis-pres-discr-L* **by** *auto*

theorem *ZObis-finished-discr-R*:
assumes *proper c* and *finished c* and $c \approx 01\ d$
shows *discr d*
using *assms ZObis-pres-discr-R[of c d]* **by** *auto*

theorem *discr-ZObis[simp]*:
assumes *proper c* and *proper d* and *discr c* and *discr d*
shows $c \approx 01\ d$
using *assms* **by** *auto*

theorem *Done-ZObis[simp]*:
Done $\approx 01\ \text{Done}$

by *simp*

theorem *Atm-ZObis*[*simp*]:
assumes *compatAtm atm*
shows *Atm atm ≈01 Atm atm*
using *assms* **by** *simp*

definition *thetaZOSeqI* **where**
thetaZOSeqI \equiv
 $\{(e ;; c, e ;; d) \mid e c d . \text{siso } e \wedge c \approx 01 d\}$

lemma *thetaZOSeqI-ZOretr*:
thetaZOSeqI \subseteq *ZOretr* (*thetaZOSeqI* *Un* *ZObis*)
unfolding *ZOretr-def matchC-LC-def* **proof** *safe*
fix *c d s t*
assume *c-d*: $(c, d) \in \text{thetaZOSeqI}$ **and** *st*: $s \approx t$
then obtain *e c1 d1* **where** *e*: *siso e* **and** *c1d1*: $c1 \approx 01 d1$
and *c*: $c = e ;; c1$ **and** *d*: $d = e ;; d1$
unfolding *thetaZOSeqI-def* **by** *auto*
let *?I0* = $\{\}$ **let** *?J0* = $\{\}$
let *?P* = $\{?I0\} \text{Un} \{\{i\} \mid i . i < \text{brn } e\}$
let *?F* = $\%I . I$
show $\exists I0 P F . \text{mC-ZOC } (\text{thetaZOSeqI } \text{Un } \text{ZObis}) c d s t I0 P F$
apply(*rule exI*[*of* - *?I0*])
apply(*rule exI*[*of* - *?P*]) **apply**(*rule exI*[*of* - *?F*])
unfolding *mC-ZOC-def* **proof** (*intro conjI*)
show *mC-ZOC-part* *c d s t ?I0 ?P ?F*
unfolding *mC-ZOC-part-def* **proof** (*intro conjI*)
show *part* $\{.. < \text{brn } c\} ?P$
unfolding *part-def* **proof** *safe*
fix *i* **assume** $i < \text{brn } c$
thus $i \in \bigcup ?P$ **using** *c e st siso-cont-indis*[*of e s t*] **by** *auto*
qed (*unfold c, simp*)

thus *part* $\{.. < \text{brn } d\} (?F ' ?P)$ **unfolding** *c d* **by** *auto*
qed *auto*

next

show *mC-ZOC-eff-cont* (*thetaZOSeqI* *Un* *ZObis*) *c d s t ?I0 ?P ?F*
unfolding *mC-ZOC-eff-cont-def* **proof**(*intro allI impI, elim conjE*)
fix *I i j*
assume *I*: $I \in ?P - \{?I0\}$ **and** *i*: $i \in I$ **and** *j*: $j \in I$
then obtain *i'* **where** $j \in ?F \{i'\}$ **and** $i' < \text{brn } e$
using *I j* **by** *auto*
thus $\text{eff } c s i \approx \text{eff } d t j \wedge (\text{cont } c s i, \text{cont } d t j) \in \text{thetaZOSeqI} \cup \text{ZObis}$
using *st c-d e i j I* **unfolding** *c d thetaZOSeqI-def*
by (*cases finished* (*cont e s i'*)) *auto*

qed
qed (*insert st c-d c, unfold m-defsAll thetaZOSeqI-def part-def, auto*)
qed

lemma *thetaZOSeqI-ZObis*:
thetaZOSeqI \subseteq *ZObis*
using *ZObis-coind thetaZOSeqI-ZOretr* **by** *blast*

theorem *Seq-siso-ZObis[simp]*:
assumes *siso e* **and** *c2 \approx 01 d2*
shows *e ;; c2 \approx 01 e ;; d2*
using *assms thetaZOSeqI-ZObis* **unfolding** *thetaZOSeqI-def* **by** *auto*

definition *thetaZOSeqD* **where**
thetaZOSeqD \equiv
 $\{(c1 ;; c2, d1 ;; d2) \mid$
 $c1 \ c2 \ d1 \ d2.$
 $\text{proper } c1 \wedge \text{proper } d1 \wedge \text{proper } c2 \wedge \text{proper } d2 \wedge$
 $\text{discr } c2 \wedge \text{discr } d2 \wedge$
 $c1 \approx 01 \ d1\}$

lemma *thetaZOSeqD-ZOretr*:
thetaZOSeqD \subseteq *ZOretr* (*thetaZOSeqD Un ZObis*)
unfolding *ZOretr-def matchC-LC-def* **proof** *safe*
fix *c d s t*
assume *c-d: (c, d) \in thetaZOSeqD* **and** *st: s \approx t*
then obtain *c1 c2 d1 d2* **where**
c1d1: proper c1 proper d1 c1 \approx 01 d1 **and**
c2d2: proper c2 proper d2 discr c2 discr d2
and *c: c = c1 ;; c2* **and** *d: d = d1 ;; d2*
unfolding *thetaZOSeqD-def* **by** *auto*
from *c1d1 st* **obtain** *P F I0*
where *match: mC-ZOC ZObis c1 d1 s t I0 P F*
using *ZObis-mC-ZOC* **by** *blast*
have *P: $\bigcup P = \{..<brn \ c1\}$* **and** *FP: $\bigcup (F \ ' \ P) = \{..<brn \ d1\}$*
using *match* **unfolding** *mC-ZOC-def mC-ZOC-part-def part-def* **by** *metis+*
show $\exists I0 \ P \ F. \ mC-ZOC \ (\text{thetaZOSeqD} \cup \ ZObis) \ c \ d \ s \ t \ I0 \ P \ F$
apply(*intro exI[of - I0] exI[of - P] exI[of - F]*)
unfolding *mC-ZOC-def* **proof** (*intro conjI*)
have *I0: I0 \in P* **using** *match* **unfolding** *m-defsAll* **by** *blast*
show *mC-ZOC-eff-cont0 (thetaZOSeqD \cup ZObis) c d s t I0 F*
unfolding *mC-ZOC-eff-cont0-def* **proof**(*intro conjI ballI*)
fix *i* **assume** *i: i \in I0*
let *?c1' = cont c1 s i* **let** *?s' = eff c1 s i*
have *i < brn c1* **using** *i I0 P* **by** *blast* **note** *i = this i*
have *c1'd1: ?c1' \approx 01 d1 proper ?c1'*
using *c1d1 i I0 match* **unfolding** *mC-ZOC-def mC-ZOC-eff-cont0-def* **by**

```

auto
  show  $s \approx \text{eff } c \ s \ i$ 
    using  $i \ \text{match unfolding } c \ mC\text{-ZOC-def } mC\text{-ZOC-eff-cont0-def}$  by simp
  show  $(\text{cont } c \ s \ i, d) \in \text{thetaZOSeqD} \cup \text{ZObis}$ 
  proof(cases finished ?c1')
    case False note  $f\text{-}c1' = \text{False}$ 
      hence  $\text{csi}: \text{cont } c \ s \ i = ?c1' ;; c2$  using  $i \ \text{unfolding } c$  by simp
      hence  $(\text{cont } c \ s \ i, d) \in \text{thetaZOSeqD}$ 
      using  $c1'd1 \ c1d1 \ c2d2 \ f\text{-}c1' \ i \ \text{match}$ 
      unfolding  $\text{csi } d \ \text{thetaZOSeqD-def } mC\text{-ZOC-def } mC\text{-ZOC-eff-cont0-def}$  by
blast
  thus ?thesis by simp
next
  case True note  $f\text{-}c1' = \text{True}$ 
    hence  $\text{csi}: \text{cont } c \ s \ i = c2$  using  $i \ \text{unfolding } c$  by simp
    have  $\text{discr } d1$  using  $f\text{-}c1' \ c1'd1 \ \text{ZObis-finished-discr-R}$  by blast
    hence  $c2 \approx 01 \ d$  using  $c2d2 \ c1d1$  unfolding  $d$  by simp
    thus ?thesis unfolding  $\text{csi}$  by simp
qed
next
  fix  $j$  assume  $j: j \in F \ I0$ 
  let  $?d1' = \text{cont } d1 \ t \ j$  let  $?t' = \text{eff } d1 \ t \ j$ 
  have  $j < \text{brn } d1$  using  $j \ I0 \ FP$  by blast note  $j = \text{this } j$ 
  have  $c1d1': c1 \approx 01 \ ?d1' \ \text{proper } ?d1'$ 
    using  $c1d1 \ j \ I0 \ \text{match unfolding } mC\text{-ZOC-def } mC\text{-ZOC-eff-cont0-def}$  by
auto
  show  $t \approx \text{eff } d \ t \ j$ 
    using  $j \ \text{match unfolding } d \ mC\text{-ZOC-def } mC\text{-ZOC-eff-cont0-def}$  by simp
  show  $(c, \text{cont } d \ t \ j) \in \text{thetaZOSeqD} \cup \text{ZObis}$ 
  proof(cases finished ?d1')
    case False note  $f\text{-}d1' = \text{False}$ 
      hence  $\text{dtj}: \text{cont } d \ t \ j = ?d1' ;; d2$  using  $j \ \text{unfolding } d$  by simp
      hence  $(c, \text{cont } d \ t \ j) \in \text{thetaZOSeqD}$ 
      using  $c1d1' \ c1d1 \ c2d2 \ f\text{-}d1' \ j \ \text{match}$ 
      unfolding  $c \ \text{dtj } \text{thetaZOSeqD-def } mC\text{-ZOC-def } mC\text{-ZOC-eff-cont0-def}$  by
blast
  thus ?thesis by simp
next
  case True note  $f\text{-}d1' = \text{True}$ 
    hence  $\text{dtj}: \text{cont } d \ t \ j = d2$  using  $j \ \text{unfolding } d$  by simp
    hence  $\text{discr } c1$  using  $f\text{-}d1' \ c1d1' \ \text{ZObis-finished-discr-L}$  by blast
    hence  $c \approx 01 \ d2$  using  $c2d2 \ c1d1$  unfolding  $c$  by simp
    thus ?thesis unfolding  $\text{dtj}$  by simp
qed
qed
next
  show  $mC\text{-ZOC-eff-cont } (\text{thetaZOSeqD} \cup \text{ZObis}) \ c \ d \ s \ t \ I0 \ P \ F$ 
  unfolding  $mC\text{-ZOC-eff-cont-def}$  proof(intro allI impI, elim conjE)
    fix  $i \ j \ I$  assume  $I : I \in P - \{I0\}$  and  $i: i \in I$  and  $j: j \in F \ I$ 

```

```

let ?c1' = cont c1 s i let ?d1' = cont d1 t j
let ?s' = eff c1 s i let ?t' = eff d1 t j
have i < brn c1 using i I P by blast note i = this i
have j < brn d1 using j I FP by blast note j = this j
have c1'd1': ?c1' ≈ 01 ?d1' proper ?c1' proper ?d1'
using c1d1 i j I match unfolding c mC-ZOC-def mC-ZOC-eff-cont-def by
auto
show eff c s i ≈ eff d t j ∧ (cont c s i, cont d t j) ∈ thetaZOSeqD ∪ ZObis
(is ?eff ∧ ?cont) proof
  show ?eff using match I i j unfolding c d m-defsAll apply simp by blast
next
show ?cont
proof(cases finished ?c1')
  case True note c1' = True
  hence csi: cont c s i = c2 using i match unfolding c m-defsAll by simp
  show ?thesis
  proof(cases finished ?d1')
    case True
    hence cont d t j = d2 using j match unfolding d m-defsAll by simp
    thus ?thesis using csi c2d2 by simp
  next
  case False
  hence dtj: cont d t j = ?d1' ;; d2
  using j match unfolding d m-defsAll by simp
  have discr ?d1' using c1'd1' c1' ZObis-finished-discr-R by blast
  thus ?thesis using c1'd1' c2d2 unfolding csi dtj by simp
qed
next
case False
  hence csi: cont c s i = ?c1' ;; c2
  using i match unfolding c m-defsAll by simp
  show ?thesis
  proof(cases finished (cont d1 t j))
    case True note d1' = True
    hence dtj: cont d t j = d2 using j match unfolding d m-defsAll by
simp
    have discr ?c1' using c1'd1' d1' ZObis-finished-discr-L by blast
    thus ?thesis using c1'd1' c2d2 unfolding csi dtj by simp
  next
  case False
  hence dtj: cont d t j = ?d1' ;; d2 using j match unfolding d m-defsAll
by simp
  thus ?thesis unfolding csi dtj thetaZOSeqD-def
  using c1'd1' c2d2 by blast
qed
qed
qed
qed
qed(insert match, unfold m-defsAll c d, auto)

```

qed

lemma *thetaZOSeqD-ZObis*:
thetaZOSeqD \subseteq *ZObis*
using *ZObis-coind thetaZOSeqD-ZOretr* by *blast*

theorem *Seq-ZObis[simp]*:
assumes *proper c1* and *proper d1* and *proper c2* and *proper d2*
and *c1 \approx_{01} d1* and *discr c2* and *discr d2*
shows *c1* ;; *c2 \approx_{01} d1* ;; *d2*
using *assms thetaZOSeqD-ZObis* unfolding *thetaZOSeqD-def* by *auto*

definition *thetaZOCh* where
thetaZOCh ch c1 c2 d1 d2 \equiv $\{(Ch\ ch\ c1\ c2, Ch\ ch\ d1\ d2)\}$

lemma *thetaZOCh-Sretr*:
assumes *compatCh ch* and *c1 \approx_{01} d1* and *c2 \approx_{01} d2*
shows *thetaZOCh ch c1 c2 d1 d2* \subseteq
 Sretr (thetaZOCh ch c1 c2 d1 d2 \cup ZObis)
(is *?th* \subseteq *Sretr (?th \cup ZObis)*)
unfolding *Sretr-def matchC-C-def* **proof** *safe*
 fix *c d s t*
 assume *c-d: (c, d) \in ?th* and *st: s \approx t*
 hence *c: c = Ch ch c1 c2 brn c = 2*
 and *d: d = Ch ch d1 d2 brn d = 2*
 unfolding *thetaZOCh-def* by *auto*
 let *?P* = $\{\{0\}, \{1\}\}$
 let *?F* = $\%I. I$
 show $\exists P\ F. mC-C\ (?th\ Un\ ZObis)\ c\ d\ s\ t\ P\ F$
 apply(*rule exI[of - ?P]*) **apply**(*rule exI[of - ?F]*)
 using *assms st c-d* unfolding *m-defsAll thetaZOCh-def part-def* by *auto*
qed

lemma *thetaZOCh-ZOretr*:
assumes *compatCh ch* and *c1 \approx_{01} d1* and *c2 \approx_{01} d2*
shows *thetaZOCh ch c1 c2 d1 d2* \subseteq
 ZOretr (thetaZOCh ch c1 c2 d1 d2 \cup ZObis)
using *thetaZOCh-Sretr[OF assms]*
by (*metis (no-types) Retr-incl subset-trans*)

lemma *thetaZOCh-ZObis*:
assumes *compatCh ch* and *c1 \approx_{01} d1* and *c2 \approx_{01} d2*
shows *thetaZOCh ch c1 c2 d1 d2* \subseteq *ZObis*
using *ZObis-coind thetaZOCh-ZOretr[OF assms]* by *blast*

theorem *Ch-iso-ZObis[simp]*:
assumes *compatCh ch* and *c1 \approx_{01} d1* and *c2 \approx_{01} d2*
shows *Ch ch c1 c2 \approx_{01} Ch ch d1 d2*

using *thetaZOCh-ZObis*[*OF assms*] **unfolding** *thetaZOCh-def* **by** *auto*

definition *theFTOne* **where**

theFTOne *cl dl* \equiv *theFT* *cl* \cup *theFT* *dl*

definition *theNFTBoth* **where**

theNFTBoth *cl dl* \equiv *theNFT* *cl* \cap *theNFT* *dl*

lemma *theFTOne-sym*: *theFTOne* *cl dl* = *theFTOne* *dl cl*

unfolding *theFTOne-def* **by** *auto*

lemma *finite-theFTOne*[*simp*]:

finite (*theFTOne* *cl dl*)

unfolding *theFTOne-def* **by** *simp*

lemma *theFTOne-length-finished*[*simp*]:

assumes *n* \in *theFTOne* *cl dl*

shows (*n* < *length* *cl* \wedge *finished* (*cl*!*n*)) \vee (*n* < *length* *dl* \wedge *finished* (*dl*!*n*))

using *assms* **unfolding** *theFTOne-def* **by** *auto*

lemma *theFTOne-length*[*simp*]:

assumes *length* *cl* = *length* *dl* **and** *n* \in *theFTOne* *cl dl*

shows *n* < *length* *cl* **and** *n* < *length* *dl*

using *assms* *theFTOne-length-finished*[*of n cl dl*] **by** *auto*

lemma *theFTOne-intro*[*intro*]:

assumes $\bigwedge n. (n < \text{length } cl \wedge \text{finished } (cl!n)) \vee (n < \text{length } dl \wedge \text{finished } (dl!n))$

shows *n* \in *theFTOne* *cl dl*

using *assms* **unfolding** *theFTOne-def* **by** *auto*

lemma *pickFT-theFTOne*[*simp*]:

assumes *WtFT* *cl* = 1

shows *pickFT* *cl* \in *theFTOne* *cl dl*

using *assms* **unfolding** *theFTOne-def* **by** *auto*

lemma *finite-theNFTBoth*[*simp*]:

finite (*theNFTBoth* *cl dl*)

unfolding *theNFTBoth-def* **by** *simp*

lemma *theNFTBoth-sym*: *theNFTBoth* *cl dl* = *theNFTBoth* *dl cl*

unfolding *theNFTBoth-def* **by** *auto*

lemma *theNFTBoth-length-finished*[*simp*]:

assumes *n* \in *theNFTBoth* *cl dl*

shows *n* < *length* *cl* **and** \neg *finished* (*cl*!*n*)

and *n* < *length* *dl* **and** \neg *finished* (*dl*!*n*)

using *assms* **unfolding** *theNFTBoth-def* **by** *auto*

lemma *theNFTBoth-intro*[*intro*]:
assumes $\bigwedge n. n < \text{length } cl \wedge \neg \text{finished } (cl!n) \wedge n < \text{length } dl \wedge \neg \text{finished } (dl!n)$
shows $n \in \text{theNFTBoth } cl \ dl$
using *assms unfolding theNFTBoth-def by auto*

lemma *theFTOne-Int-theNFTBoth*[*simp*]:
 $\text{theFTOne } cl \ dl \cap \text{theNFTBoth } cl \ dl = \{\}$
and $\text{theNFTBoth } cl \ dl \cap \text{theFTOne } cl \ dl = \{\}$
unfolding *theFTOne-def theNFTBoth-def theFT-def theNFT-def by auto*

lemma *theFT-Un-theNFT-One-Both*[*simp*]:
assumes $\text{length } cl = \text{length } dl$
shows
 $\text{theFTOne } cl \ dl \cup \text{theNFTBoth } cl \ dl = \{.. < \text{length } cl\}$ **and**
 $\text{theNFTBoth } cl \ dl \cup \text{theFTOne } cl \ dl = \{.. < \text{length } cl\}$
using *assms*
unfolding *theFTOne-def theNFTBoth-def theFT-def theNFT-def by auto*

lemma *in-theFTOne-theNFTBoth*[*simp*]:
assumes $n1 \in \text{theFTOne } cl \ dl$ **and** $n2 \in \text{theNFTBoth } cl \ dl$
shows $n1 \neq n2$ **and** $n2 \neq n1$
using *assms theFTOne-Int-theNFTBoth by blast+*

definition *BrnFT* **where**
 $\text{BrnFT } cl \ dl \equiv \bigcup n \in \text{theFTOne } cl \ dl. \{brnL \ cl \ n \ .. < + \ brn \ (cl!n)\}$

definition *BrnNFT* **where**
 $\text{BrnNFT } cl \ dl \equiv \bigcup n \in \text{theNFTBoth } cl \ dl. \{brnL \ cl \ n \ .. < + \ brn \ (cl!n)\}$

lemma *BrnFT-elim*[*elim, consumes 1, case-names Local*]:
assumes $ii \in \text{BrnFT } cl \ dl$
and $\bigwedge n \ i. \llbracket n \in \text{theFTOne } cl \ dl; i < brn \ (cl!n); ii = brnL \ cl \ n + i \rrbracket \implies phi$
shows *phi*
using *assms unfolding BrnFT-def by auto*

lemma *finite-BrnFT*[*simp*]:
 $\text{finite } (\text{BrnFT } cl \ dl)$
unfolding *BrnFT-def by auto*

lemma *BrnFT-incl-brnL*[*simp*]:
assumes $l: \text{length } cl = \text{length } dl$ **and** $cl: \text{properL } cl$
shows $\text{BrnFT } cl \ dl \subseteq \{.. < brnL \ cl \ (\text{length } cl)\}$ (**is** $?L \subseteq ?R$)
proof –
have $?L \subseteq (\bigcup n < \text{length } cl. \{brnL \ cl \ n \ .. < + \ brn \ (cl!n)\})$
using l **unfolding** *BrnFT-def theFTOne-def theFT-def by auto*

also have ... = ?R using cl brnL-UN by auto
 finally show ?thesis .
 qed

lemma BrnNFT-elim[elim, consumes 1, case-names Local]:
 assumes ii ∈ BrnNFT cl dl
 and $\bigwedge n i. \llbracket n \in \text{theNFTBoth } cl \ dl; i < \text{brn } (cl!n); ii = \text{brnL } cl \ n + i \rrbracket \implies phi$
 shows phi
 using assms unfolding BrnNFT-def by auto

lemma finite-BrnNFT[simp]:
 finite (BrnNFT cl dl)
 unfolding BrnNFT-def by auto

lemma BrnNFT-incl-brnL[simp]:
 assumes cl: properL cl
 shows BrnNFT cl dl \subseteq {..
 proof -
 have ?L \subseteq ($\bigcup n < \text{length } cl. \{ \text{brnL } cl \ n.. < + \text{brn } (cl!n) \}$)
 unfolding BrnNFT-def theNFTBoth-def theNFT-def by auto
 also have ... = ?R using cl brnL-UN by auto
 finally show ?thesis .
 qed

lemma BrnFT-Int-BrnNFT[simp]:
 assumes l: length cl = length dl
 shows
 BrnFT cl dl \cap BrnNFT cl dl = {} (is ?L)
 and BrnNFT cl dl \cap BrnFT cl dl = {} (is ?R)
 proof -
 {fix ii assume 1: ii ∈ BrnFT cl dl and 2: ii ∈ BrnNFT cl dl
 from 1 have False
 proof (cases rule: BrnFT-elim)
 case (Local n1 i1)
 hence n1: n1 < length cl n1 < length dl n1 ∈ theFTOne cl dl
 and i1: i1 < brn (cl!n1)
 and ii1: ii = brnL cl n1 + i1 using l by auto
 from 2 show ?thesis
 proof (cases rule: BrnNFT-elim)
 case (Local n2 i2)
 hence n2: n2 ∈ theNFTBoth cl dl and i2: i2 < brn (cl!n2)
 and ii2: ii = brnL cl n2 + i2 by auto
 have n12: n1 ≠ n2 using n1 n2 by simp
 show ?thesis
 proof (cases n1 < n2)
 case True hence Suc: Suc n1 ≤ n2 by simp
 have ii < brnL cl (Suc n1) unfolding ii1 using i1 n1 by simp
 also have ... ≤ brnL cl n2 using Suc by auto
 also have ... ≤ ii unfolding ii2 by simp

finally show *False* **by** *simp*
next
case *False* **hence** *Suc: Suc n2 ≤ n1* **using** *n12* **by** *simp*
have *ii < brnL cl (Suc n2)* **unfolding** *ii2* **using** *i2 n2* **by** *simp*
also have *... ≤ brnL cl n1* **using** *Suc* **by** *auto*
also have *... ≤ ii* **unfolding** *ii1* **by** *simp*
finally show *False* **by** *simp*
qed
qed
qed
}
thus *?L* **by** *blast* **thus** *?R* **by** *blast*
qed

lemma *BrnFT-Un-BrnNFT[simp]*:
assumes *l: length cl = length dl* **and** *cl: properL cl*
shows *BrnFT cl dl ∪ BrnNFT cl dl = {..
and* *BrnNFT cl dl ∪ BrnFT cl dl = {..
proof –
have *R: ?R = (∪ n < length cl. {brnL cl n..
using *cl brnL-UN* **by** *auto*
thus *?L1 = ?R* **unfolding** *R*
unfolding *l BrnFT-def BrnNFT-def*
theFTOne-def theNFTBoth-def theFT-def theNFT-def **by** *blast*
thus *?L2 = ?R* **by** *blast*
qed**

lemma *BrnFT-part*:
assumes *l: length cl = length dl*
and *P: ∧ n. n < length cl ⇒ part {..
shows *BrnFT cl dl = (∪ n ∈ theFTOne cl dl. Union (shift cl n ‘ (P n)))* **(is** *?L = ?R*
proof (*safe elim!: BrnFT-elim*)
fix *x n i* **assume** *n: n ∈ theFTOne cl dl* **and** *i: i < brn (cl ! n)*
hence *n < length cl n < length dl* **using** *l* **by** *auto* **note** *n = this n*
hence *i ∈ Union (P n)* **using** *P i* **unfolding** *part-def* **by** *auto*
then obtain *I* **where** *i ∈ I* **and** *I ∈ P n* **by** *blast*
thus *brnL cl n + i ∈ ?R* **using** *n* **unfolding** *shift-def* **by** *auto*
next
fix *n ii I* **assume** *n: n ∈ theFTOne cl dl* **and** *ii: ii ∈ shift cl n I* **and** *I: I ∈ P n*
hence *n < length cl* **using** *l* **by** *simp*
hence *I ⊆ {..
hence *ii ∈ {brnL cl n..
thus *ii ∈ ?L* **using** *n* **unfolding** *BrnFT-def* **by** *auto*
qed***

lemma *brnL-pickFT-BrnFT[simp]*:
assumes *properL cl* **and** *WtFT cl = 1*

shows $brnL\ cl\ (pickFT\ cl) \in BrnFT\ cl\ dl$
using *assms brn-gt-0-L unfolding BrnFT-def by auto*

lemma *WtFT-ParT-BrnFT[simp]*:
assumes $length\ cl = length\ dl\ properL\ cl$ **and** $WtFT\ cl = 1$
shows $sum\ (wt\ (ParT\ cl)\ s)\ (BrnFT\ cl\ dl) = 1$
proof –
have $brnL\ cl\ (pickFT\ cl) \in BrnFT\ cl\ dl$ **and**
 $BrnFT\ cl\ dl \subseteq \{..<brnL\ cl\ (length\ cl)\}$
using *assms BrnFT-incl-brnL by (simp, blast)*
thus *?thesis using assms by simp*
qed

definition *UNpart1 where*
 $UNpart1\ cl\ dl\ P \equiv \bigcup\ n \in theNFTBoth\ cl\ dl.\ shift\ cl\ n\ ' (P\ n)$

definition *UNpart01 where*
 $UNpart01\ cl\ dl\ P \equiv \{BrnFT\ cl\ dl\} \cup UNpart1\ cl\ dl\ P$

lemma *BrnFT-UNpart01[simp]*:
 $BrnFT\ cl\ dl \in UNpart01\ cl\ dl\ P$
unfolding *UNpart01-def by simp*

lemma *UNpart1-cases[elim, consumes 1, case-names Local]*:
assumes $II \in UNpart1\ cl\ dl\ P$
 $\bigwedge\ n\ I.\ \llbracket n \in theNFTBoth\ cl\ dl; I \in P\ n; II = shift\ cl\ n\ I \rrbracket \implies phi$
shows *phi*
using *assms unfolding UNpart1-def by auto*

lemma *UNpart01-cases[elim, consumes 1, case-names Local0 Local]*:
assumes $II \in UNpart01\ cl\ dl\ P$ **and** $II = BrnFT\ cl\ dl \implies phi$
 $\bigwedge\ n\ I.\ \llbracket n \in theNFTBoth\ cl\ dl; I \in P\ n; II = shift\ cl\ n\ I; II \in UNpart1\ cl\ dl\ P \rrbracket$
 $\implies phi$
shows *phi*
using *assms unfolding UNpart01-def UNpart1-def by auto*

lemma *emp-UNpart1*:
assumes $\bigwedge\ n.\ n < length\ cl \implies \{\} \notin P\ n$
shows $\{\} \notin UNpart1\ cl\ dl\ P$
using *assms unfolding UNpart1-def by auto*

lemma *emp-UNpart01*:
assumes $\bigwedge\ n.\ n < length\ cl \implies \{\} \notin P\ n$
shows $\{\} \notin UNpart01\ cl\ dl\ P - \{BrnFT\ cl\ dl\}$
using *assms emp-UNpart1 unfolding UNpart01-def by auto*

lemma *BrnFT-Int-UNpart1[simp]*:

assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and II : $II \in \text{UNpart1 } cl \ dl \ P$
shows $\text{BrnFT } cl \ dl \cap II = \{\}$
using II **proof**(*cases rule: UNpart1-cases*)
have 1 : $\text{BrnFT } cl \ dl = (\bigcup n \in \text{theFTOne } cl \ dl. \text{Union } (\text{shift } cl \ n \ ' (P \ n)))$
apply(*rule BrnFT-part*) **using** $l \ P$ **by** *auto*
case (*Local n I*)
hence n : $n < \text{length } cl \ n < \text{length } dl \ n \in \text{theNFTBoth } cl \ dl$
and I : $I \in P \ n$ **and** II : $II = \text{shift } cl \ n \ I$ **by** *auto*
{fix $n0$ **assume** $n0$: $n0 \in \text{theFTOne } cl \ dl$
hence $n0 < \text{length } cl \ n0 < \text{length } dl$ **using** l **by** *auto* **note** $n0 = \text{this } n0$
{fix JJ **assume** $JJ \in \text{shift } cl \ n0 \ ' (P \ n0)$
then obtain J **where** J : $J \in P \ n0$ **and** JJ : $JJ = \text{shift } cl \ n0 \ J$ **by** *auto*
have $n \neq n0$ **using** $n \ n0$ **by** *simp*
have $JJ \ \text{Int } II = \{\}$ **unfolding** $JJ \ II$
apply(*rule part-brn-disj3*) **using** $P \ I \ J \ n \ n0$ **by** *auto*
}
hence $\text{Union } (\text{shift } cl \ n0 \ ' (P \ n0)) \ \text{Int } II = \{\}$ **by** *blast*
}
thus *?thesis* **unfolding** $II \ 1$ **by** *blast*
qed

lemma *BrnFT-notIn-UNpart1*:
assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
shows $\text{BrnFT } cl \ dl \notin \text{UNpart1 } cl \ dl \ P$
using *assms BrnFT-Int-UNpart1 emp-UNpart1* **by** (*metis Int-absorb*)

lemma *UNpart1-UNpart01*:
assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
shows $\text{UNpart1 } cl \ dl \ P = \text{UNpart01 } cl \ dl \ P - \{\text{BrnFT } cl \ dl\}$
proof–
have $\text{BrnFT } cl \ dl \notin \text{UNpart1 } cl \ dl \ P$
apply(*rule BrnFT-notIn-UNpart1*) **using** *assms* **by** *auto*
thus *?thesis* **unfolding** *UNpart01-def* **by** *auto*
qed

lemma *part-UNpart1[simp]*:
assumes l : $\text{length } cl = \text{length } dl$
and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n)$
shows $\text{part } (\text{BrnNFT } cl \ dl) (\text{UNpart1 } cl \ dl \ P)$
unfolding *BrnNFT-def UNpart1-def* **apply**(*rule part-UN*)
using $l \ P$ **apply** *fastforce*
apply(*rule brnL-Int*) **using** l **by** *auto*

lemma *part-UNpart01*:
assumes cl : *properL cl* **and** l : $\text{length } cl = \text{length } dl$

and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
shows $\text{part } \{..< \text{brnL } cl \ (\text{length } cl)\} (UN\text{part01 } cl \ dl \ P)$
unfolding $UN\text{part01-def}$ **apply**(rule part-Un-singl2[of - - BrnNFT cl dl])
using *assms* **using** $BrnFT\text{-Int-}UN\text{part1}$ **by** (*simp*, *simp*, *blast*)

definition $UN\text{lift01}$ **where**

$UN\text{lift01 } cl \ dl \ P \ F \ II \equiv$
 if $II = BrnFT \ cl \ dl$
 then $BrnFT \ dl \ cl$
 else $\text{shift } dl \ (\text{pickT } cl \ P \ II) \ (F \ (\text{pickT } cl \ P \ II) \ (\text{back } cl \ (\text{pickT } cl \ P \ II) \ II))$

lemma $UN\text{lift01-}BrnFT[\text{simp}]$:
 $UN\text{lift01 } cl \ dl \ P \ F \ (BrnFT \ cl \ dl) = BrnFT \ dl \ cl$
unfolding $UN\text{lift01-def}$ **by** *simp*

lemma $UN\text{lift01-shift}[\text{simp}]$:

assumes $l: \text{length } cl = \text{length } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $n: n \in \text{theNFTBoth } cl \ dl$ **and** $I: I \in P \ n$
shows $UN\text{lift01 } cl \ dl \ P \ F \ (\text{shift } cl \ n \ I) = \text{shift } dl \ n \ (F \ n \ I)$

proof–

let $?N = \text{length } cl$
define II **where** $II = \text{shift } cl \ n \ I$
have $n < \text{length } cl$ **using** $n \ l$ **by** *auto* **note** $n = \text{this } n$
have $II: \text{shift } cl \ n \ I = II$ **using** $II\text{-def}$ **by** *simp*
have $II \in UN\text{part1 } cl \ dl \ P$ **unfolding** $II\text{-def}$ $UN\text{part1-def}$ **using** $n \ I$ **by** *auto*
hence $II \neq BrnFT \ cl \ dl$ **using** $BrnFT\text{-notIn-}UN\text{part1}$ [of $cl \ dl \ P$] $l \ n \ P$ **by** *auto*
hence $1: UN\text{lift01 } cl \ dl \ P \ F \ II =$
 $\text{shift } dl \ (\text{pickT } cl \ P \ II) \ (F \ (\text{pickT } cl \ P \ II) \ (\text{back } cl \ (\text{pickT } cl \ P \ II) \ II))$
unfolding $UN\text{lift01-def}$ **by** *simp*
have $n: n = \text{pickT } cl \ P \ II$ **apply**(rule *pickT-unique*)
using *assms* **unfolding** $II\text{-def}$ **by** *auto*
have $\text{back } cl \ n \ II = I$ **unfolding** $II\text{-def}$ **by** *simp*
hence $\text{shift } dl \ n \ (F \ n \ (\text{back } cl \ n \ II)) = \text{shift } dl \ n \ (F \ n \ I)$ **by** *simp*
thus $?thesis$ **unfolding** $1 \ II \ n$ [*THEN sym*].

qed

lemma $UN\text{lift01-inj-on-}UN\text{part1}$:

assumes $l: \text{length } cl = \text{length } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $FP: \bigwedge n. n < \text{length } dl \implies \text{part } \{..< \text{brn } (dl!n)\} (F \ n \ ' (P \ n)) \wedge \{\} \notin F \ n \ ' (P \ n)$
and $F: \bigwedge n. n < \text{length } cl \implies \text{inj-on } (F \ n) \ (P \ n)$
shows $\text{inj-on } (UN\text{lift01 } cl \ dl \ P \ F) \ (UN\text{part1 } cl \ dl \ P)$ (**is** $\text{inj-on } ?G \ ?Q$)
unfolding inj-on-def **proof** *clarify*
fix $III \ II2$
assume $III: III \in ?Q$ **and** $II2: II2 \in ?Q$ **and** $G: ?G \ III = ?G \ II2$

from $II1$ **show** $II1 = II2$
proof(cases rule: UNpart1-cases)
 case (Local $n1$ $I1$)
 hence $n1$: $n1 \in \text{theNFTBoth } cl \ dl \ n1 < \text{length } cl \ n1 < \text{length } dl$ **and** II : $I1 \in P \ n1$
 and III : $III = \text{shift } cl \ n1 \ I1$ **using** l **by** *auto*
 hence $G1\text{-def}$: $?G \ III = \text{shift } dl \ n1 \ (F \ n1 \ I1)$ **using** $l \ P$ **by** *simp*
 have $Pn1$: $\text{part } \{..< \text{brn } (dl!n1)\} (F \ n1 \ ' (P \ n1)) \ \{\} \notin F \ n1 \ ' (P \ n1)$
 using $n1 \ FP$ **by** *auto*
 have $F1\text{-in}$: $F \ n1 \ I1 \in F \ n1 \ ' (P \ n1)$ **using** $I1$ **by** *simp*
 hence $Fn1I1$: $F \ n1 \ I1 \neq \{\} \ F \ n1 \ I1 \subseteq \{..< \text{brn } (dl!n1)\}$
 using $Pn1$ **by** (*blast*, *unfold part-def*, *blast*)
 hence $G1$: $?G \ III \neq \{\} \ ?G \ III \subseteq \{\text{brnL } dl \ n1 \ ..<+ \text{brn } (dl!n1)\}$
 unfolding $G1\text{-def}$ **by** *simp-all*
 from $II2$ **show** *?thesis*
 proof(cases rule: UNpart1-cases)
 case (Local $n2$ $I2$)
 hence $n2$: $n2 \in \text{theNFTBoth } cl \ dl \ n2 < \text{length } cl \ n2 < \text{length } dl$
 and $I2$: $I2 \in P \ n2$ **and** $II2$: $II2 = \text{shift } cl \ n2 \ I2$ **using** l **by** *auto*
 hence $G2\text{-def}$: $?G \ II2 = \text{shift } dl \ n2 \ (F \ n2 \ I2)$ **using** $l \ P$ **by** *auto*
 have $Pn2$: $\text{part } \{..< \text{brn } (dl!n2)\} (F \ n2 \ ' (P \ n2)) \ \{\} \notin F \ n2 \ ' (P \ n2)$
 using $n2 \ FP$ **by** *auto*
 have $F2\text{-in}$: $F \ n2 \ I2 \in F \ n2 \ ' (P \ n2)$ **using** $I2$ **by** *simp*
 hence $Fn2I2$: $F \ n2 \ I2 \neq \{\} \ F \ n2 \ I2 \subseteq \{..< \text{brn } (dl!n2)\}$
 using $Pn2$ **by** (*blast*, *unfold part-def*, *blast*)
 hence $G2$: $?G \ II2 \neq \{\} \ ?G \ II2 \subseteq \{\text{brnL } dl \ n2 \ ..<+ \text{brn } (dl!n2)\}$
 unfolding $G2\text{-def}$ **by** *simp-all*

 have $n12$: $n1 = n2$ **using** $n1 \ n2 \ G1 \ G2 \ G \ \text{brnL-Int}$ **by** *blast*
 have $F \ n1 \ I1 = F \ n2 \ I2$ **using** G **unfolding** $G1\text{-def} \ G2\text{-def} \ n12$ **by** *simp*
 hence $I1 = I2$ **using** $I1 \ I2 \ n1 \ F$ **unfolding** $n12 \ \text{inj-on-def}$ **by** *blast*
 thus *?thesis* **unfolding** $II1 \ II2 \ n12$ **by** *simp*
 qed
 qed
qed

lemma *inj-on-singl*:

assumes *inj-on* $f \ A$ **and** $a0 \notin A$ **and** $\bigwedge a. a \in A \implies f \ a \neq f \ a0$

shows *inj-on* $f \ (\{a0\} \cup A)$

using *assms* **unfolding** *inj-on-def* **by** *fastforce*

lemma *UNlift01-inj-on*:

assumes l : $\text{length } cl = \text{length } dl$

and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$

and FP : $\bigwedge n. n < \text{length } dl \implies \text{part } \{..< \text{brn } (dl!n)\} (F \ n \ ' (P \ n)) \wedge \{\} \notin F \ n \ ' (P \ n)$

and F : $\bigwedge n. n < \text{length } cl \implies \text{inj-on } (F \ n) (P \ n)$

shows *inj-on* ($\text{UNlift01 } cl \ dl \ P \ F$) ($\text{UNpart01 } cl \ dl \ P$)

unfolding UNpart01-def **proof**(rule *inj-on-singl*)

```

show inj-on (UNlift01 cl dl P F) (UNpart1 cl dl P)
apply (rule UNlift01-inj-on-UNpart1) using assms by auto
next
show BrnFT cl dl  $\notin$  UNpart1 cl dl P
apply(rule BrnFT-notIn-UNpart1) using l P by auto
next
let ?Q = %n. F n ‘ (P n)
fix II assume II  $\in$  UNpart1 cl dl P
hence UNlift01 cl dl P F II  $\neq$  BrnFT dl cl
proof(cases rule: UNpart1-cases)
  case (Local n I)
    hence n: n  $\in$  theNFTBoth cl dl n  $\in$  theNFTBoth dl cl n < length cl n < length
dl
    and I: I  $\in$  P n and II: II = shift cl n I using l theNFTBoth-sym by auto
    have shift dl n (F n I)  $\in$  UNpart1 dl cl ?Q
    unfolding UNpart1-def shift-def using n I by auto
    hence shift dl n (F n I)  $\neq$  BrnFT dl cl
    using BrnFT-notIn-UNpart1[of dl cl ?Q] n l FP by auto
    thus ?thesis unfolding II using n I l P by simp
  qed
  thus UNlift01 cl dl P F II  $\neq$  UNlift01 cl dl P F (BrnFT cl dl) by simp
qed

lemma UNlift01-UNpart1:
assumes l: length cl = length dl
and P:  $\bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P n) \wedge \{\} \notin P n$ 
shows (UNlift01 cl dl P F) ‘ (UNpart1 cl dl P) = UNpart1 dl cl (%n. F n ‘ (P
n)) (is ?G ‘ ?Q = ?R)
proof safe
  fix II assume II: II  $\in$  ?Q
  thus ?G II  $\in$  ?R
  proof(cases rule: UNpart1-cases)
    case (Local n I)
      hence n: n  $\in$  theNFTBoth cl dl n  $\in$  theNFTBoth dl cl n < length cl
n < length dl and I: I  $\in$  P n
      and II: II = shift cl n I using l theNFTBoth-sym by auto
      hence G: ?G II = shift dl n (F n I) using l P by simp
      show ?thesis using n I unfolding G UNpart1-def by auto
    qed
  next
  fix JJ assume JJ: JJ  $\in$  ?R
  thus JJ  $\in$  ?G ‘ ?Q
  proof(cases rule: UNpart1-cases)
    case (Local n J)
      hence n: n  $\in$  theNFTBoth cl dl n  $\in$  theNFTBoth dl cl n < length cl n < length
dl
      and J: J  $\in$  F n ‘ (P n)
      and JJ: JJ = shift dl n J using l theNFTBoth-sym by auto
      then obtain I where I: I  $\in$  P n and J = F n I by auto

```

hence $JJ = \text{shift } dl \ n \ (F \ n \ I)$ **using** JJ **by** *simp*
 also have $\dots = \text{UNlift01 } cl \ dl \ P \ F \ (\text{shift } cl \ n \ I)$ **using** $n \ l \ P$ **by** *simp*
 finally have $JJ: JJ = \text{UNlift01 } cl \ dl \ P \ F \ (\text{shift } cl \ n \ I)$.
 show *?thesis* **using** $n \ l \ I$ **unfolding** JJ UNpart1-def **by** *auto*
qed
qed

lemma UNlift01-UNpart01 :
assumes $l: \text{length } cl = \text{length } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
shows $(\text{UNlift01 } cl \ dl \ P \ F) \ ' (\text{UNpart01 } cl \ dl \ P) = \text{UNpart01 } dl \ cl \ (\%n. F \ n \ ' (P \ n))$
using *assms* UNlift01-UNpart1 [of $cl \ dl \ P$] **unfolding** UNpart01-def **by** *auto*

lemma $\text{emp-UNlift01-UNpart1}$:
assumes $l: \text{length } cl = \text{length } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $FP: \bigwedge n. n < \text{length } dl \implies \{\} \notin F \ n \ ' (P \ n)$
shows $\{\} \notin (\text{UNlift01 } cl \ dl \ P \ F) \ ' (\text{UNpart1 } cl \ dl \ P)$ **(is** $\{\} \notin ?R$ **)**
proof –
 have $R: ?R = \text{UNpart1 } dl \ cl \ (\%n. F \ n \ ' (P \ n))$
 apply (rule UNlift01-UNpart1) **using** *assms* **by** *auto*
 show *?thesis* **unfolding** R **apply** (rule emp-UNpart1) **using** FP **by** *simp*
qed

lemma $\text{emp-UNlift01-UNpart01}$:
assumes $l: \text{length } cl = \text{length } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $FP: \bigwedge n. n < \text{length } dl \implies \{\} \notin F \ n \ ' (P \ n)$
shows $\{\} \notin (\text{UNlift01 } cl \ dl \ P \ F) \ ' (\text{UNpart01 } cl \ dl \ P - \{\text{BrnFT } cl \ dl\})$
(is $\{\} \notin ?U \ ' ?V$ **)**
proof –
 have $V: ?V = \text{UNpart1 } cl \ dl \ P$ **apply** (rule UNpart1-UNpart01 [THEN sym])
using *assms* **by** *auto*
 show *?thesis* **unfolding** V **apply** (rule $\text{emp-UNlift01-UNpart1}$)
using *assms* **by** *auto*
qed

lemma $\text{part-UNlift01-UNpart1}$:
assumes $l: \text{length } cl = \text{length } dl$ **and** $dl: \text{properL } dl$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $FP: \bigwedge n. n < \text{length } dl \implies \text{part } \{.. < \text{brn } (dl!n)\} (F \ n \ ' (P \ n))$
shows $\text{part } (\text{BrnNFT } dl \ cl) ((\text{UNlift01 } cl \ dl \ P \ F) \ ' (\text{UNpart1 } cl \ dl \ P))$ **(is** $\text{part } ?C$
 $?R$ **)**
proof –
 let $?Q = \%n. F \ n \ ' (P \ n)$
 have $R: ?R = \text{UNpart1 } dl \ cl \ ?Q$
 apply (rule UNlift01-UNpart1 [of $cl \ dl \ P \ F$]) **using** *assms* **by** *auto*
 show *?thesis* **unfolding** R **apply** (rule part-UNpart1) **using** $dl \ l \ FP$ **by** *auto*

qed

lemma *part-UNlift01-UNpart01*:

assumes l : $\text{length } cl = \text{length } dl$ **and** dl : *properL* dl

and P : $\bigwedge n. n < \text{length } cl \implies \text{part } \{..< \text{brn } (cl!n)\} (P\ n) \wedge \{\} \notin P\ n$

and FP : $\bigwedge n. n < \text{length } dl \implies \text{part } \{..< \text{brn } (dl!n)\} (F\ n\ ' (P\ n)) \wedge \{\} \notin (F\ n\ ' (P\ n))$

shows $\text{part } \{..< \text{brnL } dl\ (\text{length } dl)\} ((UNlift01\ cl\ dl\ P\ F)\ ' (UNpart01\ cl\ dl\ P))$
(**is** $\text{part } ?K\ ?R$)

proof –

let $?G = UNlift01\ cl\ dl\ P\ F$ **let** $?Q = \%n. F\ n\ ' (P\ n)$

have R : $?R = \{?G\ (BrnFT\ cl\ dl)\} \cup ?G\ ' (UNpart1\ cl\ dl\ P)$

unfolding *UNpart01-def* **by** *simp*

show *thesis* **unfolding** R **apply**(*rule* *part-Un-singl2*[*of* - - *BrnNFT* $dl\ cl$])

using *assms* *part-UNlift01-UNpart1*

apply(*force*, *force*)

using *assms* **apply** *simp* **apply**(*rule* *BrnFT-Int-UNpart1*[*of* $dl\ cl\ ?Q$])

apply(*force*, *force*) **using** *UNlift01-UNpart1* **by** *auto*

qed

lemma *diff-frac-eq-1*:

assumes $b \neq (0::\text{real})$

shows $1 - a / b = (b - a) / b$

by (*metis* *assms* *diff-divide-distrib* *divide-self-if*)

lemma *diff-frac-eq-2*:

assumes $b \neq (1::\text{real})$

shows $1 - (a - b) / (1 - b) = (1 - a) / (1 - b)$

(**is** $?L = ?R$)

proof –

have b : $1 - b \neq 0$ **using** *assms* **by** *simp*

hence $?L = (1 - b - (a - b)) / (1 - b)$ (**is** $?L = ?A / ?B$)

using *diff-frac-eq-1* **by** *blast*

also **have** $?A = 1 - a$ **by** *simp*

finally **show** *thesis* **by** *simp*

qed

lemma *triv-div-mult*:

assumes vSF : $vSF \neq (1::\text{real})$

and L : $L = (K - vSF) / (1 - vSF)$ **and** Ln : $L \neq 1$

shows $(VS / (1 - vSF) * V) / (1 - L) = (VS * V) / (1 - K)$

(**is** $?A = ?B$)

proof –

have $vSF-0$: $1 - vSF \neq 0$ **using** vSF **by** *simp*

{**assume** $K = 1$

hence $L = 1$ **using** $L\ vSF$ **by** *simp*

hence *False* **using** Ln **by** *simp*

}

hence $Kn: K \neq 1$ **by** *auto*
hence $K-0: 1 - K \neq 0$ **by** *simp*
have $1 - L = (1 - K) / (1 - vSF)$ **unfolding** L **using** vSF *diff-frac-eq-2* **by**
blast
hence $?A = (VS / (1 - vSF) * V) / ((1 - K) / (1 - vSF))$ **by** *simp*
also have $\dots = ?B$ **using** $vSF-0$ $K-0$ **by** *auto*
finally show *?thesis* .
qed

lemma *ss-wt-ParT-UNlift01*:

assumes $l: \text{length } cl = \text{length } dl$

and $cldl: \text{properL } cl \text{ properL } dl$ **and** $II: II \in \text{UNpart01 } cl \ dl \ P - \{\text{BrnFT } cl \ dl\}$

and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$

and $FP: \bigwedge n. n < \text{length } dl \implies \text{part } \{.. < \text{brn } (dl!n)\} (F \ n \ ' (P \ n))$

and $sw:$

$\bigwedge n \ I. \llbracket n < \text{length } cl; I \in P \ n \rrbracket \implies$

$\text{sum } (wt \ (cl \ ! \ n) \ s) \ I =$

$\text{sum } (wt \ (dl \ ! \ n) \ t) \ (F \ n \ I)$

and $st: s \approx t$

and $le1: \text{sum } (wt \ (\text{ParT } cl) \ s) \ (\text{BrnFT } cl \ dl) < 1$

$\text{sum } (wt \ (\text{ParT } dl) \ t) \ (\text{BrnFT } dl \ cl) < 1$

shows

$\text{sum } (wt \ (\text{ParT } cl) \ s) \ II /$

$(1 - \text{sum } (wt \ (\text{ParT } cl) \ s) \ (\text{BrnFT } cl \ dl)) =$

$\text{sum } (wt \ (\text{ParT } dl) \ t) \ (\text{UNlift01 } cl \ dl \ P \ F \ II) /$

$(1 - \text{sum } (wt \ (\text{ParT } dl) \ t) \ (\text{BrnFT } dl \ cl))$

(is $\text{sum } ?vP \ II / (1 - \text{sum } ?vP \ ?II-0) =$

$\text{sum } ?wP \ ?JJ / (1 - \text{sum } ?wP \ ?JJ-0))$

proof–

let $?N = \text{length } cl$

let $?vS = \%n. 1 / ?N$ **let** $?wS = \%n. 1 / (\text{length } dl)$

let $?vSF = \text{WtFT } cl$ **let** $?wSF = \text{WtFT } dl$

let $?ss = \%n. s$ **let** $?tt = \%n. t$

let $?v = \%n. wt \ (cl \ ! \ n) \ (?ss \ n)$ **let** $?w = \%n. wt \ (dl \ ! \ n) \ (?tt \ n)$

have $sstt: \bigwedge n. n < ?N \implies ?ss \ n \approx ?tt \ n$

using $st \ l$ **by** *auto*

have $vSwS: \bigwedge n. n < ?N \implies ?vS \ n = ?wS \ n$ **and** $sstt: \bigwedge n. n < ?N \implies ?ss \ n \approx ?tt \ n$

using $assms$ **by** *auto*

have $nf: ?vSF \neq 1 \ ?wSF \neq 1$ **using** $le1 \ cldl \ l$ **by** *auto*

have $theFT\text{-}theNFT[simp]:$

$\bigwedge n. n \in \text{theFT } dl - \text{theFT } cl \implies n < \text{length } cl \wedge \neg \text{finished } (cl \ ! \ n)$

$\bigwedge n. n \in \text{theFT } cl - \text{theFT } dl \implies n < \text{length } dl \wedge \neg \text{finished } (dl \ ! \ n)$

unfolding $theFT\text{-}def$ **using** l **by** *auto*

have $sum\text{-}v[simp]:$

$\bigwedge n. n < \text{length } cl \implies \text{sum } (?v \ n) \ \{.. < \text{brn } (cl \ ! \ n)\} = 1$

using $cldl$ **by** *auto*

```

have sum-w[simp]:
 $\bigwedge n. n < \text{length } dl \implies \text{sum } (?w \ n) \ \{.. < \text{brn } (dl \ ! \ n)\} = 1$ 
using cddl by auto
have theFTOne: theFTOne cl dl = theFT dl - theFT cl  $\cup$  theFT cl
theFTOne dl cl = theFT cl - theFT dl  $\cup$  theFT dl
unfolding theFTOne-def by blast+
have sum-vS-wS: sum ?vS (theFTOne cl dl) = sum ?wS (theFTOne dl cl)
unfolding theFTOne-sym[of cl dl] apply (rule sum.cong)
using vSwS l unfolding theFTOne-def theFT-def theNFT-def by auto

have II: II  $\in$  UNpart1 cl dl P using II l P UNpart1-UNpart01 by blast
thus ?thesis
proof(cases rule: UNpart1-cases)
  case (Local n I)
    hence n: n < ?N n < length dl n  $\in$  theNFTBoth cl dl
     $\neg$  finished (cl!n)  $\neg$  finished (dl!n)
    and I: I  $\in$  P n
    and II: II = shift cl n I using l by auto
    have I-sub: I  $\subseteq$  {.. < brn (cl!n)} using n I P unfolding part-def by blast
    hence FnI-sub: F n I  $\subseteq$  {.. < brn (dl!n)} using n I FP unfolding part-def by
blast
    have JJ: ?JJ = shift dl n (F n I)
    unfolding II using l P n I by simp

have sum ?vP ?II-0 =
sum (%n. sum ?vP {brnL cl n..<+brn (cl ! n)}) (theFTOne cl dl)
unfolding BrnFT-def apply(rule sum.UNION-disjoint)
using brnL-Int l by auto
also have ... =
sum
  (%n. sum ?vP {brnL cl n..<+brn (cl ! n)})
  ((theFT dl - theFT cl)  $\cup$  theFT cl) unfolding theFTOne-def
by (metis Un-Diff-cancel2 Un-commute)
also have ... =
sum
  (%n. sum ?vP {brnL cl n..<+brn (cl ! n)})
  (theFT dl - theFT cl) +
sum
  (%n. sum ?vP {brnL cl n..<+brn (cl ! n)})
  (theFT cl) (is ... = ?L + ?R)
apply(rule sum.union-disjoint) by auto
also have ?R = 0 apply(rule sum.neutral) using cddl nf by auto
finally have sum ?vP ?II-0 = ?L by simp
also have ?L =
sum
  (%n. ?vS n / (1 - ?vSF) * sum (?v n) {.. < brn (cl ! n)})
  (theFT dl - theFT cl)
apply(intro sum.cong) using cddl nf
using theFT-theNFT sum-wt-ParT-notWtFT-notFinished[of cl] by metis+

```

also have ... =
sum
 (%n. ?vS n * *sum* (?v n) {..
 (theFT dl - theFT cl) / (1 - ?vSF) (**is** ... = ?L / (1 - ?vSF))
unfolding times-divide-eq-left sum-divide-distrib **by** simp
also have ?L = *sum* ?vS (theFT dl - theFT cl)
apply(intro sum.cong) **by** auto
finally have
sum ?vP ?II-0 = (*sum* ?vS (theFT dl - theFT cl)) / (1 - ?vSF)
 (**is** ... = ?L / ?R) **by** simp
also have ?L = *sum* ?vS (theFTOne cl dl) - ?vSF
unfolding eq-diff-eq WtFT-def theFTOne
apply(rule sum.union-disjoint[THEN sym]) **by** auto
finally have vPII0: *sum* ?vP ?II-0 =
 (*sum* ?vS (theFTOne cl dl) - ?vSF) / (1 - ?vSF) **by** simp

have *sum* ?wP ?JJ-0 =
sum (%n. *sum* ?wP {brnL dl n..
unfolding BrnFT-def **apply**(rule sum.UNION-disjoint)
unfolding theFTOne-def theFT-def **apply** (force, force, clarify)
apply(rule brnL-Int) **using** l **by** auto
also have ... =

sum
 (%n. *sum* ?wP {brnL dl n..
 ((theFT cl - theFT dl) \cup theFT dl) **unfolding** theFTOne-def
by (metis Un-Diff-cancel2 Un-commute)

also have ... =
sum
 (%n. *sum* ?wP {brnL dl n..
 (theFT cl - theFT dl) +
sum
 (%n. *sum* ?wP {brnL dl n..
 (theFT dl) (**is** ... = ?L + ?R)
apply(rule sum.union-disjoint) **by** auto
also have ?R = 0 **apply**(rule sum.neutral) **using** cdl nf **by** auto
finally have *sum* ?wP ?JJ-0 = ?L **by** simp
also have ?L =

sum
 (%n. ?wS n / (1 - ?wSF) * *sum* (?w n) {..
 (theFT cl - theFT dl)
apply(intro sum.cong) **using** cdl nf
using theFT-theNFT sum-wt-ParT-notWtFT-notFinished[of dl] **by** metis+
also have ... =

sum
 (%n. ?wS n * *sum* (?w n) {..
 (theFT cl - theFT dl) / (1 - ?wSF) (**is** ... = ?L / (1 - ?wSF))
unfolding times-divide-eq-left sum-divide-distrib **by** simp
also have ?L = *sum* ?wS (theFT cl - theFT dl)

apply(*intro sum.cong*) **by** *auto*
finally have
 $sum\ ?wP\ ?JJ-0 = (sum\ ?wS\ (theFT\ cl - theFT\ dl)) / (1 - ?wSF)$
(is ... = ?L / ?R) by simp
also have $?L = sum\ ?wS\ (theFTOne\ dl\ cl) - ?wSF$
unfolding *eq-diff-eq WtFT-def theFTOne*
apply(*rule sum.union-disjoint[THEN sym]*) **by** *auto*
finally have *wPJJ0*: $sum\ ?wP\ ?JJ-0 =$
 $(sum\ ?wS\ (theFTOne\ dl\ cl) - ?wSF) / (1 - ?wSF)$ **by** *simp*

have $sum\ ?vP\ II / (1 - sum\ ?vP\ ?II-0) =$
 $(?vS\ n) / (1 - ?vSF) * (sum\ (?v\ n)\ I) / (1 - sum\ ?vP\ ?II-0)$
unfolding *II* **using** *n nf cldl I-sub* **by** *simp*
also have ... =
 $(?vS\ n) * (sum\ (?v\ n)\ I) / (1 - sum\ ?vS\ (theFTOne\ cl\ dl))$
using *nf(1) vPII0* **by** (*rule triv-div-mult*) (*insert le1, auto*)
also have ... =
 $(?wS\ n) * (sum\ (?w\ n)\ (F\ n\ I)) / (1 - sum\ ?wS\ (theFTOne\ dl\ cl))$
using *n vSwS[of n] sw[of n I] I* **unfolding** *sum-vS-wS* **by** *simp*
also have ... =
 $(?wS\ n) / (1 - ?wSF) * (sum\ (?w\ n)\ (F\ n\ I)) / (1 - sum\ ?wP\ ?JJ-0)$
using *nf(2) wPJJ0* **by** (*rule triv-div-mult[THEN sym]*) (*insert le1, auto*)
also have ... = $sum\ ?wP\ ?JJ / (1 - sum\ ?wP\ ?JJ-0)$
unfolding *JJ* **using** *n nf cldl FnI-sub* **by** *simp*
finally show *?thesis* .

qed
qed

definition *thetaZOParT* **where**

thetaZOParT \equiv
 $\{(ParT\ cl, ParT\ dl) \mid$
 $cl\ dl.$
 $properL\ cl \wedge properL\ dl \wedge SbisL\ cl\ dl\}$

lemma *cont-eff-ParT-BrnFT-L*:

assumes *l*: $length\ cl = length\ dl$

and *cldl*: $properL\ cl\ properL\ dl\ SbisL\ cl\ dl$

and *ii*: $ii \in BrnFT\ cl\ dl$

and *eff-cont*:

$\bigwedge n\ I\ i\ j. \llbracket n < length\ cl; I \in P\ n; i \in I; j \in F\ n\ I \rrbracket \implies$

$eff\ (cl!n)\ s\ i \approx eff\ (dl!n)\ t\ j \wedge$

$cont\ (cl!n)\ s\ i \approx_s\ cont\ (dl!n)\ t\ j$

shows

$s \approx eff\ (ParT\ cl)\ s\ ii \wedge$

$(cont\ (ParT\ cl)\ s\ ii, ParT\ dl) \in thetaZOParT$

(is $?eff \wedge ?cont$)

proof–

```

let ?N = length cl let ?p = %n. 1 / length cl let ?ss = %n. s
from ii show ?thesis
proof(cases rule: BrnFT-elim)
  case (Local n i)
    hence n: n ∈ theFTOne cl dl
    and i: i < brn (cl ! n) and ii: ii = brnL cl n + i by auto
    from n have n < length cl n < length dl using l cddl
    unfolding theFTOne-def theFT-def by auto note n = this n
    have discr: discr (cl!n)
    proof(cases finished (cl!n))
      case True
        thus ?thesis using n cddl discr-finished by auto
    next
      case False
        hence finished (dl!n) using n unfolding theFTOne-def theFT-def by auto
        moreover have proper (cl!n) and proper (dl!n) and cl!n ≈01 dl!n
        using n cddl by auto
        ultimately show ?thesis using ZObis-finished-dscr-L by blast
    qed
    hence eff: ?ss n ≈ eff (cl!n) (?ss n) i
    and cont: proper (cont (cl!n) (?ss n) i) ∧ discr (cont (cl!n) (?ss n) i)
    using i cddl n by auto
    show ?thesis
  proof
    have s ≈ eff (cl!n) (?ss n) i using eff n indis-trans by blast
    thus ?eff using i n cddl unfolding ii by simp
  next
    have cont (cl!n) (?ss n) i ≈s cl!n using discr cont cddl n by auto
    moreover have cl!n ≈s dl!n using cddl n by auto
    ultimately have cont (cl!n) (?ss n) i ≈s dl!n using Sbis-trans by blast
    thus ?cont using i n cddl unfolding ii thetaZOParT-def by auto
  qed
qed
qed

```

```

lemma cont-eff-ParT-BrnFT-R:
assumes l: length cl = length dl
and cddl: properL cl properL dl SbisL cl dl
and jj: jj ∈ BrnFT dl cl
and eff-cont:
  ∧ n I i j. [n < length cl; I ∈ P n; i ∈ I; j ∈ F n I] ⇒
    eff (cl!n) s i ≈ eff (dl!n) t j ∧ cont (cl!n) s i ≈s cont (dl!n) t j
shows
  t ≈ eff (ParT dl) t jj ∧
  (ParT cl, cont (ParT dl) t jj) ∈ thetaZOParT
(is ?eff ∧ ?cont)
proof –
  let ?N = length dl let ?q = %n. 1 / ?N let ?tt = %n. t
  from jj show ?thesis

```

proof (*cases rule: BrnFT-elim*)
case (*Local n j*)
hence $n: n \in \text{theFTOne } dl \ cl$
and $j: j < \text{brn } (dl \ ! \ n)$ **and** $jj: jj = \text{brnL } dl \ n \ + \ j$ **by** *auto*
from n **have** $n < \text{length } cl \ n < \text{length } dl$ **using** $l \ cddl$
unfolding *theFTOne-def theFT-def* **by** *auto* **note** $n = \text{this } n$
have *discr: discr (dl!n)*
proof (*cases finished (dl!n)*)
case *True*
thus *?thesis using n cddl discr-finished by auto*
next
case *False*
hence *finished (cl!n) using n unfolding theFTOne-def theFT-def by auto*
moreover **have** *proper (cl!n) and proper (dl!n) and cl!n ≈01 dl!n*
using $n \ cddl$ **by** *auto*
ultimately show *?thesis using ZObis-finished-dscr-R by blast*
qed
hence *eff: ?tt n ≈ eff (dl!n) (?tt n) j*
and *cont: proper (cont (dl!n) (?tt n) j) ∧ discr (cont (dl!n) (?tt n) j)*
using $j \ cddl \ n$ **by** *auto*
show *?thesis*
proof
have $t \approx \text{eff } (dl!n) \ (?tt \ n) \ j$ **using** *eff n indis-trans by blast*
thus *?eff using j n cddl unfolding jj by simp*
next
have $cl!n \approx_s \ dl!n$ **using** $cddl \ n$ **by** *auto*
moreover **have** $dl!n \approx_s \ \text{cont } (dl!n) \ (?tt \ n) \ j$ **using** *discr cont cddl n by auto*
ultimately have $cl!n \approx_s \ \text{cont } (dl!n) \ (?tt \ n) \ j$ **using** *Sbis-trans by blast*
thus *?cont using j n cddl unfolding jj thetaZOParT-def by simp*
qed
qed
qed

lemma *cont-eff-ParT-UNlift01:*
assumes $l: \text{length } cl = \text{length } dl$
and $cddl: \text{properL } cl \ \text{properL } dl \ \text{SbisL } cl \ dl$
and $II: II \in \text{UNpart01 } cl \ dl \ P \ - \ \{\text{BrnFT } cl \ dl\}$
and $ii: ii \in II$ **and** $jj: jj \in \text{UNlift01 } cl \ dl \ P \ F \ II$
and $P: \bigwedge n. n < \text{length } cl \implies \text{part } \{.. < \text{brn } (cl!n)\} (P \ n) \wedge \{\} \notin P \ n$
and $FP: \bigwedge n. n < \text{length } dl \implies \text{part } \{.. < \text{brn } (dl!n)\} (F \ n \ ' (P \ n))$
and *eff-cont:*
 $\bigwedge n \ I \ i \ j. \llbracket n < \text{length } cl; I \in P \ n; i \in I; j \in F \ n \ I \rrbracket \implies$
 $\text{eff } (cl!n) \ s \ i \approx$
 $\text{eff } (dl!n) \ t \ j \wedge$
 $\text{cont } (cl!n) \ s \ i \approx_s$
 $\text{cont } (dl!n) \ t \ j$
and $st: s \approx t$
shows
 $\text{eff } (\text{ParT } cl) \ s \ ii \approx \text{eff } (\text{ParT } dl) \ t \ jj \wedge$

```

(cont (ParT cl) s ii, cont (ParT dl) t jj) ∈ thetaZOParT
(is ?eff ∧ ?cont)
proof –
  let ?N = length cl
  let ?p = %n. 1 / ?N let ?q = %n. 1 / (length dl)
  let ?ss = %n. s let ?tt = %n. t
  have sstt: ∧ n. n < ?N ⇒ ?ss n ≈ ?tt n using st l by auto
  have pq: ∧ n. n < ?N ⇒ ?p n = ?q n and sstt: ∧ n. n < ?N ⇒ ?ss n ≈ ?tt
n
  using assms l by auto
  have II: II ∈ UNpart1 cl dl P using II l P UNpart1-UNpart01 by blast
  thus ?thesis
proof(cases rule: UNpart1-cases)
  case (Local n I)
  hence n: n < ?N n < length dl n ∈ theNFTBoth cl dl
  ¬ finished (cl!n) ¬ finished (dl!n)
  and I: I ∈ P n and II: II = shift cl n I using l by auto
  from ii II obtain i where i: i ∈ I and ii: ii = brnL cl n + i
  unfolding shift-def by auto
  have i < brn (cl!n) using i I n P unfolding part-def by blast note i = this i
  have jj: jj ∈ shift dl n (F n I) using jj P n I l unfolding II by simp
  from jj II obtain j where j: j ∈ F n I and jj: jj = brnL dl n + j
  unfolding shift-def by auto
  have j < brn (dl!n) using j I n FP unfolding part-def by blast note j = this
j
  show ?thesis
proof
  have eff (cl!n) (?ss n) i ≈ eff (dl!n) (?tt n) j
  using n I i j eff-cont by blast
  thus ?eff unfolding ii jj using st cddl n i j by simp
next
  have 1: cont (cl!n) (?ss n) i ≈s cont (dl!n) (?tt n) j
  using n I i j eff-cont by blast
  have (cont (ParT cl) s ii, cont (ParT dl) t jj) =
    (ParT (cl[n := cont (cl ! n) (?ss n) i]),
     ParT (dl[n := cont (dl ! n) (?tt n) j]))
  (is ?A = ?B)
  unfolding ii jj using n i j cddl by simp
  moreover have ?B ∈ thetaZOParT
  unfolding thetaZOParT-def apply (simp, safe)
  apply(intro properL-update)
  using cddl apply force
  apply(rule proper-cont) using cddl i n apply (force,force)
  apply(intro properL-update)
  using cddl apply force
  apply(rule proper-cont) using cddl j n apply (force,force)
  apply(intro SbisL-update)
  using 1 cddl n i apply (force,force)
  done

```

```

    ultimately show ?cont by auto
  qed
  qed
  qed

lemma thetaZOParT-ZOretr: thetaZOParT  $\subseteq$  ZOretr (thetaZOParT)
unfolding ZOretr-def matchC-LC-def proof safe
  fix c d s t
  assume c-d: (c, d)  $\in$  thetaZOParT and st: s  $\approx$  t
  then obtain cl dl where
    c: c = ParT cl and d: d = ParT dl and
    cdl: properL cl properL dl SbisL cl dl
  unfolding thetaZOParT-def by blast
  let ?N = length cl
  let ?ss = %n. s let ?tt = %n. t
  have N: ?N = length dl using cdl by simp
  have sstt:  $\bigwedge n. n < ?N \implies ?ss n \approx ?tt n$  using st N by auto
  let ?phi = %n PFn. mC-C Sbis (cl ! n) (dl ! n) (?ss n) (?tt n) (fst PFn) (snd
  PFn)
  {fix n assume n: n < ?N
    hence cl ! n  $\approx$  s dl ! n using cdl by auto
    hence  $\exists$  PFn. ?phi n PFn using n Sbis-mC-C sstt by fastforce
  }
  then obtain PF where phi:  $\bigwedge n. n < ?N \implies ?phi n (PF n)$ 
  using bchoice[of {.. $?N$ } ?phi] by blast
  define P F where P = fst o PF and F = snd o PF
  have m:  $\bigwedge n. n < ?N \implies mC-C Sbis (cl ! n) (dl ! n) (?ss n) (?tt n) (P n) (F
  n)$ 
  using phi unfolding P-def F-def by auto

  have brn-c: brn c = brnL cl ?N unfolding c by simp
  have brn-d: brn d = brnL dl (length dl) unfolding d by simp
  have P:  $\bigwedge n. n < ?N \implies part \{.. $brn (cl ! n)\} (P n) \wedge \{\} \notin (P n)$ 
  using m unfolding m-defsAll part-def by auto
  have FP:  $\bigwedge n. n < length dl \implies part \{.. $brn (dl ! n)\} (F n ' (P n)) \wedge \{\} \notin F
  n ' (P n)$ 
  using m N unfolding m-defsAll part-def by auto
  have F:  $\bigwedge n. n < ?N \implies inj-on (F n) (P n)$  using m unfolding m-defsAll by
  auto
  have sw:  $\bigwedge n I. \llbracket n < length cl; I \in P n \rrbracket \implies$ 
    sum (wt (cl ! n) (?ss n)) I = sum (wt (dl ! n) (?tt n)) (F n I)
  using m unfolding mC-C-def mC-C-wt-def by auto
  have eff-cont:  $\bigwedge n I i j. \llbracket n < length cl; I \in P n; i \in I; j \in F n I \rrbracket \implies$ 
    eff (cl ! n) (?ss n) i  $\approx$  eff (dl ! n) (?tt n) j  $\wedge$  cont (cl ! n) (?ss n) i  $\approx$  s cont (dl ! n)
    (?tt n) j
  using m unfolding mC-C-def mC-C-eff-cont-def by auto

  define II0 where II0 = BrnFT cl dl
  define Q G where Q = UNpart01 cl dl P and G = UNlift01 cl dl P F$$ 
```



```

note defi = II0-def Q-def G-def brn-c brn-d
show  $\exists II0 Q G. mC-ZOC (thetaZOParT) c d s t II0 Q G$ 
apply(rule exI[of - II0]) apply(rule exI[of - Q]) apply(rule exI[of - G])
unfolding mC-ZOC-def proof (intro conjI)
  show mC-ZOC-part c d s t II0 Q G unfolding mC-ZOC-part-def proof(intro
conjI)
  show  $\{\} \notin Q - \{II0\}$  unfolding defi apply(rule emp-UNpart01) using P
by simp
  show  $\{\} \notin G \text{ ' } (Q - \{II0\})$  unfolding defi
apply(rule emp-UNlift01-UNpart01) using N P FP by auto
  show  $II0 \in Q$  unfolding defi by simp
  show part  $\{..<brn c\} Q$ 
  unfolding defi apply(rule part-UNpart01) using cldl P by auto
  show part  $\{..<brn d\} (G \text{ ' } Q)$ 
  unfolding defi apply(rule part-UNlift01-UNpart01) using N cldl P FP by
auto
  qed
next
  show inj-on G Q
  unfolding defi apply(rule UNlift01-inj-on) using N P FP F by auto
next
  show mC-ZOC-wt c d s t II0 Q G
  unfolding mC-ZOC-wt-def proof (intro impI ballI, elim conjE)
  fix II assume II:  $II \in Q - \{II0\}$  and
  le1:  $sum (wt c s) II0 < 1$   $sum (wt d t) (G II0) < 1$ 
  thus
   $sum (wt c s) II / (1 - sum (wt c s) II0) =$ 
   $sum (wt d t) (G II) / (1 - sum (wt d t) (G II0))$ 
  unfolding c d defi UNlift01-BrnFT apply(intro ss-wt-ParT-UNlift01)
using N cldl II P FP sw st by auto
  qed
next
  show mC-ZOC-eff-cont0 (thetaZOParT) c d s t II0 G
  unfolding mC-ZOC-eff-cont0-def
  proof(intro conjI[OF ballI ballI])
  fix ii assume ii  $\in II0$  thus  $s \approx eff c s ii \wedge (cont c s ii, d) \in thetaZOParT$ 
  unfolding defi c d apply(intro cont-eff-ParT-BrnFT-L)
  using N cldl P FP eff-cont st by (auto intro!: )
  next
  fix jj assume jj  $\in G II0$ 
  hence  $jj \in BrnFT dl cl$  unfolding defi UNlift01-BrnFT by simp
  thus  $t \approx eff d t jj \wedge (c, cont d t jj) \in thetaZOParT$ 
  unfolding defi c d apply(intro cont-eff-ParT-BrnFT-R)
  using N cldl P FP eff-cont st by auto
  qed
next
  show mC-ZOC-eff-cont (thetaZOParT) c d s t II0 Q G
  unfolding mC-ZOC-eff-cont-def proof (intro allI impI, elim conjE)
  fix II ii jj assume II:  $II \in Q - \{II0\}$  and ii:  $ii \in II$  and jj:  $jj \in G II$ 

```

```

thus  $\text{eff } c \text{ s } ii \approx \text{eff } d \text{ t } jj \wedge (\text{cont } c \text{ s } ii, \text{cont } d \text{ t } jj) \in \text{thetaZOParT}$ 
unfolding  $\text{defi } c \text{ d}$  apply( $\text{intro cont-eff-ParT-UNlift01}$ )
using  $N \text{ cldl } P \text{ FP eff-cont st}$  by  $\text{auto}$ 
qed
qed
qed

```

```

lemma  $\text{thetaZOParT-ZObis: thetaZOParT} \subseteq \text{ZObis}$ 
using  $\text{ZObis-raw-coind thetaZOParT-ZOretr}$  by  $\text{auto}$ 

```

```

theorem  $\text{ParT-ZObis[simp]}$ :
assumes  $\text{properL } cl$  and  $\text{properL } dl$  and  $\text{SbisL } cl \text{ dl}$ 
shows  $\text{ParT } cl \approx_{01} \text{ParT } dl$ 
using  $\text{assms thetaZOParT-ZObis}$  unfolding  $\text{thetaZOParT-def}$  by  $\text{blast}$ 

```

end

end

5 Syntactic Criteria

```

theory  $\text{Syntactic-Criteria}$ 
imports  $\text{Compositionality}$ 
begin

```

```

context  $\text{PL-Indis}$ 
begin

```

```

lemma  $\text{proper-intros[intro]}$ :
 $\text{proper Done}$ 
 $\text{proper (Atm atm)}$ 
 $\text{proper } c1 \implies \text{proper } c2 \implies \text{proper (Seq } c1 \text{ } c2)$ 
 $\text{proper } c1 \implies \text{proper } c2 \implies \text{proper (Ch } ch \text{ } c1 \text{ } c2)$ 
 $\text{proper } c \implies \text{proper (While } tst \text{ } c)$ 
 $\text{properL } cs \implies \text{proper (Par } cs)$ 
 $\text{properL } cs \implies \text{proper (ParT } cs)$ 
 $(\bigwedge c. c \in \text{set } cs \implies \text{proper } c) \implies cs \neq [] \implies \text{properL } cs$ 
by  $\text{auto}$ 

```

```

lemma  $\text{discr}$ :
 $\text{discr Done}$ 
 $\text{presAtm } atm \implies \text{discr (Atm } atm)$ 
 $\text{discr } c1 \implies \text{discr } c2 \implies \text{discr (Seq } c1 \text{ } c2)$ 
 $\text{discr } c1 \implies \text{discr } c2 \implies \text{discr (Ch } ch \text{ } c1 \text{ } c2)$ 
 $\text{discr } c \implies \text{discr (While } tst \text{ } c)$ 
 $\text{properL } cs \implies (\bigwedge c. c \in \text{set } cs \implies \text{discr } c) \implies \text{discr (Par } cs)$ 

```

$properL\ cs \implies (\bigwedge c. c \in set\ cs \implies discr\ c) \implies discr\ (ParT\ cs)$
by (auto intro!: discr-Par discr-ParT)

lemma *siso*:

$compatAtm\ atm \implies siso\ (Atm\ atm)$
 $siso\ c1 \implies siso\ c2 \implies siso\ (Seq\ c1\ c2)$
 $compatCh\ ch \implies siso\ c1 \implies siso\ c2 \implies siso\ (Ch\ ch\ c1\ c2)$
 $compatTst\ tst \implies siso\ c \implies siso\ (While\ tst\ c)$
 $properL\ cs \implies (\bigwedge c. c \in set\ cs \implies siso\ c) \implies siso\ (Par\ cs)$
 $properL\ cs \implies (\bigwedge c. c \in set\ cs \implies siso\ c) \implies siso\ (ParT\ cs)$
by (auto intro!: siso-Par siso-ParT)

lemma *Sbis*:

$compatAtm\ atm \implies Atm\ atm \approx_s Atm\ atm$
 $siso\ c1 \implies c2 \approx_s c2 \implies Seq\ c1\ c2 \approx_s Seq\ c1\ c2$
 $proper\ c1 \implies proper\ c2 \implies c1 \approx_s c1 \implies discr\ c2 \implies Seq\ c1\ c2 \approx_s Seq\ c1\ c2$
 $compatCh\ ch \implies c1 \approx_s c1 \implies c2 \approx_s c2 \implies Ch\ ch\ c1\ c2 \approx_s Ch\ ch\ c1\ c2$
 $properL\ cs \implies (\bigwedge c. c \in set\ cs \implies c \approx_s c) \implies Par\ cs \approx_s Par\ cs$
by (auto intro!: Par-Sbis)

lemma *ZObis*:

$compatAtm\ atm \implies Atm\ atm \approx_{01} Atm\ atm$
 $siso\ c1 \implies c2 \approx_{01} c2 \implies Seq\ c1\ c2 \approx_{01} Seq\ c1\ c2$
 $proper\ c1 \implies proper\ c2 \implies c1 \approx_{01} c1 \implies discr\ c2 \implies Seq\ c1\ c2 \approx_{01} Seq\ c1\ c2$
 $compatCh\ ch \implies c1 \approx_{01} c1 \implies c2 \approx_{01} c2 \implies Ch\ ch\ c1\ c2 \approx_{01} Ch\ ch\ c1\ c2$
 $properL\ cs \implies (\bigwedge c. c \in set\ cs \implies c \approx_s c) \implies ParT\ cs \approx_{01} ParT\ cs$
by (auto intro!: ParT-ZObis)

lemma *discr-imp-Sbis*: $proper\ c \implies discr\ c \implies c \approx_s c$
by auto

lemma *siso-imp-Sbis*: $siso\ c \implies c \approx_s c$
by auto

lemma *Sbis-imp-ZObis*: $c \approx_s c \implies c \approx_{01} c$
by auto

fun *SC-discr* **where**

$SC-discr\ Done \iff True$
 $| SC-discr\ (Atm\ atm) \iff presAtm\ atm$
 $| SC-discr\ (Seq\ c1\ c2) \iff SC-discr\ c1 \wedge SC-discr\ c2$
 $| SC-discr\ (Ch\ ch\ c1\ c2) \iff SC-discr\ c1 \wedge SC-discr\ c2$
 $| SC-discr\ (While\ tst\ c) \iff SC-discr\ c$
 $| SC-discr\ (ParT\ cs) \iff (\forall c \in set\ cs. SC-discr\ c)$
 $| SC-discr\ (Par\ cs) \iff (\forall c \in set\ cs. SC-discr\ c)$

theorem *SC-discr-discr*[intro]: $\text{proper } c \implies \text{SC-discr } c \implies \text{discr } c$
by (*induct c*) (*auto intro!*: *discr*)

fun *SC-iso* **where**

SC-iso Done $\longleftrightarrow \text{True}$
| *SC-iso (Atm atm)* $\longleftrightarrow \text{compatAtm atm}$
| *SC-iso (Seq c1 c2)* $\longleftrightarrow \text{SC-iso } c1 \wedge \text{SC-iso } c2$
| *SC-iso (Ch ch c1 c2)* $\longleftrightarrow \text{compatCh ch} \wedge \text{SC-iso } c1 \wedge \text{SC-iso } c2$
| *SC-iso (While tst c)* $\longleftrightarrow \text{compatTst tst} \wedge \text{SC-iso } c$
| *SC-iso (Par cs)* $\longleftrightarrow (\forall c \in \text{set } cs. \text{SC-iso } c)$
| *SC-iso (ParT cs)* $\longleftrightarrow (\forall c \in \text{set } cs. \text{SC-iso } c)$

theorem *SC-iso-iso*[intro]: $\text{proper } c \implies \text{SC-iso } c \implies \text{iso } c$
by (*induct c*) (*auto intro!*: *iso*)

fun *SC-Sbis* **where**

SC-Sbis Done $\longleftrightarrow \text{True}$
| *SC-Sbis (Atm atm)* $\longleftrightarrow \text{compatAtm atm}$
| *SC-Sbis (Seq c1 c2)* $\longleftrightarrow (\text{SC-iso } c1 \wedge \text{SC-Sbis } c2) \vee$
 $(\text{SC-Sbis } c1 \wedge \text{SC-discr } c2) \vee$
 $\text{SC-discr } (\text{Seq } c1 \text{ } c2) \vee \text{SC-iso } (\text{Seq } c1 \text{ } c2)$
| *SC-Sbis (Ch ch c1 c2)* $\longleftrightarrow (\text{if } \text{compatCh } ch$
 $\text{then } \text{SC-Sbis } c1 \wedge \text{SC-Sbis } c2$
 $\text{else } (\text{SC-discr } (\text{Ch } ch \text{ } c1 \text{ } c2) \vee \text{SC-iso } (\text{Ch } ch \text{ } c1 \text{ } c2)))$
| *SC-Sbis (While tst c)* $\longleftrightarrow \text{SC-discr } (\text{While } tst \text{ } c) \vee \text{SC-iso } (\text{While } tst \text{ } c)$
| *SC-Sbis (Par cs)* $\longleftrightarrow (\forall c \in \text{set } cs. \text{SC-Sbis } c)$
| *SC-Sbis (ParT cs)* $\longleftrightarrow \text{SC-iso } (\text{ParT } cs) \vee \text{SC-discr } (\text{ParT } cs)$

theorem *SC-iso-SCbis*[intro]: $\text{SC-iso } c \implies \text{SC-Sbis } c$
by (*induct c*) *auto*

theorem *SC-discr-SCbis*[intro]: $\text{SC-discr } c \implies \text{SC-Sbis } c$
by (*induct c*) *auto*

declare *SC-iso.simps*[*simp del*]

declare *SC-discr.simps*[*simp del*]

theorem *SC-Sbis-Sbis*[intro]: $\text{proper } c \implies \text{SC-Sbis } c \implies c \approx_s c$
by (*induct c*)
(*auto intro: Sbis discr-imp-Sbis iso-imp-Sbis*
split: if-split-asm)

fun *SC-ZObis* **where**

SC-ZObis Done $\longleftrightarrow \text{True}$
| *SC-ZObis (Atm atm)* $\longleftrightarrow \text{compatAtm atm}$
| *SC-ZObis (Seq c1 c2)* $\longleftrightarrow (\text{SC-iso } c1 \wedge \text{SC-ZObis } c2) \vee$
 $(\text{SC-ZObis } c1 \wedge \text{SC-discr } c2) \vee$

```

      SC-Sbis (Seq c1 c2)
| SC-ZObis (Ch ch c1 c2)  $\longleftrightarrow$  (if compatCh ch
      then SC-ZObis c1  $\wedge$  SC-ZObis c2
      else SC-Sbis (Ch ch c1 c2))
| SC-ZObis (While tst c)  $\longleftrightarrow$  SC-Sbis (While tst c)
| SC-ZObis (Par cs)  $\longleftrightarrow$  SC-Sbis (Par cs)
| SC-ZObis (ParT cs)  $\longleftrightarrow$  ( $\forall c \in \text{set } cs. \text{SC-Sbis } c$ )

```

theorem *SC-Sbis-SC-ZObis[intro]*: $\text{SC-Sbis } c \implies \text{SC-ZObis } c$
by (*induct c*) (*auto simp: SC-iso.simps SC-discr.simps*)

declare *SC-Sbis.simps[simp del]*

theorem *SC-ZObis-ZObis: proper* $c \implies \text{SC-ZObis } c \implies c \approx_{01} c$
apply (*induct c*)
apply (*auto intro: Sbis-imp-ZObis ZObis split: if-split-asm*)
apply (*auto intro!: ZObis(5)*)
done

end

end

6 Concrete setting

theory *Concrete*
imports *Syntactic-Criteria*
begin

datatype *level* = *Lo* | *Hi*

lemma [*simp*]: $\bigwedge l. l \neq \text{Hi} \longleftrightarrow l = \text{Lo}$ **and**
 [*simp*]: $\bigwedge l. \text{Hi} \neq l \longleftrightarrow \text{Lo} = l$ **and**
 [*simp*]: $\bigwedge l. l \neq \text{Lo} \longleftrightarrow l = \text{Hi}$ **and**
 [*simp*]: $\bigwedge l. \text{Lo} \neq l \longleftrightarrow \text{Hi} = l$
by (*metis level.exhaust level.simps(2)*)**+**

lemma [*dest*]: $\bigwedge l A. \llbracket l \in A; \text{Lo} \notin A \rrbracket \implies l = \text{Hi}$ **and**
 [*dest*]: $\bigwedge l A. \llbracket l \in A; \text{Hi} \notin A \rrbracket \implies l = \text{Lo}$
by (*metis level.exhaust*)**+**

declare *level.split[split]*

instantiation *level* :: *complete-lattice*
begin
definition *top-level*: *top* $\equiv \text{Hi}$

definition *bot-level*: $bot \equiv Lo$
definition *inf-level*: $inf\ l1\ l2 \equiv if\ Lo \in \{l1, l2\}\ then\ Lo\ else\ Hi$
definition *sup-level*: $sup\ l1\ l2 \equiv if\ Hi \in \{l1, l2\}\ then\ Hi\ else\ Lo$
definition *less-eq-level*: $less-eq\ l1\ l2 \equiv (l1 = Lo \vee l2 = Hi)$
definition *less-level*: $less\ l1\ l2 \equiv l1 = Lo \wedge l2 = Hi$
definition *Inf-level*: $Inf\ L \equiv if\ Lo \in L\ then\ Lo\ else\ Hi$
definition *Sup-level*: $Sup\ L \equiv if\ Hi \in L\ then\ Hi\ else\ Lo$

instance
proof **qed** (*auto simp: top-level bot-level inf-level sup-level less-eq-level less-level Inf-level Sup-level*)

end

lemma *sup-eq-Lo[simp]*: $sup\ a\ b = Lo \longleftrightarrow a = Lo \wedge b = Lo$
by (*auto simp: sup-level*)

datatype *var* = $h \mid h' \mid l \mid l'$
datatype *exp* = $Ct\ nat \mid Var\ var \mid Plus\ exp\ exp \mid Minus\ exp\ exp$
datatype *test* = $Tr \mid Eq\ exp\ exp \mid Gt\ exp\ exp \mid Non\ test$
datatype *atom* = $Assign\ var\ exp$
type-synonym *choice* = $real + test$
type-synonym *state* = $var \Rightarrow nat$

syntax
-assign $:: 'a \Rightarrow 'a \Rightarrow 'a\ (- ::= - [1000, 61]\ 61)$

translations
 $x ::= expr == CONST\ Atm\ (CONST\ Assign\ x\ expr)$

primrec *sec* **where**
 $sec\ h = Hi$
 $| sec\ h' = Hi$
 $| sec\ l = Lo$
 $| sec\ l' = Lo$

fun *eval* **where**
 $eval\ (Ct\ n)\ s = n$
 $| eval\ (Var\ x)\ s = s\ x$
 $| eval\ (Plus\ e1\ e2)\ s = eval\ e1\ s + eval\ e2\ s$
 $| eval\ (Minus\ e1\ e2)\ s = eval\ e1\ s - eval\ e2\ s$

fun *tval* **where**
 $tval\ Tr\ s = True$
 $| tval\ (Eq\ e1\ e2)\ s = (eval\ e1\ s = eval\ e2\ s)$
 $| tval\ (Gt\ e1\ e2)\ s = (eval\ e1\ s > eval\ e2\ s)$
 $| tval\ (Non\ e)\ s = (\neg\ tval\ e\ s)$

fun *aval* **where**
 $aval\ (Assign\ x\ e)\ s = (s\ (x := eval\ e\ s))$

fun *cval* **where**

cval (*Inl* *p*) *s* = *min* 1 (*max* 0 *p*)
| *cval* (*Inr* *tst*) *s* = (*if* *tval* *tst* *s* *then* 1 *else* 0)

definition *indis* :: (*state* * *state*) **setwhere**

indis ≡ {(*s*,*t*). *ALL* *x*. *sec* *x* = *Lo* → *s* *x* = *t* *x*}

interpretation *Example-PL*: *PL-Indis* *aval* *tval* *cval* *indis*

proof

fix *ch* :: *choice* **and** *s* **show** $0 \leq \text{cval } ch \ s \wedge \text{cval } ch \ s \leq 1$
by (*cases* *ch*) *auto*
next
show *equiv UNIV indis*
unfolding *refl-on-def sym-def trans-def equiv-def indis-def* **by** *auto*
qed

fun *exprSec* **where**

exprSec (*Ct* *n*) = *Lo*
| *exprSec* (*Var* *x*) = *sec* *x*
| *exprSec* (*Plus* *e1* *e2*) = *sup* (*exprSec* *e1*) (*exprSec* *e2*)
| *exprSec* (*Minus* *e1* *e2*) = *sup* (*exprSec* *e1*) (*exprSec* *e2*)

fun *tstSec* **where**

tstSec *Tr* = *Lo*
| *tstSec* (*Eq* *e1* *e2*) = *sup* (*exprSec* *e1*) (*exprSec* *e2*)
| *tstSec* (*Gt* *e1* *e2*) = *sup* (*exprSec* *e1*) (*exprSec* *e2*)
| *tstSec* (*Non* *e*) = *tstSec* *e*

lemma *exprSec-Lo-eval-eq*: *exprSec* *expr* = *Lo* ⇒ (*s*, *t*) ∈ *indis* ⇒ *eval* *expr* *s*
= *eval* *expr* *t*

by (*induct* *expr*) (*auto simp: indis-def*)

lemma *compatAtmSyntactic[simp]*: *exprSec* *expr* = *Lo* ∨ *sec* *v* = *Hi* ⇒ *Example-PL.compatAtm* (*Assign* *v* *expr*)

unfolding *Example-PL.compatAtm-def*

by (*induct* *expr*)

(*auto simp: indis-def intro!: arg-cong2[where f=(+)] arg-cong2[where f=(-)]
exprSec-Lo-eval-eq*)

lemma *presAtmSyntactic[simp]*: *sec* *v* = *Hi* ⇒ *Example-PL.presAtm* (*Assign* *v* *expr*)

unfolding *Example-PL.presAtm-def* **by** (*simp add: indis-def*)

lemma *compatTstSyntactic[simp]*: *tstSec* *tst* = *Lo* ⇒ *Example-PL.compatTst* *tst*

unfolding *Example-PL.compatTst-def*

by (*induct* *tst*)

(*simp-all, safe del: iffI*)

intro!: $\text{arg-cong2}[\mathbf{where} f=(=)] \text{arg-cong2}[\mathbf{where} f=(<) :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}] \text{exprSec-Lo-eval-eq}$

lemma *compatPrchSyntactic*[simp]: *Example-PL.compatCh* (Inl p)
unfolding *Example-PL.compatCh-def* **by** *auto*

lemma *compatIfchSyntactic*[simp]: *Example-PL.compatCh* (Inr tst) \longleftrightarrow *Example-PL.compatTst* tst
unfolding *Example-PL.compatCh-def* *Example-PL.compatTst-def* **by** *auto*

abbreviation *Ch-half* ($Ch_{1/2}$) **where** $Ch_{1/2} \equiv Ch$ (Inl (1/2))
abbreviation *If* **where** $If\ tst \equiv Ch$ (Inr tst)

abbreviation *siso* c $\equiv Example-PL.siso$ c
abbreviation *discr* c $\equiv Example-PL.discr$ c
abbreviation *Sbis-abbrev* (**infix** \approx_s 55) **where** $c1 \approx_s c2 \equiv (c1, c2) \in Example-PL.Sbis$
abbreviation *ZObis-abbrev* (**infix** \approx_{01} 55) **where** $c1 \approx_{01} c2 \equiv (c1, c2) \in Example-PL.ZObis$

abbreviation *SC-siso* c $\equiv Example-PL.SC-siso$ c
abbreviation *SC-discr* c $\equiv Example-PL.SC-discr$ c
abbreviation *SC-Sbis* c $\equiv Example-PL.SC-Sbis$ c
abbreviation *SC-ZObis* c $\equiv Example-PL.SC-ZObis$ c

lemma *SC-discr* (h ::= Ct 0)
by (*simp add: Example-PL.SC-discr.simps*)

6.1 The secure programs from the paper's Example 3

definition [simp]: $d0 =$
 $h' ::= Ct\ 0$;;
While (*Gt* (*Var* h) (*Ct* 0))
($Ch_{1/2}$ (h ::= Ct 0)
($h' ::= Plus$ (*Var* h') (*Ct* 1)))

definition [simp]: $d1 =$
While (*Gt* (*Var* h) (*Ct* 0))
($Ch_{1/2}$ (h ::= *Minus* (*Var* h) (*Ct* 1))
(h ::= *Plus* (*Var* h) (*Ct* 1)))

definition [simp]: $d2 =$
If (*Eq* (*Var* l) (*Ct* 0))
($l' ::= Ct\ 1$)
d0

definition [simp]: $d3 =$


```
h ::= Ct 5 ;;  
ParT [d0, (l ::= Ct 1)]
```

theorem *SC-discr d0*

SC-discr d1

SC-Sbis d2

SC-ZObis d2

by (*auto simp: Example-PL.SC-discr.simps Example-PL.SC-Sbis.simps Example-PL.SC-ZObis.simps*)

theorem *discr d0*

discr d1

d2 ≈_s d2

d3 ≈₀₁ d3

by (*auto intro!: compatAtmSyntactic*

Example-PL.ZObis Example-PL.proper-intros

Example-PL.Atm-Sbis)

end

References

- [1] A. Popescu, J. Hölzl, and T. Nipkow. Formalizing probabilistic noninterference. In G. Gonthier and M. Norrish, editors, *Certified Programs and Proofs (CPP 2013)*, volume 8307 of *LNCS*, pages 259–275. Springer, 2013.
- [2] A. Popescu, J. Hölzl, and T. Nipkow. Noninterfering schedulers. In R. Heckel and S. Milius, editors, *Algebra and Coalgebra in Computer Science (CALCO 2013)*, volume 8089 of *LNCS*, pages 236–252. Springer, 2013.