

The Five Platonic Solids

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Abstract

This entry formalizes that there are exactly five Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. A Platonic solid is a convex polyhedron whose faces are congruent regular polygons, with the same number of edges meeting at each vertex. Euler’s polyhedron formula is used to show that there are at most five Platonic solids, and each of the five, defined as the convex hull of its vertices, is then shown to satisfy the required properties. The formal proof is computational (slow!) and is ported from John Harrison’s formalization in HOL Light [1]. Large language models were used to explain the intermediate steps of the HOL Light formalization and to write parts of the more complicated proofs.

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1 Convexity

theory *Convex-Euclidean-Space-More*
imports *HOL-Analysis.Starlike*
begin

lemma *connected-Int-rel-frontier*:
assumes *connected S*
and $S \subseteq \text{affine hull } T$
and $S \cap T \neq \{\}$
and $S - T \neq \{\}$
shows $S \cap \text{rel-frontier } T \neq \{\}$
 $\langle \text{proof} \rangle$

end
theory *Convex-More*
imports *HOL-Analysis.Starlike*
begin

lemma *convex-hull-union-explicit*:
fixes $S\ T :: 'a::\text{euclidean-space set}$
assumes *convex S*
and *convex T*
shows $\text{convex hull } (S \cup T) =$
 $S \cup T \cup$
 $\{(1 - u) *_R x + u *_R y \mid x\ y\ u.\ x \in S \wedge y \in T \wedge 0 \leq u \wedge u \leq 1\}$
(is ?lhs = ?rhs)
 $\langle \text{proof} \rangle$

lemma *convex-hull-union-nonempty-explicit*:
fixes $S\ T :: 'a::\text{euclidean-space set}$
assumes *convex S*
and $S \neq \{\}$
and *convex T*
and $T \neq \{\}$
shows $\text{convex hull } (S \cup T) =$
 $\{(1 - u) *_R x + u *_R y \mid x\ y\ u.\ x \in S \wedge y \in T \wedge 0 \leq u \wedge u \leq 1\}$
(is - = ?rhs)
 $\langle \text{proof} \rangle$

lemma *convex-hull-isometry*:
assumes *orthogonal-transformation f*
shows $(\lambda x.\ c + f\ x) ` (\text{convex hull } S) = \text{convex hull } ((\lambda x.\ c + f\ x) ` S)$
 $\langle \text{proof} \rangle$

end

2 Determinants

theory *Determinants-More*
imports *HOL-Analysis.Starlike*
begin

lemma *hyperplane-subset-imp-scaled*:
fixes $a :: 'a::euclidean-space$
assumes $\{x. a \cdot x = b\} \subseteq \{x. a' \cdot x = b'\}$
and $a \neq 0$
and $a' \neq 0$
shows $\exists c. a' = c *_R a \wedge b' = c * b \wedge c \neq 0$
 $\langle proof \rangle$

lemma *subset-hyperplanes*:
fixes $a :: 'a::euclidean-space$
shows $(\{x. a \cdot x = b\} \subseteq \{x. a' \cdot x = b'\}) =$
 $(\{x. a \cdot x = b\} = \{\} \vee \{x. a' \cdot x = b'\} = UNIV \vee$
 $\{x. a \cdot x = b\} = \{x. a' \cdot x = b'\})$
(is ?lhs = ?rhs)
 $\langle proof \rangle$

end

3 Polytopes

theory *Polytope-More*
imports
Convex-Euclidean-Space-More
HOL-Analysis.Analysis
begin

3.1 Invariance under isometries

lemma *facet-of-translation-eq*:
 $((+) a \text{ ` } T \text{ facet-of } (+) a \text{ ` } S) = (T \text{ facet-of } S)$
 $\langle proof \rangle$

lemma *facets-of-translation*:
shows $\{F. F \text{ facet-of } (+) a \text{ ` } S\} = (\cdot) ((+) a) \text{ ` } \{F. F \text{ facet-of } S\}$
 $\langle proof \rangle$

lemma *polyhedron-translation-eq*:
shows $\text{polyhedron } S = \text{polyhedron } ((+) x \text{ ` } S)$
 $\langle proof \rangle$

lemma *face-of-isometry-eq*:
assumes *orthogonal-transformation f*

shows $((\lambda x. c + f x) \text{ ' } T \text{ face-of } (\lambda x. c + f x) \text{ ' } S) \longleftrightarrow T \text{ face-of } S$
 $\langle \text{proof} \rangle$

lemma *extreme-point-of-isometry-eq*:
assumes *orthogonal-transformation f*
shows $(c + f T \text{ extreme-point-of } (\lambda x. c + f x) \text{ ' } S) \longleftrightarrow T \text{ extreme-point-of } S$
 $\langle \text{proof} \rangle$

lemma *aff-dim-isometry-eq*:
assumes *orthogonal-transformation f*
shows $\text{aff-dim } S = \text{aff-dim } ((\lambda x. c + f x) \text{ ' } S)$
 $\langle \text{proof} \rangle$

3.2 Distinctness of facets

lemma *facets-of-polyhedron-explicit-distinct*:
fixes $S :: 'a :: \text{euclidean-space set}$
assumes *finite: finite F*
and *seq: $S = \text{affine hull } S \cap \bigcap F$*
and *faceq: $\bigwedge h. h \in F \implies a \cdot h \neq 0 \wedge h = \{x. a \cdot h \cdot x \leq b \cdot h\}$*
and *psub: $\bigwedge F'. F' \subset F \implies S \subset \text{affine hull } S \cap \bigcap F'$*
and *h1-in: $h1 \in F$*
and *h2-in: $h2 \in F$*
and *eq: $S \cap \{x. a \cdot h1 \cdot x = b \cdot h1\} = S \cap \{x. a \cdot h2 \cdot x = b \cdot h2\}$*
shows $h1 = h2$
 $\langle \text{proof} \rangle$

3.3 Ridges lie in two facets

lemma *polyhedron-ridge-two-facets-0*:
fixes $p :: 'a :: \text{euclidean-space set}$
assumes *polyhedron p*
and *r face-of p*
and $0 \in r$
and $\text{aff-dim } r = \text{aff-dim } p - 2$
shows $\exists f1 f2. f1 \text{ face-of } p \wedge \text{aff-dim } f1 = \text{aff-dim } p - 1 \wedge$
 $f2 \text{ face-of } p \wedge \text{aff-dim } f2 = \text{aff-dim } p - 1 \wedge$
 $f1 \neq f2 \wedge r \subseteq f1 \wedge r \subseteq f2 \wedge f1 \cap f2 = r \wedge$
 $(\forall f. f \text{ face-of } p \wedge \text{aff-dim } f = \text{aff-dim } p - 1 \wedge r \subseteq f$
 $\implies f = f1 \vee f = f2)$
 $\langle \text{proof} \rangle$

lemma *polyhedron-ridge-two-facets*:
fixes $p :: 'a :: \text{euclidean-space set}$
assumes *polyhedron p*
and *r face-of p*
and $r \neq \{\}$
and $\text{aff-dim } r = \text{aff-dim } p - 2$
shows $\exists f1 f2. f1 \text{ face-of } p \wedge \text{aff-dim } f1 = \text{aff-dim } p - 1 \wedge$
 $f2 \text{ face-of } p \wedge \text{aff-dim } f2 = \text{aff-dim } p - 1 \wedge$

$$\begin{aligned}
& f1 \neq f2 \wedge r \subseteq f1 \wedge r \subseteq f2 \wedge r = f1 \cap f2 \wedge \\
& (\forall f. f \text{ face-of } p \wedge \text{aff-dim } f = \text{aff-dim } p - 1 \wedge r \subseteq f \\
& \longrightarrow f = f1 \vee f = f2)
\end{aligned}$$

$\langle \text{proof} \rangle$

3.4 Lower bounds on face counts

3.4.1 Vertices

lemma *polytope-vertex-lower-bound*:

assumes *polytope* *p*

shows $\text{aff-dim } p + 1 \leq \text{card } \{v. v \text{ face-of } p \wedge \text{aff-dim } v = 0\}$

$\langle \text{proof} \rangle$

3.4.2 Facets

lemma *polytope-facet-lower-bound-0*:

assumes *polytope* *p*

assumes $\text{aff-dim } p \neq 0$

assumes *origin-in-p*: $0 \in p$

shows $\text{aff-dim } p + 1 \leq \text{card } \{f. f \text{ facet-of } p\}$

$\langle \text{proof} \rangle$

lemma *polytope-facet-lower-bound*:

assumes *polytope* *p*

assumes $\text{aff-dim } p \neq 0$

shows $\text{aff-dim } p + 1 \leq \text{card } \{f. f \text{ facet-of } p\}$

$\langle \text{proof} \rangle$

3.5 Double counting

lemma *multiple-counting*:

fixes *R* :: 'a $\Rightarrow$ 'b $\Rightarrow$ bool

assumes *finite* *s*

assumes *finite* *t*

assumes $\bigwedge x. x \in s \implies \text{card } \{y \in t. R \ x \ y\} = m$

assumes $\bigwedge y. y \in t \implies \text{card } \{x \in s. R \ x \ y\} = n$

shows $m * \text{card } s = n * \text{card } t$

$\langle \text{proof} \rangle$

3.5.1 $mF = 2E$

Let *F* be the number of facets and *E* be the number of ridges. If there are *m* ridges per facet, then $m * F = 2 * E$.

lemma *facet-ridge-relation*:

assumes *polytope* *p*

and $\text{aff-dim } p \geq 2$

and *edges-per-face*: $\forall f. f \text{ face-of } p \wedge \text{aff-dim } f = \text{aff-dim } p - 1 \longrightarrow$

$\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = \text{aff-dim } p - 2 \wedge e \subseteq f\} = m$

shows $m * \text{card } \{f. f \text{ face-of } p \wedge \text{aff-dim } f = \text{aff-dim } p - 1\} =$

$2 * \text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = \text{aff-dim } p - 2\}$
 <proof>

3.5.2 $nV = 2E$

Let V be the number of vertices and E be the number of edges. If there are n edges per vertex, then $n*V = 2*E$.

lemma *edge-vertex-relation:*

assumes *polytope* p

and *edges-per-vert:* $\forall v. v \text{ face-of } p \wedge \text{aff-dim } v = 0 \longrightarrow$

$\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = n$

shows $n * \text{card } \{v. v \text{ face-of } p \wedge \text{aff-dim } v = 0\} =$

$2 * \text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1\}$

<proof>

hide-fact *multiple-counting*

3.6 Face containment

lemma *ridge-belongs-to-facet:*

assumes *e face-of* p

and $\text{aff-dim } e = \text{aff-dim } p - 2$

and $\text{aff-dim } p \geq 2$

and *polyhedron* p

obtains f **where** $f \text{ face-of } p \wedge \text{aff-dim } f = \text{aff-dim } p - 1 \wedge e \text{ face-of } f$

<proof>

lemma *peak-belongs-to-ridge:*

assumes *v face-of* p

and $\text{aff-dim } v = \text{aff-dim } p - 3$

and $\text{aff-dim } p \geq 3$

and *polyhedron* p

obtains e **where** $e \text{ face-of } p \wedge \text{aff-dim } e = \text{aff-dim } p - 2 \wedge v \text{ face-of } e$

<proof>

end

4 Computational helpers

theory *Computation*

imports

Convex-More

Determinants-More

HOL-Analysis.Cross3

HOL-Analysis.Polytope

HOL-Eisbach.Eisbach

Polytope-More

HOL-Examples.Sqrt
begin

4.1 Operations in $\mathbb{Q}(\sqrt{5})$

Numbers of the form $a + b\sqrt{5}$, where $a, b \in \mathbb{Q}$

abbreviation $\varphi :: \text{real}$
where $\varphi \equiv (1 + \text{sqrt } 5)/2$

lemma *real-rat5-mul*:

$(a1 + b1 * \text{sqrt } 5) * (a2 + b2 * \text{sqrt } 5)$
 $= (a1 * a2 + 5 * b1 * b2) + (a1 * b2 + a2 * b1) * \text{sqrt } 5$
 $\langle \text{proof} \rangle$

lemma *real-rat5-inv*:

assumes $a^2 \neq 5 * b^2$
shows $\text{inverse } (a + b * \text{sqrt } 5) = a / (a^2 - 5 * b^2) + -b / (a^2 - 5 * b^2)$
 $* \text{sqrt } 5$
 $\langle \text{proof} \rangle$

lemma *real-rat5-div*:

assumes $a2^2 \neq 5 * b2^2$
shows $(a1 + b1 * \text{sqrt } 5) / (a2 + b2 * \text{sqrt } 5) =$
 $((a1 * a2 - 5 * b1 * b2) / (a2^2 - 5 * b2^2)) +$
 $((a2 * b1 - a1 * b2) / (a2^2 - 5 * b2^2)) * \text{sqrt } 5$
 $\langle \text{proof} \rangle$

lemma *real-rat5-le-sqrt-cases*:

$x \leq y * \text{sqrt } 5 \longleftrightarrow$
 $(x \leq 0 \wedge 0 \leq y) \vee$
 $(0 \leq x \wedge 0 \leq y \wedge x^2 \leq 5 * y^2) \vee$
 $(x \leq 0 \wedge y \leq 0 \wedge 5 * y^2 \leq x^2)$
 $\langle \text{proof} \rangle$

lemma *real-rat5-le-comparison*:

$(a1 + b1 * \text{sqrt } 5) \leq (a2 + b2 * \text{sqrt } 5) \longleftrightarrow$
 $(a1 \leq a2 \wedge b1 \leq b2) \vee$
 $(a2 \leq a1 \wedge b1 \leq b2 \wedge (a1 - a2)^2 \leq 5 * (b2 - b1)^2) \vee$
 $(a1 \leq a2 \wedge b2 \leq b1 \wedge 5 * (b2 - b1)^2 \leq (a1 - a2)^2)$
 $\langle \text{proof} \rangle$

Wrapper definition to avoiding oversimplification when applying simp

definition *rat5* :: $\text{rat} \Rightarrow \text{rat} \Rightarrow \text{real}$

where $\text{rat5 } a \ b = \text{real-of-rat } a + \text{real-of-rat } b * \text{sqrt } 5$

lemma *rat5-add*: $\text{rat5 } a1 \ b1 + \text{rat5 } a2 \ b2 = \text{rat5 } (a1 + a2) \ (b1 + b2)$

$\langle \text{proof} \rangle$

lemma *rat5-sub*: $\text{rat5 } a1 \ b1 - \text{rat5 } a2 \ b2 = \text{rat5 } (a1 - a2) \ (b1 - b2)$

$\langle proof \rangle$

lemma *rat5-mul*: $rat5\ a1\ b1 * rat5\ a2\ b2 = rat5\ (a1 * a2 + 5 * b1 * b2)\ (a1 * b2 + b1 * a2)$
 $\langle proof \rangle$

lemma *rat5-div*:
 assumes $a2 \neq 0 \vee b2 \neq 0$
 shows $rat5\ a1\ b1 / rat5\ a2\ b2 =$
 $rat5\ ((a1 * a2 - 5 * b1 * b2) / (a2^2 - 5 * b2^2))$
 $((a2 * b1 - a1 * b2) / (a2^2 - 5 * b2^2))$
 $\langle proof \rangle$

lemma *rat5-le*:
 $(rat5\ a1\ b1 \leq rat5\ a2\ b2) =$
 $((a1 \leq a2 \wedge b1 \leq b2) \vee$
 $(a2 \leq a1 \wedge b1 \leq b2 \wedge (a1 - a2)^2 \leq 5 * (b2 - b1)^2) \vee$
 $(a1 \leq a2 \wedge b2 \leq b1 \wedge 5 * (b2 - b1)^2 \leq (a1 - a2)^2))$
 $\langle proof \rangle$

lemma *rat5-eq*: $rat5\ a1\ b1 = rat5\ a2\ b2 \longleftrightarrow a1 = a2 \wedge b1 = b2$
 $\langle proof \rangle$

lemma *rat5-0-1*:
 shows $rat5\ 0\ 0 = 0\ rat5\ 1\ 0 = 1\ \langle proof \rangle$

lemma *rat5-eq-0*:
 $rat5\ a1\ b1 = 0 \longleftrightarrow a1 = 0 \wedge b1 = 0$
 $\langle proof \rangle$

lemma *rat5-eq-1*:
 $rat5\ a1\ b1 = 1 \longleftrightarrow a1 = 1 \wedge b1 = 0$
 $\langle proof \rangle$

4.2 Operations in $\mathbb{R}^3$

lemma *vector3-add*:
 fixes $x1\ x2\ x3\ y1\ y2\ y3 :: real$
 shows $vector\ [x1, x2, x3] + vector\ [y1, y2, y3] =$
 $vector\ [x1 + y1, x2 + y2, x3 + y3]$
 $\langle proof \rangle$

lemma *vector3-sub*:
 fixes $x1\ x2\ x3\ y1\ y2\ y3 :: real$
 shows $vector\ [x1, x2, x3] - vector\ [y1, y2, y3] =$
 $vector\ [x1 - y1, x2 - y2, x3 - y3]$
 $\langle proof \rangle$

unbundle *cross3-syntax*

lemma *vector3-cross*:

fixes $x1\ x2\ x3\ y1\ y2\ y3 :: \text{real}$
shows $\text{vector } [x1, x2, x3] \times \text{vector } [y1, y2, y3] =$
 $(\text{vector } [x2 * y3 - x3 * y2, x3 * y1 - x1 * y3, x1 * y2 - x2 * y1] ::$
 $\text{real}^3)$
 $\langle \text{proof} \rangle$

lemma *vector3-eq-0*:

fixes $x1\ x2\ x3 :: \text{real}$
shows $(\text{vector } [x1, x2, x3] :: \text{real}^3) = 0 \longleftrightarrow x1 = 0 \wedge x2 = 0 \wedge x3 = 0$
 $\langle \text{proof} \rangle$

lemma *vector3-eq*:

fixes $x1\ x2\ x3\ y1\ y2\ y3 :: \text{real}$
shows $(\text{vector } [x1, x2, x3] :: \text{real}^3) = \text{vector } [y1, y2, y3] \longleftrightarrow x1 = y1 \wedge x2 = y2 \wedge x3 = y3$
 $\langle \text{proof} \rangle$

lemma *vector3-neg*:

shows $((\text{vector } [x1, x2, x3] :: \text{real}^3) \neq \text{vector } [y1, y2, y3]) =$
 $(x1 \neq y1 \vee x2 \neq y2 \vee x3 \neq y3)$
 $\langle \text{proof} \rangle$

lemma *vector3-dot*:

fixes $x1\ x2\ x3\ y1\ y2\ y3 :: \text{real}$
shows $(\text{vector } [x1, x2, x3] :: \text{real}^3) \cdot \text{vector } [y1, y2, y3] =$
 $x1*y1 + x2*y2 + x3*y3$
 $\langle \text{proof} \rangle$

4.3 Computation of faces

4.3.1 2-faces

lemma *compute-faces-2*:

fixes $F\ S :: (\text{real}^3)\ \text{set}$
assumes *finite S*
shows $(F\ \text{face-of}\ (\text{convex hull } S) \wedge \text{aff-dim } F = 2) =$
 $(\exists x \in S. \exists y \in S. \exists z \in S.$
 $\text{let } a = (z - x) \times (y - x)\ \text{in}$
 $a \neq 0 \wedge$
 $(\text{let } b = a \cdot x\ \text{in}$
 $((\forall w \in S. a \cdot w \leq b) \vee (\forall w \in S. a \cdot w \geq b)) \wedge$
 $F = \text{convex hull } (S \cap \{x. a \cdot x = b\})))$
 $(\text{is ?lhs} = \text{?rhs})$
 $\langle \text{proof} \rangle$

lemma *set-choice-decomposition-1*:

assumes *sym-yz*: $\bigwedge x\ y\ z. Q\ x\ y\ z \implies Q\ x\ z\ y$
and *sym-xy*: $\bigwedge x\ y\ z. Q\ x\ y\ z \implies Q\ y\ x\ z$

and *degen*: $\bigwedge x z. \neg Q x x z$
shows $(\exists x \in \text{insert } v S. \exists y \in \text{insert } v S. \exists z \in \text{insert } v S. Q x y z) \longleftrightarrow$
 $(\exists y \in S. \exists z \in S. Q v y z) \vee (\exists x \in S. \exists y \in S. \exists z \in S. Q x y z)$
 $\langle \text{proof} \rangle$

lemma *compute-faces-2-step-1*:
fixes $F S v$ **and** $T :: (\text{real}^3)$ *set*
defines $Q \equiv (\lambda x y z. \text{let } a = (z - x) \times (y - x) \text{ in}$
 $a \neq 0 \wedge$
 $(\text{let } b = a \cdot x \text{ in}$
 $((\forall w \in T. a \cdot w \leq b) \vee (\forall w \in T. a \cdot w \geq b)) \wedge$
 $F = \text{convex hull } (T \cap \{k. a \cdot k = b\})))$
shows $(\exists x \in (\text{insert } v S). \exists y \in (\text{insert } v S). \exists z \in (\text{insert } v S). Q x y z) =$
 $((\exists y \in S. \exists z \in S. Q v y z) \vee (\exists x \in S. \exists y \in S. \exists z \in S. Q x y z))$
 $(\text{is } ?lhs = ?rhs)$
 $\langle \text{proof} \rangle$

lemma *set-choice-decomposition-2*:
assumes *sym*: $\bigwedge x y. Q x y \implies Q y x$
and *degen*: $\bigwedge x. \neg Q x x$
shows $(\exists y \in \text{insert } u S. \exists z \in \text{insert } u S. Q y z) \longleftrightarrow$
 $(\exists z \in S. Q u z) \vee (\exists y \in S. \exists z \in S. Q y z)$
 $\langle \text{proof} \rangle$

lemma *compute-faces-2-step-2*:
fixes $F u v S$ **and** $T :: (\text{real}^3)$ *set*
defines $Q \equiv (\lambda y z. \text{let } a = (z - v) \times (y - v) \text{ in}$
 $a \neq 0 \wedge$
 $(\text{let } b = a \cdot v \text{ in}$
 $((\forall w \in T. a \cdot w \leq b) \vee (\forall w \in T. a \cdot w \geq b)) \wedge$
 $F = \text{convex hull } (T \cap \{k. a \cdot k = b\})))$
shows $(\exists y \in (\text{insert } u S). \exists z \in (\text{insert } u S). Q y z) =$
 $((\exists z \in S. Q u z) \vee (\exists y \in S. \exists z \in S. Q y z))$
 $(\text{is } ?lhs = ?rhs)$
 $\langle \text{proof} \rangle$

4.3.2 1-faces

lemma *compute-faces-1*:
fixes $F n d$ **and** $S :: (\text{real}^3)$ *set*
assumes $\bigwedge x. x \in S \implies n \cdot x = d$
and *finite* S
and $n \neq 0$
shows $(F \text{ face-of } (\text{convex hull } S) \wedge \text{aff-dim } F = 1) =$
 $(\exists x \in S. \exists y \in S.$
 $\text{let } a = n \times (y - x) \text{ in}$
 $a \neq 0 \wedge$
 $(\text{let } b = a \cdot x \text{ in}$
 $((\forall w \in S. a \cdot w \leq b) \vee (\forall w \in S. a \cdot w \geq b))) \wedge$

$F = \text{convex hull } (S \cap \{w. a \cdot w = b\})$
 (is ?lhs = ?rhs)
 <proof>

lemma *compute-faces-1-step*:
fixes $F \ n \ u \ S$ **and** $T :: (\text{real}^3) \text{ set}$
defines $Q \equiv (\lambda x \ y. \text{let } a = n \times (y - x) \text{ in}$
 $a \neq 0 \wedge$
 $(\text{let } b = a \cdot x \text{ in}$
 $((\forall w \in T. a \cdot w \leq b) \vee (\forall w \in T. a \cdot w \geq b)) \wedge$
 $F = \text{convex hull } (T \cap \{k. a \cdot k = b\}))$
shows $(\exists y \in (\text{insert } u \ S). \exists z \in (\text{insert } u \ S). Q \ y \ z) =$
 $((\exists z \in S. Q \ u \ z) \vee (\exists y \in S. \exists z \in S. Q \ y \ z))$
 (is ?lhs = ?rhs)
 <proof>

method *facet-edges* **uses** *plane n-nz defs* =
 (simp only: *compute-faces-1*[*OF plane - n-nz*] *finite-insert finite.emptyI*,
simp only: compute-faces-1-step bex-empty simp-thms(31),
simp only: empty-iff ball-insert bex-empty bex-simps(5) ball-simps(5) simp-thms(21,31),
simp add: defs vector3-sub vector3-cross vector3-dot vector3-eq-0 Let-def
rat5-mul rat5-add rat5-sub rat5-eq rat5-eq-0 rat5-le)

4.4 Full-dimensionality

lemma *polytope-3D*:
fixes $S :: (\text{real}^3) \text{ set}$
assumes $(z - x) \times (y - x) \neq 0$
and $\exists w \in S. ((z - x) \times (y - x)) \cdot w \neq ((z - x) \times (y - x)) \cdot x$
shows $\text{aff-dim } (\text{convex hull } (\text{insert } x (\text{insert } y (\text{insert } z \ S)))) = 3$
 (is *aff-dim ?hull* = 3)
 <proof>

4.5 Relations between faces of different dimensions

These lemmas are used when formalizing the number of edges per face and number of edges per vertex.

corollary *edge-belongs-to-face*:
assumes $e \text{ face-of } p$
and $\text{aff-dim } e = 1$
and $\text{aff-dim } p = 3$
and *polyhedron* p
obtains f **where** $f \text{ face-of } p \text{ aff-dim } f = 2 \ e \text{ face-of } f$
 <proof>

corollary *vertex-belongs-to-edge*:
assumes $v \text{ face-of } p$
and $\text{aff-dim } v = 0$
and $\text{aff-dim } p = 3$

and *polyhedron* p
obtains e **where** e *face-of* p $\text{aff-dim } e = 1$ v *face-of* e
 $\langle \text{proof} \rangle$

lemma *vertices-of-edge*:
assumes $e = \text{convex hull } \{a, b\}$
and v *face-of* e
and $\text{aff-dim } v = 0$
shows $v = \{a\} \vee v = \{b\}$
 $\langle \text{proof} \rangle$

lemma *vertex-edge-set-eq*:
assumes v *face-of* S
shows $\text{card } \{e. e \text{ face-of } S \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} =$
 $\text{card } \{e. e \text{ face-of } S \wedge \text{aff-dim } e = 1 \wedge v \text{ face-of } e\}$
 $\langle \text{proof} \rangle$

lemma *face-edge-set-eq*:
assumes f *face-of* S
shows $\text{card } \{e. e \text{ face-of } S \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} =$
 $\text{card } \{e. e \text{ face-of } f \wedge \text{aff-dim } e = 1\}$
 $\langle \text{proof} \rangle$

lemma *set-disj-union-distr*:
shows $\{e. (P \ e \vee Q \ e) \wedge R \ e\} = \{e. P \ e \wedge R \ e\} \cup \{e. Q \ e \wedge R \ e\}$
 $\langle \text{proof} \rangle$

lemma *v-sing-face-of-e*:
shows $(e = a \wedge \{x\} \text{ face-of } e) = (e = a \wedge x \text{ extreme-point-of } a)$
 $\langle \text{proof} \rangle$

lemma *card-3-iff2*:
shows $(\text{card } \{x, y, z\} = 3) = (x \neq y \wedge x \neq z \wedge y \neq z)$
 $\langle \text{proof} \rangle$

lemma *card-4-iff2*:
shows $(\text{card } \{w, x, y, z\} = 4) = (w \neq x \wedge w \neq y \wedge w \neq z \wedge x \neq y \wedge x \neq z \wedge$
 $y \neq z)$
 $\langle \text{proof} \rangle$

lemma *card-5-iff*:
shows $\text{card } \{u, w, x, y, z\} = 5 \longleftrightarrow (u \neq w \wedge u \neq x \wedge u \neq y \wedge u \neq z \wedge w \neq x \wedge$
 $w \neq y \wedge w \neq z \wedge x \neq y \wedge y \neq z \wedge x \neq z)$
 $\langle \text{proof} \rangle$

method *edges-per-vertex* =
 $(\text{unfold set-disj-union-distr segment-convex-hull}[\text{symmetric}] \text{ v-sing-face-of-e};$
 $\text{simp-all add: extreme-point-of-segment vector3-eq rat5-eq};$
 $\text{unfold card-3-iff2 card-4-iff2 card-5-iff closed-segment-eq};$

simp-all add: vector3-eq rat5-eq doubleton-eq-iff)

lemma *set-cases*:

shows $\{x. x = a\} = \{a\} \quad \{x. x = a \vee P x\} = \text{insert } a \ \{x. P x\}$
 $\langle \text{proof} \rangle$

4.6 Congruence

definition *congruent* :: $['a :: \text{euclidean-space set}, 'a \text{ set}] \Rightarrow \text{bool}$ (**infixr** $\langle (\text{congruent}) \rangle$ 50)

where $S \text{ congruent } T \longleftrightarrow (\exists c f. \text{orthogonal-transformation } f \wedge T = (\lambda x. c + f x) \text{ ` } S)$

lemma *congruent-refl*:

shows $S \text{ congruent } S$
 $\langle \text{proof} \rangle$

lemma *congruent-sym*:

shows $S \text{ congruent } T \longleftrightarrow T \text{ congruent } S$
 $\langle \text{proof} \rangle$

lemma *congruent-trans*:

assumes $R \text{ congruent } S$
assumes $S \text{ congruent } T$
shows $R \text{ congruent } T$
 $\langle \text{proof} \rangle$

lemma *congruent-set*:

assumes $\forall t \in (\text{insert } s \ S). \ s \text{ congruent } t$
assumes $x \in (\text{insert } s \ S)$
assumes $y \in (\text{insert } s \ S)$
shows $x \text{ congruent } y$
 $\langle \text{proof} \rangle$

4.6.1 Congruence of segments

lemma *congruent-segments*:

fixes $a \ b \ c \ d :: \text{real}^n$
assumes $\text{dist } a \ b = \text{dist } c \ d$
shows $(\text{closed-segment } a \ b) \text{ congruent } (\text{closed-segment } c \ d)$
 $\langle \text{proof} \rangle$

4.6.2 Congruence of 2-faces

lemma *congruent-simple*:

assumes $\exists A :: \text{real}^3 \rightarrow \text{real}^3. \text{orthogonal-matrix } A \wedge (\lambda x. A * v \ x) \text{ ` } S = T$
shows $(\text{convex hull } S) \text{ congruent } (\text{convex hull } T)$
 $\langle \text{proof} \rangle$

lemma *matrix-vector-mul-3*:

shows ($\text{vector} [\text{vector}[a11, a12, a13],$
 $\text{vector}[a21, a22, a23],$
 $\text{vector}[a31, a32, a33]]::\text{real}^3$) $\ast v$ $\text{vector} [x1, x2, x3] =$
 $\text{vector} [a11 \ast x1 + a12 \ast x2 + a13 \ast x3,$
 $a21 \ast x1 + a22 \ast x2 + a23 \ast x3,$
 $a31 \ast x1 + a32 \ast x2 + a33 \ast x3]$
 $\langle \text{proof} \rangle$

lemma *matrix-lemma:*

fixes $A :: \text{real}^3$
shows ($A \ast v x1 = x2 \wedge A \ast v y1 = y2 \wedge A \ast v z1 = z2$) =
 $((\text{vector} [x1, y1, z1]::\text{real}^3) \ast v (\text{row } 1 A) = \text{vector}[x2\$1, y2\$1, z2\$1] \wedge$
 $(\text{vector} [x1, y1, z1]::\text{real}^3) \ast v (\text{row } 2 A) = \text{vector}[x2\$2, y2\$2, z2\$2] \wedge$
 $(\text{vector} [x1, y1, z1]::\text{real}^3) \ast v (\text{row } 3 A) = \text{vector}[x2\$3, y2\$3, z2\$3])$
 $\langle \text{proof} \rangle$

lemma *matrix-by-cramer-lemma:*

fixes $A :: \text{real}^3$
assumes $\text{matrix-det} (\text{vector}[x1, y1, z1]::\text{real}^3) \neq 0$
shows ($A \ast v x1 = x2 \wedge A \ast v y1 = y2 \wedge A \ast v z1 = z2$) =
 $(A = (\chi \ m \ k. \text{matrix-det} ((\chi \ i \ j.$
 $\text{if } j = k$
 $\text{then } (\text{vector} [x2\$m, y2\$m, z2\$m]::\text{real}^3)\$i$
 $\text{else } (\text{vector} [x1, y1, z1]::\text{real}^3)\$i\$j)$
 $::\text{real}^3) /$
 $\text{matrix-det} (\text{vector} [x1, y1, z1]::\text{real}^3)))$
 $\langle \text{proof} \rangle$

definition $\text{mk-matrix33} :: (\text{real}^3) \Rightarrow (\text{real}^3) \Rightarrow (\text{real}^3) \Rightarrow (\text{real}^3) \Rightarrow (\text{real}^3)$
 $\Rightarrow (\text{real}^3) \Rightarrow \text{real}^3$

where $\text{mk-matrix33 } x1 \ y1 \ z1 \ x2 \ y2 \ z2 = (\text{let } d = \text{matrix-det} (\text{vector}[x1, y1,$
 $z1]::\text{real}^3) \text{ in } \text{vector} [\text{vector} [$
 $(x2\$1 \ast y1\$2 \ast z1\$3 +$
 $x1\$2 \ast y1\$3 \ast z2\$1 +$
 $x1\$3 \ast y2\$1 \ast z1\$2 -$
 $x2\$1 \ast y1\$3 \ast z1\$2 -$
 $x1\$2 \ast y2\$1 \ast z1\$3 -$
 $x1\$3 \ast y1\$2 \ast z2\$1) / d,$
 $(x1\$1 \ast y2\$1 \ast z1\$3 +$
 $x2\$1 \ast y1\$3 \ast z1\$1 +$
 $x1\$3 \ast y1\$1 \ast z2\$1 -$
 $x1\$1 \ast y1\$3 \ast z2\$1 -$
 $x2\$1 \ast y1\$1 \ast z1\$3 -$
 $x1\$3 \ast y2\$1 \ast z1\$1) / d,$
 $(x1\$1 \ast y1\$2 \ast z2\$1 +$
 $x1\$2 \ast y2\$1 \ast z1\$1 +$
 $x2\$1 \ast y1\$1 \ast z1\$2 -$
 $x1\$1 \ast y2\$1 \ast z1\$2 -$
 $x1\$2 \ast y1\$1 \ast z2\$1 -$

$$\begin{aligned}
& (x2\$1 * y1\$2 * z1\$1) / d], \text{ vector } [\\
& (x2\$2 * y1\$2 * z1\$3 + \\
& x1\$2 * y1\$3 * z2\$2 + \\
& x1\$3 * y2\$2 * z1\$2 - \\
& x2\$2 * y1\$3 * z1\$2 - \\
& x1\$2 * y2\$2 * z1\$3 - \\
& x1\$3 * y1\$2 * z2\$2) / d, \\
& (x1\$1 * y2\$2 * z1\$3 + \\
& x2\$2 * y1\$3 * z1\$1 + \\
& x1\$3 * y1\$1 * z2\$2 - \\
& x1\$1 * y1\$3 * z2\$2 - \\
& x2\$2 * y1\$1 * z1\$3 - \\
& x1\$3 * y2\$2 * z1\$1) / d, \\
& (x1\$1 * y1\$2 * z2\$2 + \\
& x1\$2 * y2\$2 * z1\$1 + \\
& x2\$2 * y1\$1 * z1\$2 - \\
& x1\$1 * y2\$2 * z1\$2 - \\
& x1\$2 * y1\$1 * z2\$2 - \\
& x2\$2 * y1\$2 * z1\$1) / d], \text{ vector } [\\
& (x2\$3 * y1\$2 * z1\$3 + \\
& x1\$2 * y1\$3 * z2\$3 + \\
& x1\$3 * y2\$3 * z1\$2 - \\
& x2\$3 * y1\$3 * z1\$2 - \\
& x1\$2 * y2\$3 * z1\$3 - \\
& x1\$3 * y1\$2 * z2\$3) / d, \\
& (x1\$1 * y2\$3 * z1\$3 + \\
& x2\$3 * y1\$3 * z1\$1 + \\
& x1\$3 * y1\$1 * z2\$3 - \\
& x1\$1 * y1\$3 * z2\$3 - \\
& x2\$3 * y1\$1 * z1\$3 - \\
& x1\$3 * y2\$3 * z1\$1) / d, \\
& (x1\$1 * y1\$2 * z2\$3 + \\
& x1\$2 * y2\$3 * z1\$1 + \\
& x2\$3 * y1\$1 * z1\$2 - \\
& x1\$1 * y2\$3 * z1\$2 - \\
& x1\$2 * y1\$1 * z2\$3 - \\
& x2\$3 * y1\$2 * z1\$1) / d]]
\end{aligned}$$

lemma *set3-neqI*:

assumes $a \notin \{x, y, z\} \vee b \notin \{x, y, z\} \vee c \notin \{x, y, z\}$
shows $\{a, b, c\} \neq \{x, y, z\}$
<proof>

lemma *set4-neqI*:

assumes $a \notin \{w, x, y, z\} \vee b \notin \{w, x, y, z\} \vee c \notin \{w, x, y, z\} \vee d \notin \{w, x, y, z\}$
shows $\{a, b, c, d\} \neq \{w, x, y, z\}$
<proof>

lemma *orthogonal-matrix33*:

fixes $Q :: \text{real}^3^3$

shows (*orthogonal-matrix* Q) =

$$\begin{aligned} & (Q_{11} * Q_{11} + Q_{21} * Q_{21} + Q_{31} * Q_{31} = 1 \wedge \\ & Q_{11} * Q_{12} + Q_{21} * Q_{22} + Q_{31} * Q_{32} = 0 \wedge \\ & Q_{11} * Q_{13} + Q_{21} * Q_{23} + Q_{31} * Q_{33} = 0 \wedge \\ & Q_{12} * Q_{11} + Q_{22} * Q_{21} + Q_{32} * Q_{31} = 0 \wedge \\ & Q_{12} * Q_{12} + Q_{22} * Q_{22} + Q_{32} * Q_{32} = 1 \wedge \\ & Q_{12} * Q_{13} + Q_{22} * Q_{23} + Q_{32} * Q_{33} = 0 \wedge \\ & Q_{13} * Q_{11} + Q_{23} * Q_{21} + Q_{33} * Q_{31} = 0 \wedge \\ & Q_{13} * Q_{12} + Q_{23} * Q_{22} + Q_{33} * Q_{32} = 0 \wedge \\ & Q_{13} * Q_{13} + Q_{23} * Q_{23} + Q_{33} * Q_{33} = 1) \end{aligned}$$

<proof>

lemma *matrix-by-cramer*:

fixes $A :: \text{real}^3^3$

assumes $A = \text{mk-matrix33 } x1 \ y1 \ z1 \ x2 \ y2 \ z2$

assumes *matrix-det* (*vector* $[x1, y1, z1] :: \text{real}^3^3$) $\neq 0$

shows $A * v \ x1 = x2 \ A * v \ y1 = y2 \ A * v \ z1 = z2$

<proof>

lemma *matrix-by-cramer-orthog*:

fixes $x1 \ y1 \ z1 \ x2 \ y2 \ z2 :: \text{real}^3$

assumes *orthogonal-matrix* (*mk-matrix33* $x1 \ y1 \ z1 \ x2 \ y2 \ z2$)

assumes *matrix-det* (*vector* $[x1, y1, z1] :: \text{real}^3^3$) $\neq 0$

shows *orthogonal-matrix* (*mk-matrix33* $x1 \ y1 \ z1 \ x2 \ y2 \ z2$) $\wedge$

$$(*v) \ (\text{mk-matrix33 } x1 \ y1 \ z1 \ x2 \ y2 \ z2) \ ' \ \{x1, y1, z1\} = \{x2, y2, z2\}$$

<proof>

Lookup table mapping a face to three vertices on that face that are adjacent (vertices that define a pair of intersecting edges)

definition *adjacency-table* :: $((\text{real}, 3) \text{ vec set} \times ((\text{real}, 3) \text{ vec} \times (\text{real}, 3) \text{ vec} \times (\text{real}, 3) \text{ vec})) \text{ list}$ **where**

adjacency-table = [

(*vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$), (*vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$),

(*vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]$, *vector* $[1, 1, 1]$), (*vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]$, *vector* $[1, 1, 1]$),

(*vector* $[-1, -1, 1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, 1]$), (*vector* $[-1, -1, 1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, 1]$),

(*vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, 1]$), (*vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, 1]$),

(*vector* $[-1, -1, -1]$, *vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]$, *vector* $[-1, 1, 1]$), (*vector* $[-1, -1, -1]$, *vector* $[-1, -1, 1]$, *vector* $[-1, 1, 1]$),

(*vector* $[-1, -1, -1]$, *vector* $[-1, -1, 1]$, *vector* $[1, -1, -1]$, *vector* $[1, -1, 1]$), (*vector* $[-1, -1, -1]$, *vector* $[-1, -1, 1]$, *vector* $[1, -1, 1]$),

(*vector* $[-1, -1, -1]$, *vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, -1]$), (*vector* $[-1, -1, -1]$, *vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]$, *vector* $[1, 1, -1]$),

[illegible]

[illegible]

$(-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)]], (\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)])),$
 $(\{\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]], (\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)])),$
 $(\{\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]], (\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)])),$
 $(\{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)]], (\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)])),$
 $(\{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)]], (\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)])),$
 $(\{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)]], (\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)])),$
 $(\{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]], (\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]))$
 $\}$

definition *adjacent-points* :: (real, 3) vec set $\Rightarrow$ (real, 3) vec $\times$ (real, 3) vec $\times$ (real, 3) vec **where**
adjacent-points $S =$ (
 $\text{case map-of adjacency-table } S \text{ of}$
 $\text{Some res} \Rightarrow \text{res}$
 $| \text{None} \Rightarrow \text{undefined}$
 $)$

lemma *set43-neqI*:

assumes $a \notin \{x, y, z\} \vee b \notin \{x, y, z\} \vee c \notin \{x, y, z\} \vee d \notin \{x, y, z\}$
shows $\{a, b, c, d\} \neq \{x, y, z\}$
 $\{x, y, z\} \neq \{a, b, c, d\}$
 $\langle \text{proof} \rangle$

lemma *solids-vectors-to-rat5*:

shows $\text{vector}[1, 1, 1] = \text{vector}[\text{rat5 } (1) \ 0, \text{rat5 } (1) \ 0, \text{rat5 } (1) \ 0]$
 $\text{vector}[1, 1, -1] = \text{vector}[\text{rat5 } (1) \ 0, \text{rat5 } (1) \ 0, \text{rat5 } (-1) \ 0]$
 $\text{vector}[1, -1, 1] = \text{vector}[\text{rat5 } (1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1) \ 0]$
 $\text{vector}[1, -1, -1] = \text{vector}[\text{rat5 } (1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0]$
 $\text{vector}[-1, 1, 1] = \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1) \ 0, \text{rat5 } (1) \ 0]$

$vector[-1, 1, -1] = vector[rat5 (-1) 0, rat5 (1) 0, rat5 (-1) 0]$
 $vector[-1, -1, 1] = vector[rat5 (-1) 0, rat5 (-1) 0, rat5 (1) 0]$
 $vector[-1, -1, -1] = vector[rat5 (-1) 0, rat5 (-1) 0, rat5 (-1) 0]$
 $vector[1, 0, 0] = vector[rat5 (1) 0, rat5 (0) 0, rat5 (0) 0]$
 $vector[-1, 0, 0] = vector[rat5 (-1) 0, rat5 (0) 0, rat5 (0) 0]$
 $vector[0, 0, 1] = vector[rat5 (0) 0, rat5 (0) 0, rat5 (1) 0]$
 $vector[0, 0, -1] = vector[rat5 (0) 0, rat5 (0) 0, rat5 (-1) 0]$
 $vector[0, 1, 0] = vector[rat5 (0) 0, rat5 (1) 0, rat5 (0) 0]$
 $vector[0, -1, 0] = vector[rat5 (0) 0, rat5 (-1) 0, rat5 (0) 0]$
 <proof>

lemma *set53-negI*:

assumes $a \notin \{x, y, z\} \vee b \notin \{x, y, z\} \vee c \notin \{x, y, z\} \vee d \notin \{x, y, z\} \vee e \notin \{x, y, z\}$
shows $\{a, b, c, d, e\} \neq \{x, y, z\}$
 $\{x, y, z\} \neq \{a, b, c, d, e\}$
 <proof>

lemma *set54-negI*:

assumes $a \notin \{w, x, y, z\} \vee b \notin \{w, x, y, z\} \vee c \notin \{w, x, y, z\} \vee d \notin \{w, x, y, z\} \vee e \notin \{w, x, y, z\}$
shows $\{a, b, c, d, e\} \neq \{w, x, y, z\}$
 $\{w, x, y, z\} \neq \{a, b, c, d, e\}$
 <proof>

lemma *set5-negI*:

assumes $a \notin \{v, w, x, y, z\} \vee b \notin \{v, w, x, y, z\} \vee c \notin \{v, w, x, y, z\} \vee d \notin \{v, w, x, y, z\} \vee e \notin \{v, w, x, y, z\}$
shows $\{a, b, c, d, e\} \neq \{v, w, x, y, z\}$
 <proof>

lemma *set3-eq-iff*:

shows $(\{a, b, c\} = \{d, e, f\}) =$
 $(a \in \{d, e, f\} \wedge b \in \{d, e, f\} \wedge c \in \{d, e, f\} \wedge$
 $d \in \{a, b, c\} \wedge e \in \{a, b, c\} \wedge f \in \{a, b, c\})$
 <proof>

lemma *set3-vector-neg*:

assumes $(a1 \neq d1 \wedge a1 \neq d2 \wedge a1 \neq d3)$
shows $\{(vector [a1, b1, c1]::real^3), vector [a2, b2, c2], vector [a3, b3, c3]\} \neq \{vector [d1, e1, f1], vector [d2, e2, f2], vector [d3, e3, f3]\}$
 <proof>

lemma *adjacent-points*:

shows *adjacent-points* $\{vector[-1, -1, 1], vector[-1, 1, -1], vector[1, -1, -1]\} = (vector [-1, -1, 1], vector [-1, 1, -1], vector [1, -1, -1])$
 $adjacent-points \{vector[-1, -1, 1], vector[-1, 1, -1], vector[1, 1, 1]\} = (vector[-1, -1, 1], vector[-1, 1, -1], vector[1, 1, 1])$
 $adjacent-points \{vector[-1, -1, 1], vector[1, -1, -1], vector[1, 1, 1]\} =$

$$\begin{aligned} & \text{adjacent-points } \{ \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)], \\ & \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0], \\ & \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } \\ & (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] \} = (\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } \\ & (1/2) \ (-1/2)], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } \\ & 1 \ 0, \text{rat5 } (-1) \ 0]) \end{aligned}$$
$$\text{adjacent-points } \{ \text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0], \text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2), \text{rat5 } 0 \text{ } 0] \} = (\text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0], \text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2), \text{rat5 } 0 \text{ } 0])$$
$$\text{adjacent-points } \{\text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2), \text{rat5 } 0 \text{ } 0], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0, \text{rat5 } (-1/2) \text{ } (-1/2)]\} = (\text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2), \text{rat5 } 0 \text{ } 0], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0, \text{rat5 } (-1/2) \text{ } (-1/2)])$$
$$\text{adjacent-points } \{ \text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2)] \} = (\text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } (-1) \text{ } 0, \text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2)])$$
$$\begin{aligned} & \text{adjacent-points } \{ \text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } \\ & 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2)], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2)] \} \\ & = (\text{vector}[\text{rat5 } (-1/2) \text{ } (-1/2), \text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } (-1) \text{ } 0, \\ & \text{rat5 } (1/2) \text{ } (1/2)], \text{vector}[\text{rat5 } 0 \text{ } 0, \text{rat5 } 1 \text{ } 0, \text{rat5 } (1/2) \text{ } (1/2)]) \end{aligned}$$

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (-1) 0], vector[rat5

$$\begin{aligned} &mk\text{-}matrix33\ (fst\ vs)\ (fst\ (snd\ vs))\ (snd\ (snd\ vs)) \\ &\quad (fst\ us)\ (fst\ (snd\ us))\ (snd\ (snd\ us)) \end{aligned}$$

method *show-congruence* =
 (((*erule forw-subst*)⁺)?;
 (*match conclusion in* *convex hull* (*s::(real³) set*) *congruent convex hull* (*t::(real³) set*) **for** *s t* $\Rightarrow$
 $\langle$ (*rule congruent-simple*), (*rule exI*[*where x=mk-matrix33-ap s t*]) $\rangle$);
 (*simp only: mk-matrix33-ap-def Let-def fst-conv snd-conv adjacent-points*);
 (*rule matrix-by-cramer-orthog | simp add: matrix-by-cramer det-3 rat5-eq-0 rat5-add rat5-mul rat5-sub*);
 (*simp only: mk-matrix33-def vector-3 det-3 rat5-add rat5-mul rat5-sub rat5-eq*);
 (*simp add: matrix-vector-mul-3 orthogonal-matrix33 rat5-add rat5-mul rat5-sub rat5-div rat5-eq rat5-eq-0 rat5-eq-1 Let-def; fast*) $\rangle$)

4.7 Equiangularity

vangle is from Manuel Eberl's AFP entry on triangles (see <https://www.isa-afp.org/thys/Triangle/Angles.html>)

definition *vangle* :: '*a::real-inner* $\Rightarrow$ '*a* $\Rightarrow$ *real* **where**
vangle *x y* = (*if* *x* = 0 $\vee$ *y* = 0 *then* $\pi / 2$ *else* $\arccos (x \cdot y / (\text{norm } x * \text{norm } y))$)

lemma *vangle-commute*:
shows *vangle* *x y* = *vangle* *y x*
 $\langle$ *proof* $\rangle$

lemma *isometry-maintains-angles-forw*:
fixes *c*
assumes *orthogonal-transformation h*
defines *g* $\equiv (\lambda x. c + h\ x)$
shows (*e1 face-of f1* $\wedge$ *aff-dim e1* = 1) $\wedge$
 (*e2 face-of f1* $\wedge$ *aff-dim e2* = 1) $\wedge$
e1 = *convex hull* {*v*, *a*} $\wedge$ *e2* = *convex hull* {*v*, *b*} $\wedge$
vangle (*a* - *v*) (*b* - *v*) = *t* $\implies$
 ((*g* '*e1 face-of g* '*f1* $\wedge$ *aff-dim* (*g* '*e1*) = 1) $\wedge$
 (*g* '*e2 face-of g* '*f1* $\wedge$ *aff-dim* (*g* '*e2*) = 1) $\wedge$
g '*e1* = *convex hull* {*g v*, *g a*} $\wedge$ *g* '*e2* = *convex hull* {*g v*, *g b*} $\wedge$
vangle (*g a* - *g v*) (*g b* - *g v*) = *t*)
 (**is** ?*lhs* $\implies$?*rhs*)
 $\langle$ *proof* $\rangle$

definition *equiangular* :: '*a::euclidean-space* *set* $\Rightarrow$ *real* $\Rightarrow$ *bool*
where *equiangular f t* $\longleftrightarrow$
 ($\forall e1\ e2\ v. (e1\ \text{face-of}\ f \wedge \text{aff-dim } e1 = 1) \wedge (e2\ \text{face-of}\ f \wedge \text{aff-dim } e2 = 1) \wedge$
 $e1 \neq e2 \wedge v\ \text{extreme-point-of}\ e1 \wedge v\ \text{extreme-point-of}\ e2 \longrightarrow$
 $(\exists a\ b. e1 = \text{convex hull } \{v, a\} \wedge e2 = \text{convex hull } \{v, b\} \wedge$

$$\text{vangle } (a - v) (b - v) = t))$$

lemma *equiangular-congruent-forw:*

assumes *f1 congruent f2*

assumes *equiangular f1 t*

shows *equiangular f2 t*

<proof>

lemma *equiangular-congruent:*

assumes *f1 congruent f2*

shows *equiangular f1 t $\longleftrightarrow$ equiangular f2 t*

<proof>

end

5 Tetrahedron

theory *Tetrahedron*

imports *Computation*

begin

5.1 Definition

definition *std-tetrahedron* :: (real^3) set

where *std-tetrahedron* $\equiv$ *convex hull*

{vector [1, 1, 1], vector [-1, -1, 1],
vector [-1, 1, -1], vector [1, -1, -1]}

lemma *tetrahedron-fulldim:*

shows *aff-dim std-tetrahedron = 3*

<proof>

lemma *tetrahedron-polyhedron:*

shows *polyhedron std-tetrahedron*

<proof>

5.2 Facets, edges, and vertices

lemma *tetrahedron-facets:*

f face-of std-tetrahedron $\wedge$ aff-dim f = 2 $\longleftrightarrow$

f = convex hull {vector[-1, -1, 1], vector[- 1, 1, - 1], vector[1, - 1, - 1]}

$\vee$

f = convex hull {vector[-1, -1, 1], vector[- 1, 1, - 1], vector[1, 1, 1]} $\vee$

f = convex hull {vector[-1, -1, 1], vector[1, - 1, - 1], vector[1, 1, 1]} $\vee$

f = convex hull {vector[-1, 1, -1], vector[1, - 1, - 1], vector[1, 1, 1]}

<proof>

lemma *tetrahedron-facet1-edges:*

defines *v1* $\equiv$ *vector[-1, -1, 1]* :: real^3

and $v2 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $n \equiv \text{vector}[-4, -4, -4] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *tetrahedron-facet2-edges:*

defines $v1 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
and $v2 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[-4, 4, 4] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *tetrahedron-facet3-edges:*

defines $v1 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
and $v2 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[4, -4, 4] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *tetrahedron-facet4-edges:*

defines $v1 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v2 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[4, 4, -4] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemmas *tetrahedron-facet-edges =*

tetrahedron-facet1-edges tetrahedron-facet2-edges
tetrahedron-facet3-edges tetrahedron-facet4-edges

lemma *tetrahedron-edges-per-face:*

assumes $f \text{ face-of std-tetrahedron}$
and $\text{aff-dim } f = 2$
shows $\text{card } \{e. e \text{ face-of std-tetrahedron} \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = 3$
<proof>

lemma *tetrahedron-edges:*

$e \text{ face-of std-tetrahedron} \wedge \text{aff-dim } e = 1 \longleftrightarrow$
 $e = \text{convex hull } \{\text{vector } [-1, -1, 1], \text{vector } [-1, 1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector } [-1, -1, 1], \text{vector } [1, -1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector } [-1, -1, 1], \text{vector } [1, 1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector } [-1, 1, -1], \text{vector } [1, -1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector } [-1, 1, -1], \text{vector } [1, 1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector } [1, -1, -1], \text{vector } [1, 1, 1]\}$
 (is ?lhs $\longleftrightarrow$?rhs)
 <proof>

lemma *tetrahedron-vertices:*

shows $(v \text{ face-of std-tetrahedron} \wedge \text{aff-dim } v = 0) =$
 $(v = \{\text{vector } [-1, -1, 1]\} \vee$
 $v = \{\text{vector } [-1, 1, -1]\} \vee$
 $v = \{\text{vector } [1, -1, -1]\} \vee$
 $v = \{\text{vector } [1, 1, 1]\})$
 (is ?lhs = ?rhs)
 <proof>

lemma *tetrahedron-edges-per-vertex:*

assumes $v \text{ face-of std-tetrahedron}$
 and $\text{aff-dim } v = 0$
 shows $\text{card } \{e. e \text{ face-of std-tetrahedron} \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = 3$
 <proof>

5.3 Regularity

lemma *tetrahedron-congruent-edges:*

assumes $e1 \text{ face-of std-tetrahedron} \wedge \text{aff-dim } e1 = 1$
 and $e2 \text{ face-of std-tetrahedron} \wedge \text{aff-dim } e2 = 1$
 shows $e1 \text{ congruent } e2$
 <proof>

lemma *tetrahedron-congruent-faces:*

assumes $f1 \text{ face-of std-tetrahedron} \wedge \text{aff-dim } f1 = 2$
 and $f2 \text{ face-of std-tetrahedron} \wedge \text{aff-dim } f2 = 2$
 shows $f1 \text{ congruent } f2$
 <proof>

lemma *tetrahedron-facet1-equiangular:*

defines $v1 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
 defines $v2 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
 defines $v3 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
 shows $\text{equiangular } (\text{convex hull } \{v1, v2, v3\}) \text{ (pi / 3)}$
 <proof>

lemma *tetrahedron-equiangular:*

assumes $f \text{ face-of std-tetrahedron} \wedge \text{aff-dim } f = 2$
 shows $\text{equiangular } f \text{ (pi / 3)}$

<proof>

end

6 Cube

theory *Cube*
 imports *Computation*
begin

6.1 Definition

definition *std-cube* :: (*real*, *3*) *vec set*
 where *std-cube* $\equiv$ *convex hull*
 {*vector* [1, 1, 1], *vector* [1, 1, -1],
 vector [1, -1, 1], *vector* [1, -1, -1],
 vector [-1, 1, 1], *vector* [-1, 1, -1],
 vector [-1, -1, 1], *vector* [-1, -1, -1]}

lemma *cube-fulldim*:
 shows *aff-dim std-cube* = 3
 <proof>

lemma *cube-polyhedron*:
 shows *polyhedron std-cube*
 <proof>

6.2 Facets, edges, and vertices

lemma *cube-facets*:
 fixes *f* :: (*real*³) *set*
 shows *f* *face-of std-cube* $\wedge$ *aff-dim f* = 2 $\longleftrightarrow$
 f = *convex hull* {*vector* [-1, -1, -1], *vector* [-1, -1, 1], *vector* [-1, 1, -1], *vector* [-1, 1, 1]} $\vee$
 f = *convex hull* {*vector* [-1, -1, -1], *vector* [-1, -1, 1], *vector* [1, -1, -1], *vector* [1, -1, 1]} $\vee$
 f = *convex hull* {*vector* [-1, -1, -1], *vector* [-1, 1, -1], *vector* [1, -1, -1], *vector* [1, 1, -1]} $\vee$
 f = *convex hull* {*vector* [-1, -1, 1], *vector* [-1, 1, 1], *vector* [1, -1, 1], *vector* [1, 1, 1]} $\vee$
 f = *convex hull* {*vector* [-1, 1, -1], *vector* [-1, 1, 1], *vector* [1, 1, -1], *vector* [1, 1, 1]} $\vee$
 f = *convex hull* {*vector* [1, -1, -1], *vector* [1, -1, 1], *vector* [1, 1, -1], *vector* [1, 1, 1]}
 <proof>

lemma *cube-facet1-edges*:
 defines *v1* $\equiv$ *vector* [-1, -1, -1] :: *real*³
 and *v2* $\equiv$ *vector* [-1, -1, 1] :: *real*³

and $v3 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v4 \equiv \text{vector}[-1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemma *cube-facet2-edges*:
defines $v1 \equiv \text{vector}[-1, -1, -1] :: \text{real}^3$
and $v2 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $v4 \equiv \text{vector}[1, -1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemma *cube-facet3-edges*:
defines $v1 \equiv \text{vector}[-1, -1, -1] :: \text{real}^3$
and $v2 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $v4 \equiv \text{vector}[1, 1, -1] :: \text{real}^3$
and $n \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemma *cube-facet4-edges*:
defines $v1 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
and $v2 \equiv \text{vector}[-1, 1, 1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, -1, 1] :: \text{real}^3$
and $v4 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemma *cube-facet5-edges*:

defines $v1 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
and $v2 \equiv \text{vector}[-1, 1, 1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, 1, -1] :: \text{real}^3$
and $v4 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemma *cube-facet6-edges*:

defines $v1 \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
and $v2 \equiv \text{vector}[1, -1, 1] :: \text{real}^3$
and $v3 \equiv \text{vector}[1, 1, -1] :: \text{real}^3$
and $v4 \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
and $n \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee$
 $e = \text{convex hull } \{v1, v3\} \vee$
 $e = \text{convex hull } \{v2, v4\} \vee$
 $e = \text{convex hull } \{v3, v4\})$
 $\langle \text{proof} \rangle$

lemmas *cube-facet-edges* =

cube-facet1-edges cube-facet2-edges cube-facet3-edges
cube-facet4-edges cube-facet5-edges cube-facet6-edges

lemma *cube-edges-per-face*:

assumes $f \text{ face-of std-cube}$
and $\text{aff-dim } f = 2$
shows $\text{card } \{e. e \text{ face-of std-cube} \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = 4$
 $\langle \text{proof} \rangle$

lemma *cube-edges*:

$e \text{ face-of std-cube} \wedge \text{aff-dim } e = 1 \longleftrightarrow$
 $e = \text{convex hull } \{\text{vector}[-1, -1, -1], \text{vector}[-1, -1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, -1, -1], \text{vector}[-1, 1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, -1, -1], \text{vector}[1, -1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, -1, 1], \text{vector}[-1, 1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, -1, 1], \text{vector}[1, -1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 1, -1], \text{vector}[-1, 1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 1, -1], \text{vector}[1, 1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 1, 1], \text{vector}[1, 1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, -1, -1], \text{vector}[1, -1, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, -1, -1], \text{vector}[1, 1, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, -1, 1], \text{vector}[1, 1, 1]\} \vee$

$e = \text{convex hull } \{\text{vector}[1, 1, -1], \text{vector}[1, 1, 1]\}$
 (is ?lhs $\longleftrightarrow$?rhs)
 <proof>

lemma *cube-vertices:*

shows $(v \text{ face-of std-cube} \wedge \text{aff-dim } v = 0) =$
 $(v = \{\text{vector } [1, 1, 1]\} \vee$
 $v = \{\text{vector } [1, 1, -1]\} \vee$
 $v = \{\text{vector } [1, -1, 1]\} \vee$
 $v = \{\text{vector } [1, -1, -1]\} \vee$
 $v = \{\text{vector } [-1, 1, 1]\} \vee$
 $v = \{\text{vector } [-1, 1, -1]\} \vee$
 $v = \{\text{vector } [-1, -1, 1]\} \vee$
 $v = \{\text{vector } [-1, -1, -1]\})$
 (is ?lhs = ?rhs)
 <proof>

lemma *cube-edges-per-vertex:*

assumes $v \text{ face-of std-cube}$
and $\text{aff-dim } v = 0$
shows $\text{card } \{e. e \text{ face-of std-cube} \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = 3$
 <proof>

6.3 Regularity

lemma *cube-congruent-edges:*

assumes $e1 \text{ face-of std-cube} \wedge \text{aff-dim } e1 = 1$
and $e2 \text{ face-of std-cube} \wedge \text{aff-dim } e2 = 1$
shows $e1 \text{ congruent } e2$
 <proof>

lemma *cube-congruent-faces:*

assumes $f1 \text{ face-of std-cube} \wedge \text{aff-dim } f1 = 2$
and $f2 \text{ face-of std-cube} \wedge \text{aff-dim } f2 = 2$
shows $f1 \text{ congruent } f2$
 <proof>

lemma *cube-facet1-equiangular:*

defines $v1 \equiv \text{vector}[-1, -1, -1] :: \text{real}^3$
defines $v2 \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
defines $v3 \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
defines $v4 \equiv \text{vector}[-1, 1, 1] :: \text{real}^3$
shows $\text{equiangular } (\text{convex hull } \{v1, v2, v3, v4\}) \text{ (pi / 2)}$
 <proof>

lemma *cube-equiangular:*

assumes $f \text{ face-of std-cube} \wedge \text{aff-dim } f = 2$
shows $\text{equiangular } f \text{ (pi / 2)}$
 <proof>

end

7 Octahedron

theory *Octahedron*
imports *Computation*
begin

7.1 Definition

definition *std-octahedron* :: (*real*, 3) *vec set*
where *std-octahedron* $\equiv$ *convex hull*
 $\{ \text{vector } [1, 0, 0], \text{vector } [-1, 0, 0],$
 $\text{vector } [0, 0, 1], \text{vector } [0, 0, -1],$
 $\text{vector } [0, 1, 0], \text{vector } [0, -1, 0] \}$

lemma *octahedron-fulldim*:
shows *aff-dim std-octahedron* = 3
 $\langle \text{proof} \rangle$

lemma *octahedron-polyhedron*:
shows *polyhedron std-octahedron*
 $\langle \text{proof} \rangle$

7.2 Facets, edges, and vertices

lemma *octahedron-facets*:
fixes *f* :: (*real*³) *set*
shows *f face-of std-octahedron* $\wedge$ *aff-dim f* = 2 $\longleftrightarrow$
 $f = \text{convex hull } \{ \text{vector } [-1, 0, 0], \text{vector } [0, -1, 0], \text{vector } [0, 0, -1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [-1, 0, 0], \text{vector } [0, -1, 0], \text{vector } [0, 0, 1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [-1, 0, 0], \text{vector } [0, 1, 0], \text{vector } [0, 0, -1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [-1, 0, 0], \text{vector } [0, 1, 0], \text{vector } [0, 0, 1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [1, 0, 0], \text{vector } [0, -1, 0], \text{vector } [0, 0, -1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [1, 0, 0], \text{vector } [0, -1, 0], \text{vector } [0, 0, 1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [1, 0, 0], \text{vector } [0, 1, 0], \text{vector } [0, 0, -1] \} \vee$
 $f = \text{convex hull } \{ \text{vector } [1, 0, 0], \text{vector } [0, 1, 0], \text{vector } [0, 0, 1] \}$
 $\langle \text{proof} \rangle$

lemma *octahedron-facet1-edges*:
defines *v1* $\equiv$ *vector* [-1, 0, 0] :: *real*³
and *v2* $\equiv$ *vector* [0, -1, 0] :: *real*³
and *v3* $\equiv$ *vector* [0, 0, -1] :: *real*³
and *n* $\equiv$ *vector* [1, 1, 1] :: *real*³
shows (*e face-of (convex hull {v1, v2, v3})* $\wedge$ *aff-dim e* = 1) =
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *octahedron-facet2-edges*:

defines $v1 \equiv \text{vector}[-1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, -1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
and $n \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *octahedron-facet3-edges*:

defines $v1 \equiv \text{vector}[-1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, -1] :: \text{real}^3$
and $n \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *octahedron-facet4-edges*:

defines $v1 \equiv \text{vector}[-1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
and $n \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *octahedron-facet5-edges*:

defines $v1 \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, -1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, -1] :: \text{real}^3$
and $n \equiv \text{vector}[1, -1, -1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *octahedron-facet6-edges*:

defines $v1 \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, -1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
and $n \equiv \text{vector}[-1, 1, -1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$

<proof>

lemma *octahedron-facet7-edges:*

defines $v1 \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, -1] :: \text{real}^3$
and $n \equiv \text{vector}[-1, -1, 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *octahedron-facet8-edges:*

defines $v1 \equiv \text{vector}[1, 0, 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[0, 1, 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[0, 0, 1] :: \text{real}^3$
and $n \equiv \text{vector}[1, 1, 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemmas *octahedron-facet-edges =*

octahedron-facet1-edges octahedron-facet2-edges
octahedron-facet3-edges octahedron-facet4-edges octahedron-facet5-edges
octahedron-facet6-edges octahedron-facet7-edges octahedron-facet8-edges

lemma *octahedron-edges-per-face:*

assumes $f \text{ face-of std-octahedron}$
and $\text{aff-dim } f = 2$
shows $\text{card } \{e. e \text{ face-of std-octahedron} \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = 3$
<proof>

lemma *octahedron-edges:*

$e \text{ face-of std-octahedron} \wedge \text{aff-dim } e = 1 \longleftrightarrow$
 $e = \text{convex hull } \{\text{vector}[-1, 0, 0], \text{vector}[0, -1, 0]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 0, 0], \text{vector}[0, 1, 0]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 0, 0], \text{vector}[0, 0, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[-1, 0, 0], \text{vector}[0, 0, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, 0, 0], \text{vector}[0, -1, 0]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, 0, 0], \text{vector}[0, 1, 0]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, 0, 0], \text{vector}[0, 0, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[1, 0, 0], \text{vector}[0, 0, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[0, -1, 0], \text{vector}[0, 0, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[0, -1, 0], \text{vector}[0, 0, 1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[0, 1, 0], \text{vector}[0, 0, -1]\} \vee$
 $e = \text{convex hull } \{\text{vector}[0, 1, 0], \text{vector}[0, 0, 1]\}$
(is ?lhs $\longleftrightarrow$?rhs)
<proof>

lemma *octahedron-vertices:*

shows $(v \text{ face-of std-octahedron} \wedge \text{aff-dim } v = 0) =$
 $(v = \{\text{vector } [1, 0, 0]\} \vee$
 $v = \{\text{vector } [-1, 0, 0]\} \vee$
 $v = \{\text{vector } [0, 0, 1]\} \vee$
 $v = \{\text{vector } [0, 0, -1]\} \vee$
 $v = \{\text{vector } [0, 1, 0]\} \vee$
 $v = \{\text{vector } [0, -1, 0]\})$
(is ?lhs = ?rhs)
 $\langle \text{proof} \rangle$

lemma *octahedron-edges-per-vertex:*

assumes $v \text{ face-of std-octahedron}$
and $\text{aff-dim } v = 0$
shows $\text{card } \{e. e \text{ face-of std-octahedron} \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = 4$
 $\langle \text{proof} \rangle$

7.3 Regularity

lemma *octahedron-congruent-edges:*

assumes $e1 \text{ face-of std-octahedron} \wedge \text{aff-dim } e1 = 1$
and $e2 \text{ face-of std-octahedron} \wedge \text{aff-dim } e2 = 1$
shows $e1 \text{ congruent } e2$
 $\langle \text{proof} \rangle$

lemma *octahedron-congruent-faces:*

assumes $f1 \text{ face-of std-octahedron} \wedge \text{aff-dim } f1 = 2$
and $f2 \text{ face-of std-octahedron} \wedge \text{aff-dim } f2 = 2$
shows $f1 \text{ congruent } f2$
 $\langle \text{proof} \rangle$

lemma *octahedron-facet1-equiangular:*

defines $v1 \equiv \text{vector}[-1, 0, 0] :: \text{real}^3$
defines $v2 \equiv \text{vector}[0, -1, 0] :: \text{real}^3$
defines $v3 \equiv \text{vector}[0, 0, -1] :: \text{real}^3$
shows $\text{equiangular } (\text{convex hull } \{v1, v2, v3\}) \text{ (pi / 3)}$
 $\langle \text{proof} \rangle$

lemma *octahedron-equiangular:*

assumes $f \text{ face-of std-octahedron} \wedge \text{aff-dim } f = 2$
shows $\text{equiangular } f \text{ (pi / 3)}$
 $\langle \text{proof} \rangle$

end

8 Dodecahedron

theory *Dodecahedron*

```

imports
  Computation
  HOL-Library.Quadratic-Discriminant
begin

```

8.1 Definition

```

definition std-dodecahedron :: (real, 3) vec set
where std-dodecahedron  $\equiv$  convex hull
  {vector [1, 1, 1], vector [1, 1, -1],
   vector [1, -1, 1], vector [1, -1, -1],
   vector [-1, 1, 1], vector [-1, 1, -1],
   vector [-1, -1, 1], vector [-1, -1, -1],
   vector [0, inverse  $\varphi$ ,  $\varphi$ ], vector [0, inverse  $\varphi$ , - $\varphi$ ],
   vector [0, -inverse  $\varphi$ ,  $\varphi$ ], vector [0, -inverse  $\varphi$ , - $\varphi$ ],
   vector [inverse  $\varphi$ ,  $\varphi$ , 0], vector [inverse  $\varphi$ , - $\varphi$ , 0],
   vector [-inverse  $\varphi$ ,  $\varphi$ , 0], vector [-inverse  $\varphi$ , - $\varphi$ , 0],
   vector [ $\varphi$ , 0, inverse  $\varphi$ ], vector [- $\varphi$ , 0, inverse  $\varphi$ ],
   vector [ $\varphi$ , 0, -inverse  $\varphi$ ], vector [- $\varphi$ , 0, -inverse  $\varphi$ ]}

```

lemma std-dodecahedron-explicit:

```

shows std-dodecahedron = convex hull
  { vector[1, 1, 1],
    vector[1, 1, -1],
    vector[1, -1, 1],
    vector[1, -1, -1],
    vector[-1, 1, 1],
    vector[-1, 1, -1],
    vector[-1, -1, 1],
    vector[-1, -1, -1],
    vector[0, -1 / 2 + 1 / 2 * sqrt(5), 1 / 2 + 1 / 2 * sqrt(5)],
    vector[0, -1 / 2 + 1 / 2 * sqrt(5), -1 / 2 + -1 / 2 * sqrt(5)],
    vector[0, 1 / 2 + -1 / 2 * sqrt(5), 1 / 2 + 1 / 2 * sqrt(5)],
    vector[0, 1 / 2 + -1 / 2 * sqrt(5), -1 / 2 + -1 / 2 * sqrt(5)],
    vector[-1 / 2 + 1 / 2 * sqrt(5), 1 / 2 + 1 / 2 * sqrt(5), 0],
    vector[-1 / 2 + 1 / 2 * sqrt(5), -1 / 2 + -1 / 2 * sqrt(5), 0],
    vector[1 / 2 + -1 / 2 * sqrt(5), 1 / 2 + 1 / 2 * sqrt(5), 0],
    vector[1 / 2 + -1 / 2 * sqrt(5), -1 / 2 + -1 / 2 * sqrt(5), 0],
    vector[1 / 2 + 1 / 2 * sqrt(5), 0, -1 / 2 + 1 / 2 * sqrt(5)],
    vector[-1 / 2 + -1 / 2 * sqrt(5), 0, -1 / 2 + 1 / 2 * sqrt(5)],
    vector[1 / 2 + 1 / 2 * sqrt(5), 0, 1 / 2 + -1 / 2 * sqrt(5)],
    vector[-1 / 2 + -1 / 2 * sqrt(5), 0, 1 / 2 + -1 / 2 * sqrt(5)] }
<proof>

```

lemma std-dodecahedron-eq:

```

shows std-dodecahedron = convex hull
  { vector[rat5 1 0, rat5 1 0, rat5 1 0],
    vector[rat5 1 0, rat5 1 0, rat5 (-1) 0],
    vector[rat5 1 0, rat5 (-1) 0, rat5 1 0],

```

$\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0],$
 $\text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0],$
 $\text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0],$
 $\text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0],$
 $\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)]\}$
 $\langle \text{proof} \rangle$

lemma *dodecahedron-fulldim:*

shows $\text{aff-dim } \text{std-dodecahedron} = 3$

$\langle \text{proof} \rangle$

lemma *dodecahedron-polyhedron:*

shows $\text{polyhedron } \text{std-dodecahedron}$

$\langle \text{proof} \rangle$

8.2 Facets, edges, and vertices

lemma *dodecahedron-facets:*

fixes $f :: (\text{real}^3) \text{ set}$

shows $f \text{ face-of } \text{std-dodecahedron} \wedge \text{aff-dim } f = 2 \longleftrightarrow$

$f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)], \text{vector}[\text{rat5 } (1/2) \ (-1/2),$
 $\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0,$
 $\text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0]\}$ $\vee$

$f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)], \text{vector}[\text{rat5 } (1/2) \ (-1/2),$
 $\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0]\}$ $\vee$

$f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2),$
 $\text{rat5 } (1/2) \ (1/2)]\}$ $\vee$

$f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2),$
 $\text{rat5 } (-1/2) \ (-1/2)]\}$ $\vee$

$f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0],$

and $v_4 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v_5 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 2 \ 0, \text{rat5 } 1 \ (-1), \text{rat5 } 0 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v_1, v_2, v_3, v_4, v_5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v_1, v_2\} \vee e = \text{convex hull } \{v_1, v_5\} \vee e = \text{convex hull } \{v_2, v_4\} \vee e = \text{convex hull } \{v_3, v_4\} \vee e = \text{convex hull } \{v_3, v_5\})$
<proof>

lemma *dodecahedron-facet3-edges:*

defines $v_1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] :: \text{real}^3$
and $v_2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v_3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v_4 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $v_5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-3) \ 1, \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v_1, v_2, v_3, v_4, v_5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v_1, v_2\} \vee e = \text{convex hull } \{v_1, v_3\} \vee e = \text{convex hull } \{v_2, v_5\} \vee e = \text{convex hull } \{v_3, v_4\} \vee e = \text{convex hull } \{v_4, v_5\})$
<proof>

lemma *dodecahedron-facet4-edges:*

defines $v_1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] :: \text{real}^3$
and $v_2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v_3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v_4 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $v_5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 3 \ (-1), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v_1, v_2, v_3, v_4, v_5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v_1, v_2\} \vee e = \text{convex hull } \{v_1, v_3\} \vee e = \text{convex hull } \{v_2, v_5\} \vee e = \text{convex hull } \{v_3, v_4\} \vee e = \text{convex hull } \{v_4, v_5\})$
<proof>

lemma *dodecahedron-facet5-edges:*

defines $v_1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v_2 \equiv \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v_3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v_4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v_5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ (-1), \text{rat5 } (-3) \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v_1, v_2, v_3, v_4, v_5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v_1, v_2\} \vee e = \text{convex hull } \{v_1, v_4\} \vee e = \text{convex hull } \{v_2, v_3\} \vee e = \text{convex hull } \{v_3, v_5\} \vee e = \text{convex hull } \{v_4, v_5\})$
<proof>

lemma *dodecahedron-facet6-edges:*

defines $v_1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v_2 \equiv \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v_3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v_4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$

and $v5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 1, \text{rat5 } (-3) \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v4\} \vee e = \text{convex hull } \{v2, v3\} \vee e = \text{convex hull } \{v3, v5\} \vee e = \text{convex hull } \{v4, v5\})$
<proof>

lemma *dodecahedron-facet7-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-2) \ 0, \text{rat5 } (-1) \ 1, \text{rat5 } 0 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v4\} \vee e = \text{convex hull } \{v1, v5\} \vee e = \text{convex hull } \{v2, v3\} \vee e = \text{convex hull } \{v2, v5\} \vee e = \text{convex hull } \{v3, v4\})$
<proof>

lemma *dodecahedron-facet8-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ (-1), \text{rat5 } 3 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v4\} \vee e = \text{convex hull } \{v2, v3\} \vee e = \text{convex hull } \{v3, v5\} \vee e = \text{convex hull } \{v4, v5\})$
<proof>

lemma *dodecahedron-facet9-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 1, \text{rat5 } 3 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v4\} \vee e = \text{convex hull } \{v2, v3\} \vee e = \text{convex hull } \{v3, v5\} \vee e = \text{convex hull } \{v4, v5\})$
<proof>

lemma *dodecahedron-facet10-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$

and $n \equiv \text{vector}[\text{rat5 } 2 \ 0, \text{rat5 } (-1) \ 1, \text{rat5 } 0 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v4\} \vee e = \text{convex hull } \{v1, v5\} \vee e = \text{convex hull } \{v2, v3\} \vee e = \text{convex hull } \{v2, v5\} \vee e = \text{convex hull } \{v3, v4\})$
<proof>

lemma *dodecahedron-facet11-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-3) \ 1, \text{rat5 } 0 \ 0, \text{rat5 } 1 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v5\} \vee e = \text{convex hull } \{v3, v4\} \vee e = \text{convex hull } \{v4, v5\})$
<proof>

lemma *dodecahedron-facet12-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v4 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $v5 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 3 \ (-1), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3, v4, v5\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v5\} \vee e = \text{convex hull } \{v3, v4\} \vee e = \text{convex hull } \{v4, v5\})$
<proof>

lemmas *dodecahedron-facet-edges =*

dodecahedron-facet1-edges dodecahedron-facet2-edges dodecahedron-facet3-edges
dodecahedron-facet4-edges dodecahedron-facet5-edges dodecahedron-facet6-edges
dodecahedron-facet7-edges dodecahedron-facet8-edges dodecahedron-facet9-edges
dodecahedron-facet10-edges dodecahedron-facet11-edges dodecahedron-facet12-edges

lemma *dodecahedron-edges-per-face:*

assumes $f \text{ face-of } \text{std-dodecahedron}$
and $\text{aff-dim } f = 2$
shows $\text{card } \{e. e \text{ face-of } \text{std-dodecahedron} \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = 5$
<proof>

lemma *dodecahedron-edges:*

$e \text{ face-of } \text{std-dodecahedron} \wedge \text{aff-dim } e = 1 \longleftrightarrow$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)],$

```

vector[rat5 (-1) 0, rat5 1 0, rat5 1 0]] ∨
e = convex hull {vector[rat5 (-1/2) (-1/2), rat5 0 0, rat5 (1/2) (-1/2)],
vector[rat5 (-1) 0, rat5 (-1) 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (-1/2) (-1/2), rat5 0 0, rat5 (1/2) (-1/2)],
vector[rat5 (-1) 0, rat5 1 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (-1/2) (-1/2), rat5 0 0],
vector[rat5 (1/2) (-1/2), rat5 (-1/2) (-1/2), rat5 0 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (-1/2) (-1/2), rat5 0 0],
vector[rat5 1 0, rat5 (-1) 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (-1/2) (-1/2), rat5 0 0],
vector[rat5 1 0, rat5 (-1) 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vec-
tor[rat5 (1/2) (-1/2), rat5 (1/2) (1/2), rat5 0 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vec-
tor[rat5 1 0, rat5 1 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (-1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vec-
tor[rat5 1 0, rat5 1 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (1/2) (-1/2), rat5 (-1/2) (-1/2), rat5 0 0],
vector[rat5 (-1) 0, rat5 (-1) 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (1/2) (-1/2), rat5 (-1/2) (-1/2), rat5 0 0],
vector[rat5 (-1) 0, rat5 (-1) 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (1/2) (-1/2), rat5 (1/2) (1/2), rat5 0 0], vec-
tor[rat5 (-1) 0, rat5 1 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (1/2) (-1/2), rat5 (1/2) (1/2), rat5 0 0], vec-
tor[rat5 (-1) 0, rat5 1 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (-1/2) (1/2)], vec-
tor[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (-1/2)]} ∨
e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (-1/2) (1/2)], vec-
tor[rat5 1 0, rat5 (-1) 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (-1/2) (1/2)], vec-
tor[rat5 1 0, rat5 1 0, rat5 1 0]} ∨
e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (-1/2)], vec-
tor[rat5 1 0, rat5 (-1) 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (-1/2)], vec-
tor[rat5 1 0, rat5 1 0, rat5 (-1) 0]} ∨
e = convex hull {vector[rat5 (-1) 0, rat5 (-1) 0, rat5 (-1) 0], vector[rat5 0
0, rat5 (1/2) (-1/2), rat5 (-1/2) (-1/2)]} ∨
e = convex hull {vector[rat5 (-1) 0, rat5 (-1) 0, rat5 1 0], vector[rat5 0 0,
rat5 (1/2) (-1/2), rat5 (1/2) (1/2)]} ∨
e = convex hull {vector[rat5 (-1) 0, rat5 1 0, rat5 (-1) 0], vector[rat5 0 0,
rat5 (-1/2) (1/2), rat5 (-1/2) (-1/2)]} ∨
e = convex hull {vector[rat5 (-1) 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5
(-1/2) (1/2), rat5 (1/2) (1/2)]} ∨
e = convex hull {vector[rat5 1 0, rat5 (-1) 0, rat5 (-1) 0], vector[rat5 0 0,
rat5 (1/2) (-1/2), rat5 (-1/2) (-1/2)]} ∨
e = convex hull {vector[rat5 1 0, rat5 (-1) 0, rat5 1 0], vector[rat5 0 0, rat5
(1/2) (-1/2), rat5 (1/2) (1/2)]} ∨
e = convex hull {vector[rat5 1 0, rat5 1 0, rat5 (-1) 0], vector[rat5 0 0, rat5
(-1/2) (1/2), rat5 (-1/2) (-1/2)]} ∨

```

$e = \text{convex hull } \{ \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)] \} \vee$
 $e = \text{convex hull } \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] \} \vee$
 $e = \text{convex hull } \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)] \}$
 (is ?lhs $\longleftrightarrow$?rhs)
 <proof>

lemma *dodecahedron-vertices:*

shows $(v \text{ face-of std-dodecahedron} \wedge \text{aff-dim } v = 0) =$
 $(v = \{ \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } 1 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1) \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1/2) \ (1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (1/2) \ (-1/2), \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) \ (1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] \} \vee$
 $v = \{ \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) \ (-1/2)] \} \}$
 (is ?lhs = ?rhs)
 <proof>

lemma *dodecahedron-edges-per-vertex:*

assumes $v \text{ face-of std-dodecahedron}$
 and $\text{aff-dim } v = 0$
 shows $\text{card } \{ e. e \text{ face-of std-dodecahedron} \wedge \text{aff-dim } e = 1 \wedge v \subseteq e \} = 3$
 <proof>

8.3 Regularity

lemma *dodecahedron-congruent-edges:*

assumes $e1 \text{ face-of std-dodecahedron} \wedge \text{aff-dim } e1 = 1$
 and $e2 \text{ face-of std-dodecahedron} \wedge \text{aff-dim } e2 = 1$
 shows $e1 \text{ congruent } e2$
 <proof>

lemma *dodecahedron-congruent-faces:*

assumes $f1$ face-of std-dodecahedron $\wedge$ aff-dim $f1 = 2$
and $f2$ face-of std-dodecahedron $\wedge$ aff-dim $f2 = 2$
shows $f1$ congruent $f2$
 <proof>

lemma dodecahedron-facet1-equiangular:
defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1/2) (1/2)] :: \text{real}^3$
defines $v2 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (1/2) (-1/2)] :: \text{real}^3$
defines $v3 \equiv \text{vector}[\text{rat5 } (1/2) (-1/2), \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
defines $v4 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
defines $v5 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
shows equiangular (convex hull $\{v1, v2, v3, v4, v5\}$) $(3 * \pi / 5)$
 <proof>

lemma dodecahedron-equiangular:
assumes f face-of std-dodecahedron $\wedge$ aff-dim $f = 2$
shows equiangular f $(3 * \pi / 5)$
 <proof>

end

9 Icosahedron

theory Icosahedron
imports
 Computation
begin

9.1 Definition

definition std-icosahedron :: (real, 3) vec set
where std-icosahedron $\equiv$ convex hull
 { vector $[0, 1, \varphi]$, vector $[0, 1, -\varphi]$,
 vector $[0, -1, \varphi]$, vector $[0, -1, -\varphi]$,
 vector $[1, \varphi, 0]$, vector $[1, -\varphi, 0]$,
 vector $[-1, \varphi, 0]$, vector $[-1, -\varphi, 0]$,
 vector $[\varphi, 0, 1]$, vector $[-\varphi, 0, 1]$,
 vector $[\varphi, 0, -1]$, vector $[-\varphi, 0, -1]$ }

lemma std-icosahedron-eq:
shows std-icosahedron = convex hull
 { vector $[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) (1/2)]$,
 vector $[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) (-1/2)]$,
 vector $[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) (1/2)]$,
 vector $[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) (-1/2)]$,
 vector $[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0]$,
 vector $[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0]$,
 vector $[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0]$,
 vector $[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0]$,

$\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0],$
 $\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0],$
 $\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0]]$
 <proof>

lemma *icosahedron-fulldim:*

shows $\text{aff-dim std-icosahedron} = 3$
 <proof>

lemma *icosahedron-polyhedron:*

shows $\text{polyhedron std-icosahedron}$
 <proof>

9.2 Facets, edges, and vertices

lemma *icosahedron-facets:*

fixes $f :: (\text{real}^3) \text{ set}$
shows $f \text{ face-of std-icosahedron} \wedge \text{aff-dim } f = 2 \longleftrightarrow$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2)$
 $(-1/2), \text{rat5 } 0 \ 0]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2)$
 $(1/2), \text{rat5 } 0 \ 0]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0,$
 $\text{rat5 } (-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 }$
 $(-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0],$
 $\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) (-1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 }$
 $(-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 }$
 $(-1) \ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2)$
 $(1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 }$
 $(-1) \ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) (1/2)]\}$
 $\vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 }$
 $0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) (1/2)]\}$
 $\vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 }$
 $(1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 \ 0]\}$
 $\vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 }$
 $(1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0], \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0], \text{vector}[\text{rat5 }$

$1\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2)]\}$
 $\vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2)], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0, \text{rat5 } 1\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0, \text{rat5 } 1\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0, \text{rat5 } 1\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } 1\ 0, \text{rat5 } (-1/2) (-1/2)]\} \vee$
 $f = \text{convex hull } \{\text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0], \text{vector}[\text{rat5 } 0\ 0, \text{rat5 } 1\ 0, \text{rat5 } (1/2) (1/2)]\}$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet1-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0, \text{rat5 } 1\ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 1\ 1, \text{rat5 } (-1)\ 1, \text{rat5 } 0\ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet2-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0, \text{rat5 } (-1)\ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1/2) (-1/2), \text{rat5 } 0\ 0, \text{rat5 } 1\ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } (-1)\ 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0\ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-1) (-1), \text{rat5 } (-1)\ 1, \text{rat5 } 0\ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet3-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 2 \ 0, \text{rat5 } 2 \ 0, \text{rat5 } 2 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet4-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-2) \ 0, \text{rat5 } 2 \ 0, \text{rat5 } (-2) \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet5-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-1) \ 1, \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet6-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-2) \ 0, \text{rat5 } (-2) \ 0, \text{rat5 } 2 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet7-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 2 \ 0, \text{rat5 } (-2) \ 0, \text{rat5 } (-2) \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet8-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 1 \ (-1), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet9-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 1 \ 1, \text{rat5 } 1 \ (-1), \text{rat5 } 0 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet10-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-1) \ (-1), \text{rat5 } 1 \ (-1), \text{rat5 } 0 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet11-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 2 \ 0, \text{rat5 } (-2) \ 0, \text{rat5 } (-2) \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle \text{proof} \rangle$

lemma *icosahedron-facet12-edges:*

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-2) \ 0, \text{rat5 } (-2) \ 0, \text{rat5 } 2 \ 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$

<proof>

lemma *icosahedron-facet13-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 0, \text{rat5 } (-1) 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } (-1) 0, \text{rat5 } (-1/2) (-1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } 1 0, \text{rat5 } (-1/2) (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-1) 1, \text{rat5 } 0 0, \text{rat5 } (-1) (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *icosahedron-facet14-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 0, \text{rat5 } 1 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } (-1) 0, \text{rat5 } (1/2) (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } (-2) 0, \text{rat5 } 2 0, \text{rat5 } (-2) 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *icosahedron-facet15-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 0, \text{rat5 } 1 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 0, \text{rat5 } (1/2) (1/2), \text{rat5 } 0 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } 1 0, \text{rat5 } (1/2) (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 2 0, \text{rat5 } 2 0, \text{rat5 } 2 0] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *icosahedron-facet16-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (1/2) (1/2), \text{rat5 } 0 0, \text{rat5 } 1 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } (-1) 0, \text{rat5 } (1/2) (1/2)] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } 1 0, \text{rat5 } (1/2) (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 1 (-1), \text{rat5 } 0 0, \text{rat5 } (-1) (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
<proof>

lemma *icosahedron-facet17-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (-1) 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 0, \text{rat5 } (-1/2) (-1/2), \text{rat5 } 0 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } (-1) 0, \text{rat5 } (-1/2) (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 0, \text{rat5 } 1 1, \text{rat5 } (-1) 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$

$\{v2, v3\}$)
 $\langle proof \rangle$

lemma *icosahedron-facet18-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ (-1), \text{rat5 } (-1) \ 1] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle proof \rangle$

lemma *icosahedron-facet19-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 1, \text{rat5 } 1 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle proof \rangle$

lemma *icosahedron-facet20-edges*:

defines $v1 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v2 \equiv \text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
and $v3 \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)] :: \text{real}^3$
and $n \equiv \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ (-1), \text{rat5 } 1 \ (-1)] :: \text{real}^3$
shows $(e \text{ face-of } (\text{convex hull } \{v1, v2, v3\}) \wedge \text{aff-dim } e = 1) =$
 $(e = \text{convex hull } \{v1, v2\} \vee e = \text{convex hull } \{v1, v3\} \vee e = \text{convex hull } \{v2, v3\})$
 $\langle proof \rangle$

lemmas *icosahedron-facet-edges* =

icosahedron-facet1-edges icosahedron-facet2-edges icosahedron-facet3-edges
icosahedron-facet4-edges icosahedron-facet5-edges icosahedron-facet6-edges
icosahedron-facet7-edges icosahedron-facet8-edges icosahedron-facet9-edges
icosahedron-facet10-edges icosahedron-facet11-edges icosahedron-facet12-edges
icosahedron-facet13-edges icosahedron-facet14-edges icosahedron-facet15-edges
icosahedron-facet16-edges icosahedron-facet17-edges icosahedron-facet18-edges
icosahedron-facet19-edges icosahedron-facet20-edges

lemma *icosahedron-edges-per-face*:

assumes $f \text{ face-of } \text{std-icosahedron}$
and $\text{aff-dim } f = 2$
shows $\text{card } \{e. e \text{ face-of } \text{std-icosahedron} \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = 3$
 $\langle proof \rangle$

lemma *icosahedron-edges*:

[illegible]

$e = \text{convex hull } \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)]\} \vee$
 $e = \text{convex hull } \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)], \text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]\}$
 $(\text{is } ?lhs \longleftrightarrow ?rhs)$
 $\langle \text{proof} \rangle$

lemma *icosahedron-vertices:*

shows $(v \text{ face-of std-icosahedron} \wedge \text{aff-dim } v = 0) =$
 $(v = \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2)]\} \vee$
 $v = \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2)]\} \vee$
 $v = \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2)]\} \vee$
 $v = \{\text{vector}[\text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2)]\} \vee$
 $v = \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } 1 \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (1/2) \ (1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0]\} \vee$
 $v = \{\text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0]\})$
 $(\text{is } ?lhs = ?rhs)$
 $\langle \text{proof} \rangle$

lemma *icosahedron-edges-per-vertex:*

assumes $v \text{ face-of std-icosahedron}$
and $\text{aff-dim } v = 0$
shows $\text{card } \{e. e \text{ face-of std-icosahedron} \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = 5$
 $\langle \text{proof} \rangle$

9.3 Regularity

lemma *icosahedron-congruent-edges:*

assumes $e1 \text{ face-of std-icosahedron} \wedge \text{aff-dim } e1 = 1$
and $e2 \text{ face-of std-icosahedron} \wedge \text{aff-dim } e2 = 1$
shows $e1 \text{ congruent } e2$
 $\langle \text{proof} \rangle$

lemma *icosahedron-congruent-faces:*

assumes $f1 \text{ face-of std-icosahedron} \wedge \text{aff-dim } f1 = 2$
and $f2 \text{ face-of std-icosahedron} \wedge \text{aff-dim } f2 = 2$

shows $f1$ congruent $f2$
 $\langle proof \rangle$

lemma *icosahedron-facet1-equiangular*:

defines $v1 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } (-1) \ 0] :: \text{real}^3$
defines $v2 \equiv \text{vector}[\text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0, \text{rat5 } 1 \ 0] :: \text{real}^3$
defines $v3 \equiv \text{vector}[\text{rat5 } (-1) \ 0, \text{rat5 } (-1/2) \ (-1/2), \text{rat5 } 0 \ 0] :: \text{real}^3$
shows *equiangular* (*convex hull* $\{v1, v2, v3\}$) ($\pi / 3$)
 $\langle proof \rangle$

lemma *icosahedron-equiangular*:

assumes f *face-of* *std-icosahedron* $\wedge$ *aff-dim* $f = 2$
shows *equiangular* f ($\pi / 3$)
 $\langle proof \rangle$

end

10 Main result

theory *Platonic-Solids*

imports

Euler-Polyhedron-Formula.Euler-Formula
Polytope-More
Tetrahedron
Cube
Octahedron
Dodecahedron
Icosahedron

begin

10.1 Possible Schläfli symbols

lemma *platonic-solid-limits*:

fixes $p :: (\text{real}^3)$ *set*
assumes *is-polytope*: *polytope* p
and *aff-dim-p*: *aff-dim* $p = 3$
and *edges-per-face*: $\forall f. f$ *face-of* $p \wedge \text{aff-dim } f = 2 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = m$
and *edges-per-vert*: $\forall v. v$ *face-of* $p \wedge \text{aff-dim } v = 0 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = n$
shows $(m = 3 \wedge n = 3) \vee$
 $(m = 4 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 4) \vee$
 $(m = 5 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 5)$
 $\langle proof \rangle$

10.2 Exactly five Platonic solids

theorem *platonic-solids-full*:

shows $(\exists p::(\text{real}^3) \text{ set. polytope } p \wedge \text{aff-dim } p = 3 \wedge$
 $(\forall f. f \text{ face-of } p \wedge \text{aff-dim } f = 2 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = m) \wedge$
 $(\forall v. v \text{ face-of } p \wedge \text{aff-dim } v = 0 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = n) \wedge$
 $(\forall f1 f2. f1 \text{ face-of } p \wedge \text{aff-dim } f1 = 2 \wedge$
 $f2 \text{ face-of } p \wedge \text{aff-dim } f2 = 2 \longrightarrow f1 \text{ congruent } f2) \wedge$
 $(\forall e1 e2. e1 \text{ face-of } p \wedge \text{aff-dim } e1 = 1 \wedge$
 $e2 \text{ face-of } p \wedge \text{aff-dim } e2 = 1 \longrightarrow e1 \text{ congruent } e2) \wedge$
 $(\exists t. \forall f. f \text{ face-of } p \wedge \text{aff-dim } f = 2 \longrightarrow \text{equiangular } f \ t)) =$
 $((m = 3 \wedge n = 3) \vee$
 $(m = 4 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 4) \vee$
 $(m = 5 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 5))$
(is ?lhs = ?rhs)
 $\langle \text{proof} \rangle$

This is the original top-level statement of the HOL Light formalization, kept here for parity. It omits the regularity condition on p .

theorem *platonic-solids*:

shows $(\exists p::(\text{real}^3) \text{ set. polytope } p \wedge \text{aff-dim } p = 3 \wedge$
 $(\forall f. f \text{ face-of } p \wedge \text{aff-dim } f = 2 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge e \subseteq f\} = m) \wedge$
 $(\forall v. v \text{ face-of } p \wedge \text{aff-dim } v = 0 \longrightarrow$
 $\text{card } \{e. e \text{ face-of } p \wedge \text{aff-dim } e = 1 \wedge v \subseteq e\} = n)) =$
 $((m = 3 \wedge n = 3) \vee$
 $(m = 4 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 4) \vee$
 $(m = 5 \wedge n = 3) \vee$
 $(m = 3 \wedge n = 5))$
(is ?lhs = ?rhs)
 $\langle \text{proof} \rangle$

end

References

- [1] J. Harrison. Proof of the Platonic solids theorem, accessed 2025. HOL Light repository.