

# Pick's Theorem

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## Abstract

We formalize Pick's theorem for finding the area of a simple polygon whose vertices are integral lattice points [1]. We are inspired by John Harrison's formalization of Pick's theorem in HOL Light [2], but tailor our proof approach to avoid a primary challenge point in his formalization, which is proving that any polygon with more than three vertices can be split (in its interior) by a line between some two vertices. Our formalization involves augmenting the existing geometry libraries in various foundational ways (e.g., by adding the definition of a polygon and formalizing some key properties thereof).

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# 1 Misc. Linear Algebra Setup

**lemma** *vec-scaleR-2*:  $(c::real) *_R ((vector [a, b])::real^2) = vector [a * c, b * c]$   
*<proof>*

**definition** *is-int* ::  $real \Rightarrow bool$  **where**  
 $is-int\ x \longleftrightarrow (\exists n::int. x = n)$

**lemma** *is-int-sum*:  $is-int\ x \wedge is-int\ y \longrightarrow is-int\ (x + y)$   
*<proof>*

**lemma** *is-int-minus*:  $is-int\ x \wedge is-int\ y \longrightarrow is-int\ (x - y)$   
*<proof>*

**lemma** *is-int-mult*:  $is-int\ x \wedge is-int\ y \longrightarrow is-int\ (x * y)$   
*<proof>*

**definition** *integral-vec* ::  $real^2 \Rightarrow bool$  **where**  
 $integral-vec\ v \longleftrightarrow (is-int\ (v\$1) \wedge is-int\ (v\$2))$

**lemma** *integral-vec-sum*:  $integral-vec\ v \wedge integral-vec\ w \longrightarrow integral-vec\ (v + w)$   
*<proof>*

**lemma** *integral-vec-minus*:  $integral-vec\ v \longrightarrow integral-vec\ (-v)$   
*<proof>*

**lemma** *real-2-inner*:  
**shows**  $((vector [a, b])::(real^2)) \cdot ((vector [c, d])::(real^2)) = a*c + b*d$   
**(is**  $?v \cdot ?w = a*c + b*d$   
*<proof>*

**lemma** *integral-vec-2*:  
**fixes**  $a\ b :: int$   
**assumes**  $v = vector [a, b]$   
**shows**  $integral-vec\ v$   
*<proof>*

**definition** *matrix-inv* ::  $real^2 \Rightarrow real^2 \Rightarrow bool$  **where**  
 $matrix-inv\ A\ A' \longleftrightarrow (A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

**lemma** *mat-vec-mult-2*:  
**fixes**  $v :: real^2$  **and**  
 $T :: real^2 \Rightarrow real^2$   
**defines**  $x: x \equiv v\$1$  **and**  $y: y \equiv v\$2$  **and**  
 $a: a \equiv T\$1\$1$  **and**  $b: b \equiv T\$1\$2$  **and**  
 $c: c \equiv T\$2\$1$  **and**  $d: d \equiv T\$2\$2$   
**shows**  $(T *v v) = vector [x*a + y*b, x*c + y*d]$   
*<proof>*

**definition** *integral-mat* ::  $\text{real}^{\mathbb{Q}}^{\mathbb{Q}} \Rightarrow \text{bool}$  **where**  
*integral-mat*  $T \longleftrightarrow (\forall v. \text{integral-vec } v \longrightarrow \text{integral-vec } (T * v))$

**definition** *integral-mat-surj* ::  $\text{real}^{\mathbb{Q}}^{\mathbb{Q}} \Rightarrow \text{bool}$  **where**  
*integral-mat-surj*  $T \longleftrightarrow (\forall v. \text{integral-vec } v \longrightarrow (\exists w. \text{integral-vec } w \wedge T * v = w))$

**definition** *integral-mat-bij* ::  $\text{real}^{\mathbb{Q}}^{\mathbb{Q}} \Rightarrow \text{bool}$  **where**  
*integral-mat-bij*  $T \longleftrightarrow \text{integral-mat } T \wedge \text{integral-mat-surj } T$

**lemma** *integral-mat-integral-vec*:  $\text{integral-mat } A \longrightarrow \text{integral-vec } v \longrightarrow \text{integral-vec } (A * v)$   
 ⟨*proof*⟩

**lemma** *integral-mat-int-entries*:  
**fixes**  $T :: \text{real}^{\mathbb{Q}}^{\mathbb{Q}}$   
**assumes** *integral-mat*  $T$   
**defines**  $a: a \equiv T\$1\$1$  **and**  $b: b \equiv T\$1\$2$  **and**  
 $c: c \equiv T\$2\$1$  **and**  $d: d \equiv T\$2\$2$   
**shows**  $\text{is-int } a \wedge \text{is-int } b \wedge \text{is-int } c \wedge \text{is-int } d$   
 ⟨*proof*⟩

## 2 Integral Bijective Matrix Determinant

**lemma** *integral-mat-int-det*:  
**fixes**  $T :: \text{real}^{\mathbb{Q}}^{\mathbb{Q}}$   
**assumes** *integral-mat*  $T$   
**shows**  $\text{is-int } (\text{det } T)$   
 ⟨*proof*⟩

**lemma** *integral-mat-bij-inv*:  
**fixes**  $T :: \text{real}^{\mathbb{Q}}^{\mathbb{Q}}$   
**assumes** *integral-mat-bij*  $T$   
**obtains**  $T_{\text{inv}}$  **where**  $\text{invertible } T \wedge \text{integral-mat-bij } T_{\text{inv}} \wedge \text{matrix-inv } T T_{\text{inv}}$   
 ⟨*proof*⟩

**lemma** *integral-mat-bij-det-pm1*:  
**fixes**  $T :: \text{real}^{\mathbb{Q}}^{\mathbb{Q}}$   
**assumes** *integral-mat-bij*  $T$   
**shows**  $\text{det } T = 1 \vee \text{det } T = -1$   
 ⟨*proof*⟩

**end**  
**theory** *Polygon-Jordan-Curve*  
**imports**  
*HOL-Analysis.Cartesian-Space*  
*HOL-Analysis.Path-Connected*

**begin**

### 3 Polygon Definitions

**type-synonym**  $R\text{-to-}R^2 = (\text{real} \Rightarrow \text{real}^2)$

**definition**  $\text{closed-path} :: R\text{-to-}R^2 \Rightarrow \text{bool}$  **where**  
 $\text{closed-path } g \longleftrightarrow \text{path } g \wedge \text{pathstart } g = \text{pathfinish } g$

**definition**  $\text{path-inside} :: R\text{-to-}R^2 \Rightarrow (\text{real}^2)$  *set* **where**  
 $\text{path-inside } g = \text{inside } (\text{path-image } g)$

**definition**  $\text{path-outside} :: R\text{-to-}R^2 \Rightarrow (\text{real}^2)$  *set* **where**  
 $\text{path-outside } g = \text{outside } (\text{path-image } g)$

**fun**  $\text{make-polygonal-path} :: (\text{real}^2)$  *list*  $\Rightarrow R\text{-to-}R^2$  **where**  
 $\text{make-polygonal-path } [] = \text{linepath } 0\ 0$   
 $| \text{make-polygonal-path } [a] = \text{linepath } a\ a$   
 $| \text{make-polygonal-path } [a,b] = \text{linepath } a\ b$   
 $| \text{make-polygonal-path } (a \# b \# xs) = (\text{linepath } a\ b) \text{ +++ } \text{make-polygonal-path } (b \# xs)$

**definition**  $\text{polygonal-path} :: R\text{-to-}R^2 \Rightarrow \text{bool}$  **where**  
 $\text{polygonal-path } g \longleftrightarrow g \in \text{make-polygonal-path}\{\text{xs} :: (\text{real}^2)$  *list*.  $\text{True}\}$

**definition**  $\text{all-integral} :: (\text{real}^2)$  *list*  $\Rightarrow \text{bool}$  **where**  
 $\text{all-integral } l = (\forall x \in \text{set } l. \text{integral-vec } x)$

**definition**  $\text{polygon} :: R\text{-to-}R^2 \Rightarrow \text{bool}$  **where**  
 $\text{polygon } g \longleftrightarrow \text{polygonal-path } g \wedge \text{simple-path } g \wedge \text{closed-path } g$

**definition**  $\text{integral-polygon} :: R\text{-to-}R^2 \Rightarrow \text{bool}$  **where**  
 $\text{integral-polygon } g \longleftrightarrow$   
 $(\text{polygon } g \wedge (\exists \text{vts}. g = \text{make-polygonal-path } \text{vts} \wedge \text{all-integral } \text{vts}))$

**definition**  $\text{make-triangle} :: \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow R\text{-to-}R^2$  **where**  
 $\text{make-triangle } a\ b\ c = \text{make-polygonal-path } [a, b, c, a]$

**definition**  $\text{polygon-of} :: R\text{-to-}R^2 \Rightarrow (\text{real}^2)$  *list*  $\Rightarrow \text{bool}$  **where**  
 $\text{polygon-of } p\ \text{vts} \longleftrightarrow \text{polygon } p \wedge p = \text{make-polygonal-path } \text{vts}$

**definition**  $\text{good-linepath} :: \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow (\text{real}^2)$  *list*  $\Rightarrow \text{bool}$  **where**  
 $\text{good-linepath } a\ b\ \text{vts} \longleftrightarrow (\text{let } p = \text{make-polygonal-path } \text{vts} \text{ in}$   
 $a \neq b \wedge \{a, b\} \subseteq \text{set } \text{vts} \wedge \text{path-image } (\text{linepath } a\ b) \subseteq \text{path-inside } p \cup \{a, b\})$

**definition**  $\text{good-polygonal-path} :: \text{real}^2 \Rightarrow (\text{real}^2)$  *list*  $\Rightarrow \text{real}^2 \Rightarrow (\text{real}^2)$  *list*

$\Rightarrow$  *bool where*  
*good-polygonal-path*  $a$  *cutvts*  $b$  *vts*  $\longleftrightarrow$  (  
*let*  $p = \text{make-polygonal-path } vts$  *in*  
*let*  $p\text{-cut} = \text{make-polygonal-path } ([a] @ \text{cutvts} @ [b])$  *in*  
 $(a \neq b \wedge \{a, b\} \subseteq \text{set } vts \wedge \text{path-image } (p\text{-cut}) \subseteq \text{path-inside } p \cup \{a, b\} \wedge$   
*loop-free*  $p\text{-cut})$ )

## 4 Jordan Curve Theorem for Polygons

**definition** *inside-outside*  $:: R\text{-to-}R^2 \Rightarrow (\text{real}^2) \text{ set} \Rightarrow (\text{real}^2) \text{ set} \Rightarrow \text{bool}$  **where**  
*inside-outside*  $p$  *ins* *outs*  $\longleftrightarrow$   
 $(ins \neq \{\}) \wedge \text{open } ins \wedge \text{connected } ins \wedge$   
 $outs \neq \{\} \wedge \text{open } outs \wedge \text{connected } outs \wedge$   
 $\text{bounded } ins \wedge \neg \text{bounded } outs \wedge$   
 $ins \cap outs = \{\} \wedge ins \cup outs = - \text{path-image } p \wedge$   
 $\text{frontier } ins = \text{path-image } p \wedge \text{frontier } outs = \text{path-image } p$ )

**lemma** *Jordan-inside-outside-real2*:

**fixes**  $p :: \text{real} \Rightarrow \text{real}^2$

**assumes** *simple-path*  $p$  *pathfinish*  $p = \text{pathstart } p$

**shows**  $\text{inside}(\text{path-image } p) \neq \{\} \wedge$   
 $\text{open}(\text{inside}(\text{path-image } p)) \wedge$   
 $\text{connected}(\text{inside}(\text{path-image } p)) \wedge$   
 $\text{outside}(\text{path-image } p) \neq \{\} \wedge$   
 $\text{open}(\text{outside}(\text{path-image } p)) \wedge$   
 $\text{connected}(\text{outside}(\text{path-image } p)) \wedge$   
 $\text{bounded}(\text{inside}(\text{path-image } p)) \wedge$   
 $\neg \text{bounded}(\text{outside}(\text{path-image } p)) \wedge$   
 $\text{inside}(\text{path-image } p) \cap \text{outside}(\text{path-image } p) = \{\} \wedge$   
 $\text{inside}(\text{path-image } p) \cup \text{outside}(\text{path-image } p) =$   
 $- \text{path-image } p \wedge$   
 $\text{frontier}(\text{inside}(\text{path-image } p)) = \text{path-image } p \wedge$   
 $\text{frontier}(\text{outside}(\text{path-image } p)) = \text{path-image } p$

*<proof>*

**lemma** *inside-outside-polygon*:

**fixes**  $p :: R\text{-to-}R^2$

**assumes** *polygon*: *polygon*  $p$

**shows** *inside-outside*  $p$  (*path-inside*  $p$ ) (*path-outside*  $p$ )

*<proof>*

**lemma** *inside-outside-unique*:

**fixes**  $p :: R\text{-to-}R^2$

**assumes** *polygon*  $p$

**assumes** *io1*: *inside-outside*  $p$  *inside1* *outside1*

**assumes** *io2*: *inside-outside*  $p$  *inside2* *outside2*

**shows**  $\text{inside1} = \text{inside2} \wedge \text{outside1} = \text{outside2}$

*<proof>*

**lemma** *polygon-jordan-curve*:

**fixes**  $p :: R\text{-to-}R^2$

**assumes** *polygon*  $p$

**obtains** *inside outside* **where**

*inside-outside*  $p$  *inside outside*

*<proof>*

**lemma** *connected-component-image*:

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow 'b::\text{euclidean-space}$

**assumes** *linear*  $f$  *bij*  $f$

**shows**  $f^{-1}(\text{connected-component-set } S \ x) = \text{connected-component-set } (f^{-1} \ S) \ (f \ x)$

*<proof>*

**lemma** *bounded-map*:

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow 'b::\text{euclidean-space}$

**assumes** *linear*  $f$  *bij*  $f$

**shows** *bounded*  $(f^{-1} \ S) = \text{bounded } S$

*<proof>*

**lemma** *inside-bijective-linear-image*:

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow 'b::\text{euclidean-space}$

**fixes**  $c :: \text{real} \Rightarrow 'a$

**assumes** *c-simple: path*  $c$

**assumes** *linear*  $f$  *bij*  $f$

**shows** *inside*  $(f^{-1}(\text{path-image } c)) = f^{-1}(\text{inside}(\text{path-image } c))$

*<proof>*

**lemma** *bij-image-intersection*:

**assumes** *path-image*  $c1 \cap \text{path-image } c2 = S$

**assumes** *bij*  $f$

**assumes**  $c \in \text{path-image } (f \circ c1) \cap \text{path-image } (f \circ c2)$

**shows**  $c \in f^{-1} \ S$

*<proof>*

**theorem** (*in* *c1-on-open-R2*) *split-inside-simple-closed-curve-locale*:

**fixes**  $c :: \text{real} \Rightarrow 'a$

**assumes** *c1-simple: simple-path*  $c1$  **and** *c1-start: pathstart*  $c1 = a$  **and** *c1-end: pathfinish*  $c1 = b$

**assumes** *c2-simple: simple-path*  $c2$  **and** *c2-start: pathstart*  $c2 = a$  **and** *c2-end: pathfinish*  $c2 = b$

**assumes** *c-simple: simple-path*  $c$  **and** *c-start: pathstart*  $c = a$  **and** *c-end: pathfinish*  $c = b$

**assumes** *a-neq-b: a*  $\neq b$

**and** *c1c2: path-image*  $c1 \cap \text{path-image } c2 = \{a, b\}$

**and** *c1c: path-image*  $c1 \cap \text{path-image } c = \{a, b\}$

**and**  $c2c$ :  $path\text{-}image\ c2 \cap path\text{-}image\ c = \{a,b\}$   
**and**  $ne\text{-}12$ :  $path\text{-}image\ c \cap inside(path\text{-}image\ c1 \cup path\text{-}image\ c2) \neq \{\}$   
**obtains**  $inside(path\text{-}image\ c1 \cup path\text{-}image\ c) \cap inside(path\text{-}image\ c2 \cup path\text{-}image\ c) = \{\}$   
 $inside(path\text{-}image\ c1 \cup path\text{-}image\ c) \cup inside(path\text{-}image\ c2 \cup path\text{-}image\ c) \cup$   
 $(path\text{-}image\ c - \{a,b\}) = inside(path\text{-}image\ c1 \cup path\text{-}image\ c2)$   
 $\langle proof \rangle$

**lemma** *split-inside-simple-closed-curve-real2*:

**fixes**  $c :: real \Rightarrow real^2$   
**assumes**  $c1\text{-}simple$ :  $simple\text{-}path\ c1$  **and**  $c1\text{-}start$ :  $pathstart\ c1 = a$  **and**  $c1\text{-}end$ :  
 $pathfinish\ c1 = b$   
**assumes**  $c2\text{-}simple$ :  $simple\text{-}path\ c2$  **and**  $c2\text{-}start$ :  $pathstart\ c2 = a$  **and**  $c2\text{-}end$ :  
 $pathfinish\ c2 = b$   
**assumes**  $c\text{-}simple$ :  $simple\text{-}path\ c$  **and**  $c\text{-}start$ :  $pathstart\ c = a$  **and**  $c\text{-}end$ :  $pathfinish\ c = b$   
**assumes**  $a\text{-}neq\text{-}b$ :  $a \neq b$   
**and**  $c1c2$ :  $path\text{-}image\ c1 \cap path\text{-}image\ c2 = \{a,b\}$   
**and**  $c1c$ :  $path\text{-}image\ c1 \cap path\text{-}image\ c = \{a,b\}$   
**and**  $c2c$ :  $path\text{-}image\ c2 \cap path\text{-}image\ c = \{a,b\}$   
**and**  $ne\text{-}12$ :  $path\text{-}image\ c \cap inside(path\text{-}image\ c1 \cup path\text{-}image\ c2) \neq \{\}$   
**obtains**  $inside(path\text{-}image\ c1 \cup path\text{-}image\ c) \cap inside(path\text{-}image\ c2 \cup path\text{-}image\ c) = \{\}$   
 $inside(path\text{-}image\ c1 \cup path\text{-}image\ c) \cup inside(path\text{-}image\ c2 \cup path\text{-}image\ c) \cup$   
 $(path\text{-}image\ c - \{a,b\}) = inside(path\text{-}image\ c1 \cup path\text{-}image\ c2)$   
 $\langle proof \rangle$

**end**

**theory** *Polygon-Lemmas*

**imports**

*Polygon-Jordan-Curve*  
*HOL-Library.Sublist*  
*HOL.Set-Interval*  
*HOL.Fun*

**begin**

## 5 Properties of make polygonal path, pathstart and pathfinish of a polygon

**lemma** *make-polygonal-path-induct*[*case-names Empty Single Two Multiple*]:

**fixes**  $ell :: (real^2) list$   
**assumes**  $empty$ :  $\bigwedge ell. ell = [] \implies P\ ell$   
**and**  $single$ :  $\bigwedge ell. \llbracket length\ ell = 1 \rrbracket \implies P\ ell$   
**and**  $two$ :  $\bigwedge ell. \llbracket length\ ell = 2 \rrbracket \implies P\ ell$   
**and**  $multiple$ :  $\bigwedge ell.$



$\llbracket \text{length } ell > 2;$   
 $P \llbracket (ell!0), (ell!1) \rrbracket;$   
 $P \llbracket (ell!1)\#(\text{drop } 2 \text{ } ell) \rrbracket \implies P \text{ } ell$   
**shows**  $P \text{ } ell$   
 $\langle \text{proof} \rangle$

**lemma** *make-polygonal-path-gives-path:*  
**fixes**  $v :: (\text{real}^2) \text{ list}$   
**shows**  $\text{path } (\text{make-polygonal-path } v)$   
 $\langle \text{proof} \rangle$

**corollary** *polygonal-path-is-path:*  
**fixes**  $g :: R\text{-to-}R^2$   
**assumes** *polygonal-path*  $g$   
**shows** *path*  $g$   
 $\langle \text{proof} \rangle$

**lemma** *polygon-to-polygonal-path:*  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes** *polygon*  $p$   
**obtains**  $ell$  **where**  $p = \text{make-polygonal-path } ell$   
 $\langle \text{proof} \rangle$

**lemma** *polygon-pathstart:*  
**fixes**  $g :: R\text{-to-}R^2$   
**assumes**  $l \neq []$   
**assumes**  $g = \text{make-polygonal-path } l$   
**shows**  $\text{pathstart } g = l!0$   
 $\langle \text{proof} \rangle$

**lemma** *polygon-pathfinish:*  
**fixes**  $g :: R\text{-to-}R^2$   
**assumes**  $l \neq []$   
**assumes**  $g = \text{make-polygonal-path } l$   
**shows**  $\text{pathfinish } g = l!(\text{length } l - 1)$   
 $\langle \text{proof} \rangle$

**lemma** *make-polygonal-path-image-property:*  
**assumes**  $\text{length } vts \geq 2$   
**assumes** *p-is-path:*  $x \in \text{path-image } (\text{make-polygonal-path } vts)$   
**shows**  $\exists k < \text{length } vts - 1. x \in \text{path-image } (\text{linepath } (vts ! k) (vts ! (k + 1)))$   
 $\langle \text{proof} \rangle$

**lemma** *linepaths-subset-make-polygonal-path-image:*  
**assumes**  $\text{length } vts \geq 2$   
**assumes**  $k < \text{length } vts - 1$   
**shows**  $\text{path-image } (\text{linepath } (vts ! k) (vts ! (k + 1))) \subseteq \text{path-image } (\text{make-polygonal-path } vts)$

*<proof>*

**lemma** *vertices-on-path-image*: **shows**  $set\ vts \subseteq path\text{-}image\ (make\text{-}polygonal\text{-}path\ vts)$

*<proof>*

**lemma** *path-image-cons-union*:

**assumes**  $p = make\text{-}polygonal\text{-}path\ vts$

**assumes**  $p' = make\text{-}polygonal\text{-}path\ vts'$

**assumes**  $vts' \neq []$

**assumes**  $vts = a \# vts' \wedge b = vts!0$

**shows**  $path\text{-}image\ p = path\text{-}image\ (linepath\ a\ b) \cup path\text{-}image\ p'$

*<proof>*

**lemma** *polygonal-path-image-linepath-union*:

**assumes**  $p = make\text{-}polygonal\text{-}path\ vts$

**assumes**  $n = length\ vts$

**assumes**  $n \geq 2$

**shows**  $path\text{-}image\ p = (\bigcup \{path\text{-}image\ (linepath\ (vts!i)\ (vts!(i+1))) \mid i.\ i \leq n - 2\})$

*<proof>*

## 6 Loop Free Properties

**lemma** *constant-linepath-is-not-loop-free*:

**shows**  $\neg(loop\text{-}free\ ((linepath\ a\ a)::real \Rightarrow real^2))$

*<proof>*

**lemma** *doubling-back-is-not-loop-free*:

**assumes**  $a \neq b$

**shows**  $\neg(loop\text{-}free\ ((make\text{-}polygonal\text{-}path\ [a,\ b,\ a])::real \Rightarrow real^2))$

*<proof>*

**lemma** *not-loop-free-first-component*:

**assumes**  $\neg(loop\text{-}free\ p1)$

**shows**  $\neg(loop\text{-}free\ (p1+++p2))$

*<proof>*

**lemma** *not-loop-free-second-component*:

**assumes**  $pathfinish\text{-}pathstart:\ pathfinish\ p1 = pathstart\ p2$

**assumes**  $\neg(loop\text{-}free\ p2)$

**shows**  $\neg(loop\text{-}free\ (p1+++p2))$

*<proof>*

**lemma** *loop-free-subpath*:

**assumes**  $path\ p$

**assumes**  $u\text{-and-}v:\ u \in \{0..1\}\ v \in \{0..1\}\ u < v$

**assumes**  $\neg(loop\text{-}free\ (subpath\ u\ v\ p))$

**shows**  $\neg(loop\text{-}free\ p)$

*<proof>*

**lemma** *loop-free-associative:*

**assumes** *path p*

**assumes** *path q*

**assumes** *path r*

**assumes** *pathfinish p = pathstart q*

**assumes** *pathfinish q = pathstart r*

**shows**  $\neg (\text{loop-free } ((p +++ q) +++ r)) \longleftrightarrow \neg (\text{loop-free } (p +++ (q +++ r)))$

*<proof>*

**lemma** *polygon-at-least-3-vertices:*

**assumes** *polygon p and*

*p = make-polygonal-path vts*

**shows**  $\text{card } (\text{set } vts) \geq 3$

*<proof>*

**lemma** *polygon-vertices-length-at-least-4:*

**assumes** *polygon p and*

*p = make-polygonal-path vts*

**shows**  $\text{length } vts \geq 4$

*<proof>*

**lemma** *linepath-loop-free:*

**assumes**  $a \neq b$

**shows** *loop-free (linepath a b)*

*<proof>*

## 7 Explicit Linepath Characterization of Polygonal Paths

**lemma** *triangle-linepath-images:*

**fixes**  $x :: \text{real}$

**assumes**  $vts = [a, b, c]$

**assumes**  $p = \text{make-polygonal-path } vts$

**shows**  $x \in \{0..1/2\} \implies p \ x = ((\text{linepath } a \ b)) \ (2*x)$

$x \in \{1/2..1\} \implies p \ x = ((\text{linepath } b \ c)) \ (2*x - 1)$

*<proof>*

**lemma** *polygon-linepath-images1:*

**fixes**  $n :: \text{nat}$

**assumes**  $n \geq 3$

**assumes**  $\text{length } ell = n$

**assumes**  $x \in \{0..1/2\}$

**shows**  $\text{make-polygonal-path } ell \ x = ((\text{linepath } (ell \ ! \ 0) \ (ell \ ! \ 1))) \ (2*x)$

*<proof>*

**lemma** *sum-insert* [*simp*]:

**assumes**  $x \notin F$  **and** *finite*  $F$

**shows**  $(\sum y \in \text{insert } x F. P y) = (\sum y \in F. P y) + P x$

*<proof>*

**lemma** *sum-of-index-diff* [*simp*]:

**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{comm-monoid-add}$

**shows**  $(\sum i \in \{a..<a+b\}. f(i-a)) = (\sum i \in \{..<b\}. f(i))$

*<proof>*

**lemma** *sum-of-index-diff2* [*simp*]:

**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{comm-monoid-add}$

**shows**  $(\sum i \in \{a+c..b+c\}. f(i)) = (\sum i \in \{a..b\}. f(i+c))$

*<proof>*

**lemma** *sum-split* [*simp*]:

**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{comm-monoid-add}$

**assumes**  $c \in \{a..b\}$

**shows**  $(\sum i \in \{a..b\}. f i) = (\sum i \in \{a..c\}. f i) + (\sum i \in \{c+1..b\}. f i)$

*<proof>*

**lemma** *summation-helper*:

**fixes**  $x :: \text{real}$

**fixes**  $k :: \text{nat}$

**assumes**  $1 \leq k$

**shows**  $(2 :: \text{real}) * (\sum i = 1..k. 1 / 2^i) - 1 = (\sum i = 1..(k-1). (1 / (2^i)))$

*<proof>*

**lemma** *polygon-linepath-images2*:

**fixes**  $n k :: \text{nat}$

**fixes**  $ell :: (\text{real}^2) \text{ list}$

**fixes**  $f :: \text{nat} \Rightarrow \text{real} \Rightarrow \text{real}$

**assumes**  $n \geq 3$

**assumes**  $0 \leq k \wedge k \leq n - 3$

**assumes**  $\text{length } ell = n$

**assumes**  $p : p = \text{make-polygonal-path } ell$

**assumes**  $f = (\lambda k x. (x - (\sum i \in \{1..k\}. 1/(2^i))) * (2^{k+1}))$

**assumes**  $x \in \{(\sum i \in \{1..k\}. 1/(2^i))..(\sum i \in \{1..(k+1)\}. 1/(2^i))\}$

**shows**  $p x = ((\text{linepath } (ell ! k) (ell ! (k+1)) (f k x)))$

*<proof>*

**lemma** *polygon-linepath-images3*:

**fixes**  $n k :: \text{nat}$

**fixes**  $ell :: (\text{real}^2) \text{ list}$

**assumes**  $n \geq 3$

**assumes**  $\text{length } ell = n$

**assumes**  $p = \text{make-polygonal-path } ell$

**assumes**  $x \in \{(\sum i \in \{1..n-2\}. 1/(2^i))..1\}$

**assumes**  $f = (\lambda x. (x - (\sum i \in \{1..n-2\}. 1/(2^i))) * (2^{n-2}))$   
**shows**  $p\ x = (\text{linepath } (ell\ !\ (n-2))\ (ell\ !\ (n-1)))\ (f\ x)$   
 <proof>

## 8 A Triangle is a Polygon

**lemma** *not-collinear-linepaths-intersect-helper*:

**assumes** *not-collinear*:  $\neg \text{collinear } \{a,b,c\}$   
**assumes**  $0 \leq k1$   
**assumes**  $k1 \leq 1$   
**assumes**  $0 \leq k2$   
**assumes**  $k2 \leq 1$   
**assumes** *eo*:  $k2 = 0 \implies k1 \neq 1$   
**shows**  $\neg ((\text{linepath } a\ b)\ k1 = (\text{linepath } b\ c)\ k2)$   
 <proof>

**lemma** *not-collinear-linepaths-intersect-helper-2*:

**assumes** *not-collinear*:  $\neg \text{collinear } \{a,b,c\}$   
**assumes**  $0 \leq k1$   
**assumes**  $k1 \leq 1$   
**assumes**  $0 \leq k2$   
**assumes**  $k2 \leq 1$   
**assumes** *eo*:  $k1 = 0 \implies k2 \neq 1$   
**shows**  $\neg ((\text{linepath } a\ b)\ k1 = (\text{linepath } c\ a)\ k2)$   
 <proof>

**lemma** *not-collinear-loopfree-path*:  $\bigwedge a\ b\ c::\text{real}^2. \neg \text{collinear } \{a,b,c\} \implies \text{loop-free}$   
 $((\text{linepath } a\ b) \text{ +++ } (\text{linepath } b\ c))$   
 <proof>

**lemma** *triangle-is-polygon*:  $\bigwedge a\ b\ c. \neg \text{collinear } \{a,b,c\} \implies \text{polygon } (\text{make-triangle } a\ b\ c)$   
 <proof>

**lemma** *have-wraparound-vertex*:

**assumes** *polygon*  $p$   
**assumes**  $p = \text{make-polygonal-path } vts$   
**shows**  $vts = (\text{take } (\text{length } vts - 1)\ vts)@[vts\ !\ 0]$   
 <proof>

**lemma** *polygon-at-least-3-vertices-wraparound*:

**assumes** *polygon*  $p$   
**assumes**  $p = \text{make-polygonal-path } vts$   
**shows**  $\text{card } (\text{set } (\text{take } (\text{length } vts - 1)\ vts)) \geq 3$   
 <proof>

## 9 Polygon Vertex Rotation

**definition** *rotate-polygon-vertices*:: 'a list  $\Rightarrow$  nat  $\Rightarrow$  'a list  
**where** *rotate-polygon-vertices* ell i =  
 (let ell1 = rotate i (butlast ell) in ell1 @ [ell1 ! 0])

**lemma** *rotate-polygon-vertices-same-set*:  
**assumes** polygon (make-polygonal-path vts)  
**shows** set (rotate-polygon-vertices vts i) = set vts  
 <proof>

**lemma** *arb-rotation-as-single-rotation*:  
**fixes** i:: nat  
**shows** rotate-polygon-vertices vts (Suc i) = rotate-polygon-vertices (rotate-polygon-vertices vts i) 1  
 <proof>

**lemma** *rotation-sum*:  
**fixes** i j :: nat  
**shows** rotate-polygon-vertices vts (i + j) = rotate-polygon-vertices (rotate-polygon-vertices vts i) j  
 <proof>

**lemma** *rotated-polygon-vertices-helper*:  
**fixes** p :: R-to-R2  
**assumes** poly-p: polygon p  
**assumes** p-is-path: p = make-polygonal-path vts  
**assumes** p'-is: p' = make-polygonal-path (rotate-polygon-vertices vts 1)  
**shows** (vts ! 0) = (rotate-polygon-vertices vts 1) ! (length (rotate-polygon-vertices vts 1) - 2)  
 (rotate-polygon-vertices vts 1) ! (length (rotate-polygon-vertices vts 1) - 1)  
 = (vts ! 1)  
 <proof>

**lemma** *rotate-polygon-vertices-same-length*:  
**fixes** vts :: (real<sup>2</sup>) list  
**assumes** length vts  $\geq$  1  
**shows** length vts = length (rotate-polygon-vertices vts i)  
 <proof>

**lemma** *rotated-polygon-vertices-helper2*:  
**assumes** len-gteq: length vts  $\geq$  2  
**assumes** i < length vts - 1  
**assumes** hd vts = last vts  
**shows** (rotate-polygon-vertices vts 1) ! i = vts ! (i+1)  
 <proof>

**lemma** *polygon-rotation-t-translation1*:  
**assumes** polygon-of p vts

**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $x' \in \{(\sum i \in \{1..k\}. 1/(2^{\widehat{i}}))..(\sum i \in \{1..k+1\}. 1/(2^{\widehat{i}}))\}$   
**assumes**  $n = \text{length } vts$   
**assumes**  $0 \leq k \wedge k \leq n - 4$   
**assumes**  $l = x' - (\sum i \in \{1..k\}. 1/(2^{\widehat{i}}))$   
**assumes**  $x = l/2 + (\sum i \in \{1..(k+1)\}. 1/(2^{\widehat{i}}))$   
**shows**  $x \in \{(\sum i \in \{1..k+1\}. 1/(2^{\widehat{i}}))..(\sum i \in \{1..k+2\}. 1/(2^{\widehat{i}}))\}$   
 $p' x' = p x$   
 <proof>

**lemma** *polygon-rotation-t-translation1-strict:*

**assumes** *polygon-of*  $p \ vts$   
**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $x' \in \{(\sum i \in \{1..k\}. 1/(2^{\widehat{i}}))..<(\sum i \in \{1..k+1\}. 1/(2^{\widehat{i}}))\}$   
**assumes**  $n = \text{length } vts$   
**assumes**  $0 \leq k \wedge k \leq n - 4$   
**assumes**  $l = x' - (\sum i \in \{1..k\}. 1/(2^{\widehat{i}}))$   
**assumes**  $x = l/2 + (\sum i \in \{1..(k+1)\}. 1/(2^{\widehat{i}}))$   
**shows**  $x \in \{(\sum i \in \{1..k+1\}. 1/(2^{\widehat{i}}))..<(\sum i \in \{1..k+2\}. 1/(2^{\widehat{i}}))\}$   
 $p' x' = p x$   
 <proof>

**lemma** *polygon-rotation-t-translation2:*

**assumes** *polygon-of*  $p \ vts$   
**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $n = \text{length } vts$   
**assumes**  $x' \in \{(\sum i \in \{1..(n-3)\}. 1/(2^{\widehat{i}}))..(\sum i \in \{1..(n-2)\}. 1/(2^{\widehat{i}}))\}$   
**assumes**  $x = x' + 1/(2^{\widehat{(n-2)}})$   
**shows**  $x \in \{(\sum i \in \{1..n-2\}. 1/(2^{\widehat{i}}))..1\}$   
 $p' x' = p x$   
 <proof>

**lemma** *polygon-rotation-t-translation2-strict:*

**assumes** *polygon-of*  $p \ vts$   
**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $n = \text{length } vts$   
**assumes**  $x' \in \{(\sum i \in \{1..(n-3)\}. 1/(2^{\widehat{i}}))..<(\sum i \in \{1..(n-2)\}. 1/(2^{\widehat{i}}))\}$   
**assumes**  $x = x' + 1/(2^{\widehat{(n-2)}})$   
**shows**  $x \in \{(\sum i \in \{1..n-2\}. 1/(2^{\widehat{i}}))..<1\}$   
 $p' x' = p x$   
 <proof>

**lemma** *polygon-rotation-t-translation3:*

**assumes** *polygon-of*  $p \ vts$

**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $x' \in \{(\sum i \in \{1..n-2\}. 1/(2^i))..1\}$   
**assumes**  $n = \text{length } vts$   
**assumes**  $l = x' - (\sum i \in \{1..n-2\}. 1/(2^i))$   
**assumes**  $x = l * (2^{n-3})$   
**shows**  $x \in \{0..1/2\}$   
 $p' \ x' = p \ x$   
 <proof>

**lemma** *polygon-rotation-t-translation3-strict*:  
**assumes** *polygon-of*  $p \ vts$   
**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
 (is  $p' = \text{make-polygonal-path } ?vts'$ )  
**assumes**  $x' \in \{(\sum i \in \{1..n-2\}. 1/(2^i))..<1\}$   
**assumes**  $n = \text{length } vts$   
**assumes**  $l = x' - (\sum i \in \{1..n-2\}. 1/(2^i))$   
**assumes**  $x = l * (2^{n-3})$   
**shows**  $x \in \{0..<1/2\}$   
 $p' \ x' = p \ x$   
 <proof>

**lemma** *f-gteq-0-sum-gt*:  $\bigwedge f :: \text{nat} \Rightarrow \text{real}. (\bigwedge i :: \text{nat}. (f \ i) > 0) \Longrightarrow a > b \Longrightarrow (\sum i = 1..a. (f \ i)) > (\sum i = 1..b. (f \ i))$  **for**  $a \ b :: \text{nat}$   
 <proof>

**lemma** *rotation-intervals-disjoint*:  
**assumes**  $k1 \neq k2$   
**shows**  $\{\sum i = 1..k1. 1 / (2^i :: \text{real})..<\sum i = 1..k1+1. 1 / 2^i\} \cap \{\sum i = 1..k2. 1 / (2^i :: \text{real})..<\sum i = 1..k2+1. 1 / 2^i\} = \{\}$   
 <proof>

**lemma** *bounding-interval-helper1*:  
**shows**  $(\sum i = 1..k. 1 / (2^i :: \text{real})) = (2^k - 1)/(2^k)$   
 <proof>

**lemma** *bounding-interval-helper2*:  
**fixes**  $x :: \text{real}$   
**assumes**  $x \in \{0..<1\}$   
**shows**  $\exists k. x < (\sum i = 1..k. 1 / (2^i :: \text{real}))$   
 <proof>

**lemma** *bounding-interval-for-reals-btw01*:  
**fixes**  $x :: \text{real}$   
**assumes**  $x \in \{0..<1\}$   
**shows**  $\exists k. x \in \{(\sum i \in \{1..k\}. 1/(2^i :: \text{real}))..<(\sum i \in \{1..(k+1)\}. 1/(2^i))\}$   
 <proof>

**lemma** *all-rotation-intervals-between-0and1*:



**shows**  $\{(\sum i \in \{1..k\}. 1/(2^{\widehat{i}}::real))..(\sum i \in \{1..(k+1)\}. 1/(2^{\widehat{i}}))\} \subseteq \{0..<1\}$   
<proof>

**lemma** *all-rotation-intervals-between-0and1-strict:*

**shows**  $\{(\sum i \in \{1..k\}. 1/(2^{\widehat{i}}::real))..<(\sum i \in \{1..(k+1)\}. 1/(2^{\widehat{i}}))\} \subseteq \{0..<1\}$   
<proof>

**lemma** *one-polygon-rotation-is-loop-free:*

**assumes** *polygon-of p vts*

**assumes**  $p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts \ 1)$   
(**is**  $p' = \text{make-polygonal-path } ?vts'$ )

**shows** *loop-free p'*

<proof>

**lemma** *one-rotation-is-polygon:*

**fixes**  $p :: R\text{-to-}R^2$

**fixes**  $i :: nat$

**assumes** *poly-p: polygon p and*

*p-is-path: p = make-polygonal-path vts and*

*p'-is: p' = make-polygonal-path (rotate-polygon-vertices vts 1)*

(**is**  $p' = \text{make-polygonal-path } ?vts'$ )

**shows** *polygon p'*

<proof>

**lemma** *rotation-is-polygon:*

**fixes**  $p :: R\text{-to-}R^2$

**fixes**  $i :: nat$

**assumes** *polygon p and*

*p = make-polygonal-path vts*

**shows** *polygon (make-polygonal-path (rotate-polygon-vertices vts i))*

<proof>

**lemma** *polygon-rotate-mod:*

**fixes**  $vts :: (real^2)$  *list*

**assumes**  $n = \text{length } vts$

**assumes**  $n \geq 2$

**assumes**  $\text{hd } vts = \text{last } vts$

**shows** *rotate-polygon-vertices vts (n - 1) = vts*

<proof>

**lemma** *polygon-rotate-mod-arb:*

**fixes**  $vts :: (real^2)$  *list*

**assumes**  $n = \text{length } vts$

**assumes**  $n \geq 2$

**assumes**  $\text{hd } vts = \text{last } vts$

**shows** *rotate-polygon-vertices vts ((n - 1) \* i) = vts*

<proof>

**lemma** *unrotation-is-polygon:*

**fixes**  $p :: R\text{-to-}R^2$   
**fixes**  $i :: \text{nat}$   
**assumes**  $\text{polygon } (\text{make-polygonal-path } (\text{rotate-polygon-vertices } vts\ i))$   
 $\quad (\text{is polygon } (\text{make-polygonal-path } ?vts'))$   
 $\quad p = \text{make-polygonal-path } vts$   
 $\quad \text{hd } vts = \text{last } vts$   
**shows**  $\text{polygon } p$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{rotated-polygon-vertices}$ :  
**assumes**  $vts' = \text{rotate-polygon-vertices } vts\ j$   
**assumes**  $\text{hd } vts = \text{last } vts$   
**assumes**  $\text{length } vts \geq 2$   
**assumes**  $j \leq i \wedge i < \text{length } vts$   
**shows**  $vts\ !\ i = vts'\ !\ (i - j)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{polygon-path-image}$ :  
**assumes**  $\text{poly-p: polygon } p$   
**assumes**  $\text{p-is-path: } p = \text{make-polygonal-path } vts$   
**shows**  $\text{path-image } p = p'\ \{0 ..< 1\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{polygon-vts-one-rotation}$ :  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes**  $\text{poly-p: polygon } p$  **and**  
 $\quad \text{p-is-path: } p = \text{make-polygonal-path } vts$  **and**  
 $\quad \text{p'-is: } p' = \text{make-polygonal-path } (\text{rotate-polygon-vertices } vts\ 1)$   
**shows**  $\text{path-image } p = \text{path-image } p'$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{polygon-vts-arb-rotation}$ :  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes**  $\text{polygon } p$  **and**  
 $\quad p = \text{make-polygonal-path } vts$   
**shows**  $\text{path-image } p = \text{path-image } (\text{make-polygonal-path } (\text{rotate-polygon-vertices } vts\ i))$   
 $\langle \text{proof} \rangle$

## 10 Translating a Polygon

**lemma**  $\text{linepath-translation}$ :  
 $\text{linepath } ((\lambda x. x + u)\ a)\ ((\lambda x. x + u)\ b) = (\lambda x. x + u) \circ (\text{linepath } a\ b)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{make-polygonal-path-translate}$ :  
**assumes**  $\text{length } vts \geq 2$   
**shows**  $\text{make-polygonal-path } (\text{map } (\lambda x. x + u)\ vts) = (\lambda x. x + u) \circ (\text{make-polygonal-path } vts)$

*<proof>*

**lemma** *translation-is-polygon:*

**assumes** *polygon-of*  $p$   $pts$

**shows** *polygon-of*  $((\lambda x. x + u) \circ p)$   $(\text{map } (\lambda x. x + u) pts)$  **(is** *polygon-of*  $?p'$   $?pts'$ )

*<proof>*

## 11 Misc. properties

**lemma** *tail-of-loop-free-polygonal-path-is-loop-free:*

**assumes** *loop-free*  $(\text{make-polygonal-path } (x\#\text{tail}))$  **(is** *loop-free*  $?p$ ) **and**  
 $\text{length tail} \geq 2$

**shows** *loop-free*  $(\text{make-polygonal-path tail})$  **(is** *loop-free*  $?p'$ )

*<proof>*

**lemma** *tail-of-simple-polygonal-path-is-simple:*

**assumes** *simple-path*  $(\text{make-polygonal-path } (x\#\text{tail}))$  **(is** *simple-path*  $?p$ ) **and**  
 $\text{length tail} \geq 2$

**shows** *simple-path*  $(\text{make-polygonal-path tail})$  **(is** *simple-path*  $?p'$ )

*<proof>*

**lemma** *interior-vertex-in-path-image-interior:*

**fixes**  $pts :: (\text{real}^2)$  *list*

**assumes**  $x \in \text{set } (\text{butlast } (\text{drop } 1 pts))$

**shows**  $\exists t. t \in \{0 < .. < 1\} \wedge (\text{make-polygonal-path } pts) t = x$

*<proof>*

**lemma** *loop-free-polygonal-path-pts-distinct:*

**assumes** *loop-free*  $(\text{make-polygonal-path } pts)$

**shows** *distinct*  $(\text{butlast } pts)$

*<proof>*

**lemma** *loop-free-polygonal-path-pts-drop1-distinct:*

**assumes** *loop-free*  $(\text{make-polygonal-path } pts)$

**shows** *distinct*  $(\text{drop } 1 pts)$

*<proof>*

**lemma** *simple-polygonal-path-pts-distinct:*

**assumes** *simple-path*  $(\text{make-polygonal-path } pts)$

**shows** *distinct*  $(\text{butlast } pts)$

*<proof>*

**lemma** *edge-subset-path-image:*

**assumes**  $p = \text{make-polygonal-path } pts$  **and**

$(i::\text{int}) \in \{0..<((\text{length } pts) - 1)\}$  **and**

$x = pts!i$  **and**

$y = vts!(i+1)$   
**shows**  $path\text{-}image (linepath\ x\ y) \subseteq path\text{-}image\ p$  (**is**  $?xy\text{-}img \subseteq ?p\text{-}img$ )  
 <proof>

## 12 Properties of Sublists of Polygonal Path Vertex Lists

**lemma** *make-polygonal-path-image-append-var:*

**assumes**  $length\ vts1 \geq 2$   
**shows**  $path\text{-}image (make\text{-}polygonal\text{-}path (vts1 @ [v])) = path\text{-}image (make\text{-}polygonal\text{-}path\ vts1\ \text{+++}\ (linepath\ (vts1\ !\ (length\ vts1 - 1))\ v))$   
 <proof>

**lemma** *make-polygonal-path-image-append-helper:*

**assumes**  $length\ vts1 \geq 1 \wedge length\ vts2 \geq 1$   
**shows**  $path\text{-}image (make\text{-}polygonal\text{-}path (vts1 @ [v] @ [v] @ vts2)) = path\text{-}image (make\text{-}polygonal\text{-}path (vts1 @ [v] @ vts2))$   
 <proof>

**lemma** *make-polygonal-path-image-append:*

**assumes**  $length\ vts1 \geq 2 \wedge length\ vts2 \geq 2$   
**shows**  $path\text{-}image (make\text{-}polygonal\text{-}path (vts1 @ vts2)) = path\text{-}image (make\text{-}polygonal\text{-}path\ vts1\ \text{+++}\ (linepath\ (vts1\ !\ (length\ vts1 - 1))\ (vts2\ !\ 0))\ \text{+++}\ make\text{-}polygonal\text{-}path\ vts2)$   
 <proof>

**lemma** *make-polygonal-path-image-append-alt:*

**assumes**  $p = make\text{-}polygonal\text{-}path\ vts$   
**assumes**  $p1 = make\text{-}polygonal\text{-}path\ vts1$   
**assumes**  $p2 = make\text{-}polygonal\text{-}path\ vts2$   
**assumes**  $last\ vts1 = hd\ vts2$   
**assumes**  $length\ vts1 \geq 2 \wedge length\ vts2 \geq 2$   
**assumes**  $vts = vts1 @ (tl\ vts2)$   
**shows**  $path\text{-}image\ p = path\text{-}image (p1\ \text{+++}\ p2)$   
 <proof>

**lemma** *cont-incr-interval-image:*

**fixes**  $f :: real \Rightarrow real$   
**assumes**  $a \leq b$   
**assumes** *continuous-on*  $\{a..b\}\ f$   
**assumes**  $\forall x \in \{a..b\}. \forall y \in \{a..b\}. x \leq y \longrightarrow f\ x \leq f\ y$   
**shows**  $f'\{a..b\} = \{f\ a..f\ b\}$   
 <proof>

**lemma** *two-x-minus-one-image:*

**assumes**  $f = (\lambda x :: real. 2 * x - 1)$   
**assumes**  $a \leq b$   
**shows**  $f'\{a..b\} = \{f\ a..f\ b\}$

*<proof>*

**lemma** *vts-split-path-image:*

**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $p1 = \text{make-polygonal-path } vts1$   
**assumes**  $p2 = \text{make-polygonal-path } vts2$   
**assumes**  $vts1 = \text{take } i \text{ } vts$   
**assumes**  $vts2 = \text{drop } (i-1) \text{ } vts$   
**assumes**  $n = \text{length } vts$   
**assumes**  $1 \leq i \wedge i < n$   
**assumes**  $x = (2^{i-1} - 1) / (2^{i-1})$   
**shows**  $\text{path-image } p1 = p\{0..x\} \wedge \text{path-image } p2 = p\{x..1\}$   
*<proof>*

**lemma** *drop-i-is-loop-free:*

**fixes**  $vts :: (\text{real}^2) \text{ list}$   
**assumes**  $m = \text{length } vts$   
**assumes**  $i \leq m - 2$   
**assumes**  $vts' = \text{drop } i \text{ } vts$   
**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $p' = \text{make-polygonal-path } vts'$   
**assumes** *loop-free*  $p$   
**shows** *loop-free*  $p'$   
*<proof>*

**lemma** *joinpaths-tl-transform:*

**assumes**  $f = (\lambda x :: \text{real}. 2*x - 1)$   
**assumes**  $\text{pathfinish } g1 = \text{pathstart } g2$   
**assumes**  $p = g1 \text{ +++ } g2$   
**assumes**  $x \geq 1/2$   
**shows**  $p \text{ } x = g2 \text{ } (f \text{ } x)$   
*<proof>*

**lemma** *joinpaths-tl-image-transform:*

**assumes**  $f = (\lambda x :: \text{real}. 2*x - 1)$   
**assumes**  $\text{pathfinish } g1 = \text{pathstart } g2$   
**assumes**  $p = g1 \text{ +++ } g2$   
**assumes**  $1/2 \leq a \wedge a \leq b$   
**shows**  $p\{a..b\} = g2\{f \text{ } a..f \text{ } b\}$   
*<proof>*

**lemma** *vts-sublist-path-image:*

**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $p' = \text{make-polygonal-path } vts'$   
**assumes**  $vts' = \text{take } j \text{ } (\text{drop } i \text{ } vts)$   
**assumes**  $m = \text{length } vts$   
**assumes**  $n = \text{length } vts'$   
**assumes**  $k = i + j$   
**assumes**  $k \leq m - 1 \wedge 2 \leq j$

**assumes**  $x1 = (2^i - 1)/(2^i)$   
**assumes**  $x2 = (2^{k-1} - 1)/(2^{k-1})$   
**shows**  $\text{path-image } p' = p'\{x1..x2\}$   
 <proof>

**lemma** *one-append-simple-path:*

**fixes**  $pts :: (\mathbb{R}^2) \text{ list}$   
**assumes**  $pts = pts' @ [z]$   
**assumes**  $n = \text{length } pts$   
**assumes**  $n \geq 3$   
**assumes**  $p = \text{make-polygonal-path } pts$   
**assumes**  $p' = \text{make-polygonal-path } pts'$   
**assumes**  $\text{simple-path } p$   
**shows**  $\text{simple-path } p'$   
 <proof>

**lemma** *take-i-is-loop-free:*

**fixes**  $pts :: (\mathbb{R}^2) \text{ list}$   
**assumes**  $n = \text{length } pts$   
**assumes**  $2 \leq i \wedge i \leq n$   
**assumes**  $pts' = \text{take } i \text{ } pts$   
**assumes**  $p = \text{make-polygonal-path } pts$   
**assumes**  $p' = \text{make-polygonal-path } pts'$   
**assumes**  $\text{loop-free } p$   
**shows**  $\text{loop-free } p'$   
 <proof>

**lemma** *sublist-is-loop-free:*

**fixes**  $pts :: (\mathbb{R}^2) \text{ list}$   
**assumes**  $p = \text{make-polygonal-path } pts$   
**assumes**  $p' = \text{make-polygonal-path } pts'$   
**assumes**  $\text{loop-free } p$   
**assumes**  $m = \text{length } pts$   
**assumes**  $n = \text{length } pts'$   
**assumes**  $\text{sublist } pts' \text{ } pts$   
**assumes**  $n \geq 2 \wedge m \geq 2$   
**shows**  $\text{loop-free } p'$   
 <proof>

**lemma** *diff-points-path-image-set-property:*

**fixes**  $a \ b :: \mathbb{R}^2$   
**assumes**  $a \neq b$   
**shows**  $\text{path-image } (\text{linepath } a \ b) \neq \{a, b\}$   
 <proof>

**lemma** *polygonal-path-vertex-t:*

**assumes**  $p = \text{make-polygonal-path } pts$   
**assumes**  $n = \text{length } pts$   
**assumes**  $n \geq 1$

**assumes**  $0 \leq i \wedge i < n - 1$   
**assumes**  $x = (2^{\widehat{i}} - 1) / (2^{\widehat{i}})$   
**shows**  $vts!i = p x$   
 <proof>

**lemma** *loop-free-split-int:*

**assumes**  $p = \text{make-polygonal-path } vts \wedge \text{loop-free } p$   
**assumes**  $vts1 = \text{take } i \text{ } vts$   
**assumes**  $vts2 = \text{drop } (i-1) \text{ } vts$   
**assumes**  $c1 = \text{make-polygonal-path } vts1$   
**assumes**  $c2 = \text{make-polygonal-path } vts2$   
**assumes**  $n = \text{length } vts$   
**assumes**  $1 \leq i \wedge i < n$   
**shows**  $(\text{path-image } c1) \cap (\text{path-image } c2) \subseteq \{\text{pathstart } c1, \text{pathstart } c2\}$   
 (is  $?C1 \cap ?C2 \subseteq \{\text{pathstart } c1, \text{pathstart } c2\}$ )  
 <proof>

**lemma** *loop-free-arc-split-int:*

**assumes**  $p = \text{make-polygonal-path } vts \wedge \text{loop-free } p \wedge \text{arc } p$   
**assumes**  $vts1 = \text{take } i \text{ } vts$   
**assumes**  $vts2 = \text{drop } (i-1) \text{ } vts$   
**assumes**  $c1 = \text{make-polygonal-path } vts1$   
**assumes**  $c2 = \text{make-polygonal-path } vts2$   
**assumes**  $n = \text{length } vts$   
**assumes**  $1 \leq i \wedge i < n$   
**shows**  $(\text{path-image } c1) \cap (\text{path-image } c2) \subseteq \{\text{pathstart } c2\}$   
 (is  $?C1 \cap ?C2 \subseteq \{\text{pathstart } c2\}$ )  
 <proof>

**lemma** *loop-free-append:*

**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $p1 = \text{make-polygonal-path } vts1$   
**assumes**  $p2 = \text{make-polygonal-path } vts2$   
**assumes**  $vts = vts1 @ (\text{tl } vts2)$   
**assumes**  $\text{loop-free } p1 \wedge \text{loop-free } p2$   
**assumes**  $\text{path-image } p1 \cap \text{path-image } p2 \subseteq \{\text{pathstart } p1, \text{pathstart } p2\}$   
**assumes**  $\text{last } vts2 \neq \text{hd } vts1 \longrightarrow \text{path-image } p1 \cap \text{path-image } p2 \subseteq \{\text{pathstart } p2\}$   
**assumes**  $\text{last } vts1 = \text{hd } vts2$   
**assumes**  $\text{arc } p1 \wedge \text{arc } p2$   
**shows**  $\text{loop-free } p$   
 <proof>

**lemma** *sublist-path-image-subset:*

**assumes**  $\text{sublist } vts1 \text{ } vts2$   
**assumes**  $\text{length } vts1 \geq 1$   
**shows**  $\text{path-image } (\text{make-polygonal-path } vts1) \subseteq \text{path-image } (\text{make-polygonal-path } vts2)$   
 <proof>

**lemma** *integral-on-edge-subset-integral-on-path*:  
**assumes**  $p = \text{make-polygonal-path } vts$  **and**  
 $(i::int) \in \{0..<((\text{length } vts) - 1)\}$  **and**  
 $x = vts!i$  **and**  
 $y = vts!(i+1)$   
**shows**  $\{v. \text{integral-vec } v \wedge v \in \text{path-image } (\text{linepath } x \ y)\}$   
 $\subseteq \{v. \text{integral-vec } v \wedge v \in \text{path-image } p\}$   
 $\langle \text{proof} \rangle$

**lemma** *sublist-pair-integral-subset-integral-on-path*:  
**assumes**  $p = \text{make-polygonal-path } vts$  **and**  
 $\text{sublist } [x, y] \ vts$   
**shows**  $\{v. \text{integral-vec } v \wedge v \in \text{path-image } (\text{linepath } x \ y)\}$   
 $\subseteq \{v. \text{integral-vec } v \wedge v \in \text{path-image } p\}$   
 $\langle \text{proof} \rangle$

**lemma** *sublist-integral-subset-integral-on-path*:  
**assumes**  $\text{length } ell \geq 2$   
**assumes**  $p = \text{make-polygonal-path } vts$  **and**  
 $\text{sublist } ell \ vts$   
**shows**  $\{v. \text{integral-vec } v \wedge v \in \text{path-image } (\text{make-polygonal-path } ell)\}$   
 $\subseteq \{v. \text{integral-vec } v \wedge v \in \text{path-image } p\}$   
 $\langle \text{proof} \rangle$

## 13 Reversing Polygonal Path Vertex List

**lemma** *rev-vts-path-image*:  
**shows**  $\text{path-image } (\text{make-polygonal-path } (\text{rev } vts)) = \text{path-image } (\text{make-polygonal-path } vts)$   
 $\langle \text{proof} \rangle$

**lemma** *rev-vts-is-loop-free*:  
**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $\text{loop-free } p$   
**shows**  $\text{loop-free } (\text{make-polygonal-path } (\text{rev } vts))$   
 $\langle \text{proof} \rangle$

**lemma** *rev-vts-is-polygon*:  
**assumes**  $\text{polygon-of } p \ vts$   
**shows**  $\text{polygon } (\text{make-polygonal-path } (\text{rev } vts))$   
 $\langle \text{proof} \rangle$

**end**  
**theory** *Linepath-Collinearity*  
**imports** *Polygon-Lemmas*

**begin**



## 14 Collinearity Properties

**lemma** *points-on-linepath-collinear*:

**assumes** *exists-c*:  $(\exists c. a - b = c *_R u)$

**assumes** *x-in-linepath*:  $x \in \text{path-image } (\text{linepath } a \ b)$

**shows**  $(\exists c. x - a = c *_R u) (\exists c. b - x = c *_R u)$

*<proof>*

**lemma** *three-points-collinear-property*:

**fixes** *a b*::  $\text{real}^2$

**assumes** *exists-c1*:  $(\exists c. a - x1 = c *_R u)$

**assumes** *exists-c2*:  $(\exists c. a - x2 = c *_R u)$

**shows**  $\exists c. x1 - x2 = c *_R u$

*<proof>*

**lemma** *in-path-image-imp-collinear*:

**fixes** *a b*::  $\text{real}^2$

**assumes** *k*  $\in \text{path-image } (\text{linepath } a \ b)$

**shows** *collinear*  $\{a, b, k\}$

*<proof>*

**lemma** *two-linepath-collinearity-property*:

**fixes** *a b c d*::  $\text{real}^2$

**assumes**  $y \neq z \wedge \{y, z\} \subseteq (\text{path-image } (\text{linepath } a \ b)) \cap (\text{path-image } (\text{linepath } c \ d))$

**shows** *collinear*  $\{a, b, c, d\}$

*<proof>*

**lemma** *polygon-vts-not-collinear*:

**assumes** *polygon-of p vts*

**shows**  $\neg \text{collinear } (\text{set } vts)$

*<proof>*

**lemma** *not-collinear-with-subset*:

**assumes** *collinear A*

**assumes**  $\neg \text{collinear } (A \cup \{x\})$

**assumes**  $\text{card } A > 2$

**assumes**  $a \in A$

**shows**  $\neg \text{collinear } ((A - \{a\}) \cup \{x\})$

*<proof>*

**lemma** *vec-diff-scale-collinear*:

**fixes** *a b c* ::  $\text{real}^2$

**assumes**  $b - a = m *_R (c - a)$

**shows** *collinear*  $\{a, b, c\}$

*<proof>*

## 15 Linepath Properties

**lemma** *good-linepath-comm*:  $\text{good-linepath } a \ b \ vts \implies \text{good-linepath } b \ a \ vts$   
 ⟨proof⟩

**lemma** *finite-set-linepaths*:  
**assumes** *polygon*:  $\text{polygon } p$   
**assumes** *polygonal-path*:  $p = \text{make-polygonal-path } vts$   
**shows** *finite*  $\{(a, b). (a, b) \in \text{set } vts \times \text{set } vts\}$   
 ⟨proof⟩

**lemma** *linepaths-intersect-once-or-collinear*:  
**fixes**  $a \ b \ c \ d :: \text{real}^2$   
**assumes**  $\text{path-image } (\text{linepath } a \ b) \cap \text{path-image } (\text{linepath } c \ d) \neq \{\}$   
**shows**  $\text{collinear } \{a, b, c, d\} \vee (\exists x. \text{path-image } (\text{linepath } a \ b) \cap \text{path-image } (\text{linepath } c \ d) = \{x\})$   
 ⟨proof⟩

**lemma** *linepaths-intersect-once-or-collinear-alt*:  
**fixes**  $a \ b \ c \ d :: \text{real}^2$   
**assumes**  $\text{path-image } (\text{linepath } a \ b) \cap \text{path-image } (\text{linepath } c \ d) \neq \{\}$   
**shows**  $\text{collinear } \{a, b, c, d\} \vee \text{card } (\text{path-image } (\text{linepath } a \ b) \cap \text{path-image } (\text{linepath } c \ d)) = 1$   
 ⟨proof⟩

**lemma** *path-image-linepath-union*:  
**fixes**  $a \ b :: 'a :: \text{euclidean-space}$   
**assumes**  $d \in \text{path-image } (\text{linepath } a \ b)$   
**shows**  $\text{path-image } (\text{linepath } a \ b) = \text{path-image } (\text{linepath } a \ d) \cup \text{path-image } (\text{linepath } d \ b)$   
 ⟨proof⟩

**lemma** *path-image-linepath-split*:  
**assumes**  $i < (\text{length } vts) - 1$   
**assumes**  $x \in \text{path-image } (\text{linepath } (vts!i) \ (vts!(i+1)))$   
**assumes**  $x \notin \text{set } vts$   
**shows**  $\text{path-image } (\text{make-polygonal-path } vts) = \text{path-image } (\text{make-polygonal-path } ((\text{take } (i+1) \ vts) @ [x] @ (\text{drop } (i+1) \ vts)))$   
 ⟨proof⟩

**lemma** *linepath-split-is-loop-free*:  
**assumes**  $d \in \text{path-image } (\text{linepath } a \ b)$   
**assumes**  $d \notin \{a, b\}$   
**shows**  $\text{loop-free } (\text{make-polygonal-path } [a, d, b])$  (is loop-free ?p)  
 ⟨proof⟩

**lemma** *loop-free-linepath-split-is-loop-free*:  
**assumes**  $p = \text{make-polygonal-path } vts$   
**assumes**  $\text{loop-free } p$

**assumes**  $n = \text{length } vts$   
**assumes**  $i < n - 1$   
**assumes**  $x \in \text{path-image } (\text{linepath } (vts!i) (vts!(i+1))) \wedge x \notin \text{set } vts$   
**assumes**  $vts' = (\text{take } (i+1) vts) @ [x] @ (\text{drop } (i+1) vts)$   
**assumes**  $p' = \text{make-polygonal-path } vts'$   
**shows**  $\text{loop-free } p' \wedge \text{path-image } p' = \text{path-image } p$   
 <proof>

**lemma** *polygon-linepath-split-is-polygon*:  
**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $i < (\text{length } vts) - 1$   
**assumes**  $a = vts!i \wedge b = vts!(i+1)$   
**assumes**  $x \in \text{path-image } (\text{linepath } a b) \wedge x \notin \text{set } vts$   
**assumes**  $vts' = (\text{take } (i+1) vts) @ [x] @ (\text{drop } (i+1) vts)$   
**shows** *polygon* ( $\text{make-polygonal-path } vts'$ )  
 <proof>

## 16 Measure of linepaths

**lemma** *linepath-is-negligible-vertical*:  
**fixes**  $a b :: \text{real}^2$   
**assumes**  $a\$1 = b\$1$   
**defines**  $p \equiv \text{linepath } a b$   
**shows** *negligible* ( $\text{path-image } p$ )  
 <proof>

**lemma** *linepath-is-negligible-non-vertical*:  
**fixes**  $a b :: \text{real}^2$   
**assumes**  $a\$1 < b\$1$   
**defines**  $p \equiv \text{linepath } a b$   
**shows** *negligible* ( $\text{path-image } p$ )  
 <proof>

**lemma** *linepath-is-negligible*:  
**fixes**  $a b :: \text{real}^2$   
**defines**  $p \equiv \text{linepath } a b$   
**shows** *negligible* ( $\text{path-image } p$ )  
 <proof>

**lemma** *linepath-has-emeasure-0*:  
 $\text{emeasure lebesgue } (\text{path-image } (\text{linepath } (a::(\text{real}^2)) (b::(\text{real}^2)))) = 0$   
 <proof>

**lemma** *linepath-has-measure-0*:  
 $\text{measure lebesgue } (\text{path-image } (\text{linepath } (a::(\text{real}^2)) (b::(\text{real}^2)))) = 0$   
 <proof>

end

**theory** *Polygon-Convex-Lemmas*

**imports**

*Polygon-Lemmas*

*Linepath-Collinearity*

**begin**

## 17 Misc. Convex Polygon Properties

**lemma** *polygon-path-image-subset-convex:*

**assumes**  $\text{length } vts > 0$

**shows**  $\text{path-image } (\text{make-polygonal-path } vts) \subseteq \text{convex hull } (\text{set } vts)$  (**is**  $\text{path-image } ?p \subseteq ?S$ )

*<proof>*

**lemma** *convex-contains-simple-closed-path-imp-contains-path-inside:*

**assumes**  $\text{convex } S$

**assumes**  $\text{simple-path } p \wedge \text{closed-path } p$

**assumes**  $\text{path-image } p \subseteq S$

**shows**  $\text{path-inside } p \subseteq S$

*<proof>*

**lemma** *convex-polygon-is-convex-hull:*

**assumes**  $\text{polygon } p$

**assumes**  $\text{convex } (\text{path-inside } p \cup \text{path-image } p)$

**assumes**  $p = \text{make-polygonal-path } vts$

**shows**  $\text{convex hull } (\text{set } vts) = \text{path-inside } p \cup \text{path-image } p$  (**is**  $?hull = ?poly$ )

*<proof>*

**lemma** *convex-polygon-inside-is-convex-hull-interior:*

**assumes**  $\text{polygon } p$

**assumes**  $\text{convex } (\text{path-inside } p)$

**assumes**  $p = \text{make-polygonal-path } vts$

**shows**  $\text{interior } (\text{convex hull } (\text{set } vts)) = \text{path-inside } p$

*<proof>*

**lemma** *convex-polygon-inside-is-convex-hull-interior2:*

**assumes**  $\text{polygon } p$

**assumes**  $\text{convex } (\text{path-inside } p \cup \text{path-image } p)$

**assumes**  $p = \text{make-polygonal-path } vts$

**shows**  $\text{interior } (\text{convex hull } (\text{set } vts)) = \text{path-inside } p$

*<proof>*

**lemma** *polygon-convex-iff:*

**assumes**  $\text{polygon } p$

**shows**  $\text{convex } (\text{path-inside } p) \iff \text{convex } (\text{path-inside } p \cup \text{path-image } p)$

*<proof>*

**lemma** *convex-polygon-frontier-is-path-image:*

**assumes** *polygon-of p vts*  
**assumes** *convex (path-inside p)*  
**shows**  $\text{frontier } (\text{convex hull } (\text{set } vts)) = \text{path-image } p$   
*<proof>*

**lemma** *convex-polygon-frontier-is-path-image2:*  
**assumes** *polygon p*  
**assumes** *convex (path-inside p)*  
**shows**  $\text{frontier } (\text{path-image } p \cup \text{path-inside } p) = \text{path-image } p$   
*<proof>*

**lemma** *convex-polygon-frontier-is-path-image3:*  
**assumes** *polygon p*  
**assumes** *convex (path-image p  $\cup$  path-inside p)*  
**shows**  $\text{frontier } (\text{path-image } p \cup \text{path-inside } p) = \text{path-image } p$   
*<proof>*

**lemma** *polygon-frontier-is-path-image:*  
**assumes** *polygon p*  
**shows**  $\text{frontier } (\text{path-inside } p) = \text{path-image } p$   
*<proof>*

**lemma** *convex-path-inside-means-convex-polygon:*  
**assumes** *polygon p*  
**assumes**  $\text{frontier } (\text{convex hull } (\text{set } vts)) = \text{path-image } p$   
**shows** *convex (path-inside p)*  
*<proof>*

**lemma** *convex-hull-of-polygon-is-convex-hull-of-vts:*  
**assumes** *polygon-of p vts*  
**shows**  $\text{convex hull } (\text{path-image } p \cup \text{path-inside } p) = \text{convex hull } (\text{set } vts)$   
*<proof>*

**lemma** *convex-hull-frontier-polygon:*  
**assumes** *polygon-of p vts*  
**assumes**  $\neg \text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\neg \text{convex } (\text{path-inside } p)$   
*<proof>*

**lemma** *frontier-int-subset:*  
**assumes**  $A \subseteq B$   
**shows**  $(\text{frontier } B) \cap A \subseteq \text{frontier } A$   
*<proof>*

**lemma** *in-frontier-in-subset:*  
**assumes**  $A \subseteq B$   
**assumes**  $x \in \text{frontier } B$   
**assumes**  $x \in A$   
**shows**  $x \in \text{frontier } A$

*<proof>*

**lemma** *in-frontier-in-subset-convex-hull:*

**assumes**  $A \subseteq B$

**assumes**  $x \in \text{frontier } (\text{convex hull } B)$

**assumes**  $x \in \text{convex hull } A$

**shows**  $x \in \text{frontier } (\text{convex hull } A)$

*<proof>*

**lemma** *convex-hull-two-extreme-points:*

**fixes**  $S :: 'a::\text{euclidean-space set}$

**assumes** *finite*  $S$

**assumes**  $\text{convex hull } S \neq \{\}$

**assumes**  $\forall x. \text{convex hull } S \neq \{x\}$

**shows**  $\text{card } \{x. x \text{ extreme-point-of } (\text{convex hull } S)\} \geq 2$  (**is**  $\text{card } ?ep \geq 2$ )

*<proof>*

**lemma** *convex-hull-two-vts-on-frontier:*

**fixes**  $S :: 'a::\text{euclidean-space set}$

**assumes**  $\text{card } S \geq 2$

**shows**  $\text{card } (S \cap \text{frontier } (\text{convex hull } S)) \geq 2$

*<proof>*

## 18 Vertices on Convex Frontier Implies Polygon is Convex

**lemma** *convex-cut-aux:*

**assumes**  $\forall v \in S. z \cdot v \leq 0$

**shows**  $\text{convex hull } S \subseteq \{x. z \cdot x \leq 0\}$

*<proof>*

**lemma** *convex-cut-aux':*

**assumes**  $\forall v \in S. z \cdot v \geq 0$

**shows**  $\text{convex hull } S \subseteq \{x. z \cdot x \geq 0\}$

*<proof>*

**lemma** *convex-cut:*

**assumes**  $z \neq 0$

**assumes**  $\{x. z \cdot x = 0\} \cap \text{interior } (\text{convex hull } S) \neq \{\}$

**obtains**  $v1\ v2$  **where**  $v1 \neq v2 \wedge \{v1, v2\} \subseteq S \wedge v1 \in \{x. z \cdot x < 0\} \wedge v2 \in \{x. z \cdot x > 0\}$

*<proof>*

**lemma** *affine-2-int-convex:*

**fixes**  $S :: 'a::\text{euclidean-space set}$

**assumes**  $\{a, b\} \subseteq S$

**assumes**  $\{a, b\} \subseteq \text{frontier } (\text{convex hull } S)$

**assumes**  $\text{affine hull } \{a, b\} \cap \text{interior } (\text{convex hull } S) \neq \{\}$

**shows**  $\text{affine hull } \{a, b\} \cap \text{convex hull } S = \text{convex hull } \{a, b\}$   
(proof)

**lemma** *halfplane-frontier-affine-hull*:

**fixes**  $b v :: \text{real}^2$   
**assumes**  $b \neq 0$   
**assumes**  $v \neq 0$   
**assumes**  $b \in \{x. v \cdot x = 0\}$   
**shows**  $\{x. v \cdot x = 0\} = \text{affine hull } \{0, b\}$   
(proof)

**lemma** *pts-on-convex-frontier-aux*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $vts!0 = 0$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{path-image } (\text{linepath } (vts!0) (vts!1)) \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
(proof)

**lemma** *pts-on-convex-frontier-aux'*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{path-image } (\text{linepath } (vts!0) (vts!1)) \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
(proof)

**lemma** *pts-on-convex-frontier*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**assumes**  $i < \text{length } vts - 1$   
**shows**  $\text{path-image } (\text{linepath } (vts!i) (vts!(i+1))) \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
(proof)

**lemma** *pts-on-frontier-means-path-image-on-frontier*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{path-image } p \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
(proof)

**lemma** *pts-on-convex-frontier-interior*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{path-inside } p = \text{interior } (\text{convex hull } (\text{set } vts))$   
(proof)

**lemma** *pts-subset-frontier*:

**assumes** *polygon-of*  $p$   $vts$   
**assumes**  $\text{set } vts \subseteq \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{convex } (\text{path-image } p \cup \text{path-inside } p)$   
(proof)

**lemma** *convex-hull-of-nonconvex-polygon-strict-subset-ep*:  
**assumes** *polygon-of p vts*  
**assumes**  $\neg$  (*convex (path-image p  $\cup$  path-inside p)*)  
**shows**  $\{v. v \text{ extreme-point-of } (\text{convex hull } (\text{set } vts))\} \subset \text{set } vts$   
 $\langle \text{proof} \rangle$

**lemma** *convex-hull-of-nonconvex-polygon-strict-subset*:  
**assumes** *polygon-of p vts*  
**assumes**  $\neg$  (*convex (path-image p  $\cup$  path-inside p)*)  
**shows**  $\exists v \in \text{set } vts. v \in \text{interior } (\text{convex hull } (\text{set } vts))$   
 $\langle \text{proof} \rangle$

**lemma** *convex-polygon-means-linepaths-inside*:  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes** *polygon-of p vts*  
**assumes** *convex-is: convex hull (set vts) = (path-inside p  $\cup$  path-image p)*  
**assumes** *a-in: a  $\in$  (path-inside p  $\cup$  path-image p)*  
**assumes** *b-in: b  $\in$  (path-inside p  $\cup$  path-image p)*  
**shows** *path-image (linepath a b)  $\subseteq$  (path-inside p  $\cup$  path-image p)*  
 $\langle \text{proof} \rangle$

**end**  
**theory** *Polygon-Splitting*  
**imports**  
*HOL-Analysis.Complete-Measure*  
*Polygon-Jordan-Curve*  
*Polygon-Convex-Lemmas*  
**begin**

## 19 Polygon Splitting

**lemma** *split-up-a-list-into-3-parts*:  
**fixes**  $i j :: \text{nat}$   
**assumes**  $i < \text{length } vts \wedge j < \text{length } vts \wedge i < j$   
**shows**  
 $vts = (\text{take } i \text{ } vts) @ ((vts ! i) \# ((\text{take } (j - i - 1) (\text{drop } (\text{Suc } i) \text{ } vts)) @ (vts ! j) \# \text{drop } (j - i) (\text{drop } (\text{Suc } i) \text{ } vts)))$   
 $\langle \text{proof} \rangle$

**definition** *is-polygon-cut* ::  $(\text{real}^2) \text{ list} \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{bool}$  **where**  
*is-polygon-cut*  $vts \ x \ y =$   
 $(x \neq y \wedge$   
 $\text{polygon } (\text{make-polygonal-path } vts) \wedge$   
 $\{x, y\} \subseteq \text{set } vts \wedge$   
 $\text{path-image } (\text{linepath } x \ y) \cap \text{path-image } (\text{make-polygonal-path } vts) = \{x, y\} \wedge$   
 $\text{path-image } (\text{linepath } x \ y) \cap \text{path-inside } (\text{make-polygonal-path } vts) \neq \{\})$

**definition** *is-polygon-cut-path* ::  $(\text{real}^2) \text{ list} \Rightarrow R\text{-to-}R^2 \Rightarrow \text{bool}$  **where**



$is\text{-polygon-cut-path } vts \text{ cutpath} =$   
 $(let\ x = pathstart\ cutpath ; y = pathfinish\ cutpath\ in$   
 $(x \neq y \wedge$   
 $poly\text{-}polygon\ (make\text{-}poly\text{-}gonal\text{-}path\ vts) \wedge$   
 $\{x, y\} \subseteq set\ vts \wedge$   
 $simple\text{-}path\ cutpath \wedge$   
 $path\text{-}image\ cutpath \cap path\text{-}image\ (make\text{-}poly\text{-}gonal\text{-}path\ vts) = \{x, y\} \wedge$   
 $path\text{-}image\ cutpath \cap path\text{-}inside\ (make\text{-}poly\text{-}gonal\text{-}path\ vts) \neq \{\})$ )

**definition**  $is\text{-polygon-split} ::$

$(real^2) \text{ list} \Rightarrow nat \Rightarrow nat \Rightarrow bool$  **where**  
 $is\text{-polygon-split } vts\ i\ j =$   
 $(i < length\ vts \wedge j < length\ vts \wedge i < j \wedge$   
 $(let\ vts1 = (take\ i\ vts)\ in$   
 $let\ vts2 = (take\ (j - i - 1)\ (drop\ (Suc\ i)\ vts))\ in$   
 $let\ vts3 = drop\ (j - i)\ (drop\ (Suc\ i)\ vts)\ in$   
 $let\ x = vts ! i\ in$   
 $let\ y = vts ! j\ in$   
 $let\ p = make\text{-}poly\text{-}gonal\text{-}path\ (vts@[vts!0])\ in$   
 $let\ p1 = make\text{-}poly\text{-}gonal\text{-}path\ (x\#\ (vts2@[y, x]))\ in$   
 $let\ p2 = make\text{-}poly\text{-}gonal\text{-}path\ (vts1\ @ [x, y]\ @ vts3\ @ [vts ! 0])\ in$   
 $let\ c1 = make\text{-}poly\text{-}gonal\text{-}path\ (x\#\ (vts2@[y]))\ in$   
 $let\ c2 = make\text{-}poly\text{-}gonal\text{-}path\ (vts1\ @ [x, y]\ @ vts3)\ in$   
 $(is\text{-polygon-cut}\ (vts@[vts!0])\ x\ y \wedge$   
 $poly\text{-}gon\ p \wedge poly\text{-}gon\ p1 \wedge poly\text{-}gon\ p2 \wedge$   
 $path\text{-}inside\ p1 \cap path\text{-}inside\ p2 = \{\} \wedge$   
 $path\text{-}inside\ p1 \cup path\text{-}inside\ p2 \cup (path\text{-}image\ (linepath\ x\ y) - \{x, y\}) =$   
 $path\text{-}inside\ p$   
 $\wedge ((path\text{-}image\ p1) - (path\text{-}image\ (linepath\ x\ y))) \cap ((path\text{-}image\ p2) -$   
 $(path\text{-}image\ (linepath\ x\ y)))$   
 $= \{\}$   
 $\wedge path\text{-}image\ p$   
 $= ((path\text{-}image\ p1) - (path\text{-}image\ (linepath\ x\ y))) \cup ((path\text{-}image\ p2) -$   
 $(path\text{-}image\ (linepath\ x\ y))) \cup \{x, y\}$   
 $)))$

**definition**  $is\text{-polygon-split-path} :: (real^2) \text{ list} \Rightarrow nat \Rightarrow nat \Rightarrow (real^2) \text{ list} \Rightarrow bool$  **where**

$is\text{-polygon-split-path } vts\ i\ j\ cutvts =$   
 $(i < length\ vts \wedge j < length\ vts \wedge i < j \wedge$   
 $(let\ vts1 = (take\ i\ vts)\ in$   
 $let\ vts2 = (take\ (j - i - 1)\ (drop\ (Suc\ i)\ vts))\ in$   
 $let\ vts3 = drop\ (j - i)\ (drop\ (Suc\ i)\ vts)\ in$   
 $let\ x = vts ! i\ in$   
 $let\ y = vts ! j\ in$   
 $let\ cutpath = make\text{-}poly\text{-}gonal\text{-}path\ (x\ #\ cutvts\ @ [y])\ in$   
 $let\ p = make\text{-}poly\text{-}gonal\text{-}path\ (vts@[vts!0])\ in$   
 $let\ p1 = make\text{-}poly\text{-}gonal\text{-}path\ (x\#\ (vts2\ @ [y]\ @ (rev\ cutvts)\ @ [x]))\ in$

let  $p2 = \text{make-polygonal-path } (vts1 \ @ \ ([x] \ @ \ \text{cutvts} \ @ \ [y]) \ @ \ vts3 \ @ \ [vts \ ! \ 0])$  in  
 let  $c1 = \text{make-polygonal-path } (x\#(vts2@[y]))$  in  
 let  $c2 = \text{make-polygonal-path } (vts1 \ @ \ ([x] \ @ \ \text{cutvts} \ @ \ [y]) \ @ \ vts3)$  in  
 ( $\text{is-polygon-cut-path } (vts@[vts!0]) \ \text{cutpath} \ \wedge$   
 $\text{polygon } p \ \wedge \ \text{polygon } p1 \ \wedge \ \text{polygon } p2 \ \wedge$   
 $\text{path-inside } p1 \ \cap \ \text{path-inside } p2 = \{\}$   $\wedge$   
 $\text{path-inside } p1 \ \cup \ \text{path-inside } p2 \ \cup \ (\text{path-image } \text{cutpath} - \{x, y\}) = \text{path-inside}$   
 $p$   
 $\wedge \ ((\text{path-image } p1) - (\text{path-image } \text{cutpath})) \cap \ ((\text{path-image } p2) - (\text{path-image}$   
 $\text{cutpath})) = \{\}$   
 $\wedge \ \text{path-image } p$   
 $= \ ((\text{path-image } p1) - (\text{path-image } \text{cutpath})) \cup \ ((\text{path-image } p2) - (\text{path-image}$   
 $\text{cutpath})) \cup \ \{x, y\}$   
 $\left. \right\}$

**lemma** *polygon-split-add-measure*:

**fixes**  $p \ p1 \ p2 :: R\text{-to-}R^2$

**assumes** *is-polygon-split*  $vts \ i \ j$

**assumes**  $vts1 = (\text{take } i \ vts)$

$vts2 = (\text{take } (j - i - 1) \ (\text{drop } (\text{Suc } i) \ vts))$

$vts3 = \text{drop } (j - i) \ (\text{drop } (\text{Suc } i) \ vts)$

$x = vts \ ! \ i$

$y = vts \ ! \ j$

$p = \text{make-polygonal-path } (vts@[vts!0])$

$p1 = \text{make-polygonal-path } (x\#(vts2@[y, x]))$

$p2 = \text{make-polygonal-path } (vts1 \ @ \ [x, y] \ @ \ vts3 \ @ \ [vts \ ! \ 0])$

**defines**  $M1 \equiv \text{measure lebesgue } (\text{path-inside } p1)$  **and**

$M2 \equiv \text{measure lebesgue } (\text{path-inside } p2)$  **and**

$M \equiv \text{measure lebesgue } (\text{path-inside } p)$

**shows**  $M1 + M2 = M$

*<proof>*

**lemma** *polygonal-paths-measurable*:

**shows**  $\text{path-image } (\text{make-polygonal-path } vts) \in \text{sets lebesgue}$

*<proof>*

**lemma** *polygonal-path-has-emeasure-0*:

**shows**  $\text{emeasure lebesgue } (\text{path-image } (\text{make-polygonal-path } vts)) = 0$

*<proof>*

**lemma** *polygon-split-path-add-measure*:

**fixes**  $p \ p1 \ p2 :: R\text{-to-}R^2$

**assumes** *is-polygon-split-path*  $vts \ i \ j \ \text{cutvts}$

**assumes**  $vts1 = (\text{take } i \ vts)$

$vts2 = (\text{take } (j - i - 1) \ (\text{drop } (\text{Suc } i) \ vts))$

$vts3 = \text{drop } (j - i) \ (\text{drop } (\text{Suc } i) \ vts)$

$x = vts \ ! \ i$

$y = vts \ ! \ j$

$p = \text{make-polygonal-path } (vts@[vts!0])$

$p1 = \text{make-polygonal-path } (x\#(vts2 \ @ \ [y] \ @ \ (\text{rev } cutvts) \ @ \ [x]))$   
 $p2 = \text{make-polygonal-path } (vts1 \ @ \ ([x] \ @ \ cutvts \ @ \ [y]) \ @ \ vts3 \ @ \ [vts \ ! \ 0])$   
**defines**  $M1 \equiv \text{measure lebesgue } (\text{path-inside } p1)$  **and**  
 $M2 \equiv \text{measure lebesgue } (\text{path-inside } p2)$  **and**  
 $M \equiv \text{measure lebesgue } (\text{path-inside } p)$   
**shows**  $M1 + M2 = M$   
 $\langle \text{proof} \rangle$

**lemma** *polygon-cut-path-to-split-path-vtx0*:  
**fixes**  $p :: R\text{-to-}R2$   
**assumes** *polygon-p*: *polygon p* **and**  
*i-gt*:  $i > 0$  **and**  
*i-lt*:  $i < \text{length } vts$  **and**  
*p-is*:  $p = \text{make-polygonal-path } (vts \ @ \ [vts \ ! \ 0])$  **and**  
*cutpath*:  $\text{cutpath} = \text{make-polygonal-path } ([vts!0] \ @ \ cutvts \ @ \ [vts!i])$  **and**  
*have-cut*: *is-polygon-cut-path*  $(vts \ @ \ [vts!0]) \ \text{cutpath}$   
**shows** *is-polygon-split-path*  $vts \ 0 \ i \ \text{cutvts}$   
 $\langle \text{proof} \rangle$

**lemma** *polygon-cut-path-to-split-path*:  
**fixes**  $p :: R\text{-to-}R2$   
**assumes** *polygon p*  
 $p = \text{make-polygonal-path } (vts \ @ \ [vts \ ! \ 0])$   
*is-polygon-cut-path*  $(vts \ @ \ [vts!0]) \ \text{cutpath}$   
 $vts1 \equiv (\text{take } i \ vts)$   
 $vts2 \equiv (\text{take } (j - i - 1) \ (\text{drop } (\text{Suc } i) \ vts))$   
 $vts3 \equiv \text{drop } (j - i) \ (\text{drop } (\text{Suc } i) \ vts)$   
 $x \equiv vts \ ! \ i$   
 $y \equiv vts \ ! \ j$   
 $\text{cutpath} = \text{make-polygonal-path } ([x] \ @ \ cutvts \ @ \ [y])$   
 $i < \text{length } vts \ \wedge \ j < \text{length } vts \ \wedge \ i < j$   
 $p1 \equiv \text{make-polygonal-path } (x\#(vts2\@([y] \ @ \ (\text{rev } cutvts) \ @ \ [x])))$  **and**  
 $p2 \equiv \text{make-polygonal-path } (vts1 \ @ \ ([x] \ @ \ cutvts \ @ \ [y]) \ @ \ vts3 \ @ \ [(vts1 \ @ \ [x]) \ ! \ 0])$   
**shows** *is-polygon-split-path*  $vts \ i \ j \ \text{cutvts}$   
 $\langle \text{proof} \rangle$

**lemma** *good-polygonal-path-implies-polygon-split-path*:  
**assumes** *polygon p*  
**assumes**  $p = \text{make-polygonal-path } (vts \ @ \ [vts!0])$   
**assumes** *good-polygonal-path*  $v1 \ \text{cutvts} \ v2 \ (vts \ @ \ [vts!0])$   
**assumes**  $i < \text{length } vts \ \wedge \ j < \text{length } vts$   
**assumes**  $vts \ ! \ i = v1$   
**assumes**  $vts \ ! \ j = v2$   
**assumes**  $i < j$   
**shows** *is-polygon-split-path*  $vts \ i \ j \ \text{cutvts}$   
 $\langle \text{proof} \rangle$

```

lemma good-path-iff:
  good-linepath a b vts  $\longleftrightarrow$  good-polygonal-path a [] b vts
  <proof>

lemma polygon-cut-iff: is-polygon-cut (vts @ [vts!0]) (vts!i) (vts!j)
   $\longleftrightarrow$  is-polygon-cut-path (vts @ [vts!0]) (linepath (vts!i) (vts!j))
  <proof>

lemma polygon-split-iff: is-polygon-split vts i j  $\longleftrightarrow$  is-polygon-split-path vts i j []
  <proof>

lemma polygon-cut-to-split-vtx0:
  fixes p :: R-to-R2
  assumes polygon-p: polygon p and
    i-gt: i > 0 and
    i-lt: i < length vts and
    p-is: p = make-polygonal-path (vts @ [vts ! 0]) and
    have-cut: is-polygon-cut (vts @ [vts!0]) (vts!0) (vts!i)
  shows is-polygon-split vts 0 i
  <proof>

lemma polygon-cut-to-split:
  fixes p :: R-to-R2
  assumes is-polygon-cut (vts @ [vts!0]) (vts!i) (vts!j)
    i < length vts  $\wedge$  j < length vts  $\wedge$  i < j
  shows is-polygon-split vts i j
  <proof>

lemma good-linepath-implies-polygon-split:
  assumes polygon p
  assumes p = make-polygonal-path (vts @ [vts!0])
  assumes good-linepath v1 v2 (vts @ [vts!0])
  assumes i < length vts  $\wedge$  j < length vts
  assumes vts ! i = v1
  assumes vts ! j = v2
  assumes i < j
  shows is-polygon-split vts i j
  <proof>

end
theory Triangle-Lemmas
imports
  Polygon-Convex-Lemmas
  Integral-Matrix
  Affine-Arithmetic.Floatarith-Expression
  HOL-Analysis.Topology-Euclidean-Space
  HOL-Analysis.Equivalence-Lebesgue-Henstock-Integration
  HOL-Analysis.Inner-Product

```

*HOL-Analysis.Line-Segment*  
*HOL-Analysis.Convex-Euclidean-Space*  
*HOL-Analysis.Change-Of-Vars*

**begin**

## 20 Triangles

**definition** *elem-triangle* ::  $\text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{bool}$  **where**

*elem-triangle*  $a\ b\ c \iff$   
 $\neg \text{collinear } \{a, b, c\}$   
 $\wedge \text{integral-vec } a \wedge \text{integral-vec } b \wedge \text{integral-vec } c$   
 $\wedge \{x. x \in \text{convex hull } \{a, b, c\} \wedge \text{integral-vec } x\} = \{a, b, c\}$

**definition** *triangle-mat* ::  $\text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \times \text{real}^2$  **where**

*triangle-mat*  $a\ b\ c = \text{transpose } (\text{vector } [b - a, c - a])$

**definition** *triangle-linear* ::  $\text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow (\text{real}^2 \Rightarrow \text{real}^2)$   
**where**

*triangle-linear*  $a\ b\ c = (\lambda x. (\text{triangle-mat } a\ b\ c) *v x)$

**definition** *triangle-affine* ::  $\text{real}^2 \Rightarrow \text{real}^2 \Rightarrow \text{real}^2 \Rightarrow (\text{real}^2 \Rightarrow \text{real}^2)$  **where**

*triangle-affine*  $a\ b\ c = (\lambda x. a + (\text{triangle-mat } a\ b\ c) *v x)$

**abbreviation** *unit-square*  $\equiv$

$(\text{convex hull } \{\text{vector } [0, 0], \text{vector } [0, 1], \text{vector } [1, 1], \text{vector } [1, 0]\})::(\text{real}^2 \text{ set})$

**abbreviation** *unit-triangle*  $\equiv$

$(\text{convex hull } \{\text{vector } [0, 0], \text{vector } [1, 0], \text{vector } [0, 1]\})::(\text{real}^2 \text{ set})$

**abbreviation** *unit-triangle'*  $\equiv$

$(\text{convex hull } \{\text{vector } [1, 1], \text{vector } [1, 0], \text{vector } [0, 1]\})::(\text{real}^2 \text{ set})$

**lemma** *triangle-inside-is-convex-hull-interior*:

**assumes** *polygon-of*  $p\ [a, b, c, a]$

**shows** *path-inside*  $p = \text{interior } (\text{convex hull } \{a, b, c\})$

*<proof>*

**lemma** *triangle-is-convex*:

**assumes**  $p = \text{make-triangle } a\ b\ c$  **and**  $\neg \text{collinear } \{a, b, c\}$

**shows** *convex*  $(\text{path-inside } p)$  (**is convex** ?s)

*<proof>*

**lemma** *affine-comp-linear-trans*: *triangle-affine*  $a\ b\ c = (\lambda x. x + a) \circ (\text{triangle-linear } a\ b\ c)$

*<proof>*

**lemma** *triangle-linear-der*:

**fixes**  $a\ b\ c :: \text{real}^2$   
**defines**  $T \equiv \text{triangle-linear } a\ b\ c$   
**shows**  $(T \text{ has-derivative } T)$  (at  $x$ )  
 <proof>

**lemma** *triangle-affine-der*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**assumes**  $S \in \text{sets lebesgue}$  **and**  $x \in S$   
**defines**  $A \equiv \text{triangle-affine } a\ b\ c$  **and**  $T \equiv \text{triangle-linear } a\ b\ c$   
**shows**  $x \in S \implies (A \text{ has-derivative } T)$  (at  $x$  within  $S$ )  
 <proof>

**lemma** *triangle-linear-inj*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**defines**  $L \equiv \text{triangle-linear } a\ b\ c$   
**shows** *inj*  $L$   
 <proof>

**lemma** *triangle-affine-inj*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**defines**  $A \equiv \text{triangle-affine } a\ b\ c$   
**shows** *inj*  $A$   
 <proof>

**lemma** *triangle-linear-integrable*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**assumes**  $S \in \text{lmeasurable}$   
**defines**  $T \equiv \text{triangle-linear } a\ b\ c$   
**shows**  $(\lambda x. \text{abs } (\det (\text{matrix } (T))))$  *integrable-on*  $S$  (**is**  $(\lambda x. ?c)$  *integrable-on*  $S$ )  
 <proof>

**lemma** *measure-differentiable-image-eq-affine*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**defines**  $A \equiv \text{triangle-affine } a\ b\ c$  **and**  $T \equiv \text{triangle-linear } a\ b\ c$   
**assumes**  $S \in \text{lmeasurable}$  **and**  $\neg \text{collinear } \{a, b, c\}$   
**shows**  $\text{measure lebesgue } (A \text{ ' } S) = \text{integral } S (\lambda x. \text{abs } (\det (\text{matrix } T)))$   
 <proof>

**lemma** *triangle-affine-img*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**defines**  $A \equiv \text{triangle-affine } a\ b\ c$   
**shows**  $\text{convex hull } \{a, b, c\} = A \text{ ' unit-triangle}$   
 <proof>

**lemma** *triangle-affine-e1-e2*:  
**fixes**  $a\ b\ c :: \text{real}^2$   
**defines**  $A \equiv \text{triangle-affine } a\ b\ c$

**shows** (*triangle-affine*  $a\ b\ c$ ) (*vector*  $[0, 0]$ ) =  $a$   
 (*triangle-affine*  $a\ b\ c$ ) (*vector*  $[1, 0]$ ) =  $b$   
 (*triangle-affine*  $a\ b\ c$ ) (*vector*  $[0, 1]$ ) =  $c$   
 ⟨*proof*⟩

**lemma** *triangle-measure-integral-of-det*:

**fixes**  $a\ b\ c :: \text{real}^2$   
**defines**  $S \equiv \text{convex hull } \{a, b, c\}$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**shows**  $\text{measure lebesgue } S =$   
 $\text{integral unit-triangle } (\lambda(x::\text{real}^2). \text{abs } (\text{det } (\text{matrix } (\text{triangle-linear } a\ b$   
 $c))))$   
 ⟨*proof*⟩

**lemma** *triangle-affine-preserves-interior*:

**assumes**  $A = \text{triangle-affine } a\ b\ c$  **and**  $L = \text{triangle-linear } a\ b\ c$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**shows**  $A \text{ ' } (\text{interior } S) = \text{interior } (A \text{ ' } S)$   
 ⟨*proof*⟩

**lemma** *triangle-affine-preserves-affine-hull*:

**assumes**  $A = \text{triangle-affine } a\ b\ c$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**shows**  $A \text{ ' } (\text{affine hull } S) = \text{affine hull } (A \text{ ' } S)$   
 ⟨*proof*⟩

**lemma** *triangle-measure-convex-hull-measure-path-inside-same*:

**assumes**  $p\text{-triangle}: p = \text{make-triangle } a\ b\ c$   
**assumes**  $\text{elem-triangle}: \text{elem-triangle } a\ b\ c$   
**shows**  $\text{measure lebesgue } (\text{convex hull } \{a, b, c\}) = \text{measure lebesgue } (\text{path-inside } p)$   
 $(\text{is measure lebesgue } ?S = \text{measure lebesgue } ?I)$   
 ⟨*proof*⟩

**lemma** *on-triangle-path-image-cases*:

**assumes**  $p = \text{make-triangle } a\ b\ c$   
**assumes**  $d \in \text{path-image } p$   
**shows**  $d \in \text{path-image } (\text{linepath } a\ b) \vee d \in \text{path-image } (\text{linepath } b\ c) \vee d \in$   
 $\text{path-image } (\text{linepath } c\ a)$   
 ⟨*proof*⟩

**lemma** *on-triangle-frontier-cases*:

**fixes**  $a\ b\ c :: \text{real}^2$   
**assumes**  $\neg \text{collinear } \{a, b, c\}$   
**assumes**  $d \in \text{frontier } (\text{convex hull } \{a, b, c\})$   
**shows**  $d \in \text{path-image } (\text{linepath } a\ b) \vee d \in \text{path-image } (\text{linepath } b\ c) \vee d \in$   
 $\text{path-image } (\text{linepath } c\ a)$   
 ⟨*proof*⟩

**lemma** *triangle-path-image-subset-convex*:  
**assumes**  $p = \text{make-triangle } a \ b \ c$   
**shows**  $\text{path-image } p \subseteq \text{convex hull } \{a, b, c\}$   
 $\langle \text{proof} \rangle$

**lemma** *triangle-convex-hull*:  
**assumes**  $p = \text{make-triangle } a \ b \ c$  **and**  $\neg \text{collinear } \{a, b, c\}$   
**shows**  $\text{convex hull } \{a, b, c\} = (\text{path-image } p) \cup (\text{path-inside } p)$   
 $\langle \text{proof} \rangle$

**end**  
**theory** *Unit-Geometry*  
**imports**  
*HOL-Analysis.Polytope*  
*Polygon-Jordan-Curve*  
*Triangle-Lemmas*

**begin**

## 21 Measure Setup

**lemma** *finite-convex-is-measurable*:  
**fixes**  $p :: (\text{real}^2) \text{ set}$   
**assumes**  $p = \text{convex hull } l$  **and** *finite*  $l$   
**shows**  $p \in \text{sets lebesgue}$   
 $\langle \text{proof} \rangle$

**lemma** *unit-square-lebesgue*:  $\text{unit-square} \in \text{sets lebesgue}$   
 $\langle \text{proof} \rangle$

**lemma** *unit-triangle-lebesgue*:  $\text{unit-triangle} \in \text{sets lebesgue}$   
 $\langle \text{proof} \rangle$

**lemma** *unit-triangle-lmeasurable*:  $\text{unit-triangle} \in \text{lmeasurable}$   
 $\langle \text{proof} \rangle$

## 22 Unit Triangle

**lemma** *unit-triangle-vts-not-collinear*:  
 $\neg \text{collinear } \{(\text{vector } [0, 0])::\text{real}^2, \text{vector } [1, 0], \text{vector } [0, 1]\}$   
**(is**  $\neg \text{collinear } \{?a, ?b, ?c\}$ **)**  
 $\langle \text{proof} \rangle$

**lemma** *unit-triangle-convex*:  
**assumes**  $p = (\text{make-polygonal-path } [\text{vector } [0, 0], \text{vector } [1, 0], \text{vector } [0, 1], \text{vector } [0, 0]])$   
**(is**  $p = \text{make-polygonal-path } [?O, ?e1, ?e2, ?O]$ **)**



**shows** *convex* (*path-inside p*)  
 ⟨*proof*⟩

**lemma** *unit-triangle-char*:  
**shows** *unit-triangle* = {*x*.  $0 \leq x \$ 1 \wedge 0 \leq x \$ 2 \wedge x \$ 1 + x \$ 2 \leq 1$ }  
 (**is** *unit-triangle* = ?*S*)  
 ⟨*proof*⟩

**lemma** *unit-triangle-interior-char*:  
**shows** *interior unit-triangle* = {*x*.  $0 < x \$ 1 \wedge 0 < x \$ 2 \wedge x \$ 1 + x \$ 2 < 1$ }  
 (**is** *interior unit-triangle* = ?*S*)  
 ⟨*proof*⟩

**lemma** *unit-triangle-is-elementary*: *elem-triangle* (*vector* [0, 0]) (*vector* [1, 0])  
 (*vector* [0, 1])  
 (**is** *elem-triangle* ?*a* ?*b* ?*c*)  
 ⟨*proof*⟩

**lemma** *unit-triangles-same-area*:  
*measure lebesgue unit-triangle'* = *measure lebesgue unit-triangle*  
 ⟨*proof*⟩

## 23 Unit Square

**lemma** *convex-hull-4*:  
*convex hull* {*a,b,c,d*} = { *u \*<sub>R</sub> a + v \*<sub>R</sub> b + w \*<sub>R</sub> c + t \*<sub>R</sub> d* | *u v w t*.  $0 \leq u \wedge 0 \leq v \wedge 0 \leq w \wedge 0 \leq t \wedge u + v + w + t = 1$  }  
 ⟨*proof*⟩

**lemma** *unit-square-characterization-helper*:  
**fixes** *a b* :: *real*  
**assumes**  $0 \leq a \wedge a \leq 1 \wedge 0 \leq b \wedge b \leq 1$  **and**  
 $a \leq b$   
**obtains** *u v w t* **where**  
 $\text{vector } [a, b] = u *_R ((\text{vector } [0, 0]))::\text{real}^2$   
 $+ v *_R (\text{vector } [0, 1])$   
 $+ w *_R (\text{vector } [1, 1])$   
 $+ t *_R (\text{vector } [1, 0])$   
 $\wedge 0 \leq u \wedge 0 \leq v \wedge 0 \leq w \wedge 0 \leq t \wedge u + v + w + t = 1$   
 ⟨*proof*⟩

**lemma** *unit-square-characterization*:  
*unit-square* = {*x*.  $0 \leq x \$ 1 \wedge x \$ 1 \leq 1 \wedge 0 \leq x \$ 2 \wedge x \$ 2 \leq 1$ } (**is** *unit-square*  
 = ?*S*)  
 ⟨*proof*⟩

**lemma** *e1e2-basis*:  
**defines** *e1* ≡ (*vector* [1, 0])::(*real*<sup>2</sup>) **and**

$e2 \equiv (\text{vector } [0, 1]) :: (\text{real}^2)$   
**shows**  $e1 = \text{axis } 1 \ (1 :: \text{real})$  **and**  $e1 \in (\text{Basis} :: ((\text{real}^2) \text{ set}))$  **and**  
 $e2 = \text{axis } 2 \ (1 :: \text{real})$  **and**  $e2 \in (\text{Basis} :: ((\text{real}^2) \text{ set}))$   
 <proof>

**lemma** *unit-square-cbox*:  $\text{unit-square} = \text{cbox } (\text{vector } [0, 0]) \ (\text{vector } [1, 1])$   
 <proof>

**lemma** *unit-square-area*:  $\text{measure lebesgue unit-square} = 1$   
 <proof>

## 24 Unit Triangle Area is 1/2

**lemma** *unit-triangle'-char*:  
**shows**  $\text{unit-triangle}' = \{x. x \$ 1 \leq 1 \wedge x \$ 2 \leq 1 \wedge x \$ 1 + x \$ 2 \geq 1\}$   
 <proof>

**lemma** *unit-square-split-diag*:  
**shows**  $\text{unit-square} = \text{unit-triangle} \cup \text{unit-triangle}'$   
 <proof>

**lemma** *unit-triangle-INT-unit-triangle'-measure*:  
 $\text{measure lebesgue } (\text{unit-triangle} \cap \text{unit-triangle}') = 0$   
 <proof>

**lemma** *unit-triangle-area*:  $\text{measure lebesgue unit-triangle} = 1/2$   
 <proof>

**end**  
**theory** *Elementary-Triangle-Area*  
**imports**  
*Unit-Geometry*

**begin**

## 25 Area of Elementary Triangle is 1/2

**lemma** *nonint-in-square-img-IMP-nonint-triangle-img*:  
**assumes**  $A = \text{triangle-affine } a \ b \ c$   
**assumes**  $x \in \text{unit-square}$   
**assumes**  $\neg \text{integral-vec } x$   
**assumes**  $\text{integral-vec } (A \ x)$   
**assumes**  $\text{elem-triangle } a \ b \ c$   
**obtains**  $x'$  **where**  $x' \in \text{unit-triangle} \wedge \neg \text{integral-vec } x' \wedge \text{integral-vec } (A \ x')$   
 <proof>

**lemma** *elem-triangle-integral-mat-bij*:  
**fixes**  $a \ b \ c :: \text{real}^2$

```

assumes elem-triangle a b c
defines L  $\equiv$  triangle-mat a b c
shows integral-mat-bij L
<proof>

lemma elem-triangle-measure-integral-of-1:
  fixes a b c :: real2
  defines S  $\equiv$  convex hull {a, b, c}
  assumes elem-triangle a b c
  shows measure lebesgue S = integral unit-triangle ( $\lambda(x::\text{real}^2). 1$ )
<proof>

lemma elem-triangle-area-is-half:
  fixes a b c :: real2
  assumes elem-triangle a b c
  defines S  $\equiv$  convex hull {a, b, c}
  shows measure lebesgue S = 1/2 (is ?S-area = 1/2)
<proof>

end
theory Pick
imports
  Polygon-Splitting
  Elementary-Triangle-Area
begin

```

## 26 Setup

### 26.1 Integral Points Cardinality Properties

```

lemma bounded-finite:
  fixes A:: (real2) set
  assumes bounded A
  shows finite {x::(real2). integral-vec x  $\wedge$  x  $\in$  A} (is finite ?A-int)
<proof>

```

```

lemma finite-path-image:
  assumes polygon p
  shows finite {x. integral-vec x  $\wedge$  x  $\in$  path-image p}
<proof>

```

```

lemma finite-path-inside:
  assumes polygon p
  shows finite {x. integral-vec x  $\wedge$  x  $\in$  path-inside p}
<proof>

```

```

lemma bounded-finite-inside:
  fixes B:: (real2) set
  assumes simple-path p

```

**shows** *bounded* (*path-inside* *p*)  
 ⟨*proof*⟩

**lemma** *finite-integral-points-path-image*:  
**assumes** *simple-path* *p*  
**shows** *finite* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-image* *p*}  
 ⟨*proof*⟩

**lemma** *finite-integral-points-path-inside*:  
**assumes** *simple-path* *p*  
**shows** *finite* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-inside* *p*}  
 ⟨*proof*⟩

## 27 Pick splitting

**lemma** *pick-split-path-union-main*:  
**assumes** *is-split*: *is-polygon-split-path* *vts* *i* *j* *cutvts*  
**assumes** *vts1* = (*take* *i* *vts*)  
**assumes** *vts2* = (*take* (*j* - *i* - 1) (*drop* (*Suc* *i*) *vts*))  
**assumes** *vts3* = *drop* (*j* - *i*) (*drop* (*Suc* *i*) *vts*)  
**assumes** *x* = *vts!**i*  
**assumes** *y* = *vts!**j*  
**assumes** *cutpath* = *make-polygonal-path* (*x* # *cutvts* @ [*y*])  
**assumes** *p*: *p* = *make-polygonal-path* (*vts*@[*vts!*0]) (**is** *p* = *make-polygonal-path* ?*p-vts*)  
**assumes** *p1*: *p1* = *make-polygonal-path* (*x*#(*vts2* @ [*y*] @ (*rev* *cutvts*) @ [*x*]))  
**(is** *p1* = *make-polygonal-path* ?*p1-vts*)  
**assumes** *p2*: *p2* = *make-polygonal-path* (*vts1* @ ([*x*] @ *cutvts* @ [*y*]) @ *vts3* @ [*vts* ! 0]) (**is** *p2* = *make-polygonal-path* ?*p2-vts*)  
**assumes** *I1*: *I1* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-inside* *p1*}  
**assumes** *B1*: *B1* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-image* *p1*}  
**assumes** *I2*: *I2* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-inside* *p2*}  
**assumes** *B2*: *B2* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-image* *p2*}  
**assumes** *I*: *I* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-inside* *p*}  
**assumes** *B*: *B* = *card* {*x*. *integral-vec* *x* ∧ *x* ∈ *path-image* *p*}  
**assumes** *all-integral-vts*: *all-integral* *vts*  
**shows** *measure lebesgue* (*path-inside* *p1*) = *I1* + *B1*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p2*) = *I2* + *B2*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p*) = *I* + *B*/2 - 1  
*measure lebesgue* (*path-inside* *p*) = *I* + *B*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p2*) = *I2* + *B2*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p1*) = *I1* + *B1*/2 - 1  
*measure lebesgue* (*path-inside* *p*) = *I* + *B*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p1*) = *I1* + *B1*/2 - 1  
 ⇒ *measure lebesgue* (*path-inside* *p2*) = *I2* + *B2*/2 - 1  
 ⟨*proof*⟩

**lemma** *pick-split-union*:  
**assumes** *is-split*: *is-polygon-split* *vts* *i* *j*

**assumes**  $vts1 = (take\ i\ vts)$   
**assumes**  $vts2 = (take\ (j - i - 1)\ (drop\ (Suc\ i)\ vts))$   
**assumes**  $vts3 = drop\ (j - i)\ (drop\ (Suc\ i)\ vts)$   
**assumes**  $x = vts!\ i$   
**assumes**  $y = vts!\ j$   
**assumes**  $p: p = make\text{-}polygonal\text{-}path\ (vts@[vts!0])$  (**is**  $p = make\text{-}polygonal\text{-}path\ ?p\text{-}vts$ )  
**assumes**  $p1: p1 = make\text{-}polygonal\text{-}path\ (x\#\ (vts2@[y, x]))$  (**is**  $p1 = make\text{-}polygonal\text{-}path\ ?p1\text{-}vts$ )  
**assumes**  $p2: p2 = make\text{-}polygonal\text{-}path\ (vts1\ @\ [x, y]\ @\ vts3\ @\ [vts!\ 0])$  (**is**  $p2 = make\text{-}polygonal\text{-}path\ ?p2\text{-}vts$ )  
**assumes**  $I1: I1 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p1\}$   
**assumes**  $B1: B1 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p1\}$   
**assumes**  $pick1: measure\ lebesgue\ (path\text{-}inside\ p1) = I1 + B1/2 - 1$   
**assumes**  $I2: I2 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p2\}$   
**assumes**  $B2: B2 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p2\}$   
**assumes**  $pick2: measure\ lebesgue\ (path\text{-}inside\ p2) = I2 + B2/2 - 1$   
**assumes**  $I: I = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p\}$   
**assumes**  $B: B = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p\}$   
**assumes**  $all\text{-}integral\text{-}vts: all\text{-}integral\ vts$   
**shows**  $measure\ lebesgue\ (path\text{-}inside\ p) = I + B/2 - 1$   
 $measure\ lebesgue\ (path\text{-}inside\ p) = measure\ lebesgue\ (path\text{-}inside\ p1) +$   
 $measure\ lebesgue\ (path\text{-}inside\ p2)$   
 $\langle proof \rangle$

**lemma** *pick-split-path-union:*

**assumes** *is-split: is-polygon-split-path*  $vts\ i\ j\ cutvts$   
**assumes**  $vts1 = (take\ i\ vts)$   
**assumes**  $vts2 = (take\ (j - i - 1)\ (drop\ (Suc\ i)\ vts))$   
**assumes**  $vts3 = drop\ (j - i)\ (drop\ (Suc\ i)\ vts)$   
**assumes**  $x = vts!\ i$   
**assumes**  $y = vts!\ j$   
**assumes**  $cutpath = make\text{-}polygonal\text{-}path\ (x\ #\ cutvts\ @\ [y])$   
**assumes**  $p: p = make\text{-}polygonal\text{-}path\ (vts@[vts!0])$  (**is**  $p = make\text{-}polygonal\text{-}path\ ?p\text{-}vts$ )  
**assumes**  $p1: p1 = make\text{-}polygonal\text{-}path\ (x\#\ (vts2\ @\ [y]\ @\ (rev\ cutvts)\ @\ [x]))$   
**(is**  $p1 = make\text{-}polygonal\text{-}path\ ?p1\text{-}vts$ )  
**assumes**  $p2: p2 = make\text{-}polygonal\text{-}path\ (vts1\ @\ ([x]\ @\ cutvts\ @\ [y])\ @\ vts3\ @\ [vts!\ 0])$  (**is**  $p2 = make\text{-}polygonal\text{-}path\ ?p2\text{-}vts$ )  
**assumes**  $I1: I1 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p1\}$   
**assumes**  $B1: B1 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p1\}$   
**assumes**  $pick1: measure\ lebesgue\ (path\text{-}inside\ p1) = I1 + B1/2 - 1$   
**assumes**  $I2: I2 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p2\}$   
**assumes**  $B2: B2 = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p2\}$   
**assumes**  $pick2: measure\ lebesgue\ (path\text{-}inside\ p2) = I2 + B2/2 - 1$   
**assumes**  $I: I = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}inside\ p\}$   
**assumes**  $B: B = card\ \{x.\ integral\text{-}vec\ x \wedge x \in path\text{-}image\ p\}$   
**assumes**  $all\text{-}integral\text{-}vts: all\text{-}integral\ vts$   
**shows**  $measure\ lebesgue\ (path\text{-}inside\ p) = I + B/2 - 1$

*<proof>*

**lemma** *pick-triangle-basic-split*:

**assumes**  $p = \text{make-triangle } a \ b \ c$  **and**  $\text{distinct } [a, b, c]$  **and**  $\neg \text{collinear } \{a, b, c\}$  **and**

*d-prop*:  $d \in \text{path-image } (\text{linepath } a \ b) \wedge d \notin \{a, b, c\}$

**shows**  $\text{good-linepath } c \ d \ [a, d, b, c, a]$

$\wedge \text{path-image } (\text{make-polygonal-path } [a, d, b, c, a]) = \text{path-image } p$

*<proof>*

## 28 Convex Hull Has Good Linepath

**lemma** *leq-2-extreme-points-means-collinear*:

**fixes**  $pts :: 'a::\text{euclidean-space set}$

**assumes** *finite pts*

**assumes**  $\text{card } \{v. v \text{ extreme-point-of } (\text{convex hull } pts)\} \leq 2$

**shows** *collinear pts*

*<proof>*

**lemma** *convex-hull-non-extreme-point-in-open-seg*:

**assumes**  $H = \text{convex hull } pts$

**assumes**  $x \in H - \{v. v \text{ extreme-point-of } H\}$

**shows**  $\exists a \ b. a \in H \wedge b \in H \wedge x \in \text{open-segment } a \ b$

*<proof>*

**lemma** *convex-hull-extreme-points-vertex-split*:

**fixes**  $pts :: (\text{real}^2) \text{ set}$

**assumes**  $H = \text{convex hull } pts$

**assumes** *finite pts*

**assumes**  $\text{card } \{v. v \text{ extreme-point-of } H\} \geq 4$

**assumes**  $\{a, b, c\} \subseteq \{v. v \text{ extreme-point-of } H\} \wedge \text{distinct } [a, b, c]$

**shows**  $\text{path-image } (\text{linepath } a \ b) \cap \text{interior } H \neq \{\}$

$\vee \text{path-image } (\text{linepath } b \ c) \cap \text{interior } H \neq \{\}$

$\vee \text{path-image } (\text{linepath } c \ a) \cap \text{interior } H \neq \{\}$

*<proof>*

**lemma** *convex-hull-has-vertex-split-helper-wlog*:

**assumes**  $p = \text{make-triangle } a \ b \ c$  **and**  $\text{distinct } [a, b, c]$  **and**  $\neg \text{collinear } \{a, b, c\}$  **and**

*d-prop*:  $d \in \text{path-image } (\text{linepath } a \ b) \wedge d \notin \{a, b, c\}$

**shows**  $\text{path-image } (\text{linepath } c \ d) \cap \text{path-inside } p \neq \{\}$

*<proof>*

**lemma** *convex-hull-has-vertex-split-helper*:

**assumes**  $p = \text{make-triangle } a \ b \ c$  **and**  $\text{distinct } [a, b, c]$  **and**  $\neg \text{collinear } \{a, b, c\}$  **and**

*d-prop*:  $d \in \text{path-image } p \wedge d \notin \{a, b, c\}$

**shows**  $\exists x \ y. \{x, y\} \subseteq \{a, b, c, d\} \wedge x \neq y \wedge \text{path-image } (\text{linepath } x \ y) \cap \text{path-inside } p \neq \{\}$

*<proof>*

**lemma** *convex-hull-has-vertex-split:*

**fixes**  $vts :: (\text{real}^2)$  *set*

**assumes**  $H = \text{convex hull } vts$

**assumes**  $\neg \text{collinear } vts$

**assumes**  $\text{card } vts > 3$

**assumes** *finite vts*

**shows**  $\exists a b. \{a, b\} \subseteq vts \wedge a \neq b \wedge \text{path-image } (\text{linepath } a b) \cap \text{interior } H \neq$

$\{\}$

*<proof>*

**lemma** *convex-polygon-has-good-linepath-helper:*

**assumes** *polygon-of p vts*

**assumes**  $\text{convex } (\text{path-inside } p \cup \text{path-image } p)$

**assumes**  $\text{card } (\text{set } vts) > 3$

**obtains**  $a b$  **where**  $\{a, b\} \subseteq \text{set } vts \wedge a \neq b \wedge \neg \text{path-image } (\text{linepath } a b) \subseteq$   
 $\text{path-image } p$

*<proof>*

**lemma** *convex-polygon-has-good-linepath:*

**assumes**  $\text{convex } (\text{path-inside } p \cup \text{path-image } p)$

**assumes** *polygon p*

**assumes**  $p = \text{make-polygonal-path } vts$

**assumes**  $\text{card } (\text{set } vts) > 3$

**shows**  $\exists a b. \text{good-linepath } a b vts$

*<proof>*

## 29 Pick's Theorem

**definition** *integral-inside:*

$\text{integral-inside } p = \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$

**definition** *integral-boundary:*

$\text{integral-boundary } p = \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$

### 29.1 Pick's Theorem Triangle Case

**definition** *pick-triangle:*

$\text{pick-triangle } p a b c \longleftrightarrow$

$p = \text{make-triangle } a b c$

$\wedge \text{all-integral } [a, b, c]$

$\wedge \text{distinct } [a, b, c]$

$\wedge \neg \text{collinear } \{a, b, c\}$

**definition** *pick-holds:*

$\text{pick-holds } p \longleftrightarrow$

*(let*  $I = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$  *in*

*let*  $B = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$  *in*

$$\text{measure lebesgue } (\text{path-inside } p) = I + B/2 - 1$$

**lemma** *pick-triangle-wlog-helper:*

**assumes** *pick-triangle*  $p$   $a$   $b$   $c$  **and**

$I = \text{card } (\text{integral-inside } p)$  **and**

$B = \text{card } (\text{integral-boundary } p)$  **and**

$\text{integral-inside } p = \{\}$  **and**

$\text{integral-vec } d \wedge d \in \text{path-image } (\text{linepath } a \ b) \wedge d \notin \{a, b, c\}$  **and**  $d \notin \{a, b, c\}$  **and**

*ih:*  $\bigwedge p' \ a' \ b' \ c'. (\text{card } (\text{integral-inside } p') + \text{card } (\text{integral-boundary } p') < I + B) \implies \text{pick-triangle } p' \ a' \ b' \ c' \implies \text{pick-holds } p'$

**shows**  $\text{measure lebesgue } (\text{path-inside } p) = I + B/2 - 1$

*<proof>*

**lemma** *pick-triangle-helper:*

**assumes** *pick-triangle*  $p$   $a$   $b$   $c$  **and**

$I = \text{card } (\text{integral-inside } p)$  **and**

$B = \text{card } (\text{integral-boundary } p)$  **and**

$\text{integral-inside } p = \{\}$  **and**

$\text{integral-vec } d \wedge d \notin \{a, b, c\}$  **and**  $d \notin \{a, b, c\}$  **and**

$d \in \text{path-image } (\text{linepath } a \ b)$

$\vee d \in \text{path-image } (\text{linepath } b \ c)$

$\vee d \in \text{path-image } (\text{linepath } c \ a)$  **and**

*ih:*  $\bigwedge p' \ a' \ b' \ c'. (\text{card } (\text{integral-inside } p') + \text{card } (\text{integral-boundary } p') < I + B) \implies \text{pick-triangle } p' \ a' \ b' \ c' \implies \text{pick-holds } p'$

**shows**  $\text{measure lebesgue } (\text{path-inside } p) = I + B/2 - 1$

*<proof>*

**lemma** *triangle-3-split-helper:*

**fixes**  $a \ b :: 'a :: \text{euclidean-space}$

**assumes**  $a \in \text{frontier } S$

**assumes**  $b \in \text{interior } S$

**assumes** *convex*  $S$

**assumes** *closed*  $S$

**shows**  $\text{path-image } (\text{linepath } a \ b) \cap \text{frontier } S = \{a\}$

*<proof>*

**lemma** *unit-triangle-interior-point-not-collinear-e1-e2:*

**assumes**  $p = \text{make-triangle } (\text{vector } [0, 0]) (\text{vector } [1, 0]) (\text{vector } [0, 1])$

(**is**  $p = \text{make-triangle } ?O \ ?e1 \ ?e2$ )

**assumes**  $z \in \text{path-inside } p$

**shows**  $\neg \text{collinear } \{?O, ?e1, z\}$

*<proof>*

**lemma** *triangle-interior-point-not-collinear-vertices-wlog-helper:*

**assumes**  $p = \text{make-triangle } a \ b \ c$

**assumes** *polygon*  $p$

**assumes**  $z \in \text{path-inside } p$

**shows**  $\neg \text{collinear } \{a, b, z\}$



*<proof>*

**lemma** *triangle-interior-point-not-collinear-vertices:*

**assumes**  $p = \text{make-triangle } a \ b \ c$

**assumes** *polygon*  $p$

**assumes**  $z \in \text{path-inside } p$

**shows**  $\neg \text{collinear } \{a, b, z\} \wedge \neg \text{collinear } \{a, c, z\} \wedge \neg \text{collinear } \{b, c, z\}$

*<proof>*

**lemma** *triangle-3-split:*

**assumes**  $p = \text{make-triangle } a \ b \ c$

**assumes** *polygon*  $p$

**assumes**  $z \in \text{path-inside } p$

**shows** *is-polygon-split-path*  $[a, b, c] \ 0 \ 1 \ [z]$

*is-polygon-split*  $[a, z, b, c] \ 1 \ 3$

$a \notin \text{path-image } (\text{make-triangle } z \ b \ c) \cup \text{path-inside } (\text{make-triangle } z \ b \ c)$

$b \notin \text{path-image } (\text{make-triangle } a \ z \ c) \cup \text{path-inside } (\text{make-triangle } a \ z \ c)$

$c \notin \text{path-image } (\text{make-triangle } a \ b \ z) \cup \text{path-inside } (\text{make-triangle } a \ b \ z)$

*<proof>*

**lemma** *smaller-triangle:*

**assumes**  $\neg \text{collinear } \{a, b, c\} \wedge \neg \text{collinear } \{a', b', c'\}$

**assumes**  $p = \text{make-triangle } a \ b \ c$

**assumes**  $p' = \text{make-triangle } a' \ b' \ c'$

**assumes** *path-inside*  $p \subseteq \text{path-inside } p'$

**assumes**  $\exists d. \text{integral-vec } d \wedge d \in \text{path-image } p' \cup \text{path-inside } p' \wedge d \notin \text{path-image } p \cup \text{path-inside } p$

**shows**  $\text{card } (\text{integral-inside } p) + \text{card } (\text{integral-boundary } p) < \text{card } (\text{integral-inside } p') + \text{card } (\text{integral-boundary } p')$

*<proof>*

**lemma** *pick-elem-triangle:*

**fixes**  $p :: R\text{-to-}R^2$

**assumes** *p-triangle:*  $p = \text{make-triangle } a \ b \ c$

**assumes** *elem-triangle:* *elem-triangle*  $a \ b \ c$

**assumes**  $I = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$  **and**

$B = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$

**shows**  $\text{measure lebesgue } (\text{path-inside } p) = I + B/2 - 1$

*<proof>*

**lemma** *pick-triangle-lemma:*

**fixes**  $p :: R\text{-to-}R^2$

**assumes**  $p = \text{make-triangle } a \ b \ c$  **and** *all-integral*  $[a, b, c]$  **and** *distinct*  $[a, b, c]$

**and**  $\neg \text{collinear } \{a, b, c\}$

$I = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$  **and**

$B = \text{card } \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$

**shows**  $\text{measure lebesgue } (\text{path-inside } p) = I + B/2 - 1$

*<proof>*

## 29.2 Pocket properties

**definition** *index-not-in-set* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  set  $\Rightarrow$  nat  $\Rightarrow$  bool  
**where** *index-not-in-set* vts A i  $\longleftrightarrow$   $i \in \{i. i < \text{length vts} \wedge \text{vts} ! i \notin A\}$

**definition** *min-index-not-in-set*::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  set  $\Rightarrow$  nat  
**where** *min-index-not-in-set* vts A = (LEAST i. *index-not-in-set* vts A i)

**definition** *nonzero-index-in-set* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  set  $\Rightarrow$  nat  $\Rightarrow$  bool  
**where**  
*nonzero-index-in-set* vts A i  $\longleftrightarrow$   $i \in \{i. 0 < i \wedge i < \text{length vts} \wedge \text{vts} ! i \in A\}$

**definition** *min-nonzero-index-in-set* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  set  $\Rightarrow$  nat **where**  
*min-nonzero-index-in-set* vts A = (LEAST i. *nonzero-index-in-set* vts A i)

**definition** *construct-pocket-0* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  set  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  list **where**  
*construct-pocket-0* vts A = take ((*min-nonzero-index-in-set* vts A) + 1) vts

**definition** *is-pocket-0* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$  bool **where**  
*is-pocket-0* vts vts'  $\longleftrightarrow$   
 polygon (make-polygonal-path vts)  
 $\wedge (\exists i. \text{vts}' = \text{take } i \text{ vts})$   
 $\wedge 3 \leq \text{length vts}' \wedge \text{length vts}' < \text{length vts}$   
 $\wedge \text{hd vts}' \in \text{frontier } (\text{convex hull } (\text{set vts})) \wedge \text{last vts}' \in \text{frontier } (\text{convex hull } (\text{set vts}))$   
 $\wedge \text{set } (\text{tl } (\text{butlast vts}')) \subseteq \text{interior } (\text{convex hull } (\text{set vts}))$

**definition** *fill-pocket-0* ::  $(\text{real}^{\mathbb{2}})$  list  $\Rightarrow$  nat  $\Rightarrow$   $(\text{real}^{\mathbb{2}})$  list **where**  
*fill-pocket-0* vts i = (hd vts) # (drop (i-1) vts)

**lemma** *min-nonzero-index-in-set-exists*:  
**assumes** set (tl vts)  $\cap$  A  $\neq$  {}  
**shows**  $\exists i. \text{nonzero-index-in-set vts A } i$   
 <proof>

**lemma** *min-nonzero-index-in-set-defined*:  
**assumes** set (tl vts)  $\cap$  A  $\neq$  {}  
**defines** i  $\equiv$  *min-nonzero-index-in-set* vts A  
**shows** *nonzero-index-in-set* vts A i  $\wedge (\forall j < i. \neg \text{nonzero-index-in-set vts A } j)$   
 <proof>

**lemma** *min-index-not-in-set-exists*:  
**assumes** set vts  $\supset$  A  
**shows**  $\exists i. \text{index-not-in-set vts A } i$   
 <proof>

**lemma** *min-index-not-in-set-defined*:  
**assumes** set vts  $\supset$  A

**defines**  $i \equiv \text{min-index-not-in-set } vts \ A$   
**shows**  $\text{index-not-in-set } vts \ A \ i \wedge (\forall j < i. \neg \text{index-not-in-set } vts \ A \ j)$   
 $\langle \text{proof} \rangle$

**lemma** *min-nonzero-index-in-set-bound*:  
**assumes**  $\text{set } (tl \ vts) \cap A \neq \{\}$   
**shows**  $\text{min-nonzero-index-in-set } vts \ A < \text{length } vts$   
 $\langle \text{proof} \rangle$

**lemma** *construct-pocket-0-subset-vts*:  
**assumes**  $\text{set } (tl \ vts) \cap A \neq \{\}$   
**shows**  $\text{set } (\text{construct-pocket-0 } vts \ A) \subseteq \text{set } vts$   
 $\langle \text{proof} \rangle$

**lemma** *min-index-not-in-set-0*:  
**assumes**  $\text{set } vts \supset A$   
**assumes**  $vts!0 \in A$   
**defines**  $i \equiv \text{min-index-not-in-set } vts \ A$   
**defines**  $r \equiv i - 1$   
**shows**  $vts!r \in A$   
 $\langle \text{proof} \rangle$

**lemma** *construct-pocket-0-last-in-set*:  
**assumes**  $\text{set } (tl \ vts) \cap A \neq \{\}$   
**assumes**  $vts!0 \in A$   
**defines**  $p \equiv \text{construct-pocket-0 } vts \ A$   
**shows**  $\text{last } p \in A$   
 $\langle \text{proof} \rangle$

**lemma** *construct-pocket-0-first-last-distinct*:  
**assumes**  $\text{card } A \geq 2$   
**assumes**  $A \subseteq \text{set } vts$   
**assumes**  $\text{distinct } (\text{butlast } vts)$   
**assumes**  $\text{hd } vts = \text{last } vts$   
**shows**  $\text{hd } (\text{construct-pocket-0 } vts \ A) \neq \text{last } (\text{construct-pocket-0 } vts \ A)$   
 $\langle \text{proof} \rangle$

**lemma** *construct-pocket-is-pocket*:  
**assumes**  $\text{polygon } (\text{make-polygonal-path } vts)$   
**assumes**  $vts!0 \in \text{frontier } (\text{convex hull } (\text{set } vts))$   
**assumes**  $vts!1 \notin \text{frontier } (\text{convex hull } (\text{set } vts))$   
**shows**  $\text{is-pocket-0 } vts \ (\text{construct-pocket-0 } vts \ (\text{set } vts \cap \text{frontier } (\text{convex hull } (\text{set } vts))))$   
 $\langle \text{proof} \rangle$

**lemma** *exists-point-above-interior*:  
**fixes**  $a :: \text{real}^2$   
**assumes**  $a \in \text{interior } (\text{convex hull } S)$

**obtains**  $x$  **where**  $x \in S \wedge x\$2 > a\$2$   
 <proof>

**lemma** *exists-point-above-convex-hull-interior*:

**fixes**  $S :: (\text{real}^2)$  *set*  
**assumes**  $S \neq \{\}$   
**assumes** *compact*  $S$   
**obtains**  $x$  **where**  $x \in S - (\text{interior } (\text{convex hull } S)) \wedge (\forall y \in \text{interior } (\text{convex hull } S). x\$2 > y\$2)$   
 <proof>

**lemma** *flip-function*:

**defines**  $M \equiv (\text{vector } [\text{vector } [1, 0], \text{vector } [0, -1]]) :: (\text{real}^2 \times \text{real}^2)$   
**defines**  $f \equiv \lambda v. M * v$   
**defines**  $g \equiv (\lambda v. \text{vector } [v\$1, -v\$2]) :: (\text{real}^2 \Rightarrow \text{real}^2)$   
**shows** *inj*  $f f = g$   
 <proof>

**lemma** *exists-point-below-convex-hull-interior*:

**fixes**  $S :: (\text{real}^2)$  *set*  
**assumes**  $S \neq \{\}$   
**assumes** *compact*  $S$   
**obtains**  $x$  **where**  $x \in S - (\text{interior } (\text{convex hull } S)) \wedge (\forall y \in \text{interior } (\text{convex hull } S). x\$2 < y\$2)$   
 <proof>

**lemma** *exists-point-above-all*:

**fixes**  $p\ q :: R\text{-to-}R^2$   
**defines**  $H \equiv \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**assumes**  $\text{path } p \wedge \text{path } q$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } H$   
**assumes**  $(p\ 0)\$2 = 0 \wedge (p\ 1)\$2 = 0$   
**assumes**  $\exists x \in p\{0 < .. < 1\}. x\$2 \geq 0$   
**obtains**  $x$  **where**  $x \in \text{path-image } q \wedge (\forall y \in \text{path-image } p. x\$2 > y\$2)$   
 <proof>

**lemma** *exists-point-below-all*:

**fixes**  $p\ q :: R\text{-to-}R^2$   
**defines**  $H \equiv \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**assumes**  $\text{path } p \wedge \text{path } q$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } H$   
**assumes**  $(p\ 0)\$2 = 0 \wedge (p\ 1)\$2 = 0$   
**assumes**  $\exists x \in \text{path-image } p \cup \text{path-image } q. x\$2 < 0$   
**obtains**  $x$  **where**  $x \in \text{path-image } q \wedge (\forall y \in \text{path-image } p. x\$2 < y\$2)$   
 <proof>

**lemma** *pocket-fill-line-int-aux*:

**fixes**  $x\ y\ z :: \text{real}^2$   
**defines**  $a \equiv y\$1$

**assumes**  $x = 0$   
**assumes**  $a > 0 \wedge y\$2 = 0$   
**assumes**  $z\$1 < 0 \vee z\$1 > a$   
**assumes**  $z\$2 = 0$   
**assumes**  $\text{convex } A \wedge \text{compact } A$   
**assumes**  $\{x, y, z\} \subseteq A$   
**assumes**  $\{x, y\} \subseteq \text{frontier } A$   
**shows**  $z \in \text{frontier } A \wedge \text{closed-segment } x y \subseteq \text{frontier } A$   
*<proof>*

**lemma** *axis-dist*:  
**fixes**  $a b :: \text{real}^2$   
**shows**  $a\$2 = b\$2 \implies \text{dist } a b = \text{dist } (a\$1) (b\$1) \quad a\$1 = b\$1 \implies \text{dist } a b = \text{dist } (a\$2) (b\$2)$   
*<proof>*

**lemma** *dist-bound-1*:  
**fixes**  $a b x :: \text{real}^2$   
**assumes**  $a\$2 = x\$2$   
**assumes**  $b \in \text{ball } x \ \varepsilon$   
**assumes**  $\varepsilon < \text{dist } a x$   
**shows**  $a\$1 < x\$1 \implies b\$1 > a\$1 \quad a\$1 > x\$1 \implies b\$1 < a\$1$   
*<proof>*

**lemma** *dist-bound-2*:  
**fixes**  $a b x :: \text{real}^2$   
**assumes**  $a\$1 = x\$1$   
**assumes**  $b \in \text{ball } x \ \varepsilon$   
**assumes**  $\varepsilon < \text{dist } a x$   
**shows**  $a\$2 < x\$2 \implies b\$2 > a\$2 \quad a\$2 > x\$2 \implies b\$2 < a\$2$   
*<proof>*

**lemma** *linepath-bound-1*:  
**fixes**  $x y :: \text{real}^2$   
**shows**  $a < x\$1 \wedge a < y\$1 \implies \forall v \in \text{path-image } (\text{linepath } x y). a < v\$1$   
 $x\$1 < b \wedge y\$1 < b \implies \forall v \in \text{path-image } (\text{linepath } x y). v\$1 < b$   
*<proof>*

**lemma** *linepath-bound-2*:  
**fixes**  $x y :: \text{real}^2$   
**shows**  $a < x\$2 \wedge a < y\$2 \implies \forall v \in \text{path-image } (\text{linepath } x y). a < v\$2$   
 $x\$2 < b \wedge y\$2 < b \implies \forall v \in \text{path-image } (\text{linepath } x y). v\$2 < b$   
*<proof>*

**lemma** *linepath-int-corner*:  
**fixes**  $x y z :: \text{real}^2$   
**assumes**  $x\$2 \neq y\$2$   
**assumes**  $y\$2 = z\$2$   
**shows**  $\text{path-image } (\text{linepath } x y) \cap \text{path-image } (\text{linepath } y z) = \{y\}$

(is path-image ?l1  $\cap$  path-image ?l2 = {y})  
 <proof>

**lemma** *linepath-int-vertical*:

**fixes** w x y z :: real<sup>2</sup>  
**assumes** w\$1  $\neq$  y\$1  
**assumes** w\$1 = x\$1  
**assumes** y\$1 = z\$1  
**shows** path-image (linepath w x)  $\cap$  path-image (linepath y z) = {}  
 <proof>

**lemma** *linepath-int-horizontal*:

**fixes** w x y z :: real<sup>2</sup>  
**assumes** w\$2  $\neq$  y\$2  
**assumes** w\$2 = x\$2  
**assumes** y\$2 = z\$2  
**shows** path-image (linepath w x)  $\cap$  path-image (linepath y z) = {}  
 <proof>

**lemma** *linepath-int-columns*:

**fixes** w x y z :: real<sup>2</sup>  
**assumes** w\$1 < y\$1  $\wedge$  w\$1 < z\$1  
**assumes** x\$1 < y\$1  $\wedge$  x\$1 < z\$1  
**shows** path-image (linepath w x)  $\cap$  path-image (linepath y z) = {}  
 (is path-image ?l1  $\cap$  path-image ?l2 = {})  
 <proof>

**lemma** *linepath-int-rows*:

**fixes** w x y z :: real<sup>2</sup>  
**assumes** w\$2 < y\$2  $\wedge$  w\$2 < z\$2  
**assumes** x\$2 < y\$2  $\wedge$  x\$2 < z\$2  
**shows** path-image (linepath w x)  $\cap$  path-image (linepath y z) = {}  
 (is path-image ?l1  $\cap$  path-image ?l2 = {})  
 <proof>

**lemma** *horizontal-segment-at-0*:

**assumes** a > 0  
**shows** closed-segment ((vector [0, 0])::(real<sup>2</sup>)) (vector [a, 0]) = {x. x\$2 = 0  
 $\wedge$  x\$1  $\in$  {0..a}}  
 (is ?l = ?s)  
 <proof>

**lemma** *horizontal-segment-at-0'*:

**fixes** x y :: real<sup>2</sup>  
**assumes** a > 0  
**assumes** x\$1 = 0  $\wedge$  x\$2 = 0  $\wedge$  y\$1 = a  $\wedge$  y\$2 = 0  
**shows** closed-segment x y = {x. x\$2 = 0  $\wedge$  x\$1  $\in$  {0..a}}  
 <proof>

**lemma** *pocket-fill-line-int-aux1*:  
**fixes**  $p\ q :: R\text{-to-}R^2$   
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1\$1$   
**defines**  $l \equiv \text{closed-segment } p0\ p1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $p0\$1 = 0 \wedge p0\$2 = 0 \wedge p1\$2 = 0$   
**assumes**  $a > 0$   
**assumes**  $\text{path-image } q \cap \{x. x\$2 = 0\} \subseteq l$   
**assumes**  $\text{path-image } p \cap \{x. x\$2 = 0\} \subseteq l$   
**assumes**  $\forall v \in \text{path-image } p. q0\$2 \leq v\$2$   
**assumes**  $\forall v \in \text{path-image } p. q1\$2 > v\$2$   
**shows**  $\text{path-image } p \cap \text{path-image } q \neq \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *pocket-fill-line-int-aux2*:  
**fixes**  $p\ q :: R\text{-to-}R^2$   
**fixes**  $A :: (\text{real}^2)\ \text{set}$   
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $a \equiv p1\$1$   
**defines**  $l \equiv \text{closed-segment } p0\ p1$   
**assumes** *simple-path*  $p$   
**assumes**  $p0\$1 = 0 \wedge p0\$2 = 0 \wedge p1\$2 = 0$   
**assumes**  $a > 0$   
**assumes** *convex*  $A \wedge \text{compact } A$   
**assumes**  $\{p0, p1\} \subseteq \text{frontier } A$   
**assumes**  $p \text{ ' } \{0 <..<1\} \subseteq \text{interior } A$   
**shows**  $\text{path-image } p \cap \{x. x\$2 = 0\} \subseteq l$   
 $\langle \text{proof} \rangle$

**lemma** *three-points-on-line*:  
**fixes**  $a\ b :: 'a::\text{real-vector}$   
**assumes**  $A = \text{affine hull } \{a, b\}$   
**assumes**  $a \neq b$   
**assumes**  $\{x, y, z\} \subseteq A$   
**assumes**  $x \neq y \wedge y \neq z \wedge x \neq z$   
**shows**  $x \in \text{open-segment } y\ z \vee y \in \text{open-segment } x\ z \vee z \in \text{open-segment } x\ y$   
 $\langle \text{proof} \rangle$

**lemma** *pocket-fill-line-int-aux3*:  
**fixes**  $A :: (\text{real}^2)\ \text{set}$   
**assumes** *convex*  $A \wedge \text{compact } A$   
**assumes**  $v \neq 0$   
**assumes**  $\text{closed-segment } 0\ w \subseteq \text{frontier } A$  (**is** *closed-segment*  $?a\ ?b \subseteq -$ )

**assumes**  $w \cdot v = 0$   
**assumes**  $w \neq 0$   
**shows**  $(A \subseteq \{x. x \cdot v \leq 0\} \vee A \subseteq \{x. x \cdot v \geq 0\})$  (**is**  $A \subseteq ?P1 \vee A \subseteq ?P2$ )  
 <proof>

**lemma** *pocket-fill-line-int-aux4*:

**fixes**  $p\ q :: R\text{-to-}R^2$   
**fixes**  $A :: (\text{real}^2)$  set  
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1\$1$   
**defines**  $l \equiv \text{closed-segment } p0\ p1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{\}$   
**assumes**  $p0\$1 = 0 \wedge p0\$2 = 0 \wedge p1\$2 = 0$   
**assumes**  $a > 0$   
**assumes**  $\forall v \in \text{path-image } p. q0\$2 \leq v\$2$   
**assumes**  $\forall v \in \text{path-image } p. q1\$2 > v\$2$   
**assumes** *convex*  $A \wedge$  *compact*  $A$   
**assumes**  $\{p0, p1\} \subseteq \text{frontier } A$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } A$   
**assumes**  $\text{path-image } q \subseteq A$   
**shows**  $l \subseteq \text{frontier } A \forall x \in (\text{path-image } p) \cup (\text{path-image } q). x\$2 \geq 0 \wedge q0\$2 = 0$   
 <proof>

**lemma** *pocket-fill-line-int-aux5*:

**fixes**  $p\ q :: R\text{-to-}R^2$   
**fixes**  $A :: (\text{real}^2)$  set  
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1\$1$   
**defines**  $l \equiv \text{closed-segment } p0\ p1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{q0, q1\}$   
**assumes**  $p0\$1 = 0 \wedge p0\$2 = 0 \wedge p1\$2 = 0$   
**assumes**  $a > 0$   
**assumes**  $A = \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**assumes**  $\{p0, p1\} \subseteq \text{frontier } A$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } A$   
**assumes**  $\text{path-image } q \subseteq A$   
**assumes**  $\exists x \in p\{0 < .. < 1\}. x\$2 \geq 0$   
**assumes**  $q0 = p1 \wedge q1 = p0$



**shows**  $l \subseteq \text{frontier } A \ \forall x \in \text{path-image } p \cup \text{path-image } q. x \geq 0$   
 <proof>

**lemma** *pocket-fill-line-int-aux6*:

**fixes**  $p \ q :: R\text{-to-}R^2$   
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1 \cdot 1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $p0 = 0 \wedge p1 \cdot 2 = 0$   
**assumes**  $a > 0$   
**assumes**  $q0 \cdot 1 \in \{0..a\} \wedge q0 \cdot 2 = 0$   
**assumes**  $\forall x \in \text{path-image } p. q1 \cdot 2 > x \cdot 2$   
**assumes**  $\forall x \in \text{path-image } p \cup \text{path-image } q. x \geq 0$   
**shows**  $\text{path-image } p \cap \text{path-image } q \neq \{\}$   
 <proof>

**lemma** *pocket-fill-line-int-aux7*:

**fixes**  $p \ q :: R\text{-to-}R^2$   
**fixes**  $A :: (\text{real}^2) \text{ set}$   
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1 \cdot 1$   
**defines**  $l \equiv \text{open-segment } p0 \ p1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{q0, q1\}$   
**assumes**  $p0 \cdot 1 = 0 \wedge p0 \cdot 2 = 0 \wedge p1 \cdot 2 = 0$   
**assumes**  $a > 0$   
**assumes**  $A = \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**assumes**  $\{p0, p1\} \subseteq \text{frontier } A$   
**assumes**  $p \cdot \{0 < .. < 1\} \subseteq \text{interior } A$   
**assumes**  $\exists x \in p \cdot \{0 < .. < 1\}. x \geq 0$   
**assumes**  $q0 = p1 \wedge q1 = p0$   
**shows**  $\text{path-image } q \cap l = \{\}$  *closed-segment*  $p0 \ p1 \subseteq \text{frontier } A$   
 <proof>

**lemma** *frontier-injective-linear-image*:

**fixes**  $f :: 'a :: \text{euclidean-space} \Rightarrow 'a :: \text{euclidean-space}$   
**assumes** *linear*  $f$  *inj*  $f$   
**shows**  $f \cdot (\text{frontier } S) = \text{frontier } (f \cdot S)$   
 <proof>

**lemma** *pocket-fill-line-int-aux8*:  
**fixes**  $p\ q :: R\text{-to-}R^2$   
**fixes**  $A :: (\text{real}^2)\ \text{set}$   
**defines**  $p0 \equiv \text{pathstart } p$   
**defines**  $p1 \equiv \text{pathfinish } p$   
**defines**  $q0 \equiv \text{pathstart } q$   
**defines**  $q1 \equiv \text{pathfinish } q$   
**defines**  $a \equiv p1\$1$   
**defines**  $l \equiv \text{open-segment } p0\ p1$   
**assumes** *simple-path*  $p$   
**assumes** *simple-path*  $q$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{q0, q1\}$   
**assumes**  $p0\$1 = 0 \wedge p0\$2 = 0 \wedge p1\$2 = 0$   
**assumes**  $a > 0$   
**assumes**  $A = \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**assumes**  $\{p0, p1\} \subseteq \text{frontier } A$   
**assumes**  $p\{0 <..<1\} \subseteq \text{interior } A$   
**assumes**  $q0 = p1 \wedge q1 = p0$   
**shows**  $\text{path-image } q \cap l = \{\} \wedge l \subseteq \text{frontier } A$   
 $\langle \text{proof} \rangle$

**lemma** *simple-path-linear-image*:  
**assumes** *simple-path*  $p$   
**assumes** *inj*  $f \wedge \text{bounded-linear } f$   
**shows** *simple-path*  $(f \circ p)$   
 $\langle \text{proof} \rangle$

**lemma** *pts-interior*:  
**fixes**  $pts$   
**defines**  $p \equiv \text{make-polygonal-path } pts$   
**assumes** *convex*  $H$   
**assumes**  $\forall j \in \{0 <..<\text{length } pts - 1\}. pts!j \notin \text{frontier } H$   
**assumes** *loop-free*  $p$   
**assumes**  $\text{path-image } p \subseteq H$   
**assumes**  $\text{length } pts \geq 3$   
**shows**  $p\{0 <..<1\} \subseteq \text{interior } H$   
 $\langle \text{proof} \rangle$

**lemma** *pocket-fill-line-int-0*:  
**assumes** *polygon-of*  $r\ pts$   
**defines**  $H \equiv \text{convex hull } (\text{set } pts)$   
**assumes**  $2 \leq i \wedge i < \text{length } pts - 1$   
**defines**  $a \equiv \text{hd } pts$   
**defines**  $b \equiv pts!i$   
**assumes**  $\{a, b\} \subseteq \text{frontier } H$   
**assumes**  $\forall j \in \{0 <..<i\}. pts!j \notin \text{frontier } H$   
**assumes**  $a = 0$   
**shows**  $\text{path-image } (\text{linepath } a\ b) \cap \text{path-image } r = \{a, b\}$   
 $\text{path-image } (\text{linepath } a\ b) \subseteq \text{frontier } H$

*<proof>*

**lemma** *linepath-translation*:  $(\lambda v. v - a) \circ (\text{linepath } x \ y) = \text{linepath } ((\lambda v. v - a) \ x) \ ((\lambda v. v - a) \ y)$   
*<proof>*

**lemma** *linepath-image-translation*:  
 $\text{path-image } ((\lambda v. v - a) \circ (\text{linepath } x \ y)) = \text{path-image } (\text{linepath } ((\lambda v. v - a) \ x) \ ((\lambda v. v - a) \ y))$   
*<proof>*

**lemma** *make-polygonal-path-translate*:  
**assumes**  $\text{length } vts \geq 1$   
**shows**  $(\lambda v. v - a) \circ (\text{make-polygonal-path } vts) = \text{make-polygonal-path } (\text{map } (\lambda v. v - a) \ vts)$   
*<proof>*

**lemma** *pocket-fill-line-int*:  
**assumes** *polygon-of*  $r \ vts$   
**defines**  $H \equiv \text{convex hull } (\text{set } vts)$   
**assumes**  $2 \leq i \wedge i < \text{length } vts - 1$   
**defines**  $a \equiv \text{hd } vts$   
**defines**  $b \equiv vts!i$   
**assumes**  $\{a, b\} \subseteq \text{frontier } H$   
**assumes**  $\forall j \in \{0 <..<i\}. vts!j \notin \text{frontier } H$   
**shows**  $\text{path-image } (\text{linepath } a \ b) \cap \text{path-image } r = \{a, b\}$   
 $\text{path-image } (\text{linepath } a \ b) \subseteq \text{frontier } H$   
*<proof>*

**lemma** *path-connected-simple-path-endless*:  
**assumes** *simple-path*  $p$   
**shows**  $\text{path-connected } (\text{path-image } p - \{\text{pathstart } p, \text{pathfinish } p\})$  (**is** *path-connected*  $?S$ )  
*<proof>*

**lemma** *simple-loop-split*:  
**assumes** *simple-path*  $p \wedge \text{closed-path } p$   
**assumes** *simple-path*  $q$   
**assumes**  $\text{path-image } q \cap \text{path-image } p = \{q \ 0, q \ 1\}$   
**assumes**  $\text{path-image } q \cap \text{path-inside } p \neq \{\}$   
**shows**  $q\{0 <..<1\} \subseteq \text{path-inside } p$   
*<proof>*

**lemma** *pocket-path-interior-arc*:  
**assumes** *simple-path*  $p \wedge \text{simple-path } q$   
**assumes** *arc*  $p \wedge \text{arc } q$   
**assumes**  $q \ 0 = p \ 1 \wedge q \ 1 = p \ 0$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{p \ 0, q \ 0\}$

**defines**  $A \equiv \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**defines**  $l \equiv \text{linepath } (p\ 0) (p\ 1)$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } A$   
**assumes**  $\text{path-image } l \subseteq \text{frontier } A$   
**assumes**  $\text{path-image } q \cap \text{path-image } l = \{l\ 0, q\ 0\}$   
**shows**  $p\{0 < .. < 1\} \cap \text{path-inside } (l\ +++\ q) \neq \{\}$   
 $\text{simple-path } (l\ +++\ q) \wedge \text{closed-path } (l\ +++\ q)$   
 $\text{path-image } p \cap \text{path-image } (l\ +++\ q) = \{p\ 0, p\ 1\}$   
*<proof>*

**lemma** *pocket-path-interior*:

**assumes**  $\text{simple-path } p \wedge \text{simple-path } q$   
**assumes**  $\text{arc } p \wedge \text{arc } q$   
**assumes**  $q\ 0 = p\ 1 \wedge q\ 1 = p\ 0$   
**assumes**  $\text{path-image } p \cap \text{path-image } q = \{p\ 0, q\ 0\}$   
**defines**  $A \equiv \text{convex hull } (\text{path-image } p \cup \text{path-image } q)$   
**defines**  $l \equiv \text{linepath } (p\ 0) (p\ 1)$   
**assumes**  $p\{0 < .. < 1\} \subseteq \text{interior } A$   
**assumes**  $\text{path-image } l \subseteq \text{frontier } A$   
**assumes**  $\text{path-image } q \cap \text{path-image } l = \{l\ 0, q\ 0\}$   
**shows**  $p\{0 < .. < 1\} \subseteq \text{path-inside } (l\ +++\ q)$   
*<proof>*

**lemma** *pocket-path-good*:

**assumes**  $\text{polygon } (\text{make-polygonal-path } vts)$   
**assumes**  $vts!0 \in \text{frontier } (\text{convex hull } (\text{set } vts))$   
**assumes**  $vts!1 \notin \text{frontier } (\text{convex hull } (\text{set } vts))$   
**assumes**  $\neg \text{convex } (\text{path-image } (\text{make-polygonal-path } vts) \cup \text{path-inside } (\text{make-polygonal-path } vts))$   
**defines**  $\text{pocket-path-vts} \equiv \text{construct-pocket-0 } vts (\text{set } vts \cap \text{frontier } (\text{convex hull } (\text{set } vts)))$   
**defines**  $\text{pocket} \equiv \text{make-polygonal-path } (\text{pocket-path-vts} @ [\text{pocket-path-vts!0}])$   
**defines**  $\text{filled-vts} \equiv \text{fill-pocket-0 } vts (\text{length } \text{pocket-path-vts})$   
**defines**  $\text{filled-p} \equiv \text{make-polygonal-path } \text{filled-vts}$   
**defines**  $a \equiv \text{hd } \text{pocket-path-vts}$   
**defines**  $b \equiv \text{last } \text{pocket-path-vts}$   
**defines**  $\text{good-pocket-path-vts} \equiv \text{tl } (\text{butlast } \text{pocket-path-vts})$   
**shows**  $\text{polygon } \text{filled-p}$   
 $\text{is-polygon-split-path } (\text{butlast } \text{filled-vts})\ 0\ 1\ \text{good-pocket-path-vts}$   
 $\text{polygon } \text{pocket}$   
 $\text{card } (\text{set } \text{pocket-path-vts}) < \text{card } (\text{set } vts)$   
 $\text{card } (\text{set } \text{filled-vts}) < \text{card } (\text{set } vts)$   
*<proof>*

### 29.3 Arbitrary Polygon Case

**lemma** *pick-rotate*:

**assumes**  $\text{polygon-of } p\ vts$   
**assumes**  $\text{all-integral } vts$

**obtains**  $p' \text{ vts}'$  **where** *polygon-of*  $p' \text{ vts}'$   
 $\wedge \text{vts}'!0 \in \text{frontier} (\text{convex hull} (\text{set vts}'))$   
 $\wedge \text{path-image } p' = \text{path-image } p$   
 $\wedge \text{all-integral vts}'$   
 $\wedge \text{set vts}' = \text{set vts}$   
 <proof>

**lemma** *pick-unrotated*:  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes** *polygon*: *polygon*  $p$   
**assumes** *polygonal-path*:  $p = \text{make-polygonal-path vts}$   
**assumes** *int-vertices*: *all-integral vts*  
**assumes** *I-is*:  $I = \text{card} \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$   
**assumes** *B-is*:  $B = \text{card} \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$   
**assumes**  $\text{vts}'!0 \in \text{frontier} (\text{convex hull} (\text{set vts}'))$   
**shows** *measure lebesgue* (*path-inside*  $p$ ) =  $I + B/2 - 1$   
 <proof>

**theorem** *pick*:  
**fixes**  $p :: R\text{-to-}R^2$   
**assumes** *polygon*  $p$   
**assumes**  $p = \text{make-polygonal-path vts}$   
**assumes** *all-integral vts*  
**assumes**  $I = \text{card} \{x. \text{integral-vec } x \wedge x \in \text{path-inside } p\}$   
**assumes**  $B = \text{card} \{x. \text{integral-vec } x \wedge x \in \text{path-image } p\}$   
**shows** *measure lebesgue* (*path-inside*  $p$ ) =  $I + B/2 - 1$   
 <proof>

end

## References

- [1] B. Grünbaum and G. C. Shephard. Pick's theorem. *The American Mathematical Monthly*, 100(2):150–161, 1993.
- [2] J. Harrison. A formal proof of Pick's theorem. *Math. Struct. Comput. Sci.*, 21(4):715–729, 2011.