The Perfect Number Theorem

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Abstract

This document presents the formal proof of the Perfect Number Theorem. The result can also be found as number 70 on the list of "top 100 mathematical theorems" [Wie]. This document was produced as result of a B.Sc. Thesis under supervision of Jaap Top and Wim H. Hesselink (University of Groningen) in 2009.

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1 Basics needed

theory *PerfectBasics*
imports *Main HOL−Computational-Algebra.Primes HOL−Algebra.Exponent*
begin

lemma *exp-is-max-div*:
  assumes *m0*: $m \neq 0$ **and** *p*: *prime p*
  shows ~ *p dvd* (*m div* (*p*^(*multiplicity p m*)))
⟨*proof*⟩

lemma *coprime-multiplicity*:
 assumes *prime* (*p*::*nat*) **and** $m > 0$
 shows *coprime p* (*m div* (*p* ^ *multiplicity p m*))
⟨*proof*⟩

theorem *simplify-sum-of-powers*: $(x - 1::nat) * (\sum i{=}0 \,..\, n \,.\, x\hat{}i) = x\hat{}(n + 1) - 1$ (**is** *?l* = *?r*)
⟨*proof*⟩

end

2 Sum of divisors function

theory *Sigma*
imports *PerfectBasics HOL−Library.Infinite-Set*
begin

definition *divisors* :: *nat* ⇒ *nat set* **where**
  divisors m ≡ {*n* . *n dvd m*}

abbreviation *sigma* :: *nat* ⇒ *nat* **where**
  sigma m ≡ ∑ (*divisors*(*m*))

lemma *divisors-eq-dvd*[*iff*]: (*a* ∈ *divisors*(*n*)) ⟷ (*a dvd n*)
⟨*proof*⟩

lemma *finite-divisors* [*simp*]:
 assumes *n*>*0* **shows** *finite* (*divisors n*)
⟨*proof*⟩

lemma *divs-of-zero-UNIV*[*simp*]: *divisors*(*0*) = *UNIV*
⟨*proof*⟩

lemma *sigma0*[*simp*]: *sigma*(*0*) = *0*
⟨*proof*⟩

lemma *sigma1*[*simp*]: *sigma*(*Suc 0*) = *1*
⟨*proof*⟩

lemma *prime-divisors*: *prime p* ⟷ *divisors p* = {*1*,*p*} ∧ *p*>*1*
⟨*proof*⟩

lemma *prime-imp-sigma*: *prime* (*p*::*nat*) ⟹ *sigma*(*p*) = *p*+*1*
⟨*proof*⟩

lemma *sigma-third-divisor*:
 assumes *1* < *a a* < *n a dvd n*
 shows *1*+*a*+*n* ≤ *sigma*(*n*)
⟨*proof*⟩

proposition *prime-iff-sigma*: *prime n* ⟷ *sigma*(*n*) = *Suc n*
⟨*proof*⟩

lemma *dvd-prime-power-iff*:
 fixes *p*::*nat*
 assumes *prime*: *prime p*
 shows {*d*. *d dvd p*^*n*} = (λ*k*. *p*^*k*) ‘ {*0*..*n*}
 ⟨*proof*⟩

lemma *rewrite-sum-of-powers*:

assumes *p*: (*p*::*nat*)>*1*
shows $\sum$ ((^) *p* ' {*0*..*n*}) = ($\sum$ *i* = *0* .. *n* . *p*^*i*) (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *sum-of-powers-int*: (*x* − *1*::*int*) * ($\sum$ *i*=*0*..*n* . *x*^*i*) = *x* ^ *Suc n* − *1*
⟨*proof*⟩

lemma *sum-of-powers-nat*: (*x* − *1*::*nat*) * ($\sum$ *i*=*0*..*n* . *x*^*i*) = *x* ^ *Suc n* − *1*
(**is** *?l* = *?r*)
⟨*proof*⟩

theorem *sigma-primepower*:
assumes *prime p*
shows (*p* − *1*) * *sigma*(*p*^*e*) = *p*^(*e*+*1*) − *1*
⟨*proof*⟩

proposition *sigma-prime-power-two*: *sigma*(*2*^*n*) = *2*^(*n*+*1*) − *1*
⟨*proof*⟩

lemma *prodsums-eq-sumprods*:
fixes *p* :: *nat* **and** *m* :: *nat*
assumes *coprime p m*
shows $\sum$ ((λ*k*. *p*^*k*) ' {*0*..*n*}) * *sigma m* = $\sum$ {*p*^*k* * *b* |*k b*. *k* ≤ *n* ∧ *b dvd m*}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *div-decomp-comp*:
fixes *a*::*nat*
shows *coprime m n* ⟹ *a dvd m*∗*n* ⟷ (∃ *b c*. *a* = *b* * *c* ∧ *b dvd m* ∧ *c dvd n*)
⟨*proof*⟩

theorem *sigma-semimultiplicative*:
assumes *p*: *prime p* **and** *cop*: *coprime p m*
shows *sigma* (*p*^*n*) * *sigma m* = *sigma* (*p*^*n* * *m*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

3 Perfect Number Theorem

theory *Perfect*
imports *Sigma*
begin

definition *perfect* :: *nat* => *bool* **where**
perfect m ≡ *m*>*0* ∧ *2*∗*m* = *sigma m*

theorem *perfect-number-theorem*:

assumes *even*: *even m* **and** *perfect*: *perfect m*
shows $\exists\ n\ .\ m = 2\hat{}n*(2\hat{}(n+1) - 1) \wedge prime\ ((2::nat)\hat{}(n+1) - 1)$
⟨*proof*⟩

theorem *Euclid-book9-prop36*:
assumes *p*: *prime* $(2\hat{}(n+1) - (1::nat))$
shows *perfect* $(2\ \hat{}\ n * (2\ \hat{}\ (n + 1) - 1))$
⟨*proof*⟩

end

References

[Wie] Freek Wiedijk. Formalizing 100 theorems. http://www.cs.ru.nl/~freek/100/.