

Partial Order Reduction

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Abstract

This entry provides a formalization of the abstract theory of ample set partial order reduction as presented in [2, 1]. The formalization includes transition systems with actions, trace theory, as well as basics on finite, infinite, and lazy sequences. We also provide a basic framework for static analysis on concurrent systems with respect to the ample set condition.

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1 List Prefixes

```

theory List-Prefixes
imports HOL-Library.Prefix-Order
begin

lemmas [intro] = prefixI strict-prefixI[folded less-eq-list-def]
lemmas [elim] = prefixE strict-prefixE[folded less-eq-list-def]

lemmas [intro?] = take-is-prefix[folded less-eq-list-def]

hide-const (open) Sublist.prefix Sublist.suffix

lemma prefix-finI-item[intro!]:
  assumes  $a = b$   $u \leq v$ 
  shows  $a \# u \leq b \# v$ 
  <proof>
lemma prefix-finE-item[elim!]:
  assumes  $a \# u \leq b \# v$ 
  obtains  $a = b$   $u \leq v$ 
  <proof>

lemma prefix-fin-append[intro]:  $u \leq u @ v$  <proof>
lemma pprefix-fin-length[dest]:
  assumes  $u < v$ 
  shows  $length\ u < length\ v$ 

```

<proof>

end

2 Lists

theory *List-Extensions*
imports *HOL-Library.Sublist*
begin

declare *remove1-idem*[*simp*]

lemma *nth-append-simps*[*simp*]:

$i < \text{length } xs \implies (xs @ ys) ! i = xs ! i$

$i \geq \text{length } xs \implies (xs @ ys) ! i = ys ! (i - \text{length } xs)$

<proof>

notation *zip* (**infixr** || 51)

abbreviation *project* $A \equiv \text{filter } (\lambda a. a \in A)$

abbreviation *select* $s w \equiv \text{nths } w s$

lemma *map-plus*[*simp*]: $\text{map } (\text{plus } n) [i ..< j] = [i + n ..< j + n]$

<proof>

lemma *singleton-list-lengthE*[*elim*]:

assumes $\text{length } xs = 1$

obtains x

where $xs = [x]$

<proof>

lemma *singleton-hd-last*: $\text{length } xs = 1 \implies \text{hd } xs = \text{last } xs$ *<proof>*

lemma *set-subsetI*[*intro*]:

assumes $\bigwedge i. i < \text{length } xs \implies xs ! i \in S$

shows $\text{set } xs \subseteq S$

<proof>

lemma *hd-take*[*simp*]:

assumes $n \neq 0 \text{ } xs \neq []$

shows $\text{hd } (\text{take } n \text{ } xs) = \text{hd } xs$

<proof>

lemma *hd-drop*[*simp*]:

assumes $n < \text{length } xs$

shows $\text{hd } (\text{drop } n \text{ } xs) = xs ! n$

<proof>

lemma *last-take*[*simp*]:

assumes $n < \text{length } xs$

shows $\text{last } (\text{take } (\text{Suc } n) \text{ } xs) = xs ! n$

<proof>

lemma *split-list-first-unique*:

assumes $u_1 @ [a] @ u_2 = v_1 @ [a] @ v_2$ $a \notin \text{set } u_1$ $a \notin \text{set } v_1$

shows $u_1 = v_1$

<proof>

end

3 Finite Prefixes of Infinite Sequences

theory *Word-Prefixes*

imports

List-Prefixes

../Extensions/List-Extensions

Transition-Systems-and-Automata.Sequence

begin

definition *prefix-fininf* :: *'a list* \Rightarrow *'a stream* \Rightarrow *bool* (**infix** \leq_{FI} 50)

where $u \leq_{FI} v \equiv \exists w. u @- w = v$

lemma *prefix-fininfI[intro]*:

assumes $u @- w = v$

shows $u \leq_{FI} v$

<proof>

lemma *prefix-fininfE[elim]*:

assumes $u \leq_{FI} v$

obtains w

where $v = u @- w$

<proof>

lemma *prefix-fininfI-empty[intro!]*: $[] \leq_{FI} w$ *<proof>*

lemma *prefix-fininfI-item[intro!]*:

assumes $a = b$ $u \leq_{FI} v$

shows $a \# u \leq_{FI} b \#\# v$

<proof>

lemma *prefix-fininfE-item[elim!]*:

assumes $a \# u \leq_{FI} b \#\# v$

obtains $a = b$ $u \leq_{FI} v$

<proof>

lemma *prefix-fininf-item[simp]*: $a \# u \leq_{FI} a \#\# v \longleftrightarrow u \leq_{FI} v$ *<proof>*

lemma *prefix-fininf-list[simp]*: $w @ u \leq_{FI} w @- v \longleftrightarrow u \leq_{FI} v$ *<proof>*

lemma *prefix-fininf-conc[intro]*: $u \leq_{FI} u @- v$ *<proof>*

lemma *prefix-fininf-prefix[intro]*: *stake* k $w \leq_{FI} w$ *<proof>*

lemma *prefix-fininf-set-range[dest]*: $u \leq_{FI} v \Longrightarrow \text{set } u \subseteq \text{sset } v$ *<proof>*

lemma *prefix-fininf-absorb*:

assumes $u \leq_{FI} v @- w$ $\text{length } u \leq \text{length } v$

shows $u \leq v$
 ⟨proof⟩

lemma *prefix-fininf-extend*:
assumes $u \leq_{FI} v @- w$ $length\ v \leq length\ u$
shows $v \leq u$
 ⟨proof⟩

lemma *prefix-fininf-length*:
assumes $u \leq_{FI} w$ $v \leq_{FI} w$ $length\ u \leq length\ v$
shows $u \leq v$
 ⟨proof⟩

lemma *prefix-fininf-append*:
assumes $u \leq_{FI} v @- w$
obtains (*absorb*) $u \leq v$ | (*extend*) z **where** $u = v @ z z \leq_{FI} w$
 ⟨proof⟩

lemma *prefix-fin-prefix-fininf-trans*[*trans, intro*]: $u \leq v \implies v \leq_{FI} w \implies u \leq_{FI} w$
 ⟨proof⟩

lemma *prefix-finE-nth*:
assumes $u \leq v$ $i < length\ u$
shows $u ! i = v ! i$
 ⟨proof⟩

lemma *prefix-fininfI-nth*:
assumes $\bigwedge i. i < length\ u \implies u ! i = w !! i$
shows $u \leq_{FI} w$
 ⟨proof⟩

definition *chain* :: $(nat \Rightarrow 'a\ list) \Rightarrow bool$
where $chain\ w \equiv mono\ w \wedge (\forall k. \exists l. k < length\ (w\ l))$

definition *limit* :: $(nat \Rightarrow 'a\ list) \Rightarrow 'a\ stream$
where $limit\ w \equiv smap\ (\lambda k. w\ (SOME\ l. k < length\ (w\ l))\ !\ k)\ nats$

lemma *chainI*[*intro?*]:
assumes $mono\ w$
assumes $\bigwedge k. \exists l. k < length\ (w\ l)$
shows $chain\ w$
 ⟨proof⟩

lemma *chainD-mono*[*dest?*]:
assumes $chain\ w$
shows $mono\ w$
 ⟨proof⟩

lemma *chainE-length*[*elim?*]:
assumes $chain\ w$
obtains l
where $k < length\ (w\ l)$
 ⟨proof⟩

lemma *chain-prefix-limit*:

assumes *chain w*

shows $w k \leq_{FI} \text{limit } w$

<proof>

lemma *chain-construct-1*:

assumes $P 0 x_0 \wedge k x. P k x \implies \exists x'. P (\text{Suc } k) x' \wedge f x \leq f x'$

assumes $\wedge k x. P k x \implies k \leq \text{length } (f x)$

obtains *Q*

where $\wedge k. P k (Q k) \text{ chain } (f \circ Q)$

<proof>

lemma *chain-construct-2*:

assumes $P 0 x_0 \wedge k x. P k x \implies \exists x'. P (\text{Suc } k) x' \wedge f x \leq f x' \wedge g x \leq g x'$

assumes $\wedge k x. P k x \implies k \leq \text{length } (f x) \wedge k x. P k x \implies k \leq \text{length } (g x)$

obtains *Q*

where $\wedge k. P k (Q k) \text{ chain } (f \circ Q) \text{ chain } (g \circ Q)$

<proof>

lemma *chain-construct-2'*:

assumes $P 0 u_0 v_0 \wedge k u v. P k u v \implies \exists u' v'. P (\text{Suc } k) u' v' \wedge u \leq u' \wedge v \leq v'$

assumes $\wedge k u v. P k u v \implies k \leq \text{length } u \wedge k u v. P k u v \implies k \leq \text{length } v$

obtains *u v*

where $\wedge k. P k (u k) (v k) \text{ chain } u \text{ chain } v$

<proof>

end

4 Sets

theory *Set-Extensions*

imports

HOL-Library.Infinite-Set

begin

declare *finite-subset[intro]*

lemma *set-not-emptyI[intro 0]*: $x \in S \implies S \neq \{\}$ *<proof>*

lemma *sets-empty-iffI[intro 0]*:

assumes $\wedge a. a \in A \implies \exists b. b \in B$

assumes $\wedge b. b \in B \implies \exists a. a \in A$

shows $A = \{\} \longleftrightarrow B = \{\}$

<proof>

lemma *disjointI[intro 0]*:

assumes $\wedge x. x \in A \implies x \in B \implies \text{False}$

shows $A \cap B = \{\}$

<proof>

lemma *range-subsetI[intro 0]*:

assumes $\wedge x. f x \in S$

shows $\text{range } f \subseteq S$

$\langle \text{proof} \rangle$

lemma *finite-imageI-range*:

assumes *finite* (*range* *f*)

shows *finite* (*f* ' *A*)

$\langle \text{proof} \rangle$

lemma *inf-img-fin-domE'*:

assumes *infinite* *A*

assumes *finite* (*f* ' *A*)

obtains *y*

where $y \in f \text{ ' } A$ *infinite* ($A \cap f \text{ ' } \{y\}$)

$\langle \text{proof} \rangle$

lemma *vimage-singleton[simp]*: $f \text{ ' } \{y\} = \{x. f \ x = y\}$ $\langle \text{proof} \rangle$

lemma *these-alt-def*: *Option.these* *S* = *Some* ' *S* $\langle \text{proof} \rangle$

lemma *the-vimage-subset*: *the* ' $\{a\} \subseteq \{None, Some \ a\}$ $\langle \text{proof} \rangle$

lemma *finite-induct-reverse[consumes 1, case-names remove]*:

assumes *finite* *S*

assumes $\bigwedge x. x \in S \implies P \ (S - \{x\}) \implies P \ S$

shows *P* *S*

$\langle \text{proof} \rangle$

lemma *zero-not-in-Suc-image[simp]*: $0 \notin \text{Suc ' } A$ $\langle \text{proof} \rangle$

lemma *Collect-split-Suc*:

$\neg P \ 0 \implies \{i. P \ i\} = \text{Suc ' } \{i. P \ (\text{Suc } i)\}$

$P \ 0 \implies \{i. P \ i\} = \{0\} \cup \text{Suc ' } \{i. P \ (\text{Suc } i)\}$

$\langle \text{proof} \rangle$

lemma *Collect-subsume[simp]*:

assumes $\bigwedge x. x \in A \implies P \ x$

shows $\{x \in A. P \ x\} = A$

$\langle \text{proof} \rangle$

lemma *Max-ge'*:

assumes *finite* *A* $A \neq \{\}$

assumes $b \in A$ $a \leq b$

shows $a \leq \text{Max } A$

$\langle \text{proof} \rangle$

abbreviation *least* *A* $\equiv \text{LEAST } k. k \in A$

lemma *least-contains[intro?, simp]*:

fixes *A* :: 'a :: *wellorder set*

assumes $k \in A$

shows *least* *A* $\in A$

$\langle proof \rangle$
lemma *least-contains'*[*intro?*, *simp*]:
fixes $A :: 'a :: wellorder\ set$
assumes $A \neq \{\}$
shows $least\ A \in A$
 $\langle proof \rangle$
lemma *least-least*[*intro?*, *simp*]:
fixes $A :: 'a :: wellorder\ set$
assumes $k \in A$
shows $least\ A \leq k$
 $\langle proof \rangle$
lemma *least-unique*:
fixes $A :: 'a :: wellorder\ set$
assumes $k \in A\ k \leq least\ A$
shows $k = least\ A$
 $\langle proof \rangle$
lemma *least-not-less*:
fixes $A :: 'a :: wellorder\ set$
assumes $k < least\ A$
shows $k \notin A$
 $\langle proof \rangle$
lemma *leastI2-order*[*simp*]:
fixes $A :: 'a :: wellorder\ set$
assumes $A \neq \{\} \wedge k. k \in A \implies (\wedge l. l \in A \implies k \leq l) \implies P\ k$
shows $P\ (least\ A)$
 $\langle proof \rangle$

lemma *least-singleton*[*simp*]:
fixes $a :: 'a :: wellorder$
shows $least\ \{a\} = a$
 $\langle proof \rangle$

lemma *least-image*[*simp*]:
fixes $f :: 'a :: wellorder \implies 'b :: wellorder$
assumes $A \neq \{\} \wedge k\ l. k \in A \implies l \in A \implies k \leq l \implies f\ k \leq f\ l$
shows $least\ (f\ ` A) = f\ (least\ A)$
 $\langle proof \rangle$

lemma *least-le*:
fixes $A\ B :: 'a :: wellorder\ set$
assumes $B \neq \{\}$
assumes $\wedge i. i \leq least\ A \implies i \leq least\ B \implies i \in B \implies i \in A$
shows $least\ A \leq least\ B$
 $\langle proof \rangle$
lemma *least-eq*:
fixes $A\ B :: 'a :: wellorder\ set$
assumes $A \neq \{\}\ B \neq \{\}$
assumes $\wedge i. i \leq least\ A \implies i \leq least\ B \implies i \in A \longleftrightarrow i \in B$
shows $least\ A = least\ B$

⟨proof⟩

lemma *least-Suc[simp]*:
 assumes $A \neq \{\}$
 shows $\text{least } (\text{Suc } 'A) = \text{Suc } (\text{least } A)$
⟨proof⟩

lemma *least-Suc-diff[simp]*: $\text{Suc } 'A - \{\text{least } (\text{Suc } 'A)\} = \text{Suc } '(A - \{\text{least } A\})$
⟨proof⟩

lemma *Max-diff-least[simp]*:
 fixes $A :: 'a :: \text{wellorder set}$
 assumes $\text{finite } A \ A - \{\text{least } A\} \neq \{\}$
 shows $\text{Max } (A - \{\text{least } A\}) = \text{Max } A$
⟨proof⟩

lemma *nat-set-card-equality-less*:
 fixes $A :: \text{nat set}$
 assumes $x \in A \ y \in A \ \text{card } \{z \in A. z < x\} = \text{card } \{z \in A. z < y\}$
 shows $x = y$
⟨proof⟩

lemma *nat-set-card-equality-le*:
 fixes $A :: \text{nat set}$
 assumes $x \in A \ y \in A \ \text{card } \{z \in A. z \leq x\} = \text{card } \{z \in A. z \leq y\}$
 shows $x = y$
⟨proof⟩

lemma *nat-set-card-mono[simp]*:
 fixes $A :: \text{nat set}$
 assumes $x \in A$
 shows $\text{card } \{z \in A. z < x\} < \text{card } \{z \in A. z < y\} \longleftrightarrow x < y$
⟨proof⟩

lemma *card-one[elim]*:
 assumes $\text{card } A = 1$
 obtains a
 where $A = \{a\}$
⟨proof⟩

lemma *image-alt-def*: $f ' A = \{f x \mid x. x \in A\}$ ⟨proof⟩

lemma *supset-mono-inductive[mono]*:
 assumes $\bigwedge x. x \in B \longrightarrow x \in C$
 shows $A \subseteq B \longrightarrow A \subseteq C$
⟨proof⟩

lemma *Collect-mono-inductive[mono]*:
 assumes $\bigwedge x. P x \longrightarrow Q x$
 shows $x \in \{x. P x\} \longrightarrow x \in \{x. Q x\}$

<proof>

lemma *image-union-split*:

assumes $f \text{ ' } (A \cup B) = g \text{ ' } C$

obtains $D E$

where $f \text{ ' } A = g \text{ ' } D$ $f \text{ ' } B = g \text{ ' } E$ $D \subseteq C$ $E \subseteq C$

<proof>

lemma *image-insert-split*:

assumes $\text{inj } g$ $f \text{ ' } \text{insert } a B = g \text{ ' } C$

obtains $d E$

where $f a = g d$ $f \text{ ' } B = g \text{ ' } E$ $d \in C$ $E \subseteq C$

<proof>

end

5 Basics

theory *Basic-Extensions*

imports *HOL-Library.Infinite-Set*

begin

5.1 Types

type-synonym $\text{'a step} = \text{'a} \Rightarrow \text{'a}$

5.2 Rules

declare *less-imp-le*[*dest, simp*]

declare *le-funI*[*intro*]

declare *le-funE*[*elim*]

declare *le-funD*[*dest*]

lemma *IdI'*[*intro*]:

assumes $x = y$

shows $(x, y) \in \text{Id}$

<proof>

lemma (**in** *order*) *order-le-cases*:

assumes $x \leq y$

obtains $(\text{eq}) x = y \mid (\text{lt}) x < y$

<proof>

lemma (**in** *linorder*) *linorder-cases'*:

obtains $(\text{le}) x \leq y \mid (\text{gt}) x > y$

<proof>

lemma *monoI-comp*[*intro*]:

assumes $\text{mono } f$ $\text{mono } g$

shows *mono* ($f \circ g$)
 ⟨*proof*⟩
lemma *strict-monoI-comp*[*intro*]:
assumes *strict-mono* f *strict-mono* g
shows *strict-mono* ($f \circ g$)
 ⟨*proof*⟩

lemma *eq-le-absorb*[*simp*]:
fixes $x\ y :: 'a :: \text{order}$
shows $x = y \wedge x \leq y \longleftrightarrow x = y\ x \leq y \wedge x = y \longleftrightarrow x = y$
 ⟨*proof*⟩

lemma *INFM-Suc*[*simp*]: $(\exists_{\infty} i. P (Suc\ i)) \longleftrightarrow (\exists_{\infty} i. P\ i)$
 ⟨*proof*⟩
lemma *INFM-plus*[*simp*]: $(\exists_{\infty} i. P (i + n :: \text{nat})) \longleftrightarrow (\exists_{\infty} i. P\ i)$
 ⟨*proof*⟩
lemma *INFM-minus*[*simp*]: $(\exists_{\infty} i. P (i - n :: \text{nat})) \longleftrightarrow (\exists_{\infty} i. P\ i)$
 ⟨*proof*⟩

5.3 Constants

definition *const* :: $'a \Rightarrow 'b \Rightarrow 'a$
where *const* $x \equiv \lambda -. x$
definition *const2* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'a$
where *const2* $x \equiv \lambda - -. x$
definition *const3* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'a$
where *const3* $x \equiv \lambda - - -. x$
definition *const4* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e \Rightarrow 'a$
where *const4* $x \equiv \lambda - - - -. x$
definition *const5* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e \Rightarrow 'f \Rightarrow 'a$
where *const5* $x \equiv \lambda - - - - -. x$

lemma *const-apply*[*simp*]: *const* $x\ y = x$ ⟨*proof*⟩
lemma *const2-apply*[*simp*]: *const2* $x\ y\ z = x$ ⟨*proof*⟩
lemma *const3-apply*[*simp*]: *const3* $x\ y\ z\ u = x$ ⟨*proof*⟩
lemma *const4-apply*[*simp*]: *const4* $x\ y\ z\ u\ v = x$ ⟨*proof*⟩
lemma *const5-apply*[*simp*]: *const5* $x\ y\ z\ u\ v\ w = x$ ⟨*proof*⟩

definition *zip-fun* :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'b \times 'c$ (**infixr** || 51)
where $f\ ||\ g \equiv \lambda x. (f\ x, g\ x)$

lemma *zip-fun-simps*[*simp*]:
 $(f\ ||\ g)\ x = (f\ x, g\ x)$
 $fst \circ (f\ ||\ g) = f$
 $snd \circ (f\ ||\ g) = g$
 $fst \circ h\ ||\ snd \circ h = h$
 $fst\ 'range\ (f\ ||\ g) = range\ f$
 $snd\ 'range\ (f\ ||\ g) = range\ g$
 ⟨*proof*⟩

lemma *zip-fun-eq*[*dest*]:
assumes $f \parallel g = h \parallel i$
shows $f = h \ g = i$
 $\langle proof \rangle$

lemma *zip-fun-range-subset*[*intro, simp*]: $range (f \parallel g) \subseteq range f \times range g$
 $\langle proof \rangle$

lemma *zip-fun-range-finite*[*elim*]:
assumes *finite* ($range (f \parallel g)$)
obtains *finite* ($range f$) *finite* ($range g$)
 $\langle proof \rangle$

lemma *zip-fun-split*:
obtains $f \ g$
where $h = f \parallel g$
 $\langle proof \rangle$

abbreviation *None-None* $\equiv (None, None)$
abbreviation *None-Some* $\equiv \lambda (y). (None, Some\ y)$
abbreviation *Some-None* $\equiv \lambda (x). (Some\ x, None)$
abbreviation *Some-Some* $\equiv \lambda (x, y). (Some\ x, Some\ y)$

abbreviation *None-None-None* $\equiv (None, None, None)$
abbreviation *None-None-Some* $\equiv \lambda (z). (None, None, Some\ z)$
abbreviation *None-Some-None* $\equiv \lambda (y). (None, Some\ y, None)$
abbreviation *None-Some-Some* $\equiv \lambda (y, z). (None, Some\ y, Some\ z)$
abbreviation *Some-None-None* $\equiv \lambda (x). (Some\ x, None, None)$
abbreviation *Some-None-Some* $\equiv \lambda (x, z). (Some\ x, None, Some\ z)$
abbreviation *Some-Some-None* $\equiv \lambda (x, y). (Some\ x, Some\ y, None)$
abbreviation *Some-Some-Some* $\equiv \lambda (x, y, z). (Some\ x, Some\ y, Some\ z)$

lemma *inj-Some2*[*simp, intro*]:
inj None-Some
inj Some-None
inj Some-Some
 $\langle proof \rangle$

lemma *inj-Some3*[*simp, intro*]:
inj None-None-Some
inj None-Some-None
inj None-Some-Some
inj Some-None-None
inj Some-None-Some
inj Some-Some-None
inj Some-Some-Some
 $\langle proof \rangle$

definition *swap* $:: 'a \times 'b \Rightarrow 'b \times 'a$

where $swap\ x \equiv (snd\ x, fst\ x)$

lemma $swap-simps[simp]$: $swap\ (a, b) = (b, a)$ $\langle proof \rangle$

lemma $swap-inj[intro, simp]$: $inj\ swap$ $\langle proof \rangle$

lemma $swap-surj[intro, simp]$: $surj\ swap$ $\langle proof \rangle$

lemma $swap-bij[intro, simp]$: $bij\ swap$ $\langle proof \rangle$

definition $push :: ('a \times 'b) \times 'c \Rightarrow 'a \times 'b \times 'c$

where $push\ x \equiv (fst\ (fst\ x), snd\ (fst\ x), snd\ x)$

definition $pull :: 'a \times 'b \times 'c \Rightarrow ('a \times 'b) \times 'c$

where $pull\ x \equiv ((fst\ x, fst\ (snd\ x)), snd\ (snd\ x))$

lemma $push-simps[simp]$: $push\ ((x, y), z) = (x, y, z)$ $\langle proof \rangle$

lemma $pull-simps[simp]$: $pull\ (x, y, z) = ((x, y), z)$ $\langle proof \rangle$

definition $label :: 'vertex \times 'label \times 'vertex \Rightarrow 'label$

where $label \equiv fst \circ snd$

lemma $label-select[simp]$: $label\ (p, a, q) = a$ $\langle proof \rangle$

5.4 Theorems for @termcurry and @termsplit

lemma $curry-split[simp]$: $curry \circ case-prod = id$ $\langle proof \rangle$

lemma $split-curry[simp]$: $case-prod \circ curry = id$ $\langle proof \rangle$

lemma $curry-le[simp]$: $curry\ f \leq curry\ g \iff f \leq g$ $\langle proof \rangle$

lemma $split-le[simp]$: $case-prod\ f \leq case-prod\ g \iff f \leq g$ $\langle proof \rangle$

lemma $mono-curry-left[simp]$: $mono\ (curry \circ h) \iff mono\ h$
 $\langle proof \rangle$

lemma $mono-split-left[simp]$: $mono\ (case-prod \circ h) \iff mono\ h$
 $\langle proof \rangle$

lemma $mono-curry-right[simp]$: $mono\ (h \circ curry) \iff mono\ h$
 $\langle proof \rangle$

lemma $mono-split-right[simp]$: $mono\ (h \circ case-prod) \iff mono\ h$
 $\langle proof \rangle$

lemma $Collect-curry[simp]$: $\{x. P\ (curry\ x)\} = case-prod\ ' \{x. P\ x\}$ $\langle proof \rangle$

lemma $Collect-split[simp]$: $\{x. P\ (case-prod\ x)\} = curry\ ' \{x. P\ x\}$ $\langle proof \rangle$

lemma $gfp-split-curry[simp]$: $gfp\ (case-prod \circ f \circ curry) = case-prod\ (gfp\ f)$
 $\langle proof \rangle$

lemma $gfp-curry-split[simp]$: $gfp\ (curry \circ f \circ case-prod) = curry\ (gfp\ f)$
 $\langle proof \rangle$

lemma $not-someI$:

assumes $\bigwedge x. P\ x \implies False$

shows $\neg P\ (SOME\ x. P\ x)$

$\langle proof \rangle$

lemma *some-ccontr*:
assumes $(\bigwedge x. \neg P x) \implies False$
shows $P (SOME x. P x)$
 $\langle proof \rangle$

end

6 Relations

theory *Relation-Extensions*

imports

Basic-Extensions

begin

abbreviation *rev-lex-prod* (**infixr** $\langle *rlex* \rangle 80$)
where $r_1 \langle *rlex* \rangle r_2 \equiv inv-image (r_2 \langle *lex* \rangle r_1) swap$

lemmas *sym-rtranclp*[*intro*] = *sym-rtrancl*[*to-pred*]

definition *liftablep* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
where $liftablep\ r\ f \equiv \forall x\ y. r\ x\ y \longrightarrow r\ (f\ x)\ (f\ y)$

lemma *liftablepI*[*intro*]:
assumes $\bigwedge x\ y. r\ x\ y \implies r\ (f\ x)\ (f\ y)$
shows $liftablep\ r\ f$
 $\langle proof \rangle$

lemma *liftablepE*[*elim*]:
assumes $liftablep\ r\ f$
assumes $r\ x\ y$
obtains $r\ (f\ x)\ (f\ y)$
 $\langle proof \rangle$

lemma *liftablep-rtranclp*:
assumes $liftablep\ r\ f$
shows $liftablep\ r^{**}\ f$
 $\langle proof \rangle$

definition *confluentp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where $confluentp\ r \equiv \forall x\ y1\ y2. r^{**}\ x\ y1 \longrightarrow r^{**}\ x\ y2 \longrightarrow (\exists z. r^{**}\ y1\ z \wedge r^{**}\ y2\ z)$

lemma *confluentpI*[*intro*]:
assumes $\bigwedge x\ y1\ y2. r^{**}\ x\ y1 \implies r^{**}\ x\ y2 \implies \exists z. r^{**}\ y1\ z \wedge r^{**}\ y2\ z$
shows $confluentp\ r$
 $\langle proof \rangle$

lemma *confluentpE*[*elim*]:
assumes $confluentp\ r$
assumes $r^{**}\ x\ y1\ r^{**}\ x\ y2$

obtains z
where $r^{**} \ y1 \ z \ r^{**} \ y2 \ z$
 $\langle proof \rangle$

lemma *confluentpI*[*intro*]:
assumes $\bigwedge x \ y1 \ y2. r^{**} \ x \ y1 \ \Longrightarrow \ r \ x \ y2 \ \Longrightarrow \ \exists z. r^{**} \ y1 \ z \ \wedge \ r^{**} \ y2 \ z$
shows *confluentp* r
 $\langle proof \rangle$

lemma *transclp-eq-implies-confluent-imp*:
assumes $r1^{**} = r2^{**}$
assumes *confluentp* $r1$
shows *confluentp* $r2$
 $\langle proof \rangle$

lemma *transclp-eq-implies-confluent-eq*:
assumes $r1^{**} = r2^{**}$
shows *confluentp* $r1 \longleftrightarrow \text{confluentp } r2$
 $\langle proof \rangle$

definition *diamondp* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow \text{bool}$
where *diamondp* $r \equiv \forall x \ y1 \ y2. r \ x \ y1 \ \longrightarrow \ r \ x \ y2 \ \longrightarrow \ (\exists z. r \ y1 \ z \ \wedge \ r \ y2 \ z)$

lemma *diamondpI*[*intro*]:
assumes $\bigwedge x \ y1 \ y2. r \ x \ y1 \ \Longrightarrow \ r \ x \ y2 \ \Longrightarrow \ \exists z. r \ y1 \ z \ \wedge \ r \ y2 \ z$
shows *diamondp* r
 $\langle proof \rangle$

lemma *diamondpE*[*elim*]:
assumes *diamondp* r
assumes $r \ x \ y1 \ r \ x \ y2$
obtains z
where $r \ y1 \ z \ r \ y2 \ z$
 $\langle proof \rangle$

lemma *diamondp-implies-confluentp*:
assumes *diamondp* r
shows *confluentp* r
 $\langle proof \rangle$

locale *wellfounded-relation* =
fixes $R :: 'a \Rightarrow 'a \Rightarrow \text{bool}$
assumes *wellfounded*: *wfP* R

end

7 Transition Systems

theory *Transition-System-Extensions*

```

imports
  Basics/Word-Prefixes
  Extensions/Set-Extensions
  Extensions/Relation-Extensions
  Transition-Systems-and-Automata.Transition-System
  Transition-Systems-and-Automata.Transition-System-Extra
  Transition-Systems-and-Automata.Transition-System-Construction
begin

  context transition-system-initial
  begin

    definition cycles :: 'state  $\Rightarrow$  'transition list set
      where cycles p  $\equiv$  {w. path w p  $\wedge$  target w p = p}

    lemma cyclesI[intro!]:
      assumes path w p target w p = p
      shows w  $\in$  cycles p
       $\langle$ proof $\rangle$ 
    lemma cyclesE[elim!]:
      assumes w  $\in$  cycles p
      obtains path w p target w p = p
       $\langle$ proof $\rangle$ 

    inductive-set executable :: 'transition set
      where executable: p  $\in$  nodes  $\implies$  enabled a p  $\implies$  a  $\in$  executable

    lemma executableI-step[intro!]:
      assumes p  $\in$  nodes enabled a p
      shows a  $\in$  executable
       $\langle$ proof $\rangle$ 
    lemma executableI-words-fin[intro!]:
      assumes p  $\in$  nodes path w p
      shows set w  $\subseteq$  executable
       $\langle$ proof $\rangle$ 
    lemma executableE[elim?]:
      assumes a  $\in$  executable
      obtains p
      where p  $\in$  nodes enabled a p
       $\langle$ proof $\rangle$ 

  end

  locale transition-system-interpreted =
    transition-system ex en
    for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
    and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
    and int :: 'state  $\Rightarrow$  'interpretation
  begin

```

definition *visible* :: 'action set
where *visible* $\equiv \{a. \exists q. en\ a\ q \wedge int\ q \neq int\ (ex\ a\ q)\}$

lemma *visibleI*[*intro*]:
assumes *en a q int q \neq int (ex a q)*
shows *a \in visible*
 $\langle proof \rangle$

lemma *visibleE*[*elim*]:
assumes *a \in visible*
obtains *q*
where *en a q int q \neq int (ex a q)*
 $\langle proof \rangle$

abbreviation *invisible* $\equiv -\ visible$

lemma *execute-fin-word-invisible*:
assumes *path w p set w \subseteq invisible*
shows *int (target w p) = int p*
 $\langle proof \rangle$

lemma *execute-inf-word-invisible*:
assumes *run w p k \leq l \wedge i. k \leq i \implies i $<$ l \implies w !! i \notin visible*
shows *int ((p ## trace w p) !! k) = int ((p ## trace w p) !! l)*
 $\langle proof \rangle$

end

locale *transition-system-complete* =
transition-system-initial ex en init +
transition-system-interpreted ex en int
for *ex :: 'action \Rightarrow 'state \Rightarrow 'state*
and *en :: 'action \Rightarrow 'state \Rightarrow bool*
and *init :: 'state \Rightarrow bool*
and *int :: 'state \Rightarrow 'interpretation*
begin

definition *language* :: 'interpretation stream set
where *language* $\equiv \{smap\ int\ (p\ ##\ trace\ w\ p)\ |p\ w. init\ p \wedge run\ w\ p\}$

lemma *languageI*[*intro!*]:
assumes *w = smap int (p ## trace v p) init p run v p*
shows *w \in language*
 $\langle proof \rangle$

lemma *languageE*[*elim!*]:
assumes *w \in language*
obtains *p v*
where *w = smap int (p ## trace v p) init p run v p*
 $\langle proof \rangle$

```

end

locale transition-system-finite-nodes =
  transition-system-initial ex en init
  for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
  and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
  and init :: 'state  $\Rightarrow$  bool
  +
  assumes reachable-finite: finite nodes

locale transition-system-cut =
  transition-system-finite-nodes ex en init
  for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
  and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
  and init :: 'state  $\Rightarrow$  bool
  +
  fixes cuts :: 'action set
  assumes cycles-cut:  $p \in nodes \Rightarrow w \in cycles\ p \Rightarrow w \neq [] \Rightarrow set\ w \cap cuts$ 
 $\neq \{\}$ 
begin

  inductive scut :: 'state  $\Rightarrow$  'state  $\Rightarrow$  bool
    where scut:  $p \in nodes \Rightarrow en\ a\ p \Rightarrow a \notin cuts \Rightarrow scut\ p\ (ex\ a\ p)$ 

  declare scut.intros[intro!]
  declare scut.cases[elim!]

  lemma scut-reachable:
    assumes scut p q
    shows  $p \in nodes\ q \in nodes$ 
     $\langle proof \rangle$ 

  lemma scut-transcl:
    assumes  $scut^{++}\ p\ q$ 
    obtains w
    where  $path\ w\ p\ target\ w\ p = q\ set\ w \cap cuts = \{\}\ w \neq []$ 
     $\langle proof \rangle$ 

  sublocale wellfounded-relation  $scut^{-1-1}$ 
   $\langle proof \rangle$ 

  lemma no-cut-scut:
    assumes  $p \in nodes\ en\ a\ p\ a \notin cuts$ 
    shows  $scut^{-1-1}\ (ex\ a\ p)\ p$ 
     $\langle proof \rangle$ 

end

locale transition-system-sticky =
  transition-system-complete ex en init int +

```

```

transition-system-cut ex en init sticky
for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
and init :: 'state  $\Rightarrow$  bool
and int :: 'state  $\Rightarrow$  'interpretation
and sticky :: 'action set
+
assumes executable-visible-sticky: executable  $\cap$  visible  $\subseteq$  sticky

end

```

8 Trace Theory

```

theory Traces
imports Basics/Word-Prefixes
begin

locale traces =
  fixes ind :: 'item  $\Rightarrow$  'item  $\Rightarrow$  bool
  assumes independence-symmetric[sym]: ind a b  $\implies$  ind b a
begin

abbreviation Ind :: 'item set  $\Rightarrow$  'item set  $\Rightarrow$  bool
  where Ind A B  $\equiv$   $\forall$  a  $\in$  A.  $\forall$  b  $\in$  B. ind a b

inductive eq-swap :: 'item list  $\Rightarrow$  'item list  $\Rightarrow$  bool (infix =S 50)
  where swap: ind a b  $\implies$  u @ [a] @ [b] @ v =S u @ [b] @ [a] @ v

declare eq-swap.intros[intro]
declare eq-swap.cases[elim]

lemma eq-swap-sym[sym]: v =S w  $\implies$  w =S v <proof>

lemma eq-swap-length[dest]: w1 =S w2  $\implies$  length w1 = length w2 <proof>
lemma eq-swap-range[dest]: w1 =S w2  $\implies$  set w1 = set w2 <proof>

lemma eq-swap-extend:
  assumes w1 =S w2
  shows u @ w1 @ v =S u @ w2 @ v
<proof>

lemma eq-swap-remove1:
  assumes w1 =S w2
  obtains (equal) remove1 c w1 = remove1 c w2 | (swap) remove1 c w1 =S
remove1 c w2
<proof>

lemma eq-swap-rev:
  assumes w1 =S w2

```

shows $rev\ w_1 =_S rev\ w_2$
 $\langle proof \rangle$

abbreviation $eq\text{-}fin :: 'item\ list \Rightarrow 'item\ list \Rightarrow bool$ (**infix** $=_F$ 50)
where $eq\text{-}fin \equiv eq\text{-}swap^{**}$

lemma $eq\text{-}fin\text{-}symp[intro, sym]: u =_F v \Longrightarrow v =_F u$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}length[dest]: w_1 =_F w_2 \Longrightarrow length\ w_1 = length\ w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}range[dest]: w_1 =_F w_2 \Longrightarrow set\ w_1 = set\ w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}remove1$:
assumes $w_1 =_F w_2$
shows $remove1\ c\ w_1 =_F remove1\ c\ w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}rev$:
assumes $w_1 =_F w_2$
shows $rev\ w_1 =_F rev\ w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}concat\text{-}eq\text{-}fin\text{-}start$:
assumes $u @ v_1 =_F u @ v_2$
shows $v_1 =_F v_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}concat$: $u @ w_1 @ v =_F u @ w_2 @ v \longleftrightarrow w_1 =_F w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}concat\text{-}start[iff]$: $w @ w_1 =_F w @ w_2 \longleftrightarrow w_1 =_F w_2$
 $\langle proof \rangle$

lemma $eq\text{-}fin\text{-}concat\text{-}end[iff]$: $w_1 @ w =_F w_2 @ w \longleftrightarrow w_1 =_F w_2$
 $\langle proof \rangle$

lemma $ind\text{-}eq\text{-}fin'$:
assumes $Ind\ \{a\}\ (set\ v)$
shows $[a] @ v =_F v @ [a]$
 $\langle proof \rangle$

lemma $ind\text{-}eq\text{-}fin[intro]$:
assumes $Ind\ (set\ u)\ (set\ v)$
shows $u @ v =_F v @ u$
 $\langle proof \rangle$

definition $le\text{-}fin :: 'item\ list \Rightarrow 'item\ list \Rightarrow bool$ (**infix** \preceq_F 50)
where $w_1 \preceq_F w_2 \equiv \exists\ v_1. w_1 @ v_1 =_F w_2$

lemma *le-finI*[*intro 0*]:
assumes $w_1 @ v_1 =_F w_2$
shows $w_1 \preceq_F w_2$
 $\langle proof \rangle$

lemma *le-finE*[*elim 0*]:
assumes $w_1 \preceq_F w_2$
obtains v_1
where $w_1 @ v_1 =_F w_2$
 $\langle proof \rangle$

lemma *le-fin-empty*[*simp*]: $[] \preceq_F w \langle proof \rangle$

lemma *le-fin-trivial*[*intro*]: $w_1 =_F w_2 \implies w_1 \preceq_F w_2$
 $\langle proof \rangle$

lemma *le-fin-length*[*dest*]: $w_1 \preceq_F w_2 \implies length\ w_1 \leq length\ w_2 \langle proof \rangle$

lemma *le-fin-range*[*dest*]: $w_1 \preceq_F w_2 \implies set\ w_1 \subseteq set\ w_2 \langle proof \rangle$

lemma *eq-fin-alt-def*: $w_1 =_F w_2 \longleftrightarrow w_1 \preceq_F w_2 \wedge w_2 \preceq_F w_1$
 $\langle proof \rangle$

lemma *le-fin-reflp*[*simp, intro*]: $w \preceq_F w \langle proof \rangle$

lemma *le-fin-transp*[*intro, trans*]:
assumes $w_1 \preceq_F w_2\ w_2 \preceq_F w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *eq-fin-le-fin-transp*[*intro, trans*]:
assumes $w_1 =_F w_2\ w_2 \preceq_F w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *le-fin-eq-fin-transp*[*intro, trans*]:
assumes $w_1 \preceq_F w_2\ w_2 =_F w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *prefix-le-fin-transp*[*intro, trans*]:
assumes $w_1 \leq w_2\ w_2 \preceq_F w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *le-fin-prefix-transp*[*intro, trans*]:
assumes $w_1 \preceq_F w_2\ w_2 \leq w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *prefix-eq-fin-transp*[*intro, trans*]:
assumes $w_1 \leq w_2\ w_2 =_F w_3$
shows $w_1 \preceq_F w_3$
 $\langle proof \rangle$

lemma *le-fin-concat-start*[*iff*]: $w @ w_1 \preceq_F w @ w_2 \longleftrightarrow w_1 \preceq_F w_2$
 $\langle proof \rangle$

lemma *le-fin-concat-end*[*dest*]:

assumes $w_1 \preceq_F w_2$
shows $w_1 \preceq_F w_2 @ w$
 $\langle proof \rangle$

definition $le\text{-fininf} :: 'item\ list \Rightarrow 'item\ stream \Rightarrow bool$ (**infix** \preceq_{FI} 50)
where $w_1 \preceq_{FI} w_2 \equiv \exists v_2. v_2 \leq_{FI} w_2 \wedge w_1 \preceq_F v_2$

lemma $le\text{-fininfI}[intro\ 0]$:
assumes $v_2 \leq_{FI} w_2\ w_1 \preceq_F v_2$
shows $w_1 \preceq_{FI} w_2$
 $\langle proof \rangle$

lemma $le\text{-fininfE}[elim\ 0]$:
assumes $w_1 \preceq_{FI} w_2$
obtains v_2
where $v_2 \leq_{FI} w_2\ w_1 \preceq_F v_2$
 $\langle proof \rangle$

lemma $le\text{-fininf-empty}[simp]$: $[] \preceq_{FI} w$ $\langle proof \rangle$

lemma $le\text{-fininf-range}[dest]$: $w_1 \preceq_{FI} w_2 \implies set\ w_1 \subseteq sset\ w_2$ $\langle proof \rangle$

lemma $eq\text{-fin-le-fininf-transp}[intro,\ trans]$:
assumes $w_1 =_F w_2\ w_2 \preceq_{FI} w_3$
shows $w_1 \preceq_{FI} w_3$
 $\langle proof \rangle$

lemma $le\text{-fin-le-fininf-transp}[intro,\ trans]$:
assumes $w_1 \preceq_F w_2\ w_2 \preceq_{FI} w_3$
shows $w_1 \preceq_{FI} w_3$
 $\langle proof \rangle$

lemma $prefix\text{-le-fininf-transp}[intro,\ trans]$:
assumes $w_1 \leq w_2\ w_2 \preceq_{FI} w_3$
shows $w_1 \preceq_{FI} w_3$
 $\langle proof \rangle$

lemma $le\text{-fin-prefix-fininf-transp}[intro,\ trans]$:
assumes $w_1 \preceq_F w_2\ w_2 \leq_{FI} w_3$
shows $w_1 \preceq_{FI} w_3$
 $\langle proof \rangle$

lemma $eq\text{-fin-prefix-fininf-transp}[intro,\ trans]$:
assumes $w_1 =_F w_2\ w_2 \leq_{FI} w_3$
shows $w_1 \preceq_{FI} w_3$
 $\langle proof \rangle$

lemma $le\text{-fininf-concat-start}[iff]$: $w @ w_1 \preceq_{FI} w @- w_2 \iff w_1 \preceq_{FI} w_2$
 $\langle proof \rangle$

lemma $le\text{-fininf-singleton}[intro,\ simp]$: $[shd\ v] \preceq_{FI} v$
 $\langle proof \rangle$

definition $le\text{-inf} :: 'item\ stream \Rightarrow 'item\ stream \Rightarrow bool$ (**infix** \preceq_I 50)

where $w_1 \preceq_I w_2 \equiv \forall v_1. v_1 \leq_{FI} w_1 \longrightarrow v_1 \preceq_{FI} w_2$

lemma *le-infI*[*intro 0*]:

assumes $\bigwedge v_1. v_1 \leq_{FI} w_1 \implies v_1 \preceq_{FI} w_2$

shows $w_1 \preceq_I w_2$

<proof>

lemma *le-infE*[*elim 0*]:

assumes $w_1 \preceq_I w_2 \ v_1 \leq_{FI} w_1$

obtains $v_1 \preceq_{FI} w_2$

<proof>

lemma *le-inf-range*[*dest*]:

assumes $w_1 \preceq_I w_2$

shows $sset\ w_1 \subseteq sset\ w_2$

<proof>

lemma *le-inf-reflp*[*simp, intro*]: $w \preceq_I w$ *<proof>*

lemma *prefix-fininf-le-inf-transp*[*intro, trans*]:

assumes $w_1 \leq_{FI} w_2 \ w_2 \preceq_I w_3$

shows $w_1 \preceq_{FI} w_3$

<proof>

lemma *le-fininf-le-inf-transp*[*intro, trans*]:

assumes $w_1 \preceq_{FI} w_2 \ w_2 \preceq_I w_3$

shows $w_1 \preceq_{FI} w_3$

<proof>

lemma *le-inf-transp*[*intro, trans*]:

assumes $w_1 \preceq_I w_2 \ w_2 \preceq_I w_3$

shows $w_1 \preceq_I w_3$

<proof>

lemma *le-infI'*:

assumes $\bigwedge k. \exists v. v \leq_{FI} w_1 \wedge k < \text{length}\ v \wedge v \preceq_{FI} w_2$

shows $w_1 \preceq_I w_2$

<proof>

lemma *le-infI-chain-left*:

assumes $\text{chain}\ w \ \bigwedge k. w\ k \preceq_{FI} v$

shows $\text{limit}\ w \preceq_I v$

<proof>

lemma *le-infI-chain-right*:

assumes $\text{chain}\ w \ \bigwedge u. u \leq_{FI} v \implies u \preceq_F w\ (l\ u)$

shows $v \preceq_I \text{limit}\ w$

<proof>

lemma *le-infI-chain-right'*:

assumes $\text{chain}\ w \ \bigwedge k. \text{stake}\ k\ v \preceq_F w\ (l\ k)$

shows $v \preceq_I \text{limit}\ w$

<proof>

definition *eq-inf* :: *'item stream* \Rightarrow *'item stream* \Rightarrow *bool* (**infix** =_I 50)

where $w_1 =_I w_2 \equiv w_1 \preceq_I w_2 \wedge w_2 \preceq_I w_1$

lemma *eq-infI*[*intro 0*]:
assumes $w_1 \preceq_I w_2$ $w_2 \preceq_I w_1$
shows $w_1 =_I w_2$
<proof>

lemma *eq-infE*[*elim 0*]:
assumes $w_1 =_I w_2$
obtains $w_1 \preceq_I w_2$ $w_2 \preceq_I w_1$
<proof>

lemma *eq-inf-range*[*dest*]: $w_1 =_I w_2 \implies \text{sset } w_1 = \text{sset } w_2$ *<proof>*

lemma *eq-inf-reflp*[*simp, intro*]: $w =_I w$ *<proof>*

lemma *eq-inf-symp*[*intro*]: $w_1 =_I w_2 \implies w_2 =_I w_1$ *<proof>*

lemma *eq-inf-transp*[*intro, trans*]:
assumes $w_1 =_I w_2$ $w_2 =_I w_3$
shows $w_1 =_I w_3$
<proof>

lemma *le-fininf-eq-inf-transp*[*intro, trans*]:
assumes $w_1 \preceq_{FI} w_2$ $w_2 =_I w_3$
shows $w_1 \preceq_{FI} w_3$
<proof>

lemma *le-inf-eq-inf-transp*[*intro, trans*]:
assumes $w_1 \preceq_I w_2$ $w_2 =_I w_3$
shows $w_1 \preceq_I w_3$
<proof>

lemma *eq-inf-le-inf-transp*[*intro, trans*]:
assumes $w_1 =_I w_2$ $w_2 \preceq_I w_3$
shows $w_1 \preceq_I w_3$
<proof>

lemma *prefix-fininf-eq-inf-transp*[*intro, trans*]:
assumes $w_1 \preceq_{FI} w_2$ $w_2 =_I w_3$
shows $w_1 \preceq_{FI} w_3$
<proof>

lemma *le-inf-concat-start*[*iff*]: $w @- w_1 \preceq_I w @- w_2 \iff w_1 \preceq_I w_2$
<proof>

lemma *eq-fin-le-inf-concat-end*[*dest*]: $w_1 =_F w_2 \implies w_1 @- w \preceq_I w_2 @- w$
<proof>

lemma *eq-inf-concat-start*[*iff*]: $w @- w_1 =_I w @- w_2 \iff w_1 =_I w_2$ *<proof>*

lemma *eq-inf-concat-end*[*dest*]: $w_1 =_F w_2 \implies w_1 @- w =_I w_2 @- w$
<proof>

lemma *le-fininf-suffixI*[*intro*]:
assumes $w =_I w_1 @- w_2$
shows $w_1 \preceq_{FI} w$
<proof>

lemma *le-fininf-suffixE[elim]*:

assumes $w_1 \preceq_{FI} w$

obtains w_2

where $w =_I w_1 @ - w_2$

$\langle proof \rangle$

lemma *subsume-fin*:

assumes $u_1 \preceq_{FI} w \ v_1 \preceq_{FI} w$

obtains w_1

where $u_1 \preceq_F w_1 \ v_1 \preceq_F w_1$

$\langle proof \rangle$

lemma *eq-fin-end*:

assumes $u_1 =_F u_2 \ u_1 @ v_1 =_F u_2 @ v_2$

shows $v_1 =_F v_2$

$\langle proof \rangle$

definition *indoc* :: 'item \Rightarrow 'item list \Rightarrow bool

where $indoc\ a\ u \equiv \exists\ u_1\ u_2. u = u_1 @ [a] @ u_2 \wedge a \notin set\ u_1 \wedge Ind\ \{a\}\ (set\ u_1)$

lemma *indoc-set*: $indoc\ a\ u \implies a \in set\ u$ $\langle proof \rangle$

lemma *indoc-appendI1[intro]*:

assumes $indoc\ a\ u$

shows $indoc\ a\ (u @ v)$

$\langle proof \rangle$

lemma *indoc-appendI2[intro]*:

assumes $a \notin set\ u \ Ind\ \{a\}\ (set\ u) \ indoc\ a\ v$

shows $indoc\ a\ (u @ v)$

$\langle proof \rangle$

lemma *indoc-appendE[elim!]*:

assumes $indoc\ a\ (u @ v)$

obtains $(first)\ a \in set\ u \ indoc\ a\ u \mid (second)\ a \notin set\ u \ Ind\ \{a\}\ (set\ u) \ indoc\ a\ v$

$\langle proof \rangle$

lemma *indoc-single*: $indoc\ a\ [b] \longleftrightarrow a = b$

$\langle proof \rangle$

lemma *indoc-append[simp]*: $indoc\ a\ (u @ v) \longleftrightarrow$

$indoc\ a\ u \vee a \notin set\ u \wedge Ind\ \{a\}\ (set\ u) \wedge indoc\ a\ v$ $\langle proof \rangle$

lemma *indoc-Nil[simp]*: $indoc\ a\ [] \longleftrightarrow False$ $\langle proof \rangle$

lemma *indoc-Cons[simp]*: $indoc\ a\ (b \# v) \longleftrightarrow a = b \vee a \neq b \wedge ind\ a\ b \wedge indoc\ a\ v$

$\langle proof \rangle$

lemma *eq-swap-indoc*: $u =_S v \implies indoc\ c\ u \implies indoc\ c\ v$ $\langle proof \rangle$

lemma *eq-fin-indoc*: $u =_F v \implies indoc\ c\ u \implies indoc\ c\ v$ $\langle proof \rangle$

lemma *eq-fin-ind'*:

assumes $[a] @ u =_F u_1 @ [a] @ u_2$ $a \notin \text{set } u_1$
shows $\text{Ind } \{a\} (\text{set } u_1)$

<proof>

lemma *eq-fin-ind*:

assumes $u @ v =_F v @ u$ $\text{set } u \cap \text{set } v = \{\}$
shows $\text{Ind } (\text{set } u) (\text{set } v)$

<proof>

lemma *le-fin-member'*:

assumes $[a] \preceq_F u @ v$ $a \in \text{set } u$
shows $[a] \preceq_F u$

<proof>

lemma *le-fin-not-member'*:

assumes $[a] \preceq_F u @ v$ $a \notin \text{set } u$
shows $[a] \preceq_F v$

<proof>

lemma *le-fininf-not-member'*:

assumes $[a] \preceq_{FI} u @- v$ $a \notin \text{set } u$
shows $[a] \preceq_{FI} v$

<proof>

lemma *le-fin-ind''*:

assumes $[a] \preceq_F w [b] \preceq_F w$ $a \neq b$
shows $\text{ind } a b$

<proof>

lemma *le-fin-ind'*:

assumes $[a] \preceq_F w v \preceq_F w$ $a \notin \text{set } v$
shows $\text{Ind } \{a\} (\text{set } v)$

<proof>

lemma *le-fininf-ind''*:

assumes $[a] \preceq_{FI} w [b] \preceq_{FI} w$ $a \neq b$
shows $\text{ind } a b$

<proof>

lemma *le-fininf-ind'*:

assumes $[a] \preceq_{FI} w v \preceq_{FI} w$ $a \notin \text{set } v$
shows $\text{Ind } \{a\} (\text{set } v)$

<proof>

lemma *indoc-alt-def*: $\text{indoc } a v \longleftrightarrow v =_F [a] @ \text{remove1 } a v$

<proof>

lemma *levi-lemma*:

assumes $t @ u =_F v @ w$

obtains $p r s q$

where $t =_F p @ r$ $u =_F s @ q$ $v =_F p @ s$ $w =_F r @ q$ $\text{Ind } (\text{set } r) (\text{set } s)$

<proof>

end

end

9 Transition Systems and Trace Theory

theory *Transition-System-Traces*

imports

Transition-System-Extensions

Traces

begin

lemma (**in** *transition-system*) *words-infI-construct*[*rule-format*, *intro?*]:

assumes $\forall v. v \leq_{FI} w \longrightarrow \text{path } v \ p$

shows *run* *w* *p*

<proof>

lemma (**in** *transition-system*) *words-infI-construct'*:

assumes $\bigwedge k. \exists v. v \leq_{FI} w \wedge k < \text{length } v \wedge \text{path } v \ p$

shows *run* *w* *p*

<proof>

lemma (**in** *transition-system*) *words-infI-construct-chain*[*intro*]:

assumes *chain* *w* $\bigwedge k. \text{path } (w \ k) \ p$

shows *run* (*limit* *w*) *p*

<proof>

lemma (**in** *transition-system*) *words-fin-blocked*:

assumes $\bigwedge w. \text{path } w \ p \implies A \cap \text{set } w = \{\} \implies A \cap \{a. \text{enabled } a \ (\text{target } w \ p)\} \subseteq A \cap \{a. \text{enabled } a \ p\}$

assumes *path* *w* *p* $A \cap \{a. \text{enabled } a \ p\} \cap \text{set } w = \{\}$

shows $A \cap \text{set } w = \{\}$

<proof>

locale *transition-system-traces* =

transition-system *ex* *en* +

traces *ind*

for *ex* :: 'action \Rightarrow 'state \Rightarrow 'state

and *en* :: 'action \Rightarrow 'state \Rightarrow bool

and *ind* :: 'action \Rightarrow 'action \Rightarrow bool

+

assumes *en*: *ind* *a* *b* \implies *en* *a* *p* \implies *en* *b* *p* \longleftrightarrow *en* *b* (*ex* *a* *p*)

assumes *ex*: *ind* *a* *b* \implies *en* *a* *p* \implies *en* *b* *p* \implies *ex* *b* (*ex* *a* *p*) = *ex* *a* (*ex* *b* *p*)

begin

lemma *diamond-bottom*:

assumes *ind* *a* *b*

assumes *en* *a* *p* *en* *b* *p*

shows *en* *a* (*ex* *b* *p*) *en* *b* (*ex* *a* *p*) *ex* *b* (*ex* *a* *p*) = *ex* *a* (*ex* *b* *p*)

$\langle proof \rangle$
lemma diamond-right:
assumes $ind\ a\ b$
assumes $en\ a\ p\ en\ b\ (ex\ a\ p)$
shows $en\ a\ (ex\ b\ p)\ en\ b\ p\ ex\ b\ (ex\ a\ p) = ex\ a\ (ex\ b\ p)$
 $\langle proof \rangle$
lemma diamond-left:
assumes $ind\ a\ b$
assumes $en\ a\ (ex\ b\ p)\ en\ b\ p$
shows $en\ a\ p\ en\ b\ (ex\ a\ p)\ ex\ b\ (ex\ a\ p) = ex\ a\ (ex\ b\ p)$
 $\langle proof \rangle$

lemma eq-swap-word:
assumes $w_1 =_S w_2\ path\ w_1\ p$
shows $path\ w_2\ p$
 $\langle proof \rangle$
lemma eq-fin-word:
assumes $w_1 =_F w_2\ path\ w_1\ p$
shows $path\ w_2\ p$
 $\langle proof \rangle$
lemma le-fin-word:
assumes $w_1 \preceq_F w_2\ path\ w_2\ p$
shows $path\ w_1\ p$
 $\langle proof \rangle$
lemma le-fininf-word:
assumes $w_1 \preceq_{FI} w_2\ run\ w_2\ p$
shows $path\ w_1\ p$
 $\langle proof \rangle$
lemma le-inf-word:
assumes $w_2 \preceq_I w_1\ run\ w_1\ p$
shows $run\ w_2\ p$
 $\langle proof \rangle$
lemma eq-inf-word:
assumes $w_1 =_I w_2\ run\ w_1\ p$
shows $run\ w_2\ p$
 $\langle proof \rangle$

lemma eq-swap-execute:
assumes $path\ w_1\ p\ w_1 =_S w_2$
shows $fold\ ex\ w_1\ p = fold\ ex\ w_2\ p$
 $\langle proof \rangle$
lemma eq-fin-execute:
assumes $path\ w_1\ p\ w_1 =_F w_2$
shows $fold\ ex\ w_1\ p = fold\ ex\ w_2\ p$
 $\langle proof \rangle$

lemma diamond-fin-word-step:
assumes $Ind\ \{a\}\ (set\ v)\ en\ a\ p\ path\ v\ p$
shows $path\ v\ (ex\ a\ p)$

```

    <proof>
lemma diamond-inf-word-step:
  assumes Ind {a} (sset w) en a p run w p
  shows run w (ex a p)
  <proof>
lemma diamond-fin-word-inf-word:
  assumes Ind (set v) (sset w) path v p run w p
  shows run w (fold ex v p)
  <proof>
lemma diamond-fin-word-inf-word':
  assumes Ind (set v) (sset w) path (u @ v) p run (u @- w) p
  shows run (u @- v @- w) p
  <proof>

end

end

```

10 Functions

```

theory Functions
imports ../Extensions/Set-Extensions
begin

  locale bounded-function =
    fixes A :: 'a set
    fixes B :: 'b set
    fixes f :: 'a  $\Rightarrow$  'b
    assumes wellformed[intro?, simp]:  $x \in A \Longrightarrow f x \in B$ 

  locale bounded-function-pair =
    f: bounded-function A B f +
    g: bounded-function B A g
    for A :: 'a set
    and B :: 'b set
    and f :: 'a  $\Rightarrow$  'b
    and g :: 'b  $\Rightarrow$  'a

  locale injection = bounded-function-pair +
    assumes left-inverse[simp]:  $x \in A \Longrightarrow g (f x) = x$ 
  begin

    lemma inj-on[intro]: inj-on f A <proof>

    lemma injective-on:
      assumes  $x \in A$   $y \in A$   $f x = f y$ 
      shows  $x = y$ 
      <proof>

```

```

end

locale injective = bounded-function +
  assumes injection:  $\exists g. \text{injection } A B f g$ 
begin

  definition g  $\equiv$  SOME g. injection A B f g

  sublocale injection A B f g  $\langle$ proof $\rangle$ 

end

locale surjection = bounded-function-pair +
  assumes right-inverse[simp]:  $y \in B \implies f (g y) = y$ 
begin

  lemma image-superset[intro]:  $f ' A \supseteq B$ 
     $\langle$ proof $\rangle$ 

  lemma image-eq[simp]:  $f ' A = B$   $\langle$ proof $\rangle$ 

end

locale surjective = bounded-function +
  assumes surjection:  $\exists g. \text{surjection } A B f g$ 
begin

  definition g  $\equiv$  SOME g. surjection A B f g

  sublocale surjection A B f g  $\langle$ proof $\rangle$ 

end

locale bijection = injection + surjection

lemma inj-on-bijection:
  assumes inj-on f A
  shows bijection A (f ' A) f (inv-into A f)
   $\langle$ proof $\rangle$ 

end

```

11 Extended Natural Numbers

```

theory ENat-Extensions
imports
  Coinductive.Coinductive-Nat
begin

```

```

declare eSuc-enat[simp]
declare iadd-Suc[simp] iadd-Suc-right[simp]
declare enat-0[simp] enat-1[simp] one-eSuc[simp]
declare enat-0-iff[iff] enat-1-iff[iff]
declare Suc-ile-eq[iff]

lemma enat-Suc0[simp]: enat (Suc 0) = eSuc 0 <proof>

lemma le-epred[iff]: l < epred k  $\longleftrightarrow$  eSuc l < k
  <proof>

lemma eq-infI[intro]:
  assumes  $\bigwedge n. \text{enat } n \leq m$ 
  shows  $m = \infty$ 
  <proof>

```

end

12 Chain-Complete Partial Orders

```

theory CCPO-Extensions
imports
  HOL-Library.Complete-Partial-Order2
  ENat-Extensions
  Set-Extensions
begin

lemma chain-split[dest]:
  assumes Complete-Partial-Order.chain ord C x  $\in$  C
  shows  $C = \{y \in C. \text{ord } x \ y\} \cup \{y \in C. \text{ord } y \ x\}$ 
  <proof>

lemma infinite-chain-below[dest]:
  assumes Complete-Partial-Order.chain ord C infinite C x  $\in$  C
  assumes finite  $\{y \in C. \text{ord } x \ y\}$ 
  shows infinite  $\{y \in C. \text{ord } y \ x\}$ 
  <proof>

lemma infinite-chain-above[dest]:
  assumes Complete-Partial-Order.chain ord C infinite C x  $\in$  C
  assumes finite  $\{y \in C. \text{ord } y \ x\}$ 
  shows infinite  $\{y \in C. \text{ord } x \ y\}$ 
  <proof>

lemma (in ccpo) ccpo-Sup-upper-inv:
  assumes Complete-Partial-Order.chain less-eq C x  $> \bigsqcup C$ 
  shows  $x \notin C$ 
  <proof>

lemma (in ccpo) ccpo-Sup-least-inv:
  assumes Complete-Partial-Order.chain less-eq C  $\bigsqcup C > x$ 

```

obtains y
where $y \in C \neg y \leq x$
 $\langle proof \rangle$

lemma *ccpo-Sup-least-inv'*:
fixes $C :: 'a :: \{ccpo, linorder\} set$
assumes *Complete-Partial-Order.chain less-eq C* $\sqcup C > x$
obtains y
where $y \in C y > x$
 $\langle proof \rangle$

lemma *mcont2mcont-lessThan*[*THEN lfp.mcont2mcont, simp, cont-intro*]:
shows *mcont-lessThan: mcont Sup less-eq Sup less-eq*
(lessThan :: 'a :: \{ccpo, linorder\} \Rightarrow 'a set)
 $\langle proof \rangle$

class *esize* =
fixes *esize* :: $'a \Rightarrow enat$

class *esize-order* = *esize* + *order* +
assumes *esize-finite*[*dest*]: *esize* $x \neq \infty \Rightarrow finite \{y. y \leq x\}$
assumes *esize-mono*[*intro*]: $x \leq y \Rightarrow esize\ x \leq esize\ y$
assumes *esize-strict-mono*[*intro*]: *esize* $x \neq \infty \Rightarrow x < y \Rightarrow esize\ x < esize\ y$
begin

lemma *infinite-chain-eSuc-esize*[*dest*]:
assumes *Complete-Partial-Order.chain less-eq C infinite C* $x \in C$
obtains y
where $y \in C esize\ y \geq eSuc\ (esize\ x)$
 $\langle proof \rangle$

lemma *infinite-chain-arbitrary-esize*[*dest*]:
assumes *Complete-Partial-Order.chain less-eq C infinite C*
obtains x
where $x \in C esize\ x \geq enat\ n$
 $\langle proof \rangle$

end

class *esize-ccpo* = *esize-order* + *ccpo*
begin

lemma *esize-cont*[*dest*]:
assumes *Complete-Partial-Order.chain less-eq C C $\neq \{\}$*
shows *esize* $(\sqcup C) = \sqcup (esize\ ` C)$
 $\langle proof \rangle$

lemma *esize-mcont: mcont Sup less-eq Sup less-eq esize*
 $\langle proof \rangle$

```

lemmas mcont2mcont-esize = esize-mcont[THEN lfp.mcont2mcont, simp, cont-intro]

end

end

```

13 Sets and Extended Natural Numbers

theory *ESet-Extensions*

imports

../Basics/Functions

Basic-Extensions

CCPO-Extensions

begin

lemma *card-lessThan-enat*[simp]: $\text{card } \{.. < \text{enat } k\} = \text{card } \{.. < k\}$

<proof>

lemma *card-atMost-enat*[simp]: $\text{card } \{.. \text{enat } k\} = \text{card } \{.. k\}$

<proof>

lemma *enat-Collect*:

assumes $\infty \notin A$

shows $\{i. \text{enat } i \in A\} = \text{the-enat } ' A$

<proof>

lemma *Collect-lessThan*: $\{i. \text{enat } i < n\} = \text{the-enat } ' \{.. < n\}$

<proof>

instantiation *set* :: (*type*) *esize-ccpo*

begin

function *esize-set* **where** $\text{finite } A \implies \text{esize } A = \text{enat } (\text{card } A) \mid \text{infinite } A \implies \text{esize } A = \infty$

<proof> **termination** *<proof>*

lemma *esize-iff-empty*[iff]: $\text{esize } A = 0 \longleftrightarrow A = \{\}$ *<proof>*

lemma *esize-iff-infinite*[iff]: $\text{esize } A = \infty \longleftrightarrow \text{infinite } A$ *<proof>*

lemma *esize-singleton*[simp]: $\text{esize } \{a\} = \text{eSuc } 0$ *<proof>*

lemma *esize-infinite-enat*[dest, simp]: $\text{infinite } A \implies \text{enat } k < \text{esize } A$ *<proof>*

instance

<proof>

end

lemma *esize-image*[simp, intro]:

assumes *inj-on* f A

shows $\text{esize } (f ' A) = \text{esize } A$

$\langle \text{proof} \rangle$
lemma *esize-insert1*[simp]: $a \notin A \implies \text{esize} (\text{insert } a \ A) = \text{eSuc} (\text{esize } A)$
 $\langle \text{proof} \rangle$
lemma *esize-insert2*[simp]: $a \in A \implies \text{esize} (\text{insert } a \ A) = \text{esize } A$
 $\langle \text{proof} \rangle$
lemma *esize-remove1*[simp]: $a \notin A \implies \text{esize} (A - \{a\}) = \text{esize } A$
 $\langle \text{proof} \rangle$
lemma *esize-remove2*[simp]: $a \in A \implies \text{esize} (A - \{a\}) = \text{epred} (\text{esize } A)$
 $\langle \text{proof} \rangle$
lemma *esize-union-disjoint*[simp]:
assumes $A \cap B = \{\}$
shows $\text{esize} (A \cup B) = \text{esize } A + \text{esize } B$
 $\langle \text{proof} \rangle$
lemma *esize-lessThan*[simp]: $\text{esize} \{.. < n\} = n$
 $\langle \text{proof} \rangle$
lemma *esize-atMost*[simp]: $\text{esize} \{.. n\} = \text{eSuc } n$
 $\langle \text{proof} \rangle$

lemma *least-eSuc*[simp]:
assumes $A \neq \{\}$
shows $\text{least} (\text{eSuc } 'A) = \text{eSuc} (\text{least } A)$
 $\langle \text{proof} \rangle$

lemma *Inf-enat-eSuc*[simp]: $\bigcap (\text{eSuc } 'A) = \text{eSuc} (\bigcap A)$ $\langle \text{proof} \rangle$

definition *lift* :: *nat set* \Rightarrow *nat set*
where $\text{lift } A \equiv \text{insert } 0 (\text{Suc } 'A)$

lemma *liftI-0*[intro, simp]: $0 \in \text{lift } A$ $\langle \text{proof} \rangle$
lemma *liftI-Suc*[intro]: $a \in A \implies \text{Suc } a \in \text{lift } A$ $\langle \text{proof} \rangle$
lemma *liftE*[elim]:
assumes $b \in \text{lift } A$
obtains $(0) \ b = 0 \mid (\text{Suc } a) \ \text{where } b = \text{Suc } a \ a \in A$
 $\langle \text{proof} \rangle$

lemma *lift-esize*[simp]: $\text{esize} (\text{lift } A) = \text{eSuc} (\text{esize } A)$ $\langle \text{proof} \rangle$
lemma *lift-least*[simp]: $\text{least} (\text{lift } A) = 0$ $\langle \text{proof} \rangle$

primrec *nth-least* :: *'a set* \Rightarrow *nat* \Rightarrow *'a* :: *wellorder*
where $\text{nth-least } A \ 0 = \text{least } A \mid \text{nth-least } A (\text{Suc } n) = \text{nth-least} (A - \{\text{least } A\}) \ n$

lemma *nth-least-wellformed*[intro?, simp]:
assumes $\text{enat } n < \text{esize } A$
shows $\text{nth-least } A \ n \in A$
 $\langle \text{proof} \rangle$

lemma *card-wellformed*[intro?, simp]:
fixes $k :: 'a :: \text{wellorder}$

assumes $k \in A$
shows $\text{enat } (\text{card } \{i \in A. i < k\}) < \text{esize } A$
 $\langle \text{proof} \rangle$

lemma *nth-least-strict-mono*:
assumes $\text{enat } l < \text{esize } A$ $k < l$
shows $\text{nth-least } A k < \text{nth-least } A l$
 $\langle \text{proof} \rangle$

lemma *nth-least-mono[intro, simp]*:
assumes $\text{enat } l < \text{esize } A$ $k \leq l$
shows $\text{nth-least } A k \leq \text{nth-least } A l$
 $\langle \text{proof} \rangle$

lemma *card-nth-least[simp]*:
assumes $\text{enat } n < \text{esize } A$
shows $\text{card } \{k \in A. k < \text{nth-least } A n\} = n$
 $\langle \text{proof} \rangle$

lemma *card-nth-least-le[simp]*:
assumes $\text{enat } n < \text{esize } A$
shows $\text{card } \{k \in A. k \leq \text{nth-least } A n\} = \text{Suc } n$
 $\langle \text{proof} \rangle$

lemma *nth-least-card*:
fixes $k :: \text{nat}$
assumes $k \in A$
shows $\text{nth-least } A (\text{card } \{i \in A. i < k\}) = k$
 $\langle \text{proof} \rangle$

interpretation *nth-least*:
bounded-function-pair $\{i. \text{enat } i < \text{esize } A\} A$ $\text{nth-least } A \lambda k. \text{card } \{i \in A. i < k\}$
 $\langle \text{proof} \rangle$

interpretation *nth-least*:
injection $\{i. \text{enat } i < \text{esize } A\} A$ $\text{nth-least } A \lambda k. \text{card } \{i \in A. i < k\}$
 $\langle \text{proof} \rangle$

interpretation *nth-least*:
surjection $\{i. \text{enat } i < \text{esize } A\} A$ $\text{nth-least } A \lambda k. \text{card } \{i \in A. i < k\}$
for $A :: \text{nat set}$
 $\langle \text{proof} \rangle$

interpretation *nth-least*:
bijection $\{i. \text{enat } i < \text{esize } A\} A$ $\text{nth-least } A \lambda k. \text{card } \{i \in A. i < k\}$
for $A :: \text{nat set}$
 $\langle \text{proof} \rangle$

lemma *nth-least-strict-mono-inverse*:

fixes $A :: \text{nat set}$

assumes $\text{enat } k < \text{esize } A \text{ enat } l < \text{esize } A \text{ nth-least } A \ k < \text{nth-least } A \ l$

shows $k < l$

<proof>

lemma *nth-least-less-card-less*:

fixes $k :: \text{nat}$

shows $\text{enat } n < \text{esize } A \wedge \text{nth-least } A \ n < k \longleftrightarrow n < \text{card } \{i \in A. i < k\}$

<proof>

lemma *nth-least-less-esize-less*:

$\text{enat } n < \text{esize } A \wedge \text{enat } (\text{nth-least } A \ n) < k \longleftrightarrow \text{enat } n < \text{esize } \{i \in A. \text{enat } i < k\}$

<proof>

lemma *nth-least-le*:

assumes $\text{enat } n < \text{esize } A$

shows $n \leq \text{nth-least } A \ n$

<proof>

lemma *nth-least-eq*:

assumes $\text{enat } n < \text{esize } A \text{ enat } n < \text{esize } B$

assumes $\bigwedge i. i \leq \text{nth-least } A \ n \implies i \leq \text{nth-least } B \ n \implies i \in A \longleftrightarrow i \in B$

shows $\text{nth-least } A \ n = \text{nth-least } B \ n$

<proof>

lemma *nth-least-restrict[simp]*:

assumes $\text{enat } i < \text{esize } \{i \in s. \text{enat } i < k\}$

shows $\text{nth-least } \{i \in s. \text{enat } i < k\} \ i = \text{nth-least } s \ i$

<proof>

lemma *least-nth-least[simp]*:

assumes $A \neq \{\} \wedge i. i \in A \implies \text{enat } i < \text{esize } B$

shows $\text{least } (\text{nth-least } B \ ' A) = \text{nth-least } B \ (\text{least } A)$

<proof>

lemma *nth-least-nth-least[simp]*:

assumes $\text{enat } n < \text{esize } A \wedge i. i \in A \implies \text{enat } i < \text{esize } B$

shows $\text{nth-least } B \ (\text{nth-least } A \ n) = \text{nth-least } (\text{nth-least } B \ ' A) \ n$

<proof>

lemma *nth-least-Max[simp]*:

assumes $\text{finite } A \ A \neq \{\}$

shows $\text{nth-least } A \ (\text{card } A - 1) = \text{Max } A$

<proof>

lemma *nth-least-le-Max*:

assumes $\text{finite } A \ A \neq \{\} \text{ enat } n < \text{esize } A$

shows $nth\text{-least } A \ n \leq \text{Max } A$
 $\langle proof \rangle$

lemma $nth\text{-least-not-contains}$:

fixes $k :: nat$

assumes $enat (Suc \ n) < esize \ A \ nth\text{-least } A \ n < k \ k < nth\text{-least } A \ (Suc \ n)$

shows $k \notin A$

$\langle proof \rangle$

lemma $nth\text{-least-Suc}[simp]$:

assumes $enat \ n < esize \ A$

shows $nth\text{-least } (Suc \ A) \ n = Suc \ (nth\text{-least } A \ n)$

$\langle proof \rangle$

lemma $nth\text{-least-lift}[simp]$:

$nth\text{-least } (lift \ A) \ 0 = 0$

$enat \ n < esize \ A \implies nth\text{-least } (lift \ A) \ (Suc \ n) = Suc \ (nth\text{-least } A \ n)$

$\langle proof \rangle$

lemma $nth\text{-least-list-card}[simp]$:

assumes $enat \ n \leq esize \ A$

shows $card \ \{k \in A. \ k < nth\text{-least } (lift \ A) \ n\} = n$

$\langle proof \rangle$

end

14 Coinductive Lists

theory $Coinductive\text{-List-Extensions}$

imports

$Coinductive.Coinductive\text{-List}$

$Coinductive.Coinductive\text{-List-Prefix}$

$Coinductive.Coinductive\text{-Stream}$

$../Extensions/List\text{-Extensions}$

$../Extensions/ESet\text{-Extensions}$

begin

hide-const **(open)** $Sublist.prefix$

hide-const **(open)** $Sublist.suffix$

declare $list\text{-of-lappend}[simp]$

declare $lnth\text{-lappend1}[simp]$

declare $lnth\text{-lappend2}[simp]$

declare $lprefix\text{-llength-le}[dest]$

declare $Sup\text{-llist-def}[simp]$

declare $length\text{-list-of}[simp]$

declare $llast\text{-linfinite}[simp]$

declare $lnth\text{-ltake}[simp]$

declare $lappend\text{-assoc}[simp]$

declare *lprefix-lappend*[simp]

lemma *lprefix-lSup-revert*: $lSup = Sup\ lprefix = less-eq$ *<proof>*

lemma *admissible-lprefixI*[cont-intro]:
assumes *mcont lub ord lSup lprefix f*
assumes *mcont lub ord lSup lprefix g*
shows *ccpo.admissible lub ord* $(\lambda x. lprefix\ (f\ x)\ (g\ x))$
<proof>

lemma *llist-lift-admissible*:
assumes *ccpo.admissible lSup lprefix P*
assumes $\bigwedge u. u \leq v \implies lfinite\ u \implies P\ u$
shows $P\ v$
<proof>

abbreviation *lfinite* $w \equiv \neg\ lfinite\ w$

notation *LNil* ($\langle \rangle$)
notation *LCons* (**infixr** $\%$ 65)
notation *lzip* (**infixr** $\|$ 51)
notation *lappend* (**infixr** $\$$ 65)
notation *lnth* (**infixl** $!$ 100)

syntax *-llist* :: *args* \Rightarrow 'a *llist* ($\langle - \rangle$)

translations
 $\langle a, x \rangle \rightleftharpoons a\ \% \langle x \rangle$
 $\langle a \rangle \rightleftharpoons a\ \% \langle \rangle$

lemma *eq-LNil-conv-lnull*[simp]: $w = \langle \rangle \longleftrightarrow lnull\ w$ *<proof>*

lemma *Collect-lnull*[simp]: $\{w. lnull\ w\} = \{\langle \rangle\}$ *<proof>*

lemma *inj-on-ltake*: *inj-on* $(\lambda k. ltake\ k\ w)\ \{..\ llength\ w\}$
<proof>

lemma *lnth-inf-llist'*[simp]: $lnth\ (inf-llist\ f) = f$ *<proof>*

lemma *not-lnull-lappend-startE*[elim]:
assumes $\neg\ lnull\ w$
obtains $a\ v$
where $w = \langle a \rangle\ \$\ v$
<proof>

lemma *not-lnull-lappend-endE*[elim]:
assumes $\neg\ lnull\ w$
obtains $a\ v$
where $w = v\ \$\ \langle a \rangle$
<proof>

lemma *llength-lappend-startE*[elim]:
assumes $llength\ w \geq eSuc\ n$
obtains $a\ v$

where $w = \langle a \rangle \$ v$ $llength\ v \geq n$
<proof>

lemma *llength-lappend-endE[elim]*:
assumes $llength\ w \geq eSuc\ n$
obtains $a\ v$
where $w = v \$ \langle a \rangle$ $llength\ v \geq n$
<proof>

lemma *llength-lappend-start'E[elim]*:
assumes $llength\ w = enat\ (Suc\ n)$
obtains $a\ v$
where $w = \langle a \rangle \$ v$ $llength\ v = enat\ n$
<proof>

lemma *llength-lappend-end'E[elim]*:
assumes $llength\ w = enat\ (Suc\ n)$
obtains $a\ v$
where $w = v \$ \langle a \rangle$ $llength\ v = enat\ n$
<proof>

lemma *ltake-llast[simp]*:
assumes $enat\ k < llength\ w$
shows $llast\ (ltake\ (enat\ (Suc\ k))\ w) = w\ ?!\ k$
<proof>

lemma *linfinite-llength[dest, simp]*:
assumes $linfinite\ w$
shows $enat\ k < llength\ w$
<proof>

lemma *llist-nth-eqI[intro]*:
assumes $llength\ u = llength\ v$
assumes $\bigwedge i. enat\ i < llength\ u \implies enat\ i < llength\ v \implies u\ ?!\ i = v\ ?!\ i$
shows $u = v$
<proof>

primcorec *lscan* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a\ llist \Rightarrow 'b \Rightarrow 'b\ llist$
where $lscan\ f\ w\ a = (case\ w\ of\ \langle \rangle \Rightarrow \langle a \rangle \mid x\ \% xs \Rightarrow a\ \% lscan\ f\ xs\ (f\ x\ a))$

lemma *lscan-simps[simp]*:
 $lscan\ f\ \langle \rangle\ a = \langle a \rangle$
 $lscan\ f\ (x\ \% xs)\ a = a\ \% lscan\ f\ xs\ (f\ x\ a)$
<proof>

lemma *lscan-lfinite[iff]*: $lfinite\ (lscan\ f\ w\ a) \longleftrightarrow lfinite\ w$
<proof>

lemma *lscan-llength[simp]*: $llength\ (lscan\ f\ w\ a) = eSuc\ (llength\ w)$
<proof>

function *lfold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a\ llist \Rightarrow 'b \Rightarrow 'b$

where $lfinite\ w \implies lfold\ f\ w = fold\ f\ (list-of\ w) \mid linfinite\ w \implies lfold\ f\ w = id$
 $\langle proof \rangle$ **termination** $\langle proof \rangle$

lemma $lfold-llist-of[simp]$: $lfold\ f\ (llist-of\ xs) = fold\ f\ xs$ $\langle proof \rangle$

lemma $finite-UNIV-llength-eq$:

assumes $finite\ (UNIV :: 'a\ set)$

shows $finite\ \{w :: 'a\ llist.\ llength\ w = enat\ n\}$

$\langle proof \rangle$

lemma $finite-UNIV-llength-le$:

assumes $finite\ (UNIV :: 'a\ set)$

shows $finite\ \{w :: 'a\ llist.\ llength\ w \leq enat\ n\}$

$\langle proof \rangle$

lemma $lprefix-ltake[dest]$: $u \leq v \implies u = ltake\ (llength\ u)\ v$

$\langle proof \rangle$

lemma $prefixes-set$: $\{v.\ v \leq w\} = \{ltake\ k\ w \mid k.\ k \leq llength\ w\}$ $\langle proof \rangle$

lemma $esize-prefixes[simp]$: $esize\ \{v.\ v \leq w\} = eSuc\ (llength\ w)$

$\langle proof \rangle$

lemma $lprefix-subsume$: $v \leq w \implies u \leq w \implies llength\ v \leq llength\ u \implies v \leq u$

$\langle proof \rangle$

lemma $ltake-infinite[simp]$: $ltake\ \infty\ w = w$ $\langle proof \rangle$

lemma $lprefix-infinite$:

assumes $u \leq v\ linfinite\ u$

shows $u = v$

$\langle proof \rangle$

instantiation $llist :: (type)\ esize-order$

begin

definition $[simp]$: $esize \equiv llength$

instance

$\langle proof \rangle$

end

14.1 Index Sets

definition $lisset :: 'a\ set \Rightarrow 'a\ llist \Rightarrow nat\ set$

where $lisset\ A\ w \equiv \{i.\ enat\ i < llength\ w \wedge w\ ?!\ i \in A\}$

lemma $lissetI[intro]$:

assumes $enat\ i < llength\ w\ w\ ?!\ i \in A$

shows $i \in lisset\ A\ w$

$\langle proof \rangle$

lemma $lissetD[dest]$:

assumes $i \in \text{liset } A \ w$
shows $\text{enat } i < \text{llength } w \ w \ ?! \ i \in A$
 $\langle \text{proof} \rangle$

lemma *liset-finite*:
assumes $\text{lfinite } w$
shows $\text{finite } (\text{liset } A \ w)$
 $\langle \text{proof} \rangle$

lemma *liset-nil[simp]*: $\text{liset } A \ <> = \{\}$ $\langle \text{proof} \rangle$

lemma *liset-cons-not-member[simp]*:
assumes $a \notin A$
shows $\text{liset } A \ (a \% w) = \text{Suc } ' \text{liset } A \ w$
 $\langle \text{proof} \rangle$

lemma *liset-cons-member[simp]*:
assumes $a \in A$
shows $\text{liset } A \ (a \% w) = \{0\} \cup \text{Suc } ' \text{liset } A \ w$
 $\langle \text{proof} \rangle$

lemma *liset-prefix*:
assumes $i \in \text{liset } A \ v \ u \leq v \ \text{enat } i < \text{llength } u$
shows $i \in \text{liset } A \ u$
 $\langle \text{proof} \rangle$

lemma *liset-suffix*:
assumes $i \in \text{liset } A \ u \ u \leq v$
shows $i \in \text{liset } A \ v$
 $\langle \text{proof} \rangle$

lemma *liset-ltake[simp]*: $\text{liset } A \ (\text{ltake } (\text{enat } k) \ w) = \text{liset } A \ w \cap \{.. < k\}$
 $\langle \text{proof} \rangle$

lemma *liset-mono[dest]*: $u \leq v \implies \text{liset } A \ u \subseteq \text{liset } A \ v$
 $\langle \text{proof} \rangle$

lemma *liset-cont[dest]*:
assumes $\text{Complete-Partial-Order.chain less-eq } C \ C \neq \{\}$
shows $\text{liset } A \ (\bigsqcup C) = (\bigcup w \in C. \text{liset } A \ w)$
 $\langle \text{proof} \rangle$

lemma *liset-mcont*: $\text{Complete-Partial-Order2.mcont lSup lprefix Sup less-eq}$
 $(\text{liset } A)$
 $\langle \text{proof} \rangle$

lemmas $\text{mcont2mcont-liset} = \text{liset-mcont}[\text{THEN lfp.mcont2mcont, simp, cont-intro}]$

14.2 Selections

abbreviation $\text{lproject } A \equiv \text{lfilter } (\lambda a. a \in A)$

abbreviation $\text{lselect } s \ w \equiv \text{lnths } w \ s$

lemma *lselect-to-lproject*: $lselect\ s\ w = lmap\ fst\ (lproject\ (UNIV \times s)\ (w\ ||\ iterates\ Suc\ 0))$

<proof>

lemma *lproject-to-lselect*: $lproject\ A\ w = lselect\ (lset\ A\ w)\ w$

<proof>

lemma *lproject-llength[simp]*: $llength\ (lproject\ A\ w) = esize\ (lset\ A\ w)$

<proof>

lemma *lproject-lfinite[simp]*: $lfinite\ (lproject\ A\ w) \longleftrightarrow finite\ (lset\ A\ w)$

<proof>

lemma *lselect-restrict-indices[simp]*: $lselect\ \{i \in s.\ enat\ i < llength\ w\}\ w = lselect\ s\ w$

<proof>

lemma *lselect-llength*: $llength\ (lselect\ s\ w) = esize\ \{i \in s.\ enat\ i < llength\ w\}$

<proof>

lemma *lselect-llength-le[simp]*: $llength\ (lselect\ s\ w) \leq esize\ s$

<proof>

lemma *least-lselect-llength*:

assumes $\neg\ lnull\ (lselect\ s\ w)$

shows $enat\ (least\ s) < llength\ w$

<proof>

lemma *lselect-lnull*: $lnull\ (lselect\ s\ w) \longleftrightarrow (\forall\ i \in s.\ enat\ i \geq llength\ w)$

<proof>

lemma *lselect-discard-start*:

assumes $\bigwedge\ i.\ i \in s \implies k \leq i$

shows $lselect\ \{i.\ k + i \in s\}\ (ldropn\ k\ w) = lselect\ s\ w$

<proof>

lemma *lselect-discard-end*:

assumes $\bigwedge\ i.\ i \in s \implies i < k$

shows $lselect\ s\ (ltake\ (enat\ k)\ w) = lselect\ s\ w$

<proof>

lemma *lselect-least*:

assumes $\neg\ lnull\ (lselect\ s\ w)$

shows $lselect\ s\ w = w\ ?!\ least\ s\ \% lselect\ (s - \{least\ s\})\ w$

<proof>

lemma *lselect-lnth[simp]*:

assumes $enat\ i < llength\ (lselect\ s\ w)$

shows $lselect\ s\ w\ ?!\ i = w\ ?!\ nth\ least\ s\ i$

<proof>

lemma *lproject-lnth[simp]*:

assumes $enat\ i < llength\ (lproject\ A\ w)$

shows $lproject\ A\ w\ ?!\ i = w\ ?!\ nth\ least\ (lset\ A\ w)\ i$

<proof>

lemma *lproject-ltake[simp]*:
assumes $enat\ k \leq llength\ (lproject\ A\ w)$
shows $lproject\ A\ (ltake\ (enat\ (nth\ least\ (lift\ (liset\ A\ w))\ k))\ w) =$
 $ltake\ (enat\ k)\ (lproject\ A\ w)$
 $\langle proof \rangle$

lemma *llength-less-llength-lselect-less*:
 $enat\ i < esize\ s \wedge enat\ (nth\ least\ s\ i) < llength\ w \iff enat\ i < llength\ (lselect$
 $s\ w)$
 $\langle proof \rangle$

lemma *lselect-lselect''*:
assumes $\bigwedge i. i \in s \implies enat\ i < llength\ w$
assumes $\bigwedge i. i \in t \implies enat\ i < llength\ (lselect\ s\ w)$
shows $lselect\ t\ (lselect\ s\ w) = lselect\ (nth\ least\ s\ 't)\ w$
 $\langle proof \rangle$

lemma *lselect-lselect'[simp]*:
assumes $\bigwedge i. i \in t \implies enat\ i < esize\ s$
shows $lselect\ t\ (lselect\ s\ w) = lselect\ (nth\ least\ s\ 't)\ w$
 $\langle proof \rangle$

lemma *lselect-lselect*:
 $lselect\ t\ (lselect\ s\ w) = lselect\ (nth\ least\ s\ ' \{i \in t. enat\ i < esize\ s\})\ w$
 $\langle proof \rangle$

lemma *lselect-lproject'*:
assumes $\bigwedge i. i \in s \implies enat\ i < llength\ w$
shows $lproject\ A\ (lselect\ s\ w) = lselect\ (s \cap liset\ A\ w)\ w$
 $\langle proof \rangle$

lemma *lselect-lproject[simp]*: $lproject\ A\ (lselect\ s\ w) = lselect\ (s \cap liset\ A\ w)\ w$
 $\langle proof \rangle$

lemma *lproject-lselect-subset[simp]*:
assumes $liset\ A\ w \subseteq s$
shows $lproject\ A\ (lselect\ s\ w) = lproject\ A\ w$
 $\langle proof \rangle$

lemma *lselect-prefix[intro]*:
assumes $u \leq v$
shows $lselect\ s\ u \leq lselect\ s\ v$
 $\langle proof \rangle$

lemma *lproject-prefix[intro]*:
assumes $u \leq v$
shows $lproject\ A\ u \leq lproject\ A\ v$
 $\langle proof \rangle$

lemma *lproject-prefix-limit[intro?]*:

```

    assumes  $\bigwedge v. v \leq w \implies \text{lfinite } v \implies \text{lproject } A \ v \leq x$ 
    shows  $\text{lproject } A \ w \leq x$ 
  <proof>
  lemma lproject-prefix-limit':
    assumes  $\bigwedge k. \exists v. v \leq w \wedge \text{enat } k < \text{llength } v \wedge \text{lproject } A \ v \leq x$ 
    shows  $\text{lproject } A \ w \leq x$ 
  <proof>

end

```

15 Prefixes on Coinductive Lists

theory *LList-Prefixes*

imports

Word-Prefixes

../Extensions/Coinductive-List-Extensions

begin

```

  lemma unfold-stream-siterate-smap:  $\text{unfold-stream } f \ g = \text{smap } f \circ \text{siterate } g$ 
  <proof>

```

```

  lemma lappend-stream-of-llist:
    assumes  $\text{lfinite } u$ 
    shows  $\text{stream-of-llist } (u \ \$ \ v) = \text{list-of } u \ @- \ \text{stream-of-llist } v$ 
  <proof>

```

```

  lemma llist-of-inf-llist-prefix[intro]:  $u \leq_{FI} v \implies \text{llist-of } u \leq \text{llist-of-stream } v$ 
  <proof>

```

```

  lemma prefix-llist-of-inf-llist[intro]:  $\text{lfinite } u \implies u \leq v \implies \text{list-of } u \leq_{FI} \text{stream-of-llist } v$ 
  <proof>

```

```

  lemma lproject-prefix-limit-chain:
    assumes  $\text{chain } w \ \bigwedge k. \text{lproject } A \ (\text{llist-of } (w \ k)) \leq x$ 
    shows  $\text{lproject } A \ (\text{llist-of-stream } (\text{limit } w)) \leq x$ 
  <proof>

```

```

  lemma lproject-eq-limit-chain:
    assumes  $\text{chain } u \ \text{chain } v \ \bigwedge k. \text{project } A \ (u \ k) = \text{project } A \ (v \ k)$ 
    shows  $\text{lproject } A \ (\text{llist-of-stream } (\text{limit } u)) = \text{lproject } A \ (\text{llist-of-stream } (\text{limit } v))$ 
  <proof>

```

end

16 Stuttering

theory *Stuttering*

imports

Stuttering-Equivalence.StutterEquivalence
LList-Prefixes

begin

function *nth-least-ext* :: *nat set* \Rightarrow *nat* \Rightarrow *nat*

where

enat $k < \text{esize } A \implies \text{nth-least-ext } A \ k = \text{nth-least } A \ k \mid$

enat $k \geq \text{esize } A \implies \text{nth-least-ext } A \ k = \text{Suc } (\text{Max } A + (k - \text{card } A))$

$\langle \text{proof} \rangle$ **termination** $\langle \text{proof} \rangle$

lemma *nth-least-ext-strict-mono*:

assumes $k < l$

shows $\text{nth-least-ext } s \ k < \text{nth-least-ext } s \ l$

$\langle \text{proof} \rangle$

definition *stutter-selection* :: *nat set* \Rightarrow 'a *llist* \Rightarrow *bool*

where *stutter-selection* $s \ w \equiv 0 \in s \wedge$

$(\forall k \ i. \text{enat } i < \text{llength } w \longrightarrow \text{enat } (\text{Suc } k) < \text{esize } s \longrightarrow$

$\text{nth-least } s \ k < i \longrightarrow i < \text{nth-least } s \ (\text{Suc } k) \longrightarrow w \ ?! \ i = w \ ?! \ \text{nth-least } s \ k) \wedge$

$(\forall i. \text{enat } i < \text{llength } w \longrightarrow \text{finite } s \longrightarrow \text{Max } s < i \longrightarrow w \ ?! \ i = w \ ?! \ \text{Max } s)$

lemma *stutter-selectionI*[*intro*]:

assumes $0 \in s$

assumes $\bigwedge k \ i. \text{enat } i < \text{llength } w \implies \text{enat } (\text{Suc } k) < \text{esize } s \implies$

$\text{nth-least } s \ k < i \implies i < \text{nth-least } s \ (\text{Suc } k) \implies w \ ?! \ i = w \ ?! \ \text{nth-least } s \ k$

assumes $\bigwedge i. \text{enat } i < \text{llength } w \implies \text{finite } s \implies \text{Max } s < i \implies w \ ?! \ i = w$

$\ ?! \ \text{Max } s$

shows *stutter-selection* $s \ w$

$\langle \text{proof} \rangle$

lemma *stutter-selectionD-0*[*dest*]:

assumes *stutter-selection* $s \ w$

shows $0 \in s$

$\langle \text{proof} \rangle$

lemma *stutter-selectionD-inside*[*dest*]:

assumes *stutter-selection* $s \ w$

assumes $\text{enat } i < \text{llength } w \ \text{enat } (\text{Suc } k) < \text{esize } s$

assumes $\text{nth-least } s \ k < i \ i < \text{nth-least } s \ (\text{Suc } k)$

shows $w \ ?! \ i = w \ ?! \ \text{nth-least } s \ k$

$\langle \text{proof} \rangle$

lemma *stutter-selectionD-infinite*[*dest*]:

assumes *stutter-selection* $s \ w$

assumes $\text{enat } i < \text{llength } w \ \text{finite } s \ \text{Max } s < i$

shows $w \ ?! \ i = w \ ?! \ \text{Max } s$

$\langle \text{proof} \rangle$

lemma *stutter-selection-stutter-sampler*[*intro*]:

assumes *linfinite* w *stutter-selection* $s \ w$

shows *stutter-sampler* $(\text{nth-least-ext } s) \ (\text{lth } w)$

$\langle \text{proof} \rangle$

```

lemma stutter-equivI-selection[intro]:
  assumes linfinite u linfinite v
  assumes stutter-selection s u stutter-selection t v
  assumes lselect s u = lselect t v
  shows lnth u ≈ lnth v
  ⟨proof⟩

definition stuttering-invariant :: 'a word set ⇒ bool
  where stuttering-invariant A ≡ ∀ u v. u ≈ v ⟶ u ∈ A ⟷ v ∈ A

lemma stuttering-invariant-complement[intro!]:
  assumes stuttering-invariant A
  shows stuttering-invariant (¬ A)
  ⟨proof⟩

lemma stutter-equiv-forw-subst[trans]: w1 = w2 ⟹ w2 ≈ w3 ⟹ w1 ≈ w3
  ⟨proof⟩

lemma stutter-sampler-build:
  assumes stutter-sampler f w
  shows stutter-sampler (0 ## (Suc ∘ f)) (a ## w)
  ⟨proof⟩
lemma stutter-extend-build:
  assumes u ≈ v
  shows a ## u ≈ a ## v
  ⟨proof⟩
lemma stutter-extend-concat:
  assumes u ≈ v
  shows w ∙ u ≈ w ∙ v
  ⟨proof⟩
lemma build-stutter: w 0 ## w ≈ w
  ⟨proof⟩
lemma replicate-stutter: replicate n (v 0) ∙ v ≈ v
  ⟨proof⟩

lemma replicate-stutter': u ∙ replicate n (v 0) ∙ v ≈ u ∙ v
  ⟨proof⟩

```

end

17 Interpreted Transition Systems and Traces

```

theory Transition-System-Interpreted-Traces
imports
  Transition-System-Traces
  Basics/Stuttering
begin

```

```

locale transition-system-interpreted-traces =
  transition-system-interpreted ex en int +
  transition-system-traces ex en ind
for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
and int :: 'state  $\Rightarrow$  'interpretation
and ind :: 'action  $\Rightarrow$  'action  $\Rightarrow$  bool
+
assumes independence-invisible:  $a \in \text{visible} \Longrightarrow b \in \text{visible} \Longrightarrow \neg \text{ind } a \ b$ 
begin

lemma eq-swap-lproject-visible:
  assumes  $u =_S v$ 
  shows  $\text{lproject visible (llist-of } u) = \text{lproject visible (llist-of } v)$ 
   $\langle \text{proof} \rangle$ 
lemma eq-fin-lproject-visible:
  assumes  $u =_F v$ 
  shows  $\text{lproject visible (llist-of } u) = \text{lproject visible (llist-of } v)$ 
   $\langle \text{proof} \rangle$ 
lemma le-fin-lproject-visible:
  assumes  $u \preceq_F v$ 
  shows  $\text{lproject visible (llist-of } u) \leq \text{lproject visible (llist-of } v)$ 
   $\langle \text{proof} \rangle$ 
lemma le-fininf-lproject-visible:
  assumes  $u \preceq_{FI} v$ 
  shows  $\text{lproject visible (llist-of } u) \leq \text{lproject visible (llist-of-stream } v)$ 
   $\langle \text{proof} \rangle$ 
lemma le-inf-lproject-visible:
  assumes  $u \preceq_I v$ 
  shows  $\text{lproject visible (llist-of-stream } u) \leq \text{lproject visible (llist-of-stream } v)$ 
   $\langle \text{proof} \rangle$ 
lemma eq-inf-lproject-visible:
  assumes  $u =_I v$ 
  shows  $\text{lproject visible (llist-of-stream } u) = \text{lproject visible (llist-of-stream } v)$ 
   $\langle \text{proof} \rangle$ 

lemma stutter-selection-lproject-visible:
  assumes  $\text{run } u \ p$ 
  shows  $\text{stutter-selection (lift (lset visible (llist-of-stream } u)))$ 
   $(\text{llist-of-stream (smap int (p \#\# \text{trace } u \ p)))$ 
   $\langle \text{proof} \rangle$ 

lemma execute-fin-visible:
  assumes  $\text{path } u \ q \ \text{path } v \ q \ u \preceq_{FI} w \ v \preceq_{FI} w$ 
  assumes  $\text{project visible } u = \text{project visible } v$ 
  shows  $\text{int (target } u \ q) = \text{int (target } v \ q)$ 
   $\langle \text{proof} \rangle$ 
lemma execute-inf-visible:
  assumes  $\text{run } u \ q \ \text{run } v \ q \ u \preceq_I w \ v \preceq_I w$ 

```

```

assumes lproject visible (llist-of-stream u) = lproject visible (llist-of-stream v)
shows snth (smap int (q ## trace u q)) ≈ snth (smap int (q ## trace v q))
⟨proof⟩

```

end

end

18 Abstract Theory of Ample Set Partial Order Reduction

theory *Ample-Abstract*

imports

Transition-System-Interpreted-Traces

Extensions/Relation-Extensions

begin

locale *ample-base* =

transition-system-interpreted-traces *ex en int ind* +

wellfounded-relation *src*

for *ex* :: 'action ⇒ 'state ⇒ 'state

and *en* :: 'action ⇒ 'state ⇒ bool

and *int* :: 'state ⇒ 'interpretation

and *ind* :: 'action ⇒ 'action ⇒ bool

and *src* :: 'state ⇒ 'state ⇒ bool

begin

definition *ample-set* :: 'state ⇒ 'action set ⇒ bool

where *ample-set* *q A* ≡

$A \subseteq \{a. \text{en } a \ q\} \wedge$

$(A \subseteq \{a. \text{en } a \ q\} \longrightarrow A \neq \{\}) \wedge$

$(\forall a. A \subseteq \{a. \text{en } a \ q\} \longrightarrow a \in A \longrightarrow \text{src } (\text{ex } a \ q) \ q) \wedge$

$(A \subseteq \{a. \text{en } a \ q\} \longrightarrow A \subseteq \text{invisible}) \wedge$

$(\forall w. A \subseteq \{a. \text{en } a \ q\} \longrightarrow \text{path } w \ q \longrightarrow A \cap \text{set } w = \{\} \longrightarrow \text{Ind } A (\text{set } w))$

lemma *ample-subset*:

assumes *ample-set* *q A*

shows $A \subseteq \{a. \text{en } a \ q\}$

⟨*proof*⟩

lemma *ample-nonempty*:

assumes *ample-set* *q A* $A \subseteq \{a. \text{en } a \ q\}$

shows $A \neq \{\}$

⟨*proof*⟩

lemma *ample-wellfounded*:
assumes *ample-set* q A $A \subset \{a. \text{en } a \ q\}$ $a \in A$
shows *src* (*ex* a q) q
 $\langle \text{proof} \rangle$

lemma *ample-invisible*:
assumes *ample-set* q A $A \subset \{a. \text{en } a \ q\}$
shows $A \subseteq \text{invisible}$
 $\langle \text{proof} \rangle$

lemma *ample-independent*:
assumes *ample-set* q A $A \subset \{a. \text{en } a \ q\}$ *path* w q $A \cap \text{set } w = \{\}$
shows *Ind* A (*set* w)
 $\langle \text{proof} \rangle$

lemma *ample-en[intro]*: *ample-set* q $\{a. \text{en } a \ q\}$ $\langle \text{proof} \rangle$

end

locale *ample-abstract* =
S?: *transition-system-complete* *ex* *en* *init* *int* +
R: *transition-system-complete* *ex* *ren* *init* *int* +
ample-base *ex* *en* *int* *ind* *src*
for *ex* :: 'action \Rightarrow 'state \Rightarrow 'state
and *en* :: 'action \Rightarrow 'state \Rightarrow bool
and *init* :: 'state \Rightarrow bool
and *int* :: 'state \Rightarrow 'interpretation
and *ind* :: 'action \Rightarrow 'action \Rightarrow bool
and *src* :: 'state \Rightarrow 'state \Rightarrow bool
and *ren* :: 'action \Rightarrow 'state \Rightarrow bool
+
assumes *reduction-ample*: $q \in \text{nodes} \Longrightarrow \text{ample-set } q \ \{a. \text{ren } a \ q\}$
begin

lemma *reduction-words-fin*:
assumes $q \in \text{nodes}$ *R.path* w q
shows *S.path* w q
 $\langle \text{proof} \rangle$

lemma *reduction-words-inf*:
assumes $q \in \text{nodes}$ *R.run* w q
shows *S.run* w q
 $\langle \text{proof} \rangle$

lemma *reduction-step*:
assumes $q \in \text{nodes}$ *run* w q
obtains
(*deferred*) a **where** *ren* a q $[a] \preceq_{FI} w$ |
(*omitted*) $\{a. \text{ren } a \ q\} \subseteq \text{invisible}$ *Ind* $\{a. \text{ren } a \ q\}$ (*sset* w)

$\langle proof \rangle$

lemma *reduction-chunk*:

assumes $q \in nodes$ $run ([a] @- v) q$

obtains $b b_1 b_2 u$

where

$R.path (b @ [a]) q$

$Ind \{a\} (set b) set b \subseteq invisible$

$b =_F b_1 @ b_2 b_1 @- u =_I v Ind (set b_2) (sset u)$

$\langle proof \rangle$

inductive *reduced-run* :: $'state \Rightarrow 'action list \Rightarrow 'action stream \Rightarrow 'action list$

\Rightarrow

$'action list \Rightarrow 'action list \Rightarrow 'action list \Rightarrow 'action stream \Rightarrow bool$

where

init: $reduced-run q [] v [] [] [] v |$

absorb: $reduced-run q v_1 ([a] @- v_2) l w w_1 w_2 u \Longrightarrow a \in set l \Longrightarrow$

$reduced-run q (v_1 @ [a]) v_2 (remove1 a l) w w_1 w_2 u |$

extend: $reduced-run q v_1 ([a] @- v_2) l w w_1 w_2 u \Longrightarrow a \notin set l \Longrightarrow$

$R.path (b @ [a]) (target w q) \Longrightarrow$

$Ind \{a\} (set b) \Longrightarrow set b \subseteq invisible \Longrightarrow$

$b =_F b_1 @ b_2 \Longrightarrow [a] @- b_1 @- u' =_I u \Longrightarrow Ind (set b_2) (sset u') \Longrightarrow$

$reduced-run q (v_1 @ [a]) v_2 (l @ b_1) (w @ b @ [a]) (w_1 @ b_1 @ [a]) (w_2 @$

$b_2) u'$

lemma *reduced-run-words-fin*:

assumes $reduced-run q v_1 v_2 l w w_1 w_2 u$

shows $R.path w q$

$\langle proof \rangle$

lemma *reduced-run-invar-2*:

assumes $reduced-run q v_1 v_2 l w w_1 w_2 u$

shows $v_2 =_I l @- u$

$\langle proof \rangle$

lemma *reduced-run-invar-1*:

assumes $reduced-run q v_1 v_2 l w w_1 w_2 u$

shows $v_1 @ l =_F w_1$

$\langle proof \rangle$

lemma *reduced-run-invisible*:

assumes $reduced-run q v_1 v_2 l w w_1 w_2 u$

shows $set w_2 \subseteq invisible$

$\langle proof \rangle$

lemma *reduced-run-ind*:

assumes $reduced-run q v_1 v_2 l w w_1 w_2 u$

shows $Ind (set\ w_2) (sset\ u)$
 $\langle proof \rangle$

lemma *reduced-run-decompose*:
assumes $reduced-run\ q\ v_1\ v_2\ l\ w\ w_1\ w_2\ u$
shows $w =_F w_1 @ w_2$
 $\langle proof \rangle$

lemma *reduced-run-project*:
assumes $reduced-run\ q\ v_1\ v_2\ l\ w\ w_1\ w_2\ u$
shows $project\ visible\ w_1 = project\ visible\ w$
 $\langle proof \rangle$

lemma *reduced-run-length-1*:
assumes $reduced-run\ q\ v_1\ v_2\ l\ w\ w_1\ w_2\ u$
shows $length\ v_1 \leq length\ w_1$
 $\langle proof \rangle$

lemma *reduced-run-length*:
assumes $reduced-run\ q\ v_1\ v_2\ l\ w\ w_1\ w_2\ u$
shows $length\ v_1 \leq length\ w$
 $\langle proof \rangle$

lemma *reduced-run-step*:
assumes $q \in nodes\ run\ (v_1 @- [a] @- v_2)\ q$
assumes $reduced-run\ q\ v_1\ ([a] @- v_2)\ l\ w\ w_1\ w_2\ u$
obtains $l'\ w'\ w_1'\ w_2'\ u'$
where $reduced-run\ q\ (v_1 @ [a])\ v_2\ l'\ (w @ w')\ (w_1 @ w_1')\ (w_2 @ w_2')\ u'$
 $\langle proof \rangle$

lemma *reduction-word*:
assumes $q \in nodes\ run\ v\ q$
obtains $u\ w$
where
 $R.run\ w\ q$
 $v =_I u\ u \preceq_I w$
 $lproject\ visible\ (lstream\ u) = lproject\ visible\ (lstream\ w)$
 $\langle proof \rangle$

lemma *reduction-equivalent*:
assumes $q \in nodes\ run\ u\ q$
obtains v
where $R.run\ v\ q\ snth\ (smap\ int\ (q\ ##\ trace\ u\ q)) \approx snth\ (smap\ int\ (q\ ##\ trace\ v\ q))$
 $\langle proof \rangle$

lemma *reduction-language-subset*: $R.language \subseteq S.language$
 $\langle proof \rangle$

```

lemma reduction-language-stuttering:
  assumes  $u \in S.\text{language}$ 
  obtains  $v$ 
  where  $v \in R.\text{language}$   $\text{snth } u \approx \text{snth } v$ 
   $\langle \text{proof} \rangle$ 

```

end

end

19 LTL Formulae

theory *Formula*

imports

Basics/Stuttering

Stuttering-Equivalence.PLTL

begin

locale *formula* =

fixes $\varphi :: 'a \text{ pltl}$

begin

definition *language* :: $'a \text{ stream set}$

where $\text{language} \equiv \{w. \text{snth } w \models_p \varphi\}$

lemma *language-entails[iff]*: $w \in \text{language} \longleftrightarrow \text{snth } w \models_p \varphi$ $\langle \text{proof} \rangle$

end

locale *formula-next-free* =

formula φ

for $\varphi :: 'a \text{ pltl}$

+

assumes *next-free*: *next-free* φ

begin

lemma *stutter-equivalent-entails[dest]*: $u \approx v \implies u \models_p \varphi \longleftrightarrow v \models_p \varphi$

$\langle \text{proof} \rangle$

end

end

20 Correctness Theorem of Partial Order Reduction

theory *Ample-Correctness*

```

imports
  Ample-Abstract
  Formula
begin

  locale ample-correctness =
    S: transition-system-complete ex en init int +
    R: transition-system-complete ex ren init int +
    F: formula-next-free  $\varphi$  +
    ample-abstract ex en init int ind src ren
    for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
    and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
    and init :: 'state  $\Rightarrow$  bool
    and int :: 'state  $\Rightarrow$  'interpretation
    and ind :: 'action  $\Rightarrow$  'action  $\Rightarrow$  bool
    and src :: 'state  $\Rightarrow$  'state  $\Rightarrow$  bool
    and ren :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
    and  $\varphi$  :: 'interpretation plll
  begin

    lemma reduction-language-indistinguishable:
      assumes R.language  $\subseteq$  F.language
      shows S.language  $\subseteq$  F.language
       $\langle$ proof $\rangle$ 

    theorem reduction-correct: S.language  $\subseteq$  F.language  $\longleftrightarrow$  R.language  $\subseteq$  F.language
       $\langle$ proof $\rangle$ 

  end

end

```

21 Static Analysis for Partial Order Reduction

```

theory Ample-Analysis
imports
  Ample-Abstract
begin

  locale transition-system-ample =
    transition-system-sticky ex en init int sticky +
    transition-system-interpreted-traces ex en int ind
    for ex :: 'action  $\Rightarrow$  'state  $\Rightarrow$  'state
    and en :: 'action  $\Rightarrow$  'state  $\Rightarrow$  bool
    and init :: 'state  $\Rightarrow$  bool
    and int :: 'state  $\Rightarrow$  'interpretation
    and sticky :: 'action set
    and ind :: 'action  $\Rightarrow$  'action  $\Rightarrow$  bool
  begin

```

sublocale *ample-base ex en int ind scut*⁻¹⁻¹ $\langle proof \rangle$

lemma *restrict-ample-set*:

assumes $s \in nodes$
assumes $A \cap \{a. en a s\} \neq \{\}$ $A \cap \{a. en a s\} \cap sticky = \{\}$
assumes $Ind (A \cap \{a. en a s\}) (executable - A)$
assumes $\bigwedge w. path w s \implies A \cap \{a. en a s\} \cap set w = \{\} \implies A \cap set w = \{\}$
shows *ample-set* $s (A \cap \{a. en a s\})$
 $\langle proof \rangle$

end

locale *transition-system-concurrent* =

transition-system-initial ex en init
for $ex :: 'action \Rightarrow 'state \Rightarrow 'state$
and $en :: 'action \Rightarrow 'state \Rightarrow bool$
and $init :: 'state \Rightarrow bool$
+
fixes $procs :: 'state \Rightarrow 'process set$
fixes $pac :: 'process \Rightarrow 'action set$
fixes $psen :: 'process \Rightarrow 'state \Rightarrow 'action set$
assumes *procs-finite*: $s \in nodes \implies finite (procs s)$
assumes *psen-en*: $s \in nodes \implies pac p \cap \{a. en a s\} \subseteq psen p s$
assumes *psen-ex*: $s \in nodes \implies a \in \{a. en a s\} - pac p \implies psen p (ex a s)$
= *psen p s*
begin

lemma *psen-fin-word*:

assumes $s \in nodes path w s pac p \cap set w = \{\}$
shows $psen p (target w s) = psen p s$
 $\langle proof \rangle$

lemma *en-fin-word*:

assumes $\bigwedge r a b. r \in nodes \implies a \in psen p s - \{a. en a s\} \implies b \in \{a. en a r\} - pac p \implies$
 $en a (ex b r) \implies en a r$
assumes $s \in nodes path w s pac p \cap set w = \{\}$
shows $pac p \cap \{a. en a (target w s)\} \subseteq pac p \cap \{a. en a s\}$
 $\langle proof \rangle$

lemma *pac-en-blocked*:

assumes $\bigwedge r a b. r \in nodes \implies a \in psen p s - \{a. en a s\} \implies b \in \{a. en a r\} - pac p \implies$
 $en a (ex b r) \implies en a r$
assumes $s \in nodes path w s pac p \cap \{a. en a s\} \cap set w = \{\}$
shows $pac p \cap set w = \{\}$
 $\langle proof \rangle$

abbreviation $proc\ a \equiv \{p. a \in pac\ p\}$
abbreviation $Proc\ A \equiv \bigcup a \in A. proc\ a$

lemma *psen-simple*:

assumes $Proc\ (psen\ p\ s) = \{p\}$

assumes $\bigwedge r\ a\ b. r \in nodes \implies a \in psen\ p\ s - \{a. en\ a\ s\} \implies en\ b\ r \implies$
 $proc\ a \cap proc\ b = \{\} \implies en\ a\ (ex\ b\ r) \implies en\ a\ r$

shows $\bigwedge r\ a\ b. r \in nodes \implies a \in psen\ p\ s - \{a. en\ a\ s\} \implies b \in \{a. en\ a$
 $r\} - pac\ p \implies$
 $en\ a\ (ex\ b\ r) \implies en\ a\ r$
 $\langle proof \rangle$

end

end

References

- [1] C.-T. Chou and D. Peled. Formal verification of a partial-order reduction technique for model checking. In T. Margaria and B. Steffen, editors, *Tools and Algorithms for the Construction and Analysis of Systems*, volume 1055 of *Lecture Notes in Computer Science*, pages 241–257. Springer Berlin Heidelberg, 1996.
- [2] D. Peled. Combining partial order reductions with on-the-fly model-checking. *Formal Methods in System Design*, 8(1):39–64, 1996.