Notes on Gödel's and Scott's Variants of the Ontological Argument (Isabelle/HOL dataset)

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Abstract

Experimental studies with Isabelle/HOL on Kurt Gödel's modal ontological argument and Dana Scott's variant of it are presented. They implicitly answer some questions that the authors have received over the last decade(s). In addition, some new results are reported.

Our contribution is explained in full detail in [2]. This document presents the corresponding Isabelle/HOL dataset (which is only slightly modified to meet AFP requirements).

Contents

1 Introduction

The Isabelle/HOL dataset associated with [2] is presented. Compared to previous work on Gödel's modal ontological argument as published in the Archive of Formal Proofs (AFP) [1, 4, 3], our dataset addresses several relevant and in some cases novel aspects, which are combined here in a single publication, including:

1. For the first time, Gödel's original manuscript [5] has been formalized as closely as possible.
2. The inconsistency of the postulates in Gödel's original manuscript is explained in detail.
3. Two different ways of eliminating this inconsistency are presented, one of which is novel.
4. Scott's variant [6] of Gödel's original manuscript is presented and compared with Gödel's original variant.
5. In addition to logics S5 and K, the above variants are also studied for logic S4.

6. The above variants are tested for various combinations of of possibilist and actualist quantifiers for individuals (which has not been done systematically in previous publications).

7. Concepts of evil are examined and the derivability of evil is critically questioned.

The purpose of this AFP publication is essentially twofold. One motivation is to make the data sources associated with [2] available in a sustainable, well-maintained way. The other motivation is to support university education in higher-order modal logic by providing a small dataset for reuse that illustrates a systematically explored philosophical argument, emphasizing in particular different notions of quantification.

Compared to [2], the Isabelle sources presented here have been slightly modified to meet some AFP requirements. This concerns the commenting out of calls to sledghemmer (to reduce computational resources) and some minor reformatting (e.g. insertion of new lines). The formalization code itself remains unchanged.

2 Interactive and automated theorem proving

2.1 SurjectiveCantor.thy (Figure 2 of [2])

The surjective Cantor theorem is used in [2] to illustrate some aspects of interactive and automated theorem proving in Isabelle/HOL as relevant for the paper. To keep the provided data material complete wrt. [2], we include these data sources also here.

theory *SurjectiveCantor* **imports** *Main*
begin

Surjective Cantor theorem: traditional interactive proof

theorem *SurjectiveCantor*: $\neg(\exists G.\forall F{::}'a{\Rightarrow}bool.\exists X{::}'a.\ G\ X = F)$
⟨*proof*⟩

Avoiding proof by contradiction (Fuenmayor & Benzmüller, 2021)

theorem *SurjectiveCantor'*: $\neg(\exists G.\forall F{::}'a{\Rightarrow}bool.\exists X{::}'a.\ G\ X = F)$
⟨*proof*⟩

Surjective Cantor theorem: automated proof by some internal/external theorem provers

theorem *SurjectiveCantor''*: $\neg(\exists G.\forall F{::}'a{\Rightarrow}bool.\exists X{::}'a.\ G\ X = F)$
nitpick[*expect=none*] — no counterexample found
— sledgehammer — most internal provers give up
— sledgehammer[remote_leo2 remote_leo3] — proof found: leo provers succeed
⟨*proof*⟩

Surjective Cantor theorem (wrong formalization attempt): the types are crucial

theorem *SurjectiveCantor'''*: $\neg(\exists\, G.\forall\, F::'b.\exists\, X::'a.\ G\ X = F)$
nitpick — counterexample found for card 'a = 1 and card 'b = 1: G=(λx. (a1 := b1)
nitpick[*satisfy*, *expect=genuine*] — model found for card 'a = 1 and card 'b = 2
nitpick[*card 'a=2*, *card 'b=3*, *expect=none*] — no counterexample found
⟨*proof*⟩
end

3 Mechanization of higher-order modal logic (HOML)

3.1 HOMLinHOL.thy (Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the logic-pluralistic LogiKEy methodology. Here logic S5 is introduced.

theory *HOMLinHOL* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities
typedecl *i* — Possible worlds
typedecl *e* — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — Modal properties

consts $R::i\Rightarrow i\Rightarrow bool$ (-**r**-) — Accessibility relation between worlds

axiomatization where
Rrefl: $\forall\, x.\ x\mathbf{r}x$ **and**
Rsymm: $\forall\, x\ y.\ x\mathbf{r}y \longrightarrow y\mathbf{r}x$ **and**
Rtrans: $\forall\, x\ y\ z.\ x\mathbf{r}y \wedge y\mathbf{r}z \longrightarrow x\mathbf{r}z$

— Logical connectives (operating on truth-sets)
abbreviation $Mbot::\sigma$ (⊥) **where** $\bot \equiv \lambda w.\ False$
abbreviation $Mtop::\sigma$ (⊤) **where** $\top \equiv \lambda w.\ True$
abbreviation $Mneg::\sigma\Rightarrow\sigma$ (¬- [*52*]*53*) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation $Mand::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** ∧ *50*) **where** $\varphi\wedge\psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation $Mor::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** ∨ *49*) **where** $\varphi\vee\psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation $Mimp::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixr** ⊃ *48*) **where** $\varphi\supset\psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation $Mequiv::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** ↔ *47*) **where** $\varphi\leftrightarrow\psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation $Mbox::\sigma\Rightarrow\sigma$ (□- [*54*]*55*) **where** $\Box\varphi \equiv \lambda w.\forall\, v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation $Mdia::\sigma\Rightarrow\sigma$ (◇- [*54*]*55*) **where** $\Diamond\varphi \equiv \lambda w.\exists\, v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$

abbreviation *Mprimeq*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-=-) **where** $x = y \equiv \lambda w.\ x = y$
abbreviation *Mprimneg*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-≠-) **where** $x \neq y \equiv \lambda w.\ x \neq y$
abbreviation *Mnegpred*::$\tau \Rightarrow \tau$ (~-) **where** $\sim\Phi \equiv \lambda x.\lambda w.\ \neg\Phi\ x\ w$
abbreviation *Mconpred*::$\tau \Rightarrow \tau \Rightarrow \tau$ (**infixl .** *50*) **where** $\Phi.\Psi \equiv \lambda x.\lambda w.\ \Phi\ x\ w \wedge \Psi\ x\ w$
abbreviation *Mexclor*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee^e$ *49*) **where** $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation *Mallposs*::$('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall$) **where** $\forall\Phi \equiv \lambda w.\forall x.\ \Phi\ x\ w$
abbreviation *Mallpossb* (**binder** $\forall$ [*8*]*9*) **where** $\forall x.\ \varphi(x) \equiv \forall\varphi$
abbreviation *Mexiposs*::$('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists$) **where** $\exists\Phi \equiv \lambda w.\exists x.\ \Phi\ x\ w$
abbreviation *Mexipossb* (**binder** $\exists$ [*8*]*9*) **where** $\exists x.\ \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)
consts *existsAt*::$e \Rightarrow \sigma$ (-@-)
abbreviation *Mallact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall^E$) **where** $\forall^E\Phi \equiv \lambda w.\forall x.\ x@w \longrightarrow \Phi\ x\ w$
abbreviation *Mallactb* (**binder** $\forall^E$ [*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E\varphi$
abbreviation *Mexiact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists^E$) **where** $\exists^E\Phi \equiv \lambda w.\exists x.\ x@w \wedge \Phi\ x\ w$
abbreviation *Mexiactb* (**binder** $\exists^E$ [*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)
abbreviation *Mleibeq*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-≡-) **where** $x \equiv y \equiv \forall P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation *Mvalid*::$\sigma \Rightarrow bool$ (⌊-⌋) **where** $\lfloor\psi\rfloor \equiv \forall w.\ \psi\ w$

end

3.2 TestsHOML.thy (Figure 4 of [2])

Tests and verifications of properties for the embedding of HOML (S5) in HOL.

theory *TestsHOML* **imports** *HOMLinHOL*
begin

— Test for S5 modal logic
lemma *axM*: $\lfloor\Box\varphi \supset \varphi\rfloor$ ⟨*proof*⟩
lemma *axD*: $\lfloor\Box\varphi \supset \Diamond\varphi\rfloor$ ⟨*proof*⟩
lemma *axB*: $\lfloor\varphi \supset \Box\Diamond\varphi\rfloor$ ⟨*proof*⟩
lemma *ax4*: $\lfloor\Box\varphi \supset \Box\Box\varphi\rfloor$ ⟨*proof*⟩
lemma *ax5*: $\lfloor\Diamond\varphi \supset \Box\Diamond\varphi\rfloor$ ⟨*proof*⟩
lemma *BarcanAct*: $\lfloor(\forall^E x.\Box(\varphi\ x)) \supset \Box(\forall^E x.(\varphi\ x))\rfloor$
  nitpick[*expect=genuine*] ⟨*proof*⟩
lemma *ConvBarcanAct*: $\lfloor\Box(\forall^E x.(\varphi\ x)) \supset (\forall^E x.\Box(\varphi\ x))\rfloor$
  nitpick[*expect=genuine*] ⟨*proof*⟩
lemma *BarcanPoss*: $\lfloor(\forall x.\Box(\varphi\ x)) \supset \Box(\forall x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩
lemma *ConvBarcanPoss*: $\lfloor\Box(\forall x.(\varphi\ x)) \supset (\forall x.\Box(\varphi\ x))\rfloor$ ⟨*proof*⟩
lemma *Hilbert-A1*: $\lfloor A \supset (B \supset A)\rfloor$ ⟨*proof*⟩

lemma *Hilbert-A2*: $\lfloor(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))\rfloor$ ⟨*proof*⟩
lemma *Hilbert-MP*: **assumes** $\lfloor A \rfloor$ **and** $\lfloor A \supset B\rfloor$ **shows** $\lfloor B\rfloor$ ⟨*proof*⟩
lemma *Quant-1*: **assumes** $\lfloor A\rfloor$ **shows** $\lfloor\forall\, x{::}'a.\ A\rfloor$ ⟨*proof*⟩
lemma *ExImPossibilist1*: $\lfloor\exists\, x{::}e.\ x = x\rfloor$ ⟨*proof*⟩
lemma *ExImPossibilist2*: $\lfloor\exists\, x{::}e.\ x \equiv x\rfloor$ ⟨*proof*⟩
lemma *ExImPossibilist3*: $\lfloor\exists\, x{::}e.\ x = t\rfloor$ ⟨*proof*⟩
lemma *ExImPossibilist4*: $\lfloor\exists\, x{::}'a.\ x \equiv t{::}'a\rfloor$ ⟨*proof*⟩
lemma *ExImPossibilist*: $\lfloor\exists\, x{::}'a.\ \top\rfloor$ ⟨*proof*⟩
lemma *Quant-2*: **assumes** $\lfloor A\rfloor$ **shows** $\lfloor\forall^E x{::}e.\ A\rfloor$ ⟨*proof*⟩
lemma *ExImActualist1*: $\lfloor\exists^E x{::}e.\ x = x\rfloor$
  nitpick[*card=1*,*expect=genuine*] ⟨*proof*⟩
lemma *ExImActualist2*: $\lfloor\exists^E x{::}e.\ x \equiv x\rfloor$
  nitpick[*card=1*,*expect=genuine*] ⟨*proof*⟩
lemma *ExImActualist3*: $\lfloor\exists^E x{::}e.\ x = t\rfloor$
  nitpick[*card=1*,*expect=genuine*] ⟨*proof*⟩
lemma *ExImActualist*: $\lfloor\exists^E x{::}e.\ \top\rfloor$
  nitpick[*card=1*,*expect=genuine*] ⟨*proof*⟩
lemma *EqRefl*: $\lfloor x = x\rfloor$ ⟨*proof*⟩
lemma *EqSym*: $\lfloor(x = y) \leftrightarrow (y = x)\rfloor$ ⟨*proof*⟩
lemma *EqTrans*: $\lfloor((x = y) \wedge (y = z)) \supset (x = z)\rfloor$ ⟨*proof*⟩
lemma *EQCong*: $\lfloor(x = y) \supset ((\varphi\ x) = (\varphi\ y))\rfloor$ ⟨*proof*⟩
lemma *EQFuncExt*: $\lfloor(\varphi = \psi) \supset (\forall\, x.\ ((\varphi\ x) = (\psi\ x)))\rfloor$ ⟨*proof*⟩
lemma *EQBoolExt1*: $\lfloor(\varphi = \psi) \supset (\varphi \leftrightarrow \psi)\rfloor$ ⟨*proof*⟩
lemma *EQBoolExt2*: $\lfloor(\varphi \leftrightarrow \psi) \supset (\varphi = \psi)\rfloor$
  nitpick[*card=2*] ⟨*proof*⟩
lemma *EQBoolExt3*: $\lfloor(\varphi \leftrightarrow \psi)\rfloor \longrightarrow \lfloor(\varphi = \psi)\rfloor$ ⟨*proof*⟩
lemma *EqPrimLeib*: $\lfloor(x = y) \leftrightarrow (x \equiv y)\rfloor$ ⟨*proof*⟩
lemma *Comprehension1*: $\lfloor\exists\,\varphi.\ \forall\, x.\ (\varphi\ x) \leftrightarrow A\rfloor$ ⟨*proof*⟩
lemma *Comprehension2*: $\lfloor\exists\,\varphi.\ \forall\, x.\ (\varphi\ x) \leftrightarrow (A\ x)\rfloor$ ⟨*proof*⟩
lemma *Comprehension3*: $\lfloor\exists\,\varphi.\ \forall\, x\ y.\ (\varphi\ x\ y) \leftrightarrow (A\ x\ y)\rfloor$ ⟨*proof*⟩
lemma *ModalCollapse*: $\lfloor\forall\,\varphi.\ \varphi \supset \Box\varphi\rfloor$
  nitpick[*card=2*,*expect=genuine*] ⟨*proof*⟩
lemma *TruePropertyAndSelfIdentity*: $\lfloor(\lambda x{::}e.\ \top) = (\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *EmptyPropertyAndSelfDifference*: $\lfloor(\lambda x{::}e.\ \bot) = (\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *EmptyProperty2*: $\lfloor\exists\, x.\ \varphi\ x\rfloor \longrightarrow \lfloor\varphi \neq (\lambda x{::}e.\ \bot)\rfloor$ ⟨*proof*⟩
lemma *EmptyProperty3*: $\lfloor\exists^E x.\ \varphi\ x\rfloor \longrightarrow \lfloor\varphi \neq (\lambda x{::}e.\ \bot)\rfloor$ ⟨*proof*⟩
lemma *EmptyProperty4*: $\lfloor\varphi \neq (\lambda x{::}e.\ \bot)\rfloor \longrightarrow \lfloor\exists\, x.\ \varphi\ x\rfloor$
  nitpick[*expect=genuine*] ⟨*proof*⟩
lemma *EmptyProperty5*: $\lfloor\varphi \neq (\lambda x{::}e.\ \bot)\rfloor \longrightarrow \lfloor\exists^E x.\ \varphi\ x\rfloor$
  nitpick[*expect=genuine*] ⟨*proof*⟩

end

3.3 ModalFilter.thy (Figure 5 of [2])

Set filter and ultrafilter formalized for our modal logic setting.

theory *ModalFilter* **imports** *HOMLinHOL*
begin

type-synonym $\tau = e \Rightarrow \sigma$
abbreviation *Element*::$\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma$ (**infix** $\in$ *90*) **where** $\varphi \in S \equiv S\ \varphi$
abbreviation *EmptySet*::τ ($\emptyset$) **where** $\emptyset \equiv \lambda x.\ \bot$
abbreviation *UniversalSet*::τ (**U**) **where** $\mathbf{U} \equiv \lambda x.\ \top$
abbreviation *Subset*::$\tau \Rightarrow \tau \Rightarrow \sigma$ (**infix** $\subseteq$ *80*)
where $\varphi \subseteq \psi \equiv \forall x.((\varphi\ x) \supset (\psi\ x))$
abbreviation *SubsetE*::$\tau \Rightarrow \tau \Rightarrow \sigma$ (**infix** $\subseteq^E$ *80*)
where $\varphi \subseteq^E \psi \equiv \forall^E x.((\varphi\ x) \supset (\psi\ x))$
abbreviation *Intersection*::$\tau \Rightarrow \tau \Rightarrow \tau$ (**infix** $\sqcap$ *91*)
where $\varphi \sqcap \psi \equiv \lambda x.((\varphi\ x) \wedge (\psi\ x))$
abbreviation *Inverse*::$\tau \Rightarrow \tau$ ($^{-1}$)
where $^{-1}\psi \equiv \lambda x.\ \neg(\psi\ x)$
abbreviation *Filter* $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge$
$(\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \varphi \subseteq^E \psi \supset \psi \in \Phi) \wedge (\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$
abbreviation *UFilter* $\Phi \equiv$ *Filter* $\Phi \wedge (\forall \varphi.\ \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$
abbreviation *FilterP* $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge (\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \varphi \subseteq \psi \supset \psi \in \Phi) \wedge$
$(\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$
abbreviation *UFilterP* $\Phi \equiv$ *FilterP* $\Phi \wedge (\forall \varphi.\ \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$

end

4 Gödel's ontological argument – 1970 manuscript

4.1 GoedelVariantHOML1.thy (Figure 6 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML1* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \mathbin{.} \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** φ *Ess.* $x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G$ *Ess.* $x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ \mathit{Ess.}\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi \rfloor$

lemma *True* **nitpick**[*satisfy*,*expect=unknown*] ⟨*proof*⟩

lemma *EmptyEssL*: $\lfloor(\lambda y.\bot)\ \mathit{Ess.}\ x\rfloor$ ⟨*proof*⟩

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
⟨*proof*⟩

end

4.2 GoedelVariantHOML2.thy (Figure 7 of [2])

After an appropriate modification of the definition of essence in Gödel's 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML2* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* ($\text{-}\supset_N\text{-}$) **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ \mathit{Ess.}\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi) \rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (*E*) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y) \rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y) \rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot) \rfloor$] ⟨*proof*⟩

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor (PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi \rfloor$

lemma *L*: $\lfloor P\ G \rfloor$ ⟨*proof*⟩

theorem *Th4*: $\lfloor \Diamond(\exists^E x.\ G\ x) \rfloor$ ⟨*proof*⟩

theorem *Th5*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x) \rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi) \rfloor$ **nitpick**[*card i=1*] ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x) \rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y \rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P \rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P \rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot) \rfloor$] ⟨*proof*⟩

end

4.3 GoedelVariantHOML3.thy (Figure 8 of [2])

After an appropriate modification of the notion of necessary property inclusion in Gödel's 1970 ontological proof, the inconsistency revealed in Figure

6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML3* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^{\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ ⟨*proof*⟩

theorem *Th4*: $\lfloor \Diamond(\exists^E x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) ⟨*proof*⟩

axiomatization where *Th4*: $\lfloor \Diamond(\exists^E x.\ G\ x) \rfloor$

theorem *Th5*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x) \rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x) \rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y \rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P \rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P \rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top) \rfloor$] ⟨*proof*⟩

end

4.4 ThereIsNoEvil1.thy (Figure 10 of [2])

Importing Gödel's modified axioms from Figure 7 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

theory *ThereIsNoEvil1* **imports** *GoedelVariantHOML2*
begin

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall \varphi.\ \neg\ P\ \varphi \supset \varphi\ x$

theorem *NecNoEvil*: $\lfloor \Box(\neg(\exists^E x.\ \ Evil\ x)) \rfloor$
— sledgehammer(Ax2a Ax4 Evil_def) — Proof found
⟨*proof*⟩

end

4.5 ThereIsNoEvil2.thy (Figure 11 of [2])

Importing Gödel's modified axioms from Figure 8 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

theory *ThereIsNoEvil2* **imports** *GoedelVariantHOML3*
begin

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall \varphi.\ \neg\ P\ \varphi \supset \varphi\ x$
theorem *NecNoEvil*: $\lfloor \Box(\neg(\exists^E x.\ \ Evil\ x)) \rfloor$
— sledgehammer(Ax1Gen Ax2a Evil_def) ⟨*proof*⟩

end

5 Scott's variant

5.1 ScottVariantHOML.thy (Figure 12 of [2])

Verification of Scott's variant of Gödel's ontological argument. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *ScottVariantHOML* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x) \rfloor$ $\langle proof \rangle$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists^E x.\ G\ x) \rfloor$ $\langle proof \rangle$

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ $\langle proof \rangle$

definition *NecExist* (NE) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top) \rfloor$] $\langle proof \rangle$

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$
— sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
$\langle proof \rangle$

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
$\langle proof \rangle$

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x) \rfloor$ $\langle proof \rangle$

lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=1*] ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x)\rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

end

5.2 HOMLinHOLonlyK.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic K is introduced.

theory *HOMLinHOLonlyK* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities
typedecl *i* — Possible worlds
typedecl *e* — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — modal properties

consts $R{::}i \Rightarrow i \Rightarrow bool$ (-**r**-) — Accessibility relation between worlds

— Logical connectives (operating on truth-sets)
abbreviation $Mbot{::}\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$
abbreviation $Mtop{::}\sigma$ ($\top$) **where** $\top \equiv \lambda w.\ True$
abbreviation $Mneg{::}\sigma\Rightarrow\sigma$ ($\neg$- [52]53) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation $Mand{::}\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\wedge$ 50) **where** $\varphi\wedge\psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation $Mor{::}\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\vee$ 49) **where** $\varphi\vee\psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation $Mimp{::}\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixr** $\supset$ 48) **where** $\varphi\supset\psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation $Mequiv{::}\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\leftrightarrow$ 47) **where** $\varphi\leftrightarrow\psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation $Mbox{::}\sigma\Rightarrow\sigma$ ($\Box$- [54]55) **where** $\Box\varphi \equiv \lambda w.\forall v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation $Mdia{::}\sigma\Rightarrow\sigma$ ($\Diamond$- [54]55) **where** $\Diamond\varphi \equiv \lambda w.\exists v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$
abbreviation $Mprimeq{::}'a\Rightarrow'a\Rightarrow\sigma$ (-**=**-) **where** $x\mathbf{=}y \equiv \lambda w.\ x{=}y$
abbreviation $Mprimneg{::}'a\Rightarrow'a\Rightarrow\sigma$ (-$\neq$-) **where** $x\neq y \equiv \lambda w.\ x\neq y$
abbreviation $Mnegpred{::}\tau\Rightarrow\tau$ ($\sim$-) **where** $\sim\Phi \equiv \lambda x.\lambda w.\ \neg\Phi\ x\ w$
abbreviation $Mconpred{::}\tau\Rightarrow\tau\Rightarrow\tau$ (**infixl .** 50) **where** $\Phi.\Psi \equiv \lambda x.\lambda w.\ \Phi\ x\ w \wedge \Psi$

$x\ w$
abbreviation *Mexclor*$::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\vee^e$ *49*) **where** $\varphi\vee^e\psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation *Mallposs*$::('a\Rightarrow\sigma)\Rightarrow\sigma$ ($\forall$) **where** $\forall\,\Phi \equiv \lambda w.\forall\, x.\ \Phi\ x\ w$
abbreviation *Mallpossb* (**binder** $\forall$ [*8*]*9*) **where** $\forall\, x.\ \varphi(x) \equiv \forall\,\varphi$
abbreviation *Mexiposs*$::('a\Rightarrow\sigma)\Rightarrow\sigma$ ($\exists$) **where** $\exists\,\Phi \equiv \lambda w.\exists\, x.\ \Phi\ x\ w$
abbreviation *Mexipossb* (**binder** $\exists$ [*8*]*9*) **where** $\exists\, x.\ \varphi(x) \equiv \exists\,\varphi$

— Actualist quantifiers (for individuals/entities)
consts *existsAt*$::e\Rightarrow\sigma$ (-@-)
abbreviation *Mallact*$::(e\Rightarrow\sigma)\Rightarrow\sigma$ ($\forall^E$) **where** $\forall^E\Phi \equiv \lambda w.\forall\, x.\ x@w \longrightarrow \Phi\ x\ w$
abbreviation *Mallactb* (**binder** $\forall^E$ [*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E\varphi$
abbreviation *Mexiact*$::(e\Rightarrow\sigma)\Rightarrow\sigma$ ($\exists^E$) **where** $\exists^E\Phi \equiv \lambda w.\exists\, x.\ x@w \wedge \Phi\ x\ w$
abbreviation *Mexiactb* (**binder** $\exists^E$ [*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)
abbreviation *Mleibeq*$::'a\Rightarrow 'a\Rightarrow\sigma$ (-$\equiv$-) **where** $x{\equiv}y \equiv \forall\, P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation *Mvalid*$::\sigma\Rightarrow bool$ ($\lfloor$-$\rfloor$) **where** $\lfloor\psi\rfloor \equiv \forall\, w.\ \psi\ w$

end

5.3 ScottVariantHOMLinK.thy (Figure 13 of [2])

Scott's variant of Gödel's argument fails for base logic K (but only it the last step).

theory *ScottVariantHOMLinK* **imports** *HOMLinHOLonlyK*
begin

consts *PositiveProperty*$::(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *A1*: $\lfloor\neg P\ \varphi \leftrightarrow P\ {\sim}\varphi\rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi\rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G\rfloor$

theorem *Coro*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$ **nitpick**[*satisfy*,*eval*=*G*] ⟨*proof*⟩

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

definition *Ess* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi$

y))

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (*NE*) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card*=*1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] ⟨*proof*⟩

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x)\rfloor$ **nitpick**[*card e*=*1*, *card i*=*2*, *eval*=*G*] ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$ **nitpick**[*card e*=*1*, *card i*=*2*, *eval*=*G*] ⟨*proof*⟩

end

5.4 ScottVariantHOMLAndersonQuant.thy (Figure 15 of [2])

Verification of Scott's variant of Gödel's argument with a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *ScottVariantHOMLAndersonQuant* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (*P*)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩

definition *God* (*G*) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists x.\ G\ x)\rfloor$ ⟨*proof*⟩

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi$
y))

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (*NE*) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card*=*1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] ⟨*proof*⟩

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x)\rfloor$
— sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi = \psi)\rfloor$ **nitpick**[*card i=2*] ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy,card=1,eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

end

6 Appendix

6.1 GoedelVariantHOML1poss.thy (Figure 16 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. In contrast to Figure 6 we here use only possibilist quantifiers and still derive falsity.

theory *GoedelVariantHOML1poss* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \boldsymbol{.}\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*expect=unknown*] ⟨*proof*⟩

lemma *EmptyEssL*: $\lfloor(\lambda y.\bot)\ Ess.\ x\rfloor$ ⟨*proof*⟩

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
⟨*proof*⟩

end

6.2 GoedelVariantHOML2poss.thy (Figure 17 of [2])

After an appropriate modification of the definition of essence, the inconsistency revealed in Figure 16 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only the modal schema B is actually needed). In contrast to Figure 7 we here use only possibilist quantifiers to obtain these results.

theory *GoedelVariantHOML2poss* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset\ G\ Ess.\ x \rfloor$ $\langle proof \rangle$

definition *NecExist* (*E*) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y) \rfloor$ $\langle proof \rangle$

theorem *Th3*: $\lfloor \Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y) \rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
$\langle proof \rangle$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset\ P\ \psi \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot) \rfloor$] $\langle proof \rangle$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset\ P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset\ \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor (PosProps\ \Phi\ \wedge\ ConjOfPropsFrom\ \varphi\ \Phi) \supset\ P\ \varphi \rfloor$

lemma *L*: $\lfloor P\ G \rfloor$ $\langle proof \rangle$

theorem *Th4*: $\lfloor \Diamond(\exists x.\ G\ x) \rfloor$ $\langle proof \rangle$

theorem *Th5*: $\lfloor \Box(\exists x.\ G\ x) \rfloor$ $\langle proof \rangle$

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
$\langle proof \rangle$

lemma *PosProps*: $\lfloor P\ (\lambda x.\top)\ \wedge\ P\ (\lambda x.\ x = x) \rfloor$ $\langle proof \rangle$
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot)\ \wedge\ \neg P(\lambda x.\ x \neq x) \rfloor$ $\langle proof \rangle$
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ $\langle proof \rangle$
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi) \rfloor$ **nitpick**[*card i=2*] $\langle proof \rangle$
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset\ y \equiv x) \rfloor$ $\langle proof \rangle$
lemma *Monotheism*: $\lfloor G\ x \wedge\ G\ y \supset\ x \equiv\ y \rfloor$ $\langle proof \rangle$
lemma *Filter*: $\lfloor FilterP\ P \rfloor$ $\langle proof \rangle$
lemma *UltraFilter*: $\lfloor UFilterP\ P \rfloor$ $\langle proof \rangle$
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot) \rfloor$] $\langle proof \rangle$

end

6.3 GoedelVariantHOML3poss.thy (Figure 18 of [2])

After an appropriate modification of the definition of necessary property implication, the inconsistency shown in Figure 16 is avoided, and the argument can be successfully verified. As shown here, this still holds when using possibilist quantifiers only.

theory *GoedelVariantHOML3poss* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi\ .\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^{\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ (\varphi\ Ess.\ x) \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ ⟨*proof*⟩

theorem *Th4*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) ⟨*proof*⟩

axiomatization where *Th4*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$

theorem *Th5*: $\lfloor \Box(\exists x.\ G\ x) \rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x) \rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi) \rfloor$ ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x) \rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y \rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor FilterP\ P \rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilterP\ P \rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\top) \rfloor$] ⟨*proof*⟩

end

6.4 ScottVariantHOMLposs.thy (Figure 19 of [2])

Scott's variant of Gödel's ontological proof is still valid when using possibilist quantifiers only.

theory *ScottVariantHOMLposs* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x) \rfloor$ ⟨*proof*⟩

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists x.\ G\ x) \rfloor$ ⟨*proof*⟩

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (NE) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

theorem *T3*: $\lfloor \Box(\exists x.\ G\ x)\rfloor$
— sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi\rfloor$
— sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=2*] ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor$*FilterP P*$\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor$*UFilterP P*$\rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

end

6.5 EvilDerivable.thy (Figure 20 of [2])

The necessary existence of an Evil-like entity proved from (controversially) modified assumptions. By rejecting Gödel's assumptions and instead postulating corresponding negative versions of them, as shown in the Figure 20, the necessary existence of Evil becomes derivable. The non-positive properties of this Evil-like entity are however identical to the positive properties of Gödel's God-like entity.

theory *EvilDerivable* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall\varphi.\ \neg\ P\ \varphi \supset \varphi\ x$

definition *Essence* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi\rfloor$

axiomatization where *A2*: $\lfloor \neg P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset \neg P\ \psi\rfloor$

axiomatization where *A4*: $\lfloor \neg P\ Evil\rfloor$

axiomatization where *A3*: $\lfloor\neg P\ \varphi \supset \Box\ (\neg P\ \varphi)\rfloor$

axiomatization where *A5*: $\lfloor\neg P\ E\rfloor$

lemma *True* **nitpick**[*satisfy*,*card i=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] ⟨*proof*⟩

theorem *T1*: $\lfloor\neg P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩

theorem *T2*: $\lfloor\Diamond(\exists^E x.\ Evil\ x)\rfloor$ ⟨*proof*⟩

theorem *T3*: $\lfloor Evil\ x \supset Evil\ Ess.\ x\rfloor$ ⟨*proof*⟩

theorem *T4*: $\lfloor\Diamond(\exists^E x.\ Evil\ x) \supset \Box(\exists^E y.\ Evil\ y)\rfloor$ ⟨*proof*⟩

theorem *T5*: $\lfloor\Box(\exists^E x.\ Evil\ x)\rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(A1 A3 T5 Evil_def Rsymm) ⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\bot) \wedge P\ (\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor\neg P(\lambda x.\top) \wedge \neg P(\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi = \psi)\rfloor$ **nitpick**[*card i=2*] ⟨*proof*⟩
lemma *Monoevilism*: $\lfloor Evil\ x \wedge Evil\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ (\lambda\ \varphi.\ \neg P\ \varphi)\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ (\lambda\ \varphi.\ \neg P\ \varphi)\rfloor$ ⟨*proof*⟩

end

7 Further Appendices

7.1 GoedelVariantHOML1AndersonQuan.thy

The same as GoedelVariantHOML1, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML1AndersonQuant* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi\ .\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall\ y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** φ *Ess.* $x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi) \rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y) \rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y) \rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi \rfloor$

lemma *True* **nitpick**[*satisfy*,*expect*=*unknown*] ⟨*proof*⟩

lemma *EmptyEssL*: $\lfloor (\lambda y.\bot)\ Ess.\ x \rfloor$ ⟨*proof*⟩

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
⟨*proof*⟩

end

7.2 GoedelVariantHOML2AndersonQuan.thy

The same as GoedelVariantHOML2, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML2AndersonQuant* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::($e \Rightarrow \sigma$)$\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \bullet \psi) \rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi \rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* ($-\supset_N-$) **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** φ *Ess.* $x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (*E*) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ ⟨*proof*⟩

theorem *Th4*: $\lfloor \Diamond(\exists^E x.\ G\ x)\rfloor$ ⟨*proof*⟩

theorem *Th5*: $\lfloor \Box(\exists^E x.\ G\ x)\rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=2*] ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

end

7.3 GoedelVariantHOML3AndersonQuan.thy

The same as GoedelVariantHOML3, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML3AndersonQuant* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \boldsymbol{.}\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^{\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall\, y{::}e.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** φ *Ess.* $x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G$ *Ess.* $x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi$ *Ess.* $x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor($*PosProps* $\Phi \wedge$ *ConjOfPropsFrom* $\varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ ⟨*proof*⟩

theorem *Th4*: $\lfloor \Diamond(\exists^E x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) ⟨*proof*⟩

axiomatization where *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$

theorem *Th5*: $\lfloor\Box(\exists^E x.\ G\ x)\rfloor$ ⟨*proof*⟩

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
⟨*proof*⟩

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x)\rfloor$ ⟨*proof*⟩
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ ⟨*proof*⟩
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ ⟨*proof*⟩
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ ⟨*proof*⟩
lemma *Filter*: $\lfloor Filter\ P\rfloor$ ⟨*proof*⟩
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ ⟨*proof*⟩
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] ⟨*proof*⟩

end

7.4 HOMLinHOLonlyS4.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic S4 is introduced.

theory *HOMLinHOLonlyS4* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities:
typedecl *i* — Possible worlds
typedecl *e* — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — modal properties

consts $R{::}i \Rightarrow i \Rightarrow bool$ (-**r**-) — Accessibility relation between worlds
axiomatization where
Rrefl: $\forall x.\ x\mathbf{r}x$ **and**
Rtrans: $\forall x\ y\ z.\ x\mathbf{r}y \wedge y\mathbf{r}z \longrightarrow x\mathbf{r}z$

— Logical connectives (operating on truth-sets)
abbreviation $Mbot{::}\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$

abbreviation *Mtop*::σ ($\top$) **where** $\top \equiv \lambda w.\ True$
abbreviation *Mneg*::$\sigma \Rightarrow \sigma$ ($\neg$- [*52*]*53*) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation *Mand*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\wedge$ *50*) **where** $\varphi \wedge \psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation *Mor*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee$ *49*) **where** $\varphi \vee \psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation *Mimp*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\supset$ *48*) **where** $\varphi \supset \psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation *Mequiv*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\leftrightarrow$ *47*) **where** $\varphi \leftrightarrow \psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation *Mbox*::$\sigma \Rightarrow \sigma$ ($\Box$- [*54*]*55*) **where** $\Box\varphi \equiv \lambda w. \forall v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation *Mdia*::$\sigma \Rightarrow \sigma$ ($\Diamond$- [*54*]*55*) **where** $\Diamond\varphi \equiv \lambda w. \exists v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$
abbreviation *Mprimeq*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-=-) **where** $x = y \equiv \lambda w.\ x = y$
abbreviation *Mprimneg*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-$\neq$-) **where** $x \neq y \equiv \lambda w.\ x \neq y$
abbreviation *Mnegpred*::$\tau \Rightarrow \tau$ ($\sim$-) **where** $\sim\Phi \equiv \lambda x. \lambda w.\ \neg\Phi\ x\ w$
abbreviation *Mconpred*::$\tau \Rightarrow \tau \Rightarrow \tau$ (**infixl** **.** *50*) **where** $\Phi.\Psi \equiv \lambda x. \lambda w.\ \Phi\ x\ w \wedge \Psi\ x\ w$
abbreviation *Mexclor*::$\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee^e$ *49*) **where** $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation *Mallposs*::$('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall$) **where** $\forall\Phi \equiv \lambda w. \forall x.\ \Phi\ x\ w$
abbreviation *Mallpossb* (**binder** $\forall$ [*8*]*9*) **where** $\forall x.\ \varphi(x) \equiv \forall\varphi$
abbreviation *Mexiposs*::$('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists$) **where** $\exists\Phi \equiv \lambda w. \exists x.\ \Phi\ x\ w$
abbreviation *Mexipossb* (**binder** $\exists$ [*8*]*9*) **where** $\exists x.\ \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)
consts *existsAt*::$e \Rightarrow \sigma$ (-@-)
abbreviation *Mallact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall^E$) **where** $\forall^E\Phi \equiv \lambda w. \forall x.\ x@w \longrightarrow \Phi\ x\ w$
abbreviation *Mallactb* (**binder** $\forall^E$ [*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E\varphi$
abbreviation *Mexiact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists^E$) **where** $\exists^E\Phi \equiv \lambda w. \exists x.\ x@w \wedge \Phi\ x\ w$
abbreviation *Mexiactb* (**binder** $\exists^E$ [*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)
abbreviation *Mleibeq*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-$\equiv$-) **where** $x \equiv y \equiv \forall P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation *Mvalid*::$\sigma \Rightarrow bool$ ($\lfloor$-$\rfloor$) **where** $\lfloor\psi\rfloor \equiv \forall w.\ \psi\ w$

end

7.5 TestsHOMLinS4.thy

Tests and verifications of properties for the embedding of HOML (S4) in HOL.

theory *TestsHOMLinS4* **imports** *HOMLinHOLonlyS4*
begin

— Test for S5 modal logic
lemma *axM*: $\lfloor\Box\varphi \supset \varphi\rfloor$ $\langle proof \rangle$
lemma *axD*: $\lfloor\Box\varphi \supset \Diamond\varphi\rfloor$ $\langle proof \rangle$
lemma *axB*: $\lfloor\varphi \supset \Box\Diamond\varphi\rfloor$
  nitpick[*expect=genuine*] $\langle proof \rangle$

lemma *ax4*: $\lfloor \Box\varphi \supset \Box\Box\varphi \rfloor$ $\langle proof \rangle$

lemma *ax5*: $\lfloor \Diamond\varphi \supset \Box\Diamond\varphi \rfloor$

nitpick[*expect=genuine*] $\langle proof \rangle$

lemma *BarcanAct*: $\lfloor (\forall^E x.\Box(\varphi\ x)) \supset \Box(\forall^E x.(\varphi\ x)) \rfloor$

nitpick[*expect=genuine*] $\langle proof \rangle$

lemma *ConvBarcanAct*: $\lfloor \Box(\forall^E x.(\varphi\ x)) \supset (\forall^E x.\Box(\varphi\ x)) \rfloor$

nitpick[*expect=genuine*] $\langle proof \rangle$

lemma *BarcanPoss*: $\lfloor (\forall x.\Box(\varphi\ x)) \supset \Box(\forall x.\ \varphi\ x) \rfloor$ $\langle proof \rangle$

lemma *ConvBarcanPoss*: $\lfloor \Box(\forall x.(\varphi\ x)) \supset (\forall x.\Box(\varphi\ x)) \rfloor$ $\langle proof \rangle$

lemma *Hilbert-A1*: $\lfloor A \supset (B \supset A) \rfloor$ $\langle proof \rangle$

lemma *Hilbert-A2*: $\lfloor (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \rfloor$ $\langle proof \rangle$

lemma *Hilbert-MP*: **assumes** $\lfloor A \rfloor$ **and** $\lfloor A \supset B \rfloor$ **shows** $\lfloor B \rfloor$ $\langle proof \rangle$

lemma *Quant-1*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall x{::}'a.\ A \rfloor$ $\langle proof \rangle$

lemma *ExImPossibilist1*: $\lfloor \exists x{::}e.\ x = x \rfloor$ $\langle proof \rangle$

lemma *ExImPossibilist2*: $\lfloor \exists x{::}e.\ x \equiv x \rfloor$ $\langle proof \rangle$

lemma *ExImPossibilist3*: $\lfloor \exists x{::}e.\ x = t \rfloor$ $\langle proof \rangle$

lemma *ExImPossibilist4*: $\lfloor \exists x{::}'a.\ x \equiv t{::}'a \rfloor$ $\langle proof \rangle$

lemma *ExImPossibilist*: $\lfloor \exists x{::}'a.\ \top \rfloor$ $\langle proof \rangle$

lemma *Quant-2*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall^E x{::}e.\ A \rfloor$ $\langle proof \rangle$

lemma *ExImActualist1*: $\lfloor \exists^E x{::}e.\ x = x \rfloor$

nitpick[*card=1,expect=genuine*] $\langle proof \rangle$

lemma *ExImActualist2*: $\lfloor \exists^E x{::}e.\ x \equiv x \rfloor$

nitpick[*card=1,expect=genuine*] $\langle proof \rangle$

lemma *ExImActualist3*: $\lfloor \exists^E x{::}e.\ x = t \rfloor$

nitpick[*card=1,expect=genuine*] $\langle proof \rangle$

lemma *ExImActualist*: $\lfloor \exists^E x{::}e.\ \top \rfloor$

nitpick[*card=1,expect=genuine*] $\langle proof \rangle$

lemma *EqRefl*: $\lfloor x = x \rfloor$ $\langle proof \rangle$

lemma *EqSym*: $\lfloor (x = y) \leftrightarrow (y = x) \rfloor$ $\langle proof \rangle$

lemma *EqTrans*: $\lfloor ((x = y) \wedge (y = z)) \supset (x = z) \rfloor$ $\langle proof \rangle$

lemma *EQCong*: $\lfloor (x = y) \supset ((\varphi\ x) = (\varphi\ y)) \rfloor$ $\langle proof \rangle$

lemma *EQFuncExt*: $\lfloor (\varphi = \psi) \supset (\forall x.\ ((\varphi\ x) = (\psi\ x))) \rfloor$ $\langle proof \rangle$

lemma *EQBoolExt1*: $\lfloor (\varphi = \psi) \supset (\varphi \leftrightarrow \psi) \rfloor$ $\langle proof \rangle$

lemma *EQBoolExt2*: $\lfloor (\varphi \leftrightarrow \psi) \supset (\varphi = \psi) \rfloor$

nitpick[*card=2*] $\langle proof \rangle$

lemma *EQBoolExt3*: $\lfloor (\varphi \leftrightarrow \psi) \rfloor \longrightarrow \lfloor (\varphi = \psi) \rfloor$ $\langle proof \rangle$

lemma *EqPrimLeib*: $\lfloor (x = y) \leftrightarrow (x \equiv y) \rfloor$ $\langle proof \rangle$

lemma *Comprehension1*: $\lfloor \exists \varphi.\ \forall x.\ (\varphi\ x) \leftrightarrow A \rfloor$ $\langle proof \rangle$

lemma *Comprehension2*: $\lfloor \exists \varphi.\ \forall x.\ (\varphi\ x) \leftrightarrow (A\ x) \rfloor$ $\langle proof \rangle$

lemma *Comprehension3*: $\lfloor \exists \varphi.\ \forall x\ y.\ (\varphi\ x\ y) \leftrightarrow (A\ x\ y) \rfloor$ $\langle proof \rangle$

lemma *ModalCollapse*: $\lfloor \forall \varphi.\ \varphi \supset \Box\varphi \rfloor$

nitpick[*card=2,expect=genuine*] $\langle proof \rangle$

lemma *TruePropertyAndSelfIdentity*: $\lfloor (\lambda x{::}e.\ \top) = (\lambda x.\ x = x) \rfloor$ $\langle proof \rangle$

lemma *EmptyPropertyAndSelfDifference*: $\lfloor (\lambda x{::}e.\ \bot) = (\lambda x.\ x \neq x) \rfloor$ $\langle proof \rangle$

lemma *EmptyProperty2*: $\lfloor \exists x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ $\langle proof \rangle$

lemma *EmptyProperty3*: $\lfloor \exists^E x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ $\langle proof \rangle$

lemma *EmptyProperty4*: $\lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor \longrightarrow \lfloor \exists x.\ \varphi\ x \rfloor$

nitpick[*expect=genuine*] $\langle proof \rangle$

lemma *EmptyProperty5*: $\lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor \longrightarrow \lfloor \exists^E x.\ \varphi\ x \rfloor$

nitpick[*expect=genuine*] ⟨*proof*⟩

end

7.6 GoedelVariantHOML1inS4.thy

The same as GoedelVariantHOML1, but now in logic S4.

theory *GoedelVariantHOML1inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* $(\text{-}Ess.\text{-})$ **where** $\varphi\ Ess.\ x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

end

7.7 GoedelVariantHOML2inS4.thy

The same as GoedelVariantHOML2, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML2inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ — nitpick sledgehammer ⟨*proof*⟩

end

7.8 GoedelVariantHOML2possInS4.thy

The same as GoedelVariantHOML2poss, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML2possInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$

abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$

axiomatization where *Ax1Gen*: $\lfloor (PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi \rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {\sim}\varphi \rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}{\supset_N}\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall\, y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi) \rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists\, x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists\, y.\ G\ y) \rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor \Diamond(\exists\, x.\ G\ x) \supset \Box(\exists\, y.\ G\ y) \rfloor$ — nitpick sledgehammer ⟨*proof*⟩

end

7.9 GoedelVariantHOML3inS4.thy

The same as GoedelVariantHOML3, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML3inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*$::(e \Rightarrow \sigma) \Rightarrow \sigma\ (P)$

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi) \rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor (PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi \rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {\sim}\varphi \rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** φ *Ess.* $x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ ⟨*proof*⟩

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ — nitpick sledgehammer ⟨*proof*⟩

end

7.10 GoedelVariantHOML3possInS4.thy

The same as GoedelVariantHOML3poss, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML3possInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \bullet \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$

abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$

axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** φ *Ess.* $x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ ⟨*proof*⟩

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ (\varphi\ Ess.\ x) \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ ⟨*proof*⟩

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ — nitpick sledgehammer ⟨*proof*⟩

end

7.11 ScottVariantHOMLinS4.thy

theory *ScottVariantHOMLinS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *A1*: $\lfloor\neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi\rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi\rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G\rfloor$

theorem *Coro*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$ ⟨*proof*⟩

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

definition *Ess* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (NE) **where** $NE\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] ⟨*proof*⟩

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x)\rfloor$ **nitpick**[*card e=1*, *card i=2*] ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi\rfloor$ **nitpick**[*card e=1*, *card i=2*] ⟨*proof*⟩

end

7.12 ScottVariantHOMLpossInS4.thy

theory *ScottVariantHOMLpossInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi\rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi\rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G\rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists x.\ G\ x)\rfloor$ ⟨*proof*⟩

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ ⟨*proof*⟩

definition *NecExist* (NE) **where** $NE\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] ⟨*proof*⟩

theorem *T3*: $\lfloor \Box(\exists x.\ G\ x)\rfloor$ **nitpick**[*card e=1*, *card i=2*] ⟨*proof*⟩

lemma *MC*: $\lfloor \varphi \supset \Box\varphi\rfloor$ **nitpick**[*card e=1*, *card i=2*] ⟨*proof*⟩

end

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