Interval Temporal Logic on Natural Numbers

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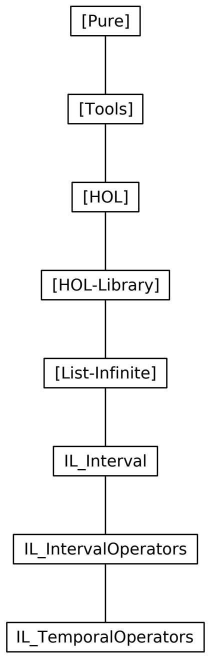
October 13, 2025

Abstract

We introduce a theory of temporal logic operators using sets of natural numbers as time domain, formalized in a shallow embedding manner. The theory comprises special natural intervals (theory IL_Interval: open and closed intervals, continuous and modulo intervals, interval traversing results), operators for shifting intervals to left/right on the number axis as well as expanding/contracting intervals by constant factors (theory IL_IntervalOperators.thy), and ultimately definitions and results for unary and binary temporal operators on arbitrary natural sets (theory IL_TemporalOperators).

Contents


[Pure]
[Tools]
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[HOL-Library]
[List-Infinite]
IL_Interval
IL_IntervalOperators
IL_TemporalOperators

1 Intervals and operations for temporal logic declarations

theory *IL-Interval*
imports
List−Infinite.InfiniteSet2
List−Infinite.SetIntervalStep
begin

1.1 Time intervals – definitions and basic lemmata

1.1.1 Definitions

type-synonym $Time = nat$

type-synonym $iT = Time\ set$

Infinite interval starting at some natural n.

definition
$iFROM :: Time \Rightarrow iT$ (‹[-...]›)
where
$[n\ldots] \equiv \{n..\}$

Finite interval starting at 0 and ending at some natural n.

definition
$iTILL :: Time \Rightarrow iT$ (‹[...-]›)
where
$[\ldots n] \equiv \{..n\}$

Finite bounded interval containing the naturals between n and $n + d$. d denotes the difference between left and right interval bound. The number of elements is $d + 1$ so that an empty interval cannot be defined.

definition
$iIN :: Time \Rightarrow nat \Rightarrow iT$ (‹[-...,-]›)
where
$[n\ldots,d] \equiv \{n..n+d\}$

Infinite modulo interval containing all naturals having the same division remainder modulo m as r, and beginning at n.

definition
$iMOD :: Time \Rightarrow nat \Rightarrow iT$ (‹[-, mod -]›)
where
$[r,\ mod\ m] \equiv \{\ x.\ x\ mod\ m = r\ mod\ m \land r \le x\}$

Finite bounded modulo interval containing all naturals having the same division remainder modulo m as r, beginning at n, and ending after c cycles at $r + m * c$. The number of elements is $c + 1$ so that an empty interval cannot be defined.

definition
  $iMODb :: Time \Rightarrow nat \Rightarrow nat \Rightarrow iT$ (‹[-, mod -, -]›)
where
  $[r, mod\ m, c] \equiv \{ x.\ x\ mod\ m = r\ mod\ m \wedge r \leq x \wedge x \leq r + m * c\}$

1.1.2 Membership in an interval

lemmas *iT-defs* = *iFROM-def iTILL-def iIN-def iMOD-def iMODb-def*

lemma *iFROM-iff*: $x \in [n\ldots] = (n \leq x)$
⟨*proof*⟩
lemma *iTILL-iff*: $x \in [\ldots n] = (x \leq n)$
⟨*proof*⟩
lemma *iIN-iff*:$x \in [n\ldots,d] = (n \leq x \wedge x \leq n + d)$
⟨*proof*⟩
lemma *iMOD-iff*: $x \in [r, mod\ m] = (x\ mod\ m = r\ mod\ m \wedge r \leq x)$
⟨*proof*⟩
lemma *iMODb-iff*: $x \in [r, mod\ m, c] =$
  $(x\ mod\ m = r\ mod\ m \wedge r \leq x \wedge x \leq r + m * c)$
⟨*proof*⟩

lemma *iFROM-D*: $x \in [n\ldots] \Longrightarrow (n \leq x)$
⟨*proof*⟩
lemma *iTILL-D*: $x \in [\ldots n] \Longrightarrow (x \leq n)$
⟨*proof*⟩
corollary *iIN-geD*: $x \in [n\ldots,d] \Longrightarrow n \leq x$
⟨*proof*⟩
corollary *iIN-leD*: $x \in [n\ldots,d] \Longrightarrow x \leq n + d$
⟨*proof*⟩
corollary *iMOD-modD*: $x \in [r, mod\ m] \Longrightarrow x\ mod\ m = r\ mod\ m$
⟨*proof*⟩
corollary *iMOD-geD*: $x \in [r, mod\ m] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-modD*: $x \in [r, mod\ m, c] \Longrightarrow x\ mod\ m = r\ mod\ m$
⟨*proof*⟩
corollary *iMODb-geD*: $x \in [r, mod\ m, c] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-leD*: $x \in [r, mod\ m, c] \Longrightarrow x \leq r + m * c$
⟨*proof*⟩

lemmas *iT-iff* = *iFROM-iff iTILL-iff iIN-iff iMOD-iff iMODb-iff*

lemmas *iT-drule* =
  iFROM-D
  iTILL-D
  iIN-geD iIN-leD
  iMOD-modD iMOD-geD

iMODb-modD iMODb-geD iMODb-leD

lemma
iFROM-I [*intro*]: $n \le x \Longrightarrow x \in [n\ldots]$ **and**
iTILL-I [*intro*]: $x \le n \Longrightarrow x \in [\ldots n]$ **and**
iIN-I [*intro*]: $n \le x \Longrightarrow x \le n + d \Longrightarrow x \in [n\ldots,d]$ **and**
iMOD-I [*intro*]: $x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \in [r, \bmod m]$ **and**
iMODb-I [*intro*]: $x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \le r + m * c \Longrightarrow x \in [r, \bmod m, c]$
⟨*proof*⟩

lemma
iFROM-E [*elim*]: $x \in [n\ldots] \Longrightarrow (n \le x \Longrightarrow P) \Longrightarrow P$ **and**
iTILL-E [*elim*]: $x \in [\ldots n] \Longrightarrow (x \le n \Longrightarrow P) \Longrightarrow P$ **and**
iIN-E [*elim*]: $x \in [n\ldots,d] \Longrightarrow (n \le x \Longrightarrow x \le n + d \Longrightarrow P) \Longrightarrow P$ **and**
iMOD-E [*elim*]: $x \in [r, \bmod m] \Longrightarrow (x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow P) \Longrightarrow P$ **and**
iMODb-E [*elim*]: $x \in [r, \bmod m, c] \Longrightarrow (x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \le r + m * c \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *iIN-Suc-insert-conv*:
insert $(Suc\ (n + d))\ [n\ldots,d] = [n\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iTILL-Suc-insert-conv*: *insert* $(Suc\ n)\ [\ldots n] = [\ldots Suc\ n]$
⟨*proof*⟩

lemma *iMODb-Suc-insert-conv*:
insert $(r + m * Suc\ c)\ [r, \bmod m, c] = [r, \bmod m, Suc\ c]$
⟨*proof*⟩

lemma *iFROM-pred-insert-conv*: *insert* $(n - Suc\ 0)\ [n\ldots] = [n - Suc\ 0\ldots]$
⟨*proof*⟩

lemma *iIN-pred-insert-conv*:
$0 < n \Longrightarrow$ *insert* $(n - Suc\ 0)\ [n\ldots,d] = [n - Suc\ 0\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iMOD-pred-insert-conv*:
$m \le r \Longrightarrow$ *insert* $(r - m)\ [r, \bmod m] = [r - m, \bmod m]$
⟨*proof*⟩

lemma *iMODb-pred-insert-conv*:
$m \le r \Longrightarrow$ *insert* $(r - m)\ [r, \bmod m, c] = [r - m, \bmod m, Suc\ c]$

⟨*proof*⟩

lemma *iFROM-Suc-pred-insert-conv*: *insert n* [*Suc n*. . .] = [*n*. . .]
⟨*proof*⟩
lemma *iIN-Suc-pred-insert-conv*: *insert n* [*Suc n*. . .,*d*] = [*n*. . .,*Suc d*]
⟨*proof*⟩
lemma *iMOD-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*] = [*r*, *mod m*]
⟨*proof*⟩
lemma *iMODb-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*, *c*] = [*r*, *mod m*, *Suc c*]
⟨*proof*⟩

lemmas *iT-Suc-insert* =
iIN-Suc-insert-conv
iTILL-Suc-insert-conv
iMODb-Suc-insert-conv
lemmas *iT-pred-insert* =
iFROM-pred-insert-conv
iIN-pred-insert-conv
iMOD-pred-insert-conv
iMODb-pred-insert-conv
lemmas *iT-Suc-pred-insert* =
iFROM-Suc-pred-insert-conv
iIN-Suc-pred-insert-conv
iMOD-Suc-pred-insert-conv
iMODb-Suc-pred-insert-conv

lemma *iMOD-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*]; *b* ∈ [*r*, *mod m*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
⟨*proof*⟩
lemma *iMODb-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*, *c*]; *b* ∈ [*r*, *mod m*, *c*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
⟨*proof*⟩

1.1.3 Interval conversions

lemma *iIN-0-iTILL-conv*:[*0*. . .,*n*] = [. . .*n*]
⟨*proof*⟩
lemma *iIN-iTILL-iTILL-conv*: *0* < *n* ⟹ [*n*. . .,*d*] = [. . .*n*+*d*] − [. . .*n* − *Suc 0*]
⟨*proof*⟩
lemma *iIN-iFROM-iTILL-conv*: [*n*. . .,*d*] = [*n*. . .] ∩ [. . .*n*+*d*]
⟨*proof*⟩
lemma *iMODb-iMOD-iTILL-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [. . .*r*+*m*∗*c*]
⟨*proof*⟩
lemma *iMODb-iMOD-iIN-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [*r*. . .,*m*∗*c*]
⟨*proof*⟩

lemma *iFROM-iTILL-iIN-conv*: *n* ≤ *n′* ⟹ [*n*. . .] ∩ [. . .*n′*] = [*n*. . .,*n′*−*n*]
⟨*proof*⟩

lemma *iMOD-iTILL-iMODb-conv*:
$r \le n \Longrightarrow [r,\ mod\ m] \cap [\ldots n] = [r,\ mod\ m,\ (n - r)\ div\ m]$
⟨*proof*⟩

lemma *iMOD-iIN-iMODb-conv*:
$[r,\ mod\ m] \cap [r\ldots,d] = [r,\ mod\ m,\ d\ div\ m]$
⟨*proof*⟩

lemma *iFROM-0*: $[0\ldots] = UNIV$
⟨*proof*⟩

lemma *iTILL-0*: $[\ldots 0] = \{0\}$
⟨*proof*⟩

lemma *iIN-0*: $[n\ldots,0] = \{n\}$
⟨*proof*⟩

lemma *iMOD-0*: $[r,\ mod\ 0] = [r\ldots,0]$
⟨*proof*⟩

lemma *iMODb-mod-0*: $[r,\ mod\ 0,\ c] = [r\ldots,0]$
⟨*proof*⟩

lemma *iMODb-0*: $[r,\ mod\ m,\ 0] = [r\ldots,0]$
⟨*proof*⟩

lemmas *iT-0* =
iFROM-0
iTILL-0
iIN-0
iMOD-0
iMODb-mod-0
iMODb-0

lemma *iMOD-1*: $[r,\ mod\ Suc\ 0] = [r\ldots]$
⟨*proof*⟩

lemma *iMODb-mod-1*: $[r,\ mod\ Suc\ 0,\ c] = [r\ldots,c]$
⟨*proof*⟩

1.1.4 Finiteness and emptiness of intervals

lemma
iFROM-not-empty: $[n\ldots] \neq \{\}$ **and**
iTILL-not-empty: $[\ldots n] \neq \{\}$ **and**
iIN-not-empty: $[n\ldots,d] \neq \{\}$ **and**
iMOD-not-empty: $[r,\ mod\ m] \neq \{\}$ **and**
iMODb-not-empty: $[r,\ mod\ m,\ c] \neq \{\}$
⟨*proof*⟩

lemmas *iT-not-empty* =
iFROM-not-empty
iTILL-not-empty
iIN-not-empty
iMOD-not-empty
iMODb-not-empty

lemma
iTILL-finite: *finite* $[\ldots n]$ **and**
iIN-finite: *finite* $[n\ldots,d]$ **and**
iMODb-finite: *finite* $[r,\ mod\ m,\ c]$ **and**
iMOD-0-finite: *finite* $[r,\ mod\ 0]$
⟨*proof*⟩

lemma *iFROM-infinite*: *infinite* $[n\ldots]$
⟨*proof*⟩

lemma *iMOD-infinite*: $0 < m \Longrightarrow$ *infinite* $[r,\ mod\ m]$
⟨*proof*⟩

lemmas *iT-finite* =
iTILL-finite
iIN-finite
iMODb-finite iMOD-0-finite

lemmas *iT-infinite* =
iFROM-infinite
iMOD-infinite

1.1.5 *Min* and *Max* element of an interval

lemma
iTILL-Min: *iMin* $[\ldots n] = 0$ **and**
iFROM-Min: *iMin* $[n\ldots] = n$ **and**
iIN-Min: *iMin* $[n\ldots,d] = n$ **and**
iMOD-Min: *iMin* $[r,\ mod\ m] = r$ **and**
iMODb-Min: *iMin* $[r,\ mod\ m,\ c] = r$
⟨*proof*⟩

lemmas *iT-Min* =
iIN-Min
iTILL-Min
iFROM-Min
iMOD-Min
iMODb-Min

lemma
iTILL-Max: *Max* $[\ldots n] = n$ **and**

iIN-Max: $Max\ [n\ldots,d] = n+d$ **and**
iMODb-Max: $Max\ [r,\ mod\ m,\ c] = r + m * c$ **and**
iMOD-0-Max: $Max\ [r,\ mod\ 0] = r$
$\langle proof \rangle$

lemmas *iT-Max* =
iTILL-Max
iIN-Max
iMODb-Max
iMOD-0-Max

lemma
iTILL-iMax: $iMax\ [\ldots n] = enat\ n$ **and**
iIN-iMax: $iMax\ [n\ldots,d] = enat\ (n+d)$ **and**
iMODb-iMax: $iMax\ [r,\ mod\ m,\ c] = enat\ (r + m * c)$ **and**
iMOD-0-iMax: $iMax\ [r,\ mod\ 0] = enat\ r$ **and**
iFROM-iMax: $iMax\ [n\ldots] = \infty$ **and**
iMOD-iMax: $0 < m \Longrightarrow iMax\ [r,\ mod\ m] = \infty$
$\langle proof \rangle$

lemmas *iT-iMax* =
iTILL-iMax
iIN-iMax
iMODb-iMax
iMOD-0-iMax
iFROM-iMax
iMOD-iMax

1.2 Adding and subtracting constants to interval elements

lemma
iFROM-plus: $x \in [n\ldots] \Longrightarrow x + k \in [n\ldots]$ **and**
iFROM-Suc: $x \in [n\ldots] \Longrightarrow Suc\ x \in [n\ldots]$ **and**
iFROM-minus: $[\![\ x \in [n\ldots];\ k \le x - n\ ]\!] \Longrightarrow x - k \in [n\ldots]$ **and**
iFROM-pred: $n < x \Longrightarrow x - Suc\ 0 \in [n\ldots]$
$\langle proof \rangle$

lemma
iTILL-plus: $[\![\ x \in [\ldots n];\ k \le n - x\ ]\!] \Longrightarrow x + k \in [\ldots n]$ **and**
iTILL-Suc: $x < n \Longrightarrow Suc\ x \in [\ldots n]$ **and**
iTILL-minus: $x \in [\ldots n] \Longrightarrow x - k \in [\ldots n]$ **and**
iTILL-pred: $x \in [\ldots n] \Longrightarrow x - Suc\ 0 \in [\ldots n]$
$\langle proof \rangle$

lemma *iIN-plus*: $[\![\ x \in [n\ldots,d];\ k \le n + d - x\ ]\!] \Longrightarrow x + k \in [n\ldots,d]$
$\langle proof \rangle$

lemma *iIN-Suc*: $[\![\ x \in [n\ldots,d];\ x < n + d\ ]\!] \Longrightarrow Suc\ x \in [n\ldots,d]$
$\langle proof \rangle$

lemma *iIN-minus*: $[\![x \in [n\ldots,d]; k \le x - n]\!] \Longrightarrow x - k \in [n\ldots,d]$
⟨*proof*⟩

lemma *iIN-pred*: $[\![x \in [n\ldots,d]; n < x]\!] \Longrightarrow x - Suc\ 0 \in [n\ldots,d]$
⟨*proof*⟩

lemma *iMOD-plus-divisor-mult*: $x \in [r,\ mod\ m] \Longrightarrow x + k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-plus-divisor*: $x \in [r,\ mod\ m] \Longrightarrow x + m \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-minus-divisor-mult*:
$[\![x \in [r,\ mod\ m]; k * m \le x - r]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-minus-divisor-mult2*:
$[\![x \in [r,\ mod\ m]; k \le (x - r)\ div\ m]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-minus-divisor*:
$[\![x \in [r,\ mod\ m]; m + r \le x]\!] \Longrightarrow x - m \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-plus*:
$x \in [r,\ mod\ m] \Longrightarrow (x + k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩
corollary *iMOD-Suc*:
$x \in [r,\ mod\ m] \Longrightarrow (Suc\ x \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMOD-minus*:
$[\![x \in [r,\ mod\ m]; k \le x - r]\!] \Longrightarrow (x - k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMOD-pred*:
$[\![x \in [r,\ mod\ m]; r < x]\!] \Longrightarrow (x - Suc\ 0 \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMODb-plus-divisor-mult*:
$[\![x \in [r,\ mod\ m,\ c]; k * m \le r + m * c - x]\!] \Longrightarrow x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-plus-divisor-mult2*:
$[\![x \in [r,\ mod\ m,\ c]; k \le c - (x - r)\ div\ m]\!] \Longrightarrow$
$x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *iMODb-plus-divisor*:

$[\![x \in [r, mod\ m, c]; x < r + m * c]\!] \Longrightarrow x + m \in [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMODb-minus-divisor-mult*:
$[\![x \in [r, mod\ m, c]; r + k * m \le x]\!] \Longrightarrow x - k * m \in [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMODb-plus*:
$[\![x \in [r, mod\ m, c]; k \le r + m * c - x]\!] \Longrightarrow$
$(x + k \in [r, mod\ m, c]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMODb-Suc*:
$[\![x \in [r, mod\ m, c]; x < r + m * c]\!] \Longrightarrow$
$(Suc\ x \in [r, mod\ m, c]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMODb-minus*:
$[\![x \in [r, mod\ m, c]; k \le x - r]\!] \Longrightarrow$
$(x - k \in [r, mod\ m, c]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMODb-pred*:
$[\![x \in [r, mod\ m, c]; r < x]\!] \Longrightarrow$
$(x - Suc\ 0 \in [r, mod\ m, c]) = (m = Suc\ 0)$
⟨*proof*⟩

lemmas *iFROM-plus-minus* =
iFROM-plus
iFROM-Suc
iFROM-minus
iFROM-pred

lemmas *iTILL-plus-minus* =
iTILL-plus
iTILL-Suc
iTILL-minus
iTILL-pred

lemmas *iIN-plus-minus* =
iIN-plus
iIN-Suc
iTILL-minus
iIN-pred

lemmas *iMOD-plus-minus-divisor* =
iMOD-plus-divisor-mult
iMOD-plus-divisor
iMOD-minus-divisor-mult

iMOD-minus-divisor-mult2
iMOD-minus-divisor

lemmas *iMOD-plus-minus* =
iMOD-plus
iMOD-Suc
iMOD-minus
iMOD-pred

lemmas *iMODb-plus-minus-divisor* =
iMODb-plus-divisor-mult
iMODb-plus-divisor-mult2
iMODb-plus-divisor
iMODb-minus-divisor-mult

lemmas *iMODb-plus-minus* =
iMODb-plus
iMODb-Suc
iMODb-minus
iMODb-pred

lemmas *iT-plus-minus* =
iFROM-plus-minus
iTILL-plus-minus
iIN-plus-minus
iMOD-plus-minus-divisor
iMOD-plus-minus
iMODb-plus-minus-divisor
iMODb-plus-minus

1.3 Relations between intervals

1.3.1 Auxiliary lemmata

lemma *Suc-in-imp-not-subset-iMOD*:
$[\![\ n \in S;\ Suc\ n \in S;\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ S \subseteq [r,\ mod\ m]$
⟨*proof*⟩

corollary *Suc-in-imp-neq-iMOD*:
$[\![\ n \in S;\ Suc\ n \in S;\ m \neq Suc\ 0\ ]\!] \Longrightarrow S \neq [r,\ mod\ m]$
⟨*proof*⟩

lemma *Suc-in-imp-not-subset-iMODb*:
$[\![\ n \in S;\ Suc\ n \in S;\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ S \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩
corollary *Suc-in-imp-neq-iMODb*:
$[\![\ n \in S;\ Suc\ n \in S;\ m \neq Suc\ 0\ ]\!] \Longrightarrow S \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

1.3.2 Subset relation between intervals

lemma
iIN-iFROM-subset-same: $[n\ldots,d] \subseteq [n\ldots]$ **and**
iIN-iTILL-subset-same: $[n\ldots,d] \subseteq [\ldots n+d]$ **and**
iMOD-iFROM-subset-same: $[r,\ mod\ m] \subseteq [r\ldots]$ **and**
iMODb-iTILL-subset-same: $[r,\ mod\ m,\ c] \subseteq [\ldots r+m*c]$ **and**
iMODb-iIN-subset-same: $[r,\ mod\ m,\ c] \subseteq [r\ldots,m*c]$ **and**
iMODb-iMOD-subset-same: $[r,\ mod\ m,\ c] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemmas *iT-subset-same* =
iIN-iFROM-subset-same
iIN-iTILL-subset-same
iMOD-iFROM-subset-same
iMODb-iTILL-subset-same
iMODb-iIN-subset-same
iMODb-iTILL-subset-same
iMODb-iMOD-subset-same

lemma *iMODb-imp-iMOD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-imp-iMODb*:
$[\![\ x \in [r,\ mod\ m];\ x \le r + m * c\ ]\!] \Longrightarrow x \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMOD-singleton-subset-conv*: $([r,\ mod\ m] \subseteq \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩
lemma *iMOD-singleton-eq-conv*: $([r,\ mod\ m] = \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩

lemma *iMODb-singleton-subset-conv*:
$([r,\ mod\ m,\ c] \subseteq \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩
lemma *iMODb-singleton-eq-conv*:
$([r,\ mod\ m,\ c] = \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩

lemma *iMODb-subset-imp-divisor-mod-0*:
$[\![\ 0 < c';\ [r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow m'\ mod\ m = 0$
⟨*proof*⟩

lemma *iMOD-subset-imp-divisor-mod-0*:
$[r',\ mod\ m'] \subseteq [r,\ mod\ m] \Longrightarrow m'\ mod\ m = 0$
⟨*proof*⟩

lemma *iMOD-subset-imp-iMODb-subset*:
$[\![\ [r',\ mod\ m'] \subseteq [r,\ mod\ m];\ r' + m' * c' \le r + m * c\ ]\!] \Longrightarrow$
$[r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]$

$\langle proof \rangle$

lemma *iMODb-subset-imp-iMOD-subset*:
$[\![\ [r', mod\ m', c'] \subseteq [r, mod\ m, c];\ 0 < c'\ ]\!] \Longrightarrow$
$[r', mod\ m'] \subseteq [r, mod\ m]$
$\langle proof \rangle$

lemma *iMODb-0-iMOD-subset-conv*:
$([r', mod\ m', 0] \subseteq [r, mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m \land r \le r')$
$\langle proof \rangle$

lemma *iFROM-subset-conv*: $([n'\ldots] \subseteq [n\ldots]) = (n \le n')$
$\langle proof \rangle$

lemma *iFROM-iMOD-subset-conv*: $([n'\ldots] \subseteq [r, mod\ m]) = (r \le n' \land m = Suc\ 0)$
$\langle proof \rangle$

lemma *iIN-subset-conv*: $([n'\ldots,d'] \subseteq [n\ldots,d]) = (n \le n' \land n'+d' \le n+d)$
$\langle proof \rangle$

lemma *iIN-iFROM-subset-conv*: $([n'\ldots,d'] \subseteq [n\ldots]) = (n \le n')$
$\langle proof \rangle$

lemma *iIN-iTILL-subset-conv*: $([n'\ldots,d'] \subseteq [\ldots n]) = (n' + d' \le n)$
$\langle proof \rangle$

lemma *iIN-iMOD-subset-conv*:
$0 < d' \Longrightarrow ([n'\ldots,d'] \subseteq [r, mod\ m]) = (r \le n' \land m = Suc\ 0)$
$\langle proof \rangle$

lemma *iIN-iMODb-subset-conv*:
$0 < d' \Longrightarrow$
$([n'\ldots,d'] \subseteq [r, mod\ m, c]) =$
$(r \le n' \land m = Suc\ 0 \land n' + d' \le r + m * c)$
$\langle proof \rangle$

lemma *iTILL-subset-conv*: $([\ldots n'] \subseteq [\ldots n]) = (n' \le n)$
$\langle proof \rangle$

lemma *iTILL-iFROM-subset-conv*: $([\ldots n'] \subseteq [n\ldots]) = (n = 0)$
$\langle proof \rangle$

lemma *iTILL-iIN-subset-conv*: $([\ldots n'] \subseteq [n\ldots,d]) = (n = 0 \land n' \le d)$
$\langle proof \rangle$

lemma *iTILL-iMOD-subset-conv*:

$0 < n' \Longrightarrow ([\ldots n'] \subseteq [r, \ mod \ m]) = (r = 0 \wedge m = Suc \ 0)$
⟨proof⟩

lemma *iTILL-iMODb-subset-conv*:
$0 < n' \Longrightarrow ([\ldots n'] \subseteq [r, \ mod \ m, \ c]) = (r = 0 \wedge m = Suc \ 0 \wedge n' \le r + m * c)$
⟨proof⟩

lemma *iMOD-iFROM-subset-conv*: $([r', \ mod \ m']) \subseteq [n\ldots] = (n \le r')$
⟨proof⟩

lemma *iMODb-iFROM-subset-conv*: $([r', \ mod \ m', \ c'] \subseteq [n\ldots]) = (n \le r')$
⟨proof⟩

lemma *iMODb-iIN-subset-conv*:
$([r', \ mod \ m', \ c'] \subseteq [n\ldots,d]) = (n \le r' \wedge r' + m' * c' \le n + d)$
⟨proof⟩

lemma *iMODb-iTILL-subset-conv*:
$([r', \ mod \ m', \ c'] \subseteq [\ldots n]) = (r' + m' * c' \le n)$
⟨proof⟩

lemma *iMOD-0-subset-conv*: $([r', \ mod \ 0] \subseteq [r, \ mod \ m]) = (r' \ mod \ m = r \ mod \ m \wedge \ r \le r')$
⟨proof⟩

lemma *iMOD-subset-conv*: $0 < m \Longrightarrow$
$([r', \ mod \ m'] \subseteq [r, \ mod \ m]) =$
$(r' \ mod \ m = r \ mod \ m \ \wedge \ r \le r' \ \wedge \ m' \ mod \ m = 0)$
⟨proof⟩

lemma *iMODb-subset-mod-0-conv*:
$([r', \ mod \ m', \ c'] \subseteq [r, \ mod \ 0, \ c\ ]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨proof⟩

lemma *iMODb-subset-0-conv*:
$([r', \ mod \ m', \ c'] \subseteq [r, \ mod \ m, \ 0\ ]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨proof⟩

lemma *iMODb-0-subset-conv*:
$([r', \ mod \ m', \ 0] \subseteq [r, \ mod \ m, \ c\ ]) = (r' \in [r, \ mod \ m, \ c])$
⟨proof⟩

lemma *iMODb-mod-0-subset-conv*:
$([r', \ mod \ 0, \ c'] \subseteq [r, \ mod \ m, \ c\ ]) = (r' \in [r, \ mod \ m, \ c])$
⟨proof⟩

lemma *iMODb-subset-conv'*: $[\![\ 0 < c; \ 0 < c' \ ]\!] \Longrightarrow$
$([r', \ mod \ m', \ c'] \subseteq [r, \ mod \ m, \ c]) =$
$(r' \ mod \ m = r \ mod \ m \ \wedge \ r \le r' \ \wedge \ m' \ mod \ m = 0 \ \wedge$

$r' + m' * c' \le r + m * c)$
⟨*proof*⟩

lemma *iMODb-subset-conv*: $[\![\ 0 < m';\ 0 < c'\ ]\!] \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \le r'\ \wedge\ m'\ mod\ m = 0\ \wedge$
$r' + m' * c' \le r + m * c)$
⟨*proof*⟩

lemma *iMODb-iMOD-subset-conv*: $0 < c' \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \le r'\ \wedge\ m'\ mod\ m = 0)$
⟨*proof*⟩

lemmas *iT-subset-conv* =
iFROM-subset-conv
iFROM-iMOD-subset-conv
iTILL-subset-conv
iTILL-iFROM-subset-conv
iTILL-iIN-subset-conv
iTILL-iMOD-subset-conv
iTILL-iMODb-subset-conv
iIN-subset-conv
iIN-iFROM-subset-conv
iIN-iTILL-subset-conv
iIN-iMOD-subset-conv
iIN-iMODb-subset-conv
iMOD-subset-conv
iMOD-iFROM-subset-conv
iMODb-subset-conv'
iMODb-subset-conv
iMODb-iFROM-subset-conv
iMODb-iIN-subset-conv
iMODb-iTILL-subset-conv
iMODb-iMOD-subset-conv

lemma *iFROM-subset*: $n \le n' \Longrightarrow [n'\ldots] \subseteq [n\ldots]$
⟨*proof*⟩

lemma *not-iFROM-iIN-subset*: $\neg\ [n'\ldots] \subseteq [n\ldots,d]$
⟨*proof*⟩

lemma *not-iFROM-iTILL-subset*: $\neg\ [n'\ldots] \subseteq [\ldots n]$
⟨*proof*⟩

lemma *not-iFROM-iMOD-subset*: $m \ne Suc\ 0 \Longrightarrow \neg\ [n'\ldots] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *not-iFROM-iMODb-subset*: $\neg\ [n'\ldots] \subseteq [r,\ mod\ m,\ c]$

⟨*proof*⟩

lemma *iIN-subset*: $\llbracket\ n \le n'; n' + d' \le n + d\ \rrbracket \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots,d]$
⟨*proof*⟩

lemma *iIN-iFROM-subset*: $n \le n' \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots]$
⟨*proof*⟩

lemma *iIN-iTILL-subset*: $n' + d' \le n \Longrightarrow [n'\ldots,d'] \subseteq [\ldots n]$
⟨*proof*⟩

lemma *not-iIN-iMODb-subset*: $\llbracket\ 0 < d'; m \ne Suc\ 0\ \rrbracket \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *not-iIN-iMOD-subset*: $\llbracket\ 0 < d'; m \ne Suc\ 0\ \rrbracket \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *iTILL-subset*: $n' \le n \Longrightarrow [\ldots n'] \subseteq [\ldots n]$
⟨*proof*⟩

lemma *iTILL-iFROM-subset*: $([\ldots n'] \subseteq [0\ldots])$
⟨*proof*⟩

lemma *iTILL-iIN-subset*: $n' \le d \Longrightarrow ([\ldots n'] \subseteq [0\ldots,d])$
⟨*proof*⟩

lemma *not-iTILL-iMOD-subset*:
$\llbracket\ 0 < n'; m \ne Suc\ 0\ \rrbracket \Longrightarrow \neg\ [\ldots n'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *not-iTILL-iMODb-subset*:
$\llbracket\ 0 < n'; m \ne Suc\ 0\ \rrbracket \Longrightarrow \neg\ [\ldots n'] \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMOD-iFROM-subset*: $n \le r' \Longrightarrow [r',\ mod\ m'] \subseteq [n\ldots]$
⟨*proof*⟩

lemma *not-iMOD-iIN-subset*: $0 < m' \Longrightarrow \neg\ [r',\ mod\ m'] \subseteq [n\ldots,d]$
⟨*proof*⟩

lemma *not-iMOD-iTILL-subset*: $0 < m' \Longrightarrow \neg\ [r',\ mod\ m'] \subseteq [\ldots n]$
⟨*proof*⟩

lemma *iMOD-subset*:
$\llbracket\ r \le r'; r'\ mod\ m = r\ mod\ m; m'\ mod\ m = 0\ \rrbracket \Longrightarrow [r',\ mod\ m'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *not-iMOD-iMODb-subset*: $0 < m' \Longrightarrow \neg\ [r', \ mod\ m'] \subseteq [r, \ mod\ m, \ c]$
$\langle proof \rangle$

lemma *iMODb-iFROM-subset*: $n \le r' \Longrightarrow [r', \ mod\ m', \ c'] \subseteq [n\ldots]$
$\langle proof \rangle$

lemma *iMODb-iTILL-subset*:
$r' + m' * c' \le n \Longrightarrow [r', \ mod\ m', \ c'] \subseteq [\ldots n]$
$\langle proof \rangle$

lemma *iMODb-iIN-subset*:
$[\![\ n \le r'; \ r' + m' * c' \le n + d\ ]\!] \Longrightarrow [r', \ mod\ m', \ c'] \subseteq [n\ldots,d]$
$\langle proof \rangle$

lemma *iMODb-iMOD-subset*:
$[\![\ r \le r'; \ r'\ mod\ m = r\ mod\ m; \ m'\ mod\ m = 0\ ]\!] \Longrightarrow [r', \ mod\ m', \ c'] \subseteq [r, \ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-subset*:
$[\![\ r \le r'; \ r'\ mod\ m = r\ mod\ m; \ m'\ mod\ m = 0; \ r' + m' * c' \le r + m * c\ ]\!] \Longrightarrow$
$[r', \ mod\ m', \ c'] \subseteq [r, \ mod\ m, \ c]$
$\langle proof \rangle$

lemma *iFROM-trans*: $[\![\ y \in [x\ldots]; \ z \in [y\ldots]\ ]\!] \Longrightarrow z \in [x\ldots]$
$\langle proof \rangle$

lemma *iTILL-trans*: $[\![\ y \in [\ldots x]; \ z \in [\ldots y]\ ]\!] \Longrightarrow z \in [\ldots x]$
$\langle proof \rangle$

lemma *iIN-trans*:
$[\![\ y \in [x\ldots,d]; \ z \in [y\ldots,d']; \ d' \le x + d - y\ ]\!] \Longrightarrow z \in [x\ldots,d]$
$\langle proof \rangle$

lemma *iMOD-trans*:
$[\![\ y \in [x, \ mod\ m]; \ z \in [y, \ mod\ m]\ ]\!] \Longrightarrow z \in [x, \ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-trans*:
$[\![\ y \in [x, \ mod\ m, \ c]; \ z \in [y, \ mod\ m, \ c']; \ m * c' \le x + m * c - y\ ]\!] \Longrightarrow$
$z \in [x, \ mod\ m, \ c]$
$\langle proof \rangle$

lemma *iMODb-trans'*:
$[\![\ y \in [x, \ mod\ m, \ c]; \ z \in [y, \ mod\ m, \ c']; \ c' \le x\ div\ m + c - y\ div\ m\ ]\!] \Longrightarrow$
$z \in [x, \ mod\ m, \ c]$
$\langle proof \rangle$

1.3.3 Equality of intervals

lemma *iFROM-eq-conv*: $([n\ldots] = [n'\ldots]) = (n = n')$
$\langle proof \rangle$

lemma *iIN-eq-conv*: $([n\ldots,d] = [n'\ldots,d']) = (n = n' \wedge d = d')$
$\langle proof \rangle$

lemma *iTILL-eq-conv*: $([\ldots n] = [\ldots n']) = (n = n')$
$\langle proof \rangle$

lemma *iMOD-0-eq-conv*: $([r,\ mod\ 0] = [r',\ mod\ m']) = (r = r' \wedge m' = 0)$
$\langle proof \rangle$

lemma *iMOD-eq-conv*: $0 < m \Longrightarrow ([r,\ mod\ m] = [r',\ mod\ m']) = (r = r' \wedge m = m')$
$\langle proof \rangle$

lemma *iMODb-mod-0-eq-conv*:
$([r,\ mod\ 0,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
$\langle proof \rangle$

lemma *iMODb-0-eq-conv*:
$([r,\ mod\ m,\ 0] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
$\langle proof \rangle$

lemma *iMODb-eq-conv*: $[\![\ 0 < m;\ 0 < c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge m = m' \wedge c = c')$
$\langle proof \rangle$

lemma *iMOD-iFROM-eq-conv*: $([n\ldots] = [r,\ mod\ m]) = (n = r \wedge m = Suc\ 0)$
$\langle proof \rangle$

lemma *iMODb-iIN-0-eq-conv*:
$([n\ldots,0] = [r,\ mod\ m,\ c]) = (n = r \wedge (m = 0 \vee c = 0))$
$\langle proof \rangle$

lemma *iMODb-iIN-eq-conv*:
$0 < d \Longrightarrow ([n\ldots,d] = [r,\ mod\ m,\ c]) = (n = r \wedge m = Suc\ 0 \wedge c = d)$
$\langle proof \rangle$

1.3.4 Inequality of intervals

lemma *iFROM-iIN-neq*: $[n'\ldots] \neq [n\ldots,d]$
$\langle proof \rangle$

corollary *iFROM-iTILL-neq*: $[n'\ldots] \neq [\ldots n]$
$\langle proof \rangle$

corollary *iFROM-iMOD-neq*: $m \neq Suc\ 0 \Longrightarrow [n\ldots] \neq [r,\ mod\ m]$

⟨*proof*⟩
corollary *iFROM-iMODb-neq*: $[n\ldots] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

corollary *iIN-iMOD-neq*: $0 < m \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m]$
⟨*proof*⟩

corollary *iIN-iMODb-neq2*: $[\![\ m \neq Suc\ 0;\ 0 < d\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iIN-iMODb-neq*: $[\![\ 2 \leq m;\ 0 < c\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iTILL-iIN-neq*: $0 < n \Longrightarrow [\ldots n'] \neq [n\ldots,d]$
⟨*proof*⟩

corollary *iTILL-iMOD-neq*: $0 < m \Longrightarrow [\ldots n] \neq [r,\ mod\ m]$
⟨*proof*⟩

corollary *iTILL-iMODb-neq*:
$[\![\ m \neq Suc\ 0;\ 0 < n\ ]\!] \Longrightarrow [\ldots n] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMOD-iMODb-neq*: $0 < m \Longrightarrow [r,\ mod\ m] \neq [r',\ mod\ m',\ c']$
⟨*proof*⟩

lemmas *iT-neq* =
iFROM-iTILL-neq iFROM-iIN-neq iFROM-iMOD-neq iFROM-iMODb-neq
iTILL-iIN-neq iTILL-iMOD-neq iTILL-iMODb-neq
iIN-iMOD-neq iIN-iMODb-neq iIN-iMODb-neq2
iMOD-iMODb-neq

1.4 Union and intersection of intervals

lemma *iFROM-union'*: $[n\ldots] \cup [n'\ldots] = [min\ n\ n'\ldots]$
⟨*proof*⟩

corollary *iFROM-union*: $n \leq n' \Longrightarrow [n\ldots] \cup [n'\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iFROM-inter'*: $[n\ldots] \cap [n'\ldots] = [max\ n\ n'\ldots]$
⟨*proof*⟩

corollary *iFROM-inter*: $n' \leq n \Longrightarrow [n\ldots] \cap [n'\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iTILL-union'*: $[\ldots n] \cup [\ldots n'] = [\ldots max\ n\ n']$
⟨*proof*⟩

corollary *iTILL-union*: $n' \le n \Longrightarrow [\ldots n] \cup [\ldots n'] = [\ldots n]$
$\langle proof \rangle$

lemma *iTILL-iFROM-union*: $n \le n' \Longrightarrow [\ldots n'] \cup [n\ldots] = UNIV$
$\langle proof \rangle$

lemma *iTILL-inter'*: $[\ldots n] \cap [\ldots n'] = [\ldots min\ n\ n']$
$\langle proof \rangle$

corollary *iTILL-inter*: $n \le n' \Longrightarrow [\ldots n] \cap [\ldots n'] = [\ldots n]$
$\langle proof \rangle$

Union and intersection for iIN only when there intersection is not empty and none of them is other's subset, for instance: .. n .. n+d .. n' .. n'+d'

lemma *iIN-union*:
$[\![n \le n'; n' \le Suc\ (n + d); n + d \le n' + d']\!] \Longrightarrow$
$[n\ldots,d] \cup [n'\ldots,d'] = [n\ldots,n' - n + d']$
$\langle proof \rangle$

lemma *iIN-append-union*:
$[n\ldots,d] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
$\langle proof \rangle$

lemma *iIN-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots,d'] = [n\ldots,Suc\ (d + d')]$
$\langle proof \rangle$

lemma *iIN-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
$\langle proof \rangle$

lemma *iIN-iFROM-union*:
$n' \le Suc\ (n + d) \Longrightarrow [n\ldots,d] \cup [n'\ldots] = [min\ n\ n'\ldots]$
$\langle proof \rangle$

lemma *iIN-iFROM-append-union*:
$[n\ldots,d] \cup [n + d\ldots] = [n\ldots]$
$\langle proof \rangle$

lemma *iIN-iFROM-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots] = [n\ldots]$
$\langle proof \rangle$

lemma *iIN-iFROM-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots] = [n\ldots]$
$\langle proof \rangle$

lemma *iIN-inter*:
$[\![n \le n'; n' \le n + d; n + d \le n' + d']\!] \Longrightarrow$
$[n\ldots,d] \cap [n'\ldots,d'] = [n'\ldots,n + d - n']$
$\langle proof \rangle$

lemma *iMOD-union*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [r,\ mod\ m]$
$\langle proof \rangle$
lemma *iMOD-union'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [min\ r\ r',\ mod\ m]$
$\langle proof \rangle$

lemma *iMOD-inter*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [r',\ mod\ m]$
$\langle proof \rangle$

lemma *iMOD-inter'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [max\ r\ r',\ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-union*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * c; r + m * c \le r' + m * c']\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [r',\ mod\ m,\ c'] = [r,\ mod\ m,\ r'\ div\ m - r\ div\ m + c']$
$\langle proof \rangle$

lemma *iMODb-append-union*:
$[r,\ mod\ m,\ c] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c + c']$
$\langle proof \rangle$

lemma *iMODb-iMOD-append-union'*:
$[\![r\ mod\ m = r'\ mod\ m; r' \le r + m * Suc\ c]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [min\ r\ r',\ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-iMOD-append-union*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * Suc\ c]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [r,\ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-append-union-Suc*:
$[r,\ mod\ m,\ c] \cup [\ r + m * Suc\ c,\ mod\ m,\ c'] = [r,\ mod\ m,\ Suc\ (c + c')]$
$\langle proof \rangle$

lemma *iMODb-append-union-pred*:
$0 < c \Longrightarrow [r,\ mod\ m,\ c - Suc\ 0] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c +$

$c']$
$\langle proof \rangle$

lemma *iMODb-inter*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * c; r + m * c \le r' + m * c']\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cap [r',\ mod\ m,\ c'] = [r',\ mod\ m,\ c - (r' - r)\ div\ m]$
$\langle proof \rangle$

lemmas *iT-union'* =
iFROM-union'
iTILL-union'
iMOD-union'
iMODb-iMOD-append-union'
lemmas *iT-union* =
iFROM-union
iTILL-union
iTILL-iFROM-union
iIN-union
iIN-iFROM-union
iMOD-union
iMODb-union
lemmas *iT-union-append* =
iIN-append-union
iIN-append-union-Suc
iIN-append-union-pred
iIN-iFROM-append-union
iIN-iFROM-append-union-Suc
iIN-iFROM-append-union-pred
iMODb-append-union
iMODb-iMOD-append-union
iMODb-append-union-Suc
iMODb-append-union-pred

lemmas *iT-inter'* =
iFROM-inter'
iTILL-inter'
iMOD-inter'
lemmas *iT-inter* =
iFROM-inter
iTILL-inter
iIN-inter
iMOD-inter
iMODb-inter

lemma *mod-partition-Union*:
$0 < m \Longrightarrow (\bigcup k.\ A \cap [k * m\ldots,m - Suc\ 0]) = A$

$\langle proof \rangle$

lemma *finite-mod-partition-Union*:
$[\![\ 0 < m;\ finite\ A\ ]\!] \Longrightarrow$
$(\bigcup k{\le}Max\ A\ div\ m.\ A \cap [k{*}m\ldots,m - Suc\ 0]) = A$
$\langle proof \rangle$

lemma *mod-partition-is-disjoint*:
$[\![\ 0 < (m::nat);\ k \neq k'\ ]\!] \Longrightarrow$
$(A \cap [k * m\ldots,m - Suc\ 0]) \cap\ (A \cap [k' * m\ldots,m - Suc\ 0]) = \{\}$
$\langle proof \rangle$

1.5 Cutting intervals

lemma *iTILL-cut-le*: $[\ldots n] \downarrow\le t = (if\ t \le n\ then\ [\ldots t]\ else\ [\ldots n])$
$\langle proof \rangle$

corollary *iTILL-cut-le1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\le t = [\ldots t]$
$\langle proof \rangle$

corollary *iTILL-cut-le2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\le t = [\ldots n]$
$\langle proof \rangle$

lemma *iFROM-cut-le*:
$[n\ldots] \downarrow\le t =$
$(if\ t < n\ then\ \{\}\ else\ [n\ldots,t{-}n])$
$\langle proof \rangle$

corollary *iFROM-cut-le1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\le t = [n\ldots,t - n]$
$\langle proof \rangle$

lemma *iIN-cut-le*:
$[n\ldots,d] \downarrow\le t = ($
$if\ t < n\ then\ \{\}\ else$
$if\ t \le n{+}d\ then\ [n\ldots,t{-}n]$
$else\ [n\ldots,d])$
$\langle proof \rangle$

corollary *iIN-cut-le1*:
$t \in [n\ldots,d] \Longrightarrow [n\ldots,d] \downarrow\le t = [n\ldots,t - n]$
$\langle proof \rangle$

lemma *iMOD-cut-le*:
$[r,\ mod\ m] \downarrow\le t = ($
$if\ t < r\ then\ \{\}$
$else\ [r,\ mod\ m,\ (t - r)\ div\ m])$
$\langle proof \rangle$

lemma *iMOD-cut-le1*:
$t \in [r, \mathit{mod}\ m] \Longrightarrow$
$[r, \mathit{mod}\ m] \downarrow\leq t = [r, \mathit{mod}\ m, (t - r)\ \mathit{div}\ m]$
⟨*proof*⟩

lemma *iMODb-cut-le*:
$[r, \mathit{mod}\ m, c] \downarrow\leq t = ($
$\mathit{if}\ t < r\ \mathit{then}\ \{\}$
$\mathit{else}\ \mathit{if}\ t < r + m * c\ \mathit{then}\ [r, \mathit{mod}\ m, (t - r)\ \mathit{div}\ m]$
$\mathit{else}\ [r, \mathit{mod}\ m, c])$
⟨*proof*⟩

lemma *iMODb-cut-le1*:
$t \in [r, \mathit{mod}\ m, c] \Longrightarrow$
$[r, \mathit{mod}\ m, c] \downarrow\leq t = [r, \mathit{mod}\ m, (t - r)\ \mathit{div}\ m]$
⟨*proof*⟩

lemma *iTILL-cut-less*:
$[\ldots n] \downarrow< t = ($
$\mathit{if}\ n < t\ \mathit{then}\ [\ldots n]\ \mathit{else}$
$\mathit{if}\ t = 0\ \mathit{then}\ \{\}$
$\mathit{else}\ [\ldots t - \mathit{Suc}\ 0])$
⟨*proof*⟩

lemma *iTILL-cut-less1*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow [\ldots n] \downarrow< t = [\ldots t - \mathit{Suc}\ 0]$
⟨*proof*⟩

lemma *iFROM-cut-less*:
$[n\ldots] \downarrow< t = ($
$\mathit{if}\ t \leq n\ \mathit{then}\ \{\}$
$\mathit{else}\ [n\ldots,t - \mathit{Suc}\ n])$
⟨*proof*⟩

lemma *iFROM-cut-less1*:
$n < t \Longrightarrow [n\ldots] \downarrow< t = [n\ldots,t - \mathit{Suc}\ n]$
⟨*proof*⟩

lemma *iIN-cut-less*:
$[n\ldots,d] \downarrow< t = ($
$\mathit{if}\ t \leq n\ \mathit{then}\ \{\}\ \mathit{else}$
$\mathit{if}\ t \leq n + d\ \mathit{then}\ [n\ldots,\ t - \mathit{Suc}\ n]$
$\mathit{else}\ [n\ldots,d])$
⟨*proof*⟩

lemma *iIN-cut-less1*:
  $[\![t \in [n\ldots,d];\ n < t]\!] \Longrightarrow [n\ldots,d] \downarrow< t = [n\ldots,\ t - Suc\ n]$
$\langle proof\rangle$

lemma *iMOD-cut-less*:
  $[r,\ mod\ m] \downarrow< t = ($
    $if\ t \le r\ then\ \{\}$
    $else\ [r,\ mod\ m,\ (t - Suc\ r)\ div\ m])$
$\langle proof\rangle$

lemma *iMOD-cut-less1*:
  $[\![t \in [r,\ mod\ m];\ r < t]\!] \Longrightarrow$
  $[r,\ mod\ m] \downarrow< t = [r,\ mod\ m,\ (t - r)\ div\ m - Suc\ 0]$
$\langle proof\rangle$

lemma *iMODb-cut-less*:
  $[r,\ mod\ m,\ c] \downarrow< t = ($
    $if\ t \le r\ then\ \{\}\ else$
    $if\ r + m * c < t\ then\ [r,\ mod\ m,\ c]$
    $else\ [r,\ mod\ m,\ (t - Suc\ r)\ div\ m])$
$\langle proof\rangle$

lemma *iMODb-cut-less1*: $[\![t \in [r,\ mod\ m,\ c];\ r < t]\!] \Longrightarrow$
  $[r,\ mod\ m,\ c] \downarrow< t = [r,\ mod\ m,\ (t - r)\ div\ m - Suc\ 0]$
$\langle proof\rangle$

lemmas *iT-cut-le* =
  iTILL-cut-le
  iFROM-cut-le
  iIN-cut-le
  iMOD-cut-le
  iMODb-cut-le

lemmas *iT-cut-le1* =
  iTILL-cut-le1
  iFROM-cut-le1
  iIN-cut-le1
  iMOD-cut-le1
  iMODb-cut-le1

lemmas *iT-cut-less* =
  iTILL-cut-less
  iFROM-cut-less
  iIN-cut-less
  iMOD-cut-less
  iMODb-cut-less

lemmas *iT-cut-less1* =
iTILL-cut-less1
iFROM-cut-less1
iIN-cut-less1
iMOD-cut-less1
iMODb-cut-less1

lemmas *iT-cut-le-less* =
iTILL-cut-le
iTILL-cut-less
iFROM-cut-le
iFROM-cut-less
iIN-cut-le
iIN-cut-less
iMOD-cut-le
iMOD-cut-less
iMODb-cut-le
iMODb-cut-less

lemmas *iT-cut-le-less1* =
iTILL-cut-le1
iTILL-cut-less1
iFROM-cut-le1
iFROM-cut-less1
iIN-cut-le1
iIN-cut-less1
iMOD-cut-le1
iMOD-cut-less1
iMODb-cut-le1
iMODb-cut-less1

lemma *iTILL-cut-ge*:
$[\ldots n] \downarrow\geq t = (\textit{if } n < t \textit{ then } \{\} \textit{ else } [t\ldots,n-t])$
$\langle proof \rangle$

corollary *iTILL-cut-ge1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = [t\ldots,n-t]$
$\langle proof \rangle$

corollary *iTILL-cut-ge2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = \{\}$
$\langle proof \rangle$

lemma *iTILL-cut-greater*:
$[\ldots n] \downarrow> t = (\textit{if } n \leq t \textit{ then } \{\} \textit{ else } [\textit{Suc } t\ldots,n - \textit{Suc } t])$
$\langle proof \rangle$

corollary *iTILL-cut-greater1*:
$t \in [\ldots n] \Longrightarrow t < n \Longrightarrow [\ldots n] \downarrow> t = [\textit{Suc } t\ldots,n - \textit{Suc } t]$
$\langle proof \rangle$

corollary *iTILL-cut-greater2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow> t = \{\}$
$\langle proof \rangle$

lemma *iFROM-cut-ge*:
$[n\ldots] \downarrow\geq t = (if\ n \leq t\ then\ [t\ldots]\ else\ [n\ldots])$
$\langle proof \rangle$
corollary *iFROM-cut-ge1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\geq t = [t\ldots]$
$\langle proof \rangle$

lemma *iFROM-cut-greater*:
$[n\ldots] \downarrow> t = (if\ n \leq t\ then\ [Suc\ t\ldots]\ else\ [n\ldots])$
$\langle proof \rangle$
corollary *iFROM-cut-greater1*:
$t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow> t = [Suc\ t\ldots]$
$\langle proof \rangle$

lemma *iIN-cut-ge*:
$[n\ldots,d] \downarrow\geq t = ($
$if\ t < n\ then\ [n\ldots,d]\ else$
$if\ t \leq n+d\ then\ [t\ldots,n+d-t]$
$else\ \{\})$
$\langle proof \rangle$

corollary *iIN-cut-ge1*: $t \in [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = [t\ldots,n+d-t]$
$\langle proof \rangle$

corollary *iIN-cut-ge2*: $t \notin [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = (if\ t < n\ then\ [n\ldots,d]\ else\ \{\})$
$\langle proof \rangle$

lemma *iIN-cut-greater*:
$[n\ldots,d] \downarrow> t = ($
$if\ t < n\ then\ [n\ldots,d]\ else$
$if\ t < n+d\ then\ [Suc\ t\ldots,n + d - Suc\ t]$
$else\ \{\})$
$\langle proof \rangle$

corollary *iIN-cut-greater1*:
$[\![\ t \in [n\ldots,d];\ t < n + d\ ]\!] \Longrightarrow$
$[n\ldots,d] \downarrow> t = [Suc\ t\ldots,n + d - Suc\ t]$
$\langle proof \rangle$

lemma *mod-cut-greater-aux-t-less*:

$\llbracket$ $0 < (m::nat)$; $r \leq t$ $\rrbracket \Longrightarrow$
$t < t + m - (t - r) \; mod \; m$
$\langle proof \rangle$

lemma *mod-cut-greater-aux-le-x*:
$\llbracket$ $(r::nat) \leq t$; $t < x$; $x \; mod \; m = r \; mod \; m \rrbracket \Longrightarrow$
$t + m - (t - r) \; mod \; m \leq x$
$\langle proof \rangle$

lemma *iMOD-cut-greater*:
$[r, \; mod \; m] \downarrow\!> t = ($
if $t < r$ *then* $[r, \; mod \; m]$ *else*
if $m = 0$ *then* $\{\}$
else $[t + m - (t - r) \; mod \; m, \; mod \; m])$
$\langle proof \rangle$

lemma *iMOD-cut-greater1*:
$t \in [r, \; mod \; m] \Longrightarrow$
$[r, \; mod \; m] \downarrow\!> t = ($
if $m = 0$ *then* $\{\}$
else $[t + m, \; mod \; m])$
$\langle proof \rangle$

lemma *iMODb-cut-greater-aux*:
$\llbracket$ $0 < m$; $t < r + m * c$; $r \leq t \rrbracket \Longrightarrow$
$(r + m * c - (t + m - (t - r) \; mod \; m)) \; div \; m =$
$c - Suc \; ((t - r) \; div \; m)$
$\langle proof \rangle$

lemma *iMODb-cut-greater*:
$[r, \; mod \; m, \; c] \downarrow\!> t = ($
if $t < r$ *then* $[r, \; mod \; m, \; c]$ *else*
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m - (t - r) \; mod \; m, \; mod \; m, \; c - Suc \; ((t-r) \; div \; m)])$
$\langle proof \rangle$

lemma *iMODb-cut-greater1*:
$t \in [r, \; mod \; m, \; c] \Longrightarrow$
$[r, \; mod \; m, \; c] \downarrow\!> t = ($
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m, \; mod \; m, \; c - Suc \; ((t-r) \; div \; m)])$
$\langle proof \rangle$

lemma *iMOD-cut-ge*:
$[r, \; mod \; m] \downarrow\!\geq t = ($
if $t \leq r$ *then* $[r, \; mod \; m]$ *else*
if $m = 0$ *then* $\{\}$

  else $[t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m])$
$\langle proof \rangle$

lemma *iMOD-cut-ge1*:
 $t \in [r,\ mod\ m] \Longrightarrow$
 $[r,\ mod\ m] \downarrow\geq t = [t,\ mod\ m]$
$\langle proof \rangle$

lemma *iMODb-cut-ge*:
 $[r,\ mod\ m,\ c] \downarrow\geq t = ($
  if $t \leq r$ *then* $[r,\ mod\ m,\ c]$ *else*
  if $r + m * c < t$ *then* $\{\}$
  else $[t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m,\ c - (t + m - Suc\ r)\ div\ m])$
$\langle proof \rangle$

lemma *iMODb-cut-ge1*:
 $t \in [r,\ mod\ m,\ c] \Longrightarrow$
 $[r,\ mod\ m,\ c] \downarrow\geq t = ($
  if $r + m * c < t$ *then* $\{\}$
  else $[t,\ mod\ m,\ c - (t - r)\ div\ m])$
$\langle proof \rangle$

lemma *iMOD-0-cut-greater*:
 $t \in [r,\ mod\ 0] \Longrightarrow [r,\ mod\ 0] \downarrow> t = \{\}$
$\langle proof \rangle$
lemma *iMODb-0-cut-greater*: $t \in [r,\ mod\ 0,\ c] \Longrightarrow$
 $[r,\ mod\ 0,\ c] \downarrow> t = \{\}$
$\langle proof \rangle$

lemmas *iT-cut-ge* =
 iTILL-cut-ge
 iFROM-cut-ge
 iIN-cut-ge
 iMOD-cut-ge
 iMODb-cut-ge

lemmas *iT-cut-ge1* =
 iTILL-cut-ge1
 iFROM-cut-ge1
 iIN-cut-ge1
 iMOD-cut-ge1
 iMODb-cut-ge1

lemmas *iT-cut-greater* =
 iTILL-cut-greater
 iFROM-cut-greater
 iIN-cut-greater

iMOD-cut-greater
iMODb-cut-greater

lemmas *iT-cut-greater1* =
iTILL-cut-greater1
iFROM-cut-greater1
iIN-cut-greater1
iMOD-cut-greater1
iMODb-cut-greater1

lemmas *iT-cut-ge-greater* =
iTILL-cut-ge
iTILL-cut-greater
iFROM-cut-ge
iFROM-cut-greater
iIN-cut-ge
iIN-cut-greater
iMOD-cut-ge
iMOD-cut-greater
iMODb-cut-ge
iMODb-cut-greater
lemmas *iT-cut-ge-greater1* =
iTILL-cut-ge1
iTILL-cut-greater1
iFROM-cut-ge1
iFROM-cut-greater1
iIN-cut-ge1
iIN-cut-greater1
iMOD-cut-ge1
iMOD-cut-greater1
iMODb-cut-ge1
iMODb-cut-greater1

1.6 Cardinality of intervals

lemma *iFROM-card*: $card\ [n\ldots] = 0$
⟨*proof*⟩

lemma *iTILL-card*: $card\ [\ldots n] = Suc\ n$
⟨*proof*⟩

lemma *iIN-card*: $card\ [n\ldots,d] = Suc\ d$
⟨*proof*⟩

lemma *iMOD-0-card*: $card\ [r,\ mod\ 0] = Suc\ 0$
⟨*proof*⟩

lemma *iMOD-card*: $0 < m \Longrightarrow card\ [r,\ mod\ m] = 0$
⟨*proof*⟩

lemma *iMOD-card-if*: *card* $[r,\ mod\ m] = (if\ m = 0\ then\ Suc\ 0\ else\ 0)$
⟨*proof*⟩

lemma *iMODb-mod-0-card*: *card* $[r,\ mod\ 0,\ c] = Suc\ 0$
⟨*proof*⟩

lemma *iMODb-card*: $0 < m \Longrightarrow card\ [r,\ mod\ m,\ c] = Suc\ c$
⟨*proof*⟩

lemma *iMODb-card-if*:
card $[r,\ mod\ m,\ c] = (if\ m = 0\ then\ Suc\ 0\ else\ Suc\ c)$
⟨*proof*⟩

lemmas *iT-card* =
iFROM-card
iTILL-card
iIN-card
iMOD-card-if
iMODb-card-if

Cardinality with *icard*

lemma *iFROM-icard*: *icard* $[n\ldots] = \infty$
⟨*proof*⟩

lemma *iTILL-icard*: *icard* $[\ldots n] = enat\ (Suc\ n)$
⟨*proof*⟩

lemma *iIN-icard*: *icard* $[n\ldots,d] = enat\ (Suc\ d)$
⟨*proof*⟩

lemma *iMOD-0-icard*: *icard* $[r,\ mod\ 0] = eSuc\ 0$
⟨*proof*⟩

lemma *iMOD-icard*: $0 < m \Longrightarrow icard\ [r,\ mod\ m] = \infty$
⟨*proof*⟩

lemma *iMOD-icard-if*: *icard* $[r,\ mod\ m] = (if\ m = 0\ then\ eSuc\ 0\ else\ \infty)$
⟨*proof*⟩

lemma *iMODb-mod-0-icard*: *icard* $[r,\ mod\ 0,\ c] = eSuc\ 0$
⟨*proof*⟩

lemma *iMODb-icard*: $0 < m \Longrightarrow icard\ [r,\ mod\ m,\ c] = enat\ (Suc\ c)$
⟨*proof*⟩

lemma *iMODb-icard-if*: *icard* $[r,\ mod\ m,\ c] = enat\ (if\ m = 0\ then\ Suc\ 0\ else\ Suc\ c)$
⟨*proof*⟩

lemmas *iT-icard* =
iFROM-icard
iTILL-icard
iIN-icard
iMOD-icard-if
iMODb-icard-if

1.7 Functions *inext* and *iprev* with intervals

lemma
iFROM-inext: $t \in [n\ldots] \Longrightarrow inext\ t\ [n\ldots] = Suc\ t$ **and**
iTILL-inext: $t < n \Longrightarrow inext\ t\ [\ldots n] = Suc\ t$ **and**
iIN-inext: $[\![\ n \le t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] = Suc\ t$
⟨*proof*⟩

lemma
iFROM-iprev': $t \in [n\ldots] \Longrightarrow iprev\ (Suc\ t)\ [n\ldots] = t$ **and**
iFROM-iprev: $n < t \Longrightarrow iprev\ t\ [n\ldots] = t - Suc\ 0$ **and**
iTILL-iprev: $t \in [\ldots n] \Longrightarrow iprev\ t\ [\ldots n] = t - Suc\ 0$ **and**
iIN-iprev: $[\![\ n < t;\ t \le n + d\ ]\!] \Longrightarrow iprev\ t\ [n\ldots,d] = t - Suc\ 0$ **and**
iIN-iprev': $[\![\ n \le t;\ t < n + d\ ]\!] \Longrightarrow iprev\ (Suc\ t)\ [n\ldots,d] = t$
⟨*proof*⟩

lemma *iMOD-inext*: $t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] = t + m$
⟨*proof*⟩
lemma *iMOD-iprev*: $[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow iprev\ t\ [r,\ mod\ m] = t - m$
⟨*proof*⟩

lemma *iMOD-iprev'*: $t \in [r,\ mod\ m] \Longrightarrow iprev\ (t + m)\ [r,\ mod\ m] = t$
⟨*proof*⟩

lemma *iMODb-inext*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$inext\ t\ [r,\ mod\ m,\ c] = t + m$
⟨*proof*⟩

lemma *iMODb-iprev*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow$
$iprev\ t\ [r,\ mod\ m,\ c] = t - m$
⟨*proof*⟩

lemma *iMODb-iprev'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$iprev\ (t + m)\ [r,\ mod\ m,\ c] = t$
⟨*proof*⟩

lemmas *iT-inext* =
iFROM-inext

iTILL-inext
iIN-inext
iMOD-inext
iMODb-inext
lemmas *iT-iprev* =
iFROM-iprev′
iFROM-iprev
iTILL-iprev
iIN-iprev
iIN-iprev′
iMOD-iprev
iMOD-iprev′
iMODb-iprev
iMODb-iprev′

lemma *iFROM-inext-if*:
$inext\ t\ [n\ldots] = (if\ t \in [n\ldots]\ then\ Suc\ t\ else\ t)$
⟨*proof*⟩

lemma *iTILL-inext-if*:
$inext\ t\ [\ldots n] = (if\ t < n\ then\ Suc\ t\ else\ t)$
⟨*proof*⟩

lemma *iIN-inext-if*:
$inext\ t\ [n\ldots,d] = (if\ n \le t \wedge t < n + d\ then\ Suc\ t\ else\ t)$
⟨*proof*⟩

lemma *iMOD-inext-if*:
$inext\ t\ [r,\ mod\ m] = (if\ t \in [r,\ mod\ m]\ then\ t + m\ else\ t)$
⟨*proof*⟩

lemma *iMODb-inext-if*:
$inext\ t\ [r,\ mod\ m,\ c] =$
$(if\ t \in [r,\ mod\ m,\ c] \wedge t < r + m * c\ then\ t + m\ else\ t)$
⟨*proof*⟩

lemmas *iT-inext-if* =
iFROM-inext-if
iTILL-inext-if
iIN-inext-if
iMOD-inext-if
iMODb-inext-if

lemma *iFROM-iprev-if*:
$iprev\ t\ [n\ldots] = (if\ n < t\ then\ t - Suc\ 0\ else\ t)$
⟨*proof*⟩
lemma *iTILL-iprev-if*:
$iprev\ t\ [\ldots n] = (if\ t \in [\ldots n]\ then\ t - Suc\ 0\ else\ t)$
⟨*proof*⟩

lemma *iIN-iprev-if*:
$iprev\ t\ [n\ldots,d] = (if\ n < t \wedge t \leq n + d\ \ then\ t - Suc\ 0\ else\ t)$
⟨*proof*⟩
lemma *iMOD-iprev-if*:
$iprev\ t\ [r,\ mod\ m] =$
$(if\ t \in [r,\ mod\ m] \wedge r < t\ then\ t - m\ else\ t)$
⟨*proof*⟩
lemma *iMODb-iprev-if*:
$iprev\ t\ [r,\ mod\ m,\ c] =$
$(if\ t \in [r,\ mod\ m,\ c] \wedge r < t\ then\ t - m\ else\ t)$
⟨*proof*⟩

lemmas *iT-iprev-if* =
iFROM-iprev-if
iTILL-iprev-if
iIN-iprev-if
iMOD-iprev-if
iMODb-iprev-if

The difference between an element and the next/previous element is constant if the element is different from Min/Max of the interval

lemma *iFROM-inext-diff-const*:
$t \in [n\ldots] \Longrightarrow inext\ t\ [n\ldots] - t = Suc\ 0$
⟨*proof*⟩

lemma *iFROM-iprev-diff-const*:
$n < t \Longrightarrow t - iprev\ t\ [n\ldots] = Suc\ 0$
⟨*proof*⟩

lemma *iFROM-iprev-diff-const'*:
$t \in [n\ldots] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots] = Suc\ 0$
⟨*proof*⟩

lemma *iTILL-inext-diff-const*:
$t < n \Longrightarrow inext\ t\ [\ldots n] - t = Suc\ 0$
⟨*proof*⟩
lemma *iTILL-iprev-diff-const*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow t - iprev\ t\ [\ldots n] = Suc\ 0$
⟨*proof*⟩

lemma *iIN-inext-diff-const*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] - t = Suc\ 0$
⟨*proof*⟩

lemma *iIN-iprev-diff-const*:
$[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow t - iprev\ t\ [n\ldots,d] = Suc\ 0$
⟨*proof*⟩
lemma *iIN-iprev-diff-const'*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots,d] = Suc\ 0$

$\langle proof \rangle$

lemma *iMOD-inext-diff-const*:
$t \in [r, mod\ m] \Longrightarrow inext\ t\ [r, mod\ m] - t = m$
$\langle proof \rangle$

lemma *iMOD-iprev-diff-const'*:
$t \in [r, mod\ m] \Longrightarrow (t + m) - iprev\ (t + m)\ [r, mod\ m] = m$
$\langle proof \rangle$

lemma *iMOD-iprev-diff-const*:
$[\![\ t \in [r, mod\ m]; r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r, mod\ m] = m$
$\langle proof \rangle$

lemma *iMODb-inext-diff-const*:
$[\![\ t \in [r, mod\ m, c]; t < r + m * c\ ]\!] \Longrightarrow inext\ t\ [r, mod\ m, c] - t = m$
$\langle proof \rangle$

lemma *iMODb-iprev-diff-const'*:
$[\![\ t \in [r, mod\ m, c]; t < r + m * c\ ]\!] \Longrightarrow (t + m) - iprev\ (t + m)\ [r, mod\ m, c] = m$
$\langle proof \rangle$

lemma *iMODb-iprev-diff-const*:
$[\![\ t \in [r, mod\ m, c]; r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r, mod\ m, c] = m$
$\langle proof \rangle$

lemmas *iT-inext-diff-const* =
iFROM-inext-diff-const
iTILL-inext-diff-const
iIN-inext-diff-const
iMOD-inext-diff-const
iMODb-inext-diff-const
lemmas *iT-iprev-diff-const* =
iFROM-iprev-diff-const
iFROM-iprev-diff-const'
iTILL-iprev-diff-const
iIN-iprev-diff-const
iIN-iprev-diff-const'
iMOD-iprev-diff-const'
iMOD-iprev-diff-const
iMODb-iprev-diff-const'
iMODb-iprev-diff-const

1.7.1 Mirroring of intervals

lemma
iIN-mirror-elem: $mirror\text{-}elem\ x\ [n\ldots,d] = n + n + d - x$ **and**
iTILL-mirror-elem: $mirror\text{-}elem\ x\ [\ldots n] = n - x$ **and**

iMODb-mirror-elem: $mirror\text{-}elem\ x\ [r,\ mod\ m,\ c] = r + r + m * c - x$
⟨*proof*⟩

lemma *iMODb-imirror-bounds*:
$r' + m' * c' \le l + r \Longrightarrow$
$imirror\text{-}bounds\ [r',\ mod\ m',\ c']\ l\ r = [l + r - r' - m' * c',\ mod\ m',\ c']$
⟨*proof*⟩

lemma *iIN-imirror-bounds*:
$n + d \le l + r \Longrightarrow imirror\text{-}bounds\ [n\ldots,d]\ l\ r = [l + r - n - d\ldots,d]$
⟨*proof*⟩

lemma *iTILL-imirror-bounds*:
$n \le l + r \Longrightarrow imirror\text{-}bounds\ [\ldots n]\ l\ r = [l + r - n\ldots,n]$
⟨*proof*⟩

lemmas *iT-imirror-bounds* =
iTILL-imirror-bounds
iIN-imirror-bounds
iMODb-imirror-bounds

lemma *iMODb-imirror-ident*: $imirror\ [r,\ mod\ m,\ c] = [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *iIN-imirror-ident*: $imirror\ [n\ldots,d] = [n\ldots,d]$
⟨*proof*⟩
lemma *iTILL-imirror-ident*: $imirror\ [\ldots n] = [\ldots n]$
⟨*proof*⟩

lemmas *iT-imirror-ident* =
iTILL-imirror-ident
iIN-imirror-ident
iMODb-imirror-ident

1.7.2 Functions *inext-nth* and *iprev-nth* on intervals

lemma *iFROM-inext-nth* : $[n\ldots] \rightarrow a = n + a$
⟨*proof*⟩

lemma *iIN-inext-nth* : $a \le d \Longrightarrow [n\ldots,d] \rightarrow a = n + a$
⟨*proof*⟩

lemma *iIN-iprev-nth*: $a \le d \Longrightarrow [n\ldots,d] \leftarrow a = n + d - a$
⟨*proof*⟩

lemma *iIN-inext-nth-if* :
$[n\ldots,d] \rightarrow a = (if\ a \le d\ then\ n + a\ else\ n + d)$
⟨*proof*⟩

lemma *iIN-iprev-nth-if*:
$[n\ldots,d] \leftarrow a = (\mathit{if}\ a \le d\ \mathit{then}\ n + d - a\ \mathit{else}\ n)$
⟨*proof*⟩

lemma *iTILL-inext-nth* : $a \le n \Longrightarrow [\ldots n] \rightarrow a = a$
⟨*proof*⟩

lemma *iTILL-inext-nth-if* :
$[\ldots n] \rightarrow a = (\mathit{if}\ a \le n\ \mathit{then}\ a\ \mathit{else}\ n)$
⟨*proof*⟩

lemma *iTILL-iprev-nth*: $a \le n \Longrightarrow [\ldots n] \leftarrow a = n - a$
⟨*proof*⟩

lemma *iTILL-iprev-nth-if*:
$[\ldots n] \leftarrow a = (\mathit{if}\ a \le n\ \mathit{then}\ n - a\ \mathit{else}\ \mathit{0})$
⟨*proof*⟩

lemma *iMOD-inext-nth*: $[r,\ \mathit{mod}\ m] \rightarrow a = r + m * a$
⟨*proof*⟩

lemma *iMODb-inext-nth*: $a \le c \Longrightarrow [r,\ \mathit{mod}\ m,\ c] \rightarrow a = r + m * a$
⟨*proof*⟩

lemma *iMODb-inext-nth-if*:
$[r,\ \mathit{mod}\ m,\ c] \rightarrow a = (\mathit{if}\ a \le c\ \mathit{then}\ r + m * a\ \mathit{else}\ r + m * c)$
⟨*proof*⟩

lemma *iMODb-iprev-nth*:
$a \le c \Longrightarrow [r,\ \mathit{mod}\ m,\ c] \leftarrow a = r + m * (c - a)$
⟨*proof*⟩

lemma *iMODb-iprev-nth-if*:
$[r,\ \mathit{mod}\ m,\ c] \leftarrow a = (\mathit{if}\ a \le c\ \mathit{then}\ r + m * (c - a)\ \mathit{else}\ r)$
⟨*proof*⟩

lemma *iIN-iFROM-inext-nth*:
$a \le d \Longrightarrow [n\ldots,d] \rightarrow a = [n\ldots] \rightarrow a$
⟨*proof*⟩

lemma *iIN-iFROM-inext*:
$a < n + d \Longrightarrow \mathit{inext}\ a\ [n\ldots,d] = \mathit{inext}\ a\ [n\ldots]$
⟨*proof*⟩

lemma *iMOD-iMODb-inext-nth*:
$a \le c \Longrightarrow [r,\ \mathit{mod}\ m,\ c] \rightarrow a = [r,\ \mathit{mod}\ m] \rightarrow a$
⟨*proof*⟩

lemma *iMOD-iMODb-inext*:
$a < r + m * c \Longrightarrow inext\ a\ [r,\ mod\ m,\ c] = inext\ a\ [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inext-nth-Suc-diff*:
$([r,\ mod\ m] \rightarrow (Suc\ n)) - ([r,\ mod\ m] \rightarrow n) = m$
⟨*proof*⟩

lemma *iMOD-inext-nth-diff*:
$([r,\ mod\ m] \rightarrow a) - ([r,\ mod\ m] \rightarrow b) = (a - b) * m$
⟨*proof*⟩

lemma *iMODb-inext-nth-diff*: $[\![\ a \le c;\ b \le c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] \rightarrow a) - ([r,\ mod\ m,\ c] \rightarrow b) = (a - b) * m$
⟨*proof*⟩

1.8 Induction with intervals

lemma *iFROM-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots];\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iIN-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots,d];\ k \ne n + d;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots,d]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iTILL-induct*:
$[\![\ P\ 0;\ \bigwedge k.\ [\![\ k \in [\ldots n];\ k \ne n;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [\ldots n]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMOD-induct*:
$[\![\ P\ r;\ \bigwedge k.\ [\![\ k \in [r,\ mod\ m];\ P\ k\ ]\!] \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-induct*:
$[\![\ P\ r;\ \bigwedge k.\ [\![\ k \in [r,\ mod\ m,\ c];\ k \ne r + m * c;\ P\ k\ ]\!] \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iIN-rev-induct*:
$[\![\ P\ (n + d);\ \bigwedge k.\ [\![\ k \in [n\ldots,d];\ k \ne n;\ P\ k\ ]\!] \Longrightarrow P\ (k - Suc\ 0);\ a \in [n\ldots,d]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iTILL-rev-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [\ldots n];\ 0 < k;\ P\ k\ ]\!] \Longrightarrow P\ (k - Suc\ 0);\ a \in [\ldots n]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-rev-induct*:
$[\![P\ (r + m * c);\ \bigwedge k.\ [\![k \in [r,\ mod\ m,\ c];\ k \neq r;\ P\ k\ ]\!] \Longrightarrow P\ (k - m);\ a \in [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

end

2 Arithmetic operators on natural intervals

theory *IL-IntervalOperators*
imports *IL-Interval*
begin

2.1 Arithmetic operations with intervals

2.1.1 Addition of and multiplication by constants

definition *iT-Plus* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** ‹⊕› *55*)
where $I \oplus k \equiv (\lambda n.(n + k))$ ‘ I

definition *iT-Mult* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** ‹⊗› *55*)
where *iT-Mult-def* : $I \otimes k \equiv (\lambda n.(n * k))$ ‘ I

lemma *iT-Plus-image-conv*: $I \oplus k = (\lambda n.(n + k))$ ‘ I
⟨*proof*⟩

lemma *iT-Mult-image-conv*: $I \otimes k = (\lambda n.(n * k))$ ‘ I
⟨*proof*⟩

lemma *iT-Plus-empty*: $\{\} \oplus k = \{\}$
⟨*proof*⟩

lemma *iT-Mult-empty*: $\{\} \otimes k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-not-empty*: $I \neq \{\} \Longrightarrow I \oplus k \neq \{\}$
⟨*proof*⟩

lemma *iT-Mult-not-empty*: $I \neq \{\} \Longrightarrow I \otimes k \neq \{\}$
⟨*proof*⟩

lemma *iT-Plus-empty-iff*: $(I \oplus k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Mult-empty-iff*: $(I \otimes k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Plus-mono*: $A \subseteq B \Longrightarrow A \oplus k \subseteq B \oplus k$
⟨*proof*⟩

lemma *iT-Mult-mono*: $A \subseteq B \Longrightarrow A \otimes k \subseteq B \otimes k$
⟨*proof*⟩

lemma *iT-Mult-0*: $I \neq \{\} \Longrightarrow I \otimes 0 = [\ldots 0]$
⟨*proof*⟩

corollary *iT-Mult-0-if*: $I \otimes 0 = (\mathit{if}\ I = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ [\ldots 0])$
⟨*proof*⟩

lemma *iT-Plus-mem-iff*: $x \in (I \oplus k) = (k \leq x \wedge (x - k) \in I)$
⟨*proof*⟩

lemma *iT-Plus-mem-iff2*: $x + k \in (I \oplus k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff-0*: $x \in (I \otimes 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff*:
$0 < k \Longrightarrow x \in (I \otimes k) = (x\ \mathit{mod}\ k = 0 \wedge x\ \mathit{div}\ k \in I)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff2*: $0 < k \Longrightarrow x * k \in (I \otimes k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-singleton*: $\{a\} \oplus k = \{a + k\}$
⟨*proof*⟩

lemma *iT-Mult-singleton*: $\{a\} \otimes k = \{a * k\}$
⟨*proof*⟩

lemma *iT-Plus-Un*: $(A \cup B) \oplus k = (A \oplus k) \cup (B \oplus k)$
⟨*proof*⟩

lemma *iT-Mult-Un*: $(A \cup B) \otimes k = (A \otimes k) \cup (B \otimes k)$
⟨*proof*⟩

lemma *iT-Plus-Int*: $(A \cap B) \oplus k = (A \oplus k) \cap (B \oplus k)$
⟨*proof*⟩

lemma *iT-Mult-Int*: $0 < k \Longrightarrow (A \cap B) \otimes k = (A \otimes k) \cap (B \otimes k)$
⟨*proof*⟩

lemma *iT-Plus-image*: f ‘ $I \oplus n = (\lambda x.\ f\ x + n)$ ‘ I
⟨*proof*⟩

lemma *iT-Mult-image*: f ‘ $I \otimes n = (\lambda x.\ f\ x * n)$ ‘ I
⟨*proof*⟩

lemma *iT-Plus-commute*: $I \oplus a \oplus b = I \oplus b \oplus a$
⟨*proof*⟩

lemma *iT-Mult-commute*: $I \otimes a \otimes b = I \otimes b \otimes a$
⟨*proof*⟩

lemma *iT-Plus-assoc*:$I \oplus a \oplus b = I \oplus (a + b)$
⟨*proof*⟩

lemma *iT-Mult-assoc*:$I \otimes a \otimes b = I \otimes (a * b)$
⟨*proof*⟩

lemma *iT-Plus-Mult-distrib*: $I \oplus n \otimes m = I \otimes m \oplus n * m$
⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩
lemma *iT-Plus-finite-iff*: *finite* $(I \oplus k)$ = *finite* I
⟨*proof*⟩

lemma *iT-Mult-0-finite*: *finite* $(I \otimes 0)$
⟨*proof*⟩

lemma *iT-Mult-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \otimes k)$ = *finite* I
⟨*proof*⟩

lemma *iT-Plus-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oplus k) = iMin\ I + k$
⟨*proof*⟩

lemma *iT-Mult-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \otimes k) = iMin\ I * k$
⟨*proof*⟩

lemma *iT-Plus-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \oplus k) = Max\ I + k$
⟨*proof*⟩

lemma *iT-Mult-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \otimes k) = Max\ I * k$
⟨*proof*⟩

corollary
iMOD-mult-0: $[r,\ mod\ m] \otimes 0 = [\ldots 0]$ **and**
iMODb-mult-0: $[r,\ mod\ m,\ c] \otimes 0 = [\ldots 0]$ **and**
iFROM-mult-0: $[n\ldots] \otimes 0 = [\ldots 0]$ **and**
iIN-mult-0: $[n\ldots,d] \otimes 0 = [\ldots 0]$ **and**
iTILL-mult-0: $[\ldots n] \otimes 0 = [\ldots 0]$
⟨*proof*⟩

lemmas *iT-mult-0* =
iTILL-mult-0
iFROM-mult-0
iIN-mult-0
iMOD-mult-0
iMODb-mult-0

lemma *iT-Plus-0*: $I \oplus 0 = I$
⟨*proof*⟩

lemma *iT-Mult-1*: $I \otimes Suc\ 0 = I$
⟨*proof*⟩

corollary
iFROM-add-Min: $iMin\ ([n\ldots] \oplus k) = n + k$ **and**
iIN-add-Min: $iMin\ ([n\ldots,d] \oplus k) = n + k$ **and**
iTILL-add-Min: $iMin\ ([\ldots n] \oplus k) = k$ **and**
iMOD-add-Min: $iMin\ ([r,\ mod\ m] \oplus k) = r + k$ **and**
iMODb-add-Min: $iMin\ ([r,\ mod\ m,\ c] \oplus k) = r + k$
⟨*proof*⟩

corollary
iFROM-mult-Min: $iMin\ ([n\ldots] \otimes k) = n * k$ **and**
iIN-mult-Min: $iMin\ ([n\ldots,d] \otimes k) = n * k$ **and**
iTILL-mult-Min: $iMin\ ([\ldots n] \otimes k) = 0$ **and**
iMOD-mult-Min: $iMin\ ([r,\ mod\ m] \otimes k) = r * k$ **and**
iMODb-mult-Min: $iMin\ ([r,\ mod\ m,\ c] \otimes k) = r * k$
⟨*proof*⟩

lemmas *iT-add-Min* =
iIN-add-Min
iTILL-add-Min
iFROM-add-Min
iMOD-add-Min
iMODb-add-Min

lemmas *iT-mult-Min* =
iIN-mult-Min
iTILL-mult-Min
iFROM-mult-Min
iMOD-mult-Min
iMODb-mult-Min

lemma *iFROM-add*: $[n\ldots] \oplus k = [n+k\ldots]$
⟨*proof*⟩

lemma *iIN-add*: $[n\ldots,d] \oplus k = [n+k\ldots,d]$

⟨*proof*⟩

lemma *iTILL-add*: $[\ldots i] \oplus k = [k\ldots,i]$
⟨*proof*⟩

lemma *iMOD-add*: $[r,\ mod\ m] \oplus k = [r + k,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-add*: $[r,\ mod\ m,\ c] \oplus k = [r + k,\ mod\ m,\ c]$
⟨*proof*⟩

lemmas *iT-add* =
iMOD-add
iMODb-add
iFROM-add
iIN-add
iTILL-add
iT-Plus-singleton

lemma *iFROM-mult*: $[n\ldots] \otimes k = [\ n * k,\ mod\ k\ ]$
⟨*proof*⟩

lemma *iIN-mult*: $[n\ldots,d] \otimes k = [\ n * k,\ mod\ k,\ d\ ]$
⟨*proof*⟩

lemma *iTILL-mult*: $[\ldots n] \otimes k = [\ 0,\ mod\ k,\ n\ ]$
⟨*proof*⟩

lemma *iMOD-mult*: $[r,\ mod\ m\ ] \otimes k = [\ r * k,\ mod\ m * k\ ]$
⟨*proof*⟩

lemma *iMODb-mult*:
$[r,\ mod\ m,\ c\ ] \otimes k = [\ r * k,\ mod\ m * k,\ c\ ]$
⟨*proof*⟩

lemmas *iT-mult* =
iTILL-mult
iFROM-mult
iIN-mult
iMOD-mult
iMODb-mult
iT-Mult-singleton

2.1.2 Some conversions between intervals using constant addition and multiplication

lemma *iFROM-conv*: $[n\ldots] = UNIV \oplus n$
⟨*proof*⟩

lemma *iIN-conv*: $[n\ldots,d] = [\ldots d] \oplus n$
⟨*proof*⟩

lemma *iMOD-conv*: $[r,\ mod\ m] = [0\ldots] \otimes m \oplus r$
⟨*proof*⟩

lemma *iMODb-conv*: $[r,\ mod\ m,\ c] = [\ldots c] \otimes m \oplus r$
⟨*proof*⟩

Some examples showing the utility of iMODb_conv

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[r,\ mod\ m,\ 4] = \{r,\ r+m,\ r+2*m,\ r+3*m,\ r+4*m\}$
⟨*proof*⟩

lemma $[2,\ mod\ 10,\ 4] = \{2,\ 12,\ 22,\ 32,\ 42\}$
⟨*proof*⟩

2.1.3 Subtraction of constants

definition *iT-Plus-neg* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** ‹⊕−› *55*) **where**
  $I \oplus- k \equiv \{x.\ x + k \in I\}$

lemma *iT-Plus-neg-mem-iff*: $(x \in I \oplus- k) = (x + k \in I)$
⟨*proof*⟩

lemma *iT-Plus-neg-mem-iff2*: $k \le x \Longrightarrow (x - k \in I \oplus- k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-neg-image-conv*: $I \oplus- k = (\lambda n.(n - k))$ ‘ $(I \downarrow\geq k)$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-eq*: $t \le k \Longrightarrow (I \downarrow\geq t) \oplus- k = I \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-mono*: $A \subseteq B \Longrightarrow A \oplus- k \subseteq B \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-empty*: $\{\} \oplus- k = \{\}$
⟨*proof*⟩
lemma *iT-Plus-neg-Max-less-empty*:

$[\![$ *finite* I; *Max* $I < k$ $]\!] \Longrightarrow I \oplus- k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-neg-not-empty-iff*: $(I \oplus- k \neq \{\}) = (\exists x \in I.\ k \leq x)$
⟨*proof*⟩

lemma *iT-Plus-neg-empty-iff*:
$(I \oplus- k = \{\}) = (I = \{\} \vee (\textit{finite}\ I \wedge \textit{Max}\ I < k))$
⟨*proof*⟩

lemma *iT-Plus-neg-assoc*: $(I \oplus- a) \oplus- b = I \oplus- (a + b)$
⟨*proof*⟩

lemma *iT-Plus-neg-commute*: $I \oplus- a \oplus- b = I \oplus- b \oplus- a$
⟨*proof*⟩

lemma *iT-Plus-neg-0*: $I \oplus- 0 = I$
⟨*proof*⟩

lemma *iT-Plus-Plus-neg-assoc*: $b \leq a \Longrightarrow I \oplus a \oplus- b = I \oplus (a - b)$
⟨*proof*⟩

lemma *iT-Plus-Plus-neg-assoc2*: $a \leq b \Longrightarrow I \oplus a \oplus- b = I \oplus- (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-le-cut-eq*:
$a \leq b \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\geq a) \oplus (b - a)$
⟨*proof*⟩

corollary *iT-Plus-neg-Plus-le-Min-eq*:
$[\![a \leq b; a \leq \textit{iMin}\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-ge-cut-eq*:
$b \leq a \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\geq a) \oplus- (a - b)$
⟨*proof*⟩

corollary *iT-Plus-neg-Plus-ge-Min-eq*:
$[\![b \leq a; a \leq \textit{iMin}\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus- (a - b)$
⟨*proof*⟩

lemma *iT-Plus-neg-Mult-distrib*:
$0 < m \Longrightarrow I \oplus- n \otimes m = I \otimes m \oplus- n * m$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-le-inverse*: $k \leq \textit{iMin}\ I \Longrightarrow I \oplus- k \oplus k = I$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-inverse*: $I \oplus- k \oplus k = I \downarrow\geq k$

$\langle proof \rangle$

lemma *iT-Plus-Plus-neg-inverse*: $I \oplus k \oplus- k = I$
$\langle proof \rangle$

lemma *iT-Plus-neg-Un*: $(A \cup B) \oplus- k = (A \oplus- k) \cup (B \oplus- k)$
$\langle proof \rangle$

lemma *iT-Plus-neg-Int*: $(A \cap B) \oplus- k = (A \oplus- k) \cap (B \oplus- k)$
$\langle proof \rangle$

lemma *iT-Plus-neg-Max-singleton*: $[\![\mathit{finite}\ I;\ I \neq \{\}]\!] \Longrightarrow I \oplus- \mathit{Max}\ I= \{0\}$
$\langle proof \rangle$

lemma *iT-Plus-neg-singleton*: $\{a\} \oplus- k = (\mathit{if}\ k \le a\ \mathit{then}\ \{a - k\}\ \mathit{else}\ \{\})$
$\langle proof \rangle$

corollary *iT-Plus-neg-singleton1*: $k \le a \Longrightarrow \{a\} \oplus- k = \{a-k\}$
$\langle proof \rangle$

corollary *iT-Plus-neg-singleton2*: $a < k \Longrightarrow \{a\} \oplus- k= \{\}$
$\langle proof \rangle$

lemma *iT-Plus-neg-finite-iff*: $\mathit{finite}\ (I \oplus- k) = \mathit{finite}\ I$
$\langle proof \rangle$

lemma *iT-Plus-neg-Min*:
$I \oplus- k \neq \{\} \Longrightarrow \mathit{iMin}\ (I \oplus- k) = \mathit{iMin}\ (I \downarrow\ge k) - k$
$\langle proof \rangle$

lemma *iT-Plus-neg-Max*:
$[\![\mathit{finite}\ I;\ I \oplus- k \neq \{\}]\!] \Longrightarrow \mathit{Max}\ (I \oplus- k) = \mathit{Max}\ I - k$
$\langle proof \rangle$

Subtractions of constants from intervals

lemma *iFROM-add-neg*: $[n\ldots] \oplus- k = [n - k\ldots]$
$\langle proof \rangle$

lemma *iTILL-add-neg*: $[\ldots n] \oplus- k = (\mathit{if}\ k \le n\ \mathit{then}\ [\ldots n - k]\ \mathit{else}\ \{\})$
$\langle proof \rangle$
lemma *iTILL-add-neg1*: $k \le n \Longrightarrow [\ldots n] \oplus- k = [\ldots n-k]$
$\langle proof \rangle$
lemma *iTILL-add-neg2*: $n < k \Longrightarrow [\ldots n] \oplus- k = \{\}$
$\langle proof \rangle$

lemma *iIN-add-neg*:
$[n\ldots,d] \oplus- k = ($
$\mathit{if}\ k \le n\ \mathit{then}\ [n - k\ldots,d]$

$\quad$ *else if* $k \leq n + d$ *then* $[\ldots n + d - k]$ *else* $\{\})$
⟨*proof*⟩

lemma *iIN-add-neg1*: $k \leq n \Longrightarrow [n\ldots,d] \oplus- k = [n - k\ldots,d]$
⟨*proof*⟩

lemma *iIN-add-neg2*: $[\![n \leq k; k \leq n + d]\!] \Longrightarrow [n\ldots,d] \oplus- k = [\ldots n + d - k]$
⟨*proof*⟩

lemma *iIN-add-neg3*: $n + d < k \Longrightarrow [n\ldots,d] \oplus- k = \{\}$
⟨*proof*⟩

lemma *iMOD-0-add-neg*: $[r, \mathit{mod}\ 0] \oplus- k = \{r\} \oplus- k$
⟨*proof*⟩

lemma *iMOD-gr0-add-neg*:
$0 < m \Longrightarrow$
$[r, \mathit{mod}\ m] \oplus- k = ($
$\quad$ *if* $k \leq r$ *then* $[r - k, \mathit{mod}\ m]$
$\quad$ *else* $[(m + r\ \mathit{mod}\ m - k\ \mathit{mod}\ m)\ \mathit{mod}\ m, \mathit{mod}\ m])$
⟨*proof*⟩

lemma *iMOD-add-neg*:
$[r, \mathit{mod}\ m] \oplus- k = ($
$\quad$ *if* $k \leq r$ *then* $[r - k, \mathit{mod}\ m]$
$\quad$ *else if* $0 < m$ *then* $[(m + r\ \mathit{mod}\ m - k\ \mathit{mod}\ m)\ \mathit{mod}\ m, \mathit{mod}\ m]$ *else* $\{\})$
⟨*proof*⟩

corollary *iMOD-add-neg1*:
$k \leq r \Longrightarrow [r, \mathit{mod}\ m] \oplus- k = [r - k, \mathit{mod}\ m]$
⟨*proof*⟩

lemma *iMOD-add-neg2*:
$[\![0 < m; r < k]\!] \Longrightarrow [r, \mathit{mod}\ m] \oplus- k = [(m + r\ \mathit{mod}\ m - k\ \mathit{mod}\ m)\ \mathit{mod}\ m,$
$\mathit{mod}\ m]$
⟨*proof*⟩

lemma *iMODb-mod-0-add-neg*: $[r, \mathit{mod}\ 0, c] \oplus- k = \{r\} \oplus- k$
⟨*proof*⟩

lemma *iMODb-add-neg*:
$[r, \mathit{mod}\ m, c] \oplus- k = ($
$\quad$ *if* $k \leq r$ *then* $[r - k, \mathit{mod}\ m, c]$
$\quad$ *else*

$\quad$ *if* $k \le r + m * c$ *then*
$\quad$ $[(m + r \textit{ mod } m - k \textit{ mod } m) \textit{ mod } m, \textit{ mod } m, (r + m * c - k) \textit{ div } m]$
$\quad$ *else* $\{\}$)
$\langle proof \rangle$

lemma *iMODb-add-neg'*:
$[r, \textit{mod } m, c] \oplus - k = ($
if $k \le r$ *then* $[r - k, \textit{mod } m, c]$
else if $k \le r + m * c$ *then*
if $k \textit{ mod } m \le r \textit{ mod } m$
then $[\ r \textit{ mod } m - k \textit{ mod } m, \textit{mod } m, c + r \textit{ div } m - k \textit{ div } m]$
else $[\ m + r \textit{ mod } m - k \textit{ mod } m, \textit{mod } m, c + r \textit{ div } m - \textit{Suc } (k \textit{ div } m)\ ]$
else $\{\}$)
$\langle proof \rangle$

corollary *iMODb-add-neg1*:
$k \le r \Longrightarrow [r, \textit{mod } m, c] \oplus - k = [r - k, \textit{mod } m, c]$
$\langle proof \rangle$

corollary *iMODb-add-neg2*:
$[\![\ r < k; k \le r + m * c\ ]\!] \Longrightarrow$
$[r, \textit{mod } m, c] \oplus - k =$
$[(m + r \textit{ mod } m - k \textit{ mod } m) \textit{ mod } m, \textit{mod } m, (r + m * c - k) \textit{ div } m]$
$\langle proof \rangle$

corollary *iMODb-add-neg2-mod-le*:
$[\![\ r < k; k \le r + m * c; k \textit{ mod } m \le r \textit{ mod } m\ ]\!] \Longrightarrow$
$[r, \textit{mod } m, c] \oplus - k =$
$[\ r \textit{ mod } m - k \textit{ mod } m, \textit{mod } m, c + r \textit{ div } m - k \textit{ div } m]$
$\langle proof \rangle$

corollary *iMODb-add-neg2-mod-less*:
$[\![\ r < k; k \le r + m * c; r \textit{ mod } m < k \textit{ mod } m]\!] \Longrightarrow$
$[r, \textit{mod } m, c] \oplus - k =$
$[\ m + r \textit{ mod } m - k \textit{ mod } m, \textit{mod } m, c + r \textit{ div } m - \textit{Suc } (k \textit{ div } m)\ ]$
$\langle proof \rangle$

lemma *iMODb-add-neg3*: $r + m * c < k \Longrightarrow [r, \textit{mod } m, c] \oplus - k = \{\}$
$\langle proof \rangle$

lemmas *iT-add-neg* =
iFROM-add-neg
iIN-add-neg
iTILL-add-neg
iMOD-add-neg
iMODb-add-neg
iT-Plus-neg-singleton

2.1.4 Subtraction of intervals from constants

definition *iT-Minus* :: *Time* ⇒ *iT* ⇒ *iT* (**infixl** ‹⊖› *55*)
 where $k \ominus I \equiv \{x.\ x \le k \wedge (k - x) \in I\}$

lemma *iT-Minus-mem-iff*: $(x \in k \ominus I) = (x \le k \wedge k - x \in I)$
⟨*proof*⟩

lemma *iT-Minus-mono*: $A \subseteq B \Longrightarrow k \ominus A \subseteq k \ominus B$
⟨*proof*⟩

lemma *iT-Minus-image-conv*: $k \ominus I = (\lambda x.\ k - x)$ ‘ $(I \downarrow\le k)$
⟨*proof*⟩

lemma *iT-Minus-cut-eq*: $k \le t \Longrightarrow k \ominus (I \downarrow\le t) = k \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Minus-cut-eq*: $k \ominus (k \ominus (I \downarrow\le k)) = I \downarrow\le k$
⟨*proof*⟩

lemma $10 \ominus [\ldots 3] = [7\ldots,3]$
⟨*proof*⟩

lemma *iT-Minus-empty*: $k \ominus \{\} = \{\}$
⟨*proof*⟩

lemma *iT-Minus-0*: $k \ominus \{0\} = \{k\}$
⟨*proof*⟩

lemma *iT-Minus-bound*: $x \in k \ominus I \Longrightarrow x \le k$
⟨*proof*⟩

lemma *iT-Minus-finite*: *finite* $(k \ominus I)$
⟨*proof*⟩

lemma *iT-Minus-less-Min-empty*: $k <$ *iMin* $I \Longrightarrow k \ominus I = \{\}$
⟨*proof*⟩

lemma *iT-Minus-Min-singleton*: $I \ne \{\} \Longrightarrow$ (*iMin* I) $\ominus I = \{0\}$
⟨*proof*⟩

lemma *iT-Minus-empty-iff*: $(k \ominus I = \{\}) = (I = \{\} \vee k <$ *iMin* $I)$
⟨*proof*⟩

lemma *iT-Minus-imirror-conv*:
 $k \ominus I =$ *imirror* $(I \downarrow\le k) \oplus k \oplus-$ (*iMin* I + *Max* $(I \downarrow\le k)$)
⟨*proof*⟩

lemma *iT-Minus-imirror-conv'*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ (I \downarrow\leq k) + Max\ (I \downarrow\leq k))$
⟨*proof*⟩

lemma *iT-Minus-Max*:
$[\![I \neq \{\};\ iMin\ I \leq k]\!] \Longrightarrow Max\ (k \ominus I) = k - (iMin\ I)$
⟨*proof*⟩

lemma *iT-Minus-Min*:
$[\![I \neq \{\};\ iMin\ I \leq k]\!] \Longrightarrow iMin\ (k \ominus I) = k - (Max\ (I \downarrow\leq k))$
⟨*proof*⟩

lemma *iT-Minus-Minus-eq*: $[\![finite\ I;\ Max\ I \leq k]\!] \Longrightarrow k \ominus (k \ominus I) = I$
⟨*proof*⟩

lemma *iT-Minus-Minus-eq2*: $I \subseteq [\ldots k] \Longrightarrow k \ominus (k \ominus I) = I$
⟨*proof*⟩

lemma *iT-Minus-Minus*: $a \ominus (b \ominus I) = (I \downarrow\leq b) \oplus a \oplus- b$
⟨*proof*⟩

lemma *iT-Minus-Plus-empty*: $k < n \Longrightarrow k \ominus (I \oplus n) = \{\}$
⟨*proof*⟩
lemma *iT-Minus-Plus-commute*: $n \leq k \Longrightarrow k \ominus (I \oplus n) = (k - n) \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Plus-cut-assoc*: $(k \ominus I) \oplus n = (k + n) \ominus (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-Plus-assoc*:
$[\![finite\ I;\ Max\ I \leq k]\!] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩
lemma *iT-Minus-Plus-assoc2*:
$I \subseteq [\ldots k] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Un*: $k \ominus (A \cup B) = (k \ominus A) \cup (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-Int*: $k \ominus (A \cap B) = (k \ominus A) \cap (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-singleton*: $k \ominus \{a\} = (if\ a \leq k\ then\ \{k - a\}\ else\ \{\})$
⟨*proof*⟩
corollary *iT-Minus-singleton1*: $a \leq k \Longrightarrow k \ominus \{a\} = \{k-a\}$
⟨*proof*⟩
corollary *iT-Minus-singleton2*: $k < a \Longrightarrow k \ominus \{a\} = \{\}$

$\langle proof \rangle$

lemma *iMOD-sub*:
$k \ominus [r, \ mod \ m] =$
$(if \ r \le k \ then \ [(k - r) \ mod \ m, \ mod \ m, \ (k - r) \ div \ m] \ else \ \{\})$
$\langle proof \rangle$

corollary *iMOD-sub1*:
$r \le k \Longrightarrow k \ominus [r, \ mod \ m] = [(k - r) \ mod \ m, \ mod \ m, \ (k - r) \ div \ m]$
$\langle proof \rangle$

corollary *iMOD-sub2*: $k < r \Longrightarrow k \ominus [r, \ mod \ m] = \{\}$
$\langle proof \rangle$

lemma *iTILL-sub*: $k \ominus [\ldots n] = (if \ n \le k \ then \ [k - n\ldots,n] \ else \ [\ldots k])$
$\langle proof \rangle$

corollary *iTILL-sub1*: $n \le k \Longrightarrow k \ominus [\ldots n] = [k - n\ldots,n]$
$\langle proof \rangle$

corollary *iTILL-sub2*: $k \le n \Longrightarrow k \ominus [\ldots n] = [\ldots k]$
$\langle proof \rangle$

lemma *iMODb-sub*:
$k \ominus [r, \ mod \ m, \ c] = ($
$if \ r + m * c \le k \ then \ [\ k - r - m * c, \ mod \ m, \ c] \ else$
$if \ r \le k \ then \ [\ (k - r) \ mod \ m, \ mod \ m, \ (k - r) \ div \ m] \ else \ \{\})$
$\langle proof \rangle$

corollary *iMODb-sub1*:
$[\![\ r \le k; \ k \le r + m * c\ ]\!] \Longrightarrow$
$k \ominus [r, \ mod \ m, \ c] = [(k - r) \ mod \ m, \ mod \ m, \ (k - r) \ div \ m]$
$\langle proof \rangle$

corollary *iMODb-sub2*: $k < r \Longrightarrow k \ominus [r, \ mod \ m, \ c] = \{\}$
$\langle proof \rangle$

corollary *iMODb-sub3*:
$r + m * c \le k \Longrightarrow k \ominus [r, \ mod \ m, \ c] = [\ k - r - m * c, \ mod \ m, \ c]$
$\langle proof \rangle$

lemma *iFROM-sub*: $k \ominus [n\ldots] = (if \ n \le k \ then \ [\ldots k - n] \ else \ \{\})$
$\langle proof \rangle$

corollary *iFROM-sub1*: $n \le k \Longrightarrow k \ominus [n\ldots] = [\ldots k{-}n]$
$\langle proof \rangle$

corollary *iFROM-sub-empty*: $k < n \Longrightarrow k \ominus [n\ldots] = \{\}$
$\langle proof \rangle$

lemma *iIN-sub*:
$k \ominus [n\ldots,d] = ($
$\mathit{if}\ n + d \le k\ \mathit{then}\ [k - (n + d)\ldots,d]$
$\mathit{else}\ \mathit{if}\ n \le k\ \mathit{then}\ [\ldots k - n]\ \mathit{else}\ \{\})$
$\langle proof \rangle$

lemma *iIN-sub1*: $n + d \le k \Longrightarrow k \ominus [n\ldots,d] = [k - (n + d)\ldots,d]$
$\langle proof \rangle$

lemma *iIN-sub2*: $[\![\ n \le k;\ k \le n + d\ ]\!] \Longrightarrow k \ominus [n\ldots,d] = [\ldots k - n]$
$\langle proof \rangle$

lemma *iIN-sub3*: $k < n \Longrightarrow k \ominus [n\ldots,d] = \{\}$
$\langle proof \rangle$

lemmas *iT-sub* =
iFROM-sub
iIN-sub
iTILL-sub
iMOD-sub
iMODb-sub
iT-Minus-singleton

2.1.5 Division of intervals by constants

Monotonicity and injectivity of artithmetic operators

lemma *iMOD-div-right-strict-mono-on*:
$[\![\ 0 < k;\ k \le m\ ]\!] \Longrightarrow \mathit{strict\text{-}mono\text{-}on}\ (\lambda x.\ x\ \mathit{div}\ k)\ [r,\ \mathit{mod}\ m]$
$\langle proof \rangle$

corollary *iMOD-div-right-inj-on*:
$[\![\ 0 < k;\ k \le m\ ]\!] \Longrightarrow \mathit{inj\text{-}on}\ (\lambda x.\ x\ \mathit{div}\ k)\ [r,\ \mathit{mod}\ m]$
$\langle proof \rangle$

lemma *iMOD-mult-div-right-inj-on*:
$\mathit{inj\text{-}on}\ (\lambda x.\ x\ \mathit{div}\ (k{::}\mathit{nat}))\ [r,\ \mathit{mod}\ (k * m)]$
$\langle proof \rangle$

lemma *iMOD-mult-div-right-inj-on2*:
$m\ \mathit{mod}\ k = 0 \Longrightarrow \mathit{inj\text{-}on}\ (\lambda x.\ x\ \mathit{div}\ k)\ [r,\ \mathit{mod}\ m]$
$\langle proof \rangle$

lemma *iMODb-div-right-strict-mono-on*:

$\llbracket\ 0 < k;\ k \leq m\ \rrbracket \Longrightarrow$ *strict-mono-on* $(\lambda x.\ x\ div\ k)\ [r,\ mod\ m,\ c]$
⟨*proof*⟩

corollary *iMODb-div-right-inj-on*:
$\llbracket\ 0 < k;\ k \leq m\ \rrbracket \Longrightarrow$ *inj-on* $(\lambda x.\ x\ div\ k)\ [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-mult-div-right-inj-on*:
inj-on $(\lambda x.\ x\ div\ (k::nat))\ [r,\ mod\ (k * m),\ c]$
⟨*proof*⟩

corollary *iMODb-mult-div-right-inj-on2*:
$m\ mod\ k = 0 \Longrightarrow$ *inj-on* $(\lambda x.\ x\ div\ k)\ [r,\ mod\ m,\ c]$
⟨*proof*⟩

definition *iT-Div* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** ‹⊘› *55*)
where $I \oslash k \equiv (\lambda n.(n\ div\ k))$ ‘ I

lemma *iT-Div-image-conv*: $I \oslash k = (\lambda n.(n\ div\ k))$ ‘ I
⟨*proof*⟩

lemma *iT-Div-mono*: $A \subseteq B \Longrightarrow A \oslash k \subseteq B \oslash k$
⟨*proof*⟩

lemma *iT-Div-empty*: $\{\} \oslash k = \{\}$
⟨*proof*⟩
lemma *iT-Div-not-empty*: $I \neq \{\} \Longrightarrow I \oslash k \neq \{\}$
⟨*proof*⟩

lemma *iT-Div-empty-iff*: $(I \oslash k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Div-0*: $I \neq \{\} \Longrightarrow I \oslash 0 = [\ldots 0]$
⟨*proof*⟩
corollary *iT-Div-0-if*: $I \oslash 0 = ($*if* $I = \{\}$ *then* $\{\}$ *else* $[\ldots 0])$
⟨*proof*⟩
corollary
iFROM-div-0: $[n\ldots] \oslash 0 = [\ldots 0]$ **and**
iTILL-div-0: $[\ldots n] \oslash 0 = [\ldots 0]$ **and**
iIN-div-0: $[n\ldots,d] \oslash 0 = [\ldots 0]$ **and**
iMOD-div-0: $[r,\ mod\ m] \oslash 0 = [\ldots 0]$ **and**
iMODb-div-0: $[r,\ mod\ m,\ c] \oslash 0 = [\ldots 0]$
⟨*proof*⟩

lemmas *iT-div-0* =
iTILL-div-0
iFROM-div-0

iIN-div-0
iMOD-div-0
iMODb-div-0

lemma *iT-Div-1*: $I \oslash Suc\ 0 = I$
⟨*proof*⟩

lemma *iT-Div-mem-iff-0*: $x \in (I \oslash 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩

lemma *iT-Div-mem-iff*:
$0 < k \Longrightarrow x \in (I \oslash k) = (\exists y \in I.\ y\ div\ k = x)$
⟨*proof*⟩

lemma *iT-Div-mem-iff2*:
$0 < k \Longrightarrow x\ div\ k \in (I \oslash k) = (\exists y \in I.\ y\ div\ k = x\ div\ k)$
⟨*proof*⟩

lemma *iT-Div-mem-iff-Int*:
$0 < k \Longrightarrow x \in (I \oslash k) = (I \cap [x * k\ldots,k - Suc\ 0] \neq \{\})$
⟨*proof*⟩

lemma *iT-Div-imp-mem*:
$0 < k \Longrightarrow x \in I \Longrightarrow x\ div\ k \in (I \oslash k)$
⟨*proof*⟩

lemma *iT-Div-singleton*: $\{a\} \oslash k = \{a\ div\ k\}$
⟨*proof*⟩

lemma *iT-Div-Un*: $(A \cup B) \oslash k = (A \oslash k) \cup (B \oslash k)$
⟨*proof*⟩

lemma *iT-Div-insert*: $(insert\ n\ I) \oslash k = insert\ (n\ div\ k)\ (I \oslash k)$
⟨*proof*⟩

lemma *not-iT-Div-Int*: $\neg\ (\forall\ k\ A\ B.\ (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k))$
⟨*proof*⟩

lemma *subset-iT-Div-Int*: $A \subseteq B \Longrightarrow (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k)$
⟨*proof*⟩

lemma *iFROM-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A\ ]\!] \Longrightarrow (A \cap [n\ldots]) \oslash k = (A \oslash k) \cap ([n\ldots] \oslash k)$
⟨*proof*⟩

lemma *iIN-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A;\ \forall x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d\ ]\!] \Longrightarrow$

$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
$\langle proof \rangle$
corollary *iTILL-iT-Div-Int*:
$[\![0 < k; \forall x{\in}A.\ x\ div\ k \leq n\ div\ k \longrightarrow x \leq n]\!] \Longrightarrow$
$(A \cap [\ldots n]) \oslash k = (A \oslash k) \cap ([\ldots n] \oslash k)$
$\langle proof \rangle$
lemma *iIN-iT-Div-Int-mod-0*:
$[\![0 < k;\ n\ mod\ k = 0; \forall x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d]\!] \Longrightarrow$
$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
$\langle proof \rangle$

lemma *mod-partition-iT-Div-Int*:
$[\![0 < k;\ 0 < d]\!] \Longrightarrow$
$(A \cap [n * k\ldots,d * k - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n * k\ldots,d * k - Suc\ 0] \oslash k)$
$\langle proof \rangle \langle proof \rangle$
corollary *mod-partition-iT-Div-Int2*:
$[\![0 < k;\ 0 < d;\ n\ mod\ k = 0;\ d\ mod\ k = 0]\!] \Longrightarrow$
$(A \cap [n\ldots,d - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n\ldots,d - Suc\ 0] \oslash k)$
$\langle proof \rangle$

corollary *mod-partition-iT-Div-Int-one-segment*:
$0 < k \Longrightarrow$
$(A \cap [n * k\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n * k\ldots,k - Suc\ 0] \oslash k)$
$\langle proof \rangle$

corollary *mod-partition-iT-Div-Int-one-segment2*:
$[\![0 < k;\ n\ mod\ k = 0]\!] \Longrightarrow$
$(A \cap [n\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n\ldots,k - Suc\ 0] \oslash k)$
$\langle proof \rangle$

lemma *iT-Div-assoc*:$I \oslash a \oslash b = I \oslash (a * b)$
$\langle proof \rangle$

lemma *iT-Div-commute*: $I \oslash a \oslash b = I \oslash b \oslash a$
$\langle proof \rangle$

lemma *iT-Mult-Div-self*: $0 < k \Longrightarrow I \otimes k \oslash k = I$
$\langle proof \rangle$
lemma *iT-Mult-Div*:
$[\![0 < d;\ \ k\ mod\ d = 0]\!] \Longrightarrow I \otimes k \oslash d = I \otimes (k\ div\ d)$
$\langle proof \rangle$

lemma *iT-Div-Mult-self*:
$0 < k \Longrightarrow I \oslash k \otimes k = \{y.\ \exists x \in I.\ y = x - x\ mod\ k\}$
$\langle proof \rangle$

lemma *iT-Plus-Div-distrib-mod-less*:
$\forall x \in I.\ x\ mod\ m + n\ mod\ m < m \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩
corollary *iT-Plus-Div-distrib-mod-0*:
$n\ mod\ m = 0 \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩

lemma *iT-Div-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oslash k) = iMin\ I\ div\ k$
⟨*proof*⟩

corollary
iFROM-div-Min: $iMin\ ([n\ldots] \oslash k) = n\ div\ k$ **and**
iIN-div-Min: $iMin\ ([n\ldots,d] \oslash k) = n\ div\ k$ **and**
iTILL-div-Min: $iMin\ ([\ldots n] \oslash k) = 0$ **and**
iMOD-div-Min: $iMin\ ([r,\ mod\ m] \oslash k) = r\ div\ k$ **and**
iMODb-div-Min: $iMin\ ([r,\ mod\ m,\ c] \oslash k) = r\ div\ k$
⟨*proof*⟩

lemmas *iT-div-Min* =
iFROM-div-Min
iIN-div-Min
iTILL-div-Min
iMOD-div-Min
iMODb-div-Min

lemma *iT-Div-Max*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oslash k) = Max\ I\ div\ k$
⟨*proof*⟩

corollary
iIN-div-Max: $Max\ ([n\ldots,d] \oslash k) = (n + d)\ div\ k$ **and**
iTILL-div-Max: $Max\ ([\ldots n] \oslash k) = n\ div\ k$ **and**
iMODb-div-Max: $Max\ ([r,\ mod\ m,\ c] \oslash k) = (r + m * c)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-0-finite*: $finite\ (I \oslash 0)$
⟨*proof*⟩

lemma *iT-Div-infinite-iff*: $0 < k \Longrightarrow infinite\ (I \oslash k) = infinite\ I$
⟨*proof*⟩
lemma *iT-Div-finite-iff*: $0 < k \Longrightarrow finite\ (I \oslash k) = finite\ I$
⟨*proof*⟩

lemma *iFROM-div*: $0 < k \Longrightarrow [n\ldots] \oslash k = [n\ div\ k\ldots]$
⟨*proof*⟩

lemma *iIN-div*:
$0 < k \Longrightarrow$
$[n\ldots,d] \oslash k = [n\ div\ k\ldots,\ d\ div\ k + (n\ mod\ k + d\ mod\ k)\ div\ k\ ]$

$\langle proof \rangle$

corollary *iIN-div-if*:
$0 < k \Longrightarrow [n\ldots,d] \oslash k =$
$[n\ div\ k\ldots,\ d\ div\ k + (if\ n\ mod\ k + d\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
$\langle proof \rangle$

corollary *iIN-div-eq1*:
$[\![\ 0 < k;\ n\ mod\ k + d\ mod\ k < k\ ]\!] \Longrightarrow$
$[n\ldots,d] \oslash k = [n\ div\ k\ldots,d\ div\ k]$
$\langle proof \rangle$

corollary *iIN-div-eq2*:
$[\![\ 0 < k;\ k \le n\ mod\ k + d\ mod\ k\ ]\!] \Longrightarrow$
$[n\ldots,d] \oslash k = [n\ div\ k\ldots,\ Suc\ (d\ div\ k)]$
$\langle proof \rangle$

corollary *iIN-div-mod-eq-0*:
$[\![\ 0 < k;\ n\ mod\ k = 0\ ]\!] \Longrightarrow [n\ldots,d] \oslash k = [n\ div\ k\ldots,d\ div\ k]$
$\langle proof \rangle$

lemma *iTILL-div*:
$0 < k \Longrightarrow [\ldots n] \oslash k = [\ldots n\ div\ k]$
$\langle proof \rangle$

lemma *iMOD-div-ge*:
$[\![\ 0 < m;\ m \le k\ ]\!] \Longrightarrow [r,\ mod\ m] \oslash k = [r\ div\ k\ldots]$
$\langle proof \rangle$
corollary *iMOD-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m] \oslash m = [r\ div\ m\ldots]$
$\langle proof \rangle$

lemma *iMOD-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k)\ ]$
$\langle proof \rangle$

lemma *iMODb-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m,\ c] \oslash m = [r\ div\ m\ldots,c]$
$\langle proof \rangle$

lemma *iMODb-div-ge*:
$[\![\ 0 < m;\ m \le k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k\ldots,(r + m * c)\ div\ k - r\ div\ k]$
$\langle proof \rangle$

corollary *iMODb-div-ge-if*:
$[\![\ 0 < m;\ m \le k\ ]\!] \Longrightarrow$

$[r, \mathit{mod}\ m,\ c] \oslash k =$
$[r\ \mathit{div}\ k\ldots,\ m * c\ \mathit{div}\ k + (\mathit{if}\ r\ \mathit{mod}\ k + m * c\ \mathit{mod}\ k < k\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)]$
$\langle \mathit{proof} \rangle$

lemma *iMODb-div*:
$[\![\ 0 < k;\ m\ \mathit{mod}\ k = 0\ ]\!] \Longrightarrow$
$[r, \mathit{mod}\ m,\ c] \oslash k = [r\ \mathit{div}\ k,\ \mathit{mod}\ (m\ \mathit{div}\ k),\ c\ ]$
$\langle \mathit{proof} \rangle$

lemmas *iT-div* =
iTILL-div
iFROM-div
iIN-div
iMOD-div
iMODb-div
iT-Div-singleton

This lemma is valid for all $k \leq m$,i. e., also for k with $m\ \mathit{mod}\ k \neq 0$.

lemma *iMODb-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ k \leq c;\ [r',\ \mathit{mod}\ m',\ c'] = [r,\ \mathit{mod}\ m,\ c] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ \mathit{div}\ k \wedge m' = m\ \mathit{div}\ k \wedge c' = c$
$\langle \mathit{proof} \rangle$

lemma *iMODb-div-mod-gr0-is-0-not-ex0*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ \mathit{mod}\ k;\ k \leq c;\ r\ \mathit{mod}\ k = 0\ ]\!] \Longrightarrow$
$\neg(\exists\, r'\ m'\ c'.\ [r',\ \mathit{mod}\ m',\ c'] = [r,\ \mathit{mod}\ m,\ c] \oslash k)$
$\langle \mathit{proof} \rangle$

lemma *iMODb-div-mod-gr0-not-ex--arith-aux1*:
$[\![\ (0{::}\mathit{nat}) < k;\ k < m;\ 0 < x1\ ]\!] \Longrightarrow$
$x1 * m + x2 - x\ \mathit{mod}\ k + x3 + x\ \mathit{mod}\ k = x1 * m + x2 + x3$
$\langle \mathit{proof} \rangle$

lemma *iMODb-div-mod-gr0-not-ex*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ \mathit{mod}\ k;\ k \leq c\ ]\!] \Longrightarrow$
$\neg(\exists\, r'\ m'\ c'.\ [r',\ \mathit{mod}\ m',\ c'] = [r,\ \mathit{mod}\ m,\ c] \oslash k)$
$\langle \mathit{proof} \rangle$

lemma *iMOD-div-eq-imp-iMODb-div-eq*:
$[\![\ 0 < k;\ k \leq m;\ [r',\ \mathit{mod}\ m'] = [r,\ \mathit{mod}\ m] \oslash k\ ]\!] \Longrightarrow$
$[r',\ \mathit{mod}\ m',\ c] = [r,\ \mathit{mod}\ m,\ c] \oslash k$
$\langle \mathit{proof} \rangle$

lemma *iMOD-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ [r',\ \mathit{mod}\ m'] = [r,\ \mathit{mod}\ m] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ \mathit{div}\ k \wedge m' = m\ \mathit{div}\ k$

$\langle proof \rangle$

lemma *iMOD-div-mod-gr0-not-ex*:
$[\![0 < k; k < m; 0 < m \ mod \ k]\!] \Longrightarrow$
$\neg (\exists r' \ m'. [r', mod \ m'] = [r, mod \ m] \oslash k)$
$\langle proof \rangle$

2.2 Interval cut operators with arithmetic interval operators

lemma
iT-Plus-cut-le2: $(I \oplus k) \downarrow\leq (t + k) = (I \downarrow\leq t) \oplus k$ **and**
iT-Plus-cut-less2: $(I \oplus k) \downarrow< (t + k) = (I \downarrow< t) \oplus k$ **and**
iT-Plus-cut-ge2: $(I \oplus k) \downarrow\geq (t + k) = (I \downarrow\geq t) \oplus k$ **and**
iT-Plus-cut-greater2: $(I \oplus k) \downarrow> (t + k) = (I \downarrow> t) \oplus k$
$\langle proof \rangle$

lemma *iT-Plus-cut-le*:
$(I \oplus k) \downarrow\leq t = (if \ t < k \ then \ \{\} \ else \ I \downarrow\leq (t - k) \oplus k)$
$\langle proof \rangle$

lemma *iT-Plus-cut-less*: $(I \oplus k) \downarrow< t = I \downarrow< (t - k) \oplus k$
$\langle proof \rangle$

lemma *iT-Plus-cut-ge*: $(I \oplus k) \downarrow\geq t = I \downarrow\geq (t - k) \oplus k$
$\langle proof \rangle$

lemma *iT-Plus-cut-greater*:
$(I \oplus k) \downarrow> t = (if \ t < k \ then \ I \oplus k \ else \ I \downarrow> (t - k) \oplus k)$
$\langle proof \rangle$

lemma
iT-Mult-cut-le2: $0 < k \Longrightarrow (I \otimes k) \downarrow\leq (t * k) = (I \downarrow\leq t) \otimes k$ **and**
iT-Mult-cut-less2: $0 < k \Longrightarrow (I \otimes k) \downarrow< (t * k) = (I \downarrow< t) \otimes k$ **and**
iT-Mult-cut-ge2: $0 < k \Longrightarrow (I \otimes k) \downarrow\geq (t * k) = (I \downarrow\geq t) \otimes k$ **and**
iT-Mult-cut-greater2: $0 < k \Longrightarrow (I \otimes k) \downarrow> (t * k) = (I \downarrow> t) \otimes k$
$\langle proof \rangle$

lemma *iT-Mult-cut-le*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\leq t = (I \downarrow\leq (t \ div \ k)) \otimes k$
$\langle proof \rangle$

lemma *iT-Mult-cut-less*:
$0 < k \Longrightarrow (I \otimes k) \downarrow< t =$
$(if \ t \ mod \ k = 0 \ then \ (I \downarrow< (t \ div \ k)) \ else \ I \downarrow< Suc \ (t \ div \ k)) \otimes k$
$\langle proof \rangle$

lemma *iT-Mult-cut-greater*:
$0 < k \Longrightarrow (I \otimes k) \downarrow> t = (I \downarrow> (t \ div \ k)) \otimes k$

⟨*proof*⟩

lemma *iT-Mult-cut-ge*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\geq t =$
$(\mathit{if}\ t\ \mathit{mod}\ k = 0\ \mathit{then}\ (I \downarrow\geq (t\ \mathit{div}\ k))\ \mathit{else}\ I \downarrow\geq \mathit{Suc}\ (t\ \mathit{div}\ k)) \otimes k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le2*: $k \leq t \Longrightarrow (I \oplus- k) \downarrow\leq (t - k) = (I \downarrow\leq t) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-less2*: $(I \oplus- k) \downarrow< (t - k) = (I \downarrow< t) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-ge2*: $(I \oplus- k) \downarrow\geq (t - k) = (I \downarrow\geq t) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-greater2*: $k \leq t \Longrightarrow (I \oplus- k) \downarrow> (t - k) = (I \downarrow> t) \oplus-$
k
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le*: $(I \oplus- k) \downarrow\leq t = I \downarrow\leq (t + k) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-less*: $(I \oplus- k) \downarrow< t = I \downarrow< (t + k) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-ge*: $(I \oplus- k) \downarrow\geq t = I \downarrow\geq (t + k) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-greater*: $(I \oplus- k) \downarrow> t = I \downarrow> (t + k) \oplus- k$
⟨*proof*⟩

lemma *iT-Minus-cut-le2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow\leq (k - t) = k \ominus (I \downarrow\geq t)$
⟨*proof*⟩

lemma *iT-Minus-cut-less2*: $(k \ominus I) \downarrow< (k - t) = k \ominus (I \downarrow> t)$
⟨*proof*⟩

lemma *iT-Minus-cut-ge2*: $(k \ominus I) \downarrow\geq (k - t) = k \ominus (I \downarrow\leq t)$
⟨*proof*⟩

lemma *iT-Minus-cut-greater2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow> (k - t) = k \ominus (I \downarrow< t)$
⟨*proof*⟩

lemma *iT-Minus-cut-le*: $(k \ominus I) \downarrow\leq t = k \ominus (I \downarrow\geq (k - t))$
⟨*proof*⟩

lemma *iT-Minus-cut-less*:

$(k \ominus I) \downarrow< t = (\mathit{if}\ t \le k\ \mathit{then}\ k \ominus (I \downarrow> (k - t))\ \mathit{else}\ k \ominus I)$
⟨*proof*⟩

lemma *iT-Minus-cut-ge*:
$(k \ominus I) \downarrow\ge t = (\mathit{if}\ t \le k\ \mathit{then}\ k \ominus (I \downarrow\le (k - t))\ \mathit{else}\ \{\})$
⟨*proof*⟩

lemma *iT-Minus-cut-greater*: $(k \ominus I) \downarrow> t = k \ominus (I \downarrow< (k - t))$
⟨*proof*⟩

lemma *iT-Div-cut-le*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\le t = I \downarrow\le (t * k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-less*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< t = I \downarrow< (t * k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-ge*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\ge t = I \downarrow\ge (t * k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-greater*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> t = I \downarrow> (t * k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-le2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\le (t\ \mathit{div}\ k) = I \downarrow\le (t - t\ \mathit{mod}\ k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-less2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< (t\ \mathit{div}\ k) = I \downarrow< (t - t\ \mathit{mod}\ k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-ge2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\ge (t\ \mathit{div}\ k) = I \downarrow\ge (t - t\ \mathit{mod}\ k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-greater2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> (t\ \mathit{div}\ k) = I \downarrow> (t - t\ \mathit{mod}\ k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

2.3 *inext* and *iprev* with interval operators

lemma *iT-Plus-inext*: $\mathit{inext}\ (n + k)\ (I \oplus k) = (\mathit{inext}\ n\ I) + k$
⟨*proof*⟩

lemma *iT-Plus-iprev*: $iprev\ (n + k)\ (I \oplus k) = (iprev\ n\ I) + k$
$\langle proof \rangle$

lemma *iT-Plus-inext2*: $k \le n \Longrightarrow inext\ n\ (I \oplus k) = (inext\ (n - k)\ I) + k$
$\langle proof \rangle$
lemma *iT-Plus-prev2*: $k \le n \Longrightarrow iprev\ n\ (I \oplus k) = (iprev\ (n - k)\ I) + k$
$\langle proof \rangle$

lemma *iT-Mult-inext*: $inext\ (n * k)\ (I \otimes k) = (inext\ n\ I) * k$
$\langle proof \rangle$

lemma *iT-Mult-iprev*: $iprev\ (n * k)\ (I \otimes k) = (iprev\ n\ I) * k$
$\langle proof \rangle$

lemma *iT-Mult-inext2-if*:
$inext\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (inext\ (n\ div\ k)\ I) * k\ else\ n)$
$\langle proof \rangle$

lemma *iT-Mult-iprev2-if*:
$iprev\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (iprev\ (n\ div\ k)\ I) * k\ else\ n)$
$\langle proof \rangle$

corollary *iT-Mult-inext2*:
$n\ mod\ k = 0 \Longrightarrow inext\ n\ (I \otimes k) = (inext\ (n\ div\ k)\ I) * k$
$\langle proof \rangle$

corollary *iT-Mult-iprev2*:
$n\ mod\ k = 0 \Longrightarrow iprev\ n\ (I \otimes k) = (iprev\ (n\ div\ k)\ I) * k$
$\langle proof \rangle$

lemma *iT-Plus-neg-inext*:
$k \le n \Longrightarrow inext\ (n - k)\ (I \oplus- k) = inext\ n\ I - k$
$\langle proof \rangle$

lemma *iT-Plus-neg-iprev*:
$iprev\ (n - k)\ (I \oplus- k) = iprev\ n\ (I \downarrow\ge k) - k$
$\langle proof \rangle$

corollary *iT-Plus-neg-inext2*: $inext\ n\ (I \oplus- k) = inext\ (n + k)\ I - k$
$\langle proof \rangle$

corollary *iT-Plus-neg-iprev2*: $iprev\ n\ (I \oplus- k) = iprev\ (n + k)\ (I \downarrow\ge k) - k$
$\langle proof \rangle$

lemma *iT-Minus-inext*:
$[\![\ k \ominus I \ne \{\};\ n \le k\ ]\!] \Longrightarrow inext\ (k - n)\ (k \ominus I) = k - iprev\ n\ I$
$\langle proof \rangle$

corollary *iT-Minus-inext2*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow \mathit{inext}\ n\ (k \ominus I) = k - \mathit{iprev}\ (k - n)\ I$
$\langle \mathit{proof} \rangle$

lemma *iT-Minus-iprev*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow \mathit{iprev}\ (k - n)\ (k \ominus I) = k - \mathit{inext}\ n\ (I \downarrow\leq k)$
$\langle \mathit{proof} \rangle$

lemma *iT-Minus-iprev2*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow \mathit{iprev}\ n\ (k \ominus I) = k - \mathit{inext}\ (k - n)\ (I \downarrow\leq k)$
$\langle \mathit{proof} \rangle$

lemma *iT-Plus-inext-nth*: $I \neq \{\} \Longrightarrow (I \oplus k) \rightarrow n = (I \rightarrow n) + k$
$\langle \mathit{proof} \rangle$

lemma *iT-Plus-iprev-nth*: $[\![\mathit{finite}\ I; I \neq \{\}]\!] \Longrightarrow (I \oplus k) \leftarrow n = (I \leftarrow n) + k$
$\langle \mathit{proof} \rangle$

lemma *iT-Mult-inext-nth*: $I \neq \{\} \Longrightarrow (I \otimes k) \rightarrow n = (I \rightarrow n) * k$
$\langle \mathit{proof} \rangle$

lemma *iT-Mult-iprev-nth*: $[\![\mathit{finite}\ I; I \neq \{\}]\!] \Longrightarrow (I \otimes k) \leftarrow n = (I \leftarrow n) * k$
$\langle \mathit{proof} \rangle$

lemma *iT-Plus-neg-inext-nth*:
$I \oplus- k \neq \{\} \Longrightarrow (I \oplus- k) \rightarrow n = (I \downarrow\geq k \rightarrow n) - k$
$\langle \mathit{proof} \rangle$

lemma *iT-Plus-neg-iprev-nth*:
$[\![\mathit{finite}\ I; I \oplus- k \neq \{\}]\!] \Longrightarrow (I \oplus- k) \leftarrow n = (I \downarrow\geq k \leftarrow n) - k$
$\langle \mathit{proof} \rangle$

lemma *iT-Minus-inext-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \rightarrow n = k - ((I \downarrow\leq k) \leftarrow n)$
$\langle \mathit{proof} \rangle$

lemma *iT-Minus-iprev-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \leftarrow n = k - ((I \downarrow\leq k) \rightarrow n)$
$\langle \mathit{proof} \rangle$

lemma *iT-Div-ge-inext-nth*:
$[\![I \neq \{\}; \forall x \in I.\ \forall y \in I.\ x < y \longrightarrow x + k \leq y]\!] \Longrightarrow$
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ \mathit{div}\ k$
$\langle \mathit{proof} \rangle$

lemma *iT-Div-mod-inext-nth*:
$[\![I \neq \{\}; \forall x \in I.\ \forall y \in I.\ x\ \mathit{mod}\ k = y\ \mathit{mod}\ k]\!] \Longrightarrow$

$(I \oslash k) \rightarrow n = (I \rightarrow n)$ *div* k
⟨*proof*⟩

lemma *iT-Div-ge-iprev-nth*:
$[\![$ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y\ ]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)$ *div* k
⟨*proof*⟩

lemma *iT-Div-mod-iprev-nth*:
$[\![$ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x$ *mod* $k = y$ *mod* $k\ ]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)$ *div* k
⟨*proof*⟩

2.4 Cardinality of intervals with interval operators

lemma *iT-Plus-card*: *card* $(I \oplus k) =$ *card* I
⟨*proof*⟩
lemma *iT-Mult-card*: $0 < k \Longrightarrow$ *card* $(I \otimes k) =$ *card* I
⟨*proof*⟩

lemma *iT-Plus-neg-card*: *card* $(I \oplus- k) =$ *card* $(I \downarrow\geq k)$
⟨*proof*⟩

lemma *iT-Plus-neg-card-le*: *card* $(I \oplus- k) \leq$ *card* I
⟨*proof*⟩

lemma *iT-Minus-card*: *card* $(k \ominus I) =$ *card* $(I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-card-le*: *finite* $I \Longrightarrow$ *card* $(k \ominus I) \leq$ *card* I
⟨*proof*⟩

lemma *iT-Div-0-card-if*:
card $(I \oslash 0) =$ (*if* $I = \{\}$ *then* 0 *else* *Suc* 0)
⟨*proof*⟩

lemma *Int-empty-sum*:
$(\sum k{\leq}(n{::}nat).$ *if* $\{\} \cap (I\ k) = \{\}$ *then* 0 *else* *Suc* $0) = 0$
⟨*proof*⟩

lemma *iT-Div-mod-partition-card*:
card $(I \cap [n * d\ldots,d -$ *Suc* $0] \oslash d) =$
(*if* $I \cap [n * d\ldots,d -$ *Suc* $0] = \{\}$ *then* 0 *else* *Suc* 0)
⟨*proof*⟩

lemma *iT-Div-conv-count*:
$0 < d \Longrightarrow I \oslash d = \{k.\ I \cap [k * d\ldots,d -$ *Suc* $0] \neq \{\}\}$
⟨*proof*⟩

lemma *iT-Div-conv-count2*:
$[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n\ ]\!] \Longrightarrow$
$I \oslash d = \{k.\ k \le n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\}$
⟨*proof*⟩

lemma *mod-partition-count-Suc*:
$\{k.\ k \le Suc\ n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\} =$
$\{k.\ k \le n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\} \cup$
$(if\ I \cap [Suc\ n * d\ldots,d - Suc\ 0] \neq \{\}\ then\ \{Suc\ n\}\ else\ \{\})$
⟨*proof*⟩

lemma *iT-Div-card*:
$\bigwedge I.[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n]\!] \Longrightarrow$
$card\ (I \oslash d) = (\sum k{\le}n.$
$if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

corollary *iT-Div-card-Suc*:
$\bigwedge I.[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n]\!] \Longrightarrow$
$card\ (I \oslash d) = (\sum k{<}Suc\ n.$
$if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩
corollary *iT-Div-Max-card*: $[\![\ 0 < d;\ finite\ I\ ]\!] \Longrightarrow$
$card\ (I \oslash d) = (\sum k{\le}Max\ I\ div\ d.$
$if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Div-card-le*: $0 < k \Longrightarrow card\ (I \oslash k) \le card\ I$
⟨*proof*⟩

lemma *iT-Div-card-inj-on*:
$inj\text{-}on\ (\lambda n.\ n\ div\ k)\ I \Longrightarrow card\ (I \oslash k) = card\ I$
⟨*proof*⟩

lemma *mod-Suc′*:
$0 < n \Longrightarrow Suc\ m\ mod\ n = (if\ m\ mod\ n < n - Suc\ 0\ then\ Suc\ (m\ mod\ n)\ else\ 0)$
⟨*proof*⟩

lemma *div-Suc*:
$0 < n \Longrightarrow Suc\ m\ div\ n = (if\ Suc\ (m\ mod\ n) = n\ then\ Suc\ (m\ div\ n)\ else\ m\ div\ n)$
⟨*proof*⟩

lemma *div-Suc′*:
$0 < n \Longrightarrow Suc\ m\ div\ n = (if\ m\ mod\ n < n - Suc\ 0\ then\ m\ div\ n\ else\ Suc\ (m\ div\ n))$
⟨*proof*⟩

lemma *iT-Div-card-ge-aux*:
$\bigwedge I.\ [\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n\ ]\!] \Longrightarrow$
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \le card\ (I \oslash d)$
⟨*proof*⟩

lemma *iT-Div-card-ge*:
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \le card\ (I \oslash d)$
⟨*proof*⟩

corollary *iT-Div-card-ge-div*: $card\ I\ div\ d \le card\ (I \oslash d)$
⟨*proof*⟩

There is no better lower bound function f for $i \oslash d$ with $card\ i$ and d as arguments.

lemma *iT-Div-card-ge--is-maximal-lower-bound*:
$\forall I\ d.\ card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \le f\ (card\ I)\ d\ \wedge$
$f\ (card\ I)\ d \le card\ (I \oslash d) \Longrightarrow$
$f\ (card\ (I{::}nat\ set))\ d = card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Plus-icard*: $icard\ (I \oplus k) = icard\ I$
⟨*proof*⟩

lemma *iT-Mult-icard*: $0 < k \Longrightarrow icard\ (I \otimes k) = icard\ I$
⟨*proof*⟩

lemma *iT-Plus-neg-icard*: $icard\ (I \oplus-\ k) = icard\ (I \downarrow\ge k)$
⟨*proof*⟩

lemma *iT-Plus-neg-icard-le*: $icard\ (I \oplus-\ k) \le icard\ I$
⟨*proof*⟩

lemma *iT-Minus-icard*: $icard\ (k \ominus I) = icard\ (I \downarrow\le k)$
⟨*proof*⟩

lemma *iT-Minus-icard-le*: $icard\ (k \ominus I) \le icard\ I$
⟨*proof*⟩

lemma *iT-Div-0-icard-if*: $icard\ (I \oslash 0) = enat\ (if\ I = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Div-mod-partition-icard*:
$icard\ (I \cap [n * d\ldots,d - Suc\ 0] \oslash d) =$

$enat\ (if\ I \cap [n * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
$\langle proof \rangle$

lemma *iT-Div-icard*:
$[\![\ 0 < d;\ finite\ I \Longrightarrow Max\ I\ div\ d \le n]\!] \Longrightarrow$
$icard\ (I \oslash d) =$
$(if\ finite\ I\ then\ enat\ (\sum k{\le}n.\ if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)\ else\ \infty)$
$\langle proof \rangle$

corollary *iT-Div-Max-icard*: $0 < d \Longrightarrow$
$icard\ (I \oslash d) = (if\ finite\ I$
$then\ enat\ (\sum k{\le}Max\ I\ div\ d.\ if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)\ else\ \infty)$
$\langle proof \rangle$

lemma *iT-Div-icard-le*: $0 < k \Longrightarrow icard\ (I \oslash k) \le icard\ I$
$\langle proof \rangle$

lemma *iT-Div-icard-inj-on*: $inj\text{-}on\ (\lambda n.\ n\ div\ k)\ I \Longrightarrow icard\ (I \oslash k) = icard\ I$
$\langle proof \rangle$

lemma *iT-Div-icard-ge*: $icard\ I\ div\ (enat\ d) + enat\ (if\ icard\ I\ mod\ (enat\ d) = 0\ then\ 0\ else\ Suc\ 0) \le icard\ (I \oslash d)$
$\langle proof \rangle$

corollary *iT-Div-icard-ge-div*: $icard\ I\ div\ (enat\ d) \le icard\ (I \oslash d)$
$\langle proof \rangle$

lemma *iT-Div-icard-ge--is-maximal-lower-bound*:
$\forall I\ d.\ icard\ I\ div\ (enat\ d) + enat\ (if\ icard\ I\ mod\ (enat\ d) = 0\ then\ 0\ else\ Suc\ 0)$
$\le f\ (icard\ I)\ d\ \wedge$
$f\ (icard\ I)\ d \le icard\ (I \oslash d) \Longrightarrow$
$f\ (icard\ (I{::}nat\ set))\ d =$
$icard\ I\ div\ (enat\ d) + enat\ (if\ icard\ I\ mod\ (enat\ d) = 0\ then\ 0\ else\ Suc\ 0)$
$\langle proof \rangle$

2.5 Results about sets of intervals

2.5.1 Set of intervals without and with empty interval

definition *iFROM-UN-set* :: $(nat\ set)\ set$
where $iFROM\text{-}UN\text{-}set \equiv \bigcup n.\ \{[n\ldots]\}$

definition *iTILL-UN-set* :: $(nat\ set)\ set$
where $iTILL\text{-}UN\text{-}set \equiv \bigcup n.\ \{[\ldots n]\}$

definition *iIN-UN-set* :: $(nat\ set)\ set$
where $iIN\text{-}UN\text{-}set \equiv \bigcup n\ d.\ \{[n\ldots,d]\}$

definition *iMOD-UN-set* :: (*nat set*) *set*
where *iMOD-UN-set* $\equiv \bigcup r\ m.\ \{[r,\ mod\ m]\}$

definition *iMODb-UN-set* :: (*nat set*) *set*
where *iMODb-UN-set* $\equiv \bigcup r\ m\ c.\ \{[r,\ mod\ m,\ c]\}$

definition *iFROM-set* :: (*nat set*) *set*
where *iFROM-set* $\equiv \{[n\ldots]\ |n.\ True\}$

definition *iTILL-set* :: (*nat set*) *set*
where *iTILL-set* $\equiv \{[\ldots n]\ |n.\ True\}$

definition *iIN-set* :: (*nat set*) *set*
where *iIN-set* $\equiv \{[n\ldots,d]\ |n\ d.\ True\}$

definition *iMOD-set* :: (*nat set*) *set*
where *iMOD-set* $\equiv \{[r,\ mod\ m]\ |r\ m.\ True\}$

definition *iMODb-set* :: (*nat set*) *set*
where *iMODb-set* $\equiv \{[r,\ mod\ m,\ c]\ |r\ m\ c.\ True\}$

definition *iMOD2-set* :: (*nat set*) *set*
where *iMOD2-set* $\equiv \{[r,\ mod\ m]\ |r\ m.\ 2 \le m\}$

definition *iMODb2-set* :: (*nat set*) *set*
where *iMODb2-set* $\equiv \{[r,\ mod\ m,\ c]\ |r\ m\ c.\ 2 \le m \wedge 1 \le c\}$

definition *iMOD2-UN-set* :: (*nat set*) *set*
where *iMOD2-UN-set* $\equiv \bigcup r.\ \bigcup m\in\{2..\}.\ \{[r,\ mod\ m]\}$

definition *iMODb2-UN-set* :: (*nat set*) *set*
where *iMODb2-UN-set* $\equiv \bigcup r.\ \bigcup m\in\{2..\}.\ \bigcup c\in\{1..\}.\ \{[r,\ mod\ m,\ c]\}$

lemmas *i-set-defs* =
iFROM-set-def iTILL-set-def iIN-set-def
iMOD-set-def iMODb-set-def
iMOD2-set-def iMODb2-set-def
lemmas *i-UN-set-defs* =
iFROM-UN-set-def iTILL-UN-set-def iIN-UN-set-def
iMOD-UN-set-def iMODb-UN-set-def
iMOD2-UN-set-def iMODb2-UN-set-def

lemma *iFROM-set-UN-set-eq*: *iFROM-set* = *iFROM-UN-set*
⟨*proof*⟩

lemma
iTILL-set-UN-set-eq: *iTILL-set* = *iTILL-UN-set* **and**
iIN-set-UN-set-eq: *iIN-set* = *iIN-UN-set* **and**
iMOD-set-UN-set-eq: *iMOD-set* = *iMOD-UN-set* **and**
iMODb-set-UN-set-eq: *iMODb-set* = *iMODb-UN-set*
⟨*proof*⟩

lemma *iMOD2-set-UN-set-eq*: *iMOD2-set* = *iMOD2-UN-set*
⟨*proof*⟩

lemma *iMODb2-set-UN-set-eq*: *iMODb2-set* = *iMODb2-UN-set*
⟨*proof*⟩

lemmas *i-set-i-UN-set-sets-eq* =
iFROM-set-UN-set-eq
iTILL-set-UN-set-eq
iIN-set-UN-set-eq
iMOD-set-UN-set-eq
iMODb-set-UN-set-eq
iMOD2-set-UN-set-eq
iMODb2-set-UN-set-eq

lemma *iMOD2-set-iMOD-set-subset*: *iMOD2-set* ⊆ *iMOD-set*
⟨*proof*⟩

lemma *iMODb2-set-iMODb-set-subset*: *iMODb2-set* ⊆ *iMODb-set*
⟨*proof*⟩

definition *i-set* :: (*nat set*) *set*
where *i-set* ≡ *iFROM-set* ∪ *iTILL-set* ∪ *iIN-set* ∪ *iMOD-set* ∪ *iMODb-set*

definition *i-UN-set* :: (*nat set*) *set*
where *i-UN-set* ≡ *iFROM-UN-set* ∪ *iTILL-UN-set* ∪ *iIN-UN-set* ∪ *iMOD-UN-set* ∪ *iMODb-UN-set*

Minimal definitions for *i-set* and *i-set*

definition *i-set-min* :: (*nat set*) *set*
where *i-set-min* ≡ *iFROM-set* ∪ *iIN-set* ∪ *iMOD2-set* ∪ *iMODb2-set*

definition *i-UN-set-min* :: (*nat set*) *set*
where *i-UN-set-min* ≡ *iFROM-UN-set* ∪ *iIN-UN-set* ∪ *iMOD2-UN-set* ∪ *iMODb2-UN-set*

definition *i-set0* :: (*nat set*) *set*
where *i-set0* ≡ *insert* {} *i-set*

lemma *i-set-i-UN-set-eq*: *i-set* = *i-UN-set*
⟨*proof*⟩

lemma *i-set-min-i-UN-set-min-eq*: *i-set-min* = *i-UN-set-min*
⟨*proof*⟩

lemma *i-set-min-eq*: *i-set* = *i-set-min*
⟨*proof*⟩

corollary *i-UN-set-i-UN-min-set-eq*: *i-UN-set* = *i-UN-set-min*
⟨*proof*⟩

lemma *i-set-min-is-minimal-let*:
  let s1 = iFROM-set; *s2= iIN-set*; *s3= iMOD2-set*; *s4= iMODb2-set in*
  s1 ∩ *s2* = {} ∧ *s1* ∩ *s3* = {} ∧ *s1* ∩ *s4* = {} ∧
  s2 ∩ *s3* = {} ∧ *s2* ∩ *s4* = {} ∧ *s3* ∩ *s4* = {}
⟨*proof*⟩

lemmas *i-set-min-is-minimal* = *i-set-min-is-minimal-let*[*simplified*]

inductive-set *i-set-ind*:: (*nat set*) *set*
where
  i-set-ind-FROM[*intro*!]: [*n*. . .] ∈ *i-set-ind*
| *i-set-ind-TILL*[*intro*!]: [. . .*n*] ∈ *i-set-ind*
| *i-set-ind-IN*[*intro*!]: [*n*. . .,*d*] ∈ *i-set-ind*
| *i-set-ind-MOD*[*intro*!]: [*r*, *mod m*] ∈ *i-set-ind*
| *i-set-ind-MODb*[*intro*!]: [*r*, *mod m*, *c*] ∈ *i-set-ind*

inductive-set *i-set0-ind* :: (*nat set*) *set*
where
  i-set0-ind-empty[*intro*!] : {} ∈ *i-set0-ind*
| *i-set0-ind-i-set*[*intro*]: *I* ∈ *i-set-ind* ⟹ *I* ∈ *i-set0-ind*

The introduction rule *i-set0-ind-i-set* is not declared a safe introduction rule, because it would disturb the correct usage of the *safe* method.

lemma *i-set-ind-subset-i-set0-ind*: *i-set-ind* ⊆ *i-set0-ind*
⟨*proof*⟩

lemma
  i-set0-ind-FROM[*intro*!] : [*n*. . .] ∈ *i-set0-ind* **and**
  i-set0-ind-TILL[*intro*!] : [. . .*n*] ∈ *i-set0-ind* **and**
  i-set0-ind-IN[*intro*!] : [*n*. . .,*d*] ∈ *i-set0-ind* **and**
  i-set0-ind-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set0-ind* **and**
  i-set0-ind-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set0-ind*
⟨*proof*⟩

lemmas *i-set0-ind-intros2* =
  i-set0-ind-empty
  i-set0-ind-FROM
  i-set0-ind-TILL

i-set0-ind-IN
i-set0-ind-MOD
i-set0-ind-MODb

lemma *i-set-i-set-ind-eq*: *i-set* = *i-set-ind*
⟨*proof*⟩

lemma *i-set0-i-set0-ind-eq*: *i-set0* = *i-set0-ind*
⟨*proof*⟩

lemma *i-set-imp-not-empty*: $I \in$ *i-set* $\Longrightarrow I \neq \{\}$
⟨*proof*⟩

lemma *i-set0-i-set-mem-conv*: ($I \in$ *i-set0*) = ($I \in$ *i-set* $\vee I = \{\}$)
⟨*proof*⟩

lemma *i-set-i-set0-mem-conv*: ($I \in$ *i-set*) = ($I \in$ *i-set0* $\wedge I \neq \{\}$)
⟨*proof*⟩

lemma *i-set0-i-set-conv*: *i-set0* $- \{\{\}\}$ = *i-set*
⟨*proof*⟩

corollary *i-set-subset-i-set0*: *i-set* $\subseteq$ *i-set0*
⟨*proof*⟩

lemma *i-set-singleton*: $\{a\} \in$ *i-set*
⟨*proof*⟩

lemma *i-set0-singleton*: $\{a\} \in$ *i-set0*
⟨*proof*⟩

corollary
i-set-FROM[*intro*!] : [n...] $\in$ *i-set* **and**
i-set-TILL[*intro*!] : [...n] $\in$ *i-set* **and**
i-set-IN[*intro*!] : [n...,d] $\in$ *i-set* **and**
i-set-MOD[*intro*!] : [r, *mod* m] $\in$ *i-set* **and**
i-set-MODb[*intro*!] : [r, *mod* m, c] $\in$ *i-set*
⟨*proof*⟩

lemmas *i-set-intros* =
i-set-FROM
i-set-TILL
i-set-IN
i-set-MOD
i-set-MODb

lemma
i-set0-empty[*intro*!]: $\{\} \in$ *i-set0* **and**
i-set0-FROM[*intro*!] : [n...] $\in$ *i-set0* **and**

$i\text{-}set0\text{-}TILL[intro!]$: $[\ldots n] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}IN[intro!]$: $[n\ldots,d] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}MOD[intro!]$: $[r,\ mod\ m] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}MODb[intro!]$: $[r,\ mod\ m,\ c] \in i\text{-}set0$
$\langle proof \rangle$

lemmas $i\text{-}set0\text{-}intros$ =
$i\text{-}set0\text{-}empty$
$i\text{-}set0\text{-}FROM$
$i\text{-}set0\text{-}TILL$
$i\text{-}set0\text{-}IN$
$i\text{-}set0\text{-}MOD$
$i\text{-}set0\text{-}MODb$

lemma $i\text{-}set\text{-}infinite\text{-}as\text{-}iMOD$:
$[\![\ I \in i\text{-}set;\ infinite\ I\ ]\!] \Longrightarrow \exists\, r\ m.\ I = [r,\ mod\ m]$
$\langle proof \rangle$

lemma $i\text{-}set\text{-}finite\text{-}as\text{-}iMODb$:
$[\![\ I \in i\text{-}set;\ finite\ I\ ]\!] \Longrightarrow \exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c]$
$\langle proof \rangle$

lemma $i\text{-}set\text{-}as\text{-}iMOD\text{-}iMODb$:
$I \in i\text{-}set \Longrightarrow (\exists\, r\ m.\ I = [r,\ mod\ m]) \vee (\exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c])$
$\langle proof \rangle$

2.5.2 Interval sets are closed under cutting

lemma $i\text{-}set\text{-}cut\text{-}le\text{-}ge\text{-}closed\text{-}disj$:
$[\![\ I \in i\text{-}set;\ t \in I;\ cut\text{-}op = (\downarrow\le) \vee cut\text{-}op = (\downarrow\ge)\ ]\!] \Longrightarrow$
$cut\text{-}op\ I\ t \in i\text{-}set$
$\langle proof \rangle$

corollary
$i\text{-}set\text{-}cut\text{-}le\text{-}closed$: $[\![\ I \in i\text{-}set;\ t \in I\ ]\!] \Longrightarrow I \downarrow\le t \in i\text{-}set$ **and**
$i\text{-}set\text{-}cut\text{-}ge\text{-}closed$: $[\![\ I \in i\text{-}set;\ t \in I\ ]\!] \Longrightarrow I \downarrow\ge t \in i\text{-}set$
$\langle proof \rangle$

lemmas $i\text{-}set\text{-}cut\text{-}le\text{-}ge\text{-}closed = i\text{-}set\text{-}cut\text{-}le\text{-}closed\ i\text{-}set\text{-}cut\text{-}ge\text{-}closed$

lemma $i\text{-}set0\text{-}cut\text{-}closed\text{-}disj$:
$[\![\ I \in i\text{-}set0;$
$cut\text{-}op = (\downarrow\le) \vee cut\text{-}op = (\downarrow\ge) \vee$
$cut\text{-}op = (\downarrow<) \vee cut\text{-}op = (\downarrow>)\ ]\!] \Longrightarrow$
$cut\text{-}op\ I\ t \in i\text{-}set0$
$\langle proof \rangle$

corollary

$\quad$ *i-set0-cut-le-closed*: $I \in \textit{i-set0} \Longrightarrow I \downarrow\leq t \in \textit{i-set0}$ **and**
$\quad$ *i-set0-cut-less-closed*: $I \in \textit{i-set0} \Longrightarrow I \downarrow< t \in \textit{i-set0}$ **and**
$\quad$ *i-set0-cut-ge-closed*: $I \in \textit{i-set0} \Longrightarrow I \downarrow\geq t \in \textit{i-set0}$ **and**
$\quad$ *i-set0-cut-greater-closed*: $I \in \textit{i-set0} \Longrightarrow I \downarrow> t \in \textit{i-set0}$
⟨*proof*⟩

lemmas *i-set0-cut-closed* =
i-set0-cut-le-closed
i-set0-cut-less-closed
i-set0-cut-ge-closed
i-set0-cut-greater-closed

2.5.3 Interval sets are closed under addition and multiplication

lemma *i-set-Plus-closed*: $I \in \textit{i-set} \Longrightarrow I \oplus k \in \textit{i-set}$
⟨*proof*⟩

lemma *i-set-Mult-closed*: $I \in \textit{i-set} \Longrightarrow I \otimes k \in \textit{i-set}$
⟨*proof*⟩

lemma *i-set0-Plus-closed*: $I \in \textit{i-set0} \Longrightarrow I \oplus k \in \textit{i-set0}$
⟨*proof*⟩

lemma *i-set0-Mult-closed*: $I \in \textit{i-set0} \Longrightarrow I \otimes k \in \textit{i-set0}$
⟨*proof*⟩

2.5.4 Interval sets are closed with certain conditions under subtraction

lemma *i-set-Plus-neg-closed*:
$\llbracket\ I \in \textit{i-set};\ \exists x{\in}I.\ k \leq x\ \rrbracket \Longrightarrow I \oplus- k \in \textit{i-set}$
⟨*proof*⟩

lemma *i-set-Minus-closed*:
$\llbracket\ I \in \textit{i-set};\ \textit{iMin}\ I \leq k\ \rrbracket \Longrightarrow k \ominus I \in \textit{i-set}$
⟨*proof*⟩

lemma *i-set0-Plus-neg-closed*: $I \in \textit{i-set0} \Longrightarrow I \oplus- k \in \textit{i-set0}$
⟨*proof*⟩

lemma *i-set0-Minus-closed*: $I \in \textit{i-set0} \Longrightarrow k \ominus I \in \textit{i-set0}$
⟨*proof*⟩

lemmas *i-set-IntOp-closed* =
i-set-Plus-closed
i-set-Mult-closed
i-set-Plus-neg-closed
i-set-Minus-closed

lemmas *i-set0-IntOp-closed* =
i-set0-Plus-closed
i-set0-Mult-closed
i-set0-Plus-neg-closed
i-set0-Minus-closed

2.5.5 Interval sets are not closed under division

lemma *iMOD-div-mod-gr0-not-in-i-set*:
$[\![0 < k; k < m; 0 < m \ mod \ k]\!] \Longrightarrow [r, mod \ m] \oslash k \notin i\text{-}set$
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-not-in-i-set*:
$[\![0 < k; k < m; 0 < m \ mod \ k; k \le c]\!] \Longrightarrow [r, mod \ m, c] \oslash k \notin i\text{-}set$
⟨*proof*⟩

lemma $[0, mod \ 3] \oslash 2 \notin i\text{-}set$
⟨*proof*⟩

lemma *i-set-Div-not-closed*: $Suc \ 0 < k \Longrightarrow \exists I \in i\text{-}set. \ I \oslash k \notin i\text{-}set$
⟨*proof*⟩
lemma *i-set0-Div-not-closed*: $Suc \ 0 < k \Longrightarrow \exists I \in i\text{-}set0. \ I \oslash k \notin i\text{-}set0$
⟨*proof*⟩

2.5.6 Sets of intervals closed under division

inductive-set *NatMultiples* :: *nat set* ⇒ *nat set*
for *F* :: *nat set*
where
NatMultiples-Factor: $k \in F \Longrightarrow k \in NatMultiples \ F$
| *NatMultiples-Product*: $[\![k \in F; m \in NatMultiples \ F]\!] \Longrightarrow k * m \in NatMultiples \ F$

lemma *NatMultiples-ex-divisor*: $m \in NatMultiples \ F \Longrightarrow \exists k \in F. \ m \ mod \ k = 0$
⟨*proof*⟩

lemma *NatMultiples-product-factor*: $[\![a \in F; b \in F]\!] \Longrightarrow a * b \in NatMultiples \ F$
⟨*proof*⟩

lemma *NatMultiples-product-factor-multiple*:
$[\![a \in F; b \in NatMultiples \ F]\!] \Longrightarrow a * b \in NatMultiples \ F$
⟨*proof*⟩

lemma *NatMultiples-product-multiple-factor*:
$[\![a \in NatMultiples \ F; b \in F]\!] \Longrightarrow a * b \in NatMultiples \ F$
⟨*proof*⟩

lemma *NatMultiples-product-multiple*:
$[\![a \in NatMultiples \ F; b \in NatMultiples \ F]\!] \Longrightarrow a * b \in NatMultiples \ F$
⟨*proof*⟩

Subset of *i-set* containing only continuous intervals, i. e., without *iMOD* and *iMODb*.

inductive-set *i-set-cont* :: (*nat set*) *set*
where
  i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont*
| *i-set-cont-TILL*[*intro*]: $[\ldots n] \in$ *i-set-cont*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont*

definition *i-set0-cont* :: (*nat set*) *set*
  where *i-set0-cont* $\equiv$ *insert* {} *i-set-cont*

lemma *i-set-cont-subset-i-set*: *i-set-cont* $\subseteq$ *i-set*
⟨*proof*⟩

lemma *i-set0-cont-subset-i-set0*: *i-set0-cont* $\subseteq$ *i-set0*
⟨*proof*⟩

Minimal definition of *i-set-cont*

inductive-set *i-set-cont-min* :: (*nat set*) *set*
where
  i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont-min*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont-min*

definition *i-set0-cont-min* :: (*nat set*) *set*
  where *i-set0-cont-min* $\equiv$ *insert* {} *i-set-cont-min*

lemma *i-set-cont-min-eq*: *i-set-cont* = *i-set-cont-min*
⟨*proof*⟩

inext and *iprev* with continuous intervals

lemma *i-set-cont-inext*:
  $[\![\ I \in \textit{i-set-cont};\ n \in I;\ \textit{finite}\ I \Longrightarrow n < \textit{Max}\ I\ ]\!] \Longrightarrow \textit{inext}\ n\ I = \textit{Suc}\ n$
⟨*proof*⟩

lemma *i-set-cont-iprev*:
  $[\![\ I \in \textit{i-set-cont};\ n \in I;\ \textit{iMin}\ I < n\ ]\!] \Longrightarrow \textit{iprev}\ n\ I = n - \textit{Suc}\ 0$
⟨*proof*⟩

lemma *i-set-cont-inext--less*:
  $[\![\ I \in \textit{i-set-cont};\ n \in I;\ n < n0;\ n0 \in I\ ]\!] \Longrightarrow \textit{inext}\ n\ I = \textit{Suc}\ n$
⟨*proof*⟩

lemma *i-set-cont-iprev--greater*:
  $[\![\ I \in \textit{i-set-cont};\ n \in I;\ n0 < n;\ n0 \in I\ ]\!] \Longrightarrow \textit{iprev}\ n\ I = n - \textit{Suc}\ 0$
⟨*proof*⟩

Sets of modulo intervals

inductive-set *i-set-mult* :: *nat* $\Rightarrow$ (*nat set*) *set*

for k :: *nat*
where
i-set-mult-FROM[*intro*!]: [n...] ∈ *i-set-mult* k
| *i-set-mult-TILL*[*intro*!]: [...n] ∈ *i-set-mult* k
| *i-set-mult-IN*[*intro*!]: [n...,d] ∈ *i-set-mult* k
| *i-set-mult-MOD*[*intro*!]: [r, *mod* $m * k$] ∈ *i-set-mult* k
| *i-set-mult-MODb*[*intro*!]: [r, *mod* $m * k$, c] ∈ *i-set-mult* k

definition *i-set0-mult* :: *nat* ⇒ (*nat set*) *set*
where *i-set0-mult* k ≡ *insert* {} (*i-set-mult* k)

lemma
i-set0-mult-empty[*intro*!]: {} ∈ *i-set0-mult* k **and**
i-set0-mult-FROM[*intro*!] : [n...] ∈ *i-set0-mult* k **and**
i-set0-mult-TILL[*intro*!] : [...n] ∈ *i-set0-mult* k **and**
i-set0-mult-IN[*intro*!] : [n...,d] ∈ *i-set0-mult* k **and**
i-set0-mult-MOD[*intro*!] : [r, *mod* $m * k$] ∈ *i-set0-mult* k **and**
i-set0-mult-MODb[*intro*!] : [r, *mod* $m * k$, c] ∈ *i-set0-mult* k
⟨*proof*⟩

lemmas *i-set0-mult-intros* =
i-set0-mult-empty
i-set0-mult-FROM
i-set0-mult-TILL
i-set0-mult-IN
i-set0-mult-MOD
i-set0-mult-MODb

lemma *i-set-mult-subset-i-set0-mult*: *i-set-mult* k ⊆ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set-mult-subset-i-set*: *i-set-mult* k ⊆ *i-set*
⟨*proof*⟩

lemma *i-set0-mult-subset-i-set0*: *i-set0-mult* k ⊆ *i-set0*
⟨*proof*⟩

lemma *i-set-mult-0-eq-i-set-cont*: *i-set-mult* *0* = *i-set-cont*
⟨*proof*⟩

lemma *i-set0-mult-0-eq-i-set0-cont*: *i-set0-mult* *0* = *i-set0-cont*
⟨*proof*⟩

lemma *i-set-mult-1-eq-i-set*: *i-set-mult* (*Suc* *0*) = *i-set*
⟨*proof*⟩

lemma *i-set0-mult-1-eq-i-set0*: *i-set0-mult* (*Suc* *0*) = *i-set0*
⟨*proof*⟩

lemma *i-set-mult-imp-not-empty*: $I \in$ *i-set-mult* $k \Longrightarrow I \neq \{\}$
⟨*proof*⟩

lemma *iMOD-in-i-set-mult-imp-divisor-mod-0*:
$[\![$ $m \neq$ *Suc 0*; $[r,$ *mod* $m] \in$ *i-set-mult* k $]\!] \Longrightarrow m$ *mod* $k = 0$
⟨*proof*⟩

lemma
divisor-mod-0-imp-iMOD-in-i-set-mult: m *mod* $k = 0 \Longrightarrow [r,$ *mod* $m] \in$ *i-set-mult* k **and**
divisor-mod-0-imp-iMODb-in-i-set-mult: m *mod* $k = 0 \Longrightarrow [r,$ *mod* $m,$ $c] \in$ *i-set-mult* k
⟨*proof*⟩

lemma *iMOD-in-i-set-mult--divisor-mod-0-conv*:
$m \neq$ *Suc 0* $\Longrightarrow ([r,$ *mod* $m] \in$ *i-set-mult* $k) = (m$ *mod* $k = 0)$
⟨*proof*⟩

lemma *i-set-mult-neq1-subset-i-set*: $k \neq$ *Suc 0* $\Longrightarrow$ *i-set-mult* $k \subset$ *i-set*
⟨*proof*⟩

lemma *mod-0-imp-i-set-mult-subset*:
a *mod* $b = 0 \Longrightarrow$ *i-set-mult* $a \subseteq$ *i-set-mult* b
⟨*proof*⟩

lemma *i-set-mult-subset-imp-mod-0*:
$[\![$ $a \neq$ *Suc 0*; (*i-set-mult* $a \subseteq$ *i-set-mult* b) $]\!] \Longrightarrow a$ *mod* $b = 0$
⟨*proof*⟩

lemma *i-set-mult-subset-conv*:
$a \neq$ *Suc 0* $\Longrightarrow$ (*i-set-mult* $a \subseteq$ *i-set-mult* b) $= (a$ *mod* $b = 0)$
⟨*proof*⟩

lemma *i-set-mult-mod-0-div*:
$[\![$ $I \in$ *i-set-mult* k; k *mod* $d = 0$ $]\!] \Longrightarrow I \oslash d \in$ *i-set-mult* (k *div* d)
⟨*proof*⟩

Intervals from *i-set-mult k* remain in *i-set* after division by d a divisor of k.

corollary *i-set-mult-mod-0-div-i-set*:
$[\![$ $I \in$ *i-set-mult* k; k *mod* $d = 0$ $]\!] \Longrightarrow I \oslash d \in$ *i-set*
⟨*proof*⟩
corollary *i-set-mult-div-self-i-set*:
$I \in$ *i-set-mult* $k \Longrightarrow I \oslash k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-mod-0-mult-in-i-set-mult*:
$[\![$ $I \in$ *i-set*; m *mod* $k = 0$ $]\!] \Longrightarrow I \otimes m \in$ *i-set-mult* k

⟨*proof*⟩

lemma *i-set-self-mult-in-i-set-mult*:
$I \in \textit{i-set} \Longrightarrow I \otimes k \in \textit{i-set-mult } k$
⟨*proof*⟩

lemma *i-set-mult-mod-gr0-div-not-in-i-set*:
$[\![\ 0 < k;\ 0 < d;\ 0 < k \textit{ mod } d\ ]\!] \Longrightarrow \exists I{\in}\textit{i-set-mult } k.\ I \oslash d \notin \textit{i-set}$
⟨*proof*⟩

lemma *i-set0-mult-mod-0-div*:
$[\![\ I \in \textit{i-set0-mult } k;\ k \textit{ mod } d = 0\ ]\!] \Longrightarrow I \oslash d \in \textit{i-set0-mult } (k \textit{ div } d)$
⟨*proof*⟩

corollary *i-set0-mult-mod-0-div-i-set0*:
$[\![\ I \in \textit{i-set0-mult } k;\ k \textit{ mod } d = 0\ ]\!] \Longrightarrow I \oslash d \in \textit{i-set0}$
⟨*proof*⟩

corollary *i-set0-mult-div-self-i-set0*:
$I \in \textit{i-set0-mult } k \Longrightarrow I \oslash k \in \textit{i-set0}$
⟨*proof*⟩

lemma *i-set0-mod-0-mult-in-i-set0-mult*:
$[\![\ I \in \textit{i-set0};\ m \textit{ mod } k = 0\ ]\!] \Longrightarrow I \otimes m \in \textit{i-set0-mult } k$
⟨*proof*⟩

lemma *i-set0-self-mult-in-i-set0-mult*:
$I \in \textit{i-set0} \Longrightarrow I \otimes k \in \textit{i-set0-mult } k$
⟨*proof*⟩

lemma *i-set0-mult-mod-gr0-div-not-in-i-set0*:
$[\![\ 0 < k;\ 0 < d;\ 0 < k \textit{ mod } d\ ]\!] \Longrightarrow \exists I{\in}\textit{i-set0-mult } k.\ I \oslash d \notin \textit{i-set0}$
⟨*proof*⟩

end

3 Temporal logic operators on natural intervals

theory *IL-TemporalOperators*
imports *IL-IntervalOperators*
begin

Bool : some additional properties

instantiation *bool* :: {*ord*, *zero*, *one*, *plus*, *times*, *order*}
begin

definition *Zero-bool-def* [*simp*]: $0 \equiv \textit{False}$
definition *One-bool-def* [*simp*]: $1 \equiv \textit{True}$
definition *add-bool-def*: $a + b \equiv a \vee b$

definition *mult-bool-def*: $a * b \equiv a \wedge b$

instance ⟨*proof*⟩

end

value *False* $<$ *True*
value *True* $<$ *True*
value *True* $\leq$ *True*

lemmas *bool-op-rel-defs* =
add-bool-def
mult-bool-def
less-bool-def
le-bool-def

instance *bool* :: *wellorder*
⟨*proof*⟩

instance *bool* :: *comm-semiring-1*
⟨*proof*⟩

3.1 Basic definitions

lemma *UNIV-nat*: $\mathbb{N} = (UNIV::nat\ set)$
⟨*proof*⟩

Universal temporal operator: Always/Globally

definition *iAll* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Always
where $iAll\ I\ P \equiv \forall t{\in}I.\ P\ t$

Existential temporal operator: Eventually/Finally

definition *iEx* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Eventually
where $iEx\ I\ P \equiv \exists t{\in}I.\ P\ t$

syntax
-iAll :: $Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ (‹(3□ - -./ -)› [0, 0, 10] 10)
-iEx :: $Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ (‹(3◇ - -./ -)› [0, 0, 10] 10)
syntax-consts
-iAll ⇌ *iAll* **and**
-iEx ⇌ *iEx*
translations
□ $t\ I.\ P$ ⇌ *CONST iAll* $I\ (\lambda t.\ P)$
◇ $t\ I.\ P$ ⇌ *CONST iEx* $I\ (\lambda t.\ P)$

Future temporal operator: Next

definition *iNext* :: $Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Next
where $iNext\ t0\ I\ P \equiv P\ (inext\ t0\ I)$

Past temporal operator: Last/Previous

definition *iLast* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Last
where *iLast t0 I P* ≡ *P* (*iprev t0 I*)

syntax
-*iNext* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3○ - - -./ -)› [0, 0, 10] 10)
-*iLast* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3⊖ - - -./ -)› [0, 0, 10] 10)
syntax-consts
-*iNext* ⇌ *iNext* **and**
-*iLast* ⇌ *iLast*
translations
○ *t t0 I*. *P* ⇌ *CONST iNext t0 I* (λ*t*. *P*)
⊖ *t t0 I*. *P* ⇌ *CONST iLast t0 I* (λ*t*. *P*)

lemma ○ *t 10* [*0*…]. (*t* + *10* > *10*)
⟨*proof*⟩

The following versions of Next and Last operator differ in the cases where no next/previous element exists or specified time point is not in interval: the weak versions return *True* and the strong versions return *False*.

definition *iNextWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Next
where *iNextWeak t0 I P* ≡ (□ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iNextStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Next
where *iNextStrong t0 I P* ≡ (◇ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iLastWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Last
where *iLastWeak t0 I P* ≡ (□ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

definition *iLastStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Last
where *iLastStrong t0 I P* ≡ (◇ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

syntax
-*iNextWeak* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$○_W$ - - -./ -)› [0, 0, 10] 10)
-*iNextStrong* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$○_S$ - - -./ -)› [0, 0, 10] 10)
-*iLastWeak* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$⊖_W$ - - -./ -)› [0, 0, 10] 10)
-*iLastStrong* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$⊖_S$ - - -./ -)› [0, 0, 10] 10)
syntax-consts
-*iNextWeak* ⇌ *iNextWeak* **and**
-*iNextStrong* ⇌ *iNextStrong* **and**
-*iLastWeak* ⇌ *iLastWeak* **and**
-*iLastStrong* ⇌ *iLastStrong*

translations
$\bigcirc_W\ t\ t0\ I.\ P \rightleftharpoons CONST\ iNextWeak\ t0\ I\ (\lambda t.\ P)$
$\bigcirc_S\ t\ t0\ I.\ P \rightleftharpoons CONST\ iNextStrong\ t0\ I\ (\lambda t.\ P)$
$\ominus_W\ t\ t0\ I.\ P \rightleftharpoons CONST\ iLastWeak\ t0\ I\ (\lambda t.\ P)$
$\ominus_S\ t\ t0\ I.\ P \rightleftharpoons CONST\ iLastStrong\ t0\ I\ (\lambda t.\ P)$

Some examples for Next and Last operator

lemma $\bigcirc\ t\ 5\ [0\ldots,10].\ ([0{::}int,10,20,30,40,50,60,70,80,90]\ !\ t < 80)$
⟨*proof*⟩

lemma $\ominus\ t\ 5\ [0\ldots,10].\ ([0{::}int,10,20,30,40,50,60,70,80,90]\ !\ t < 80)$
⟨*proof*⟩

Temporal Until operator

definition $iUntil :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Until
where $iUntil\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t')$

Temporal Since operator (past operator corresponding to Until)

definition $iSince :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Since
where $iSince\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t')$

syntax
$\text{-}iUntil ::\ Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - (3$\mathcal{U}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
$\text{-}iSince ::\ Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - (3$\mathcal{S}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
syntax-consts
-iUntil ⇌ *iUntil* **and**
-iSince ⇌ *iSince*
translations
$P.\ t\ \mathcal{U}\ t'\ I.\ Q \rightleftharpoons CONST\ iUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{S}\ t'\ I.\ Q \rightleftharpoons CONST\ iSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition $iWeakUntil :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Until/Wating-for/Unless
where $iWeakUntil\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t'))$

definition $iWeakSince :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Since/Back-to
where $iWeakSince\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t'))$

syntax
$\text{-}iWeakUntil ::\ Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$

(‹(-./ - (3$\mathcal{W}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
$\text{-}iWeakSince ::\ Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$

(‹(-./ - (3$\mathcal{B}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
syntax-consts

$\text{-}iWeakUntil \rightleftharpoons iWeakUntil$ **and**
$\text{-}iWeakSince \rightleftharpoons iWeakSince$
translations
$P.\ t\ \mathcal{W}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{B}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition $iRelease :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Release
where $iRelease\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \lor (\Diamond\ t\ I.\ P\ t \land (\Box\ t'\ (I \downarrow\leq t).\ Q\ t'))$

definition $iTrigger :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Trigger
where $iTrigger\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \lor (\Diamond\ t\ I.\ P\ t \land (\Box\ t'\ (I \downarrow\geq t).\ Q\ t'))$

syntax
$\text{-}iRelease :: Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$(\text{‹}(\text{-}./\ \text{-}\ (3\mathcal{R}\ \text{-}\ \text{-})./\ \text{-})\text{›}\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
$\text{-}iTrigger :: Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$(\text{‹}(\text{-}./\ \text{-}\ (3\mathcal{T}\ \text{-}\ \text{-})./\ \text{-})\text{›}\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
syntax-consts
$\text{-}iRelease \rightleftharpoons iRelease$ **and**
$\text{-}iTrigger \rightleftharpoons iTrigger$
translations
$P.\ t\ \mathcal{R}\ t'\ I.\ Q \rightleftharpoons CONST\ iRelease\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{T}\ t'\ I.\ Q \rightleftharpoons CONST\ iTrigger\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

lemmas *iTL-Next-defs* =
iNext-def iLast-def
iNextWeak-def iLastWeak-def
iNextStrong-def iLastStrong-def
lemmas *iTL-defs* =
iAll-def iEx-def
iUntil-def iSince-def
iWeakUntil-def iWeakSince-def
iRelease-def iTrigger-def
iTL-Next-defs

⟨*ML*⟩

3.2 Basic lemmata for temporal operators

3.2.1 Intro/elim rules

lemma
iexI[*intro*]: $[\![\ P\ t;\ t \in I\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
rev-iexI[*intro?*]: $[\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
iexE[*elim!*]: $[\![\ \Diamond\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$

$\langle proof \rangle$

lemma
iallI[*intro*!]: $(\bigwedge t.\ t \in I \Longrightarrow P\ t) \Longrightarrow \Box\ t\ I.\ P\ t$ **and**
ispec[*dest?*]: $[\![\ \Box\ t\ I.\ P\ t;\ t \in I\ ]\!] \Longrightarrow P\ t$ **and**
iallE[*elim*]: $[\![\ \Box\ t\ I.\ P\ t;\ P\ t \Longrightarrow Q;\ t \notin I \Longrightarrow Q\ ]\!] \Longrightarrow Q$
$\langle proof \rangle$

lemma
iuntilI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$ **and**
rev-iuntilI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
$\langle proof \rangle$

lemma
iuntilE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{U}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
$\langle proof \rangle$

lemma
isinceI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$ **and**
rev-isinceI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
$\langle proof \rangle$
lemma
isinceE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{S}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
$\langle proof \rangle$

3.2.2 Rewrite rules for trivial simplification

lemma *iall-triv*[*simp*]: $(\Box\ t\ I.\ P) = ((\exists\, t.\ t \in I) \longrightarrow P)$
$\langle proof \rangle$

lemma *iex-triv*[*simp*]: $(\Diamond\ t\ I.\ P) = ((\exists\, t.\ t \in I) \wedge P)$
$\langle proof \rangle$

lemma *iex-conjL1*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
$\langle proof \rangle$

lemma *iex-conjR1*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
$\langle proof \rangle$

lemma *iex-conjL2*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iex-conjR2*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iex-commute*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P\ t1\ t2) =$
$(\Diamond\ t2\ I2.\ \Diamond\ t1\ I1.\ P\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conjL1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conjR1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iall-conjL2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conjR2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iall-commute*:
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P\ t1\ t2) =$
$(\Box\ t2\ I2.\ \Box\ t1\ I1.\ P\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conj-distrib*:
$(\Box\ t\ I.\ P\ t \wedge Q\ t) = ((\Box\ t\ I.\ P\ t) \wedge (\Box\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iex-disj-distrib*:

$(\Diamond\ t\ I.\ P\ t \vee Q\ t) = ((\Diamond\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iall-conj-distrib2*:
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2 \wedge Q\ t1\ t2) =$
$((\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2) \wedge (\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩

lemma *iex-disj-distrib2*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2 \vee Q\ t1\ t2) =$
$((\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2) \vee (\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩

lemma *iUntil-disj-distrib*:
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (Q1\ t2 \vee Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q1\ t2) \vee (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩

lemma *iSince-disj-distrib*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ (Q1\ t2 \vee Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q1\ t2) \vee (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩

lemma
iNext-iff: $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iff: $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩

lemma
iNext-iEx-iff: $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iEx-iff: $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩

lemma *inext-singleton-cut-greater-not-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 \neq \{\}) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *iprev-singleton-cut-less-not-empty-iff*:
$(\{iprev\ t0\ I\} \downarrow< t0 \neq \{\}) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *inext-singleton-cut-greater-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 = \{\}) = (I \downarrow> t0 = \{\} \vee t0 \notin I)$
⟨*proof*⟩

lemma *iprev-singleton-cut-less-empty-iff*:
$(\{iprev\ t0\ I\} \downarrow< t0 = \{\}) = (I \downarrow< t0 = \{\} \vee t0 \notin I)$
⟨*proof*⟩

lemma *iNextWeak-iff* : $(\bigcirc_W\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \vee (I \downarrow> t0 = \{\}) \vee t0 \notin I)$
⟨*proof*⟩

lemma *iNextStrong-iff* : $(\bigcirc_S\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \wedge (I \downarrow> t0 \neq \{\}) \wedge t0 \in I)$
⟨*proof*⟩

lemma *iLastWeak-iff* : $(\ominus_W\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \vee (I \downarrow< t0 = \{\}) \vee t0 \notin I)$
⟨*proof*⟩

lemma *iLastStrong-iff* : $(\ominus_S\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \wedge (I \downarrow< t0 \neq \{\}) \wedge t0 \in I)$
⟨*proof*⟩

lemmas *iTL-Next-iff* =
iNext-iff iLast-iff
iNextWeak-iff iNextStrong-iff
iLastWeak-iff iLastStrong-iff

lemma
iNext-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{iprev\ t0\ I\}.\ P\ t)$
⟨*proof*⟩
lemmas *iNextLast-iff-singleton* = *iNext-iff-singleton iLast-iff-singleton*

lemma
iNext-iEx-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iEx-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{iprev\ t0\ I\}.\ P\ t)$
⟨*proof*⟩

lemma
iNextWeak-iAll-conv: $(\bigcirc_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iNextStrong-iEx-conv: $(\bigcirc_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iLastWeak-iAll-conv: $(\ominus_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$ **and**
iLastStrong-iEx-conv: $(\ominus_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$
⟨*proof*⟩

lemma
iAll-True[*simp*]: $\Box\ t\ I.\ True$ **and**
iAll-False[*simp*]: $(\Box\ t\ I.\ False) = (I = \{\})$ **and**
iEx-True[*simp*]: $(\Diamond\ t\ I.\ True) = (I \neq \{\})$ **and**
iEx-False[*simp*]: $\neg\ (\Diamond\ t\ I.\ False)$
⟨*proof*⟩

lemma *empty-iff-iAll-False*: $(I = \{\}) = (\Box\ t\ I.\ False)$ ⟨*proof*⟩
lemma *not-empty-iff-iEx-True*: $(I \neq \{\}) = (\Diamond\ t\ I.\ True)$ ⟨*proof*⟩

lemma
iNext-True: $\bigcirc\ t\ t0\ I.\ True$ **and**
iNextWeak-True: $(\bigcirc_W\ t\ t0\ I.\ True)$ **and**
iNext-False: $\neg\ (\bigcirc\ t\ t0\ I.\ False)$ **and**
iNextStrong-False: $\neg\ (\bigcirc_S\ t\ t0\ I.\ False)$
⟨*proof*⟩

lemma
iNextStrong-True: $(\bigcirc_S\ t\ t0\ I.\ True) = (I \downarrow\!>\ t0 \neq \{\} \wedge t0 \in I)$ **and**
iNextWeak-False: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ False)) = (I \downarrow\!>\ t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma
iLast-True: $\ominus\ t\ t0\ I.\ True$ **and**
iLastWeak-True: $(\ominus_W\ t\ t0\ I.\ True)$ **and**
iLast-False: $\neg\ (\ominus\ t\ t0\ I.\ False)$ **and**
iLastStrong-False: $\neg\ (\ominus_S\ t\ t0\ I.\ False)$
⟨*proof*⟩

lemma
iLastStrong-True: $(\ominus_S\ t\ t0\ I.\ True) = (I \downarrow\!<\ t0 \neq \{\} \wedge t0 \in I)$ **and**
iLastWeak-False: $(\neg\ (\ominus_W\ t\ t0\ I.\ False)) = (I \downarrow\!<\ t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *iUntil-True-left*[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iUntil-True*[*simp*]: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$
⟨*proof*⟩

lemma *iUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (I \neq \{\} \wedge Q\ (iMin\ I))$
⟨*proof*⟩

lemma *iUntil-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ False)$
⟨*proof*⟩

lemma *iSince-True-left*[*simp*]: $(True.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iSince-True-if*:
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) =$
$(if\ finite\ I\ then\ I \neq \{\}\ else\ \Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow\!>\ t1).\ P\ t2)$
⟨*proof*⟩

corollary *iSince-True-finite*[*simp*]: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) = (I \neq \{\})$
$\langle proof \rangle$

lemma *iSince-False-left*[*simp*]: $(False.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (finite\ I \wedge I \neq \{\} \wedge Q\ (Max\ I))$
$\langle proof \rangle$

lemma *iSince-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ False)$
$\langle proof \rangle$

lemma *iWeakUntil-True-left*[*simp*]: $True.\ t'\ \mathcal{W}\ t\ I.\ Q\ t$
$\langle proof \rangle$

lemma *iWeakUntil-True*[*simp*]: $P\ t'.\ t'\ \mathcal{W}\ t\ I.\ True$
$\langle proof \rangle$

lemma *iWeakUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = (I = \{\} \vee Q\ (iMin\ I))$
$\langle proof \rangle$

lemma *iWeakUntil-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ False) = (\Box\ t\ I.\ P\ t)$
$\langle proof \rangle$

lemma *iWeakSince-True-left*[*simp*]: $True.\ t'\ \mathcal{B}\ t\ I.\ Q\ t$
$\langle proof \rangle$

lemma *iWeakSince-True-disj*:
$(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ True) =$
$(I = \{\} \vee (\Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow> t1).\ P\ t2))$
$\langle proof \rangle$

lemma *iWeakSince-True-finite*[*simp*]: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{B}\ t\ I.\ True)$
$\langle proof \rangle$

lemma *iWeakSince-False-left*[*simp*]: $(False.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = (I = \{\} \vee (finite\ I \wedge Q\ (Max\ I)))$
$\langle proof \rangle$

lemma *iWeakSince-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ False) = (\Box\ t\ I.\ P\ t)$
$\langle proof \rangle$

lemma *iRelease-True-left*[*simp*]: $(True.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (I = \{\} \vee Q\ (iMin\ I))$
$\langle proof \rangle$

lemma *iRelease-True*[*simp*]: $P\ t'.\ t'\ \mathcal{R}\ t\ I.\ True$
$\langle proof \rangle$

lemma *iRelease-False-left*[*simp*]: $(False.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
$\langle proof \rangle$

lemma *iRelease-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ \mathit{False}) = (I = \{\})$
⟨*proof*⟩

lemma *iTrigger-True-left*[*simp*]: $(\mathit{True}.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (I = \{\} \vee (\Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow\geq t1).\ Q\ t2))$
⟨*proof*⟩

lemma *iTrigger-True*[*simp*]: $P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \mathit{True}$
⟨*proof*⟩

lemma *iTrigger-False-left*[*simp*]: $(\mathit{False}.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iTrigger-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \mathit{False}) = (I = \{\})$
⟨*proof*⟩

lemma
iUntil-TrueTrue[*simp*]: $(\mathit{True}.\ t'\ \mathcal{U}\ t\ I.\ \mathit{True}) = (I \neq \{\})$ **and**
iSince-TrueTrue[*simp*]: $(\mathit{True}.\ t'\ \mathcal{S}\ t\ I.\ \mathit{True}) = (I \neq \{\})$ **and**
iWeakUntil-TrueTrue[*simp*]: $\mathit{True}.\ t'\ \mathcal{W}\ t\ I.\ \mathit{True}$ **and**
iWeakSince-TrueTrue[*simp*]: $\mathit{True}.\ t'\ \mathcal{B}\ t\ I.\ \mathit{True}$ **and**
iRelease-TrueTrue[*simp*]: $\mathit{True}.\ t'\ \mathcal{R}\ t\ I.\ \mathit{True}$ **and**
iTrigger-TrueTrue[*simp*]: $\mathit{True}.\ t'\ \mathcal{T}\ t\ I.\ \mathit{True}$
⟨*proof*⟩

3.2.3 Empty sets and singletons

lemma *iAll-empty*[*simp*]: $\Box\ t\ \{\}.\ P\ t$ ⟨*proof*⟩
lemma *iEx-empty*[*simp*]: $\neg\ (\Diamond\ t\ \{\}.\ P\ t)$ ⟨*proof*⟩

lemma *iUntil-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{U}\ t1\ \{\}.\ Q\ t1)$ ⟨*proof*⟩
lemma *iSince-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{S}\ t1\ \{\}.\ Q\ t1)$ ⟨*proof*⟩
lemma *iWeakUntil-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{W}\ t1\ \{\}.\ Q\ t1$ ⟨*proof*⟩
lemma *iWeakSince-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{B}\ t1\ \{\}.\ Q\ t1$ ⟨*proof*⟩

lemma *iRelease-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{R}\ t1\ \{\}.\ Q\ t1$ ⟨*proof*⟩
lemma *iTrigger-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{T}\ t1\ \{\}.\ Q\ t1$ ⟨*proof*⟩

lemmas *iTL-empty* =
iAll-empty iEx-empty
iUntil-empty iSince-empty
iWeakUntil-empty iWeakSince-empty
iRelease-empty iTrigger-empty

lemma *iAll-singleton*[*simp*]: $(\Box\ t'\ \{t\}.\ P\ t') = P\ t$ ⟨*proof*⟩
lemma *iEx-singleton*[*simp*]: $(\Diamond\ t'\ \{t\}.\ P\ t') = P\ t$ ⟨*proof*⟩

lemma *iUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{U}\ t1\ \{t\}.\ Q\ t1) = Q\ t$

⟨*proof*⟩

lemma *iSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{S}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
⟨*proof*⟩

lemma *iWeakUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{W}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
⟨*proof*⟩

lemma *iWeakSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{B}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
⟨*proof*⟩

lemma *iRelease-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{R}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
⟨*proof*⟩

lemma *iTrigger-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{T}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
⟨*proof*⟩

lemmas *iTL-singleton* =
iAll-singleton iEx-singleton
iUntil-singleton iSince-singleton
iWeakUntil-singleton iWeakSince-singleton
iRelease-singleton iTrigger-singleton

3.2.4 Conversions between temporal operators

lemma *iAll-iEx-conv*: $(\Box\ t\ I.\ P\ t) = (\neg\ (\Diamond\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩
lemma *iEx-iAll-conv*: $(\Diamond\ t\ I.\ P\ t) = (\neg\ (\Box\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩

lemma *not-iAll*[*simp*]: $(\neg\ (\Box\ t\ I.\ P\ t)) = (\Diamond\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩
lemma *not-iEx*[*simp*]: $(\neg\ (\Diamond\ t\ I.\ P\ t)) = (\Box\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩

lemma *iUntil-iEx-conv*: $(True.\ t'\ \mathcal{U}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩
lemma *iSince-iEx-conv*: $(True.\ t'\ \mathcal{S}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩

lemma *iRelease-iAll-conv*: $(False.\ t'\ \mathcal{R}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iTrigger-iAll-conv*: $(False.\ t'\ \mathcal{T}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iWeakUntil-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩

lemma *iWeakSince-iSince-conv*: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩

lemma *iUntil-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) \wedge (\Diamond$

$t\ I.\ Q\ t)$)
⟨*proof*⟩

lemma *iSince-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iRelease-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{W}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩

lemma *iRelease-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{U}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iTrigger-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{B}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩

lemma *iTrigger-iSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{S}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iRelease-not-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩
lemma *iUntil-not-iRelease-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{R}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

The Trigger operator $\mathcal{T}$is a past operator, so that it is used for time intervals, that are bounded by a current time point, and thus are finite. For an infinite interval the stated relation to the Since operator $\mathcal{S}$would not be fulfilled.

lemma *iTrigger-not-iSince-conv*: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

lemma *iSince-not-iTrigger-conv*: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

lemma *not-iUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I \downarrow< t).\ \neg\ P\ t')))$
⟨*proof*⟩

lemma *not-iSince*:

$(\neg\ (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{>}\ t).\ \neg\ P\ t')))$
⟨*proof*⟩

lemma *iWeakUntil-conj-iUntil-conv*:
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ (P\ t2\ \wedge\ Q\ t2)) = (\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ \neg\ P\ t2))$
⟨*proof*⟩

lemma *iUntil-disj-iUntil-conv*:
$(P\ t1\ \vee\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) =$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-disj-iWeakUntil-conv*:
$(P\ t1\ \vee\ Q\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-iUntil-conj-conv*:
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (\neg\ P\ t2\ \wedge\ \neg\ Q\ t2)))$
⟨*proof*⟩

Negation and temporal operators

lemma
not-iNext[*simp*]: $(\neg\ (\bigcirc\ t\ t0\ I.\ P\ t)) = (\bigcirc\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextWeak[*simp*]: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ P\ t)) = (\bigcirc_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextStrong[*simp*]: $(\neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)) = (\bigcirc_W\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLast[*simp*]: $(\neg\ (\ominus\ t\ t0\ I.\ P\ t)) = (\ominus\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastWeak[*simp*]: $(\neg\ (\ominus_W\ t\ t0\ I.\ P\ t)) = (\ominus_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastStrong[*simp*]: $(\neg\ (\ominus_S\ t\ t0\ I.\ P\ t)) = (\ominus_W\ t\ t0\ I.\ \neg\ P\ t)$
⟨*proof*⟩

lemma *not-iWeakUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{<}\ t).\ \neg\ P\ t')))\ \wedge\ (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩
lemma *not-iWeakSince*:
$(\neg\ (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{>}\ t).\ \neg\ P\ t')))\ \wedge\ (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩

lemma *not-iRelease*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t)\ \wedge\ (\Box\ t\ I.\ P\ t \longrightarrow (\Diamond\ t\ I\ {\downarrow}{\le}\ t.\ \neg\ Q\ t)))$
⟨*proof*⟩

lemma *not-iRelease-iUntil-conv*:

$(\neg (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t)$
⟨*proof*⟩

lemma *not-iTrigger*:
$(\neg (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ \neg\ P\ t \vee (\Diamond\ t\ I \downarrow\geq t.\ \neg\ Q\ t)))$
⟨*proof*⟩

lemma *not-iTrigger-iSince-conv*:
$\mathit{finite}\ I \Longrightarrow (\neg (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t)$
⟨*proof*⟩

3.2.5 Some implication results

lemma *all-imp-iall*: $\forall x.\ P\ x \Longrightarrow \Box\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *bex-imp-lex*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \exists x.\ P\ x$ ⟨*proof*⟩

lemma *iAll-imp-iEx*: $I \neq \{\} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *i-set-iAll-imp-iEx*: $I \in \mathit{i\text{-}set} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$
⟨*proof*⟩

lemmas *iT-iAll-imp-iEx* = *iT-not-empty*[*THEN iAll-imp-iEx*]

lemma *iUntil-imp-iEx*: $P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
⟨*proof*⟩

lemma *iSince-imp-iEx*: $P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
⟨*proof*⟩

lemma *iall-subset-imp-iall*: $[\![\ \Box\ t\ B.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
⟨*proof*⟩

lemma *iex-subset-imp-iex*: $[\![\ \Diamond\ t\ A.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
⟨*proof*⟩

lemma *iall-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ I.\ Q\ t$ ⟨*proof*⟩
lemma *iex-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Diamond\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ Q\ t$ ⟨*proof*⟩

lemma *iUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2$
⟨*proof*⟩

lemma *iSince-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2$
⟨*proof*⟩

lemma *iWeakUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$

$\langle proof \rangle$

lemma *iWeakSince-imp*:
$[\![P1\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2$
$\langle proof \rangle$

lemma *iRelease-imp*:
$[\![P1\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2$
$\langle proof \rangle$

lemma *iTrigger-imp*:
$[\![P1\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2$
$\langle proof \rangle$

lemma *iMin-imp-iUntil*:
$[\![I \neq \{\};\ Q\ (iMin\ I)\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
$\langle proof \rangle$

lemma *Max-imp-iSince*:
$[\![finite\ I;\ I \neq \{\};\ Q\ (Max\ I)\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
$\langle proof \rangle$

3.2.6 Congruence rules for temporal operators' predicates

lemma *iAll-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Box\ t\ I.\ P\ (f\ t)\ t) = (\Box\ t\ I.\ P\ (g\ t)\ t)$
$\langle proof \rangle$

lemma *iEx-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Diamond\ t\ I.\ P\ (f\ t)\ t) = (\Diamond\ t\ I.\ P\ (g\ t)\ t)$
$\langle proof \rangle$

lemma *iUntil-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
$\langle proof \rangle$

lemma *iUntil-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (g\ t2)\ t2)$
$\langle proof \rangle$

lemma *iSince-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
$\langle proof \rangle$

lemma *iSince-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (g\ t2)\ t2)$

$\langle proof \rangle$

lemma *bex-subst*:
$\forall x{\in}A.\ P\ x \longrightarrow (Q\ x = Q'\ x) \Longrightarrow$
$(\exists x{\in}A.\ P\ x \wedge Q\ x) = (\exists x{\in}A.\ P\ x \wedge Q'\ x)$
$\langle proof \rangle$

lemma *iEx-subst*:
$\Box\ t\ I.\ P\ t \longrightarrow (Q\ t = Q'\ t) \Longrightarrow$
$(\Diamond\ t\ I.\ P\ t \wedge Q\ t) = (\Diamond\ t\ I.\ P\ t \wedge Q'\ t)$
$\langle proof \rangle$

3.2.7 Temporal operators with set unions/intersections and subsets

lemma *iAll-subset*: $[\![\ A \subseteq B;\ \Box\ t\ B.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
$\langle proof \rangle$

lemma *iEx-subset*: $[\![\ A \subseteq B;\ \Diamond\ t\ A.\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
$\langle proof \rangle$

lemma *iUntil-append*:
$[\![\ \mathit{finite}\ A;\ \mathit{Max}\ A \le \mathit{iMin}\ B\ ]\!] \Longrightarrow$
$P\ t1.\ t1\ \mathcal{U}\ t2\ A.\ Q\ t2 \Longrightarrow P\ t1.\ t1\ \mathcal{U}\ t2\ (A \cup B).\ Q\ t2$
$\langle proof \rangle$

lemma *iSince-prepend*:
$[\![\ \mathit{finite}\ A;\ \mathit{Max}\ A \le \mathit{iMin}\ B\ ]\!] \Longrightarrow$
$P\ t1.\ t1\ \mathcal{S}\ t2\ B.\ Q\ t2 \Longrightarrow P\ t1.\ t1\ \mathcal{S}\ t2\ (A \cup B).\ Q\ t2$
$\langle proof \rangle$

lemma
iAll-union: $[\![\ \Box\ t\ A.\ P\ t;\ \Box\ t\ B.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ (A \cup B).\ P\ t$ **and**
iAll-union-conv: $(\Box\ t\ A \cup B.\ P\ t) = ((\Box\ t\ A.\ P\ t) \wedge (\Box\ t\ B.\ P\ t))$
$\langle proof \rangle$

lemma
iEx-union: $(\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t) \Longrightarrow \Diamond\ t\ (A \cup B).\ P\ t$ **and**
iEx-union-conv: $(\Diamond\ t\ (A \cup B).\ P\ t) = ((\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t))$
$\langle proof \rangle$

lemma *iAll-inter*: $(\Box\ t\ A.\ P\ t) \vee (\Box\ t\ B.\ P\ t) \Longrightarrow \Box\ t\ (A \cap B).\ P\ t$ $\langle proof \rangle$

lemma *not-iEx-inter*:
$\exists\ A\ B\ P.\ (\Diamond\ t\ A.\ P\ t) \wedge (\Diamond\ t\ B.\ P\ t) \wedge \neg\ (\Diamond\ t\ (A \cap B).\ P\ t)$
$\langle proof \rangle$

lemma

iAll-insert: $[\![P\ t;\ \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t'\ (\mathit{insert}\ t\ I).\ P\ t'$ **and**
iAll-insert-conv: $(\Box\ t'\ (\mathit{insert}\ t\ I).\ P\ t') = (P\ t \wedge (\Box\ t'\ I.\ P\ t'))$
⟨*proof*⟩

lemma
iEx-insert: $[\![P\ t \vee (\Diamond\ t\ I.\ P\ t)\ ]\!] \Longrightarrow \Diamond\ t'\ (\mathit{insert}\ t\ I).\ P\ t'$ **and**
iEx-insert-conv: $(\Diamond\ t'\ (\mathit{insert}\ t\ I).\ P\ t') = (P\ t \vee (\Diamond\ t'\ I.\ P\ t'))$
⟨*proof*⟩

3.3 Further results for temporal operators

lemma *Collect-minI-iEx*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow\!<\ t).\ \neg\ P\ t')$
⟨*proof*⟩

lemma *iUntil-disj-conv1*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (Q\ (\mathit{iMin}\ I) \vee (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t \wedge \mathit{iMin}\ I < t))$
⟨*proof*⟩

lemma *iSince-disj-conv1*:
$[\![\mathit{finite}\ I;\ I \neq \{\}\ ]\!] \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (Q\ (\mathit{Max}\ I) \vee (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t \wedge t < \mathit{Max}\ I))$
⟨*proof*⟩

lemma *iUntil-next*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) =$
$(Q\ (\mathit{iMin}\ I) \vee (P\ (\mathit{iMin}\ I) \wedge (P\ t'.\ t'\ \mathcal{U}\ t\ (I \downarrow\!>\ (\mathit{iMin}\ I)).\ Q\ t)))$
⟨*proof*⟩

lemma *iSince-prev*: $[\![\mathit{finite}\ I;\ I \neq \{\}\ ]\!] \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) =$
$(Q\ (\mathit{Max}\ I) \vee (P\ (\mathit{Max}\ I) \wedge (P\ t'.\ t'\ \mathcal{S}\ t\ (I \downarrow\!<\ \mathit{Max}\ I).\ Q\ t)))$
⟨*proof*⟩

lemma *iNext-induct-rule*:
$[\![P\ (\mathit{iMin}\ I);\ \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'));\ t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩

lemma *iNext-induct*:
$[\![P\ (\mathit{iMin}\ I);\ \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'))\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$
⟨*proof*⟩

lemma *iLast-induct-rule*:
$[\![P\ (\mathit{Max}\ I);\ \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t'));\ \mathit{finite}\ I;\ t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩

lemma *iLast-induct*:
$[\![P\ (\mathit{Max}\ I);\ \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t'));\ \mathit{finite}\ I\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$

⟨*proof*⟩

lemma *iUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iSince-conj-not*: *finite* $I \Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakSince-conj-not*: *finite* $I \Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iNextStrong-imp-iNextWeak*: $(\bigcirc_S\ t\ t0\ I.\ P\ t) \longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *iLastStrong-imp-iLastWeak*: $(\ominus_S\ t\ t0\ I.\ P\ t) \longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma *infin-imp-iNextWeak-iNextStrong-eq-iNext*:
$[\![$ *infinite* $I;\ t0 \in I\ ]\!] \Longrightarrow$
$((\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)) \wedge ((\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t))$
⟨*proof*⟩

lemma *infin-imp-iNextWeak-eq-iNext*: $[\![$ *infinite* $I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNext*: $[\![$ *infinite* $I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNextWeak*: $[\![$ *infinite* $I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
not-in-iNext-eq: $t0 \notin I \Longrightarrow (\bigcirc\ t\ t0\ I.\ P\ t) = (P\ t0)$ **and**
not-in-iLast-eq: $t0 \notin I \Longrightarrow (\ominus\ t\ t0\ I.\ P\ t) = (P\ t0)$
⟨*proof*⟩

lemma
not-in-iNextWeak-eq: $t0 \notin I \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastWeak-eq: $t0 \notin I \Longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
not-in-iNextStrong-eq: $t0 \notin I \Longrightarrow \neg (\bigcirc_S\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastStrong-eq: $t0 \notin I \Longrightarrow \neg (\ominus_S\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
iNext-UNIV: $(\bigcirc\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextWeak-UNIV: $(\bigcirc_W\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextStrong-UNIV: $(\bigcirc_S\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$
⟨*proof*⟩

lemma
iLast-UNIV: $(\ominus\ t\ t0\ UNIV.\ P\ t) = P\ (t0 - Suc\ 0)$ **and**
iLastWeak-UNIV: $(\ominus_W\ t\ t0\ UNIV.\ P\ t) = (\mathit{if}\ 0 < t0\ \mathit{then}\ P\ (t0 - Suc\ 0)\ \mathit{else}\ True)$ **and**
iLastStrong-UNIV: $(\ominus_S\ t\ t0\ UNIV.\ P\ t) = (\mathit{if}\ 0 < t0\ \mathit{then}\ P\ (t0 - Suc\ 0)\ \mathit{else}\ False)$
⟨*proof*⟩

lemmas *iTL-Next-UNIV* =
iNext-UNIV iNextWeak-UNIV iNextStrong-UNIV
iLast-UNIV iLastWeak-UNIV iLastStrong-UNIV

lemma *inext-nth-iNext-Suc*: $(\bigcirc\ t\ (I \to n)\ I.\ P\ t) = P\ (I \to Suc\ n)$
⟨*proof*⟩

lemma *iprev-nth-iLast-Suc*: $(\ominus\ t\ (I \leftarrow n)\ I.\ P\ t) = P\ (I \leftarrow Suc\ n)$
⟨*proof*⟩

3.4 Temporal operators and arithmetic interval operators

Shifting intervals through addition and subtraction of constants. Mirroring intervals through subtraction of intervals from constants. Expanding and compressing intervals through multiplication and division by constants.

Always operator

lemma *iT-Plus-iAll-conv*: $(\Box\ t\ I \oplus k.\ P\ t) = (\Box\ t\ I.\ P\ (t + k))$
⟨*proof*⟩

lemma *iT-Mult-iAll-conv*: $(\Box\ t\ I \otimes k.\ P\ t) = (\Box\ t\ I.\ P\ (t * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iAll-conv*: $(\Box\ t\ I \oplus\!-\ k.\ P\ t) = (\Box\ t\ (I \downarrow\!\geq k).\ P\ (t - k))$
⟨*proof*⟩

lemma *iT-Minus-iAll-conv*: $(\Box\ t\ k \ominus I.\ P\ t) = (\Box\ t\ (I \downarrow\!\leq k).\ P\ (k - t))$
⟨*proof*⟩

lemma *iT-Div-iAll-conv*: $(\square\ t\ I \oslash k.\ P\ t) = (\square\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

lemmas *iT-arith-iAll-conv* =
iT-Plus-iAll-conv
iT-Mult-iAll-conv
iT-Plus-neg-iAll-conv
iT-Minus-iAll-conv
iT-Div-iAll-conv

Eventually operator

lemma
iT-Plus-iEx-conv: $(\Diamond\ t\ I \oplus k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t + k))$ **and**
iT-Mult-iEx-conv: $(\Diamond\ t\ I \otimes k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t * k))$ **and**
iT-Plus-neg-iEx-conv: $(\Diamond\ t\ I \oplus- k.\ P\ t) = (\Diamond\ t\ (I \downarrow\geq k).\ P\ (t - k))$ **and**
iT-Minus-iEx-conv: $(\Diamond\ t\ k \ominus I.\ P\ t) = (\Diamond\ t\ (I \downarrow\leq k).\ P\ (k - t))$ **and**
iT-Div-iEx-conv: $(\Diamond\ t\ I \oslash k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

Until and Since operators

lemma *iT-Plus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{U}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Until and Since operators can be converted into each other through substraction of intervals from constants

lemma *iUntil-iSince-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iSince-iUntil-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$

$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Plus-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{S}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iSince-conv*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Weak Until and Weak Since operators

lemma *iT-Plus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{W}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{B}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

lemma *iT-Plus-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iWeakSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{B}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iWeakSince-conv*:
$(P\ t1.\ t1\ \mathcal{B}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{W}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iWeakSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Release and Trigger operators

lemma *iT-Plus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{R}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{T}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

lemma *iT-Plus-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iTrigger-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 -$

k). $t1$ $\mathcal{T}$ $t2$ (I $\downarrow\geq$ k). Q ($t2$ $-$ k))
⟨*proof*⟩

lemma *iT-Minus-iTrigger-conv*:
(P $t1$. $t1$ $\mathcal{T}$ $t2$ ($k \ominus I$). Q $t2$) $=$ (P (k $-$ $t1$). $t1$ $\mathcal{R}$ $t2$ (I $\downarrow\leq$ k). Q (k $-$ $t2$))
⟨*proof*⟩

lemma *iT-Div-iTrigger-conv*:
$0 < k \Longrightarrow$ (P $t1$. $t1$ $\mathcal{T}$ $t2$ ($I \oslash k$). Q $t2$) $=$ (P ($t1$ *div* k). $t1$ $\mathcal{T}$ $t2$ I. Q ($t2$ *div* k))
⟨*proof*⟩

end