

# Nash Equilibria for Finite Games in Isabelle/HOL

Arthur Freitas Ramos      David Barros Hulak  
Ruy J. G. B. de Queiroz

June 25, 2026

## Abstract

This development formalizes Nash equilibria for finite strategic-form games, following Nash's equilibrium concept [5, 4]. It gives reusable definitions of profiles, unilateral deviations, best responses, dominant strategies, and pure Nash equilibria; proves existence for finite ordinal potential games [3] and games with dominant strategies; and verifies matching pennies as a finite game with no pure Nash equilibrium. It also formalizes mixed-strategy profiles for finite games, support lemmas for equilibrium strategies, Dirac embeddings of pure profiles, and the existence of a mixed Nash equilibrium using Brouwer's fixed point theorem. Worked examples cover the Prisoner's Dilemma, a coordination game, matching pennies, and rock-paper-scissors. AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

## Contributions and Scope

This entry gives a reusable Isabelle/HOL formalization of pure and mixed Nash equilibria for finite strategic-form games. The pure-game locale supports a finite player set with player-indexed finite strategy sets and is used to formalize unilateral deviations, best responses, dominant strategies, and ordinal potential games. The mixed-game development proves Nash's finite-game existence theorem by defining an excess-payoff map on a compact convex product of simplices and applying Brouwer's fixed point theorem from HOL-Analysis.

The mixed-game locale is intentionally less general than the pure-game locale: players and pure strategies are represented by finite HOL types, so every player uses the same finite pure-strategy type. This choice makes mixed profiles Cartesian vectors indexed by player/strategy pairs, which gives direct access to the compactness, convexity, continuity, and fixed-point infrastructure needed for the Brouwer proof. The entry also includes support lemmas for mixed equilibria, Dirac embeddings of pure profiles, and checked

examples of the Prisoner’s Dilemma, a coordination game, matching pennies, and rock-paper-scissors.

## Related Work

Le Roux, Martin-Dorel, and Smaus formalized a Nash-equilibrium existence theorem in both Coq and Isabelle for finite-outcome games derived from win/lose games [2]. Bagnall, Merten, and Stewart developed an Ss-reflect/Coq library for algorithmic game theory, including pure and mixed Nash equilibria, potential games, smooth games, approximate equilibria, and applications to routing and congestion games [1]. The present entry is narrower in mathematical scope but focuses on an AFP-style Isabelle/HOL development for finite strategic-form games, with a Brouwer-based mixed-equilibrium existence proof and small canonical examples.

## Contents

<b>1</b>	<b>Pure Nash Equilibria</b>	<b>3</b>
<b>2</b>	<b>Existence for Finite Potential Games</b>	<b>5</b>
<b>3</b>	<b>Dominant Strategies as a Degenerate Existence Result</b>	<b>6</b>
<b>4</b>	<b>Matching Pennies</b>	<b>6</b>
<b>5</b>	<b>Mixed Nash Equilibria</b>	<b>7</b>
<b>6</b>	<b>Prisoner’s Dilemma</b>	<b>12</b>
<b>7</b>	<b>Coordination Game</b>	<b>13</b>
<b>8</b>	<b>Two-Player Profile Sums</b>	<b>14</b>
<b>9</b>	<b>Matching Pennies as a Mixed Equilibrium</b>	<b>14</b>
<b>10</b>	<b>Rock-Paper-Scissors</b>	<b>15</b>

```

theory Nash-Equilibrium
  imports Main
begin

```

## 1 Pure Nash Equilibria

Nash's equilibrium concept says that a strategy profile is stable when no player can improve by changing only her own strategy. This entry formalizes the pure-strategy version for finite strategic-form games with a common strategy type.

The central locale below fixes a finite set of players, a finite nonempty strategy set for each player, and a payoff function. The player and strategy types are finite; this keeps the profile space finite while still allowing player-indexed strategy restrictions.

The development is intended as a reusable finite-game layer: the basic locale gives pure Nash equilibria and best responses, later locales derive existence from ordinal potentials and dominant strategies, and the companion mixed theory proves the finite mixed-equilibrium theorem using HOL-Analysis.

```

locale finite-game =
  fixes players :: 'p::finite set
    and strategies :: 'p  $\Rightarrow$  's::finite set
    and payoff :: 'p  $\Rightarrow$  ('p  $\Rightarrow$  's)  $\Rightarrow$  'u::preorder
  assumes nonempty-strategies:  $i \in \text{players} \implies \text{strategies } i \neq \{\}$ 
begin

```

```

definition profiles :: ('p  $\Rightarrow$  's) set where
  profiles = {s.  $\forall i \in \text{players}. s \ i \in \text{strategies } i$ }

```

```

definition deviation :: ('p  $\Rightarrow$  's)  $\Rightarrow$  'p  $\Rightarrow$  's  $\Rightarrow$  ('p  $\Rightarrow$  's) where
  deviation s i x = s(i := x)

```

```

definition Nash-equilibrium :: ('p  $\Rightarrow$  's)  $\Rightarrow$  bool where
  Nash-equilibrium s  $\longleftrightarrow$ 
    s  $\in$  profiles  $\wedge$ 
    ( $\forall i \in \text{players}. \forall x \in \text{strategies } i.
      \text{payoff } i (\text{deviation } s \ i \ x) \leq \text{payoff } i \ s$ )

```

```

definition best-response-to :: ('p  $\Rightarrow$  's)  $\Rightarrow$  'p  $\Rightarrow$  's  $\Rightarrow$  bool where
  best-response-to s i x  $\longleftrightarrow$ 
    i  $\in$  players  $\wedge$  x  $\in$  strategies i  $\wedge$ 
    ( $\forall y \in \text{strategies } i. \text{payoff } i (\text{deviation } s \ i \ y) \leq \text{payoff } i (\text{deviation } s \ i \ x)$ )

```

```

definition dominant-strategy :: 'p  $\Rightarrow$  's  $\Rightarrow$  bool where
  dominant-strategy i x  $\longleftrightarrow$ 
    i  $\in$  players  $\wedge$  x  $\in$  strategies i  $\wedge$ 
    ( $\forall s \in \text{profiles}. \forall y \in \text{strategies } i.
      \text{payoff } i (\text{deviation } s \ i \ x) \geq \text{payoff } i (\text{deviation } s \ i \ y)$ )

```

$$\text{payoff } i (\text{deviation } s \ i \ y) \leq \text{payoff } i (\text{deviation } s \ i \ x)$$

**lemma** *profiles-iff*:

$s \in \text{profiles} \longleftrightarrow (\forall i \in \text{players}. s \ i \in \text{strategies } i)$   
 ⟨proof⟩

**lemma** *profile-strategy*:

**assumes**  $s \in \text{profiles } i \in \text{players}$   
**shows**  $s \ i \in \text{strategies } i$   
 ⟨proof⟩

**lemma** *finite-profiles* [*simp, intro*]: *finite profiles*

⟨proof⟩

**lemma** *profiles-nonempty*:  $\text{profiles} \neq \{\}$

⟨proof⟩

**lemma** *deviation-apply* [*simp*]:

$\text{deviation } s \ i \ x \ j = (\text{if } j = i \ \text{then } x \ \text{else } s \ j)$   
 ⟨proof⟩

**lemma** *deviation-self* [*simp*]:  $\text{deviation } s \ i \ (s \ i) = s$

⟨proof⟩

**lemma** *deviation-in-profiles*:

**assumes**  $s \in \text{profiles } i \in \text{players } x \in \text{strategies } i$   
**shows**  $\text{deviation } s \ i \ x \in \text{profiles}$   
 ⟨proof⟩

**lemma** *Nash-equilibriumI*:

**assumes**  $s \in \text{profiles}$   
**and**  $\bigwedge i \ x. i \in \text{players} \implies x \in \text{strategies } i \implies$   
 $\text{payoff } i (\text{deviation } s \ i \ x) \leq \text{payoff } i \ s$   
**shows** *Nash-equilibrium*  $s$   
 ⟨proof⟩

**lemma** *Nash-equilibrium-profile*:

**assumes** *Nash-equilibrium*  $s$   
**shows**  $s \in \text{profiles}$   
 ⟨proof⟩

**lemma** *Nash-equilibriumD*:

**assumes** *Nash-equilibrium*  $s \ i \in \text{players } x \in \text{strategies } i$   
**shows**  $\text{payoff } i (\text{deviation } s \ i \ x) \leq \text{payoff } i \ s$   
 ⟨proof⟩

**lemma** *Nash-equilibrium-iff-best-responses*:

**assumes**  $s \in \text{profiles}$   
**shows** *Nash-equilibrium*  $s \longleftrightarrow (\forall i \in \text{players}. \text{best-response-to } s \ i \ (s \ i))$

*<proof>*

**lemma** *dominant-strategy-profile-is-Nash:*

**assumes**  $s \in \text{profiles}$

**and** *dominant:*  $\bigwedge i. i \in \text{players} \implies \text{dominant-strategy } i (s \ i)$

**shows** *Nash-equilibrium*  $s$

*<proof>*

**end**

## 2 Existence for Finite Potential Games

Pure Nash equilibria need not exist in arbitrary finite games. A standard positive result is that every finite game with an ordinal potential has a pure Nash equilibrium: choose a profile whose potential is maximal. The definition used here is the one needed for the argument: every strict unilateral payoff improvement strictly increases the potential. Exact and ordinal potential games in the sense of Monderer and Shapley satisfy this assumption.

**locale** *finite-potential-game =*

*finite-game players strategies payoff*

**for**  $\text{players} :: 'p::\text{finite set}$

**and**  $\text{strategies} :: 'p \Rightarrow 's::\text{finite set}$

**and**  $\text{payoff} :: 'p \Rightarrow ('p \Rightarrow 's) \Rightarrow 'u::\text{linorder} +$

**fixes**  $\text{potential} :: ('p \Rightarrow 's) \Rightarrow 'v::\text{linorder}$

**assumes** *potential-increases:*

$\llbracket s \in \text{profiles}; i \in \text{players}; x \in \text{strategies } i;$

$\text{payoff } i \ s < \text{payoff } i \ (\text{deviation } s \ i \ x)\rrbracket$

$\implies \text{potential } s < \text{potential } (\text{deviation } s \ i \ x)$

**begin**

**lemma** *maximal-potential-profile:*

**obtains**  $s$  **where**

$s \in \text{profiles}$

$\bigwedge t. t \in \text{profiles} \implies \text{potential } t \leq \text{potential } s$

*<proof>*

**theorem** *exists-Nash-equilibrium:*

$\exists s \in \text{profiles}. \text{Nash-equilibrium } s$

*<proof>*

**end**

### 3 Dominant Strategies as a Degenerate Existence Result

Dominant strategies give another elementary source of equilibria. The next locale packages the hypothesis that every player has a distinguished dominant strategy and derives existence by constructing the corresponding profile.

```
locale finite-dominant-strategy-game =  
  finite-game players strategies payoff  
  for players :: 'p::finite set  
    and strategies :: 'p  $\Rightarrow$  's::finite set  
    and payoff :: 'p  $\Rightarrow$  ('p  $\Rightarrow$  's)  $\Rightarrow$  'u::preorder +  
  fixes dominant :: 'p  $\Rightarrow$  's  
  assumes dominant:  $i \in \text{players} \implies \text{dominant-strategy } i \text{ (dominant } i)$   
begin
```

```
definition dominant-profile :: 'p  $\Rightarrow$  's where  
  dominant-profile  $i = (\text{if } i \in \text{players} \text{ then } \text{dominant } i \text{ else undefined})$ 
```

```
lemma dominant-profile-in-profiles:  
  dominant-profile  $\in \text{profiles}$   
   $\langle \text{proof} \rangle$ 
```

```
theorem dominant-profile-is-Nash:  
  Nash-equilibrium dominant-profile  
   $\langle \text{proof} \rangle$ 
```

**end**

### 4 Matching Pennies

The following two-player zero-sum game shows why an existence theorem for pure Nash equilibria needs additional hypotheses. The row player wants the two coins to match; the column player wants them to differ. After every pure profile, exactly one player can improve by switching sides.

```
datatype penny-player = Row | Column
```

```
datatype coin-side = Heads | Tails
```

```
instantiation penny-player :: finite  
begin
```

```
instance  
 $\langle \text{proof} \rangle$ 
```

**end**

```

instantiation coin-side :: finite
begin

instance
  ⟨proof⟩

end

definition matching-pennies-payoff :: penny-player ⇒ (penny-player ⇒ coin-side)
  ⇒ int where
  matching-pennies-payoff p s =
    (case p of
      Row ⇒ if s Row = s Column then 1 else 0
    | Column ⇒ if s Row = s Column then 0 else 1)

interpretation matching-pennies:
  finite-game UNIV λ-. UNIV matching-pennies-payoff
  ⟨proof⟩

definition switch-coin :: coin-side ⇒ coin-side where
  switch-coin x = (case x of Heads ⇒ Tails | Tails ⇒ Heads)

lemma switch-coin-neq [simp]: switch-coin x ≠ x
  ⟨proof⟩

lemma coin-side-switch:
  fixes x :: coin-side
  obtains y where y ≠ x
  ⟨proof⟩

lemma matching-pennies-no-pure-Nash:
  ¬ matching-pennies.Nash-equilibrium s
  ⟨proof⟩

end
theory Mixed-Nash-Equilibrium
  imports
    Nash-Equilibrium
    HOL-Analysis.Cartesian-Euclidean-Space
    HOL-Analysis.Brouwer-Fixpoint
begin

```

## 5 Mixed Nash Equilibria

This theory develops the mixed-strategy version of Nash equilibrium for finite games whose players and pure strategies are represented by finite HOL types. A mixed profile is a Cartesian vector indexed by player/strategy pairs.

This locale is deliberately more restrictive than the pure-game locale: ev-

ery player has the same finite pure-strategy type. In return, the profile space is a finite Cartesian product of real coordinates, so compactness, convexity, continuity, and Brouwer's fixed point theorem can be applied directly.

**type-synonym**  $(\text{'p}, \text{'s})$  *mixed-profile* =  $\text{real} \wedge (\text{'p} \times \text{'s})$

**locale** *finite-type-game* =  
**fixes** *payoff* ::  $\text{'p}::\text{finite} \Rightarrow (\text{'p} \Rightarrow \text{'s}::\text{finite}) \Rightarrow \text{real}$   
**begin**

**definition** *prob* ::  $(\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{'p} \Rightarrow \text{'s} \Rightarrow \text{real}$  **where**  
*prob*  $m$   $i$   $x$  =  $m$  \$  $(i, x)$

**definition** *mixed-profiles* ::  $(\text{'p}, \text{'s})$  *mixed-profile set* **where**  
*mixed-profiles* =  
 $\{m \in \text{cbox } 0 \ 1. \forall i. (\sum_{x \in \text{UNIV}} \text{prob } m \ i \ x) = 1\}$

**definition** *uniform-mixed-profile* ::  $(\text{'p}, \text{'s})$  *mixed-profile* **where**  
*uniform-mixed-profile* =  $(\chi \ i x. 1 / \text{real } \text{CARD}(\text{'s}))$

**definition** *opponent-weight* ::  $\text{'p} \Rightarrow (\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow (\text{'p} \Rightarrow \text{'s}) \Rightarrow \text{real}$  **where**  
*opponent-weight*  $i$   $m$   $s$  =  $(\prod_{j \in \text{UNIV} - \{i\}} \text{prob } m \ j \ (s \ j))$

**definition** *pure-deviation-payoff* ::  
 $\text{'p} \Rightarrow \text{'s} \Rightarrow (\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{real}$  **where**  
*pure-deviation-payoff*  $i$   $x$   $m$  =  
 $(\sum_{s \in \{s. s \ i = x\}} \text{opponent-weight } i \ m \ s * \text{payoff } i \ s)$

**definition** *mixed-payoff* ::  $\text{'p} \Rightarrow (\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{real}$  **where**  
*mixed-payoff*  $i$   $m$  =  
 $(\sum_{x \in \text{UNIV}} \text{prob } m \ i \ x * \text{pure-deviation-payoff } i \ x \ m)$

**definition** *mixed-Nash-equilibrium* ::  $(\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{bool}$  **where**  
*mixed-Nash-equilibrium*  $m \longleftrightarrow$   
 $m \in \text{mixed-profiles} \wedge$   
 $(\forall i \ x. \text{pure-deviation-payoff } i \ x \ m \leq \text{mixed-payoff } i \ m)$

**definition** *excess* ::  $\text{'p} \Rightarrow \text{'s} \Rightarrow (\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{real}$  **where**  
*excess*  $i$   $x$   $m$  =  $\max \ 0 \ (\text{pure-deviation-payoff } i \ x \ m - \text{mixed-payoff } i \ m)$

**definition** *excess-sum* ::  $\text{'p} \Rightarrow (\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow \text{real}$  **where**  
*excess-sum*  $i$   $m$  =  $(\sum_{x \in \text{UNIV}} \text{excess } i \ x \ m)$

**definition** *nash-map* ::  $(\text{'p}, \text{'s})$  *mixed-profile*  $\Rightarrow (\text{'p}, \text{'s})$  *mixed-profile* **where**  
*nash-map*  $m$  =  $(\chi \ i x. (\text{prob } m \ (\text{fst } i x) \ (\text{snd } i x) + \text{excess } (\text{fst } i x) \ (\text{snd } i x) \ m) /$   
 $(1 + \text{excess-sum } (\text{fst } i x) \ m))$

**lemma** *prob-nash-map*:  
 $\text{prob } (\text{nash-map } m) \ i \ x = (\text{prob } m \ i \ x + \text{excess } i \ x \ m) / (1 + \text{excess-sum } i \ m)$   
 $\langle \text{proof} \rangle$

**lemma** *mixed-profiles-prob-nonneg*:

**assumes**  $m \in \text{mixed-profiles}$

**shows**  $0 \leq \text{prob } m \ i \ x$

*<proof>*

**lemma** *mixed-profiles-prob-le-one*:

**assumes**  $m \in \text{mixed-profiles}$

**shows**  $\text{prob } m \ i \ x \leq 1$

*<proof>*

**lemma** *mixed-profiles-sum-prob*:

**assumes**  $m \in \text{mixed-profiles}$

**shows**  $(\sum_{x \in \text{UNIV.}} \text{prob } m \ i \ x) = 1$

*<proof>*

**lemma** *prob-uniform-mixed-profile* [simp]:

*prob uniform-mixed-profile i x = 1 / real CARD('s)*

*<proof>*

**lemma** *uniform-mixed-profile-in-mixed-profiles* [simp]:

*uniform-mixed-profile  $\in$  mixed-profiles*

*<proof>*

**lemma** *excess-nonneg* [simp]:  $0 \leq \text{excess } i \ x \ m$

*<proof>*

**lemma** *excess-sum-nonneg* [simp]:  $0 \leq \text{excess-sum } i \ m$

*<proof>*

**lemma** *denom-pos* [simp]:  $0 < 1 + \text{excess-sum } i \ m$

*<proof>*

**lemma** *nash-map-in-mixed-profiles*:

**assumes**  $m: m \in \text{mixed-profiles}$

**shows** *nash-map*  $m \in \text{mixed-profiles}$

*<proof>*

**lemma** *continuous-prob*:

*continuous-on S ( $\lambda m. \text{prob } m \ i \ x$ )*

*<proof>*

**lemma** *continuous-opponent-weight*:

*continuous-on S ( $\lambda m. \text{opponent-weight } i \ m \ s$ )*

*<proof>*

**lemma** *continuous-pure-deviation-payoff*:

*continuous-on S ( $\lambda m. \text{pure-deviation-payoff } i \ x \ m$ )*

*<proof>*

**lemma** *continuous-mixed-payoff*:  
*continuous-on S* ( $\lambda m. \text{mixed-payoff } i \ m$ )  
 $\langle \text{proof} \rangle$

**lemma** *continuous-excess*:  
*continuous-on S* ( $\lambda m. \text{excess } i \ x \ m$ )  
 $\langle \text{proof} \rangle$

**lemma** *continuous-excess-sum*:  
*continuous-on S* ( $\lambda m. \text{excess-sum } i \ m$ )  
 $\langle \text{proof} \rangle$

**lemma** *continuous-nash-map*:  
*continuous-on mixed-profiles nash-map*  
 $\langle \text{proof} \rangle$

**lemma** *mixed-profiles-closed*: *closed mixed-profiles*  
 $\langle \text{proof} \rangle$

**lemma** *mixed-profiles-compact*: *compact mixed-profiles*  
 $\langle \text{proof} \rangle$

**lemma** *mixed-profiles-convex*: *convex mixed-profiles*  
 $\langle \text{proof} \rangle$

**lemma** *mixed-profiles-nonempty*: *mixed-profiles*  $\neq \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *mixed-Nash-equilibrium-profile*:  
**assumes** *mixed-Nash-equilibrium*  $m$   
**shows**  $m \in \text{mixed-profiles}$   
 $\langle \text{proof} \rangle$

**lemma** *mixed-Nash-equilibriumD*:  
**assumes** *mixed-Nash-equilibrium*  $m$   
**shows**  $\text{pure-deviation-payoff } i \ x \ m \leq \text{mixed-payoff } i \ m$   
 $\langle \text{proof} \rangle$

**lemma** *mixed-Nash-support-payoff-eq*:  
**assumes**  $ne: \text{mixed-Nash-equilibrium } m$  **and**  $px: \text{prob } m \ i \ x > 0$   
**shows**  $\text{pure-deviation-payoff } i \ x \ m = \text{mixed-payoff } i \ m$   
 $\langle \text{proof} \rangle$

**lemma** *mixed-Nash-zero-probability-if-less*:  
**assumes**  $ne: \text{mixed-Nash-equilibrium } m$   
**and**  $less: \text{pure-deviation-payoff } i \ x \ m < \text{mixed-payoff } i \ m$   
**shows**  $\text{prob } m \ i \ x = 0$   
 $\langle \text{proof} \rangle$

**definition** *dirac-mixed-profile* :: ('p ⇒ 's) ⇒ ('p, 's) *mixed-profile* **where**  
*dirac-mixed-profile* s = (χ ix. if snd ix = s (fst ix) then 1 else 0)

**lemma** *prob-dirac-mixed-profile* [simp]:  
 prob (dirac-mixed-profile s) i x = (if x = s i then 1 else 0)  
 ⟨proof⟩

**lemma** *dirac-mixed-profile-in-mixed-profiles* [simp]:  
 dirac-mixed-profile s ∈ mixed-profiles  
 ⟨proof⟩

**lemma** *opponent-weight-dirac-mixed-profile*:  
 opponent-weight i (dirac-mixed-profile s) t =  
 (if ∀ j. j ≠ i → t j = s j then 1 else 0)  
 ⟨proof⟩

**lemma** *pure-deviation-payoff-dirac-mixed-profile*:  
 pure-deviation-payoff i x (dirac-mixed-profile s) = payoff i (s(i := x))  
 ⟨proof⟩

**lemma** *mixed-payoff-dirac-mixed-profile*:  
 mixed-payoff i (dirac-mixed-profile s) = payoff i s  
 ⟨proof⟩

**lemma** *dirac-mixed-Nash-equilibrium*:  
 assumes  $\bigwedge i x. \text{payoff } i (s(i := x)) \leq \text{payoff } i s$   
 shows *mixed-Nash-equilibrium* (dirac-mixed-profile s)  
 ⟨proof⟩

**lemma** *fixed-point-imp-excess-zero*:  
 assumes  $m: m \in \text{mixed-profiles}$  and  $fp: \text{nash-map } m = m$   
 shows  $\text{excess } i x m = 0$   
 ⟨proof⟩

**theorem** *exists-mixed-Nash-equilibrium*:  
 $\exists m \in \text{mixed-profiles}. \text{mixed-Nash-equilibrium } m$   
 ⟨proof⟩

**end**

**end**

**theory** *Nash-Equilibrium-Examples*  
 imports *Mixed-Nash-Equilibrium*  
 begin

**lemma** *UNIV-penny-player*: (UNIV :: penny-player set) = {Row, Column}  
 ⟨proof⟩

**lemma** *UNIV-coin-side* [simp]: (*UNIV* :: *coin-side set*) = {*Heads*, *Tails*}  
⟨*proof*⟩

**lemma** *players-except-Row* [simp]: (*UNIV* :: *penny-player set*) - {*Row*} = {*Column*}  
⟨*proof*⟩

**lemma** *players-except-Column* [simp]: (*UNIV* :: *penny-player set*) - {*Column*} =  
{*Row*}  
⟨*proof*⟩

**lemma** *player-insert-except-Row* [simp]: {*Row*, *Column*} - {*Row*} = {*Column*}  
⟨*proof*⟩

**lemma** *player-insert-except-Column* [simp]: {*Row*, *Column*} - {*Column*} = {*Row*}  
⟨*proof*⟩

## 6 Prisoner's Dilemma

The Prisoner's Dilemma gives a small example using the dominant-strategy existence result. Defection is dominant for both players, hence the all-defect profile is a pure Nash equilibrium.

**datatype** *prisoner* = *Prisoner1* | *Prisoner2*

**datatype** *prisoner-move* = *Cooperate* | *Defect*

**instantiation** *prisoner* :: *finite*  
**begin**

**instance**  
⟨*proof*⟩

**end**

**instantiation** *prisoner-move* :: *finite*  
**begin**

**instance**  
⟨*proof*⟩

**end**

**fun** *other-prisoner* :: *prisoner* ⇒ *prisoner* **where**  
  *other-prisoner Prisoner1* = *Prisoner2*  
  | *other-prisoner Prisoner2* = *Prisoner1*

**definition** *prisoners-dilemma-payoff* :: *prisoner* ⇒ (*prisoner* ⇒ *prisoner-move*)  
⇒ *int* **where**  
  *prisoners-dilemma-payoff p s* =

(case (s p, s (other-prisoner p)) of  
 (Cooperate, Cooperate)  $\Rightarrow$  3  
 | (Defect, Cooperate)  $\Rightarrow$  5  
 | (Cooperate, Defect)  $\Rightarrow$  0  
 | (Defect, Defect)  $\Rightarrow$  1)

**interpretation** prisoners-dilemma:

finite-game UNIV  $\lambda$ -. UNIV prisoners-dilemma-payoff  
 <proof>

**interpretation** prisoners-dilemma-dominant:

finite-dominant-strategy-game UNIV  $\lambda$ -. UNIV prisoners-dilemma-payoff  $\lambda$ -. De-  
 fect  
 <proof>

**lemma** prisoners-dilemma-defect-defect-Nash:

prisoners-dilemma.Nash-equilibrium ( $\lambda$ -. Defect)  
 <proof>

## 7 Coordination Game

A two-player coordination game has two pure equilibria. Both players receive payoff one when their choices agree and zero otherwise.

**datatype** coordination-choice = Choice-A | Choice-B

**instantiation** coordination-choice :: finite  
 begin

**instance**

<proof>

**end**

**definition** coordination-payoff ::

penny-player  $\Rightarrow$  (penny-player  $\Rightarrow$  coordination-choice)  $\Rightarrow$  int **where**  
 coordination-payoff p s = (if s Row = s Column then 1 else 0)

**interpretation** coordination:

finite-game UNIV  $\lambda$ -. UNIV coordination-payoff  
 <proof>

**lemma** coordination-A-A-Nash:

coordination.Nash-equilibrium ( $\lambda$ -. Choice-A)  
 <proof>

**lemma** coordination-B-B-Nash:

coordination.Nash-equilibrium ( $\lambda$ -. Choice-B)  
 <proof>

## 8 Two-Player Profile Sums

**definition** *two-player-profile* :: 'a  $\Rightarrow$  'a  $\Rightarrow$  penny-player  $\Rightarrow$  'a **where**  
*two-player-profile* r c p = (case p of Row  $\Rightarrow$  r | Column  $\Rightarrow$  c)

**lemma** *two-player-profile-simps* [simp]:  
*two-player-profile* r c Row = r  
*two-player-profile* r c Column = c  
 ⟨proof⟩

**lemma** *sum-profiles-fixed-Row*:  
**fixes** F :: (penny-player  $\Rightarrow$  'a::finite)  $\Rightarrow$  'b::comm-monoid-add  
**shows** ( $\sum s \in \{s. s \text{ Row} = x\}. F s$ ) =  
 ( $\sum y \in UNIV. F (\text{two-player-profile } x y)$ )  
 ⟨proof⟩

**lemma** *sum-profiles-fixed-Column*:  
**fixes** F :: (penny-player  $\Rightarrow$  'a::finite)  $\Rightarrow$  'b::comm-monoid-add  
**shows** ( $\sum s \in \{s. s \text{ Column} = x\}. F s$ ) =  
 ( $\sum y \in UNIV. F (\text{two-player-profile } y x)$ )  
 ⟨proof⟩

## 9 Matching Pennies as a Mixed Equilibrium

**definition** *matching-pennies-payoff-real* ::  
 penny-player  $\Rightarrow$  (penny-player  $\Rightarrow$  coin-side)  $\Rightarrow$  real **where**  
*matching-pennies-payoff-real* p s = real-of-int (matching-pennies-payoff p s)

**interpretation** *matching-pennies-mixed*:  
 finite-type-game *matching-pennies-payoff-real* ⟨proof⟩

**lemma** *matching-pennies-pure-deviation-Row*:  
*matching-pennies-mixed.pure-deviation-payoff* Row x m =  
 ( $\sum y \in UNIV.$   
*matching-pennies-mixed.prob* m Column y \*  
*matching-pennies-payoff-real* Row (two-player-profile x y))  
 ⟨proof⟩

**lemma** *matching-pennies-pure-deviation-Column*:  
*matching-pennies-mixed.pure-deviation-payoff* Column x m =  
 ( $\sum y \in UNIV.$   
*matching-pennies-mixed.prob* m Row y \*  
*matching-pennies-payoff-real* Column (two-player-profile y x))  
 ⟨proof⟩

**lemma** *matching-pennies-uniform-pure-deviation-payoff*:  
*matching-pennies-mixed.pure-deviation-payoff* i x  
*matching-pennies-mixed.uniform-mixed-profile* = 1 / 2  
 ⟨proof⟩

**lemma** *matching-pennies-uniform-mixed-payoff*:  
*matching-pennies-mixed.mixed-payoff i matching-pennies-mixed.uniform-mixed-profile*  
 $= 1 / 2$   
 ⟨proof⟩

**lemma** *matching-pennies-uniform-mixed-Nash*:  
*matching-pennies-mixed.mixed-Nash-equilibrium*  
*matching-pennies-mixed.uniform-mixed-profile*  
 ⟨proof⟩

## 10 Rock-Paper-Scissors

**datatype** *rps* = *Rock* | *Paper* | *Scissors*

**instantiation** *rps* :: *finite*  
**begin**

**instance**  
 ⟨proof⟩

**end**

**lemma** *UNIV-rps [simp]*: (*UNIV* :: *rps* set) = {*Rock*, *Paper*, *Scissors*}  
 ⟨proof⟩

**fun** *beats* :: *rps* ⇒ *rps* ⇒ *bool* **where**  
*beats Rock Scissors* = *True*  
 | *beats Paper Rock* = *True*  
 | *beats Scissors Paper* = *True*  
 | *beats - -* = *False*

**definition** *rps-payoff* :: *penny-player* ⇒ (*penny-player* ⇒ *rps*) ⇒ *real* **where**  
*rps-payoff* *p* *s* =  
 (if *s* *Row* = *s* *Column* then 0  
 else if *beats* (*s* *Row*) (*s* *Column*)  
 then if *p* = *Row* then 1 else -1  
 else if *p* = *Row* then -1 else 1)

**interpretation** *rps-mixed*:  
*finite-type-game rps-payoff* ⟨proof⟩

**lemma** *rps-pure-deviation-Row*:  
*rps-mixed.pure-deviation-payoff* *Row* *x* *m* =  
 ( $\sum_{y \in \text{UNIV}} \text{rps-mixed.prob } m \text{ } \text{Column } y * \text{rps-payoff } \text{Row } (\text{two-player-profile } x \ y)$ )  
 ⟨proof⟩

**lemma** *rps-pure-deviation-Column*:

*rps-mixed.pure-deviation-payoff* Column  $x$   $m =$   
( $\sum_{y \in UNIV} rps-mixed.prob\ m\ Row\ y * rps-payoff\ Column\ (two-player-profile\ y\ x)$ )  
{proof}

**lemma** *rps-uniform-pure-deviation-payoff*:  
*rps-mixed.pure-deviation-payoff*  $i\ x\ rps-mixed.uniform-mixed-profile = 0$   
{proof}

**lemma** *rps-uniform-mixed-payoff*:  
*rps-mixed.mixed-payoff*  $i\ rps-mixed.uniform-mixed-profile = 0$   
{proof}

**lemma** *rps-uniform-mixed-Nash*:  
*rps-mixed.mixed-Nash-equilibrium* *rps-mixed.uniform-mixed-profile*  
{proof}

end

## References

- [1] A. Bagnall, S. Merten, and G. Stewart. A library for algorithmic game theory in ssreflect/coq. *Journal of Formalized Reasoning*, 10(1):67–95, 2017. DOI: <https://doi.org/10.6092/issn.1972-5787/7235>.
- [2] S. Le Roux, E. Martin-Dorel, and J.-G. Smaus. An existence theorem of nash equilibrium in coq and isabelle. In *Proceedings Eighth International Symposium on Games, Automata, Logics and Formal Verification*, volume 256 of *Electronic Proceedings in Theoretical Computer Science*, pages 46–60, 2017. DOI: <https://doi.org/10.4204/EPTCS.256.4>.
- [3] D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 14(1):124–143, 1996. DOI: <https://doi.org/10.1006/game.1996.0044>.
- [4] J. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951. DOI: <https://doi.org/10.2307/1969529>.
- [5] J. F. Nash. Equilibrium points in  $n$ -person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950. DOI: <https://doi.org/10.1073/pnas.36.1.48>.