

# Inner Structure, Determinism and Modal Algebra of Multirelations

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## Abstract

Binary multirelations form a model of alternating nondeterminism useful for analysing games, interactions of computing systems with their environments or abstract interpretations of probabilistic programs. We investigate this alternating structure in a relational language based on power allegories extended with specific operations on multirelations. We develop algebras of modal operators over multirelations, related to concurrent dynamic logics, in this language.

## Contents

<b>1</b>	<b>Properties of Binary Relations</b>	<b>3</b>
1.1	Univalence, totality, determinism, and related properties . . .	3
1.2	Inverse image and the diagonal and graph functors . . . . .	6
1.3	Relational domain, codomain and modalities . . . . .	6
1.4	Residuation . . . . .	8
1.5	Symmetric quotient . . . . .	10
<b>2</b>	<b>Properties of Power Allegories</b>	<b>11</b>
2.1	Power transpose, epsilon, epsilonoff . . . . .	11
2.2	Relational image functor . . . . .	13
2.3	Unit and multiplication of powerset monad . . . . .	13
2.4	Subset relation . . . . .	15
2.5	Complementation relation . . . . .	16
2.6	Kleisli lifting and Kleisli composition . . . . .	17
2.7	Relational box . . . . .	18
<b>3</b>	<b>Basic Properties of Multirelations</b>	<b>19</b>
3.1	Peleg composition, parallel composition (inner union) and units	19
3.2	Tests . . . . .	22
3.3	Parallel subidentities . . . . .	24
3.4	Domain . . . . .	28

3.5	Vectors . . . . .	33
3.6	Up-closure and Parikh composition . . . . .	34
3.7	Nonterminal and terminal multirelations . . . . .	36
3.8	Powers . . . . .	41
3.9	Star . . . . .	42
3.10	Omega . . . . .	44
<b>4</b>	<b>Multirelational Properties of Power Allegories</b>	<b>45</b>
4.1	Peleg lifting . . . . .	45
4.2	Fusion and fission . . . . .	47
4.3	Galois connections for multirelations . . . . .	47
4.4	Properties of alpha, fission and fusion . . . . .	48
4.5	Properties of fusion, fission, nu and tau . . . . .	49
4.6	Box and diamond . . . . .	50
<b>5</b>	<b>Inner Structure</b>	<b>51</b>
5.1	Inner union, inner intersection and inner complement . . . . .	51
5.2	Dual . . . . .	56
5.3	Co-composition . . . . .	57
5.4	Inner order . . . . .	59
5.5	Up-closure, down-closure and convex-closure . . . . .	59
<b>6</b>	<b>Powerdomain Preorders</b>	<b>65</b>
6.1	Functional properties of multirelations . . . . .	75
6.2	Equivalences induced by powerdomain preorders . . . . .	80
6.3	Further results for convex-closure . . . . .	88
<b>7</b>	<b>Fusion and Fission</b>	<b>92</b>
7.1	Atoms and co-atoms . . . . .	92
7.2	Inner-functional properties . . . . .	93
7.3	Fusion . . . . .	96
7.4	Fission . . . . .	98
7.5	Co-fusion and co-fission . . . . .	100
<b>8</b>	<b>Modalities</b>	<b>103</b>
8.1	Tests . . . . .	103
8.2	Domain and antidomain . . . . .	110
8.3	Left residual . . . . .	113
8.4	Modal operations . . . . .	116
8.5	Goldblatt's axioms without star . . . . .	122
8.6	Goldblatt's axioms with star . . . . .	127
8.7	Propositional Hoare logic . . . . .	129

The theories formally verify results in [3, 1, 2]. See these papers for further details and related work.

The basic algebra of homogeneous binary multirelations is formalised in [4]. The present theories consider heterogeneous binary multirelations, which may have different source and target sets. While homogeneous multirelations arise as a special case where source and target sets coincide, we do not attempt to generalise the algebras of [4] to the heterogeneous case but study new concepts instead. Thus the present theories and [4] are complementary. A unification of the two approaches based on category theory is possible future work.

Algebraic structures for multirelations with Parikh composition are formalised in [5].

## 1 Properties of Binary Relations

**theory** *Relational-Properties*

**imports** *Main*

**begin**

**no-notation** *set-difference* (**infixl**  $\setminus$  65)

This is a general-purpose theory for enrichments of Rel, which is still quite basic, but helpful for developing properties of multirelations.

**notation** *relcomp* (**infixl**  $\langle ; \rangle$  75)

**notation** *converse* ( $\langle \smile \rangle$  [1000] 999)

**type-synonym** ( $'a, 'b$ ) *rel* = ( $'a \times 'b$ ) *set*

**lemma** *modular-law*:  $R ; S \cap T \subseteq (R \cap T ; S \smile) ; S$   
*<proof>*

**lemma** *compl-conv*:  $-(R \smile) = (-R) \smile$   
*<proof>*

**definition** *top* :: ( $'a, 'b$ ) *rel* **where**  
 $top = \{(a, b) \mid a \ b. \ True\}$

**abbreviation** *neg*  $R \equiv Id \cap -R$

### 1.1 Univalence, totality, determinism, and related properties

**definition** *univalent* :: ( $'a, 'b$ ) *rel*  $\Rightarrow$  *bool* **where**

*univalent*  $R = (R^\smile ; R \subseteq Id)$

**definition** *total* :: ('a,'b) rel  $\Rightarrow$  bool **where**  
*total*  $R = (Id \subseteq R ; R^\smile)$

**definition** *injective* :: ('a,'b) rel  $\Rightarrow$  bool **where**  
*injective*  $R = (R ; R^\smile \subseteq Id)$

**definition** *surjective* :: ('a,'b) rel  $\Rightarrow$  bool **where**  
*surjective*  $R = (Id \subseteq R^\smile ; R)$

**definition** *deterministic* :: ('a,'b) rel  $\Rightarrow$  bool **where**  
*deterministic*  $R = (\text{univalent } R \wedge \text{total } R)$

**definition** *bijective* :: ('a,'b) rel  $\Rightarrow$  bool **where**  
*bijective*  $R = (\text{injective } R \wedge \text{surjective } R)$

**lemma** *univalent-set*: *univalent*  $R = (\forall a b c. (a,b) \in R \wedge (a,c) \in R \longrightarrow b = c)$   
*<proof>*

[Univalent relations feature as single-valued relations in Main.](#)

**lemma** *univ-single-valued*: *univalent*  $R = \text{single-valued } R$   
*<proof>*

**lemma** *total-set*: *total*  $R = (\forall a. \exists b. (a,b) \in R)$   
*<proof>*

**lemma** *total-var*: *total*  $R = (R ; \text{top} = \text{top})$   
*<proof>*

**lemma** *deterministic-set*: *deterministic*  $R = (\forall a. \exists ! B. (a,B) \in R)$   
*<proof>*

**lemma** *deterministic-var1*: *deterministic*  $R = (R ; -Id = -R)$   
*<proof>*

**lemma** *deterministic-var2*: *deterministic*  $R = (\forall S. R ; -S = -(R ; S))$   
*<proof>*

**lemma** *inj-univ*: *injective*  $R = \text{univalent } (R^\smile)$   
*<proof>*

**lemma** *injective-set*: *injective*  $S = (\forall a b c. (a,c) \in S \wedge (b,c) \in S \longrightarrow a = b)$   
*<proof>*

**lemma** *surj-tot*: *surjective*  $R = \text{total } (R^\smile)$   
*<proof>*

**lemma** *surjective-set*: *surjective*  $S = (\forall b. \exists a. (a,b) \in S)$

*<proof>*

**lemma** *surj-var*: *surjective*  $R = (R^\sim ; \text{top} = \text{top})$   
*<proof>*

**lemma** *bij-det*: *bijjective*  $R = \text{deterministic } (R^\sim)$   
*<proof>*

**lemma** *univ-relcomp*: *univalent*  $R \implies \text{univalent } S \implies \text{univalent } (R ; S)$   
*<proof>*

**lemma** *tot-relcomp*: *total*  $R \implies \text{total } S \implies \text{total } (R ; S)$   
*<proof>*

**lemma** *det-relcomp*: *deterministic*  $R \implies \text{deterministic } S \implies \text{deterministic } (R ; S)$   
*<proof>*

**lemma** *inj-relcomp*: *injective*  $R \implies \text{injective } S \implies \text{injective } (R ; S)$   
*<proof>*

**lemma** *surj-relcomp*: *surjective*  $R \implies \text{surjective } S \implies \text{surjective } (R ; S)$   
*<proof>*

**lemma** *bij-relcomp*: *bijjective*  $R \implies \text{bijjective } S \implies \text{bijjective } (R ; S)$   
*<proof>*

**lemma** *det-Id*: *deterministic*  $\text{Id}$   
*<proof>*

**lemma** *bij-Id*: *bijjective*  $\text{Id}$   
*<proof>*

**lemma** *tot-top*: *total*  $\text{top}$   
*<proof>*

**lemma** *tot-surj*: *surjective*  $\text{top}$   
*<proof>*

**lemma** *det-meet-distl*: *univalent*  $R \implies R ; (S \cap T) = R ; S \cap R ; T$   
*<proof>*

**lemma** *inj-meet-distr*: *injective*  $T \implies (R \cap S) ; T = R ; T \cap S ; T$   
*<proof>*

**lemma** *univ-modular*: *univalent*  $S \implies R ; S \cap T = (R \cap T ; S^\sim) ; S$   
*<proof>*

## 1.2 Inverse image and the diagonal and graph functors

**definition**  $Invim :: ('a, 'b) \text{ rel} \Rightarrow 'b \text{ set} \Rightarrow 'a \text{ set}$  **where**

$$Invim R = Image (R^\smile)$$

**definition**  $Delta :: 'a \text{ set} \Rightarrow ('a, 'a) \text{ rel}$  ( $\langle \Delta \rangle$ ) **where**

$$\Delta P = \{(p, p) \mid p. p \in P\}$$

**definition**  $Grph :: ('a \Rightarrow 'b) \Rightarrow ('a, 'b) \text{ rel}$  **where**

$$Grph f = \{(x, y). y = f x\}$$

**lemma**  $Image\text{-}Grph$  [*simp*]:  $Image \circ Grph = image$

*\langle proof \rangle*

## 1.3 Relational domain, codomain and modalities

Domain and codomain (range) maps have been defined in Main, but they return sets instead of relations.

**definition**  $dom :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$  **where**

$$dom R = Id \cap R ; R^\smile$$

**definition**  $cod :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$  **where**

$$cod R = dom (R^\smile)$$

**definition**  $rel\text{-}fdia :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$  ( $\langle (|-) \rangle$ ) [61,81] 82)

**where**

$$|R\rangle Q = dom (R ; dom Q)$$

**definition**  $rel\text{-}bdia :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$  ( $\langle \langle (|-) \rangle \rangle$ ) [61,81] 82)

**where**

$$rel\text{-}bdia R = rel\text{-}fdia (R^\smile)$$

**definition**  $rel\text{-}fbox :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$  ( $\langle (|-) \rangle$ ) [61,81] 82)

**where**

$$|R\rangle Q = neg (dom (R ; neg (dom Q)))$$

**definition**  $rel\text{-}bbox :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$  ( $\langle \langle (|-) \rangle \rangle$ ) [61,81] 82)

**where**

$$rel\text{-}bbox R = rel\text{-}fbox (R^\smile)$$

**lemma**  $rel\text{-}bdia\text{-}def\text{-}var$ :  $rel\text{-}bdia = rel\text{-}fdia \circ converse$

*\langle proof \rangle*

**lemma**  $dom\text{-}set$ :  $dom R = \{(a, a) \mid a. \exists b. (a, b) \in R\}$

*\langle proof \rangle*

**lemma**  $dom\text{-}Domain$ :  $dom = \Delta \circ Domain$

*\langle proof \rangle*

**lemma** *cod-set*:  $\text{cod } R = \{(b,b) \mid b. \exists a. (a,b) \in R\}$   
*<proof>*

**lemma** *cod-Range*:  $\text{cod} = \Delta \circ \text{Range}$   
*<proof>*

**lemma** *rel-fdia-set*:  $|R\rangle Q = \{(a,a) \mid a. \exists b. (a,b) \in R \wedge (b,b) \in \text{dom } Q\}$   
*<proof>*

**lemma** *rel-bdia-set*:  $\langle R| P = \{(b,b) \mid b. \exists a. (a,b) \in R \wedge (a,a) \in \text{dom } P\}$   
*<proof>*

**lemma** *rel-fbox-set*:  $|R] Q = \{(a,a) \mid a. \forall b. (a,b) \in R \longrightarrow (b,b) \in \text{dom } Q\}$   
*<proof>*

**lemma** *rel-bbox-set*:  $[R| P = \{(b,b) \mid b. \forall a. (a,b) \in R \longrightarrow (a,a) \in \text{dom } P\}$   
*<proof>*

**lemma** *dom-alt-def*:  $\text{dom } R = \text{Id} \cap R ; \text{top}$   
*<proof>*

**lemma** *dom-gla*:  $(\text{dom } R \subseteq \text{Id} \cap S) = (R \subseteq (\text{Id} \cap S) ; R)$   
*<proof>*

**lemma** *dom-gla-top*:  $(\text{dom } R \subseteq \text{Id} \cap S) = (R \subseteq (\text{Id} \cap S) ; \text{top})$   
*<proof>*

**lemma** *dom-subid*:  $(\text{dom } R = R) = (R = \text{Id} \cap R)$   
*<proof>*

**lemma** *dom-cod*:  $(\text{dom } R = R) = (\text{cod } R = R)$   
*<proof>*

**lemma** *dom-top*:  $R ; \text{top} = \text{dom } R ; \text{top}$   
*<proof>*

**lemma** *top-dom*:  $\text{dom } R = \text{dom } (R ; \text{top})$   
*<proof>*

**lemma** *cod-top*:  $\text{cod } R = \text{Id} \cap \text{top} ; R$   
*<proof>*

**lemma** *dom-conv* [*simp*]:  $(\text{dom } R)^\smile = \text{dom } R$   
*<proof>*

**lemma** *total-dom*:  $\text{total } R = (\text{dom } R = \text{Id})$   
*<proof>*

**lemma** *surj-cod*:  $\text{surjective } R = (\text{cod } R = \text{Id})$

$\langle proof \rangle$

**lemma** *fdia-demod*:  $(|R\rangle P \subseteq dom Q) = (R ; dom P \subseteq dom Q ; R)$   
 $\langle proof \rangle$

**lemma** *bbox-demod*:  $(dom P \subseteq [R| Q) = (R ; dom P \subseteq dom Q ; R)$   
 $\langle proof \rangle$

**lemma** *bdia-demod*:  $(\langle R| P \subseteq dom Q) = (dom P ; R \subseteq R ; dom Q)$   
 $\langle proof \rangle$

**lemma** *fbox-demod*:  $(dom P \subseteq |R] Q) = (dom P ; R \subseteq R ; dom Q)$   
 $\langle proof \rangle$

**lemma** *fdia-demod-top*:  $(|R\rangle P \subseteq dom Q) = (R ; dom P ; top \subseteq dom Q ; top)$   
 $\langle proof \rangle$

**lemma** *bbox-demod-top*:  $(dom P \subseteq [R| Q) = (R ; dom P ; top \subseteq dom Q ; top)$   
 $\langle proof \rangle$

**lemma** *fdia-bbox-galois*:  $(|R\rangle P \subseteq dom Q) = (dom P \subseteq [R| Q)$   
 $\langle proof \rangle$

**lemma** *bdia-fbox-galois*:  $(\langle R| P \subseteq dom Q) = (dom P \subseteq |R] Q)$   
 $\langle proof \rangle$

**lemma** *fdia-bdia-conjugation*:  $(|R\rangle P \subseteq neg (dom Q)) = (\langle R| Q \subseteq neg (dom P))$   
 $\langle proof \rangle$

**lemma** *bfox-bbox-conjugation*:  $(neg (dom Q) \subseteq |R] P) = (neg (dom P) \subseteq [R| Q)$   
 $\langle proof \rangle$

## 1.4 Residuation

**definition** *lres* ::  $(a',c) rel \Rightarrow (b',c) rel \Rightarrow (a',b) rel$  (**infixl**  $\langle // \rangle$  75)  
**where**  $R // S = \{(a,b). \forall c. (b,c) \in S \longrightarrow (a,c) \in R\}$

**definition** *rres* ::  $(c',a) rel \Rightarrow (c',b) rel \Rightarrow (a',b) rel$  (**infixl**  $\langle \rangle$  75)  
**where**  $R \setminus S = \{(b,a). \forall c. (c,b) \in R \longrightarrow (c,a) \in S\}$

**lemma** *rres-lres-conv*:  $R \setminus S = (S^\smile // R^\smile)^\smile$   
 $\langle proof \rangle$

**lemma** *lres-galois*:  $(R ; S \subseteq T) = (R \subseteq T // S)$   
 $\langle proof \rangle$

**lemma** *rres-galois*:  $(R ; S \subseteq T) = (S \subseteq R \setminus T)$   
 $\langle proof \rangle$

**lemma** *lres-compl*:  $R \parallel S = -(-R ; S^\smile)$   
 ⟨proof⟩

**lemma** *rres-compl*:  $R \setminus S = -(R^\smile ; -S)$   
 ⟨proof⟩

**lemma** *lres-simp [simp]*:  $(R \parallel R) ; R = R$   
 ⟨proof⟩

**lemma** *rres-simp [simp]*:  $R ; (R \setminus R) = R$   
 ⟨proof⟩

**lemma** *lres-curry*:  $R \parallel (T ; S) = (R \parallel S) \parallel T$   
 ⟨proof⟩

**lemma** *rres-curry*:  $(R ; S) \setminus T = S \setminus (R \setminus T)$   
 ⟨proof⟩

**lemma** *lres-Id*:  $Id \subseteq R \parallel R$   
 ⟨proof⟩

**lemma** *det-lres*: *deterministic*  $R \implies (R ; S) \parallel S = R ; (S \parallel S)$   
 ⟨proof⟩

**lemma** *det-rres*: *deterministic*  $(R^\smile) \implies S \setminus (S ; R) = (S \setminus S) ; R$   
 ⟨proof⟩

**lemma** *rres-bij*: *bijective*  $S \implies (R \setminus T) ; S = R \setminus (T ; S)$   
 ⟨proof⟩

**lemma** *lres-bij*: *bijective*  $S \implies (R \parallel T^\smile) ; S = R \parallel (T ; S)^\smile$   
 ⟨proof⟩

**lemma** *dom-rres-top*:  $(\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top})) = (\text{dom } P ; \text{top} \subseteq R \setminus (\text{dom } Q ; \text{top}))$   
 ⟨proof⟩

**lemma** *dom-rres-top-var*:  $(\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top})) = (P ; \text{top} \subseteq R \setminus (Q ; \text{top}))$   
 ⟨proof⟩

**lemma** *fdia-rres-top*:  $(|R\rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top}))$   
 ⟨proof⟩

**lemma** *fdia-rres-top-var*:  $(|R\rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq R \setminus (Q ; \text{top}))$   
 ⟨proof⟩

**lemma** *dom-galois-var2*:  $(|R\rangle (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q) ; \text{top}))$

$\langle proof \rangle$

**lemma** *rres-top*:  $R \setminus (dom\ Q ; top) ; top = R \setminus (dom\ Q ; top)$   
 $\langle proof \rangle$

**lemma** *ddd-var*:  $(\ |R\rangle P \subseteq dom\ Q) = (dom\ P \subseteq dom\ ((R \setminus (dom\ Q ; top)) ; top))$   
 $\langle proof \rangle$

**lemma** *wlp-prop*:  $dom\ ((R \setminus (dom\ Q ; top)) ; top) = neg\ (cod\ (neg\ (dom\ Q); R))$   
 $\langle proof \rangle$

**lemma** *wlp-prop-var*:  $dom\ ((R \setminus (dom\ Q ; top))) = neg\ (cod\ ((neg\ (dom\ Q)); R))$   
 $\langle proof \rangle$

**lemma** *dom-demod*:  $(\ |R\rangle (Id \cap P) \subseteq Id \cap Q) = (R ; (Id \cap P) \subseteq (Id \cap Q) ; R)$   
 $\langle proof \rangle$

**lemma** *fdia-bbox-galois-var*:  $(\ |R\rangle (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq Id \cap -\ cod\ ((Id \cap -Q); R))$   
 $\langle proof \rangle$

**lemma** *dom-demod-var2*:  $(\ |R\rangle (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q) ; R))$   
 $\langle proof \rangle$

## 1.5 Symmetric quotient

**definition** *syq* ::  $(c, 'a)\ rel \Rightarrow (c, 'b)\ rel \Rightarrow (c, 'b)\ rel$  (**infixl**  $\div$  75)  
where  $R \div S = (R \setminus S) \cap (R^\smile \parallel S^\smile)$

**lemma** *syq-set*:  $R \div S = \{(a, b). \forall c. (c, a) \in R \longleftrightarrow (c, b) \in S\}$   
 $\langle proof \rangle$

**lemma** *converse-syq* [*simp*]:  $(R \div S)^\smile = S \div R$   
 $\langle proof \rangle$

**lemma** *syq-compl*:  $R \div S = - (R^\smile ; -S) \cap - (- (R^\smile) ; S)$   
 $\langle proof \rangle$

**lemma** *syq-compl2* [*simp*]:  $-R \div -S = R \div S$   
 $\langle proof \rangle$

**lemma** *syq-expand1*:  $R ; (R \div S) = S \cap (top ; (R \div S))$   
 $\langle proof \rangle$

**lemma** *syq-expand2*:  $(R \div S) ; S^\smile = R^\smile \cap ((R \div S) ; top)$   
 $\langle proof \rangle$

**lemma** *syq-comp1*:  $(R \div S) ; (S \div T) = (R \div T) \cap (top ; (S \div T))$   
 ⟨proof⟩

**lemma** *syq-comp2*:  $(R \div S) ; (S \div T) = (R \div T) \cap ((R \div S) ; top)$   
 ⟨proof⟩

**lemma** *syq-bij*: *bijjective*  $T \implies (R \div S) ; T = R \div (S ; T)$   
 ⟨proof⟩

**end**

## 2 Properties of Power Allegories

**theory** *Power-Allegories-Properties*

**imports** *Relational-Properties*

**begin**

### 2.1 Power transpose, epsilon, epsiloff

**definition** *Lambda* ::  $('a, 'b \text{ set}) \text{ rel} \Rightarrow ('a, 'b \text{ set}) \text{ rel} (\langle \Lambda \rangle)$  **where**  
 $\Lambda R = \{(a, B) \mid a \in B. B = \{b. (a, b) \in R\}\}$

**definition** *epsilon* ::  $('a, 'a \text{ set}) \text{ rel}$  **where**  
 $\epsilon = \{(a, A). a \in A\}$

**definition** *epsiloff* =  $\{(A, a). a \in A\}$

**definition** *alpha* ::  $('a, 'b \text{ set}) \text{ rel} \Rightarrow ('a, 'b) \text{ rel} (\langle \alpha \rangle)$  **where**  
 $\alpha R = R ; \text{epsiloff}$

alpha can be seen as a relational approximation of a multirelation. The next lemma provides a relational definition of Lambda.

**lemma** *Lambda-syq*:  $\Lambda R = R^\smile \div \epsilon$   
 ⟨proof⟩

**lemma** *epsiloff-epsilon*:  $\text{epsiloff} = \epsilon^\smile$   
 ⟨proof⟩

**lemma** *alpha-set*:  $\alpha R = \{(a, b) \mid a \in b. b \in \bigcup \{B. (a, B) \in R\}\}$   
 ⟨proof⟩

**lemma** *alpha-relcomp* [*simp*]:  $\alpha (R ; S) = R ; \alpha S$   
 ⟨proof⟩

**lemma** *Lambda-epsiloff-up1*:  $f = \Lambda R \implies R = \alpha f$   
 ⟨proof⟩

**lemma** *Lambda-epsiloff-up2: deterministic  $f \implies R = \alpha f \implies f = \Lambda R$*   
*\langle proof \rangle*

**lemma** *Lambda-epsiloff-up:*  
**assumes** *deterministic  $f$*   
**shows**  $(R = \alpha f) = (f = \Lambda R)$   
*\langle proof \rangle*

**lemma** *det-lambda: deterministic  $(\Lambda R)$*   
*\langle proof \rangle*

**lemma** *Lambda-alpha-canc: deterministic  $f \implies \Lambda (\alpha f) = f$*   
*\langle proof \rangle*

**lemma** *alpha-Lambda-canc [simp]:  $\alpha (\Lambda R) = R$*   
*\langle proof \rangle*

**lemma** *alpha-cancel:*  
**assumes** *deterministic  $f$*   
**and** *deterministic  $g$*   
**shows**  $\alpha f = \alpha g \implies f = g$   
*\langle proof \rangle*

**lemma** *Lambda-fusion:*  
**assumes** *deterministic  $f$*   
**shows**  $\Lambda (f ; R) = f ; \Lambda R$   
*\langle proof \rangle*

**lemma** *Lambda-fusion-var:  $\Lambda (\Lambda R ; S) = \Lambda R ; \Lambda S$*   
*\langle proof \rangle*

**lemma** *Lambda-epsiloff [simp]:  $\Lambda \text{epsiloff} = \text{Id}$*   
*\langle proof \rangle*

**lemma** *alpha-epsiloff [simp]:  $\alpha \text{Id} = \text{epsiloff}$*   
*\langle proof \rangle*

**lemma** *alpha-Sup-pres:  $\alpha (\bigcup \mathcal{R}) = (\bigcup R \in \mathcal{R}. \alpha R)$*   
*\langle proof \rangle*

**lemma** *alpha-ord-pres:  $R \subseteq S \implies \alpha R \subseteq \alpha S$*   
*\langle proof \rangle*

**lemma** *alpha-inf-pres:  $\alpha \{(a,A). \exists B C. A = B \cap C \wedge (a,B) \in R \wedge (a,C) \in S\}$*   
 $= \alpha R \cap \alpha S$   
*\langle proof \rangle*

## 2.2 Relational image functor

**definition**  $\text{pow} :: ('a, 'b) \text{ rel} \Rightarrow ('a \text{ set}, 'b \text{ set}) \text{ rel} (\langle \mathcal{P} \rangle)$  **where**  
 $\mathcal{P} R = \Lambda (\text{epsiloff} ; R)$

**lemma**  $\text{pow-set}$ :  $\mathcal{P} R = \{(A,B). B = \text{Image } R A\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{pow-set-var}$ :  $\mathcal{P} R = \{(A,B). B = \{b. \exists a \in A. (a,b) \in R\}\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{pow-converse-set}$ :  $\mathcal{P} (R^\sim) = \{(Q,P). P = \{a. \exists b. (a,b) \in R \wedge b \in Q\}\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{det-pow}$ : *deterministic*  $(\mathcal{P} R)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{Lambda-pow}$ :  $\Lambda (R ; S) = \Lambda R ; \mathcal{P} S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{pow-func1}$  [*simp*]:  $\mathcal{P} \text{Id} = \text{Id}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{pow-func2}$ :  $\mathcal{P} (R ; S) = \mathcal{P} R ; \mathcal{P} S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{Grph-Image}$  [*simp*]:  $\text{Grph} \circ \text{Image} = \mathcal{P}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lambda-alpha-idem}$  [*simp*]:  $\Lambda (\alpha (\Lambda (\alpha R))) = \Lambda (\alpha R)$   
 $\langle \text{proof} \rangle$

## 2.3 Unit and multiplication of powerset monad

**definition**  $\text{eta} :: ('a, 'a \text{ set}) \text{ rel} (\langle \eta \rangle)$  **where**  
 $\eta = \Lambda \text{Id}$

**definition**  $\text{mu} :: ('a \text{ set set}, 'a \text{ set}) \text{ rel} (\langle \mu \rangle)$  **where**  
 $\mu = \text{pow epsiloff}$

**lemma**  $\text{eta-set}$ :  $\eta = \{(a, \{a\}) \mid a. \text{True}\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{alpha-eta}$  [*simp*]:  $\alpha \eta = \text{Id}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{det-eta}$ : *deterministic*  $\eta$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{mu-set}$ :  $\mu = \{(A,B). B = \{b. \exists C. C \in A \wedge b \in C\}\}$

*<proof>*

**lemma** *det-mu: deterministic  $\mu$*   
*<proof>*

**lemma** *Lambda-eta:*  
**assumes** *deterministic  $R$*   
**shows**  $\Lambda R = R ; \eta$   
*<proof>*

**lemma** *eta-nat-trans:*  
**assumes** *deterministic  $R$*   
**shows**  $\eta ; \mathcal{P} R = R ; \eta$   
*<proof>*

**lemma** *mu-nat-trans:*  
**assumes** *deterministic  $R$*   
**shows**  $\mathcal{P} (\mathcal{P} R) ; \mu = \mu ; \mathcal{P} R$   
*<proof>*

The standard axioms for the powerset monad are derivable.

**lemma** *pow-monad1 [simp]:  $\mathcal{P} \mu ; \mu = \mu ; \mu$*   
*<proof>*

**lemma** *pow-monad2 [simp]:  $\mathcal{P} \eta ; \mu = Id$*   
*<proof>*

**lemma** *pow-monad3 [simp]:  $\eta ; \mu = Id$*   
*<proof>*

**lemma** *Lambda-mu:*  
**assumes** *deterministic  $R$*   
**shows**  $\Lambda(R) ; \mu = R$   
*<proof>*

**lemma** *pow-Lambda-mu [simp]:  $\mathcal{P} (\Lambda R) ; \mu = \mathcal{P} R$*   
*<proof>*

**lemma** *lambda-alpha-mu:  $\Lambda (\alpha R) = \Lambda R ; \mu$*   
*<proof>*

**lemma** *alpha-eta-pow [simp]:  $\alpha (\eta ; \mathcal{P} R) = R$*   
*<proof>*

**lemma** *eta-pow-Lambda [simp]:  $\eta ; \mathcal{P} R = \Lambda R$*   
*<proof>*

**lemma** *pow-prop1:  $\mathcal{P} R \subseteq S \implies R \subseteq \alpha (\eta ; S)$*   
*<proof>*

**lemma** *pow-prop-2*:  $R \subseteq \mathcal{P} S \implies \alpha (\eta ; R) \subseteq S$   
 ⟨proof⟩

**lemma** *pow-prop*:  $R = \mathcal{P} S \implies \alpha (\eta ; R) = S$   
 ⟨proof⟩

**lemma** *alpha-eta-id* [simp]:  $\alpha (R ; \eta) = R$   
 ⟨proof⟩

**lemma** *eta-alpha-idem* [simp]:  $\alpha (\alpha R ; \eta) ; \eta = \alpha R ; \eta$   
 ⟨proof⟩

**lemma** *lambda-eta-alpha* [simp]:  $\Lambda (\alpha (\alpha R ; \eta)) = \Lambda (\alpha R)$   
 ⟨proof⟩

**lemma** *eta-lambda-idem* [simp]:  $\alpha (\Lambda (\alpha R)) ; \eta = \alpha R ; \eta$   
 ⟨proof⟩

**lemma** *Grph-eta* [simp]:  $Grph (\lambda x. \{x\}) = \eta$   
 ⟨proof⟩

**lemma** *Grph-epsiloff* [simp]:  $Grph (\lambda x. \{x\}) ; \text{epsiloff} = Id$   
 ⟨proof⟩

**lemma** *Image-epsiloff* [simp]:  $Image \text{epsiloff} \circ (\lambda x. \{x\}) = id$   
 ⟨proof⟩

## 2.4 Subset relation

**definition** *Omega* :: ('a set, 'a set) rel ( $\langle \Omega \rangle$ ) **where**  
 $\Omega = \text{epsilon} \setminus \text{epsiloff}$

**lemma** *Omega-set*:  $\Omega = \{(A,B). A \subseteq B\}$   
 ⟨proof⟩

**lemma** *conv-Omega*:  $\Omega^\sim = \text{epsiloff} // \text{epsiloff}$   
 ⟨proof⟩

**lemma** *epsilon-eta-Omega* [simp]:  $\eta ; \Omega = \text{epsilon}$   
 ⟨proof⟩

**lemma** *epsiloff-eta-Omega* [simp]:  $\Omega^\sim ; \eta^\sim = \text{epsiloff}$   
 ⟨proof⟩

**lemma** *epsilon-Omega* [simp]:  $\text{epsilon} ; \Omega = \text{epsilon}$   
 ⟨proof⟩

**lemma** *conv-Omega-epsiloff* [simp]:  $\Omega^\sim ; \text{epsiloff} = \text{epsiloff}$

*<proof>*

**lemma** *Lambda-conv* [simp]:  $(\Lambda R)^\sim = \text{epsilon} \div R^\sim$   
*<proof>*

**lemma** *Lambda-Omega*:  $\Lambda R ; \Omega = R^\sim \setminus \text{epsilon}$   
*<proof>*

**lemma** *syq-epsiloff-prop* [simp]:  $\Omega^\sim ; (\text{epsilon} \div R) = \text{epsiloff} \parallel R^\sim$   
*<proof>*

**lemma** *pow-semicom*:  $((P, Q) \in \mathcal{P} R ; \Omega) = (\Delta P ; R \subseteq R ; \Delta Q)$   
*<proof>*

## 2.5 Complementation relation

**definition** *Compl* :: ('a set, 'a set) rel (*<C>*) **where**  
 $\mathcal{C} = \text{epsilon} \div \text{-epsilon}$

**lemma** *Compl-set*:  $\mathcal{C} = \{(A, -A) \mid A. \text{True}\}$   
*<proof>*

**lemma** *Compl-Compl* [simp]:  $\mathcal{C} ; \mathcal{C} = \text{Id}$   
*<proof>*

**lemma** *Compl-def-var*:  $\mathcal{C} = \Lambda (\text{-epsiloff})$   
*<proof>*

**lemma** *converse-Compl* [simp]:  $\mathcal{C}^\sim = \mathcal{C}$   
*<proof>*

**lemma** *det-Compl*: *deterministic*  $\mathcal{C}$   
*<proof>*

**lemma** *bij-Compl*: *bijjective*  $\mathcal{C}$   
*<proof>*

**lemma** *Compl-compl-epsiloff* [simp]:  $\mathcal{C} ; \text{-epsiloff} = \text{epsiloff}$   
*<proof>*

**lemma** *Compl-epsiloff* [simp]:  $\mathcal{C} ; \text{epsiloff} = \text{-epsiloff}$   
*<proof>*

**lemma** *compl-epsilon-Compl* [simp]:  $\text{-epsilon} ; \mathcal{C} = \text{epsilon}$   
*<proof>*

**lemma** *epsilon-Compl* [simp]:  $\text{epsilon} ; \mathcal{C} = \text{-epsilon}$   
*<proof>*

**lemma** *Lambda-Compl-var*:  $\Lambda R ; \mathcal{C} = R^\sim \div -\text{epsilon}$   
 $\langle \text{proof} \rangle$

**lemma** *Lambda-Compl*:  $\Lambda R ; \mathcal{C} = \Lambda (-R)$   
 $\langle \text{proof} \rangle$

## 2.6 Kleisli lifting and Kleisli composition

**definition** *klift* ::  $(\text{'a, 'b set}) \text{ rel} \Rightarrow (\text{'a set, 'b set}) \text{ rel} \langle \text{-P} \rangle$  [1000] 999) **where**  
 $(R)_{\mathcal{P}} = \mathcal{P} (\alpha R)$

**definition** *kcomp* ::  $(\text{'a, 'b set}) \text{ rel} \Rightarrow (\text{'b, 'c set}) \text{ rel} \Rightarrow (\text{'a, 'c set}) \text{ rel}$  (**infixl**  $\langle \text{-P} \rangle$  70) **where**  
 $R \cdot_{\mathcal{P}} S = R ; (S)_{\mathcal{P}}$

**lemma** *klift-var*:  $(R)_{\mathcal{P}} = \Lambda (\text{epsiloff} ; R ; \text{epsiloff})$   
 $\langle \text{proof} \rangle$

**lemma** *klift-set*:  $(R)_{\mathcal{P}} = \{(A, B). B = \bigcup (\text{Image } R A)\}$   
 $\langle \text{proof} \rangle$

**lemma** *klift-set-var*:  $(R)_{\mathcal{P}} = \{(A, B). B = \bigcup \{C. \exists a \in A. (a, C) \in R\}\}$   
 $\langle \text{proof} \rangle$

**lemma** *klift-mu*:  $(R)_{\mathcal{P}} = \mathcal{P} R ; \mu$   
 $\langle \text{proof} \rangle$

**lemma** *klift-empty*:  $(\{\}, A) \in (R)_{\mathcal{P}} \longleftrightarrow A = \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *klift-ext1*:  $(R ; (S)_{\mathcal{P}})_{\mathcal{P}} = (R)_{\mathcal{P}} ; (S)_{\mathcal{P}}$   
 $\langle \text{proof} \rangle$

**lemma** *klift-ext2*: *deterministic*  $R \Longrightarrow \eta ; (R)_{\mathcal{P}} = R$   
 $\langle \text{proof} \rangle$

**lemma** *klift-ext3* [*simp*]:  $(\eta)_{\mathcal{P}} = \text{Id}$   
 $\langle \text{proof} \rangle$

**lemma** *pow-klift* [*simp*]:  $(R ; \eta)_{\mathcal{P}} = \mathcal{P} R$   
 $\langle \text{proof} \rangle$

**lemma** *mu-klift* [*simp*]:  $(\text{Id})_{\mathcal{P}} = \mu$   
 $\langle \text{proof} \rangle$

**lemma** *kcomp-var*:  $R \cdot_{\mathcal{P}} S = R ; \mathcal{P} S ; \mu$   
 $\langle \text{proof} \rangle$

**lemma** *kcomp-assoc*:  $R \cdot_{\mathcal{P}} (S \cdot_{\mathcal{P}} T) = (R \cdot_{\mathcal{P}} S) \cdot_{\mathcal{P}} T$

*<proof>*

**lemma** *kcomp-oner*:  $R \cdot_{\mathcal{P}} \eta = R$   
*<proof>*

**lemma** *kcomp-onel*: *deterministic*  $R \implies \eta \cdot_{\mathcal{P}} R = R$   
*<proof>*

## 2.7 Relational box

**definition** *rbox* ::  $( 'a, 'b ) \text{ rel} \Rightarrow ( 'b \text{ set}, 'a \text{ set} ) \text{ rel}$  **where**  
 $rbox\ R = \Lambda (\text{epsiloff} \ //\ R)$

**lemma** *rbox-set*:  $rbox\ R = \{(Q,P). P = \{a. \forall b. (a,b) \in R \longrightarrow b \in Q\}\}$   
*<proof>*

**lemma** *rbox-exp*:  $((Q,P) \in (rbox\ (R::('a,'b)\ \text{rel}))) = (P = -\{a. \exists b. (a,b) \in R \wedge b \in -Q\})$   
*<proof>*

**lemma** *rbox-subset*:  $rbox\ R ; \Omega^{\smile} = \{(Q,P). P \subseteq \{a. \forall b. (a,b) \in R \longrightarrow b \in Q\}\}$   
*<proof>*

**lemma** *rbox-semicomm*:  $(Q,P) \in rbox\ R ; \Omega^{\smile} = (\Delta\ P ; R \subseteq R ; \Delta\ Q)$   
*<proof>*

**lemma** *rbox-semicomm-var*:  $(Q,P) \in rbox\ R ; \Omega^{\smile} = (\Delta\ P \subseteq (R ; \Delta\ Q) \ //\ R)$   
*<proof>*

**lemma** *rbox-omega*:  $rbox\ \text{epsiloff} = \Lambda (\Omega^{\smile})$   
*<proof>*

**lemma** *Omega-rbox*:  $\Omega = (\alpha (rbox\ \text{epsiloff}))^{\smile}$   
*<proof>*

**lemma** *pow-rbox*:  $((Q,P) \in rbox\ R ; \Omega^{\smile}) = ((P,Q) \in \mathcal{P}\ R ; \Omega)$   
*<proof>*

**lemma** *rbox-pow-Compl*:  $rbox\ R = \mathcal{C} ; \mathcal{P}\ (R^{\smile}) ; \mathcal{C}$   
*<proof>*

**lemma** *pow-rbox-Compl*:  $\mathcal{P}\ R = \mathcal{C} ; rbox\ (R^{\smile}) ; \mathcal{C}$   
*<proof>*

**lemma** *pow-conjugation*:  $\mathcal{C} ; (\mathcal{P}\ (R^{\smile}) ; \Omega)^{\smile} = \mathcal{P}\ R ; \mathcal{C} ; \Omega^{\smile}$   
*<proof>*

**lemma** *pow-rbox-eq*:  $rbox\ R ; \Omega^{\smile} = (\mathcal{P}\ R ; \Omega)^{\smile}$   
*<proof>*

end

### 3 Basic Properties of Multirelations

theory *Multirelations-Basics*

imports *Power-Allegories-Properties*

begin

This theory extends a previous AFP entry for multirelations with one single objects to proper multirelations in Rel.

#### 3.1 Peleg composition, parallel composition (inner union) and units

type-synonym  $(\prime a, \prime b) \text{ mrel} = (\prime a, \prime b \text{ set}) \text{ rel}$

definition  $s\text{-prod} :: (\prime a, \prime b) \text{ mrel} \Rightarrow (\prime b, \prime c) \text{ mrel} \Rightarrow (\prime a, \prime c) \text{ mrel}$  (infixl  $\langle \cdot \rangle$  75)

where

$R \cdot S = \{(a, A). (\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge A = \bigcup (f \cdot B)))\}$

definition  $s\text{-id} :: (\prime a, \prime a) \text{ mrel}$  ( $\langle 1_\sigma \rangle$ ) where

$1_\sigma = (\bigcup a. \{(a, \{a\})\})$

definition  $p\text{-prod} :: (\prime a, \prime b) \text{ mrel} \Rightarrow (\prime a, \prime b) \text{ mrel} \Rightarrow (\prime a, \prime b) \text{ mrel}$  (infixl  $\langle \parallel \rangle$  70)

where

$R \parallel S = \{(a, A). (\exists B C. A = B \cup C \wedge (a, B) \in R \wedge (a, C) \in S)\}$

definition  $p\text{-id} :: (\prime a, \prime b) \text{ mrel}$  ( $\langle 1_\pi \rangle$ ) where

$1_\pi = (\bigcup a. \{(a, \{\})\})$

definition  $U :: (\prime a, \prime b) \text{ mrel}$  where

$U = \{(a, A) \mid a \in A. \text{True}\}$

abbreviation  $NC \equiv U - 1_\pi$

named-theorems *mr-simp*

declare  $s\text{-prod-def}$  [*mr-simp*]  $p\text{-prod-def}$  [*mr-simp*]  $s\text{-id-def}$  [*mr-simp*]  $p\text{-id-def}$  [*mr-simp*]  $U\text{-def}$  [*mr-simp*]

lemma  $s\text{-prod-idl}$  [*simp*]:  $1_\sigma \cdot R = R$

*<proof>*

lemma  $s\text{-prod-idr}$  [*simp*]:  $R \cdot 1_\sigma = R$

*<proof>*

**lemma** *p-prod-ild* [*simp*]:  $1_\pi \parallel R = R$   
*<proof>*

**lemma** *c-prod-idr* [*simp*]:  $R \parallel 1_\pi = R$   
*<proof>*

**lemma** *cl7* [*simp*]:  $1_\sigma \parallel 1_\sigma = 1_\sigma$   
*<proof>*

**lemma** *p-prod-assoc*:  $R \parallel S \parallel T = R \parallel (S \parallel T)$   
*<proof>*

**lemma** *p-prod-comm*:  $R \parallel S = S \parallel R$   
*<proof>*

**lemma** *subidem-par*:  $R \subseteq R \parallel R$   
*<proof>*

**lemma** *meet-le-par*:  $R \cap S \subseteq R \parallel S$   
*<proof>*

**lemma** *s-prod-distr*:  $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$   
*<proof>*

**lemma** *s-prod-sup-distr*:  $(\bigcup X) \cdot S = (\bigcup R \in X. R \cdot S)$   
*<proof>*

**lemma** *s-prod-subdistl*:  $R \cdot S \cup R \cdot T \subseteq R \cdot (S \cup T)$   
*<proof>*

**lemma** *s-prod-sup-subdistl*:  $X \neq \{\} \implies (\bigcup S \in X. R \cdot S) \subseteq R \cdot \bigcup X$   
*<proof>*

**lemma** *s-prod-isol*:  $R \subseteq S \implies R \cdot T \subseteq S \cdot T$   
*<proof>*

**lemma** *s-prod-isor*:  $R \subseteq S \implies T \cdot R \subseteq T \cdot S$   
*<proof>*

**lemma** *s-prod-zero* [*simp*]:  $\{\} \cdot R = \{\}$   
*<proof>*

**lemma** *s-prod-wzeror*:  $R \cdot \{\} \subseteq R$   
*<proof>*

**lemma** *p-prod-zero* [*simp*]:  $R \parallel \{\} = \{\}$   
*<proof>*

**lemma** *s-prod-p-idl* [*simp*]:  $1_\pi \cdot R = 1_\pi$

*<proof>*

**lemma** *p-id-st*:  $R \cdot 1_\pi = \{(a, \{\}) \mid a. \exists B. (a, B) \in R\}$   
*<proof>*

**lemma** *c6*:  $R \cdot 1_\pi \subseteq 1_\pi$   
*<proof>*

**lemma** *p-prod-distl*:  $R \parallel (S \cup T) = R \parallel S \cup R \parallel T$   
*<proof>*

**lemma** *p-prod-sup-distl*:  $R \parallel (\bigcup X) = (\bigcup S \in X. R \parallel S)$   
*<proof>*

**lemma** *p-prod-isol*:  $R \subseteq S \implies R \parallel T \subseteq S \parallel T$   
*<proof>*

**lemma** *p-prod-isor*:  $R \subseteq S \implies T \parallel R \subseteq T \parallel S$   
*<proof>*

**lemma** *s-prod-assoc1*:  $(R \cdot S) \cdot T \subseteq R \cdot (S \cdot T)$   
*<proof>*

**lemma** *seq-conc-subdistr*:  $(R \parallel S) \cdot T \subseteq R \cdot T \parallel S \cdot T$   
*<proof>*

**lemma** *U-U [simp]*:  $U \cdot U = U$   
*<proof>*

**lemma** *U-par-idem [simp]*:  $U \parallel U = U$   
*<proof>*

**lemma** *p-id-NC*:  $R - 1_\pi = R \cap NC$   
*<proof>*

**lemma** *NC-NC [simp]*:  $NC \cdot NC = NC$   
*<proof>*

**lemma** *nc-par-idem [simp]*:  $NC \parallel NC = NC$   
*<proof>*

**lemma** *cl4*:  
  **assumes**  $T \parallel T \subseteq T$   
  **shows**  $R \cdot T \parallel S \cdot T \subseteq (R \parallel S) \cdot T$   
*<proof>*

**lemma** *cl3*:  $R \cdot (S \parallel T) \subseteq R \cdot S \parallel R \cdot T$   
*<proof>*

**lemma** *p-id-assoc1*:  $(1_\pi \cdot R) \cdot S = 1_\pi \cdot (R \cdot S)$   
*<proof>*

**lemma** *p-id-assoc2*:  $(R \cdot 1_\pi) \cdot T = R \cdot (1_\pi \cdot T)$   
*<proof>*

**lemma** *cl1* [*simp*]:  $R \cdot 1_\pi \cup R \cdot NC = R \cdot U$   
*<proof>*

**lemma** *tarski-aux*:  
assumes  $R - 1_\pi \neq \{\}$   
and  $(a, A) \in NC$   
shows  $(a, A) \in NC \cdot ((R - 1_\pi) \cdot NC)$   
*<proof>*

**lemma** *tarski*:  
assumes  $R - 1_\pi \neq \{\}$   
shows  $NC \cdot ((R - 1_\pi) \cdot NC) = NC$   
*<proof>*

**lemma** *tarski-var*:  
assumes  $R \cap NC \neq \{\}$   
shows  $NC \cdot ((R \cap NC) \cdot NC) = NC$   
*<proof>*

**lemma** *s-le-nc*:  $1_\sigma \subseteq NC$   
*<proof>*

**lemma** *U-nc* [*simp*]:  $U \cdot NC = U$   
*<proof>*

**lemma** *x-y-split* [*simp*]:  $(R \cap NC) \cdot S \cup R \cdot \{\} = R \cdot S$   
*<proof>*

**lemma** *c-nc-comp1* [*simp*]:  $1_\pi \cup NC = U$   
*<proof>*

## 3.2 Tests

**lemma** *s-id-st*:  $R \cap 1_\sigma = \{(a, \{a\}) \mid a. (a, \{a\}) \in R\}$   
*<proof>*

**lemma** *subid-aux2*:  
assumes  $(a, A) \in R \cap 1_\sigma$   
shows  $A = \{a\}$   
*<proof>*

**lemma** *s-prod-test-aux1*:  
assumes  $(a, A) \in R \cdot (P \cap 1_\sigma)$

**shows**  $((a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma)))$   
 $\langle proof \rangle$

**lemma** *s-prod-test-aux2*:  
**assumes**  $(a,A) \in R$   
**and**  $\forall a \in A. (a,\{a\}) \in S$   
**shows**  $(a,A) \in R \cdot S$   
 $\langle proof \rangle$

**lemma** *s-prod-test*:  $(a,A) \in R \cdot (P \cap 1_\sigma) \longleftrightarrow (a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma))$   
 $\langle proof \rangle$

**lemma** *s-prod-test-var*:  $R \cdot (P \cap 1_\sigma) = \{(a,A). (a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma))\}$   
 $\langle proof \rangle$

**lemma** *test-s-prod-aux1*:  
**assumes**  $(a,A) \in (P \cap 1_\sigma) \cdot R$   
**shows**  $(a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$   
 $\langle proof \rangle$

**lemma** *test-s-prod-aux2*:  
**assumes**  $(a,A) \in R$   
**and**  $(a,\{a\}) \in P$   
**shows**  $(a,A) \in P \cdot R$   
 $\langle proof \rangle$

**lemma** *test-s-prod*:  $(a,A) \in (P \cap 1_\sigma) \cdot R \longleftrightarrow (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$   
 $\langle proof \rangle$

**lemma** *test-s-prod-var*:  $(P \cap 1_\sigma) \cdot R = \{(a,A). (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R\}$   
 $\langle proof \rangle$

**lemma** *test-assoc1*:  $(R \cdot (P \cap 1_\sigma)) \cdot S = R \cdot ((P \cap 1_\sigma) \cdot S)$   
 $\langle proof \rangle$

**lemma** *test-assoc2*:  $((P \cap 1_\sigma) \cdot R) \cdot S = (P \cap 1_\sigma) \cdot (R \cdot S)$   
 $\langle proof \rangle$

**lemma** *test-assoc3*:  $(R \cdot S) \cdot (P \cap 1_\sigma) = R \cdot (S \cdot (P \cap 1_\sigma))$   
 $\langle proof \rangle$

**lemma** *s-distl-test*:  $(P \cap 1_\sigma) \cdot (S \cup T) = (P \cap 1_\sigma) \cdot S \cup (P \cap 1_\sigma) \cdot T$   
 $\langle proof \rangle$

**lemma** *s-distl-sup-test*:  $(P \cap 1_\sigma) \cdot \bigcup X = (\bigcup S \in X. (P \cap 1_\sigma) \cdot S)$   
 $\langle proof \rangle$

**lemma** *subid-par-idem* [simp]:  $(P \cap 1_\sigma) \parallel (P \cap 1_\sigma) = (P \cap 1_\sigma)$   
 ⟨proof⟩

**lemma** *seq-conc-subdistrl*:  $(P \cap 1_\sigma) \cdot (S \parallel T) = ((P \cap 1_\sigma) \cdot S) \parallel ((P \cap 1_\sigma) \cdot T)$   
 ⟨proof⟩

**lemma** *test-s-prod-is-meet* [simp]:  $(P \cap 1_\sigma) \cdot (Q \cap 1_\sigma) = P \cap Q \cap 1_\sigma$   
 ⟨proof⟩

**lemma** *test-p-prod-is-meet* [simp]:  $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cap (Q \cap 1_\sigma)$   
 ⟨proof⟩

**lemma** *test-multiplicativer*:  $(P \cap Q \cap 1_\sigma) \cdot T = ((P \cap 1_\sigma) \cdot T) \cap ((Q \cap 1_\sigma) \cdot T)$   
 ⟨proof⟩

**lemma** *cl9* [simp]:  $(R \cap 1_\sigma) \cdot 1_\pi \parallel 1_\sigma = R \cap 1_\sigma$   
 ⟨proof⟩

**lemma** *s-subid-closed* [simp]:  $R \cap NC \cap 1_\sigma = R \cap 1_\sigma$   
 ⟨proof⟩

**lemma** *sub-id-le-nc*:  $R \cap 1_\sigma \subseteq NC$   
 ⟨proof⟩

**lemma** *x-y-prop*:  $1_\sigma \cap ((R \cap NC) \cdot S) = 1_\sigma \cap R \cdot S$   
 ⟨proof⟩

**lemma** *s-nc-U*:  $1_\sigma \cap R \cdot NC = 1_\sigma \cap R \cdot U$   
 ⟨proof⟩

**lemma** *sid-le-nc-var*:  $1_\sigma \cap R \subseteq 1_\sigma \cap (R \parallel NC)$   
 ⟨proof⟩

**lemma** *s-nc-par-U*:  $1_\sigma \cap (R \parallel NC) = 1_\sigma \cap (R \parallel U)$   
 ⟨proof⟩

**lemma** *s-id-par-s-prod*:  $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cdot (Q \cap 1_\sigma)$   
 ⟨proof⟩

### 3.3 Parallel subidentities

**lemma** *p-id-zero-st*:  $R \cap 1_\pi = \{(a, \{\}) \mid a. (a, \{\}) \in R\}$   
 ⟨proof⟩

**lemma** *p-subid-iff*:  $R \subseteq 1_\pi \iff R \cdot 1_\pi = R$   
 ⟨proof⟩

**lemma** *p-subid-iff-var*:  $R \subseteq 1_\pi \iff R \cdot \{\} = R$

$\langle proof \rangle$

**lemma** *term-par-idem* [*simp*]:  $(R \cap 1_\pi) \parallel (R \cap 1_\pi) = (R \cap 1_\pi)$   
 $\langle proof \rangle$

**lemma** *c1* [*simp*]:  $R \cdot 1_\pi \parallel R = R$   
 $\langle proof \rangle$

**lemma** *p-id-zero*:  $R \cap 1_\pi = R \cdot \{\}$   
 $\langle proof \rangle$

**lemma** *cl5*:  $(R \cdot S) \cdot (T \cdot \{\}) = R \cdot (S \cdot (T \cdot \{\}))$   
 $\langle proof \rangle$

**lemma** *c4*:  $(R \cdot S) \cdot 1_\pi = R \cdot (S \cdot 1_\pi)$   
 $\langle proof \rangle$

**lemma** *c3*:  $(R \parallel S) \cdot 1_\pi = R \cdot 1_\pi \parallel S \cdot 1_\pi$   
 $\langle proof \rangle$

**lemma** *p-id-idem* [*simp*]:  $(R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$   
 $\langle proof \rangle$

**lemma** *x-c-par-idem* [*simp*]:  $R \cdot 1_\pi \parallel R \cdot 1_\pi = R \cdot 1_\pi$   
 $\langle proof \rangle$

**lemma** *x-zero-le-c*:  $R \cdot \{\} \subseteq 1_\pi$   
 $\langle proof \rangle$

**lemma** *p-subid-lb1*:  $R \cdot \{\} \parallel S \cdot \{\} \subseteq R \cdot \{\}$   
 $\langle proof \rangle$

**lemma** *p-subid-lb2*:  $R \cdot \{\} \parallel S \cdot \{\} \subseteq S \cdot \{\}$   
 $\langle proof \rangle$

**lemma** *p-subid-idem* [*simp*]:  $R \cdot \{\} \parallel R \cdot \{\} = R \cdot \{\}$   
 $\langle proof \rangle$

**lemma** *p-subid-glb*:  $T \cdot \{\} \subseteq R \cdot \{\} \implies T \cdot \{\} \subseteq S \cdot \{\} \implies T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$   
 $\langle proof \rangle$

**lemma** *p-subid-glb-iff*:  $T \cdot \{\} \subseteq R \cdot \{\} \wedge T \cdot \{\} \subseteq S \cdot \{\} \iff T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$   
 $\langle proof \rangle$

**lemma** *x-c-glb*:  $(T::('a, 'b) \text{ mrel}) \cdot 1_\pi \subseteq (R::('a, 'b) \text{ mrel}) \cdot 1_\pi \implies T \cdot 1_\pi \subseteq (S::('a, 'b) \text{ mrel}) \cdot 1_\pi \implies T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$   
 $\langle proof \rangle$

**lemma** *x-c-lb1*:  $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq R \cdot 1_\pi$   
 ⟨proof⟩

**lemma** *x-c-lb2*:  $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq S \cdot 1_\pi$   
 ⟨proof⟩

**lemma** *x-c-glb-iff*:  $(T::('a,'b) \text{ mrel}) \cdot 1_\pi \subseteq (R::('a,'b) \text{ mrel}) \cdot 1_\pi \wedge T \cdot 1_\pi \subseteq (S::('a,'b) \text{ mrel}) \cdot 1_\pi \iff T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$   
 ⟨proof⟩

**lemma** *nc-iff1*:  $R \subseteq NC \iff R \cap 1_\pi = \{\}$   
 ⟨proof⟩

**lemma** *nc-iff2*:  $R \subseteq NC \iff R \cdot \{\} = \{\}$   
 ⟨proof⟩

**lemma** *zero-assoc3*:  $(R \cdot S) \cdot \{\} = R \cdot (S \cdot \{\})$   
 ⟨proof⟩

**lemma** *x-zero-interr*:  $R \cdot \{\} \parallel S \cdot \{\} = (R \parallel S) \cdot \{\}$   
 ⟨proof⟩

**lemma** *p-subid-interr*:  $R \cdot T \cdot 1_\pi \parallel S \cdot T \cdot 1_\pi = (R \parallel S) \cdot T \cdot 1_\pi$   
 ⟨proof⟩

**lemma** *cl2 [simp]*:  $1_\pi \cap (R \cup NC) = R \cdot \{\}$   
 ⟨proof⟩

**lemma** *cl6 [simp]*:  $R \cdot \{\} \cdot S = R \cdot \{\}$   
 ⟨proof⟩

**lemma** *cl11 [simp]*:  $(R \cap NC) \cdot 1_\pi \parallel NC = (R \cap NC) \cdot NC$   
 ⟨proof⟩

**lemma** *x-split [simp]*:  $(R \cap NC) \cup (R \cap 1_\pi) = R$   
 ⟨proof⟩

**lemma** *x-split-var [simp]*:  $(R \cap NC) \cup R \cdot \{\} = R$   
 ⟨proof⟩

**lemma** *s-x-c [simp]*:  $1_\sigma \cap R \cdot 1_\pi = \{\}$   
 ⟨proof⟩

**lemma** *s-x-zero [simp]*:  $1_\sigma \cap R \cdot \{\} = \{\}$   
 ⟨proof⟩

**lemma** *c-nc [simp]*:  $R \cdot 1_\pi \cap NC = \{\}$   
 ⟨proof⟩

**lemma** *zero-nc* [*simp*]:  $R \cdot \{\} \cap NC = \{\}$   
*<proof>*

**lemma** *nc-zero* [*simp*]:  $(R \cap NC) \cdot \{\} = \{\}$   
*<proof>*

**lemma** *c-def* [*simp*]:  $U \cdot \{\} = 1_\pi$   
*<proof>*

**lemma** *U-c* [*simp*]:  $U \cdot 1_\pi = 1_\pi$   
*<proof>*

**lemma** *nc-c* [*simp*]:  $NC \cdot 1_\pi = 1_\pi$   
*<proof>*

**lemma** *nc-U* [*simp*]:  $NC \cdot U = U$   
*<proof>*

**lemma** *x-c-nc-split* [*simp*]:  $((R \cap NC) \cdot NC) \cup (R \cdot \{\} \parallel NC) = (R \cdot 1_\pi) \parallel NC$   
*<proof>*

**lemma** *x-c-U-split* [*simp*]:  $R \cdot U \cup (R \cdot \{\} \parallel U) = R \cdot 1_\pi \parallel U$   
*<proof>*

**lemma** *p-subid-par-eq-meet* [*simp*]:  $R \cdot \{\} \parallel S \cdot \{\} = R \cdot \{\} \cap S \cdot \{\}$   
*<proof>*

**lemma** *p-subid-par-eq-meet-var* [*simp*]:  $R \cdot 1_\pi \parallel S \cdot 1_\pi = R \cdot 1_\pi \cap S \cdot 1_\pi$   
*<proof>*

**lemma** *x-zero-add-closed*:  $R \cdot \{\} \cup S \cdot \{\} = (R \cup S) \cdot \{\}$   
*<proof>*

**lemma** *x-zero-meet-closed*:  $R \cdot \{\} \cap S \cdot \{\} = (R \cap S) \cdot \{\}$   
*<proof>*

**lemma** *scomp-univalent-pres*: *univalent*  $R \implies$  *univalent*  $S \implies$  *univalent*  $(R \cdot S)$   
*<proof>*

**lemma** *univalent s-id*  
*<proof>*

**lemma** *det-peleg*: *deterministic*  $R \implies$  *deterministic*  $S \implies$  *deterministic*  $(R \cdot S)$   
*<proof>*

**lemma** *deterministic-sid*: *deterministic*  $1_\sigma$   
*<proof>*

### 3.4 Domain

**definition**  $Dom :: ('a, 'b) mrel \Rightarrow ('a, 'a) mrel$  **where**  
 $Dom R = \{(a, \{a\}) \mid a. \exists B. (a, B) \in R\}$

**named-theorems** *mrd-simp*

**declare** *mr-simp* [*mrd-simp*] *Dom-def* [*mrd-simp*]

**lemma** *d-def-expl*:  $Dom R = R \cdot 1_\pi \parallel 1_\sigma$   
 $\langle proof \rangle$

**lemma** *s-subid-iff2*:  $(R \cap 1_\sigma = R) = (Dom R = R)$   
 $\langle proof \rangle$

**lemma** *cl8-var*:  $Dom R \cdot S = R \cdot 1_\pi \parallel S$   
 $\langle proof \rangle$

**lemma** *cl8* [*simp*]:  $R \cdot 1_\pi \parallel 1_\sigma \cdot S = R \cdot 1_\pi \parallel S$   
 $\langle proof \rangle$

**lemma** *cl10-var*:  $Dom (R - 1_\pi) = 1_\sigma \cap ((R - 1_\pi) \cdot NC)$   
 $\langle proof \rangle$

**lemma** *c10*:  $(R \cap NC) \cdot 1_\pi \parallel 1_\sigma = 1_\sigma \cap ((R \cap NC) \cdot NC)$   
 $\langle proof \rangle$

**lemma** *cl9-var* [*simp*]:  $Dom (R \cap 1_\sigma) = R \cap 1_\sigma$   
 $\langle proof \rangle$

**lemma** *d-s-id* [*simp*]:  $Dom R \cap 1_\sigma = Dom R$   
 $\langle proof \rangle$

**lemma** *d-s-id-ax*:  $Dom R \subseteq 1_\sigma$   
 $\langle proof \rangle$

**lemma** *d-assoc1*:  $Dom R \cdot (S \cdot T) = (Dom R \cdot S) \cdot T$   
 $\langle proof \rangle$

**lemma** *d-meet-distr-var*:  $(Dom R \cap Dom S) \cdot T = Dom R \cdot T \cap Dom S \cdot T$   
 $\langle proof \rangle$

**lemma** *d-idem* [*simp*]:  $Dom (Dom R) = Dom R$   
 $\langle proof \rangle$

**lemma** *cd-2-var*:  $Dom (R \cdot 1_\pi) \cdot S = R \cdot 1_\pi \parallel S$   
 $\langle proof \rangle$

**lemma** *dc-prop1* [*simp*]:  $Dom R \cdot 1_\pi = R \cdot 1_\pi$   
 $\langle proof \rangle$

**lemma** *dc-prop2* [*simp*]:  $Dom (R \cdot 1_\pi) = Dom R$   
*<proof>*

**lemma** *ds-prop* [*simp*]:  $Dom R \parallel 1_\sigma = Dom R$   
*<proof>*

**lemma** *dc* [*simp*]:  $Dom 1_\pi = 1_\sigma$   
*<proof>*

**lemma** *cd-iso* [*simp*]:  $Dom (R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$   
*<proof>*

**lemma** *dc-iso* [*simp*]:  $Dom (Dom R \cdot 1_\pi) = Dom R$   
*<proof>*

**lemma** *d-s-id-inter* [*simp*]:  $Dom R \cdot Dom S = Dom R \cap Dom S$   
*<proof>*

**lemma** *d-conc6*:  $Dom (R \parallel S) = Dom R \parallel Dom S$   
*<proof>*

**lemma** *d-conc-inter* [*simp*]:  $Dom R \parallel Dom S = Dom R \cap Dom S$   
*<proof>*

**lemma** *d-conc-s-prod-ax*:  $Dom R \parallel Dom S = Dom R \cdot Dom S$   
*<proof>*

**lemma** *d-rest-ax* [*simp*]:  $Dom R \cdot R = R$   
*<proof>*

**lemma** *d-loc-ax* [*simp*]:  $Dom (R \cdot Dom S) = Dom (R \cdot S)$   
*<proof>*

**lemma** *assoc-p-subid*:  $(R \cdot S) \cdot (T \cdot 1_\pi) = R \cdot (S \cdot (T \cdot 1_\pi))$   
*<proof>*

**lemma** *d-exp-ax* [*simp*]:  $Dom (Dom R \cdot S) = Dom R \cdot Dom S$   
*<proof>*

**lemma** *d-comm-ax*:  $Dom R \cdot Dom S = Dom S \cdot Dom R$   
*<proof>*

**lemma** *d-s-id-prop* [*simp*]:  $Dom 1_\sigma = 1_\sigma$   
*<proof>*

**lemma** *d-s-prod-closed* [*simp*]:  $Dom (Dom R \cdot Dom S) = Dom R \cdot Dom S$   
*<proof>*

**lemma** *d-p-prod-closed* [*simp*]:  $Dom (Dom R \parallel Dom S) = Dom R \parallel Dom S$

*<proof>*

**lemma** *d-idem2* [*simp*]:  $Dom R \cdot Dom R = Dom R$   
*<proof>*

**lemma** *d-assoc*:  $(Dom R \cdot Dom S) \cdot Dom T = Dom R \cdot (Dom S \cdot Dom T)$   
*<proof>*

**lemma** *iso-1* [*simp*]:  $Dom R \cdot 1_\pi \parallel 1_\sigma = Dom R$   
*<proof>*

**lemma** *d-idem-par* [*simp*]:  $Dom R \parallel Dom R = Dom R$   
*<proof>*

**lemma** *d-inter-r*:  $Dom R \cdot (S \parallel T) = Dom R \cdot S \parallel Dom R \cdot T$   
*<proof>*

**lemma** *d-add-ax*:  $Dom (R \cup S) = Dom R \cup Dom S$   
*<proof>*

**lemma** *d-sup-add*:  $Dom (\bigcup X) = (\bigcup R \in X. Dom R)$   
*<proof>*

**lemma** *d-distl*:  $Dom R \cdot (S \cup T) = Dom R \cdot S \cup Dom R \cdot T$   
*<proof>*

**lemma** *d-sup-distl*:  $Dom R \cdot \bigcup X = (\bigcup S \in X. Dom R \cdot S)$   
*<proof>*

**lemma** *d-zero-ax* [*simp*]:  $Dom \{\} = \{\}$   
*<proof>*

**lemma** *d-absorb1* [*simp*]:  $Dom R \cup Dom R \cdot Dom S = Dom R$   
*<proof>*

**lemma** *d-absorb2* [*simp*]:  $Dom R \cdot (Dom R \cup Dom S) = Dom R$   
*<proof>*

**lemma** *d-dist1*:  $Dom R \cdot (Dom S \cup Dom T) = Dom R \cdot Dom S \cup Dom R \cdot Dom T$   
*<proof>*

**lemma** *d-dist2*:  $Dom R \cup (Dom S \cdot Dom T) = (Dom R \cup Dom S) \cdot (Dom R \cup Dom T)$   
*<proof>*

**lemma** *d-add-prod-closed* [*simp*]:  $Dom (Dom R \cup Dom S) = Dom R \cup Dom S$   
*<proof>*

**lemma** *x-zero-prop*:  $R \cdot \{\} \parallel S = \text{Dom} (R \cdot \{\}) \cdot S$   
*<proof>*

**lemma** *cda-add-ax*:  $\text{Dom} ((R \cup S) \cdot T) = \text{Dom} (R \cdot T) \cup \text{Dom} (S \cdot T)$   
*<proof>*

**lemma** *d-x-zero*:  $\text{Dom} (R \cdot \{\}) = R \cdot \{\} \parallel 1_\sigma$   
*<proof>*

**lemma** *cda-ax2*:  
**assumes**  $(R \parallel S) \cdot \text{Dom} T = R \cdot \text{Dom} T \parallel S \cdot \text{Dom} T$   
**shows**  $\text{Dom} ((R \parallel S) \cdot T) = \text{Dom} (R \cdot T) \cdot \text{Dom} (S \cdot T)$   
*<proof>*

**lemma** *d-lb1*:  $\text{Dom} R \cdot \text{Dom} S \subseteq \text{Dom} R$   
*<proof>*

**lemma** *d-lb2*:  $\text{Dom} R \cdot \text{Dom} S \subseteq \text{Dom} S$   
*<proof>*

**lemma** *d-glb*:  $\text{Dom} T \subseteq \text{Dom} R \wedge \text{Dom} T \subseteq \text{Dom} S \implies \text{Dom} T \subseteq \text{Dom} R \cdot \text{Dom} S$   
*<proof>*

**lemma** *d-glb-iff*:  $\text{Dom} T \subseteq \text{Dom} R \wedge \text{Dom} T \subseteq \text{Dom} S \iff \text{Dom} T \subseteq \text{Dom} R \cdot \text{Dom} S$   
*<proof>*

**lemma** *d-interr*:  $R \cdot \text{Dom} P \parallel S \cdot \text{Dom} P = (R \parallel S) \cdot \text{Dom} P$   
*<proof>*

**lemma** *cl10-d*:  $\text{Dom} (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot NC$   
*<proof>*

**lemma** *cl11-d [simp]*:  $\text{Dom} (R \cap NC) \cdot NC = (R \cap NC) \cdot NC$   
*<proof>*

**lemma** *cl10-d-var1*:  $\text{Dom} (R \cap NC) = 1_\sigma \cap R \cdot NC$   
*<proof>*

**lemma** *cl10-d-var2*:  $\text{Dom} (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot U$   
*<proof>*

**lemma** *cl10-d-var3*:  $\text{Dom} (R \cap NC) = 1_\sigma \cap R \cdot U$   
*<proof>*

**lemma** *d-U [simp]*:  $\text{Dom} U = 1_\sigma$   
*<proof>*

**lemma** *d-nc* [*simp*]:  $Dom\ NC = 1_\sigma$   
*<proof>*

**lemma** *alt-d-def-nc-nc*:  $Dom\ (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel NC)$   
*<proof>*

**lemma** *alt-d-def-nc-U*:  $Dom\ (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel U)$   
*<proof>*

**lemma** *d-def-split* [*simp*]:  $Dom\ (R \cap NC) \cup Dom\ (R \cdot \{\}) = Dom\ R$   
*<proof>*

**lemma** *d-def-split-var* [*simp*]:  $Dom\ (R \cap NC) \cup ((R \cdot \{\}) \parallel 1_\sigma) = Dom\ R$   
*<proof>*

**lemma** *ax7* [*simp*]:  $(1_\sigma \cap R \cdot U) \cup (R \cdot \{\}) \parallel 1_\sigma = Dom\ R$   
*<proof>*

**lemma** *dom12-d*:  $Dom\ R = 1_\sigma \cap (R \cdot 1_\pi \parallel NC)$   
*<proof>*

**lemma** *dom12-d-U*:  $Dom\ R = 1_\sigma \cap (R \cdot 1_\pi \parallel U)$   
*<proof>*

**lemma** *dom-def-var*:  $Dom\ R = (R \cdot U \cap 1_\pi) \parallel 1_\sigma$   
*<proof>*

**lemma** *ax5-d* [*simp*]:  $Dom\ (R \cap NC) \cdot U = (R \cap NC) \cdot U$   
*<proof>*

**lemma** *ax5-0* [*simp*]:  $Dom\ (R \cdot \{\}) \cdot U = R \cdot \{\} \parallel U$   
*<proof>*

**lemma** *x-c-U-split2*:  $Dom\ R \cdot NC = (R \cap NC) \cdot NC \cup (R \cdot \{\}) \parallel NC$   
*<proof>*

**lemma** *x-c-U-split3*:  $Dom\ R \cdot U = (R \cap NC) \cdot U \cup (R \cdot \{\}) \parallel U$   
*<proof>*

**lemma** *x-c-U-split-d*:  $Dom\ R \cdot U = R \cdot U \cup (R \cdot \{\}) \parallel U$   
*<proof>*

**lemma** *x-U-prop2*:  $R \cdot NC = Dom\ (R \cap NC) \cdot NC \cup R \cdot \{\}$   
*<proof>*

**lemma** *x-U-prop3*:  $R \cdot U = Dom\ (R \cap NC) \cdot U \cup R \cdot \{\}$   
*<proof>*

**lemma** *d-x-nc* [*simp*]:  $Dom\ (R \cdot NC) = Dom\ R$

*<proof>*

**lemma** *d-x-U* [*simp*]:  $Dom (R \cdot U) = Dom R$   
*<proof>*

**lemma** *d-llp1*:  $Dom R \subseteq Dom S \implies R \subseteq Dom S \cdot R$   
*<proof>*

**lemma** *d-llp2*:  $R \subseteq Dom S \cdot R \implies Dom R \subseteq Dom S$   
*<proof>*

**lemma** *demod1*:  $Dom (R \cdot S) \subseteq Dom T \implies R \cdot Dom S \subseteq Dom T \cdot R$   
*<proof>*

**lemma** *demod2*:  $R \cdot Dom S \subseteq Dom T \cdot R \implies Dom (R \cdot S) \subseteq Dom T$   
*<proof>*

**lemma** *d-meet-closed* [*simp*]:  $Dom (Dom x \cap Dom y) = Dom x \cap Dom y$   
*<proof>*

**lemma** *d-add-var*:  $Dom P \cdot R \cup Dom Q \cdot R = Dom (P \cup Q) \cdot R$   
*<proof>*

**lemma** *d-interr-U*:  $Dom x \cdot U \parallel Dom y \cdot U = Dom (x \parallel y) \cdot U$   
*<proof>*

**lemma** *d-meet*:  $Dom x \cdot z \cap Dom y \cdot z = (Dom x \cap Dom y) \cdot z$   
*<proof>*

**lemma** *cs-hom-meet*:  $Dom (x \cdot 1_\pi \cap y \cdot 1_\pi) = Dom (x \cdot 1_\pi) \cap Dom (y \cdot 1_\pi)$   
*<proof>*

**lemma** *iso3* [*simp*]:  $Dom (Dom x \cdot U) = Dom x$   
*<proof>*

**lemma** *iso4* [*simp*]:  $Dom (x \cdot 1_\pi \parallel U) \cdot U = x \cdot 1_\pi \parallel U$   
*<proof>*

**lemma** *iso3-sharp* [*simp*]:  $Dom (Dom (x \cap NC) \cdot NC) = Dom (x \cap NC)$   
*<proof>*

**lemma** *iso4-sharp* [*simp*]:  $Dom ((x \cap NC) \cdot NC) \cdot NC = (x \cap NC) \cdot NC$   
*<proof>*

### 3.5 Vectors

**lemma** *vec-iff1*:

**assumes**  $\forall a. (\exists A. (a, A) \in R) \longrightarrow (\forall A. (a, A) \in R)$

**shows**  $R \cdot 1_\pi \parallel U = R$

*<proof>*

**lemma** *vec-iff2*:

**assumes**  $R \cdot 1_\pi \parallel U = R$

**shows**  $(\forall a. (\exists A. (a,A) \in R) \longrightarrow (\forall A. (a,A) \in R))$

*<proof>*

**lemma** *vec-iff*:  $(\forall a. (\exists A. (a,A) \in R) \longrightarrow (\forall A. (a,A) \in R)) \longleftrightarrow R \cdot 1_\pi \parallel U = R$

*<proof>*

**lemma** *U-par-zero* [*simp*]:  $\{\} \cdot R \parallel U = \{\}$

*<proof>*

**lemma** *U-par-s-id* [*simp*]:  $1_\sigma \cdot 1_\pi \parallel U = U$

*<proof>*

**lemma** *U-par-p-id* [*simp*]:  $1_\pi \cdot 1_\pi \parallel U = U$

*<proof>*

**lemma** *U-par-nc* [*simp*]:  $NC \cdot 1_\pi \parallel U = U$

*<proof>*

### 3.6 Up-closure and Parikh composition

**definition** *s-prod-pa* ::  $('a,'b) \text{ mrel} \Rightarrow ('b,'c) \text{ mrel} \Rightarrow ('a,'c) \text{ mrel}$  (**infixl**  $\langle \otimes \rangle$  75)

**where**

$R \otimes S = \{(a,A). (\exists B. (a,B) \in R \wedge (\forall b \in B. (b,A) \in S))\}$

**lemma** *U-par-st*:  $(a,A) \in R \parallel U \longleftrightarrow (\exists B. B \subseteq A \wedge (a,B) \in R)$

*<proof>*

**lemma** *p-id-U*:  $R \parallel U = \{(a,B). \exists A. (a,A) \in R \wedge A \subseteq B\}$

*<proof>*

**lemma** *ucl-iff*:  $(\forall a A B. (a,A) \in R \wedge A \subseteq B \longrightarrow (a,B) \in R) \longleftrightarrow R \parallel U = R$

*<proof>*

**lemma** *upclosed-ext*:  $R \subseteq R \parallel U$

*<proof>*

**lemma** *onelem*:  $R \cdot S \parallel U \subseteq R \otimes (S \parallel U)$

*<proof>*

**lemma** *twolem*:  $R \otimes (S \parallel U) \subseteq R \cdot S \parallel U$

*<proof>*

**lemma** *pe-pa-sim*:  $R \cdot S \parallel U = R \otimes (S \parallel U)$

*<proof>*

**lemma** *pe-pa-sim-var*:  $(R \parallel U) \cdot (S \parallel U) \parallel U = (R \parallel U) \otimes (S \parallel U)$   
 ⟨proof⟩

**lemma** *pa-assoc1*:  $((R \parallel U) \otimes (S \parallel U)) \otimes (T \parallel U) \subseteq (R \parallel U) \otimes ((S \parallel U) \otimes (T \parallel U))$   
 ⟨proof⟩

**lemma** *up-closed-par-is-meet*:  $(R \parallel U) \parallel (S \parallel U) = (R \parallel U) \cap (S \parallel U)$   
 ⟨proof⟩

**lemma** *U-nc-par [simp]*:  $NC \parallel U = NC$   
 ⟨proof⟩

**lemma** *uc-par-meet*:  $(R \parallel U) \cap (S \parallel U) = R \parallel U \parallel S \parallel U$   
 ⟨proof⟩

**lemma** *uc-unc [simp]*:  $R \parallel U \parallel R \parallel U = R \parallel U$   
 ⟨proof⟩

**lemma** *uc-interr*:  $(R \parallel S) \cdot (T \parallel U) = R \cdot (T \parallel U) \parallel S \cdot (T \parallel U)$   
 ⟨proof⟩

**lemma** *iso5 [simp]*:  $(R \cdot 1_\pi \parallel U) \cdot 1_\pi = R \cdot 1_\pi$   
 ⟨proof⟩

**lemma** *iso6 [simp]*:  $(R \cdot 1_\pi \parallel U) \cdot 1_\pi \parallel U = R \cdot 1_\pi \parallel U$   
 ⟨proof⟩

**lemma** *sv-hom-par*:  $(R \parallel S) \cdot U = R \cdot U \parallel S \cdot U$   
 ⟨proof⟩

**lemma** *vs-hom-meet*:  $Dom ((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) = Dom (R \cdot 1_\pi \parallel U) \cap Dom (S \cdot 1_\pi \parallel U)$   
 ⟨proof⟩

**lemma** *cv-hom-meet*:  $(R \cdot 1_\pi \cap S \cdot 1_\pi) \parallel U = (R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)$   
 ⟨proof⟩

**lemma** *cv-hom-par [simp]*:  $R \parallel U \parallel S \parallel U = (R \parallel S) \parallel U$   
 ⟨proof⟩

**lemma** *vc-hom-meet*:  $((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cap ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$   
 ⟨proof⟩

**lemma** *vc-hom-seq*:  $((R \cdot 1_\pi \parallel U) \cdot (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cdot ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$   
 ⟨proof⟩

### 3.7 Nonterminal and terminal multirelations

**definition**  $\tau :: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} (\langle \tau \rangle)$  **where**  
 $\tau R = R \cdot \{\}$

**definition**  $\nu :: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} (\langle \nu \rangle)$  **where**  
 $\nu R = R \cap NC$

**lemma**  $nc\text{-s}$  [*simp*]:  $\nu 1_\sigma = 1_\sigma$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-scomp-closed}$ :  $\nu R \cdot \nu S \subseteq NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-scomp-closed-alt}$  [*simp*]:  $\nu (\nu R \cdot \nu S) = \nu R \cdot \nu S$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-ccomp-closed}$ :  $\nu R \parallel \nu S \subseteq NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-ccomp-closed-alt}$  [*simp*]:  $\nu (R \parallel \nu S) = R \parallel \nu S$   
 $\langle \text{proof} \rangle$

**lemma**  $tarski\text{-prod}$ :  $(\nu R \cdot NC) \cdot (\nu S \cdot NC) = (\text{if } \nu S = \{\} \text{ then } \{\} \text{ else } \nu R \cdot NC)$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-prod-aux}$  [*simp*]:  $(\nu R \cdot NC) \cdot NC = \nu R \cdot NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-vec-add-closed}$ :  $(\nu R \cdot NC \cup \nu S \cdot NC) \cdot NC = \nu R \cdot NC \cup \nu S \cdot NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-vec-par-is-meet}$ :  $\nu R \cdot NC \parallel \nu S \cdot NC = \nu R \cdot NC \cap \nu S \cdot NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-vec-meet-closed}$ :  $(\nu R \cdot NC \cap \nu S \cdot NC) \cdot NC = \nu R \cdot NC \cap \nu S \cdot NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-vec-par-closed}$ :  $(\nu R \cdot NC \parallel \nu S \cdot NC) \cdot NC = \nu R \cdot NC \parallel \nu S \cdot NC$   
 $\langle \text{proof} \rangle$

**lemma**  $nc\text{-vec-seq-closed}$ :  $((\nu R \cdot NC) \cdot (\nu S \cdot NC)) \cdot NC = (\nu R \cdot NC) \cdot (\nu S \cdot NC)$   
 $\langle \text{proof} \rangle$

**lemma**  $iso5\text{-sharp}$  [*simp*]:  $(\nu R \cdot 1_\pi \parallel NC) \cdot 1_\pi = \nu R \cdot 1_\pi$   
 $\langle \text{proof} \rangle$

**lemma**  $iso6\text{-sharp}$  [*simp*]:  $(\nu R \cdot NC \cdot 1_\pi) \parallel NC = \nu R \cdot NC$

*<proof>*

**lemma** *nsv-hom-par*:  $(R \parallel S) \cdot NC = R \cdot NC \parallel S \cdot NC$   
*<proof>*

**lemma** *nvs-hom-meet*:  $Dom (\nu R \cdot NC \cap \nu S \cdot NC) = Dom (\nu R \cdot NC) \cap Dom (\nu S \cdot NC)$   
*<proof>*

**lemma** *ncv-hom-meet*:  $R \cdot 1_\pi \cap S \cdot 1_\pi \parallel NC = (R \cdot 1_\pi \parallel NC) \cap (S \cdot 1_\pi \parallel NC)$   
*<proof>*

**lemma** *ncv-hom-par*:  $(R \parallel S) \parallel NC = R \parallel NC \parallel S \parallel NC$   
*<proof>*

**lemma** *nvc-hom-meet*:  $(\nu R \cdot NC \cap \nu S \cdot NC) \cdot 1_\pi = (\nu R \cdot NC) \cdot 1_\pi \cap (\nu S \cdot NC) \cdot 1_\pi$   
*<proof>*

**lemma** *tau-int*:  $\tau R \leq R$   
*<proof>*

**lemma** *nu-int*:  $\nu R \leq R$   
*<proof>*

**lemma** *tau-ret [simp]*:  $\tau (\tau R) = \tau R$   
*<proof>*

**lemma** *nu-ret [simp]*:  $\nu (\nu R) = \nu R$   
*<proof>*

**lemma** *tau-iso*:  $R \leq S \implies \tau R \leq \tau S$   
*<proof>*

**lemma** *nu-iso*:  $R \leq S \implies \nu R \leq \nu S$   
*<proof>*

**lemma** *tau-zero [simp]*:  $\tau \{\} = \{\}$   
*<proof>*

**lemma** *nu-zero [simp]*:  $\nu \{\} = \{\}$   
*<proof>*

**lemma** *tau-s [simp]*:  $\tau 1_\sigma = \{\}$   
*<proof>*

**lemma** *tau-c [simp]*:  $\tau 1_\pi = 1_\pi$   
*<proof>*

**lemma** *nu-c* [*simp*]:  $\nu 1_\pi = \{\}$   
*<proof>*

**lemma** *tau-nc* [*simp*]:  $\tau NC = \{\}$   
*<proof>*

**lemma** *nu-nc* [*simp*]:  $\nu NC = NC$   
*<proof>*

**lemma** *tau-U* [*simp*]:  $\tau U = 1_\pi$   
*<proof>*

**lemma** *nu-U* [*simp*]:  $\nu U = NC$   
*<proof>*

**lemma** *tau-add* [*simp*]:  $\tau (R \cup S) = \tau R \cup \tau S$   
*<proof>*

**lemma** *nu-add* [*simp*]:  $\nu (R \cup S) = \nu R \cup \nu S$   
*<proof>*

**lemma** *tau-meet* [*simp*]:  $\tau (R \cap S) = \tau R \cap \tau S$   
*<proof>*

**lemma** *nu-meet* [*simp*]:  $\nu (R \cap S) = \nu R \cap \nu S$   
*<proof>*

**lemma** *tau-seq*:  $\tau (R \cdot S) = \tau R \cup \nu R \cdot \tau S$   
*<proof>*

**lemma** *tau-par* [*simp*]:  $\tau (R \parallel S) = \tau R \parallel \tau S$   
*<proof>*

**lemma** *nu-par-aux1*:  $R \parallel \tau S = \text{Dom} (\tau S) \cdot R$   
*<proof>*

**lemma** *nu-par-aux3* [*simp*]:  $\nu (\nu R \parallel \tau S) = \nu R \parallel \tau S$   
*<proof>*

**lemma** *nu-par-aux4* [*simp*]:  $\nu (\tau R \parallel \tau S) = \{\}$   
*<proof>*

**lemma** *nu-par*:  $\nu (R \parallel S) = \text{Dom} (\tau R) \cdot \nu S \cup \text{Dom} (\tau S) \cdot \nu R \cup (\nu R \parallel \nu S)$   
*<proof>*

**lemma** *sprod-tau-nu*:  $R \cdot S = \tau R \cup \nu R \cdot S$   
*<proof>*

**lemma** *pprod-tau-nu*:  $R \parallel S = (\nu R \parallel \nu S) \cup \text{Dom} (\tau R) \cdot \nu S \cup \text{Dom} (\tau S) \cdot \nu R$

$R \cup (\tau R \parallel \tau S)$   
*<proof>*

**lemma** *tau-idem* [*simp*]:  $\tau R \cdot \tau R = \tau R$   
*<proof>*

**lemma** *tau-interr*:  $(R \parallel S) \cdot \tau T = R \cdot \tau T \parallel S \cdot \tau T$   
*<proof>*

**lemma** *tau-le-c*:  $\tau R \leq 1_\pi$   
*<proof>*

**lemma** *c-le-tauc*:  $1_\pi \leq \tau 1_\pi$   
*<proof>*

**lemma** *x-alpha-tau* [*simp*]:  $\nu R \cup \tau R = R$   
*<proof>*

**lemma** *alpha-tau-zero* [*simp*]:  $\nu (\tau R) = \{\}$   
*<proof>*

**lemma** *tau-alpha-zero* [*simp*]:  $\tau (\nu R) = \{\}$   
*<proof>*

**lemma** *sprod-tau-nu-var* [*simp*]:  $\nu (\nu R \cdot S) = \nu (R \cdot S)$   
*<proof>*

**lemma** *tau-s-prod* [*simp*]:  $\tau (R \cdot S) = R \cdot \tau S$   
*<proof>*

**lemma** *alpha-fp*:  $\nu R = R \longleftrightarrow R \cdot \{\} = \{\}$   
*<proof>*

**lemma** *p-prod-tau-alpha*:  $R \parallel S = (R \parallel \nu S) \cup (\nu R \parallel S) \cup (\tau R \parallel \tau S)$   
*<proof>*

**lemma** *p-prod-tau-alpha-var*:  $R \parallel S = (R \parallel \nu S) \cup (\nu R \parallel S) \cup \tau (R \parallel S)$   
*<proof>*

**lemma** *alpha-par*:  $\nu (R \parallel S) = (\nu R \parallel S) \cup (R \parallel \nu S)$   
*<proof>*

**lemma** *alpha-tau* [*simp*]:  $\nu (R \cdot \tau S) = \{\}$   
*<proof>*

**lemma** *nu-par-prop*:  $\nu R = R \implies \nu (R \parallel S) = R \parallel S$   
*<proof>*

**lemma** *tau-seq-prop*:  $\tau R = R \implies R \cdot S = R$

*<proof>*

**lemma tau-seq-prop2:**  $\tau R = R \implies \tau (R \cdot S) = R \cdot S$   
*<proof>*

**lemma d-nu:**  $\nu (Dom R \cdot S) = Dom R \cdot \nu S$   
*<proof>*

**lemma nu-ideal1:**  $\nu R = R \implies S \leq R \implies \nu S = S$   
*<proof>*

**lemma tau-ideal1:**  $\tau R = R \implies S \leq R \implies \tau S = S$   
*<proof>*

**lemma nu-ideal2:**  $\nu R = R \implies \nu S = S \implies \nu (R \cup S) = R \cup S$   
*<proof>*

**lemma tau-ideal2:**  $\tau R = R \implies \tau S = S \implies \tau (R \cup S) = R \cup S$   
*<proof>*

**lemma tau-add-precong:**  $\tau R \leq \tau S \implies \tau (R \cup T) \leq \tau (S \cup T)$   
*<proof>*

**lemma tau-meet-precong:**  $\tau R \leq \tau S \implies \tau (R \cap T) \leq \tau (S \cap T)$   
*<proof>*

**lemma tau-par-precong:**  $\tau R \leq \tau S \implies \tau (R \parallel T) \leq \tau (S \parallel T)$   
*<proof>*

**lemma tau-seq-precongl:**  $\tau R \leq \tau S \implies \tau (T \cdot R) \leq \tau (T \cdot S)$   
*<proof>*

**lemma nu-add-precong:**  $\nu R \leq \nu S \implies \nu (R \cup T) \leq \nu (S \cup T)$   
*<proof>*

**lemma nu-meet-precong:**  $\nu R \leq \nu S \implies \nu (R \cap T) \leq \nu (S \cap T)$   
*<proof>*

**lemma nu-seq-precongr:**  $\nu R \leq \nu S \implies \nu (R \cdot T) \leq \nu (S \cdot T)$   
*<proof>*

**definition**

$$tcg R S = (\tau R \leq \tau S \wedge \tau S \leq \tau R)$$

**definition**

$$ncg R S = (\nu R \leq \nu S \wedge \nu S \leq \nu R)$$

**lemma tcg-refl:**  $tcg R R$   
*<proof>*

**lemma** *tcg-trans*:  $tcg\ R\ S \implies tcg\ S\ T \implies tcg\ R\ T$   
*<proof>*

**lemma** *tcg-sym*:  $tcg\ R\ S \implies tcg\ S\ R$   
*<proof>*

**lemma** *ncg-refl*:  $ncg\ R\ R$   
*<proof>*

**lemma** *ncg-trans*:  $ncg\ R\ S \implies ncg\ S\ T \implies ncg\ R\ T$   
*<proof>*

**lemma** *ncg-sym*:  $ncg\ R\ S \implies ncg\ S\ R$   
*<proof>*

**lemma** *tcg-alt*:  $tcg\ R\ S = (\tau\ R = \tau\ S)$   
*<proof>*

**lemma** *ncg-alt*:  $ncg\ R\ S = (\nu\ R = \nu\ S)$   
*<proof>*

**lemma** *tcg-add*:  $\tau\ R = \tau\ S \implies \tau\ (R \cup T) = \tau\ (S \cup T)$   
*<proof>*

**lemma** *tcg-meet*:  $\tau\ R = \tau\ S \implies \tau\ (R \cap T) = \tau\ (S \cap T)$   
*<proof>*

**lemma** *tcg-par*:  $\tau\ R = \tau\ S \implies \tau\ (R \parallel T) = \tau\ (S \parallel T)$   
*<proof>*

**lemma** *tcg-seql*:  $\tau\ R = \tau\ S \implies \tau\ (T \cdot R) = \tau\ (T \cdot S)$   
*<proof>*

**lemma** *ncg-add*:  $\nu\ R = \nu\ S \implies \nu\ (R \cup T) = \nu\ (S \cup T)$   
*<proof>*

**lemma** *ncg-meet*:  $\nu\ R = \nu\ S \implies \nu\ (R \cap T) = \nu\ (S \cap T)$   
*<proof>*

**lemma** *ncg-seqr*:  $\nu\ R = \nu\ S \implies \nu\ (R \cdot T) = \nu\ (S \cdot T)$   
*<proof>*

### 3.8 Powers

**primrec** *p-power* ::  $('a, 'a)\ mrel \Rightarrow nat \Rightarrow ('a, 'a)\ mrel$  **where**  
 $p\text{-power}\ R\ 0 = 1_\sigma \mid$   
 $p\text{-power}\ R\ (Suc\ n) = R \cdot p\text{-power}\ R\ n$

**primrec** *power-rd* :: ('a,'a) mrel  $\Rightarrow$  nat  $\Rightarrow$  ('a,'a) mrel **where**  
*power-rd* R 0 = {} |  
*power-rd* R (Suc n) =  $1_\sigma \cup R \cdot \text{power-rd } R \ n$

**primrec** *power-sq* :: ('a,'a) mrel  $\Rightarrow$  nat  $\Rightarrow$  ('a,'a) mrel **where**  
*power-sq* R 0 =  $1_\sigma$  |  
*power-sq* R (Suc n) =  $1_\sigma \cup R \cdot \text{power-sq } R \ n$

**lemma** *power-rd-chain*: *power-rd* R n  $\leq$  *power-rd* R (n + 1)  
 <proof>

**lemma** *power-sq-chain*: *power-sq* R n  $\leq$  *power-sq* R (n + 1)  
 <proof>

**lemma** *pow-chain*: *p-power* ( $1_\sigma \cup R$ ) n  $\leq$  *p-power* ( $1_\sigma \cup R$ ) (n + 1)  
 <proof>

**lemma** *pow-prop*: *p-power* ( $1_\sigma \cup R$ ) (n + 1) =  $1_\sigma \cup R \cdot \text{p-power } (1_\sigma \cup R) \ n$   
 <proof>

**lemma** *power-rd-le-sq*: *power-rd* R n  $\leq$  *power-sq* R n  
 <proof>

**lemma** *power-sq-le-rd*: *power-sq* R n  $\leq$  *power-rd* R (Suc n)  
 <proof>

**lemma** *power-sq-power*: *power-sq* R n = *p-power* ( $1_\sigma \cup R$ ) n  
 <proof>

### 3.9 Star

**lemma** *iso-prop*: mono ( $\lambda X. S \cup R \cdot X$ )  
 <proof>

**lemma** *gfp-lfp-prop*: *gfp* ( $\lambda X. R \cdot X$ )  $\cup$  *lfp* ( $\lambda X. S \cup R \cdot X$ )  $\subseteq$  *gfp* ( $\lambda X. S \cup R \cdot X$ )  
 <proof>

**definition** *star* :: ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel **where**  
*star* R = *lfp* ( $\lambda X. s\text{-id} \cup R \cdot X$ )

**lemma** *star-unfold*:  $1_\sigma \cup R \cdot \text{star } R \leq \text{star } R$   
 <proof>

**lemma** *star-induct*:  $1_\sigma \cup R \cdot S \leq S \implies \text{star } R \leq S$   
 <proof>

**lemma** *star-refl*:  $1_\sigma \leq \text{star } R$   
 <proof>

**lemma** *star-unfold-part*:  $R \cdot \text{star } R \leq \text{star } R$   
*<proof>*

**lemma** *star-ext-aux*:  $R \leq R \cdot \text{star } R$   
*<proof>*

**lemma** *star-ext*:  $R \leq \text{star } R$   
*<proof>*

**lemma** *star-co-trans*:  $\text{star } R \leq \text{star } R \cdot \text{star } R$   
*<proof>*

**lemma** *star-iso*:  $R \leq S \implies \text{star } R \leq \text{star } S$   
*<proof>*

**lemma** *star-unfold-eq* [*simp*]:  $1_\sigma \cup R \cdot \text{star } R = \text{star } R$   
*<proof>*

**lemma** *nu-star1*:  
**assumes**  $\bigwedge(R::('a, 'a) \text{ mrel}) (S::('a, 'a) \text{ mrel}) (T::('a, 'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$   
**shows**  $\text{star } (R::('a, 'a) \text{ mrel}) \leq \text{star } (\nu R) \cdot (1_\sigma \cup \tau R)$   
*<proof>*

**lemma** *nu-star2*:  
**assumes**  $\bigwedge(R::('a, 'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$   
**shows**  $\text{star } (\nu (R::('a, 'a) \text{ mrel})) \cdot (1_\sigma \cup \tau R) \leq \text{star } R$   
*<proof>*

**lemma** *nu-star*:  
**assumes**  $\bigwedge(R::('a, 'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$   
**and**  $\bigwedge(R::('a, 'a) \text{ mrel}) (S::('a, 'a) \text{ mrel}) (T::('a, 'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$   
**shows**  $\text{star } (\nu (R::('a, 'a) \text{ mrel})) \cdot (1_\sigma \cup \tau R) = \text{star } R$   
*<proof>*

**lemma** *tau-star*:  $\text{star } (\tau R) = 1_\sigma \cup \tau R$   
*<proof>*

**lemma** *tau-star-var*:  
**assumes**  $\bigwedge(R::('a, 'a) \text{ mrel}) (S::('a, 'a) \text{ mrel}) (T::('a, 'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$   
**and**  $\bigwedge(R::('a, 'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$   
**shows**  $\tau (\text{star } (R::('a, 'a) \text{ mrel})) = \text{star } (\nu R) \cdot \tau R$   
*<proof>*

**lemma** *nu-star-sub*:  $\text{star } (\nu R) \leq \nu (\text{star } R)$   
*<proof>*

**lemma** *nu-star-nu* [simp]:  $\nu (\text{star } (\nu R)) = \text{star } (\nu R)$   
 ⟨proof⟩

**lemma** *nu-star-tau* [simp]:  $\nu (\text{star } (\tau R)) = 1_\sigma$   
 ⟨proof⟩

**lemma** *tau-star-tau* [simp]:  $\tau (\text{star } (\tau R)) = \tau R$   
 ⟨proof⟩

**lemma** *tau-star-nu* [simp]:  $\tau (\text{star } (\nu R)) = \{\}$   
 ⟨proof⟩

**lemma** *d-star-unfold* [simp]:  $\text{Dom } S \cup \text{Dom } (R \cdot \text{Dom } (\text{star } R \cdot S)) = \text{Dom } (\text{star } R \cdot S)$   
 ⟨proof⟩

**lemma** *d-star-sim1*:

**assumes**  $\bigwedge R S T. \text{Dom } (T::('a,'b) \text{ mrel}) \cup (R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel})$   
 $\leq S \implies \text{star } R \cdot \text{Dom } T \leq S$   
**shows**  $(R::('a,'a) \text{ mrel}) \cdot \text{Dom } (T::('a,'b) \text{ mrel}) \leq \text{Dom } T \cdot (S::('a,'a) \text{ mrel})$   
 $\implies \text{star } R \cdot \text{Dom } T \leq \text{Dom } T \cdot \text{star } S$   
 ⟨proof⟩

**lemma** *d-star-induct*:

**assumes**  $\bigwedge R S T. \text{Dom } (T::('a,'b) \text{ mrel}) \cup (R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel})$   
 $\leq S \implies \text{star } R \cdot \text{Dom } T \leq S$   
**shows**  $\text{Dom } ((R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel})) \leq \text{Dom } S \implies \text{Dom } (\text{star } R \cdot S) \leq \text{Dom } S$   
 ⟨proof⟩

### 3.10 Omega

**definition** *omega* ::  $('a,'a) \text{ mrel} \Rightarrow ('a,'a) \text{ mrel}$  **where**  
 $\text{omega } R \equiv \text{gfp } (\lambda X. R \cdot X)$

**lemma** *om-unfold*:  $\text{omega } R \leq R \cdot \text{omega } R$   
 ⟨proof⟩

**lemma** *om-coinduct*:  $S \leq R \cdot S \implies S \leq \text{omega } R$   
 ⟨proof⟩

**lemma** *om-unfold-eq* [simp]:  $R \cdot \text{omega } R = \text{omega } R$   
 ⟨proof⟩

**lemma** *om-iso*:  $R \leq S \implies \text{omega } R \leq \text{omega } S$   
 ⟨proof⟩

**lemma** *zero-om* [simp]:  $\text{omega } \{\} = \{\}$

*<proof>*

**lemma** *s-id-om* [*simp*]:  $\omega 1_\sigma = U$   
*<proof>*

**lemma** *p-id-om* [*simp*]:  $\omega 1_\pi = 1_\pi$   
*<proof>*

**lemma** *nc-om* [*simp*]:  $\omega NC = U$   
*<proof>*

**lemma** *U-om* [*simp*]:  $\omega U = U$   
*<proof>*

**lemma** *tau-om1*:  $\tau R \leq \tau (\omega R)$   
*<proof>*

**lemma** *tau-om2* [*simp*]:  $\omega (\tau R) = \tau R$   
*<proof>*

**lemma** *tau-om3*:  $\omega (\tau R) \leq \tau (\omega R)$   
*<proof>*

**lemma** *om-nu-tau*:  $\omega (\nu R) \cup \text{star } (\nu R) \cdot \tau R \leq \omega R$   
*<proof>*

**end**

## 4 Multirelational Properties of Power Allegories

**theory** *Power-Allegories-Multirelations*

**imports** *Multirelations-Basics*

**begin**

*We start with random little properties.*

**lemma** *eta-s-id*:  $\eta = s\text{-id}$   
*<proof>*

**lemma** *Lambda-empty* [*simp*]:  $\Lambda \{\} = p\text{-id}$   
*<proof>*

**lemma** *alpha-pid* [*simp*]:  $\alpha p\text{-id} = \{\}$   
*<proof>*

### 4.1 Peleg lifting

**definition** *plift* ::  $(a, b) \text{ mrel} \Rightarrow (a \text{ set}, b \text{ set}) \text{ rel}$  (*<-\**) [1000] 999) **where**

$$R_* = \{(A,B). \exists f. (\forall a \in A. (a, f(a)) \in R) \wedge B = \bigcup (f \text{ ` } A)\}$$

**lemma** *pcomp-pltft*:  $R \cdot S = R ; S_*$   
 ⟨proof⟩

**lemma** *det-pltft-klift*: *deterministic*  $R \implies R_* = (R)_{\mathcal{P}}$   
 ⟨proof⟩

**lemma** *pltft-ext2* [*simp*]:  $\eta ; R_* = R$   
 ⟨proof⟩

**lemma** *pltftext-3* [*simp*]:  $\eta_* = Id$   
 ⟨proof⟩

**lemma** *d-dom-pltft*:  $(Dom R)_* = dom (R_*)$   
 ⟨proof⟩

**lemma** *d-pid-pltft*:  $(Dom R)_* \subseteq Id$   
 ⟨proof⟩

**lemma** *d-pltft-sub*:  $A \subseteq B \implies (B,B) \in (Dom R)_* \implies (A,A) \in (Dom R)_*$   
 ⟨proof⟩

**lemma** *pltft-empty*:  $(\{\}, A) \in R_* \iff A = \{\}$   
 ⟨proof⟩

**lemma** *univ-pltft-klift*:  
**assumes** *univalent*  $R$   
**shows**  $R_* = (Dom R)_* ; (R)_{\mathcal{P}}$   
 ⟨proof⟩

**lemma** *pltft-ext1*:  
**assumes** *univalent*  $f$   
**shows**  $(R ; f_*)_* = R_* ; f_*$   
 ⟨proof⟩

**lemma** *pltft-assoc-univ*: *univalent*  $f \implies (R \cdot S) \cdot f = R \cdot (S \cdot f)$   
 ⟨proof⟩

**lemma** *Lambda-funct*:  $\Lambda (R ; S) = \Lambda R \cdot \Lambda S$   
 ⟨proof⟩

**lemma** *eta-funct*:  $R ; S ; \eta = (R ; \eta) \cdot (S ; \eta)$   
 ⟨proof⟩

**lemma** *alpha-funct-det*: *deterministic*  $R \implies$  *deterministic*  $S \implies \alpha (R \cdot S) = \alpha R ; \alpha S$   
 ⟨proof⟩

**lemma** *pcomp-det*: *deterministic*  $S \implies R \cdot S = R ; (S)_{\mathcal{P}}$   
 ⟨proof⟩

**lemma** *pcomp-det2*: *deterministic*  $R \implies$  *deterministic*  $S \implies (R \cdot S)_{\mathcal{P}} = (R)_{\mathcal{P}} ; (S)_{\mathcal{P}}$   
 ⟨proof⟩

**lemma** *pcomp-alpha*:  $\alpha (R \cdot S) = R ; \alpha ((S)_*)$   
 ⟨proof⟩

## 4.2 Fusion and fission

**definition** *fus* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$  **where**  
 $fus R = \Lambda (\alpha R)$

**definition** *fis* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$  **where**  
 $fis R = \alpha R ; \eta$

**lemma** *fus-set*:  $fus R = \{(a, B) \mid a B. B = \bigcup (\text{Image } R \{a\})\}$   
 ⟨proof⟩

**lemma** *fis-set*:  $fis R = \{(a, \{b\}) \mid a b. b \in \bigcup (\text{Image } R \{a\})\}$   
 ⟨proof⟩

**lemma** *fis-det-comp*: *deterministic*  $R \implies$  *deterministic*  $S \implies fis (R \cdot S) = fis R \cdot fis S$   
 ⟨proof⟩

**lemma** *fis-fix-det*: *deterministic*  $R = (fus R = R)$   
 ⟨proof⟩

## 4.3 Galois connections for multirelations

**lemma** *sub-subh*:  $R \subseteq S \implies R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$   
 ⟨proof⟩

**lemma** *alpha-Lambda-galois*:  $(\alpha R \subseteq S) = (R \subseteq \Lambda S ; (\text{epsiloff} \parallel \text{epsiloff}))$   
 ⟨proof⟩

**lemma** *alpha-Lambda-galois-set*:  $(\alpha R \subseteq S) = (R \subseteq \{(a, A). \exists B. (a, B) \in \Lambda S \wedge A \subseteq B\})$   
 ⟨proof⟩

**lemma** *epsiloff-eta-lres*:  $\text{epsiloff} ; \eta \subseteq \text{epsiloff} \parallel \text{epsiloff}$   
 ⟨proof⟩

**lemma** *eta-alpha-galois*:  $(R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})) = (R \subseteq \alpha S)$   
 ⟨proof⟩

**lemma** *eta-alpha-galois-set*:  $(R ; \eta \subseteq \{(a,A). \exists B. (a,B) \in S \wedge A \subseteq B\}) = (R \subseteq \alpha S)$

*<proof>*

**lemma** *Lambda-iso*:  $R \subseteq S \implies \Lambda R \subseteq \Lambda S ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *eta-iso*:  $R \subseteq S \implies R ; \eta \subseteq S ; \eta ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *alpha-iso*:  $R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff}) \implies \alpha R \subseteq \alpha S$

*<proof>*

**lemma** *Lambda-canc-dcl*:  $R \subseteq \Lambda (\alpha R) ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *eta-canc-dcl*:  $\alpha R ; \eta \subseteq R ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *alpha-surj*: *surj*  $\alpha$

*<proof>*

**lemma** *Lambda-inj*: *inj*  $\Lambda$

*<proof>*

**lemma** *eta-inj*: *inj*  $(\lambda x. x ; \eta)$

*<proof>*

**lemma** *fus-least-odet*:

**assumes**  $\Lambda (\alpha S) = S$

**and**  $R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$

**shows**  $\Lambda (\alpha R) \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *fis-greatest-idet*:

**assumes**  $\alpha S ; \eta = S$

**and**  $S \subseteq R ; (\text{epsiloff} \parallel \text{epsiloff})$

**shows**  $S \subseteq \alpha R ; \eta ; (\text{epsiloff} \parallel \text{epsiloff})$

*<proof>*

**lemma** *fis-fus-galois*:  $(\alpha R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})) = (R \subseteq \Lambda (\alpha S) ;$

$(\text{epsiloff} \parallel \text{epsiloff}))$

*<proof>*

#### 4.4 Properties of alpha, fission and fusion

**lemma** *alpha-lax*:  $\alpha (R \cdot S) \subseteq \alpha R ; \alpha S$

*<proof>*

**lemma** *alpha-down* [*simp*]:  $\alpha (R ; \Omega^\smile) = \alpha R$   
 ⟨*proof*⟩

**lemma** *fis-fis* [*simp*]:  $fis \circ fis = fis$   
 ⟨*proof*⟩

**lemma** *fus-fus* [*simp*]:  $fus \circ fus = fus$   
 ⟨*proof*⟩

**lemma** *fis-fus* [*simp*]:  $fis \circ fus = fis$   
 ⟨*proof*⟩

**lemma** *fus-fis* [*simp*]:  $fus \circ fis = fus$   
 ⟨*proof*⟩

**lemma** *fis-alpha*:  $fis R \cdot S = \alpha R ; S$   
 ⟨*proof*⟩

**lemma** *fis-lax*:  $fis (R \cdot S) \subseteq fis R \cdot fis S$   
 ⟨*proof*⟩

**lemma** *klift-fus*:  $(R)_{\mathcal{P}} = fus (epsilon\text{loff} ; R)$   
 ⟨*proof*⟩

**lemma** *fus-eta-klift*:  $fus R = \eta ; (R)_{\mathcal{P}}$   
 ⟨*proof*⟩

**lemma** *fus-Lambda-mu*:  $fus R = \Lambda R ; \mu$   
 ⟨*proof*⟩

## 4.5 Properties of fusion, fission, nu and tau

**lemma** *alpha-tau* [*simp*]:  $\alpha (\tau R) = \{\}$   
 ⟨*proof*⟩

**lemma** *alpha-nu* [*simp*]:  $\alpha (\nu R) = \alpha R$   
 ⟨*proof*⟩

**lemma** *nu-fis* [*simp*]:  $\nu (fis R) = fis R$   
 ⟨*proof*⟩

**lemma** *nu-fis-var*:  $\nu (fis R) = fis (\nu R)$   
 ⟨*proof*⟩

**lemma** *tau-fis* [*simp*]:  $\tau (fis R) = \{\}$   
 ⟨*proof*⟩

### Properties of tests and domain

**lemma** *subid-plit*:  $(P \cap \eta)_* = \{(A,A) \mid A. \forall a \in A. (a, \{a\}) \in (P \cap \eta)\}$   
 ⟨*proof*⟩

**lemma** *U-subid*:  $R ; (P \cap \eta)_* = R \cap U ; (P \cap \eta)_*$   
 ⟨proof⟩

**lemma** *subid-plift-down*:  $U ; (P \cap \eta)_* ; \Omega^\smile = U ; (P \cap \eta)_*$   
 ⟨proof⟩

**lemma** *nu-subid-plift*:  $\nu (R ; (P \cap \eta)_*) = \nu R ; (P \cap \eta)_*$   
 ⟨proof⟩

**lemma** *dom-fis1*:  $\text{dom} (fis R) = \text{dom} (\alpha R)$   
 ⟨proof⟩

**lemma** *dom-fis2*:  $\text{dom} (fis R) = \text{dom} (\alpha (\nu R))$   
 ⟨proof⟩

**lemma** *dom-fis3*:  $\text{dom} (fis R) = \text{dom} (\nu R)$   
 ⟨proof⟩

**lemma** *dom-fis4*:  $\text{dom} (fis R) = \text{dom} (\nu (fus R))$   
 ⟨proof⟩

**lemma** *dom-alpha*:  $\text{dom} (\alpha R ; (P \cap \eta)) = \text{dom} (\nu (R ; \Omega^\smile) ; (P \cap \eta)_*)$   
 ⟨proof⟩

## 4.6 Box and diamond

**definition** *box* :: ('a, 'b) mrel  $\Rightarrow$  ('b set, 'a set) rel **where**  
 $box R = rbox (\alpha R)$

**definition** *dia* :: ('a, 'b) mrel  $\Rightarrow$  ('b set, 'a set) rel **where**  
 $dia R = \mathcal{P} ((\alpha R)^\smile)$

**lemma** *box-set*:  $box R = \{(B, A). A = \{a. \forall C. (a, C) \in R \longrightarrow C \subseteq B\}\}$   
 ⟨proof⟩

**lemma** *dia-set*:  $dia R = \{(B, A). A = \{a. \exists C. (a, C) \in R \wedge C \cap B \neq \{\}\}\}$   
 ⟨proof⟩

**lemma** *box-Omega*:  $box R = \Lambda (\Omega^\smile // R)$   
 ⟨proof⟩

**end**

**theory** *Multirelations*

**imports** *Power-Allegories-Multirelations*

**begin**

**lemma** *nonempty-set-card*:  
**assumes** *finite S*  
**shows**  $S \neq \{\}$   $\longleftrightarrow$   $\text{card } S \geq 1$   
 $\langle \text{proof} \rangle$

**no-notation** *one-class.one*  $\langle 1 \rangle$   
**no-notation** *times-class.times* (**infixl**  $\langle * \rangle$  70)

**no-notation** *rel-fdia*  $\langle (| - ) \rangle$  [61,81] 82)  
**no-notation** *rel-bdia*  $\langle ( \langle - | - \rangle ) \rangle$  [61,81] 82)  
**no-notation** *rel-fbox*  $\langle (| - ] \rangle$  [61,81] 82)  
**no-notation** *rel-bbox*  $\langle ( [ - | \rangle ) \rangle$  [61,81] 82)

**declare** *s-prod-pa-def* [*mr-simp*]

**notation** *s-prod* (**infixl**  $\langle * \rangle$  70)  
**notation** *s-id*  $\langle 1 \rangle$

**lemma** *sp-oi-subdist*:  
 $(P \cap Q) * (R \cap S) \subseteq P * R$   
 $\langle \text{proof} \rangle$

**lemma** *sp-oi-subdist-2*:  
 $(P \cap Q) * (R \cap S) \subseteq (P * R) \cap (Q * S)$   
 $\langle \text{proof} \rangle$

## 5 Inner Structure

### 5.1 Inner union, inner intersection and inner complement

**abbreviation** *inner-union* (**infixl**  $\langle \cup \cup \rangle$  65)  
**where** *inner-union*  $\equiv$  *p-prod*

**definition** *inner-intersection*  $:: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$  (**infixl**  $\langle \cap \cap \rangle$  65) **where**  
 $R \cap \cap S \equiv \{ (a, B) . \exists C D . B = C \cap D \wedge (a, C) \in R \wedge (a, D) \in S \}$

**definition** *inner-complement*  $:: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$  ( $\langle \sim \rightarrow$  [80] 80) **where**  
 $\sim R \equiv \{ (a, B) . (a, -B) \in R \}$

**abbreviation** *iu-unit*  $\langle 1_{\cup \cup} \rangle$   
**where**  $1_{\cup \cup} \equiv$  *p-id*

**definition** *ii-unit*  $:: ('a, 'a) \text{ mrel}$   $\langle 1_{\cap \cap} \rangle$   
**where**  $1_{\cap \cap} \equiv \{ (a, UNIV) \mid a . \text{True} \}$

**declare** *inner-intersection-def* [*mr-simp*] *inner-complement-def* [*mr-simp*]  
*ii-unit-def* [*mr-simp*]

**lemma** *iu-assoc*:  
 $(R \cup S) \cup T = R \cup (S \cup T)$   
*<proof>*

**lemma** *iu-commute*:  
 $R \cup S = S \cup R$   
*<proof>*

**lemma** *iu-unit*:  
 $R \cup 1 \cup = R$   
*<proof>*

**lemma** *ii-assoc*:  
 $(R \cap S) \cap T = R \cap (S \cap T)$   
*<proof>*

**lemma** *ii-commute*:  
 $R \cap S = S \cap R$   
*<proof>*

**lemma** *ii-unit [simp]*:  
 $R \cap 1 \cap = R$   
*<proof>*

**lemma** *pa-ic*:  
 $\sim(R \otimes \sim S) = R \otimes S$   
*<proof>*

**lemma** *ic-involutive [simp]*:  
 $\sim \sim R = R$   
*<proof>*

**lemma** *ic-injective*:  
 $\sim R = \sim S \implies R = S$   
*<proof>*

**lemma** *ic-antidist-iu*:  
 $\sim(R \cup S) = \sim R \cap \sim S$   
*<proof>*

**lemma** *ic-antidist-ii*:  
 $\sim(R \cap S) = \sim R \cup \sim S$   
*<proof>*

**lemma** *ic-iu-unit [simp]*:  
 $\sim 1 \cup = 1 \cap$   
*<proof>*

**lemma** *ic-ii-unit [simp]*:

$$\sim 1 \cap \cap = 1 \cup \cup$$

*<proof>*

**lemma** *ii-unit-split-iu* [simp]:

$$1 \cup \cup \sim 1 = 1 \cap \cap$$

*<proof>*

**lemma** *aux-1*:

$$B = \{a\} \cap D \implies \neg D = \{a\} \implies B = \{\}$$

*<proof>*

**lemma** *iu-unit-split-ii* [simp]:

$$1 \cap \cap \sim 1 = 1 \cup \cup$$

*<proof>*

**lemma** *iu-right-dist-ou*:

$$(R \cup S) \cup \cup T = (R \cup \cup T) \cup (S \cup \cup T)$$

*<proof>*

**lemma** *ii-right-dist-ou*:

$$(R \cup S) \cap \cap T = (R \cap \cap T) \cup (S \cap \cap T)$$

*<proof>*

**lemma** *iu-left-isotone*:

$$R \subseteq S \implies R \cup \cup T \subseteq S \cup \cup T$$

*<proof>*

**lemma** *iu-right-isotone*:

$$R \subseteq S \implies T \cup \cup R \subseteq T \cup \cup S$$

*<proof>*

**lemma** *iu-isotone*:

$$R \subseteq S \implies P \subseteq Q \implies R \cup \cup P \subseteq S \cup \cup Q$$

*<proof>*

**lemma** *ii-left-isotone*:

$$R \subseteq S \implies R \cap \cap T \subseteq S \cap \cap T$$

*<proof>*

**lemma** *ii-right-isotone*:

$$R \subseteq S \implies T \cap \cap R \subseteq T \cap \cap S$$

*<proof>*

**lemma** *ii-isotone*:

$$R \subseteq S \implies P \subseteq Q \implies R \cap \cap P \subseteq S \cap \cap Q$$

*<proof>*

**lemma** *iu-right-subdist-ii*:

$$(R \cap \cap S) \cup \cup T \subseteq (R \cup \cup T) \cap \cap (S \cup \cup T)$$

$\langle proof \rangle$

**lemma** *ii-right-subdist-iu*:

$$(R \cup\cup S) \cap\cap T \subseteq (R \cap\cap T) \cup\cup (S \cap\cap T)$$

$\langle proof \rangle$

**lemma** *ic-isotone*:

$$R \subseteq S \implies \sim R \subseteq \sim S$$

$\langle proof \rangle$

**lemma** *ic-bot* [*simp*]:

$$\sim\{\} = \{\}$$

$\langle proof \rangle$

**lemma** *ic-top* [*simp*]:

$$\sim U = U$$

$\langle proof \rangle$

**lemma** *ic-dist-ou*:

$$\sim(R \cup S) = \sim R \cup \sim S$$

$\langle proof \rangle$

**lemma** *ic-dist-oi*:

$$\sim(R \cap S) = \sim R \cap \sim S$$

$\langle proof \rangle$

**lemma** *ic-dist-oc*:

$$\sim\sim R = \sim(\sim R)$$

$\langle proof \rangle$

**lemma** *ii-sub-idempotent*:

$$R \subseteq R \cap\cap R$$

$\langle proof \rangle$

**definition** *inner-Union* :: ( $'i \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'b) \text{ mrel}$ ) ( $\langle \cup\cup \text{-}| \rightarrow [80, 80] \ 80 \rangle$ ) **where**

$$\cup\cup X|I \equiv \{ (a, B) . \exists f . B = (\cup i \in I . f i) \wedge (\forall i \in I . (a, f i) \in X i) \}$$

**definition** *inner-Intersection* :: ( $'i \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'b) \text{ mrel}$ ) ( $\langle \cap\cap \text{-}| \rightarrow [80, 80] \ 80 \rangle$ ) **where**

$$\cap\cap X|I \equiv \{ (a, B) . \exists f . B = (\cap i \in I . f i) \wedge (\forall i \in I . (a, f i) \in X i) \}$$

**declare** *inner-Union-def* [*mr-simp*] *inner-Intersection-def* [*mr-simp*]

**lemma** *iU-empty*:

$$\cup\cup X|\{\} = 1_{\cup\cup}$$

$\langle proof \rangle$

**lemma** *iI-empty*:

$$\bigcap \bigcap X | \{\} = 1_{\cap \cap}$$

*<proof>*

**lemma** *ic-antidist-iU*:

$$\sim \bigcup \bigcup X | I = \bigcap \bigcap (\text{inner-complement o } X) | I$$

*<proof>*

**lemma** *ic-antidist-iI*:

$$\sim \bigcap \bigcap X | I = \bigcup \bigcup (\text{inner-complement o } X) | I$$

*<proof>*

**lemma** *iu-right-dist-oU*:

$$\bigcup X \cup \cup T = (\bigcup R \in X . R \cup \cup T)$$

*<proof>*

**lemma** *ii-right-dist-oU*:

$$\bigcup X \cap \cap T = (\bigcup R \in X . R \cap \cap T)$$

*<proof>*

**lemma** *iu-right-subdist-iI*:

$$\bigcap \bigcap X | I \cup \cup T \subseteq \bigcap \bigcap (\lambda i . X i \cup \cup T) | I$$

*<proof>*

**lemma** *ii-right-subdist-iU*:

$$\bigcup \bigcup X | I \cap \cap T \subseteq \bigcup \bigcup (\lambda i . X i \cap \cap T) | I$$

*<proof>*

**lemma** *ic-dist-oU*:

$$\sim \bigcup X = \bigcup (\text{inner-complement ' } X)$$

*<proof>*

**lemma** *ic-dist-oI*:

$$\sim \bigcap X = \bigcap (\text{inner-complement ' } X)$$

*<proof>*

**lemma** *sp-left-subdist-iU*:

$$R * (\bigcup \bigcup X | I) \subseteq \bigcup \bigcup (\lambda i . R * X i) | I$$

*<proof>*

**lemma** *sp-right-subdist-iU*:

$$(\bigcup \bigcup X | I) * R \subseteq \bigcup \bigcup (\lambda i . X i * R) | I$$

*<proof>*

**lemma** *sp-right-dist-iU*:

**assumes**  $\forall J :: 'a \text{ set} . J \neq \{\} \longrightarrow (\bigcup \bigcup (\lambda j . R) | J) \subseteq R$

**shows**  $(\bigcup \bigcup X | I) * R = \bigcup \bigcup (\lambda i . X i * R) | (I :: 'a \text{ set})$

*<proof>*

## 5.2 Dual

**abbreviation**  $dual :: ('a, 'b) mrel \Rightarrow ('a, 'b) mrel \langle \cdot^d \rangle [100] 100$   
**where**  $R^d \equiv \sim -R$

**lemma** *dual*:

$$R^d = \{ (a, B) . (a, -B) \notin R \}$$

*<proof>*

**declare** *dual* [*mr-simp*]

**lemma** *dual-antitone*:

$$R \subseteq S \Longrightarrow S^d \subseteq R^d$$

*<proof>*

**lemma** *ic-oc-dual*:

$$\sim R = -R^d$$

*<proof>*

**lemma** *dual-involutive* [*simp*]:

$$R^{dd} = R$$

*<proof>*

**lemma** *dual-antidist-ou*:

$$(R \cup S)^d = R^d \cap S^d$$

*<proof>*

**lemma** *dual-antidist-oi*:

$$(R \cap S)^d = R^d \cup S^d$$

*<proof>*

**lemma** *dual-dist-oc*:

$$(-R)^d = -R^d$$

*<proof>*

**lemma** *dual-dist-ic*:

$$(\sim R)^d = \sim R^d$$

*<proof>*

**lemma** *dual-antidist-oU*:

$$(\bigcup X)^d = \bigcap (dual \ ' X)$$

*<proof>*

**lemma** *dual-antidist-oI*:

$$(\bigcap X)^d = \bigcup (dual \ ' X)$$

*<proof>*

### 5.3 Co-composition

**definition** *co-prod* :: ('a,'b) mrel  $\Rightarrow$  ('b,'c) mrel  $\Rightarrow$  ('a,'c) mrel (**infixl**  $\odot$  70)

**where**

$$R \odot S \equiv \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap \{ f b \mid b . b \in B \}) \}$$

**lemma** *co-prod-im*:

$$R \odot S = \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap ((\lambda x . f x) ` B)) \}$$

*<proof>*

**lemma** *co-prod-iff*:

$$(a,C) \in (R \odot S) \iff (\exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap \{ f b \mid b . b \in B \}))$$

*<proof>*

**declare** *co-prod-im* [*mr-simp*]

**lemma** *co-prod*:

$$R \odot S = \sim(R * \sim S)$$

*<proof>*

**lemma** *cp-left-isotone*:

$$R \subseteq S \implies R \odot T \subseteq S \odot T$$

*<proof>*

**lemma** *cp-right-isotone*:

$$R \subseteq S \implies T \odot R \subseteq T \odot S$$

*<proof>*

**lemma** *cp-isotone*:

$$R \subseteq S \implies P \subseteq Q \implies R \odot P \subseteq S \odot Q$$

*<proof>*

**lemma** *ic-dist-cp*:

$$\sim(R \odot S) = R * \sim S$$

*<proof>*

**lemma** *ic-dist-sp*:

$$\sim(R * S) = R \odot \sim S$$

*<proof>*

**lemma** *ic-cp-ic-unit*:

$$\sim R = R \odot \sim 1$$

*<proof>*

**lemma** *cp-left-zero* [*simp*]:

$$\{\} \odot R = \{\}$$

*<proof>*

**lemma** *cp-left-unit* [*simp*]:

$$1 \odot R = R$$

*<proof>*

**lemma** *cp-ic-unit* [*simp*]:

$$\sim 1 \odot \sim 1 = 1$$

*<proof>*

**lemma** *cp-right-dist-ou*:

$$(R \cup S) \odot T = (R \odot T) \cup (S \odot T)$$

*<proof>*

**lemma** *cp-left-iu-unit* [*simp*]:

$$1_{\cup\cup} \odot R = 1_{\cap\cap}$$

*<proof>*

**lemma** *cp-right-ii-unit*:

$$R \odot 1_{\cap\cap} \subseteq R \cup\cup \sim R$$

*<proof>*

**lemma** *sp-right-iu-unit*:

$$R * 1_{\cup\cup} \subseteq R \cap\cap \sim R$$

*<proof>*

**lemma** *cp-left-subdist-ii*:

$$R \odot (S \cap\cap T) \subseteq (R \odot S) \cap\cap (R \odot T)$$

*<proof>*

**lemma** *cp-right-subantidist-iu*:

$$(R \cup\cup S) \odot T \subseteq (R \odot T) \cap\cap (S \odot T)$$

*<proof>*

**lemma** *cp-right-antidist-iu*:

$$\text{assumes } T \cap\cap T \subseteq T$$

$$\text{shows } (R \cup\cup S) \odot T = (R \odot T) \cap\cap (S \odot T)$$

*<proof>*

**lemma** *cp-right-dist-oU*:

$$\bigcup X \odot T = \bigcup_{R \in X} R \odot T$$

*<proof>*

**lemma** *cp-left-subdist-iI*:

$$R \odot (\bigcap \bigcap X | I) \subseteq \bigcap \bigcap (\lambda i . R \odot X i) | I$$

*<proof>*

**lemma** *cp-right-subantidist-iU*:

$$(\bigcup \bigcup X | I) \odot R \subseteq \bigcap \bigcap (\lambda i . X i \odot R) | I$$

*<proof>*

**lemma** *cp-right-antidist-iU*:  
**assumes**  $\forall J :: 'a \text{ set} . J \neq \{\}$   $\longrightarrow (\bigcap \bigcap (\lambda j . R)|J) \subseteq R$   
**shows**  $(\bigcup \bigcup X|I) \odot R = \bigcap \bigcap (\lambda i . X i \odot R)|(I :: 'a \text{ set})$   
*<proof>*

## 5.4 Inner order

**definition** *inner-order-iu* ::  $'a \times 'b \text{ set} \Rightarrow 'a \times 'b \text{ set} \Rightarrow \text{bool}$  (**infix**  $\langle \preceq_{\cup\cup} \rangle$  50)

**where**

$$x \preceq_{\cup\cup} y \equiv \text{fst } x = \text{fst } y \wedge \text{snd } x \subseteq \text{snd } y$$

**definition** *inner-order-ii* ::  $'a \times 'b \text{ set} \Rightarrow 'a \times 'b \text{ set} \Rightarrow \text{bool}$  (**infix**  $\langle \preceq_{\cap\cap} \rangle$  50)

**where**

$$x \preceq_{\cap\cap} y \equiv \text{fst } x = \text{fst } y \wedge \text{snd } x \supseteq \text{snd } y$$

**lemma** *inner-order-dual*:

$$x \preceq_{\cup\cup} y \longleftrightarrow y \preceq_{\cap\cap} x$$

*<proof>*

**interpretation** *inner-order-iu*: *order*  $(\preceq_{\cup\cup}) \lambda x y . x \preceq_{\cup\cup} y \wedge x \neq y$

*<proof>*

## 5.5 Up-closure, down-closure and convex-closure

**abbreviation** *up* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$   $(\langle \uparrow \rangle [100] 100)$

**where**  $R \uparrow \equiv R \cup\cup U$

**abbreviation** *down* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$   $(\langle \downarrow \rangle [100] 100)$

**where**  $R \downarrow \equiv R \cap\cap U$

**abbreviation** *convex* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$   $(\langle \updownarrow \rangle [100] 100)$

**where**  $R \updownarrow \equiv R \uparrow \cap R \downarrow$

**lemma** *up*:

$$R \uparrow = \{ (a, B) . \exists C . (a, C) \in R \wedge C \subseteq B \}$$

*<proof>*

**lemma** *down*:

$$R \downarrow = \{ (a, B) . \exists C . (a, C) \in R \wedge B \subseteq C \}$$

*<proof>*

**lemma** *convex*:

$$R \updownarrow = \{ (a, B) . \exists C D . (a, C) \in R \wedge (a, D) \in R \wedge C \subseteq B \wedge B \subseteq D \}$$

*<proof>*

**declare** *up* [*mr-simp*] *down* [*mr-simp*] *convex* [*mr-simp*]

**lemma** *ic-up*:

$$\sim(R \uparrow) = (\sim R) \downarrow$$

$\langle proof \rangle$

**lemma** *ic-down*:

$$\sim(R\downarrow) = (\sim R)\uparrow$$

$\langle proof \rangle$

**lemma** *ic-convex*:

$$\sim(R\updownarrow) = (\sim R)\updownarrow$$

$\langle proof \rangle$

**lemma** *up-isotone*:

$$R \subseteq S \implies R\uparrow \subseteq S\uparrow$$

$\langle proof \rangle$

**lemma** *up-increasing*:

$$R \subseteq R\uparrow$$

$\langle proof \rangle$

**lemma** *up-idempotent* [*simp*]:

$$R\uparrow\uparrow = R\uparrow$$

$\langle proof \rangle$

**lemma** *up-dist-ou*:

$$(R \cup S)\uparrow = R\uparrow \cup S\uparrow$$

$\langle proof \rangle$

**lemma** *up-dist-iu*:

$$(R \cup\cup S)\uparrow = R\uparrow \cup\cup S\uparrow$$

$\langle proof \rangle$

**lemma** *up-dist-ii*:

$$(R \cap\cap S)\uparrow = R\uparrow \cap\cap S\uparrow$$

$\langle proof \rangle$

**lemma** *down-isotone*:

$$R \subseteq S \implies R\downarrow \subseteq S\downarrow$$

$\langle proof \rangle$

**lemma** *down-increasing*:

$$R \subseteq R\downarrow$$

$\langle proof \rangle$

**lemma** *down-idempotent* [*simp*]:

$$R\downarrow\downarrow = R\downarrow$$

$\langle proof \rangle$

**lemma** *down-dist-ou*:

$$(R \cup S)\downarrow = R\downarrow \cup S\downarrow$$

$\langle proof \rangle$

**lemma** *down-dist-iu*:

$$(R \cup\cup S)\downarrow = R\downarrow \cup\cup S\downarrow$$

*<proof>*

**lemma** *down-dist-ii*:

$$(R \cap\cap S)\downarrow = R\downarrow \cap\cap S\downarrow$$

*<proof>*

**lemma** *convex-isotone*:

$$R \subseteq S \implies R\uparrow \subseteq S\uparrow$$

*<proof>*

**lemma** *convex-increasing*:

$$R \subseteq R\uparrow$$

*<proof>*

**lemma** *convex-idempotent [simp]*:

$$R\uparrow\uparrow = R\uparrow$$

*<proof>*

**lemma** *down-sp*:

$$R\downarrow = R * (1_{\cup\cup} \cup 1)$$

*<proof>*

**lemma** *up-cp*:

$$R\uparrow = \sim R \odot (1_{\cap\cap} \cup \sim 1)$$

*<proof>*

**lemma** *down-dist-sp*:

$$(R * S)\downarrow = R * S\downarrow$$

*<proof>*

**lemma** *up-dist-cp*:

$$(R \odot S)\uparrow = R \odot S\uparrow$$

*<proof>*

**lemma** *iu-up-oi*:

$$R\uparrow \cup\cup S\uparrow = R\uparrow \cap S\uparrow$$

*<proof>*

**lemma** *ii-down-oi*:

$$R\downarrow \cap\cap S\downarrow = R\downarrow \cap S\downarrow$$

*<proof>*

**lemma** *down-dist-ii-oi*:

$$R\downarrow \cap S\downarrow = (R \cap\cap S)\downarrow$$

*<proof>*

**lemma** *up-dist-iu-oi*:

$$R\uparrow \cap S\uparrow = (R \cup S)\uparrow$$

*<proof>*

**lemma** *oi-down-sub-up*:

$$R\downarrow \cap S\uparrow \subseteq (R\downarrow \cap S)\uparrow$$

*<proof>*

**lemma** *oi-down-up*:

$$R\downarrow \cap S = \{\} \implies R \cap S\uparrow = \{\}$$

*<proof>*

**lemma** *oi-down-up-iff*:

$$R\downarrow \cap S = \{\} \iff R \cap S\uparrow = \{\}$$

*<proof>*

**lemma** *down-double-complement-up*:

$$R\downarrow \subseteq S \iff R \subseteq -((-S)\uparrow)$$

*<proof>*

**lemma** *up-double-complement-down*:

$$R\uparrow \subseteq S \iff R \subseteq -((-S)\downarrow)$$

*<proof>*

**lemma** *below-up-oi-down*:

$$R \subseteq R\uparrow \cap R\downarrow$$

*<proof>*

**lemma** *cp-pa-sim*:

$$(R \odot S)\downarrow = R \otimes S\downarrow$$

*<proof>*

**lemma** *domain-up-down-conjugate*:

$$(R\uparrow \cap S) * 1_{\cup\cup} = (R \cap S\downarrow) * 1_{\cup\cup}$$

*<proof>*

**lemma** *down-below-sp-top*:

$$R\downarrow \subseteq R * U$$

*<proof>*

**lemma** *down-oi-up-closed*:

**assumes**  $Q\uparrow = Q$

**shows**  $R\downarrow \cap Q \subseteq (R \cap Q)\downarrow$

*<proof>*

**lemma** *up-dist-oU*:

$$(\bigcup X)\uparrow = \bigcup (\text{up } X)$$

*<proof>*

**lemma** *up-dist-iU*:  
**assumes**  $I \neq \{\}$   
**shows**  $(\bigcup\bigcup X|I)\uparrow = \bigcup\bigcup (up \ o \ X)|I$   
 $\langle proof \rangle$

**lemma** *up-dist-iI*:  
 $(\bigcap\bigcap X|I)\uparrow = \bigcap\bigcap (up \ o \ X)|I$   
 $\langle proof \rangle$

**lemma** *down-dist-oU*:  
 $(\bigcup X)\downarrow = \bigcup (down \ ' \ X)$   
 $\langle proof \rangle$

**lemma** *down-dist-iU*:  
 $(\bigcup\bigcup X|I)\downarrow = \bigcup\bigcup (down \ o \ X)|I$   
 $\langle proof \rangle$

**lemma** *down-dist-iI*:  
**assumes**  $I \neq \{\}$   
**shows**  $(\bigcap\bigcap X|I)\downarrow = \bigcap\bigcap (down \ o \ X)|I$   
 $\langle proof \rangle$

**lemma** *iU-up-oI*:  
**assumes**  $I \neq \{\}$   
**shows**  $\bigcup\bigcup (up \ o \ X)|I = \bigcap (up \ ' \ X \ ' \ I)$   
 $\langle proof \rangle$

**lemma** *iI-down-oI*:  
**assumes**  $I \neq \{\}$   
**shows**  $\bigcap\bigcap (down \ o \ X)|I = \bigcap (down \ ' \ X \ ' \ I)$   
 $\langle proof \rangle$

**lemma** *down-dist-iI-oI*:  
 $\bigcap (down \ ' \ X \ ' \ I) = (\bigcap\bigcap X|I)\downarrow$   
 $\langle proof \rangle$

**lemma** *up-dist-iU-oI*:  
 $\bigcap (up \ ' \ X \ ' \ I) = (\bigcup\bigcup X|I)\uparrow$   
 $\langle proof \rangle$

**lemma** *iu-up*:  
 $(R \cup\cup R)\uparrow = R\uparrow$   
 $\langle proof \rangle$

**lemma** *ii-down*:  
 $(R \cap\cap R)\downarrow = R\downarrow$   
 $\langle proof \rangle$

**lemma** *iU-up*:  
 assumes  $I \neq \{\}$   
 shows  $(\bigcup (\lambda j . R)|I)\uparrow = R\uparrow$   
 $\langle proof \rangle$

**lemma** *iI-down*:  
 assumes  $I \neq \{\}$   
 shows  $(\bigcap (\lambda j . R)|I)\downarrow = R\downarrow$   
 $\langle proof \rangle$

**lemma** *iu-unit-up*:  
 $1_{UU}\uparrow = U$   
 $\langle proof \rangle$

**lemma** *iu-unit-down*:  
 $1_{UU}\downarrow = 1_{UU}$   
 $\langle proof \rangle$

**lemma** *iu-unit-convex*:  
 $1_{UU}\updownarrow = 1_{UU}$   
 $\langle proof \rangle$

**lemma** *ii-unit-up*:  
 $1_{II}\uparrow = 1_{II}$   
 $\langle proof \rangle$

**lemma** *ii-unit-down*:  
 $1_{II}\downarrow = U$   
 $\langle proof \rangle$

**lemma** *ii-unit-convex*:  
 $1_{II}\updownarrow = 1_{II}$   
 $\langle proof \rangle$

**lemma** *sp-unit-down*:  
 $1\downarrow = 1 \cup 1_{UU}$   
 $\langle proof \rangle$

**lemma** *sp-unit-convex*:  
 $1\updownarrow = 1$   
 $\langle proof \rangle$

**lemma** *top-up*:  
 $U\uparrow = U$   
 $\langle proof \rangle$

**lemma** *top-down*:  
 $U\downarrow = U$   
 $\langle proof \rangle$

**lemma** *top-convex*:

$$U\downarrow = U$$

*<proof>*

**lemma** *bot-up*:

$$\{\}\uparrow = \{\}$$

*<proof>*

**lemma** *bot-down*:

$$\{\}\downarrow = \{\}$$

*<proof>*

**lemma** *bot-convex*:

$$\{\}\downarrow = \{\}$$

*<proof>*

**lemma** *down-oi-up-convex*:

$$(R\downarrow \cap S\uparrow)\downarrow = R\downarrow \cap S\uparrow$$

*<proof>*

**lemma** *convex-iff-down-oi-up*:

$$Q = Q\downarrow \longleftrightarrow (\exists R S . Q = R\downarrow \cap S\uparrow)$$

*<proof>*

**lemma** *convex-closed-oI*:

$$(\bigcap R \in X . R\downarrow)\downarrow = (\bigcap R \in X . R\downarrow)$$

*<proof>*

**lemma** *convex-closed-oi*:

$$(R\downarrow \cap S\downarrow)\downarrow = R\downarrow \cap S\downarrow$$

*<proof>*

**lemma**

$$(R\downarrow \cup S\downarrow)\downarrow = R\downarrow \cup S\downarrow$$

**nitpick***[expect=genuine,card=1,3]*  
*<proof>*

## 6 Powerdomain Preorders

**abbreviation** *lower-less-eq* :: ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  bool (**infixl**  $\sqsubseteq\downarrow$  50)

**where**

$$R \sqsubseteq\downarrow S \equiv R \subseteq S\downarrow$$

**abbreviation** *upper-less-eq* :: ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  bool (**infixl**  $\sqsubseteq\uparrow$  50) **where**

$$R \sqsubseteq\uparrow S \equiv S \subseteq R\uparrow$$

**abbreviation** *convex-less-eq* :: ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  bool (**infixl**  $\sqsubseteq\downarrow\uparrow$ )

50) **where**

$$R \sqsubseteq\updownarrow S \equiv R \sqsubseteq\downarrow S \wedge R \sqsubseteq\uparrow S$$

**abbreviation** *Convex-less-eq* :: ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  bool (**infixl**  $\langle\sqsubseteq\updownarrow\rangle$ )

50) **where**

$$R \sqsubseteq\updownarrow S \equiv R \subseteq S\updownarrow$$

**lemma** *lower-less-eq*:

$$R \sqsubseteq\downarrow S \longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow (\exists C . (a,C) \in S \wedge B \subseteq C))$$

*<proof>*

**lemma** *upper-less-eq*:

$$R \sqsubseteq\uparrow S \longleftrightarrow (\forall a C . (a,C) \in S \longrightarrow (\exists B . (a,B) \in R \wedge B \subseteq C))$$

*<proof>*

**lemma** *Convex-less-eq*:

$$R \sqsubseteq\updownarrow S \longleftrightarrow (\forall a C . (a,C) \in R \longrightarrow (\exists B D . (a,B) \in S \wedge (a,D) \in S \wedge B \subseteq C \wedge C \subseteq D))$$

*<proof>*

**lemma** *Convex-lower-upper*:

$$R \sqsubseteq\updownarrow S \longleftrightarrow R \sqsubseteq\downarrow S \wedge S \sqsubseteq\uparrow R$$

*<proof>*

**lemma** *lower-reflexive*:

$$R \sqsubseteq\downarrow R$$

*<proof>*

**lemma** *upper-reflexive*:

$$R \sqsubseteq\uparrow R$$

*<proof>*

**lemma** *convex-reflexive*:

$$R \sqsubseteq\updownarrow R$$

*<proof>*

**lemma** *Convex-reflexive*:

$$R \sqsubseteq\updownarrow R$$

*<proof>*

**lemma** *lower-transitive*:

$$R \sqsubseteq\downarrow S \Longrightarrow S \sqsubseteq\downarrow T \Longrightarrow R \sqsubseteq\downarrow T$$

*<proof>*

**lemma** *upper-transitive*:

$$R \sqsubseteq\uparrow S \Longrightarrow S \sqsubseteq\uparrow T \Longrightarrow R \sqsubseteq\uparrow T$$

*<proof>*

**lemma** *convex-transitive*:

$$R \sqsubseteq\uparrow S \implies S \sqsubseteq\uparrow T \implies R \sqsubseteq\uparrow T$$

*<proof>*

**lemma** *Convex-transitive:*

$$R \sqsubseteq\uparrow S \implies S \sqsubseteq\uparrow T \implies R \sqsubseteq\uparrow T$$

*<proof>*

**lemma** *bot-lower-least:*

$$\{\} \sqsubseteq\downarrow R$$

*<proof>*

**lemma** *top-upper-least:*

$$U \sqsubseteq\uparrow R$$

*<proof>*

**lemma** *bot-Convex-least:*

$$\{\} \sqsubseteq\uparrow R$$

*<proof>*

**lemma** *top-lower-greatest:*

$$R \sqsubseteq\downarrow U$$

*<proof>*

**lemma** *bot-upper-greatest:*

$$R \sqsubseteq\uparrow \{\}$$

*<proof>*

**lemma** *top-Convex-greatest:*

$$R \sqsubseteq\uparrow U$$

*<proof>*

**lemma** *lower-ii-increasing:*

$$R \sqsubseteq\downarrow R \cup\cup R$$

*<proof>*

**lemma** *upper-ii-increasing:*

$$R \sqsubseteq\uparrow R \cup\cup S$$

*<proof>*

**lemma** *convex-ii-increasing:*

$$R \sqsubseteq\uparrow R \cup\cup R$$

*<proof>*

**lemma** *Convex-ii-increasing:*

$$R \sqsubseteq\uparrow R \cup\cup R$$

*<proof>*

**lemma** *lower-ii-decreasing:*

$$R \cap\cap S \sqsubseteq\downarrow R$$

$\langle proof \rangle$

**lemma** *upper-ii-decreasing:*

$$R \cap R \sqsubseteq \uparrow R$$

$\langle proof \rangle$

**lemma** *convex-ii-decreasing:*

$$R \cap R \sqsubseteq \downarrow R$$

$\langle proof \rangle$

**lemma** *Convex-ii-increasing:*

$$R \sqsubseteq \downarrow R \cap R$$

$\langle proof \rangle$

**lemma** *iu-lower-left-isotone:*

$$R \sqsubseteq \downarrow S \implies R \cup T \sqsubseteq \downarrow S \cup T$$

$\langle proof \rangle$

**lemma** *iu-upper-left-isotone:*

$$R \sqsubseteq \uparrow S \implies R \cup T \sqsubseteq \uparrow S \cup T$$

$\langle proof \rangle$

**lemma** *iu-convex-left-isotone:*

$$R \sqsubseteq \downarrow S \implies R \cup T \sqsubseteq \downarrow S \cup T$$

$\langle proof \rangle$

**lemma** *iu-Convex-left-isotone:*

$$R \sqsubseteq \downarrow S \implies R \cup T \sqsubseteq \downarrow S \cup T$$

$\langle proof \rangle$

**lemma** *iu-lower-right-isotone:*

$$R \sqsubseteq \downarrow S \implies T \cup R \sqsubseteq \downarrow T \cup S$$

$\langle proof \rangle$

**lemma** *iu-upper-right-isotone:*

$$R \sqsubseteq \uparrow S \implies T \cup R \sqsubseteq \uparrow T \cup S$$

$\langle proof \rangle$

**lemma** *iu-convex-right-isotone:*

$$R \sqsubseteq \downarrow S \implies T \cup R \sqsubseteq \downarrow T \cup S$$

$\langle proof \rangle$

**lemma** *iu-Convex-right-isotone:*

$$R \sqsubseteq \downarrow S \implies T \cup R \sqsubseteq \downarrow T \cup S$$

$\langle proof \rangle$

**lemma** *iu-lower-isotone:*

$$R \sqsubseteq \downarrow S \implies P \sqsubseteq \downarrow Q \implies R \cup P \sqsubseteq \downarrow S \cup Q$$

$\langle proof \rangle$

**lemma** *iu-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies P \sqsubseteq\uparrow Q \implies R \sqcup\sqcup P \sqsubseteq\uparrow S \sqcup\sqcup Q$$

*<proof>*

**lemma** *iu-convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcup\sqcup P \sqsubseteq\downarrow S \sqcup\sqcup Q$$

*<proof>*

**lemma** *iu-Convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcup\sqcup P \sqsubseteq\downarrow S \sqcup\sqcup Q$$

*<proof>*

**lemma** *ii-lower-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \sqcap\sqcap T \sqsubseteq\downarrow S \sqcap\sqcap T$$

*<proof>*

**lemma** *ii-upper-left-isotone*:

$$R \sqsubseteq\uparrow S \implies R \sqcap\sqcap T \sqsubseteq\uparrow S \sqcap\sqcap T$$

*<proof>*

**lemma** *ii-convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \sqcap\sqcap T \sqsubseteq\downarrow S \sqcap\sqcap T$$

*<proof>*

**lemma** *ii-Convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \sqcap\sqcap T \sqsubseteq\downarrow S \sqcap\sqcap T$$

*<proof>*

**lemma** *ii-lower-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \sqcap\sqcap R \sqsubseteq\downarrow T \sqcap\sqcap S$$

*<proof>*

**lemma** *ii-upper-right-isotone*:

$$R \sqsubseteq\uparrow S \implies T \sqcap\sqcap R \sqsubseteq\uparrow T \sqcap\sqcap S$$

*<proof>*

**lemma** *ii-convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \sqcap\sqcap R \sqsubseteq\downarrow T \sqcap\sqcap S$$

*<proof>*

**lemma** *ii-Convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \sqcap\sqcap R \sqsubseteq\downarrow T \sqcap\sqcap S$$

*<proof>*

**lemma** *ii-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap\sqcap P \sqsubseteq\downarrow S \sqcap\sqcap Q$$

*<proof>*

**lemma** *ii-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies P \sqsubseteq\uparrow Q \implies R \sqcap P \sqsubseteq\uparrow S \sqcap Q$$

*<proof>*

**lemma** *ii-convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap P \sqsubseteq\downarrow S \sqcap Q$$

*<proof>*

**lemma** *ii-Convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap P \sqsubseteq\downarrow S \sqcap Q$$

*<proof>*

**lemma** *ou-lower-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

*<proof>*

**lemma** *ou-upper-left-isotone*:

$$R \sqsubseteq\uparrow S \implies R \cup T \sqsubseteq\uparrow S \cup T$$

*<proof>*

**lemma** *ou-convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

*<proof>*

**lemma** *ou-Convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

*<proof>*

**lemma** *ou-lower-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \cup R \sqsubseteq\downarrow T \cup S$$

*<proof>*

**lemma** *ou-upper-right-isotone*:

$$R \sqsubseteq\uparrow S \implies T \cup R \sqsubseteq\uparrow T \cup S$$

*<proof>*

**lemma** *ou-convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \cup R \sqsubseteq\downarrow T \cup S$$

*<proof>*

**lemma** *ou-Convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \cup R \sqsubseteq\downarrow T \cup S$$

*<proof>*

**lemma** *ou-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \cup P \sqsubseteq\downarrow S \cup Q$$

*<proof>*

**lemma** *ou-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies P \sqsubseteq\uparrow Q \implies R \cup P \sqsubseteq\uparrow S \cup Q$$

*<proof>*

**lemma** *ou-convex-isotone:*

$$R \sqsubseteq\updownarrow S \implies P \sqsubseteq\updownarrow Q \implies R \cup P \sqsubseteq\updownarrow S \cup Q$$

*<proof>*

**lemma** *ou-Convex-isotone:*

$$R \sqsubseteq\updownarrow S \implies P \sqsubseteq\updownarrow Q \implies R \cup P \sqsubseteq\updownarrow S \cup Q$$

*<proof>*

**lemma** *sp-lower-left-isotone:*

$$R \sqsubseteq\downarrow S \implies T * R \sqsubseteq\downarrow T * S$$

*<proof>*

**lemma** *sp-upper-left-isotone:*

$$R \sqsubseteq\uparrow S \implies T * R \sqsubseteq\uparrow T * S$$

*<proof>*

**lemma** *sp-convex-left-isotone:*

$$R \sqsubseteq\updownarrow S \implies T * R \sqsubseteq\updownarrow T * S$$

*<proof>*

**lemma** *sp-Convex-left-isotone:*

$$R \sqsubseteq\updownarrow S \implies T * R \sqsubseteq\updownarrow T * S$$

*<proof>*

**lemma** *cp-lower-left-isotone:*

$$R \sqsubseteq\downarrow S \implies T \odot R \sqsubseteq\downarrow T \odot S$$

*<proof>*

**lemma** *cp-upper-left-isotone:*

$$R \sqsubseteq\uparrow S \implies T \odot R \sqsubseteq\uparrow T \odot S$$

*<proof>*

**lemma** *cp-convex-left-isotone:*

$$R \sqsubseteq\updownarrow S \implies T \odot R \sqsubseteq\updownarrow T \odot S$$

*<proof>*

**lemma** *cp-Convex-left-isotone:*

$$R \sqsubseteq\updownarrow S \implies T \odot R \sqsubseteq\updownarrow T \odot S$$

*<proof>*

**lemma** *lower-ic-upper:*

$$R \sqsubseteq\downarrow S \iff \sim S \sqsubseteq\uparrow \sim R$$

*<proof>*

**lemma** *upper-ic-lower:*

$$R \sqsubseteq\uparrow S \iff \sim S \sqsubseteq\downarrow \sim R$$

*<proof>*

**lemma** *convex-ic:*

$$R \sqsubseteq\Downarrow S \iff \sim S \sqsubseteq\Downarrow \sim R$$

*<proof>*

**lemma** *Convex-ic:*

$$R \sqsubseteq\Downarrow S \iff \sim R \sqsubseteq\Downarrow \sim S$$

*<proof>*

**lemma** *up-lower-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\uparrow \sqsubseteq\Downarrow S\uparrow$$

*<proof>*

**lemma** *up-upper-isotone:*

$$R \sqsubseteq\Uparrow S \implies R\uparrow \sqsubseteq\Uparrow S\uparrow$$

*<proof>*

**lemma** *up-convex-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\uparrow \sqsubseteq\Downarrow S\uparrow$$

*<proof>*

**lemma** *up-Convex-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\uparrow \sqsubseteq\Downarrow S\uparrow$$

*<proof>*

**lemma** *down-lower-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

*<proof>*

**lemma** *down-upper-isotone:*

$$R \sqsubseteq\Uparrow S \implies R\downarrow \sqsubseteq\Uparrow S\downarrow$$

*<proof>*

**lemma** *down-convex-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

*<proof>*

**lemma** *down-Convex-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

*<proof>*

**lemma** *convex-lower-isotone:*

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

*<proof>*

**lemma** *convex-upper-isotone:*

$$R \sqsubseteq\Uparrow S \implies R\downarrow \sqsubseteq\Uparrow S\downarrow$$

*<proof>*

**lemma** *convex-convex-isotone*:

$$R \sqsubseteq\uparrow S \implies R\uparrow \sqsubseteq\uparrow S\uparrow$$

*<proof>*

**lemma** *convex-Convex-isotone*:

$$R \sqsubseteq\uparrow S \implies R\uparrow \sqsubseteq\uparrow S\uparrow$$

*<proof>*

**lemma** *subset-lower*:

$$R \subseteq S \implies R \sqsubseteq\downarrow S$$

*<proof>*

**lemma** *subset-upper*:

$$R \subseteq S \implies S \sqsubseteq\uparrow R$$

*<proof>*

**lemma** *subset-Convex*:

$$R \subseteq S \implies R \sqsubseteq\uparrow S$$

*<proof>*

**lemma** *oi-subset-lower-left-isotone*:

$$R \subseteq S \implies R \cap T \sqsubseteq\downarrow S \cap T$$

*<proof>*

**lemma** *oi-subset-upper-left-antitone*:

$$R \subseteq S \implies S \cap T \sqsubseteq\uparrow R \cap T$$

*<proof>*

**lemma** *oi-subset-Convex-left-isotone*:

$$R \subseteq S \implies R \cap T \sqsubseteq\uparrow S \cap T$$

*<proof>*

**lemma** *oi-subset-lower-right-isotone*:

$$R \subseteq S \implies T \cap R \sqsubseteq\downarrow T \cap S$$

*<proof>*

**lemma** *oi-subset-upper-right-antitone*:

$$R \subseteq S \implies T \cap S \sqsubseteq\uparrow T \cap R$$

*<proof>*

**lemma** *oi-subset-Convex-right-isotone*:

$$R \subseteq S \implies T \cap R \sqsubseteq\uparrow T \cap S$$

*<proof>*

**lemma** *oi-subset-lower-isotone*:

$$R \subseteq S \implies P \subseteq Q \implies R \cap P \sqsubseteq\downarrow S \cap Q$$

*<proof>*

**lemma** *oi-subset-upper-antitone*:

$$R \subseteq S \implies P \subseteq Q \implies S \cap Q \sqsubseteq\uparrow R \cap P$$

*<proof>*

**lemma** *oi-subset-Convex-isotone*:

$$R \subseteq S \implies P \subseteq Q \implies R \cap P \sqsubseteq\Downarrow S \cap Q$$

*<proof>*

**lemma** *sp-ii-unit-lower*:

$$R * 1_{\cup\cup} \sqsubseteq\downarrow R$$

*<proof>*

**lemma** *cp-ii-unit-upper*:

$$R \sqsubseteq\uparrow R \odot 1_{\cap\cap}$$

*<proof>*

**lemma** *lower-ii-down*:

$$R \sqsubseteq\downarrow S \iff R\downarrow = (R \cap\cap S)\downarrow$$

*<proof>*

**lemma** *lower-ii-lower-bound*:

$$R \sqsubseteq\downarrow S \iff R \subseteq R \cap\cap S$$

*<proof>*

**lemma** *upper-ii-up*:

$$R \sqsubseteq\uparrow S \iff S\uparrow = (R \cup\cup S)\uparrow$$

*<proof>*

**lemma** *upper-ii-upper-bound*:

$$R \sqsubseteq\uparrow S \iff S \subseteq R \cup\cup S$$

*<proof>*

**lemma**

$$R \sqsubseteq\downarrow S \iff R = R \cap\cap S$$

**nitpick***[expect=genuine,card=1]*  
*<proof>*

**lemma**

$$R \sqsubseteq\uparrow S \iff S = R \cup\cup S$$

**nitpick***[expect=genuine,card=1]*  
*<proof>*

**lemma** *convex-oi-Convex-ii*:

$$R\downarrow \cap S\downarrow \sqsubseteq\Downarrow R \cup\cup S$$

*<proof>*

**lemma** *convex-oi-Convex-ii*:

$$R\downarrow \cap S\downarrow \sqsubseteq\Downarrow R \cap\cap S$$

*<proof>*

**lemma** *convex-oi-iu-ii*:

$$R\downarrow \cap S\uparrow = (R \cup\cup S)\uparrow \cap (R \cap\cap S)\downarrow$$

*<proof>*

**lemma** *ii-lower-iu*:

$$R \cap\cap S \sqsubseteq\downarrow R \cup\cup S$$

*<proof>*

**lemma** *ii-upper-iu*:

$$R \cap\cap S \sqsubseteq\uparrow R \cup\cup S$$

*<proof>*

**lemma** *ii-convex-iu*:

$$R \cap\cap S \sqsubseteq\updownarrow R \cup\cup S$$

*<proof>*

**lemma** *convex-oi-iu-ii-convex*:

$$R\downarrow \cap S\uparrow = (R \cup\cup S)\downarrow \cap (R \cap\cap S)\updownarrow$$

*<proof>*

## 6.1 Functional properties of multirelations

**lemma** *id-one-converse*:

$$Id = 1 ; 1\sim$$

*<proof>*

**lemma** *dom-explicit*:

$$Dom R = R ; U \cap 1$$

*<proof>*

**lemma** *dom-explicit-2*:

$$Dom R = R ; top \cap 1$$

*<proof>*

**lemma** *total-dom*:

$$total R \longleftrightarrow Dom R = 1$$

*<proof>*

**lemma** *total-eq*:

$$total R \longleftrightarrow 1_{\cup\cup} = R * 1_{\cup\cup}$$

*<proof>*

**lemma** *domain-pointwise*:

$$x \in R * 1_{\cup\cup} \longleftrightarrow (\exists a B . (a,B) \in R \wedge x = (a,\{\}))$$

*<proof>*

*card only works for finite sets*

**lemma** *univalent-2*:

*univalent*  $R \longleftrightarrow (\forall a . \text{finite } \{ B . (a,B) \in R \} \wedge \text{card } \{ B . (a,B) \in R \} \leq \text{one-class.one})$   
 ⟨proof⟩

**lemma** *univalent-3*:

*univalent*  $R \longleftrightarrow (\forall S . R * 1_{\cup\cup} = S * 1_{\cup\cup} \wedge S \subseteq R \longrightarrow S = R)$   
 ⟨proof⟩

**lemma** *total-2*:

*total*  $R \longleftrightarrow (\forall a . \{ B . (a,B) \in R \} \neq \{\})$   
 ⟨proof⟩

**lemma** *total-3*:

*total*  $R \longleftrightarrow (\forall a . \text{finite } \{ B . (a,B) \in R \} \longrightarrow \text{card } \{ B . (a,B) \in R \} \geq \text{one-class.one})$   
 ⟨proof⟩

**lemma** *total-4*: *total*  $R \longleftrightarrow 1_{\cup\cup} \subseteq R * 1_{\cup\cup}$

⟨proof⟩

**lemma** *deterministic-2*:

*deterministic*  $R \longleftrightarrow (\forall a . \text{card } \{ B . (a,B) \in R \} = \text{one-class.one})$   
 ⟨proof⟩

**lemma** *univalent-convex*:

**assumes** *univalent*  $S$

**shows**  $S = S \downarrow$

⟨proof⟩

**lemma** *univalent-iu-idempotent*:

**assumes** *univalent*  $S$

**shows**  $S = S \cup\cup S$

⟨proof⟩

**lemma** *univalent-ii-idempotent*:

**assumes** *univalent*  $S$

**shows**  $S = S \cap\cap S$

⟨proof⟩

**lemma** *univalent-down-iu-idempotent*:

**assumes** *univalent*  $S$

**shows**  $S = S \downarrow \cup\cup S$

⟨proof⟩

**lemma** *univalent-up-ii-idempotent*:

**assumes** *univalent*  $S$

**shows**  $S = S \uparrow \cap\cap S$

⟨proof⟩

**lemma** *univalent-convex-iu-idempotent*:

**assumes** *univalent S*

**shows**  $S = S \Downarrow \cupcup S$

*<proof>*

**lemma** *univalent-convex-ii-idempotent*:

**assumes** *univalent S*

**shows**  $S = S \Downarrow \capcap S$

*<proof>*

**lemma** *univalent-iu-closed*:

*univalent R*  $\implies$  *univalent S*  $\implies$  *univalent (R  $\cupcup$  S)*

*<proof>*

**lemma** *univalent-ii-closed*:

*univalent R*  $\implies$  *univalent S*  $\implies$  *univalent (R  $\capcap$  S)*

*<proof>*

**lemma** *total-lower*:

*total R*  $\longleftrightarrow$   $1 \cupcup \sqsubseteq\downarrow R$

*<proof>*

**lemma** *total-upper*:

*total R*  $\longleftrightarrow$   $R \sqsubseteq\uparrow 1 \capcap$

*<proof>*

**lemma** *total-lower-iu*:

**assumes** *total T*

**shows**  $R \sqsubseteq\downarrow R \cupcup T$

*<proof>*

**lemma** *total-upper-ii*:

**assumes** *total T*

**shows**  $R \capcap T \sqsubseteq\uparrow R$

*<proof>*

**lemma** *total-univalent-lower-iu*:

**assumes** *total T*

**and** *univalent S*

**and**  $T \sqsubseteq\downarrow S$

**shows**  $T \cupcup S = S$

*<proof>*

**lemma** *total-iu-closed*:

*total R*  $\implies$  *total S*  $\implies$  *total (R  $\cupcup$  S)*

*<proof>*

**lemma** *total-ii-closed*:

*total R*  $\implies$  *total S*  $\implies$  *total (R  $\capcap$  S)*

*<proof>*

**lemma** *deterministic-lower:*

**assumes** *deterministic V*

**shows**  $R \sqsubseteq\downarrow V \iff (\forall a B C . (a,B) \in R \wedge (a,C) \in V \longrightarrow B \subseteq C)$   
*<proof>*

**lemma** *deterministic-upper:*

**assumes** *deterministic V*

**shows**  $V \sqsubseteq\uparrow R \iff (\forall a B C . (a,B) \in R \wedge (a,C) \in V \longrightarrow C \subseteq B)$   
*<proof>*

**lemma** *deterministic-iu-closed:*

*deterministic R*  $\implies$  *deterministic S*  $\implies$  *deterministic (R  $\cup\cup$  S)*

*<proof>*

**lemma** *deterministic-ii-closed:*

*deterministic R*  $\implies$  *deterministic S*  $\implies$  *deterministic (R  $\cap\cap$  S)*

*<proof>*

**lemma** *total-univalent-lower-implies-upper:*

**assumes** *total T*

**and** *univalent S*

**and**  $T \sqsubseteq\downarrow S$

**shows**  $T \sqsubseteq\uparrow S$

*<proof>*

**lemma** *total-univalent-lower-implies-convex:*

**assumes** *total T*

**and** *univalent S*

**and**  $T \sqsubseteq\downarrow S$

**shows**  $T \sqsubseteq\updownarrow S$

*<proof>*

**lemma** *total-univalent-upper-implies-lower:*

**assumes** *total T*

**and** *univalent S*

**and**  $S \sqsubseteq\uparrow T$

**shows**  $S \sqsubseteq\downarrow T$

*<proof>*

**lemma** *total-univalent-upper-implies-convex:*

**assumes** *total T*

**and** *univalent S*

**and**  $S \sqsubseteq\uparrow T$

**shows**  $S \sqsubseteq\updownarrow T$

*<proof>*

**lemma** *deterministic-lower-upper:*

**assumes** *deterministic T*  
**and** *deterministic S*  
**shows**  $S \sqsubseteq\downarrow T \longleftrightarrow S \sqsubseteq\uparrow T$   
*<proof>*

**lemma** *deterministic-lower-convex:*  
**assumes** *deterministic T*  
**and** *deterministic S*  
**shows**  $S \sqsubseteq\downarrow T \longleftrightarrow S \sqsubseteq\uparrow\downarrow T$   
*<proof>*

**lemma** *deterministic-upper-convex:*  
**assumes** *deterministic T*  
**and** *deterministic S*  
**shows**  $S \sqsubseteq\uparrow T \longleftrightarrow S \sqsubseteq\downarrow\uparrow T$   
*<proof>*

**lemma** *total-down-sp-sp-down:*  
**assumes** *total T*  
**shows**  $R\downarrow * T \subseteq R * T\downarrow$   
*<proof>*

**lemma** *total-down-sp-semi-commute:*  
*total T*  $\implies R\downarrow * T \subseteq (R * T)\downarrow$   
*<proof>*

**lemma** *total-down-dist-sp:*  
*total T*  $\implies (R * T)\downarrow = R\downarrow * T\downarrow$   
*<proof>*

**lemma** *univalent-ic-closed:*  
*univalent R*  $\longleftrightarrow$  *univalent ( $\sim R$ )*  
*<proof>*

**lemma** *total-ic-closed:*  
*total R*  $\longleftrightarrow$  *total ( $\sim R$ )*  
*<proof>*

**lemma** *deterministic-ic-closed:*  
*deterministic R*  $\longleftrightarrow$  *deterministic ( $\sim R$ )*  
*<proof>*

**lemma** *iu-unit-deterministic:*  
*deterministic ( $1_{\cup\cup}$ )*  
*<proof>*

**lemma** *ii-unit-deterministic:*  
*deterministic ( $1_{\cap\cap}$ )*  
*<proof>*

**lemma** *univalent-upper-iu*:  
**assumes** *univalent R*  
**shows**  $(R \sqsubseteq\uparrow S) \longleftrightarrow (R \cup\cup S = S)$   
*<proof>*

**lemma** *univalent-lower-ii*:  
**assumes** *univalent S*  
**shows**  $(R \sqsubseteq\downarrow S) = (R \cap\cap S = R)$   
*<proof>*

## 6.2 Equivalences induced by powerdomain preorders

**abbreviation** *lower-eq* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$  (**infixl**  $\langle =\downarrow \rangle$  50)  
**where**  
 $R =\downarrow S \equiv R \sqsubseteq\downarrow S \wedge S \sqsubseteq\downarrow R$

**abbreviation** *upper-eq* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$  (**infixl**  $\langle =\uparrow \rangle$  50)  
**where**  
 $R =\uparrow S \equiv R \sqsubseteq\uparrow S \wedge S \sqsubseteq\uparrow R$

**abbreviation** *convex-eq* ::  $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$  (**infixl**  $\langle =\Downarrow \rangle$  50)  
**where**  
 $R =\Downarrow S \equiv R \sqsubseteq\Downarrow S \wedge S \sqsubseteq\Downarrow R$

**lemma** *Convex-eq*:  
 $R =\Downarrow S \equiv R \sqsubseteq\Downarrow S \wedge S \sqsubseteq\Downarrow R$   
*<proof>*

**lemma** *convex-lower-upper*:  
 $R =\Downarrow S \longleftrightarrow R =\downarrow S \wedge R =\uparrow S$   
*<proof>*

**lemma** *lower-eq-down*:  
 $R =\downarrow S \longleftrightarrow R\downarrow = S\downarrow$   
*<proof>*

**lemma** *upper-eq-up*:  
 $R =\uparrow S \longleftrightarrow R\uparrow = S\uparrow$   
*<proof>*

**lemma** *convex-eq-convex*:  
 $R =\Downarrow S \longleftrightarrow R\Downarrow = S\Downarrow$   
*<proof>*

**lemma** *lower-eq*:  
 $R =\downarrow S \longleftrightarrow (\forall a B . (\exists C . (a, C) \in R \wedge B \subseteq C) \longleftrightarrow (\exists C . (a, C) \in S \wedge B \subseteq C))$   
*<proof>*

**lemma** *upper-eq*:

$$R =\uparrow S \iff (\forall a C . (\exists B . (a,B) \in R \wedge B \subseteq C) \iff (\exists B . (a,B) \in S \wedge B \subseteq C))$$

*<proof>*

**lemma** *lower-eq-reflexive*:

$$R =\downarrow R$$

*<proof>*

**lemma** *upper-eq-reflexive*:

$$R =\uparrow R$$

*<proof>*

**lemma** *convex-eq-reflexive*:

$$R =\Downarrow R$$

*<proof>*

**lemma** *lower-eq-symmetric*:

$$R =\downarrow S \implies S =\downarrow R$$

*<proof>*

**lemma** *upper-eq-symmetric*:

$$R =\uparrow S \implies S =\uparrow R$$

*<proof>*

**lemma** *convex-eq-symmetric*:

$$R =\Downarrow S \implies S =\Downarrow R$$

*<proof>*

**lemma** *lower-eq-transitive*:

$$R =\downarrow S \implies S =\downarrow T \implies R =\downarrow T$$

*<proof>*

**lemma** *upper-eq-transitive*:

$$R =\uparrow S \implies S =\uparrow T \implies R =\uparrow T$$

*<proof>*

**lemma** *convex-eq-transitive*:

$$R =\Downarrow S \implies S =\Downarrow T \implies R =\Downarrow T$$

*<proof>*

**lemma** *ou-lower-eq-left-congruence*:

$$R =\downarrow S \implies R \cup T =\downarrow S \cup T$$

*<proof>*

**lemma** *ou-upper-eq-left-congruence*:

$$R =\uparrow S \implies R \cup T =\uparrow S \cup T$$

*<proof>*

**lemma** *ou-convex-eq-left-congruence*:

$$R = \Downarrow S \implies R \cup T = \Downarrow S \cup T$$

*<proof>*

**lemma** *ou-lower-eq-right-congruence*:

$$R = \Downarrow S \implies T \cup R = \Downarrow T \cup S$$

*<proof>*

**lemma** *ou-upper-eq-right-congruence*:

$$R = \Uparrow S \implies T \cup R = \Uparrow T \cup S$$

*<proof>*

**lemma** *ou-convex-eq-right-congruence*:

$$R = \Downarrow S \implies T \cup R = \Downarrow T \cup S$$

*<proof>*

**lemma** *ou-lower-eq-congruence*:

$$R = \Downarrow S \implies P = \Downarrow Q \implies R \cup P = \Downarrow S \cup Q$$

*<proof>*

**lemma** *ou-upper-eq-congruence*:

$$R = \Uparrow S \implies P = \Uparrow Q \implies R \cup P = \Uparrow S \cup Q$$

*<proof>*

**lemma** *ou-convex-eq-congruence*:

$$R = \Downarrow S \implies P = \Downarrow Q \implies R \cup P = \Downarrow S \cup Q$$

*<proof>*

**lemma** *iu-lower-eq-left-congruence*:

$$R = \Downarrow S \implies R \cup\cup T = \Downarrow S \cup\cup T$$

*<proof>*

**lemma** *iu-upper-eq-left-congruence*:

$$R = \Uparrow S \implies R \cup\cup T = \Uparrow S \cup\cup T$$

*<proof>*

**lemma** *iu-convex-eq-left-congruence*:

$$R = \Downarrow S \implies R \cup\cup T = \Downarrow S \cup\cup T$$

*<proof>*

**lemma** *iu-lower-eq-right-congruence*:

$$R = \Downarrow S \implies T \cup\cup R = \Downarrow T \cup\cup S$$

*<proof>*

**lemma** *iu-upper-eq-right-congruence*:

$$R = \Uparrow S \implies T \cup\cup R = \Uparrow T \cup\cup S$$

*<proof>*

**lemma** *iu-convex-eq-right-congruence*:

$$R =\Downarrow S \implies T \cup\cup R =\Downarrow T \cup\cup S$$

*<proof>*

**lemma** *iu-lower-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \cup\cup P =\Downarrow S \cup\cup Q$$

*<proof>*

**lemma** *iu-upper-eq-congruence*:

$$R =\Uparrow S \implies P =\Uparrow Q \implies R \cup\cup P =\Uparrow S \cup\cup Q$$

*<proof>*

**lemma** *iu-convex-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \cup\cup P =\Downarrow S \cup\cup Q$$

*<proof>*

**lemma** *ii-lower-eq-left-congruence*:

$$R =\Downarrow S \implies R \cap\cap T =\Downarrow S \cap\cap T$$

*<proof>*

**lemma** *ii-upper-eq-left-congruence*:

$$R =\Uparrow S \implies R \cap\cap T =\Uparrow S \cap\cap T$$

*<proof>*

**lemma** *ii-convex-eq-left-congruence*:

$$R =\Downarrow S \implies R \cap\cap T =\Downarrow S \cap\cap T$$

*<proof>*

**lemma** *ii-lower-eq-right-congruence*:

$$R =\Downarrow S \implies T \cap\cap R =\Downarrow T \cap\cap S$$

*<proof>*

**lemma** *ii-upper-eq-right-congruence*:

$$R =\Uparrow S \implies T \cap\cap R =\Uparrow T \cap\cap S$$

*<proof>*

**lemma** *ii-convex-eq-right-congruence*:

$$R =\Downarrow S \implies T \cap\cap R =\Downarrow T \cap\cap S$$

*<proof>*

**lemma** *ii-lower-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \cap\cap P =\Downarrow S \cap\cap Q$$

*<proof>*

**lemma** *ii-upper-eq-congruence*:

$$R =\Uparrow S \implies P =\Uparrow Q \implies R \cap\cap P =\Uparrow S \cap\cap Q$$

*<proof>*

**lemma** *ii-convex-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \sqcap P =\Downarrow S \sqcap Q$$

*<proof>*

**lemma** *sp-lower-eq-left-congruence:*

$$R =\Downarrow S \implies T * R =\Downarrow T * S$$

*<proof>*

**lemma** *sp-upper-eq-left-congruence:*

$$R =\Uparrow S \implies T * R =\Uparrow T * S$$

*<proof>*

**lemma** *sp-convex-eq-left-congruence:*

$$R =\Downarrow S \implies T * R =\Downarrow T * S$$

*<proof>*

**lemma** *cp-lower-eq-left-congruence:*

$$R =\Downarrow S \implies T \odot R =\Downarrow T \odot S$$

*<proof>*

**lemma** *cp-upper-eq-left-congruence:*

$$R =\Uparrow S \implies T \odot R =\Uparrow T \odot S$$

*<proof>*

**lemma** *cp-convex-eq-left-congruence:*

$$R =\Downarrow S \implies T \odot R =\Downarrow T \odot S$$

*<proof>*

**lemma** *lower-eq-ic-upper:*

$$R =\Downarrow S \iff \sim R =\Uparrow \sim S$$

*<proof>*

**lemma** *upper-eq-ic-lower:*

$$R =\Uparrow S \iff \sim R =\Downarrow \sim S$$

*<proof>*

**lemma** *convex-eq-ic-lower:*

$$R =\Downarrow S \iff \sim R =\Downarrow \sim S$$

*<proof>*

**lemma** *up-lower-eq-congruence:*

$$R =\Downarrow S \implies R\uparrow =\Downarrow S\uparrow$$

*<proof>*

**lemma** *up-upper-eq-congruence:*

$$R =\Uparrow S \implies R\uparrow =\Uparrow S\uparrow$$

*<proof>*

**lemma** *up-convex-eq-congruence:*

$$R =\Downarrow S \implies R\uparrow =\Downarrow S\uparrow$$

*<proof>*

**lemma** *down-lower-eq-congruence:*

$$R = \downarrow S \implies R \downarrow = \downarrow S \downarrow$$

*<proof>*

**lemma** *down-upper-eq-congruence:*

$$R = \uparrow S \implies R \downarrow = \uparrow S \downarrow$$

*<proof>*

**lemma** *down-convex-eq-congruence:*

$$R = \Downarrow S \implies R \downarrow = \Downarrow S \downarrow$$

*<proof>*

**lemma** *convex-lower-eq-congruence:*

$$R = \downarrow S \implies R \Downarrow = \downarrow S \Downarrow$$

*<proof>*

**lemma** *convex-upper-eq-congruence:*

$$R = \uparrow S \implies R \Downarrow = \uparrow S \Downarrow$$

*<proof>*

**lemma** *convex-convex-eq-congruence:*

$$R = \Downarrow S \implies R \Downarrow = \Downarrow S \Downarrow$$

*<proof>*

**lemma** *univalent-lower-eq-subset:*

**assumes** *univalent*  $S$

**and**  $S = \downarrow R$

**shows**  $S \subseteq R$

*<proof>*

**lemma** *univalent-lower-eq:*

**assumes** *univalent*  $R$

**and** *univalent*  $S$

**and**  $R = \downarrow S$

**shows**  $R = S$

*<proof>*

**lemma** *univalent-lower-eq-iff:*

**assumes** *univalent*  $R$

**and** *univalent*  $S$

**shows**  $(R = \downarrow S) \iff (R = S)$

*<proof>*

**lemma** *univalent-upper-eq-subset:*

**assumes** *univalent*  $S$

**and**  $S = \uparrow R$

**shows**  $S \subseteq R$

$\langle proof \rangle$

**lemma** *univalent-upper-eq*:  
 **assumes** *univalent*  $R$   
 **and** *univalent*  $S$   
 **and**  $R = \uparrow S$   
 **shows**  $R = S$   
  $\langle proof \rangle$

**lemma** *univalent-upper-eq-iff*:  
 **assumes** *univalent*  $R$   
 **and** *univalent*  $S$   
 **shows**  $(R = \uparrow S) \longleftrightarrow (R = S)$   
  $\langle proof \rangle$

**lemma** *univalent-convex-eq-iff*:  
 **assumes** *univalent*  $R$   
 **and** *univalent*  $S$   
 **shows**  $(R = \Downarrow S) \longleftrightarrow (R = S)$   
  $\langle proof \rangle$

**lemma** *total-univalent-upper-ii*:  
 **assumes** *total*  $T$   
 **and** *univalent*  $S$   
 **and**  $S \sqsubseteq \uparrow T$   
 **shows**  $T \cap \cap S = S$   
  $\langle proof \rangle$

**lemma** *lower-eq-down-closed*:  
  $R = \downarrow R \downarrow$   
  $\langle proof \rangle$

**lemma** *upper-eq-up-closed*:  
  $R = \uparrow R \uparrow$   
  $\langle proof \rangle$

**lemma** *convex-eq-up-closed*:  
  $R = \Downarrow R \Downarrow$   
  $\langle proof \rangle$

**lemma** *lower-join*:  
  $(\forall P . Q \sqsubseteq \downarrow P \longleftrightarrow R \sqsubseteq \downarrow P \wedge S \sqsubseteq \downarrow P) \longleftrightarrow Q = \downarrow R \cup S$   
  $\langle proof \rangle$

**lemma** *lower-meet*:  
  $(\forall P . P \sqsubseteq \downarrow Q \longleftrightarrow P \sqsubseteq \downarrow R \wedge P \sqsubseteq \downarrow S) \longleftrightarrow Q = \downarrow R \cap \cap S$   
  $\langle proof \rangle$

**lemma** *upper-join*:

$$(\forall P . Q \sqsubseteq\uparrow P \iff R \sqsubseteq\uparrow P \wedge S \sqsubseteq\uparrow P) \iff Q =\uparrow R \cup\cup S$$

*<proof>*

**lemma** *upper-meet:*

$$(\forall P . P \sqsubseteq\uparrow Q \iff P \sqsubseteq\uparrow R \wedge P \sqsubseteq\uparrow S) \iff Q =\uparrow R \cup S$$

*<proof>*

**lemma** *lower-ii-idempotent:*

$$R \cap\cap R =\downarrow R$$

*<proof>*

**lemma** *upper-iu-idempotent:*

$$R \cup\cup R =\uparrow R$$

*<proof>*

**lemma** *lower-iI-idempotent:*

$$I \neq \{\} \implies (\cap\cap(\lambda j . R)|I) =\downarrow R$$

*<proof>*

**lemma** *upper-iU-idempotent:*

$$I \neq \{\} \implies (\cup\cup(\lambda j . R)|I) =\uparrow R$$

*<proof>*

**lemma** *down-closed-intersection-closed:*

$$R = R\downarrow \implies \forall I . I \neq \{\} \longrightarrow (\cap\cap(\lambda j . R)|I) \subseteq R$$

*<proof>*

**lemma** *up-closed-union-closed:*

$$R = R\uparrow \implies \forall I . I \neq \{\} \longrightarrow (\cup\cup(\lambda j . R)|I) \subseteq R$$

*<proof>*

**lemma** *ou-down-lower-eq-ou:*

$$R\downarrow \cup S\downarrow =\downarrow R \cup S$$

*<proof>*

**lemma** *oi-down-lower-eq-ii:*

$$R\downarrow \cap S\downarrow =\downarrow R \cap\cap S$$

*<proof>*

**lemma** *ou-up-upper-eq-ou:*

$$R\uparrow \cup S\uparrow =\uparrow R \cup S$$

*<proof>*

**lemma** *oi-up-upper-eq-iu:*

$$R\uparrow \cap S\uparrow =\uparrow R \cup\cup S$$

*<proof>*

**lemma** *oU-down-lower-eq-oU:*

$$(\cup R \in X . R\downarrow) =\downarrow \cup X$$

*<proof>*

**lemma** *oI-down-lower-eq-iI*:

$$(\bigcap_{i \in I} . X \downarrow) = \downarrow \bigcap \bigcap X | I$$

*<proof>*

**lemma** *oU-up-upper-eq-oU*:

$$(\bigcup_{R \in X} . R \uparrow) = \uparrow \bigcup X$$

*<proof>*

**lemma** *oI-up-upper-eq-iI*:

$$(\bigcap_{i \in I} . X \uparrow) = \uparrow \bigcup \bigcup X | I$$

*<proof>*

**lemma** *down-order-lower*:

$$R \downarrow \subseteq S \downarrow \longleftrightarrow R \sqsubseteq \downarrow S$$

*<proof>*

**lemma** *up-order-upper*:

$$R \uparrow \subseteq S \uparrow \longleftrightarrow S \sqsubseteq \uparrow R$$

*<proof>*

**lemma** *convex-order-lower-upper*:

$$R \updownarrow \subseteq S \updownarrow \longleftrightarrow R \sqsubseteq \downarrow S \wedge S \sqsubseteq \uparrow R$$

*<proof>*

**lemma** *convex-order-Convex*:

$$R \updownarrow \subseteq S \updownarrow \longleftrightarrow R \sqsubseteq \updownarrow S$$

*<proof>*

### 6.3 Further results for convex-closure

**lemma** *convex-down*:

$$R \updownarrow \downarrow = R \downarrow$$

*<proof>*

**lemma** *convex-up*:

$$R \updownarrow \uparrow = R \uparrow$$

*<proof>*

**lemma** *iu-dist-oi-convex*:

**assumes**  $R = R \updownarrow$

**and**  $S = S \updownarrow$

**and**  $T = T \updownarrow$

**shows**  $(R \cap S) \cup T = (R \cup T) \cap (S \cup T)$

**nitpick** $[expect=genuine,card=1]$

*<proof>*

**lemma** *ii-dist-oi-convex*:

**assumes**  $R = R\uparrow$   
**and**  $S = S\uparrow$   
**and**  $T = T\uparrow$   
**shows**  $(R \cap S) \cap T = (R \cap T) \cap (S \cap T)$   
**nitpick**[*expect=genuine,card=1*]  
 <proof>

**lemma** *oI-up-closed*:  
**assumes**  $\forall R \in X . R\uparrow = R$   
**shows**  $(\bigcap X)\uparrow = \bigcap X$   
 <proof>

**lemma** *oI-down-closed*:  
**assumes**  $\forall R \in X . R\downarrow = R$   
**shows**  $(\bigcap X)\downarrow = \bigcap X$   
 <proof>

**lemma** *oI-convex-closed*:  
**assumes**  $\forall R \in X . R\uparrow\downarrow = R$   
**shows**  $(\bigcap X)\uparrow\downarrow = \bigcap X$   
 <proof>

**lemma** *up-dist-Union*:  
 $(\bigcup X)\uparrow = \bigcup \{ R\uparrow \mid R . R \in X \}$   
 <proof>

**lemma** *down-dist-Union*:  
 $(\bigcup X)\downarrow = \bigcup \{ R\downarrow \mid R . R \in X \}$   
 <proof>

**lemma** *convex-dist-Union*:  
 $(\bigcup X)\uparrow\downarrow = \bigcup \{ R\uparrow\downarrow \mid R . R \in X \}$   
**nitpick**[*expect=genuine,card=1,2*]  
 <proof>

**lemma** *up-dist-Inter*:  
 $(\bigcap X)\uparrow = \bigcap \{ R\uparrow \mid R . R \in X \}$   
**nitpick**[*expect=genuine,card=1*]  
 <proof>

**lemma** *down-dist-Inter*:  
 $(\bigcap X)\downarrow = \bigcap \{ R\downarrow \mid R . R \in X \}$   
**nitpick**[*expect=genuine,card=1*]  
 <proof>

**lemma** *convex-dist-Inter*:  
 $(\bigcap X)\uparrow\downarrow = \bigcap \{ R\uparrow\downarrow \mid R . R \in X \}$   
**nitpick**[*expect=genuine,card=1,2*]  
 <proof>

**lemma** *Inter-convex-closed*:

$$(\bigcap X)\downarrow = \bigcap X$$

**nitpick**[*expect=genuine,card=1,2*]

*<proof>*

**abbreviation** *convex-iu* (**infixl**  $\langle \cup\cup\downarrow \rangle$  70)

**where**  $R \cup\cup\downarrow S \equiv (R \cup\cup S)\downarrow$

**lemma** *convex-iu*:

$$R \cup\cup\downarrow S = (R\downarrow \cup\cup S\downarrow) \cap R\uparrow \cap S\uparrow$$

*<proof>*

**lemma** *convex-iu-sub*:

$$R\downarrow \cup\cup S \subseteq R \cup\cup\downarrow S$$

*<proof>*

**lemma** *convex-iu-convex-left*:

$$R \cup\cup\downarrow S = R\downarrow \cup\cup\downarrow S$$

*<proof>*

**lemma** *convex-iu-convex-right*:

$$R \cup\cup\downarrow S = R \cup\cup\downarrow S\downarrow$$

*<proof>*

**lemma** *convex-iu-convex*:

$$R \cup\cup\downarrow S = R\downarrow \cup\cup\downarrow S\downarrow$$

*<proof>*

**lemma** *convex-iu-assoc*:

$$(R \cup\cup\downarrow S) \cup\cup\downarrow T = R \cup\cup\downarrow (S \cup\cup\downarrow T)$$

*<proof>*

**lemma** *convex-iu-comm*:

$$R \cup\cup\downarrow S = S \cup\cup\downarrow R$$

*<proof>*

**lemma** *convex-iu-unit*:

$$R = R\downarrow \implies R \cup\cup\downarrow 1_{\cup\cup} = R$$

*<proof>*

**abbreviation** *convex-ii* (**infixl**  $\langle \cap\cap\downarrow \rangle$  70)

**where**  $R \cap\cap\downarrow S \equiv (R \cap\cap S)\downarrow$

**lemma** *convex-ii*:

$$R \cap\cap\downarrow S = (R\uparrow \cap\cap S\uparrow) \cap R\downarrow \cap S\downarrow$$

*<proof>*

**lemma** *convex-ii-sub*:

$$R\downarrow \cap\cap S \subseteq R \cap\cap\downarrow S$$

*<proof>*

**lemma** *convex-ii-convex-left:*

$$R \cap\cap\downarrow S = R\downarrow \cap\cap\downarrow S$$

*<proof>*

**lemma** *convex-ii-convex-right:*

$$R \cap\cap\downarrow S = R \cap\cap\downarrow S\downarrow$$

*<proof>*

**lemma** *convex-ii-convex:*

$$R \cap\cap\downarrow S = R\downarrow \cap\cap\downarrow S\downarrow$$

*<proof>*

**lemma** *convex-ii-assoc:*

$$(R \cap\cap\downarrow S) \cap\cap\downarrow T = R \cap\cap\downarrow (S \cap\cap\downarrow T)$$

*<proof>*

**lemma** *convex-ii-comm:*

$$R \cap\cap\downarrow S = S \cap\cap\downarrow R$$

*<proof>*

**lemma** *convex-ii-unit:*

$$R = R\downarrow \implies R \cap\cap\downarrow 1_{\cap\cap} = R$$

*<proof>*

**lemma** *convex-ii-ic:*

$$\sim(R \cup\cup\downarrow S) = \sim R \cap\cap\downarrow \sim S$$

*<proof>*

**lemma** *convex-ii-ic:*

$$\sim(R \cap\cap\downarrow S) = \sim R \cup\cup\downarrow \sim S$$

*<proof>*

**abbreviation** *convex-sup* :: ('a,'b) mrel set  $\Rightarrow$  ('a,'b) mrel ( $\cup\downarrow$ ) **where**

$$\cup\downarrow X \equiv (\cup X)\downarrow$$

**lemma** *convex-sup-convex:*

$$\cup\downarrow X = (\cup\downarrow X)\downarrow$$

*<proof>*

**lemma** *convex-sup-inter:*

$$\cup\downarrow X = \cap\{ Y . Y = Y\downarrow \wedge \cup X \subseteq Y \}$$

*<proof>*

**lemma** *convex-ii-dist-convex-sup:*

$$\cup\downarrow X \cup\cup\downarrow S = \cup\downarrow\{ R \cup\cup\downarrow S \mid R . R \in X \}$$

*<proof>*

**lemma** *convex-ii-dist-convex-sup*:  
 $\bigcup\downarrow X \text{ } \text{ } \text{ } \text{ } S = \bigcup\downarrow\{ R \text{ } \text{ } \text{ } S \mid R . R \in X \}$   
*<proof>*

**lemma** *convex-dist-sup*:  
 $(\bigcup X)\downarrow = \bigcup\downarrow\{ R\downarrow \mid R . R \in X \}$   
*<proof>*

## 7 Fusion and Fission

### 7.1 Atoms and co-atoms

**definition** *atoms* :: ('a,'b) mrel ( $\langle A_{\cup\cup} \rangle$ )  
**where**  $A_{\cup\cup} \equiv \{ (a, \{b\}) \mid a \text{ } b . \text{True} \}$

**definition** *co-atoms* :: ('a,'b) mrel ( $\langle A_{\cap\cap} \rangle$ )  
**where**  $A_{\cap\cap} \equiv \{ (a, \text{UNIV} - \{b\}) \mid a \text{ } b . \text{True} \}$

**declare** *atoms-def* [*mr-simp*] *co-atoms-def* [*mr-simp*]

**lemma** *atoms-solution*:  
 $A_{\cup\cup}\uparrow = -1_{\cup\cup}$   
*<proof>*

**lemma** *atoms-least-solution*:  
**assumes**  $R\uparrow = -1_{\cup\cup}$   
**shows**  $A_{\cup\cup} \subseteq R$   
*<proof>*

**lemma** *ic-atoms*:  
 $\sim A_{\cup\cup} = A_{\cap\cap}$   
*<proof>*

**lemma** *ic-co-atoms*:  
 $\sim A_{\cap\cap} = A_{\cup\cup}$   
*<proof>*

**lemma** *co-atoms-solution*:  
 $A_{\cap\cap}\downarrow = -1_{\cap\cap}$   
*<proof>*

**lemma** *co-atoms-least-solution*:  
**assumes**  $R\downarrow = -1_{\cap\cap}$   
**shows**  $A_{\cap\cap} \subseteq R$   
*<proof>*

**lemma** *iu-unit-atoms-disjoint*:  
 $1_{\cup\cup} \cap A_{\cup\cup} = \{\}$

*<proof>*

**lemma** *ii-unit-co-atoms-disjoint*:

$$1_{\cap\cap} \cap A_{\cap\cap} = \{\}$$

*<proof>*

**lemma** *atoms-sp-idempotent*:

$$A_{\cup\cup} * A_{\cup\cup} = A_{\cup\cup}$$

*<proof>*

**lemma** *atoms-sp-cp*:

$$(R \cap A_{\cup\cup}) * S = (R \cap A_{\cup\cup}) \odot S$$

*<proof>*

## 7.2 Inner-functional properties

**abbreviation** *inner-univalent* :: ('a,'b) mrel  $\Rightarrow$  bool **where**

$$\textit{inner-univalent } R \equiv R \subseteq 1_{\cup\cup} \cup A_{\cup\cup}$$

**abbreviation** *inner-total* :: ('a,'b) mrel  $\Rightarrow$  bool **where**

$$\textit{inner-total } R \equiv R \subseteq -1_{\cup\cup}$$

**abbreviation** *inner-deterministic* :: ('a,'b) mrel  $\Rightarrow$  bool **where**

$$\textit{inner-deterministic } R \equiv \textit{inner-total } R \wedge \textit{inner-univalent } R$$

**lemma** *inner-deterministic-atoms*:

$$\textit{inner-deterministic } R \longleftrightarrow R \subseteq A_{\cup\cup}$$

*<proof>*

**lemma** *inner-univalent*:

$$\textit{inner-univalent } R \longleftrightarrow (\forall a b c B . (a,B) \in R \wedge b \in B \wedge c \in B \longrightarrow b = c)$$

*<proof>*

**lemma** *inner-univalent-2*:

$$\textit{inner-univalent } R \longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow \textit{finite } B \wedge \textit{card } B \leq \textit{one-class.one})$$

*<proof>*

**lemma** *inner-total*:

$$\textit{inner-total } R \longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow (\exists b . b \in B))$$

*<proof>*

**lemma** *inner-total-2*:

$$\textit{inner-total } R \longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow B \neq \{\})$$

*<proof>*

**lemma** *inner-total-3*:

$$\textit{inner-total } R \longleftrightarrow (\forall a B . (a,B) \in R \wedge \textit{finite } B \longrightarrow \textit{card } B \geq \textit{one-class.one})$$

*<proof>*

**lemma** *inner-deterministic:*

*inner-deterministic*  $R \iff (\forall a B . (a,B) \in R \implies (\exists! b . b \in B))$   
*<proof>*

**lemma** *inner-deterministic-2:*

*inner-deterministic*  $R \iff (\forall a B . (a,B) \in R \implies \text{card } B = \text{one-class.one})$   
*<proof>*

**lemma** *inner-deterministic-sp-unit:*

*inner-deterministic*  $1$   
*<proof>*

**lemma** *inner-univalent-down:*

**assumes** *inner-univalent*  $S$   
**shows**  $S\downarrow \subseteq S \cup 1_{\cup\cup}$   
*<proof>*

**lemma** *inner-deterministic-lower-eq:*

**assumes** *inner-deterministic*  $V$   
**and** *inner-deterministic*  $W$   
**and**  $V =\downarrow W$   
**shows**  $V = W$   
*<proof>*

**lemma** *inner-total-down-closed:*

*inner-total*  $T \implies R \subseteq T \implies \text{inner-total } R$   
*<proof>*

**lemma** *inner-univalent-down-closed:*

*inner-univalent*  $T \implies R \subseteq T \implies \text{inner-univalent } R$   
*<proof>*

**lemma** *inner-deterministic-down-closed:*

*inner-deterministic*  $T \implies R \subseteq T \implies \text{inner-deterministic } R$   
*<proof>*

**lemma** *inner-univalent-conver:*

**assumes** *inner-univalent*  $R$   
**shows**  $R = R\downarrow$   
*<proof>*

**lemma** *inner-deterministic-alt-closure:*

*inner-deterministic*  $R = (R \text{ O } \text{converse } 1 \text{ O } 1 = R)$   
*<proof>*

**lemma** *inner-deterministic-s-id-conv-epsiloff:*

*inner-deterministic*  $R \implies R \text{ O } \text{converse } s\text{-id} = R \text{ O } \text{epsiloff}$   
*<proof>*

**lemma** *inner-deterministic-lower-iff*:

**assumes** *inner-deterministic R*  
**and** *inner-deterministic S*  
**shows**  $(R \sqsubseteq\downarrow S) \longleftrightarrow (R \subseteq S)$   
*<proof>*

**lemma** *inner-deterministic-upper-iff*:

**assumes** *inner-deterministic R*  
**and** *inner-deterministic S*  
**shows**  $(R \sqsubseteq\uparrow S) \longleftrightarrow (S \subseteq R)$   
*<proof>*

**lemma** *inner-deterministic-lower-eq-iff*:

**assumes** *inner-deterministic R*  
**and** *inner-deterministic S*  
**shows**  $(R =\downarrow S) \longleftrightarrow (R = S)$   
*<proof>*

**lemma** *inner-deterministic-upper-eq-iff*:

**assumes** *inner-deterministic R*  
**and** *inner-deterministic S*  
**shows**  $(R =\uparrow S) \longleftrightarrow (R = S)$   
*<proof>*

**lemma** *inner-deterministic-convex-eq-iff*:

**assumes** *inner-deterministic R*  
**and** *inner-deterministic S*  
**shows**  $(R =\updownarrow S) \longleftrightarrow (R = S)$   
*<proof>*

**lemma**

*inner-univalent R*  $\implies$  *inner-univalent S*  $\implies$  *inner-univalent (R  $\cup\cup$  S)*  
**nitpick**[*expect=genuine,card=1,2*]  
*<proof>*

**lemma** *inner-univalent-ii-closed*:

*inner-univalent R*  $\implies$  *inner-univalent S*  $\implies$  *inner-univalent (R  $\cap\cap$  S)*  
*<proof>*

**lemma** *inner-total-iu-closed*:

*inner-total R*  $\implies$  *inner-total S*  $\implies$  *inner-total (R  $\cup\cup$  S)*  
*<proof>*

**lemma**

*inner-total R*  $\implies$  *inner-total S*  $\implies$  *inner-total (R  $\cap\cap$  S)*  
**nitpick**[*expect=genuine,card=1,2*]  
*<proof>*

### 7.3 Fusion

**lemma** *fusion-set*:

$$\text{fus } R \equiv \{ (a, B) . B = \bigcup \{ C . (a, C) \in R \} \}$$

*<proof>*

**declare** *fusion-set* [*mr-simp*]

**lemma** *fusion-lower-increasing*:

$$R \sqsubseteq\downarrow \text{fus } R$$

*<proof>*

**lemma** *fusion-deterministic*:

$$\text{deterministic } (\text{fus } R)$$

*<proof>*

**lemma** *fusion-least*:

**assumes**  $R \sqsubseteq\downarrow S$   
**and** *deterministic*  $S$   
**shows**  $\text{fus } R \sqsubseteq\downarrow S$   
*<proof>*

**lemma** *fusion-unique*:

**assumes**  $\forall R . R \sqsubseteq\downarrow f R$   
**and**  $\forall R . \text{deterministic } (f R)$   
**and**  $\forall R S . R \sqsubseteq\downarrow S \wedge \text{deterministic } S \longrightarrow f R \sqsubseteq\downarrow S$   
**shows**  $f T = \text{fus } T$   
*<proof>*

**lemma** *fusion-down-char*:

$$(\text{fus } R)\downarrow = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow)$$

*<proof>*

**lemma** *fusion-up-char*:

$$(\text{fus } R)\uparrow = -((\sim(R\downarrow) \cap A_{\cap\cap})\downarrow)$$

*<proof>*

**lemma** *fusion-up-char-2*:

$$(\text{fus } R)\uparrow = -(((R\downarrow \cap A_{\cup\cup}) * \sim I)\downarrow)$$

*<proof>*

**lemma** *fusion-char*:

$$\text{fus } R = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow) \cap -((\sim(R\downarrow) \cap A_{\cap\cap})\downarrow)$$

*<proof>*

**lemma** *fusion-char-2*:

$$\text{fus } R = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow) \cap -(((R\downarrow \cap A_{\cup\cup}) * \sim I)\downarrow)$$

*<proof>*

**lemma** *fusion-lower-isotone*:

$R \sqsubseteq\downarrow S \implies \text{fus } R \sqsubseteq\downarrow \text{fus } S$   
 $\langle \text{proof} \rangle$

**lemma** *fusion-iu-idempotent*:  
 $\text{fus } R \cup\cup \text{fus } R = \text{fus } R$   
 $\langle \text{proof} \rangle$

**lemma** *fusion-down*:  
 $\text{fus } R = \text{fus } (R\downarrow)$   
 $\langle \text{proof} \rangle$

**lemma** *fusion-iu-total*:  
 $\text{total } T \implies T \cup\cup \text{fus } T = \text{fus } T$   
 $\langle \text{proof} \rangle$

**lemma** *fusion-deterministic-fixpoint*:  
 $\text{deterministic } R \longleftrightarrow R = \text{fus } R$   
 $\langle \text{proof} \rangle$

**abbreviation** *non-empty* ::  $(a, 'b) \text{ mrel} \Rightarrow (a, 'b) \text{ mrel } (\langle \text{ne } \rightarrow [100] 100 \rangle)$   
**where**  $\text{ne } R \equiv R \cap -1 \cup\cup$

**lemma** *non-empty*:  
 $\text{ne } R = \{ (a, B) \mid a \ B . (a, B) \in R \wedge B \neq \{\} \}$   
 $\langle \text{proof} \rangle$

**lemma** *ne-equality*:  
 $\text{ne } R = R \longleftrightarrow R \subseteq -1 \cup\cup$   
 $\langle \text{proof} \rangle$

**lemma** *ne-dist-ou*:  
 $\text{ne } (R \cup S) = \text{ne } R \cup \text{ne } S$   
 $\langle \text{proof} \rangle$

**lemma** *ne-down-idempotent*:  
 $\text{ne } ((\text{ne } (R\downarrow))\downarrow) = \text{ne } (R\downarrow)$   
 $\langle \text{proof} \rangle$

**lemma** *ne-up*:  
 $(\text{ne } R)\uparrow = \text{ne } R * 1\uparrow$   
 $\langle \text{proof} \rangle$

**lemma** *ne-dist-down-sp*:  
 $\text{ne } (R\downarrow * S) = \text{ne } (R\downarrow) * \text{ne } S$   
 $\langle \text{proof} \rangle$

**lemma** *total-ne-down-dist-sp*:  
 $\text{total } T \implies \text{ne } ((R * T)\downarrow) = \text{ne } (R\downarrow) * \text{ne } (T\downarrow)$   
 $\langle \text{proof} \rangle$

**lemma** *inner-univalent-char*:

*inner-univalent*  $S \iff (\forall R . \text{fus } R = \text{fus } S \wedge R \sqsubseteq\downarrow S \implies \text{ne } R = \text{ne } S)$   
*<proof>*

**lemma** *ne-dist-oU*:

$\text{ne } (\bigcup X) = \bigcup (\text{non-empty } X)$   
*<proof>*

## 7.4 Fission

**lemma** *fission-set*:

$\text{fis } R = \{ (a, \{b\}) \mid a \text{ } b . \exists B . (a, B) \in R \wedge b \in B \}$   
*<proof>*

**declare** *fission-set* [*mr-simp*]

**lemma** *fission-var*:

$\text{fis } R = R\downarrow \cap A_{\cup\cup}$   
*<proof>*

**lemma** *fission-lower-decreasing*:

$\text{fis } R \sqsubseteq\downarrow R$   
*<proof>*

**lemma** *fission-inner-deterministic*:

*inner-deterministic* ( $\text{fis } R$ )  
*<proof>*

**lemma** *fission-greatest*:

**assumes**  $S \sqsubseteq\downarrow R$   
**and** *inner-deterministic*  $S$   
**shows**  $S \sqsubseteq\downarrow \text{fis } R$   
*<proof>*

**lemma** *fission-unique*:

**assumes**  $\forall R . f R \sqsubseteq\downarrow R$   
**and**  $\forall R . \text{inner-deterministic } (f R)$   
**and**  $\forall R S . S \sqsubseteq\downarrow R \wedge \text{inner-deterministic } S \implies S \sqsubseteq\downarrow f R$   
**shows**  $f T = \text{fis } T$   
*<proof>*

**lemma** *fission-lower-isotone*:

$R \sqsubseteq\downarrow S \implies \text{fis } R \sqsubseteq\downarrow \text{fis } S$   
*<proof>*

**lemma** *fission-idempotent*:

$\text{fis } (\text{fis } R) = \text{fis } R$   
*<proof>*

**lemma** *fission-top*:

$$\text{fis } U = A_{\cup\cup}$$

*<proof>*

**lemma** *fission-down*:

$$\text{fis } R = \text{fis } (R \downarrow)$$

*<proof>*

**lemma** *fission-ne-fixpoint*:

$$\text{fis } R = \text{ne } (\text{fis } R)$$

*<proof>*

**lemma** *fission-down-ne-fixpoint*:

$$\text{fis } R = \text{ne } ((\text{fis } R) \downarrow)$$

*<proof>*

**lemma** *fission-inner-deterministic-fixpoint*:

$$\text{inner-deterministic } R \longleftrightarrow R = \text{fis } R$$

*<proof>*

**lemma** *fission-sp-subdist*:

$$\text{fis } (R * S) \subseteq \text{fis } R * \text{fis } S$$

*<proof>*

**lemma** *fission-sp-total-dist*:

**assumes** *total*  $T$

**shows**  $\text{fis } (R * T) = \text{fis } R * \text{fis } T$

*<proof>*

**lemma** *fission-dist-ou*:

$$\text{fis } (R \cup S) = \text{fis } R \cup \text{fis } S$$

*<proof>*

**lemma** *fission-sp-iu-unit*:

$$\text{fis } (R * 1_{\cup\cup}) = \{\}$$

*<proof>*

**lemma** *fission-fusion-lower-decreasing*:

$$\text{fis } (\text{fus } R) \sqsubseteq \downarrow R$$

*<proof>*

**lemma** *fission-fusion-lower-increasing*:

$$R \sqsubseteq \downarrow \text{fus } (\text{fis } R)$$

*<proof>*

**lemma** *fission-fusion-galois*:

$$\text{fis } R \sqsubseteq \downarrow S \longleftrightarrow R \sqsubseteq \downarrow \text{fus } S$$

*<proof>*

**lemma** *fission-fusion*:

$$\text{fis } (\text{fus } R) = \text{fis } R$$

*<proof>*

**lemma** *fusion-fission*:

$$\text{fus } (\text{fis } R) = \text{fus } R$$

*<proof>*

**lemma** *same-fusion-fission-lower*:

$$\text{fus } R = \text{fus } S \implies \text{fis } R \sqsubseteq \downarrow S$$

*<proof>*

**lemma** *fission-below-ne-down-fusion*:

$$\text{fis } R \subseteq \text{ne } ((\text{fus } R)\downarrow)$$

*<proof>*

**lemma** *ne-fusion-fission*:

$$(\text{ne } ((\text{fus } R)\downarrow))\uparrow = (\text{fis } R)\uparrow$$

*<proof>*

**lemma** *fission-up-ne-down-up*:

$$(\text{fis } R)\uparrow = (\text{ne } (R\downarrow))\uparrow$$

*<proof>*

**lemma** *fusion-idempotent*:

$$\text{fus } (\text{fus } R) = \text{fus } R$$

*<proof>*

**lemma** *fission-dist-oU*:

$$\text{fis } (\bigcup X) = \bigcup (\text{fis } ' X)$$

*<proof>*

## 7.5 Co-fusion and co-fission

**definition** *co-fusion* :: ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel ( $\langle \prod \prod \rightarrow [80] 80 \rangle$ ) **where**

$$\prod \prod R \equiv \{ (a,B) . B = \bigcap \{ C . (a,C) \in R \} \}$$

**declare** *co-fusion-def* [*mr-simp*]

**lemma** *co-fusion-upper-decreasing*:

$$\prod \prod R \sqsubseteq \uparrow R$$

*<proof>*

**lemma** *co-fusion-deterministic*:

$$\text{deterministic } (\prod \prod R)$$

*<proof>*

**lemma** *co-fusion-greatest*:

**assumes**  $S \sqsubseteq \uparrow R$   
**and** *deterministic*  $S$   
**shows**  $S \sqsubseteq \uparrow \sqcap \sqcap R$   
*<proof>*

**lemma** *co-fusion-unique*:  
**assumes**  $\forall R . f R \sqsubseteq \uparrow R$   
**and**  $\forall R . \text{deterministic } (f R)$   
**and**  $\forall R S . S \sqsubseteq \uparrow R \wedge \text{deterministic } S \longrightarrow S \sqsubseteq \uparrow f R$   
**shows**  $f T = \sqcap \sqcap T$   
*<proof>*

**lemma** *co-fusion-up-char*:  
 $(\sqcap \sqcap R) \uparrow = -((-(R \uparrow) \cap A_{\cap \cap}) \downarrow)$   
*<proof>*

**lemma** *co-fusion-down-char*:  
 $(\sqcap \sqcap R) \downarrow = -((\sim(R \uparrow) \cap A_{\cup \cup}) \uparrow)$   
*<proof>*

**lemma** *co-fusion-down-char-2*:  
 $(\sqcap \sqcap R) \downarrow = -(((R \uparrow) \cap A_{\cap \cap}) \odot \sim I) \uparrow$   
*<proof>*

**lemma** *co-fusion-char*:  
 $\sqcap \sqcap R = -((-(R \uparrow) \cap A_{\cap \cap}) \downarrow) \cap -((\sim(R \uparrow) \cap A_{\cup \cup}) \uparrow)$   
*<proof>*

**lemma** *co-fusion-char-2*:  
 $\sqcap \sqcap R = -((-(R \uparrow) \cap A_{\cap \cap}) \downarrow) \cap -(((R \uparrow) \cap A_{\cap \cap}) \odot \sim I) \uparrow$   
*<proof>*

**lemma** *co-fusion-upper-isotone*:  
 $R \sqsubseteq \uparrow S \implies \sqcap \sqcap R \sqsubseteq \uparrow \sqcap \sqcap S$   
*<proof>*

**lemma** *co-fusion-ii-idempotent*:  
 $\sqcap \sqcap R \cap \sqcap \sqcap R = \sqcap \sqcap R$   
*<proof>*

**lemma** *co-fusion-up*:  
 $\sqcap \sqcap R = \sqcap \sqcap (R \uparrow)$   
*<proof>*

**lemma** *co-fusion-ii-total*:  
 $\text{total } T \implies T \cap \sqcap \sqcap T = \sqcap \sqcap T$   
*<proof>*

**lemma** *co-fusion-deterministic-fixpoint*:

*deterministic*  $R \longleftrightarrow R = \prod \prod R$   
 ⟨proof⟩

**abbreviation** *co-fission* ::  $(a, b) \text{ mrel} \Rightarrow (a, b) \text{ mrel} (\text{at}_{\cap\cap} \rightarrow [80] 80)$  **where**  
 $\text{at}_{\cap\cap} R \equiv R \uparrow \cap A_{\cap\cap}$

**lemma** *co-fission*:  
 $\text{at}_{\cap\cap} R = \{ (a, B) \mid a B . (\exists b . -B = \{b\}) \wedge (\exists C . (a, C) \in R \wedge C \subseteq B) \}$   
 ⟨proof⟩

**declare** *co-fission* [*mr-simp*]

**lemma** *co-fission-upper-increasing*:  
 $R \sqsubseteq \uparrow \text{at}_{\cap\cap} R$   
 ⟨proof⟩

**lemma** *co-fission-ic-inner-deterministic*:  
*inner-deterministic*  $(\sim \text{at}_{\cap\cap} R)$   
 ⟨proof⟩

**lemma** *co-fission-least*:  
**assumes**  $R \sqsubseteq \uparrow S$   
**and** *inner-deterministic*  $(\sim S)$   
**shows**  $\text{at}_{\cap\cap} R \sqsubseteq \uparrow S$   
 ⟨proof⟩

**lemma** *co-fission-unique*:  
**assumes**  $\forall R . R \sqsubseteq \uparrow f R$   
**and**  $\forall R . \text{inner-deterministic} (\sim f R)$   
**and**  $\forall R S . R \sqsubseteq \uparrow S \wedge \text{inner-deterministic} (\sim S) \longrightarrow f R \sqsubseteq \uparrow S$   
**shows**  $f T = \text{at}_{\cap\cap} T$   
 ⟨proof⟩

**lemma** *co-fission-upper-isotone*:  
 $R \sqsubseteq \uparrow S \implies \text{at}_{\cap\cap} R \sqsubseteq \uparrow \text{at}_{\cap\cap} S$   
 ⟨proof⟩

**lemma** *co-fission-idempotent*:  
 $\text{at}_{\cap\cap} (\text{at}_{\cap\cap} R) = \text{at}_{\cap\cap} R$   
 ⟨proof⟩

**lemma** *co-fission-top*:  
 $\text{at}_{\cap\cap} U = A_{\cap\cap}$   
 ⟨proof⟩

**lemma** *co-fission-up*:  
 $\text{at}_{\cap\cap} R = \text{at}_{\cap\cap} (R \uparrow)$   
 ⟨proof⟩

**lemma** *co-fission-ic-inner-deterministic-fixpoint:*  
*inner-deterministic*  $(\sim R) \longleftrightarrow R = at_{\cap\cap} R$   
 ⟨proof⟩

**lemma** *co-fusion-co-fission-upper-decreasing:*  
 $\Box\Box(at_{\cap\cap} R) \sqsubseteq\uparrow R$   
 ⟨proof⟩

**lemma** *co-fission-co-fusion-upper-increasing:*  
 $R \sqsubseteq\uparrow at_{\cap\cap} (\Box\Box R)$   
 ⟨proof⟩

**lemma** *co-fusion-co-fission-galois:*  
 $\Box\Box R \sqsubseteq\uparrow S \longleftrightarrow R \sqsubseteq\uparrow at_{\cap\cap} S$   
 ⟨proof⟩

**lemma** *co-fission-co-fusion:*  
 $at_{\cap\cap} (\Box\Box R) = at_{\cap\cap} R$   
 ⟨proof⟩

**lemma** *co-fusion-co-fission:*  
 $\Box\Box(at_{\cap\cap} R) = \Box\Box R$   
 ⟨proof⟩

**lemma** *same-co-fusion-co-fission-upper:*  
 $\Box\Box R = \Box\Box S \implies S \sqsubseteq\uparrow at_{\cap\cap} R$   
 ⟨proof⟩

**lemma** *co-fusion-idempotent:*  
 $\Box\Box(\Box\Box R) = \Box\Box R$   
 ⟨proof⟩

## 8 Modalities

### 8.1 Tests

**abbreviation** *test* ::  $(\prime a, \prime a) \text{ mrel} \Rightarrow \text{bool}$  where  
 $\text{test } R \equiv R \subseteq 1$

**lemma** *test:*  
 $\text{test } R \longleftrightarrow (\forall a B . (a, B) \in R \longrightarrow B = \{a\})$   
 ⟨proof⟩

**lemma** *test-fix:*  $\text{test } R \equiv R \cap 1_{\sigma} = R$   
 ⟨proof⟩

**lemma** *test-ou-closed:*  
 $\text{test } p \implies \text{test } q \implies \text{test } (p \cup q)$   
 ⟨proof⟩

**lemma** *test-oi-closed*:

$test\ p \implies test\ (p \cap q)$   
*<proof>*

**abbreviation** *test-complement* :: ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel ( $\lrcorner \rightarrow$  [80] 80)

**where**

$\lrcorner R \equiv -R \cap 1$

**lemma** *test-complement-closed*:

$test\ (\lrcorner p)$   
*<proof>*

**lemma** *test-double-complement*:

$test\ p \longleftrightarrow p = \lrcorner \lrcorner p$   
*<proof>*

**lemma** *test-complement*:

$(a, \{a\}) \in \lrcorner p \longleftrightarrow \neg (a, \{a\}) \in p$   
*<proof>*

**declare** *test-complement* [*mr-simp*]

**lemma** *test-complement-antitone*:

**assumes** *test p*  
**shows**  $p \subseteq q \longleftrightarrow \lrcorner q \subseteq \lrcorner p$   
*<proof>*

**lemma** *test-complement-huntington*:

$test\ p \implies p = \lrcorner (\lrcorner p \cup \lrcorner q) \cup \lrcorner (\lrcorner p \cup q)$   
*<proof>*

**abbreviation** *test-implication* :: ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel

(**infixl**  $\rightarrow$  65) **where**

$p \rightarrow q \equiv \lrcorner p \cup q$

**lemma** *test-implication-closed*:

$test\ q \implies test\ (p \rightarrow q)$   
*<proof>*

**lemma** *test-implication*:

$(a, \{a\}) \in p \rightarrow q \longleftrightarrow ((a, \{a\}) \in p \longrightarrow (a, \{a\}) \in q)$   
*<proof>*

**declare** *test-implication* [*mr-simp*]

**lemma** *test-implication-left-antitone*:

**assumes** *test p*  
**shows**  $p \subseteq r \implies r \rightarrow q \subseteq p \rightarrow q$

*<proof>*

**lemma** *test-implication-right-isotone:*

**assumes** *test p*

**shows**  $q \subseteq r \implies p \rightarrow q \subseteq p \rightarrow r$

*<proof>*

**lemma** *test-sp-idempotent:*

*test p*  $\implies p * p = p$

*<proof>*

**lemma** *test-sp:*

**assumes** *test p*

**shows**  $p * R = (p * U) \cap R$

*<proof>*

**lemma** *sp-test:*

*test p*  $\implies R * p = R \cap (U * p)$

*<proof>*

**lemma** *sp-test-dist-oi:*

*test p*  $\implies (R \cap S) * p = (R * p) \cap (S * p)$

*<proof>*

**lemma** *sp-test-dist-oi-left:*

*test p*  $\implies (R \cap S) * p = (R * p) \cap S$

*<proof>*

**lemma** *sp-test-dist-oi-right:*

*test p*  $\implies (R \cap S) * p = R \cap (S * p)$

*<proof>*

**lemma** *sp-test-sp-oi-left:*

*test p*  $\implies (R \cap (U * p)) * T = R * p * T$

*<proof>*

**lemma** *sp-test-sp-oi-right:*

*test p*  $\implies R * ((p * U) \cap T) = R * p * T$

*<proof>*

**lemma** *test-sp-ne:*

*test p*  $\implies p * ne R = ne (p * R)$

*<proof>*

**lemma** *ne-sp-test:*

*test p*  $\implies ne R * p = ne (R * p)$

*<proof>*

**lemma** *top-sp-test-down-closed:*

**assumes** *test p*  
**shows**  $U * p = (U * p)\downarrow$   
 $\langle proof \rangle$

**lemma** *oc-top-sp-test-up-closed*:  
 $test\ p \implies -(U * p) = (-(U * p))\uparrow$   
 $\langle proof \rangle$

**lemma** *top-sp-test*:  
 $test\ p \implies (a, B) \in U * p \iff (\forall b \in B . (b, \{b\}) \in p)$   
 $\langle proof \rangle$

**lemma** *oc-top-sp-test*:  
 $test\ p \implies (a, B) \in -(U * p) \iff (\exists b \in B . (b, \{b\}) \notin p)$   
 $\langle proof \rangle$

**declare** *top-sp-test* [*mr-simp*] *oc-top-sp-test* [*mr-simp*]

**lemma** *oc-top-sp-test-0*:  
 $-1_{\cup\cup} * \wr p = ne\ (U * \wr p)$   
 $\langle proof \rangle$

**lemma** *oc-top-sp-test-1*:  
**assumes** *test p*  
**shows**  $-(U * p) = (ne\ (U * \wr p))\uparrow$   
 $\langle proof \rangle$

**lemma** *oc-top-sp-test-2*:  
 $test\ p \implies -(U * p) = (-1_{\cup\cup} * \wr p)\uparrow$   
 $\langle proof \rangle$

**lemma** *split-sp-test*:  
**assumes** *test p*  
**shows**  $R = (R * p) \cup (ne\ R \cap (ne\ (R\downarrow * \wr p))\uparrow)$   
 $\langle proof \rangle$

**lemma** *top-sp-test-down-iff-1*:  
**assumes** *test p*  
**shows**  $R \subseteq U * p \iff R\downarrow \subseteq U * p$   
 $\langle proof \rangle$

**lemma** *test-ne*:  
 $test\ p \implies ne\ p = p$   
 $\langle proof \rangle$

**lemma** *ne-test-up*:  
 $test\ p \implies ne\ (p\uparrow) = p\uparrow$   
 $\langle proof \rangle$

**lemma** *ne-sp-test-up*:

$$\text{test } p \implies (ne (R * p))\uparrow = ne R * p\uparrow$$

*<proof>*

**lemma** *ne-down-sp-test-up*:

$$\text{test } p \implies ne (R\downarrow * p\uparrow) = ne (R\downarrow) * p\uparrow$$

*<proof>*

**lemma** *test-up-sp*:

$$\text{test } p \implies p\uparrow = p * 1\uparrow$$

*<proof>*

**lemma** *top-test-oi-top-complement*:

$$\text{test } p \implies (U * p) \cap (U * \imath p) = 1_{\cup\cup}$$

*<proof>*

**lemma** *sp-test-oi-complement*:

$$\text{test } p \implies (R * p) \cap (R * \imath p) = R \cap 1_{\cup\cup}$$

*<proof>*

**lemma** *ne-top-sp-test-complement*:

**assumes** *test p*  
**shows**  $ne (U * p) * \imath p = \{\}$

*<proof>*

**lemma** *complement-test-sp-top*:

**assumes** *test p*  
**shows**  $-(p * U) = \imath p * U$

*<proof>*

**lemma** *top-sp-test-shunt*:

**assumes** *test p*  
**shows**  $R \subseteq U * p \longrightarrow R * \imath p \subseteq 1_{\cup\cup}$

*<proof>*

**lemma** *top-sp-test-down-iff-2*:

**assumes** *test p*  
**shows**  $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow * \imath p \subseteq 1_{\cup\cup}$

*<proof>*

**lemma** *top-sp-test-down-iff-3*:

$$R\downarrow * \imath p \subseteq 1_{\cup\cup} \longleftrightarrow ne (R\downarrow) * \imath p \subseteq \{\}$$

*<proof>*

**lemma** *top-sp-test-down-iff-4*:

**assumes** *test p*  
**shows**  $R\downarrow \cap (U * \imath p) \subseteq 1_{\cup\cup} \longleftrightarrow R\downarrow \subseteq 1_{\cup\cup} \cup (U * p)$

*<proof>*

**lemma** *top-sp-test-down-iff-5*:

**assumes**  $test\ p$

**shows**  $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow \subseteq 1_{UU} \cup (U * p)$

*<proof>*

**lemma** *iu-test-sp-left-zero*:

**assumes**  $q \subseteq 1_{UU}$

**shows**  $q * R = q$

*<proof>*

**lemma** *test-iu-test-split*:

$t \subseteq 1 \cup 1_{UU} \longleftrightarrow (\exists p\ q . p \subseteq 1 \wedge q \subseteq 1_{UU} \wedge t = p \cup q)$

*<proof>*

**lemma** *test-iu-test-sp-assoc-1*:

$t \subseteq 1 \cup 1_{UU} \implies t * (R * S) = (t * R) * S$

*<proof>*

**lemma** *test-iu-test-sp-assoc-2*:

$t \subseteq 1_{UU} \implies R * (t * S) = (R * t) * S$

*<proof>*

**lemma** *test-iu-test-sp-assoc-3*:

**assumes**  $t \subseteq 1 \cup 1_{UU}$

**shows**  $R * (t * S) = (R * t) * S$

*<proof>*

**lemma** *test-iu-test-sp-assoc-4*:

$t \subseteq 1_{UU} \implies R * (S * t) = (R * S) * t$

*<proof>*

**lemma** *test-iu-test-sp-assoc-5*:

**assumes**  $t \subseteq 1 \cup 1_{UU}$

**shows**  $R * (S * t) = (R * S) * t$

*<proof>*

**lemma** *inner-deterministic-sp-assoc*:

**assumes** *inner-univalent*  $t$

**shows**  $t * (R * S) = (t * R) * S$

*<proof>*

**lemma** *iu-unit-below-top-sp-test*:

$1_{UU} \subseteq U * R$

*<proof>*

**lemma** *ne-oi-complement-top-sp-test-1*:

$ne\ (R \cap -(U * S)) = R \cap -(U * S)$

*<proof>*

**lemma** *ne-oi-complement-top-sp-test-2*:  
 $ne\ R \cap -(U * S) = R \cap -(U * S)$   
 ⟨proof⟩

**lemma** *schroeder-test*:  
**assumes** *test p*  
**shows**  $R * p \subseteq S \longleftrightarrow -S * p \subseteq -R$   
 ⟨proof⟩

**lemma** *complement-test-sp-test*:  
 $test\ p \implies -p * p \subseteq -1$   
 ⟨proof⟩

**lemma** *test-sp-commute*:  
 $test\ p \implies test\ q \implies p * q = q * p$   
 ⟨proof⟩

**lemma** *test-shunting*:  
**assumes** *test p*  
**and** *test q*  
**and** *test r*  
**shows**  $p * q \subseteq r \longleftrightarrow p \subseteq r \cup \setminus q$   
 ⟨proof⟩

**lemma** *test-sp-is-iu*:  
 $test\ p \implies test\ q \implies p * q = p \cup \cup q$   
 ⟨proof⟩

**lemma** *test-set*:  
 $test\ p \implies p = \{ (a, \{a\}) \mid a . (a, \{a\}) \in p \}$   
 ⟨proof⟩

**lemma** *test-sp-is-ii*:  
**assumes** *test p*  
**and** *test q*  
**shows**  $p * q = p \cap \cap q$   
 ⟨proof⟩

**lemma** *test-galois-1*:  
**assumes** *test p*  
**and** *test q*  
**shows**  $p * q \subseteq r \longleftrightarrow q \subseteq p \rightarrow r$   
 ⟨proof⟩

**lemma** *test-sp-shunting*:  
**assumes** *test p*  
**shows**  $\setminus p * R \subseteq \{\} \longleftrightarrow R \subseteq p * R$   
 ⟨proof⟩

**lemma** *test-oU-closed*:  
 $\forall p \in X . \text{test } p \implies \text{test } (\bigcup X)$   
 $\langle \text{proof} \rangle$

**lemma** *test-oI-closed*:  
 $\exists p \in X . \text{test } p \implies \text{test } (\bigcap X)$   
 $\langle \text{proof} \rangle$

**lemma** *sp-test-dist-oI*:  
**assumes** *test p*  
**and**  $X \neq \{\}$   
**shows**  $(\bigcap X) * p = (\bigcap R \in X . R * p)$   
 $\langle \text{proof} \rangle$

**lemma** *test-iU-is-iI*:  
**assumes**  $\forall i \in I . \text{test } (X \ i)$   
**and**  $I \neq \{\}$   
**shows**  $\bigcup \bigcup X | I = \bigcap \bigcap X | I$   
 $\langle \text{proof} \rangle$

**lemma** *test-iU-is-oI*:  
**assumes**  $\forall i \in I . \text{test } (X \ i)$   
**and**  $I \neq \{\}$   
**shows**  $\bigcup \bigcup X | I = \bigcap (X \ ' \ I)$   
 $\langle \text{proof} \rangle$

## 8.2 Domain and antidomain

**declare** *Dom-def* [*mr-simp*]

**abbreviation** *aDom* ::  $(\ 'a, 'b) \text{ mrel} \Rightarrow (\ 'a, 'a) \text{ mrel}$  **where**  
 $aDom \ R \equiv \lambda \text{ Dom } R$

**lemma** *ad-set*:  $aDom \ R = \{(a, \{a\}) \mid a. \neg(\exists A. (a, A) \in R)\}$   
 $\langle \text{proof} \rangle$

**lemma** *d-test*:  
 $\text{test } (Dom \ R)$   
 $\langle \text{proof} \rangle$

**lemma** *ad-test*:  
 $\text{test } (aDom \ R)$   
 $\langle \text{proof} \rangle$

**lemma** *ad-expl*:  
 $aDom \ R = -((R * 1_{\cup\cup}) \cup\cup 1) \cap 1$   
 $\langle \text{proof} \rangle$

**lemma** *ad-expl-2*:

$aDom (R::('a,'b) mrel) = -((R * (1_{UU}::('b,'a) mrel))\uparrow) \cap (1::('a,'a) mrel)$   
 ⟨proof⟩

**lemma** *aDom*:

$aDom R = \{ (a, \{a\}) \mid a . \neg(\exists B . (a, B) \in R) \}$   
 ⟨proof⟩

**declare** *aDom* [*mr-simp*]

**lemma** *d-down-oi-up-1*:

$Dom (R\downarrow \cap S) = Dom (R \cap S\uparrow)$   
 ⟨proof⟩

**lemma** *d-down-oi-up-2*:

$Dom (R\downarrow \cap S) = Dom (R\downarrow \cap S\uparrow)$   
 ⟨proof⟩

**lemma** *d-ne-down-dp-complement-test*:

**assumes** *test p*

**shows**  $Dom (R \cap -(U * p)) = Dom (ne (R\downarrow) * \wr p)$   
 ⟨proof⟩

**lemma** *d-strict*:

$R = \{\} \longleftrightarrow Dom R = \{\}$   
 ⟨proof⟩

**lemma** *d-sp-strict*:

$R * S = \{\} \longleftrightarrow R * Dom S = \{\}$   
 ⟨proof⟩

**lemma** *d-complement-ad*:

$Dom R = \wr aDom R$   
 ⟨proof⟩

**lemma** *down-sp-below-iu-unit*:

$R\downarrow * S \subseteq 1_{UU} \longleftrightarrow R \subseteq U * aDom (ne S)$   
 ⟨proof⟩

**lemma** *ad-sp-bot*:

$aDom R * R = \{\}$   
 ⟨proof⟩

**lemma** *sp-top-d*:

$R * U \subseteq Dom R * U$   
 ⟨proof⟩

**lemma** *d-sp-top*:

$Dom (R * U) = Dom R$   
 ⟨proof⟩

**lemma** *d-down*:

$$\text{Dom } (R\downarrow) = \text{Dom } R$$

*<proof>*

**lemma** *d-up*:

$$\text{Dom } (R\uparrow) = \text{Dom } R$$

*<proof>*

**lemma** *d-isotone*:

$$R \subseteq S \implies \text{Dom } R \subseteq \text{Dom } S$$

*<proof>*

**lemma** *ad-antitone*:

$$R \subseteq S \implies a\text{Dom } S \subseteq a\text{Dom } R$$

*<proof>*

**lemma** *d-dist-ou*:

$$\text{Dom } (R \cup S) = \text{Dom } R \cup \text{Dom } S$$

*<proof>*

**lemma** *d-dist-iu*:

$$\text{Dom } (R \cup\cup S) = \text{Dom } R * \text{Dom } S$$

*<proof>*

**lemma** *d-dist-ii*:

$$\text{Dom } (R \cap\cap S) = \text{Dom } R * \text{Dom } S$$

*<proof>*

**lemma** *d-loc*:

$$\text{Dom } (R * \text{Dom } S) = \text{Dom } (R * S)$$

*<proof>*

**lemma** *ad-loc*:

$$a\text{Dom } (R * \text{Dom } S) = a\text{Dom } (R * S)$$

*<proof>*

**lemma** *d-ne-down*:

$$\text{Dom } (ne (R\downarrow)) = \text{Dom } (ne R)$$

*<proof>*

**lemma** *ne-sp-iu-unit-up*:

$$ne R = R \implies (R * 1_{\cup\cup})\uparrow = R * U$$

*<proof>*

**lemma** *ne-d-expl*:

$$ne R = R \implies \text{Dom } R = R * U \cap 1$$

*<proof>*

**lemma** *ne-a-expl*:

$$ne\ R = R \implies aDom\ R = -(R * U) \cap 1$$

*<proof>*

**lemma** *d-dist-oU*:

$$Dom\ (\bigcup X) = \bigcup (Dom\ ' X)$$

*<proof>*

**lemma** *d-dist-iU-iI*:

$$Dom\ (\bigcup \bigcup X|I) = Dom\ (\bigcap \bigcap X|I)$$

*<proof>*

**lemma** *d-dist-iU-oI*:

**assumes**  $I \neq \{\}$   
**shows**  $Dom\ (\bigcup \bigcup X|I) = \bigcap (Dom\ ' X\ ' I)$

*<proof>*

### 8.3 Left residual

**definition** *sp-lres* ::  $('a, 'c)\ mrel \Rightarrow ('b, 'c)\ mrel \Rightarrow ('a, 'b)\ mrel$  (**infixl**  $\langle \circ \rangle$  65)

**where**

$$Q \circ R \equiv \{ (a, B) . \forall f . (\forall b \in B . (b, f\ b) \in R) \longrightarrow (a, \bigcup \{ f\ b \mid b . b \in B \}) \in Q \}$$

**declare** *sp-lres-def* [*mr-simp*]

**lemma** *sp-lres-galois*:

$$S * R \subseteq Q \longleftrightarrow S \subseteq Q \circ R$$

*<proof>*

**lemma** *sp-lres-expl*:

$$Q \circ R = \bigcup \{ S . S * R \subseteq Q \}$$

*<proof>*

**lemma** *bot-sp-lres-d*:

$$\{\} \circ R = \{\} \circ Dom\ R$$

*<proof>*

**lemma** *bot-sp-lres-expl*:

$$\{\} \circ R = -(U * Dom\ R)$$

*<proof>*

**lemma** *sp-lres-sp-below*:

$$(Q \circ R) * R \subseteq Q$$

*<proof>*

**lemma** *sp-lres-left-isotone*:

$$Q \subseteq S \implies Q \circ R \subseteq S \circ R$$

*<proof>*

**lemma** *sp-lres-right-antitone*:  
 $S \subseteq R \implies Q \otimes R \subseteq Q \otimes S$   
 ⟨proof⟩

**lemma** *sp-lres-down-closed-1*:  
 $Q \downarrow \otimes R = Q \downarrow \otimes R \downarrow$   
 ⟨proof⟩

**lemma** *sp-lres-down-closed-2*:  
 assumes  $R \downarrow = R$   
 and *total*  $T$   
 shows  $(R \otimes T) \downarrow = R \otimes T$   
 ⟨proof⟩

**lemma** *down-sp-sp*:  
 $R \downarrow * S = R * (1_{UU} \cup S)$   
 ⟨proof⟩

**lemma** *iu-unit-sp-lres-iu-unit-ou*:  
 $U * aDom (ne R) = 1_{UU} \otimes (1_{UU} \cup R)$   
 ⟨proof⟩

**lemma** *bot-sl-below-complement-d*:  
 $\{\} \otimes R \subseteq - Dom R$   
 ⟨proof⟩

**lemma** *sp-unit-oi-bot-sp-lres*:  
 $1 \cap - Dom R = 1 \cap (\{\} \otimes R)$   
 ⟨proof⟩

**lemma** *ad-explicit-d*:  
 $aDom R = -(U * Dom R) \cap 1$   
 ⟨proof⟩

**lemma** *top-test-sp-lres-total-expl-1*:  
 assumes *test*  $p$   
 shows  $\forall S . S \downarrow \subseteq (U * p) \otimes R \iff S \subseteq U * aDom (R \cap -(U * p))$   
 ⟨proof⟩

**lemma** *top-test-sp-lres-total-expl-2*:  
 assumes *test*  $p$   
 and *total*  $T$   
 shows  $(U * p) \otimes T = U * aDom (T \cap -(U * p))$   
 ⟨proof⟩

**lemma** *top-test-sp-lres-total-expl-3*:  
 assumes *test*  $p$   
 shows  $((U * p) \otimes R) \cap 1 = aDom (R \cap -(U * p))$

*<proof>*

**lemma** *top-test-sp-lres-total-expl-4:*

**assumes** *test p*

**shows**  $aDom (ne (R\downarrow) * \wr p) = ((U * p) \circlearrowleft R) \cap 1$

*<proof>*

**lemma** *oi-complement-top-sp-test-top-1:*

**assumes** *test p*

**shows**  $(R \cap -(U * p)) * U = (R\downarrow \cap -(U * p)) * U$

*<proof>*

**lemma** *oi-complement-top-sp-test-top-2:*

**assumes** *test p*

**shows**  $(R\downarrow \cap -(U * p)) * U = ne (R\downarrow) * \wr p * U$

*<proof>*

**lemma** *oi-complement-top-sp-test-top-3:*

**assumes** *test p*

**shows**  $(R\downarrow \cap -(U * p)) * U = ne (R\downarrow) * -(p * U)$

*<proof>*

**lemma** *split-sp-test-2:*

*test p*  $\implies R \subseteq R * p \cup ne (R\downarrow) * (\wr p)\uparrow$

*<proof>*

**lemma** *split-sp-test-3:*

*test p*  $\implies R \subseteq R * p \cup R\downarrow * (\wr p)\uparrow$

*<proof>*

**lemma** *split-sp-test-4:*

**assumes** *test p*

**and** *test q*

**shows**  $R * (p \cup q) \subseteq R * p \cup ne (R\downarrow) * q\uparrow$

*<proof>*

**lemma** *split-sp-test-5:*

**assumes** *test p*

**and** *test q*

**shows**  $R * (p \cup q) \subseteq R * p \cup R\downarrow * q\uparrow$

*<proof>*

**lemma** *split-sp-test-6:*

**assumes** *test p*

**and** *test q*

**shows**  $Dom (R * (p \cup q)) \subseteq Dom (R * p \cup ne (R\downarrow) * q)$

*<proof>*

**lemma** *split-sp-test-7:*

**assumes** *test p*  
**and** *test q*  
**shows**  $Dom (ne (R\downarrow) * (p \cup q)) = Dom (ne (R\downarrow) * p \cup ne (R\downarrow) * q)$   
 $\langle proof \rangle$

**lemma** *test-sp-left-dist-iu-1*:  
 $test\ p \implies p * (R \cup\cup S) = p * R \cup\cup S$   
 $\langle proof \rangle$

**lemma** *test-sp-left-dist-iu-2*:  
 $test\ p \implies p * (R \cup\cup S) = R \cup\cup p * S$   
 $\langle proof \rangle$

**lemma** *d-sp-below-iu-down*:  
 $Dom\ R * S \subseteq (R \cup\cup S)\downarrow$   
 $\langle proof \rangle$

**lemma** *d-sp-ne-down-below-ne-iu-down*:  
 $Dom\ R * ne (S\downarrow) \subseteq ne ((R \cup\cup S)\downarrow)$   
 $\langle proof \rangle$

**lemma** *top-test*:  
 $test\ p \implies U * p = \{ (a, B) . (\forall b \in B . (b, \{b\}) \in p) \}$   
 $\langle proof \rangle$

**lemma** *iu-oi-complement-top-test-ou-up*:  
 $test\ p \implies (R \cup\cup S) \cap -(U * p) \subseteq ((R \cup S) \cap -(U * p))\uparrow$   
 $\langle proof \rangle$

**lemma** *d-ne-iu-down-sp-test-ou*:  
**assumes** *test p*  
**shows**  $Dom (ne ((R \cup\cup S)\downarrow) * p) \subseteq Dom ((ne (R\downarrow) \cup ne (S\downarrow)) * p)$   
 $\langle proof \rangle$

**lemma** *test-sp-left-dist-iU*:  
**assumes** *test p*  
**and**  $I \neq \{\}$   
**shows**  $p * (\cup\cup X|I) = \cup\cup (\lambda i . p * X\ i)|I$   
 $\langle proof \rangle$

## 8.4 Modal operations

**definition** *adia* ::  $('a, 'b) mrel \implies ('b, 'b) mrel \implies ('a, 'a) mrel (\langle | - \rangle \rightarrow [50, 90] 95)$   
**where**

$$|R\rangle p \equiv \{ (a, \{a\}) \mid a . \exists B . (a, B) \in R \wedge (\forall b \in B . (b, \{b\}) \in p) \}$$

**definition** *abox* ::  $('a, 'b) mrel \implies ('b, 'b) mrel \implies ('a, 'a) mrel (\langle | - \rangle \rightarrow [50, 90] 95)$   
**where**

$$|R\rangle p \equiv \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall b \in B . (b, \{b\}) \in p) \}$$

**definition** *edia* :: ('a,'b) mrel  $\Rightarrow$  ('b,'b) mrel  $\Rightarrow$  ('a,'a) mrel ( $\langle \cdot \mid - \rangle$ )  $\rightarrow$  [50,90] 95) **where**

$|R\rangle\rangle p \equiv \{ (a,\{a\}) \mid a . \exists B . (a,B) \in R \wedge (\exists b \in B . (b,\{b\}) \in p) \}$

**definition** *ebox* :: ('a,'b) mrel  $\Rightarrow$  ('b,'b) mrel  $\Rightarrow$  ('a,'a) mrel ( $\langle \cdot \mid - \rangle$ )  $\rightarrow$  [50,90] 95) **where**

$|R\rangle\rangle p \equiv \{ (a,\{a\}) \mid a . \forall B . (a,B) \in R \longrightarrow (\exists b \in B . (b,\{b\}) \in p) \}$

**declare** *adia-def* [*mr-simp*] *abox-def* [*mr-simp*] *edia-def* [*mr-simp*] *ebox-def* [*mr-simp*]

**lemma** *adia*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{Dom } (R * p)$

$\langle \text{proof} \rangle$

**lemma** *abox-1*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{aDom } (R \cap -(U * p))$

$\langle \text{proof} \rangle$

**lemma** *abox*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{aDom } (ne (R\downarrow) * \wr p)$

$\langle \text{proof} \rangle$

**lemma** *edia-1*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{Dom } (R \cap -(U * \wr p))$

$\langle \text{proof} \rangle$

**lemma** *edia*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{Dom } (ne (R\downarrow) * p)$

$\langle \text{proof} \rangle$

**lemma** *ebox*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = \text{aDom } (R * \wr p)$

$\langle \text{proof} \rangle$

**lemma** *abox-2*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = -((R \cap -(U * p)) * U) \cap 1$

$\langle \text{proof} \rangle$

**lemma** *abox-3*:

**assumes** *test p*

**shows**  $|R]p = -(ne (R\downarrow) * \wr p * U) \cap 1$   
*<proof>*

**lemma** *abox-4*:  
**assumes** *test p*  
**shows**  $|R]p = ((U * p) \otimes R) \cap 1$   
*<proof>*

**lemma** *abox-ebox*:  
**assumes** *test p*  
**shows**  $|R]p = |ne (R\downarrow)]p$   
*<proof>*

**lemma** *abox-edia*:  
**assumes** *test p*  
**shows**  $|R]p = \wr |R\rangle(\wr p)$   
*<proof>*

**lemma** *abox-adia*:  
**assumes** *test p*  
**shows**  $|R]p = \wr |ne (R\downarrow)\rangle(\wr p)$   
*<proof>*

**lemma** *edia-adia*:  
**assumes** *test p*  
**shows**  $|R\rangle p = |ne (R\downarrow)\rangle p$   
*<proof>*

**lemma** *edia-abox*:  
**assumes** *test p*  
**shows**  $|R\rangle p = \wr |R](\wr p)$   
*<proof>*

**lemma** *edia-ebox*:  
**assumes** *test p*  
**shows**  $|R\rangle p = \wr |ne (R\downarrow)](\wr p)$   
*<proof>*

**lemma** *abox-ne-down*:  
**assumes** *test p*  
**shows**  $|R]p = |ne (R\downarrow)]p$   
*<proof>*

**lemma** *edia-ne-down*:  
**assumes** *test p*  
**shows**  $|R\rangle p = |ne (R\downarrow)\rangle p$   
*<proof>*

**lemma** *adia-up*:

**assumes**  $test\ p$   
**shows**  $|R\rangle p = |R\uparrow\rangle p$   
 $\langle proof \rangle$

**lemma** *ebox-up*:  
**assumes**  $test\ p$   
**shows**  $|R]]p = |R\uparrow]]p$   
 $\langle proof \rangle$

**lemma** *adia-ebox*:  
**assumes**  $test\ p$   
**shows**  $|R\rangle p = \iota |R]](\iota\ p)$   
 $\langle proof \rangle$

**lemma** *ebox-adia*:  
**assumes**  $test\ p$   
**shows**  $|R]]p = \iota |R\rangle(\iota\ p)$   
 $\langle proof \rangle$

**lemma** *abox-down*:  
**assumes**  $test\ p$   
**shows**  $|R]p = |R\downarrow]p$   
 $\langle proof \rangle$

**lemma** *edia-down*:  
**assumes**  $test\ p$   
**shows**  $|R\rangle\rangle p = |R\downarrow\rangle\rangle p$   
 $\langle proof \rangle$

**lemma** *fusion-oi-complement-top-test-up*:  
 $test\ p \implies fus\ R \cap -(U * p) \subseteq (R \cap -(U * p))\uparrow$   
 $\langle proof \rangle$

**lemma** *adia-left-isotone*:  
 $test\ p \implies R \subseteq S \implies |R\rangle p \subseteq |S\rangle p$   
 $\langle proof \rangle$

**lemma** *adia-right-isotone*:  
 $test\ p \implies test\ q \implies p \subseteq q \implies |R\rangle p \subseteq |R\rangle q$   
 $\langle proof \rangle$

**lemma** *abox-left-antitone*:  
 $test\ p \implies R \subseteq S \implies |S]p \subseteq |R]p$   
 $\langle proof \rangle$

**lemma** *abox-right-isotone*:  
 $test\ p \implies test\ q \implies p \subseteq q \implies |R]p \subseteq |R]q$   
 $\langle proof \rangle$

**lemma** *edia-left-isotone*:

$$\text{test } p \implies R \subseteq S \implies |R\rangle\rangle p \subseteq |S\rangle\rangle p$$

*<proof>*

**lemma** *edia-right-isotone*:

$$\text{test } p \implies \text{test } q \implies p \subseteq q \implies |R\rangle\rangle p \subseteq |R\rangle\rangle q$$

*<proof>*

**lemma** *ebox-left-antitone*:

$$\text{test } p \implies R \subseteq S \implies |S]]p \subseteq |R]]p$$

*<proof>*

**lemma** *ebox-right-isotone*:

$$\text{test } p \implies \text{test } q \implies p \subseteq q \implies |R]]p \subseteq |R]]q$$

*<proof>*

**lemma** *edia-fusion*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = |fus\ R\rangle\rangle p$

*<proof>*

**lemma** *abox-fusion*:

**assumes** *test p*

**shows**  $|R]p = |fus\ R]p$

*<proof>*

**lemma** *abox-fission*:

**assumes** *test p*

**shows**  $|R]p = |fis\ R]p$

*<proof>*

**lemma** *edia-fission*:

**assumes** *test p*

**shows**  $|R\rangle\rangle p = |fis\ R\rangle\rangle p$

*<proof>*

**lemma** *fission-below*:

$$fis\ R \subseteq S \iff (\forall a\ b\ B . (a,B) \in R \wedge b \in B \implies (a,\{b\}) \in S)$$

*<proof>*

**lemma** *below-fission-up*:

$$S \subseteq (fis\ R)\uparrow \iff (\forall a\ B . (a,B) \in S \implies (\exists C . (a,C) \in R \wedge C \cap B \neq \{\}))$$

*<proof>*

**lemma** *ebox-below-abox*:

**assumes** *test p*

**and**  $fis\ R \subseteq S$

**shows**  $|S]]p \subseteq |R]]p$

*<proof>*

**lemma** *abox-below-ebox:*

**assumes** *test p*  
**and**  $S \subseteq (fis\ R)\uparrow$   
**shows**  $|R]p \subseteq |S]]p$   
*<proof>*

**lemma** *abox-eq-ebox:*

**assumes** *test p*  
**and**  $fis\ R \subseteq S$   
**and**  $S \subseteq (fis\ R)\uparrow$   
**shows**  $|R]p = |S]]p$   
*<proof>*

**lemma** *abox-eq-ebox-sufficient:*

$S = fis\ R \vee S = ne\ (R\downarrow) \vee S = (ne\ (R\downarrow))\uparrow \longrightarrow fis\ R \subseteq S \wedge S \subseteq (fis\ R)\uparrow$   
*<proof>*

**lemma** *ebox-fission-abox:*

*test p*  $\implies |R]p = |fis\ R]]p$   
*<proof>*

**lemma** *ebox-down-ne-up-abox:*

*test p*  $\implies |R]p = |(ne\ (R\downarrow))\uparrow]]p$   
*<proof>*

**lemma** *same-fusion:*

**assumes**  $fis\ R \sqsubseteq\downarrow S$   
**and**  $S \sqsubseteq\downarrow fus\ R$   
**shows**  $fis\ R = fis\ S$   
*<proof>*

**lemma** *same-abox:*

**assumes**  $fis\ R \sqsubseteq\downarrow S$   
**and**  $S \sqsubseteq\downarrow fus\ R$   
**and** *test p*  
**shows**  $|R]p = |S]p$   
*<proof>*

**lemma** *abox-ebox-inner-deterministic:*

**assumes** *test p*  
**and** *inner-deterministic R*  
**shows**  $|R]p = |R]]p$   
*<proof>*

**lemma** *adia-edia-inner-deterministic:*

**assumes** *test p*  
**and** *inner-deterministic R*  
**shows**  $|R\rangle p = |R\rangle\rangle p$

$\langle proof \rangle$

**lemma** *abox-adia-deterministic*:

**assumes** *test p*  
**and** *deterministic R*  
**shows**  $|R]p = |R\rangle p$   
 $\langle proof \rangle$

**lemma** *ebox-edia-deterministic*:

**assumes** *test p*  
**and** *deterministic R*  
**shows**  $|R]]p = |R\rangle\rangle p$   
 $\langle proof \rangle$

**lemma** *abox-ebox-fusion*:

**assumes** *test p*  
**shows**  $|fis R]p = |fis R]]p$   
 $\langle proof \rangle$

**lemma** *abox-fission-edia-fusion*:

**assumes** *test p*  
**shows**  $|fis R]p = |fus R\rangle p$   
 $\langle proof \rangle$

**lemma** *abox-adia-fusion*:

**assumes** *test p*  
**shows**  $|fus R]p = |fus R\rangle p$   
 $\langle proof \rangle$

## 8.5 Goldblatt's axioms without star

**lemma** *abox-sp-unit*:

$|R]1 = 1$   
 $\langle proof \rangle$

**lemma** *ou-unit-abox*:

$test\ p \implies |\{\}p = 1$   
 $\langle proof \rangle$

**lemma** *ou-unit-test-implication*:

$test\ p \implies \{\} \rightarrow p = 1$   
 $\langle proof \rangle$

**lemma** *sp-unit-abox*:

$test\ p \implies |1]p = p$   
 $\langle proof \rangle$

**lemma** *sp-unit-test-implication*:

$test\ p \implies 1 \rightarrow p = p$

$\langle proof \rangle$

**lemma** *test-abox-ebox*:

$test\ p \implies test\ q \implies |q]p = |q]]p$

$\langle proof \rangle$

**lemma** *test-abox*:

$test\ p \implies test\ q \implies |q]p = q \rightarrow p$

$\langle proof \rangle$

**lemma** *abox-ou-adia-sp-unit*:

**assumes** *test p*

**shows**  $|R]p \cup |R]1 = 1$

$\langle proof \rangle$

**lemma** *d-test-sp*:

$test\ p \implies Dom\ (p * R) = p * Dom\ R$

$\langle proof \rangle$

**lemma** *ad-test-sp*:

$test\ p \implies aDom\ (p * R) = \wr p \cup aDom\ R$

$\langle proof \rangle$

**lemma** *adia-test-sp*:

$test\ p \implies test\ q \implies |p * R\rangle q = p * |R\rangle q$

$\langle proof \rangle$

**lemma** *ebox-test-sp*:

$test\ p \implies test\ q \implies |p * R]]q = \wr p \cup |R]]q$

$\langle proof \rangle$

**lemma** *abox-test-sp*:

**assumes** *test p*

**and** *test q*

**shows**  $|p * R]q = \wr p \cup |R]q$

$\langle proof \rangle$

**lemma** *abox-test-sp-2*:

$test\ p \implies test\ q \implies p \cup |R]q = |\wr p * R]q$

$\langle proof \rangle$

**lemma** *abox-test-sp-3*:

$test\ p \implies test\ q \implies p \rightarrow |R]q = |p * R]q$

$\langle proof \rangle$

**lemma** *fission-sp-dist*:

$fis\ (R * S) = fis\ (R * Dom\ S) * fis\ S$

$\langle proof \rangle$

**lemma** *abox-test*:

*test*  $p \implies \text{test} ( |R\rangle p )$

*<proof>*

**lemma** *adia-test*:

*test*  $p \implies \text{test} ( |R\rangle p )$

*<proof>*

**lemma** *ebox-test*:

*test*  $p \implies \text{test} ( |R\rangle\!\rangle p )$

*<proof>*

**lemma** *edia-test*:

*test*  $p \implies \text{test} ( |R\rangle\!\rangle p )$

*<proof>*

**lemma** *abox-sp*:

**assumes** *test*  $p$

**and** *test*  $q$

**shows**  $|R\rangle(p * q) = |R\rangle p * |R\rangle q$

*<proof>*

**lemma** *adia-ou-below-ne-down*:

**assumes** *test*  $p$

**shows**  $|R\rangle(p \cup \wr q) \subseteq |R\rangle p \cup |ne (R\downarrow)\rangle(\wr q)$

*<proof>*

**lemma** *abox-adia-mp*:

**assumes** *test*  $p$

**and** *test*  $q$

**shows**  $|R\rangle(p \rightarrow q) * |R\rangle p \subseteq |R\rangle q$

*<proof>*

**lemma** *adia-abox-mp*:

**assumes** *test*  $p$

**and** *test*  $q$

**shows**  $|R\rangle p * |R\rangle(p \rightarrow q) \subseteq |R\rangle q$

*<proof>*

**lemma** *abox-implication-adia*:

**assumes** *test*  $p$

**and** *test*  $q$

**shows**  $|R\rangle(p \rightarrow q) \subseteq |R\rangle p \rightarrow |R\rangle q$

*<proof>*

**lemma** *abox-adia-implication*:

**assumes** *test*  $p$

**and** *test*  $q$

**shows**  $|R\rangle p \subseteq |R\rangle q \rightarrow |R\rangle(p * q)$

$\langle proof \rangle$

**lemma** *abox-mp*:

**assumes** *test p*

**and** *test q*

**shows**  $|R]p * |R](p \rightarrow q) \subseteq |R]q$

$\langle proof \rangle$

**lemma** *abox-implication*:

**assumes** *test p*

**and** *test q*

**shows**  $|R](p \rightarrow q) \subseteq |R]p \rightarrow |R]q$

$\langle proof \rangle$

**lemma** *ebox-left-dist-ou*:

**assumes** *test p*

**shows**  $|R \cup S]]p = |R]]p * |S]]p$

$\langle proof \rangle$

**lemma** *abox-left-dist-ou*:

**assumes** *test p*

**shows**  $|R \cup S]p = |R]p * |S]p$

$\langle proof \rangle$

**lemma** *adia-left-dist-ou*:

**assumes** *test p*

**shows**  $|R \cup S\rangle p = |R\rangle p \cup |S\rangle p$

$\langle proof \rangle$

**lemma** *edia-left-dist-ou*:

**assumes** *test p*

**shows**  $|R \cup S\rangle\rangle p = |R\rangle\rangle p \cup |S\rangle\rangle p$

$\langle proof \rangle$

**lemma** *abox-dist-iu-1*:

**assumes** *test p*

**shows**  $|R \cupcup S]p = |Dom R * ne (S\downarrow)]]p * |Dom S * ne (R\downarrow)]]p$

$\langle proof \rangle$

**lemma** *abox-dist-iu-2*:

**assumes** *test p*

**shows**  $|R \cupcup S]p = |Dom R * S]p * |Dom S * R]p$

$\langle proof \rangle$

**lemma** *abox-dist-iu-3*:

**assumes** *test p*

**shows**  $|R \cupcup S]p = ( |R]1 \rightarrow |S]p ) * ( |S]1 \rightarrow |R]p )$

$\langle proof \rangle$

**lemma** *abox-adia-sp-one-set*:

$$|R||S\rangle 1 = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall b \in B . \exists D . (b, D) \in S) \}$$

*<proof>*

**lemma** *abox-abox-set*:

$$|R||S\rangle p = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall C . (\exists b \in B . (b, C) \in S) \longrightarrow (\forall c \in C . (c, \{c\}) \in p)) \}$$

*<proof>*

**lemma** *sp-abox-set*:

$$|R * S\rangle p = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall C . (\exists f . (\forall b \in B . (b, f b) \in S) \wedge C = \bigcup \{ f b \mid b . b \in B \}) \longrightarrow (\forall c \in C . (c, \{c\}) \in p)) \}$$

*<proof>*

**lemma** *abox-sp-1*:

**assumes** *test p*

**shows**  $|R||S\rangle 1 * |R * S\rangle p \subseteq |R||S\rangle p$

*<proof>*

**lemma** *abox-sp-2*:

**assumes** *test p*

**shows**  $|R||S\rangle p = |R\downarrow * S\rangle p$

*<proof>*

**lemma** *abox-sp-3*:

**assumes** *test p*

**shows**  $|R||S\rangle p \subseteq |R * S\rangle p$

*<proof>*

**lemma** *abox-sp-4*:

**assumes** *test p*

**shows**  $|R * S\rangle p \subseteq |R||S\rangle 1 \rightarrow |R||S\rangle p$

*<proof>*

**lemma** *abox-sp-5*:

**assumes** *test p*

**shows**  $|R||S\rangle 1 * |R * S\rangle p = |R||S\rangle 1 * |R||S\rangle p$

*<proof>*

**lemma** *abox-sp-6*:

**assumes** *test p*

**shows**  $|R||S\rangle 1 \rightarrow |R * S\rangle p = |R||S\rangle 1 \rightarrow |R||S\rangle p$

*<proof>*

**lemma** *abox-sp-7*:

**assumes** *test p*

**and** *total S*

**shows**  $|R * S\rangle p = |R||S\rangle p$

*<proof>*

**lemma** *adia-sp-associative*:  
**assumes** *test p*  
**shows**  $|Q * (R * S)\rangle p = |(Q * R) * S\rangle p$   
 $\langle proof \rangle$

**lemma** *ebox-sp-associative*:  
**assumes** *test p*  
**shows**  $|Q * (R * S)] p = |(Q * R) * S] p$   
 $\langle proof \rangle$

**lemma** *edia-sp-associative*:  
**assumes** *test p*  
**shows**  $|Q * (R * S)\rangle\rangle p = |(Q * R) * S\rangle\rangle p$   
 $\langle proof \rangle$

**lemma** *abox-sp-associative*:  
**assumes** *test p*  
**shows**  $|Q * (R * S)] p = |(Q * R) * S] p$   
 $\langle proof \rangle$

**lemma** *abox-oI*:  
**assumes**  $X \neq \{\}$   
**shows**  $|R] \cap X = (\bigcap p \in X . |R] p)$   
 $\langle proof \rangle$

**lemma** *ebox-left-dist-oU*:  
**assumes**  $X \neq \{\}$   
**shows**  $|\bigcup X]] p = (\bigcap R \in X . |R]] p)$   
 $\langle proof \rangle$

**lemma** *abox-left-dist-oU*:  
**assumes**  $X \neq \{\}$   
**shows**  $|\bigcup X] p = (\bigcap R \in X . |R] p)$   
 $\langle proof \rangle$

**lemma** *adia-left-dist-oU*:  
 $|\bigcup X\rangle p = (\bigcup R \in X . |R\rangle p)$   
 $\langle proof \rangle$

**lemma** *edia-left-dist-oU*:  
 $|\bigcup X\rangle\rangle p = (\bigcup R \in X . |R\rangle\rangle p)$   
 $\langle proof \rangle$

## 8.6 Goldblatt's axioms with star

**unbundle** *no rtrancl-syntax*  
**notation** *star*  $\langle \leftarrow^* \rangle$  [1000] 999

**lemma** *star-induct-1*:

**assumes**  $1 \subseteq X$   
**and**  $R * X \subseteq X$   
**shows**  $R^* \subseteq X$   
*<proof>*

**lemma** *star-induct*:

**assumes**  $S \subseteq 1 \cup 1_{\cup\cup}$   
**and**  $S \subseteq X$   
**and**  $R * X \subseteq X$   
**shows**  $R^* * S \subseteq X$   
*<proof>*

**lemma** *star-total*:

*total* ( $R^*$ )  
*<proof>*

**lemma** *star-down*:

$R^* \downarrow = (R \downarrow)^* \cup 1_{\cup\cup}$   
*<proof>*

**lemma** *ne-star-down*:

$ne (R^* \downarrow) = ne ((R \downarrow)^*)$   
*<proof>*

**lemma** *ne-down-star*:

$ne ((R \downarrow)^*) = (ne (R \downarrow))^*$   
*<proof>*

**lemma** *abox-star-unfold*:

$test p \implies |R^*]p = p * |R]|R^*]p$   
*<proof>*

**lemma** *star-sp-test-commute*:

**assumes**  $S \subseteq 1 \cup 1_{\cup\cup}$   
**and**  $Q * S \subseteq S * R$   
**shows**  $Q^* * S \subseteq S * R^*$   
*<proof>*

**lemma** *adia-star-induct*:

**assumes** *test*  $p$   
**shows**  $|R]p \subseteq p \longleftrightarrow |R^*]p \subseteq p$   
*<proof>*

**lemma** *ebox-star-induct*:

**assumes** *test*  $p$   
**shows**  $p \subseteq |R]]p \longleftrightarrow p \subseteq |R^*]]p$   
*<proof>*

**lemma** *abox-star-induct*:  
**assumes** *test p*  
**shows**  $p \subseteq |R]p \longleftrightarrow p \subseteq |R^*]p$   
 $\langle proof \rangle$

**lemma** *edia-star-induct*:  
**assumes** *test p*  
**shows**  $|R\rangle\rangle p \subseteq p \longleftrightarrow |R^*\rangle\rangle p \subseteq p$   
 $\langle proof \rangle$

**lemma** *abox-star-induct-1*:  
**assumes** *test p*  
**and** *test q*  
**and**  $q \subseteq p * |R]q$   
**shows**  $q \subseteq |R^*]p$   
 $\langle proof \rangle$

**lemma** *adia-star-induct-1*:  
**assumes** *test p*  
**and** *test q*  
**and**  $p \cup |R\rangle q \subseteq q$   
**shows**  $|R^*\rangle p \subseteq q$   
 $\langle proof \rangle$

**lemma** *abox-segerberg*:  
**assumes** *test p*  
**shows**  $|R^*](p \rightarrow |R]p) \subseteq p \rightarrow |R^*]p$   
 $\langle proof \rangle$

**lemma** *abox-segerberg-adia*:  
**assumes** *test p*  
**shows**  $|R^*](|R\rangle p \rightarrow p) \subseteq |R^*\rangle p \rightarrow p$   
 $\langle proof \rangle$

**lemma** *s-p-id-sp*:  
 $(s-id \cup p-id) * R = R \cup p-id$   
 $\langle proof \rangle$

## 8.7 Propositional Hoare logic

**abbreviation** *hoare*  $:: ('a,'a) mrel \Rightarrow ('a,'b) mrel \Rightarrow ('b,'b) mrel \Rightarrow bool$  ( $\langle -\{\} - \rangle$ )  
 $[50,60,50] 95)$   
**where**  $p\{\!|R|\!\}q \equiv p \subseteq |R]q$

**abbreviation** *if-then-else*  $:: ('a,'a) mrel \Rightarrow ('a,'b) mrel \Rightarrow ('a,'b) mrel \Rightarrow ('a,'b) mrel$   
**where** *if-then-else*  $p R S \equiv p * R \cup \wr p * S$

**abbreviation** *while-do*  $:: ('a,'a) mrel \Rightarrow ('a,'a) mrel \Rightarrow ('a,'a) mrel$

where  $\text{while-do } p \ R \equiv (p * R)^* * \wr p$

**lemma** *hoare-skip*:

assumes  $\text{test } p$   
shows  $p \{I\} p$   
*<proof>*

**lemma** *hoare-cons*:

assumes  $\text{test } s$   
and  $r \subseteq p$   
and  $q \subseteq s$   
and  $p \{R\} q$   
shows  $r \{R\} s$   
*<proof>*

**lemma** *hoare-seq*:

assumes  $\text{test } q$   
and  $\text{test } r$   
and  $p \{R\} q$   
and  $q \{S\} r$   
shows  $p \{R * S\} r$   
*<proof>*

**lemma** *hoare-if*:

assumes  $\text{test } p$   
and  $\text{test } q$   
and  $\text{test } r$   
and  $(p * q) \{R\} r$   
and  $((\wr p) * q) \{S\} r$   
shows  $q \{\text{if-then-else } p \ R \ S\} r$   
*<proof>*

**lemma** *hoare-while*:

assumes  $\text{test } p$   
and  $\text{test } q$   
and  $(p * q) \{R\} q$   
shows  $q \{\text{while-do } p \ R\} (q * (\wr p))$   
*<proof>*

**lemma** *hoare-par*:

assumes  $\text{test } q$   
and  $p \{R\} q$   
and  $p \{S\} q$   
shows  $p \{R \cup S\} q$   
*<proof>*

## 9 Counterexamples

locale *counterexamples*

**begin**

**lemma counter-01:**

$\neg ((U::('a,'b) \text{ mrel}) * \neg((U::('b,'c) \text{ mrel}) * (R::('c,'d) \text{ mrel})) \subseteq \neg(U * R))$   
*<proof>*

**abbreviation a-1**  $\equiv$  *finite-1.a<sub>1</sub>*

**lemma counter-02:**

$\exists R::(\text{Enum.finite-1}, \text{Enum.finite-1}) \text{ mrel} . \exists p . \neg (\text{test } p \longrightarrow (R \cap \neg(U * p)) * U = R * \neg(p * U))$   
*<proof>*

**lemma counter-03:**

$\exists R::(\text{Enum.finite-1}, \text{Enum.finite-1}) \text{ mrel} . \exists p . \neg (\text{test } p \longrightarrow (R \cap \neg(U * p)) * 1_{\cup\cup} = R * (\neg(p * U) \cap 1_{\cup\cup}))$   
*<proof>*

**abbreviation b-1**  $\equiv$  *finite-2.a<sub>1</sub>*

**abbreviation b-2**  $\equiv$  *finite-2.a<sub>2</sub>*

**abbreviation b-1-0**  $\equiv$   $(b-1, \{\})$

**abbreviation b-1-1**  $\equiv$   $(b-1, \{b-1\})$

**abbreviation b-1-2**  $\equiv$   $(b-1, \{b-2\})$

**abbreviation b-1-3**  $\equiv$   $(b-1, \{b-1, b-2\})$

**abbreviation b-2-0**  $\equiv$   $(b-2, \{\})$

**abbreviation b-2-1**  $\equiv$   $(b-2, \{b-1\})$

**abbreviation b-2-2**  $\equiv$   $(b-2, \{b-2\})$

**abbreviation b-2-3**  $\equiv$   $(b-2, \{b-1, b-2\})$

**lemma counter-04:**

$\exists R::(\text{Enum.finite-2}, \text{Enum.finite-2}) \text{ mrel} . \exists p q . \neg (\text{test } p \longrightarrow \text{test } q \longrightarrow |R * p]q = |R][p]q)$   
*<proof>*

**lemma counter-05:**

$\neg (\exists f . \forall R p . \text{test } p \longrightarrow |R]p = |f R]p)$   
*<proof>*

**lemma counter-06:**

$\neg (\exists f . \forall R p . \text{test } p \longrightarrow |R]]p = |f R]p)$   
*<proof>*

**lemma counter-07:**

$\neg (\exists f . \text{mono } f \wedge (\forall R . \text{fus } R = \text{lfp } (\lambda X . f R X)))$   
*<proof>*

**abbreviation c-1**  $\equiv$  *finite-3.a<sub>1</sub>*

**abbreviation c-2**  $\equiv$  *finite-3.a<sub>2</sub>*

**abbreviation c-3**  $\equiv$  *finite-3.a<sub>3</sub>*

**lemma counter-08:**

$\neg (\sim(1::(\text{Enum.finite-3}, \text{Enum.finite-3}) \text{ mrel}) * \sim 1 \in \{1, \sim 1\})$   
 $\langle \text{proof} \rangle$

**lemma counter-09:**

$\neg (\sim(1::(\text{Enum.finite-3}, \text{Enum.finite-3}) \text{ mrel}) \odot 1 \in \{1, \sim 1\})$   
 $\langle \text{proof} \rangle$

**lemma ex-2-cases:**

$\exists b. b = b-1 \vee b = b-2$   
 $\langle \text{proof} \rangle$

**lemma all-2-cases:**

$(\forall b. b = b-2 \wedge b = b-1) = \text{False}$   
 $\langle \text{proof} \rangle$

**lemma impl-2-cases:**

$\bigcup \{ X . \exists b. (b = b-1 \longrightarrow X = Y) \wedge (b = b-2 \longrightarrow X = Z) \} = Y \cup Z$   
 $\langle \text{proof} \rangle$

**lemma ex-2-set-cases:**

$(\exists B::\text{Enum.finite-2 set} . P B) \longleftrightarrow P \{ \} \vee P \{ b-1 \} \vee P \{ b-2 \} \vee P \{ b-1, b-2 \}$   
 $\langle \text{proof} \rangle$

**abbreviation**  $B-0 \equiv \{ \}::\text{Enum.finite-2 set}$

**abbreviation**  $B-1 \equiv \{ b-1 \}$

**abbreviation**  $B-2 \equiv \{ b-2 \}$

**abbreviation**  $B-3 \equiv \{ b-1, b-2 \}$

**abbreviation**  $\text{mkf } x \ y \equiv \lambda z . \text{ if } z = b-1 \text{ then } x \text{ else } y$

**lemma mkf:**

$f = \text{mkf } (f \ b-1) \ (f \ b-2)$   
 $\langle \text{proof} \rangle$

**lemma mkf2:**

$f \ b-1 = X \wedge f \ b-2 = Y \implies f = \text{mkf } X \ Y$   
 $\langle \text{proof} \rangle$

**lemma ex-2-mrel-cases:**

$(\exists f::\text{Enum.finite-2} \implies \text{Enum.finite-2 set} . P f) \longleftrightarrow$   
 $P (\text{mkf } B-0 \ B-0) \vee P (\text{mkf } B-0 \ B-1) \vee P (\text{mkf } B-0 \ B-2) \vee P (\text{mkf } B-0 \ B-3) \vee$   
 $P (\text{mkf } B-1 \ B-0) \vee P (\text{mkf } B-1 \ B-1) \vee P (\text{mkf } B-1 \ B-2) \vee P (\text{mkf } B-1 \ B-3) \vee$   
 $P (\text{mkf } B-2 \ B-0) \vee P (\text{mkf } B-2 \ B-1) \vee P (\text{mkf } B-2 \ B-2) \vee P (\text{mkf } B-2 \ B-3) \vee$   
 $P (\text{mkf } B-3 \ B-0) \vee P (\text{mkf } B-3 \ B-1) \vee P (\text{mkf } B-3 \ B-2) \vee P (\text{mkf } B-3 \ B-3)$   
 $\langle \text{proof} \rangle$

**lemma counter-10:**

$\exists R::(\text{Enum.finite-2}, \text{Enum.finite-2}) \text{ mrel} . \neg (U::(\text{Enum.finite-2}, \text{Enum.finite-2}))$

$mrel) * (U * R) \subseteq U * R$   
*<proof>*

**lemma** *counter-11*:

$\exists (R::(Enum.finite-2,Enum.finite-2) mrel) (s::(Enum.finite-2,Enum.finite-2) mrel) (t::(Enum.finite-2,Enum.finite-2) mrel) . \neg (inner-univalent s \wedge inner-univalent t \longrightarrow R * (s * t) = (R * s) * t)$   
*<proof>*

**lemma** *counter-12*:

$\neg(\exists S . 1_{UU} \odot S = 1_{UU})$   
*<proof>*

**lemma** *counter-13*:

$\neg(\exists S . \forall R . R \odot S = R)$   
*<proof>*

**end**

**end**

## References

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