Inner Structure, Determinism and Modal Algebra of Multirelations

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Abstract

Binary multirelations form a model of alternating nondeterminism useful for analysing games, interactions of computing systems with their environments or abstract interpretations of probabilistic programs. We investigate this alternating structure in a relational language based on power allegories extended with specific operations on multirelations. We develop algebras of modal operators over multirelations, related to concurrent dynamic logics, in this language.

Contents

The theories formally verify results in [3, 1, 2]. See these papers for further details and related work.

The basic algebra of homogeneous binary multirelations is formalised in [4]. The present theories consider heterogeneous binary multirelations, which may have different source and target sets. While homogeneous multirelations arise as a special case where source and target sets coincide, we do not attempt to generalise the algebras of [4] to the heterogeneous case but study new concepts instead. Thus the present theories and [4] are complementary. A unification of the two approaches based on category theory is possible future work.

Algebraic structures for multirelations with Parikh composition are formalised in [5].

1 Properties of Binary Relations

theory *Relational-Properties*

imports *Main*

begin

This is a general-purpose theory for enrichments of Rel, which is still quite basic, but helpful for developing properties of multirelations.

notation *relcomp* (**infixl** ; *75*)
notation *converse* ($-^{\smile}$ [*1000*] *999*)

type-synonym $('a,'b)$ *rel* $= ('a \times 'b)$ *set*

lemma *modular-law*: $R \mathbin{;} S \cap T \subseteq (R \cap T \mathbin{;} S^{\smile}) \mathbin{;} S$
⟨*proof*⟩

lemma *compl-conv*: $-(R^{\smile}) = (-R)^{\smile}$
⟨*proof*⟩

definition *top* :: $('a,'b)$ *rel* **where**
top $= \{(a,b) \mid a\ b.\ True\}$

abbreviation *neg* $R \equiv Id \cap -R$

1.1 Univalence, totality, determinism, and related properties

definition *univalent* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
univalent $R = (R^{\smile} \mathbin{;} R \subseteq Id)$

definition *total* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
total $R = (Id \subseteq R \ ; \ R^{\smile})$

definition *injective* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
injective $R = (R \ ; \ R^{\smile} \subseteq Id)$

definition *surjective* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
surjective $R = (Id \subseteq R^{\smile} \ ; \ R)$

definition *deterministic* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
deterministic $R = (\mathit{univalent}\ R \wedge \mathit{total}\ R)$

definition *bijective* :: $('a,'b)$ *rel* $\Rightarrow$ *bool* **where**
bijective $R = (\mathit{injective}\ R \wedge \mathit{surjective}\ R)$

lemma *univalent-set*: *univalent* $R = (\forall\, a\ b\ c.\ (a,b) \in R \wedge (a,c) \in R \longrightarrow b = c)$
⟨*proof*⟩

Univalent relations feature as single-valued relations in Main.

lemma *univ-single-valued*: *univalent* R = *single-valued* R
⟨*proof*⟩

lemma *total-set*: *total* $R = (\forall\, a.\ \exists\, b.\ (a,b) \in R)$
⟨*proof*⟩

lemma *total-var*: *total* $R = (R \ ; \ \mathit{top} = \mathit{top})$
⟨*proof*⟩

lemma *deterministic-set*: *deterministic* $R = (\forall\, a\ .\ \exists!B\ .\ (a,B) \in R)$
⟨*proof*⟩

lemma *deterministic-var1*: *deterministic* $R = (R \ ; \ -Id = -R)$
⟨*proof*⟩

lemma *deterministic-var2*: *deterministic* $R = (\forall\, S.\ R \ ; \ -S = -(R \ ; \ S))$
⟨*proof*⟩

lemma *inj-univ*: *injective* R = *univalent* $(R^{\smile})$
⟨*proof*⟩

lemma *injective-set*: *injective* $S = (\forall\, a\ b\ c.\ (a,c) \in S \wedge (b,c) \in S \longrightarrow a = b)$
⟨*proof*⟩

lemma *surj-tot*: *surjective* R = *total* $(R^{\smile})$
⟨*proof*⟩

lemma *surjective-set*: *surjective* $S = (\forall\, b.\ \exists\, a.\ (a,b) \in S)$
⟨*proof*⟩

lemma *surj-var*: *surjective* $R = (R^{\smile}$; $top = top)$
⟨*proof*⟩

lemma *bij-det*: *bijective* R = *deterministic* $(R^{\smile})$
⟨*proof*⟩

lemma *univ-relcomp*: *univalent* $R \Longrightarrow$ *univalent* $S \Longrightarrow$ *univalent* $(R ; S)$
⟨*proof*⟩

lemma *tot-relcomp*: *total* $R \Longrightarrow$ *total* $S \Longrightarrow$ *total* $(R ; S)$
⟨*proof*⟩

lemma *det-relcomp*: *deterministic* $R \Longrightarrow$ *deterministic* $S \Longrightarrow$ *deterministic* $(R ; S)$
⟨*proof*⟩

lemma *inj-relcomp*: *injective* $R \Longrightarrow$ *injective* $S \Longrightarrow$ *injective* $(R ; S)$
⟨*proof*⟩

lemma *surj-relcomp*: *surjective* $R \Longrightarrow$ *surjective* $S \Longrightarrow$ *surjective* $(R ; S)$
⟨*proof*⟩

lemma *bij-relcomp*: *bijective* $R \Longrightarrow$ *bijective* $S \Longrightarrow$ *bijective* $(R ; S)$
⟨*proof*⟩

lemma *det-Id*: *deterministic Id*
⟨*proof*⟩

lemma *bij-Id*: *bijective Id*
⟨*proof*⟩

lemma *tot-top*: *total top*
⟨*proof*⟩

lemma *tot-surj*: *surjective top*
⟨*proof*⟩

lemma *det-meet-distl*: *univalent* $R \Longrightarrow R ; (S \cap T) = R ; S \cap R ; T$
⟨*proof*⟩

lemma *inj-meet-distr*: *injective* $T \Longrightarrow (R \cap S) ; T = R ; T \cap S ; T$
⟨*proof*⟩

lemma *univ-modular*: *univalent* $S \Longrightarrow R ; S \cap T = (R \cap T ; S^{\smile}) ; S$
⟨*proof*⟩

1.2 Inverse image and the diagonal and graph functors

definition *Invim* :: $('a,'b)$ *rel* $\Rightarrow$ *'b set* $\Rightarrow$ *'a set* **where**

Invim R = Image $(R^\smile)$

definition *Delta* :: *'a set* ⇒ *('a,'a) rel* (Δ) **where**
Δ *P* = {(*p*,*p*) |*p*. *p* ∈ *P*}

definition *Grph* :: *('a* ⇒ *'b)* ⇒ *('a,'b) rel* **where**
Grph f = {(*x*,*y*). *y* = *f x*}

lemma *Image-Grph* [*simp*]: *Image* ∘ *Grph* = *image*
⟨*proof*⟩

1.3 Relational domain, codomain and modalities

Domain and codomain (range) maps have been defined in Main, but they return sets instead of relations.

definition *dom* :: *('a,'b) rel* ⇒ *('a,'a) rel* **where**
dom R = *Id* ∩ *R* ; $R^\smile$

definition *cod* :: *('a,'b) rel* ⇒ *('b,'b) rel* **where**
cod R = *dom* $(R^\smile)$

definition *rel-fdia* :: *('a,'b) rel* ⇒ *('b,'b) rel* ⇒ *('a,'a) rel* ((|-⟩-) [*61*,*81*] *82*)
where
|*R*⟩ *Q* = *dom* (*R* ; *dom Q*)

definition *rel-bdia* :: *('a,'b) rel* ⇒ *('a,'a) rel* ⇒ *('b,'b) rel* ((⟨-|-) [*61*,*81*] *82*)
where
rel-bdia R = *rel-fdia* $(R^\smile)$

definition *rel-fbox* :: *('a,'b) rel* ⇒ *('b,'b) rel* ⇒ *('a,'a) rel* ((|-]-) [*61*,*81*] *82*)
where
|*R*]*Q* = *neg* (*dom* (*R* ; *neg* (*dom Q*)))

definition *rel-bbox* :: *('a,'b) rel* ⇒ *('a,'a) rel* ⇒ *('b,'b) rel* (([-|-) [*61*,*81*] *82*)
where
rel-bbox R = *rel-fbox* $(R^\smile)$

lemma *rel-bdia-def-var*: *rel-bdia* = *rel-fdia* ∘ *converse*
⟨*proof*⟩

lemma *dom-set*: *dom R* = {(*a*,*a*) |*a*. ∃ *b*. (*a*,*b*) ∈ *R*}
⟨*proof*⟩

lemma *dom-Domain*: *dom* = Δ ∘ *Domain*
⟨*proof*⟩

lemma *cod-set*: *cod R* = {(*b*,*b*) |*b*. ∃ *a*. (*a*,*b*) ∈ *R*}
⟨*proof*⟩

lemma *cod-Range*: $cod = \Delta \circ Range$
$\langle proof \rangle$

lemma *rel-fdia-set*: $|R\rangle\ Q = \{(a,a)\ |a.\ \exists\, b.\ (a,b) \in R \wedge (b,b) \in dom\ Q\}$
$\langle proof \rangle$

lemma *rel-bdia-set*: $\langle R|\ P = \{(b,b)\ |b.\ \exists\, a.\ (a,b) \in R \wedge (a,a) \in dom\ P\}$
$\langle proof \rangle$

lemma *rel-fbox-set*: $|R]\ Q = \{(a,a)\ |a.\ \forall\, b.\ (a,b) \in R \longrightarrow (b,b) \in dom\ Q\}$
$\langle proof \rangle$

lemma *rel-bbox-set*: $[R|\ P = \{(b,b)\ |b.\ \forall\, a.\ (a,b) \in R \longrightarrow (a,a) \in dom\ P\}$
$\langle proof \rangle$

lemma *dom-alt-def*: $dom\ R = Id \cap R\ ;\ top$
$\langle proof \rangle$

lemma *dom-gla*: $(dom\ R \subseteq Id \cap S) = (R \subseteq (Id \cap S)\ ;\ R)$
$\langle proof \rangle$

lemma *dom-gla-top*: $(dom\ R \subseteq Id \cap S) = (R \subseteq (Id \cap S)\ ;\ top)$
$\langle proof \rangle$

lemma *dom-subid*: $(dom\ R = R) = (R = Id \cap R)$
$\langle proof \rangle$

lemma *dom-cod*: $(dom\ R = R) = (cod\ R = R)$
$\langle proof \rangle$

lemma *dom-top*: $R\ ;\ top = dom\ R\ ;\ top$
$\langle proof \rangle$

lemma *top-dom*: $dom\ R = dom\ (R\ ;\ top)$
$\langle proof \rangle$

lemma *cod-top*: $cod\ R = Id \cap top\ ;\ R$
$\langle proof \rangle$

lemma *dom-conv* [*simp*]: $(dom\ R)^{\smile} = dom\ R$
$\langle proof \rangle$

lemma *total-dom*: $total\ R = (dom\ R = Id)$
$\langle proof \rangle$

lemma *surj-cod*: $surjective\ R = (cod\ R = Id)$
$\langle proof \rangle$

lemma *fdia-demod*: $(\ |R\rangle\ P \subseteq dom\ Q) = (R\ ;\ dom\ P \subseteq dom\ Q\ ;\ R)$

⟨*proof*⟩

lemma *bbox-demod*: $(dom\ P \subseteq [R|\ Q) = (R\ ;\ dom\ P \subseteq dom\ Q\ ;\ R)$
⟨*proof*⟩

lemma *bdia-demod*: $(\langle R|\ P \subseteq dom\ Q) = (dom\ P\ ;\ R \subseteq R\ ;\ dom\ Q)$
⟨*proof*⟩

lemma *fbox-demod*: $(dom\ P \subseteq |R]\ Q) = (dom\ P\ ;\ R \subseteq R\ ;\ dom\ Q)$
⟨*proof*⟩

lemma *fdia-demod-top*: $(\ |R\rangle\ P \subseteq dom\ Q) = (R\ ;\ dom\ P\ ;\ top \subseteq dom\ Q\ ;\ top)$
⟨*proof*⟩

lemma *bbox-demod-top*: $(dom\ P \subseteq [R|\ Q) = (R\ ;\ dom\ P\ ;\ top \subseteq dom\ Q\ ;\ top)$
⟨*proof*⟩

lemma *fdia-bbox-galois*: $(\ |R\rangle\ P \subseteq dom\ Q) = (dom\ P \subseteq [R|\ Q)$
⟨*proof*⟩

lemma *bdia-fbox-galois*: $(\langle R|\ P \subseteq dom\ Q) = (dom\ P \subseteq |R]\ Q)$
⟨*proof*⟩

lemma *fdia-bdia-conjugation*: $(\ |R\rangle\ P \subseteq neg\ (dom\ Q)) = (\langle R|\ Q \subseteq neg\ (dom\ P))$
⟨*proof*⟩

lemma *bfox-bbox-conjugation*: $(neg\ (dom\ Q) \subseteq |R]\ P) = (neg\ (dom\ P) \subseteq [R|\ Q)$
⟨*proof*⟩

1.4 Residuation

definition *lres* :: $('a,'c)\ rel \Rightarrow ('b,'c)\ rel \Rightarrow ('a,'b)\ rel$ (**infixl** $/\!/$ *75*)
where $R /\!/ S = \{(a,b).\ \forall\, c.\ (b,c) \in S \longrightarrow (a,c) \in R\}$

definition *rres* :: $('c,'a)\ rel \Rightarrow ('c,'b)\ rel \Rightarrow ('a,'b)\ rel$ (**infixl** $\backslash$ *75*)
where $R \backslash S = \{(b,a).\ \forall\, c.\ (c,b) \in R \longrightarrow (c,a) \in S\}$

lemma *rres-lres-conv*: $R \backslash S = (S^{\smile} /\!/ R^{\smile})^{\smile}$
⟨*proof*⟩

lemma *lres-galois*: $(R\ ;\ S \subseteq T) = (R \subseteq T /\!/ S)$
⟨*proof*⟩

lemma *rres-galois*: $(R\ ;\ S \subseteq T) = (S \subseteq R \backslash T)$
⟨*proof*⟩

lemma *lres-compl*: $R /\!/ S = -(-R\ ;\ S^{\smile})$
⟨*proof*⟩

lemma *rres-compl*: $R \setminus S = -(R^{\smile} ; -S)$
  $\langle proof \rangle$

lemma *lres-simp* [*simp*]: $(R \mathbin{/\!/} R) ; R = R$
  $\langle proof \rangle$

lemma *rres-simp* [*simp*]: $R ; (R \setminus R) = R$
  $\langle proof \rangle$

lemma *lres-curry*: $R \mathbin{/\!/} (T ; S) = (R \mathbin{/\!/} S) \mathbin{/\!/} T$
  $\langle proof \rangle$

lemma *rres-curry*: $(R ; S) \setminus T = S \setminus (R \setminus T)$
  $\langle proof \rangle$

lemma *lres-Id*: $Id \subseteq R \mathbin{/\!/} R$
  $\langle proof \rangle$

lemma *det-lres*: *deterministic* $R \Longrightarrow (R ; S) \mathbin{/\!/} S = R ; (S \mathbin{/\!/} S)$
  $\langle proof \rangle$

lemma *det-rres*: *deterministic* $(R^{\smile}) \Longrightarrow S \setminus (S ; R) = (S \setminus S) ; R$
  $\langle proof \rangle$

lemma *rres-bij*: *bijective* $S \Longrightarrow (R \setminus T) ; S = R \setminus (T ; S)$
  $\langle proof \rangle$

lemma *lres-bij*: *bijective* $S \Longrightarrow (R \mathbin{/\!/} T^{\smile}) ; S = R \mathbin{/\!/} (T ; S)^{\smile}$
  $\langle proof \rangle$

lemma *dom-rres-top*: $(dom\ P \subseteq R \setminus (dom\ Q ; top)) = (dom\ P ; top \subseteq R \setminus (dom\ Q ; top))$
  $\langle proof \rangle$

lemma *dom-rres-top-var*: $(dom\ P \subseteq R \setminus (dom\ Q ; top)) = (P ; top \subseteq R \setminus (Q ; top))$
  $\langle proof \rangle$

lemma *fdia-rres-top*: $(|R\rangle P \subseteq dom\ Q) = (dom\ P \subseteq R \setminus (dom\ Q ; top))$
  $\langle proof \rangle$

lemma *fdia-rres-top-var*: $(|R\rangle\ P \subseteq dom\ Q) = (dom\ P \subseteq R \setminus (Q ; top))$
  $\langle proof \rangle$

lemma *dom-galois-var2*: $(|R\rangle\ (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q) ; top))$
  $\langle proof \rangle$

lemma *rres-top*: $R \setminus (dom\ Q ; top) ; top = R \setminus (dom\ Q ; top)$

$\langle proof \rangle$

lemma *ddd-var*: $(\ |R\rangle\ P \subseteq dom\ Q) = (dom\ P \subseteq dom\ ((R \setminus (dom\ Q\ ;\ top))\ ;\ top))$
$\langle proof \rangle$

lemma *wlp-prop*: $dom\ ((R \setminus (dom\ Q\ ;\ top))\ ;\ top) = neg\ (cod\ (neg\ (dom\ Q);\ R))$
$\langle proof \rangle$

lemma *wlp-prop-var*: $dom\ ((R \setminus (dom\ Q\ ;\ top))) = neg\ (cod\ ((neg\ (dom\ Q));\ R))$
$\langle proof \rangle$

lemma *dom-demod*: $(\ |R\rangle\ (Id \cap P) \subseteq Id \cap Q) = (R\ ;\ (Id \cap P) \subseteq (Id \cap Q)\ ;\ R)$
$\langle proof \rangle$

lemma *fdia-bbox-galois-var*: $(\ |R\rangle\ (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq Id \cap -\ cod\ ((Id \cap -Q);\ R))$
$\langle proof \rangle$

lemma *dom-demod-var2*: $(\ |R\rangle\ (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q)\ ;\ R))$
$\langle proof \rangle$

1.5 Symmetric quotient

definition $syq :: ('c,'a)\ rel \Rightarrow ('c,'b)\ rel \Rightarrow ('a,'b)\ rel$ (**infixl** $\div$ *75*)
where $R \div S = (R \setminus S) \cap (R^{\smile} /\!\!/ S^{\smile})$

lemma *syq-set*: $R \div S = \{(a,b).\ \forall c.\ (c,a) \in R \longleftrightarrow (c,b) \in S\}$
$\langle proof \rangle$

lemma *converse-syq* [*simp*]: $(R \div S)^{\smile} = S \div R$
$\langle proof \rangle$

lemma *syq-compl*: $R \div S = -\ (R^{\smile}\ ;\ -S) \cap -\ (-(R^{\smile})\ ;\ S)$
$\langle proof \rangle$

lemma *syq-compl2* [*simp*]: $-R \div -S = R \div S$
$\langle proof \rangle$

lemma *syq-expand1*: $R\ ;\ (R \div S) = S \cap (top\ ;\ (R \div S))$
$\langle proof \rangle$

lemma *syq-expand2*: $(R \div S)\ ;\ S^{\smile} = R^{\smile} \cap ((R \div S)\ ;\ top)$
$\langle proof \rangle$

lemma *syq-comp1*: $(R \div S)\ ;\ (S \div T) = (R \div T) \cap (top\ ;\ (S \div T))$
$\langle proof \rangle$

lemma *syq-comp2*: $(R \div S) ; (S \div T) = (R \div T) \cap ((R \div S) ; top)$
⟨*proof*⟩

lemma *syq-bij*: *bijective* $T \Longrightarrow (R \div S) ; T = R \div (S ; T)$
⟨*proof*⟩

end

2 Properties of Power Allegories

theory *Power-Allegories-Properties*

imports *Relational-Properties*

begin

2.1 Power transpose, epsilon, epsiloff

definition *Lambda* :: $('a,'b)$ *rel* $\Rightarrow$ $('a,'b$ *set*$)$ *rel* (Λ) **where**
$\Lambda\ R = \{(a,B) \mid a\ B.\ B = \{b.\ (a,b) \in R\}\}$

definition *epsilon* :: $('a,'a$ *set*$)$ *rel* **where**
$epsilon = \{(a,A).\ a \in A\}$

definition *epsiloff* $= \{(A,a).\ a \in A\}$

definition *alpha* :: $('a,'b$ *set*$)$ *rel* $\Rightarrow$ $('a,'b)$ *rel* (α) **where**
$\alpha\ R = R ; epsiloff$

alpha can be seen as a relational approximation of a multirelation. The next lemma provides a relational definition of Lambda.

lemma *Lambda-syq*: $\Lambda\ R = R^{\smile} \div epsilon$
⟨*proof*⟩

lemma *epsiloff-epsilon*: $epsiloff = epsilon^{\smile}$
⟨*proof*⟩

lemma *alpha-set*: $\alpha\ R = \{(a,b) \mid a\ b.\ b \in \bigcup\{B.\ (a,B) \in R\}\}$
⟨*proof*⟩

lemma *alpha-relcomp* [*simp*]: $\alpha\ (R ; S) = R ; \alpha\ S$
⟨*proof*⟩

lemma *Lambda-epsiloff-up1*: $f = \Lambda\ R \Longrightarrow R = \alpha\ f$
⟨*proof*⟩

lemma *Lambda-epsiloff-up2*: *deterministic* $f \Longrightarrow R = \alpha\ f \Longrightarrow f = \Lambda\ R$
⟨*proof*⟩

lemma *Lambda-epsiloff-up*:
assumes *deterministic f*
shows $(R = \alpha\ f) = (f = \Lambda\ R)$
⟨*proof*⟩

lemma *det-lambda*: *deterministic* $(\Lambda\ R)$
⟨*proof*⟩

lemma *Lambda-alpha-canc*: *deterministic f* $\Longrightarrow \Lambda\ (\alpha\ f) = f$
⟨*proof*⟩

lemma *alpha-Lambda-canc* [*simp*]: $\alpha\ (\Lambda\ R) = R$
⟨*proof*⟩

lemma *alpha-cancel*:
assumes *deterministic f*
and *deterministic g*
shows $\alpha\ f = \alpha\ g \Longrightarrow f = g$
⟨*proof*⟩

lemma *Lambda-fusion*:
assumes *deterministic f*
shows $\Lambda\ (f\ ;\ R) = f\ ;\ \Lambda\ R$
⟨*proof*⟩

lemma *Lambda-fusion-var*: $\Lambda\ (\Lambda\ R\ ;\ S) = \Lambda\ R\ ;\ \Lambda\ S$
⟨*proof*⟩

lemma *Lambda-epsiloff* [*simp*]: Λ *epsiloff* = *Id*
⟨*proof*⟩

lemma *alpha-epsiloff* [*simp*]: α *Id* = *epsiloff*
⟨*proof*⟩

lemma *alpha-Sup-pres*: $\alpha\ (\bigcup \mathcal{R}) = (\bigcup R \in \mathcal{R}.\ \alpha\ R)$
⟨*proof*⟩

lemma *alpha-ord-pres*: $R \subseteq S \Longrightarrow$ *alpha* $R \subseteq$ *alpha* S
⟨*proof*⟩

lemma *alpha-inf-pres*: $\alpha\ \{(a,A).\ \exists B\ C.\ A = B \cap C \wedge (a,B) \in R \wedge (a,C) \in S\}$
$= \alpha\ R \cap \alpha\ S$
⟨*proof*⟩

2.2 Relational image functor

definition *pow* :: $('a,\ 'b)$ *rel* $\Rightarrow$ $('a\ set,\ 'b\ set)$ *rel* $(\mathcal{P})$ **where**
$\mathcal{P}\ R = \Lambda$ (*epsiloff* ; R)

lemma *pow-set*: $\mathcal{P}\ R = \{(A,B).\ B = Image\ R\ A\}$
⟨*proof*⟩

lemma *pow-set-var*: $\mathcal{P}\ R = \{(A,B).\ B = \{b.\ \exists\, a \in A.\ (a,b) \in R\}\}$
⟨*proof*⟩

lemma *pow-converse-set*: $\mathcal{P}\ (R^{\smile}) = \{(Q,P).\ P = \{a.\ \exists\, b.\ (a,b) \in R \wedge b \in Q\}\}$
⟨*proof*⟩

lemma *det-pow*: *deterministic* $(\mathcal{P}\ R)$
⟨*proof*⟩

lemma *Lambda-pow*: $\Lambda\ (R\ ;\ S) = \Lambda\ R\ ;\ \mathcal{P}\ S$
⟨*proof*⟩

lemma *pow-func1* [*simp*]: $\mathcal{P}\ Id = Id$
⟨*proof*⟩

lemma *pow-func2*: $\mathcal{P}\ (R\ ;\ S) = \mathcal{P}\ R\ ;\ \mathcal{P}\ S$
⟨*proof*⟩

lemma *Grph-Image* [*simp*]: $Grph \circ Image = \mathcal{P}$
⟨*proof*⟩

lemma *lambda-alpha-idem* [*simp*]: $\Lambda\ (\alpha\ (\Lambda\ (\alpha\ R))) = \Lambda\ (\alpha\ R)$
⟨*proof*⟩

2.3 Unit and multiplication of powerset monad

definition *eta* :: $('a,'a\ set)\ rel\ (\eta)$ **where**
$\eta = \Lambda\ Id$

definition *mu* :: $('a\ set\ set,\ 'a\ set)\ rel\ (\mu)$ **where**
$\mu = pow\ epsiloff$

lemma *eta-set*: $\eta = \{(a,\{a\})\ |a.\ True\}$
⟨*proof*⟩

lemma *alpha-eta* [*simp*]: $\alpha\ \eta = Id$
⟨*proof*⟩

lemma *det-eta*: *deterministic* η
⟨*proof*⟩

lemma *mu-set*: $\mu = \{(A,B).\ B = \{b.\ \exists\, C.\ C \in A \wedge b \in C\}\}$
⟨*proof*⟩

lemma *det-mu*: *deterministic* μ
⟨*proof*⟩

lemma *Lambda-eta*:
  assumes *deterministic* R
  shows $\Lambda\ R = R\ ;\ \eta$
$\langle proof \rangle$

lemma *eta-nat-trans*:
  assumes *deterministic* R
  shows $\eta\ ;\ \mathcal{P}\ R = R\ ;\ \eta$
$\langle proof \rangle$

lemma *mu-nat-trans*:
  assumes *deterministic* R
  shows $\mathcal{P}\ (\mathcal{P}\ R)\ ;\ \mu = \mu\ ;\ \mathcal{P}\ R$
  $\langle proof \rangle$

The standard axioms for the powerset monad are derivable.

lemma *pow-monad1* [*simp*]: $\mathcal{P}\ \mu\ ;\ \mu = \mu\ ;\ \mu$
  $\langle proof \rangle$

lemma *pow-monad2* [*simp*]: $\mathcal{P}\ \eta\ ;\ \mu = \mathit{Id}$
  $\langle proof \rangle$

lemma *pow-monad3* [*simp*]: $\eta\ ;\ \mu = \mathit{Id}$
  $\langle proof \rangle$

lemma *Lambda-mu*:
  assumes *deterministic* R
  shows $\Lambda(R)\ ;\ \mu = R$
$\langle proof \rangle$

lemma *pow-Lambda-mu* [*simp*]: $\mathcal{P}\ (\Lambda\ R)\ ;\ \mu = \mathcal{P}\ R$
  $\langle proof \rangle$

lemma *lambda-alpha-mu*: $\Lambda\ (\alpha\ R) = \Lambda\ R\ ;\ \mu$
  $\langle proof \rangle$

lemma *alpha-eta-pow* [*simp*]: $\alpha\ (\eta\ ;\ \mathcal{P}\ R) = R$
$\langle proof \rangle$

lemma *eta-pow-Lambda* [*simp*]: $\eta\ ;\ \mathcal{P}\ R = \Lambda\ R$
  $\langle proof \rangle$

lemma *pow-prop1*: $\mathcal{P}\ R \subseteq S \Longrightarrow R \subseteq \alpha\ (\eta\ ;\ S)$
  $\langle proof \rangle$

lemma *pow-prop-2*: $R \subseteq \mathcal{P}\ S \Longrightarrow \alpha\ (\eta\ ;\ R) \subseteq S$
  $\langle proof \rangle$

lemma *pow-prop*: $R = \mathcal{P}\ S \Longrightarrow \alpha\ (\eta\ ;\ R) = S$
⟨*proof*⟩

lemma *alpha-eta-id* [*simp*]: $\alpha\ (R\ ;\ \eta) = R$
⟨*proof*⟩

lemma *eta-alpha-idem* [*simp*]: $\alpha\ (\alpha\ R;\ \eta)\ ;\ \eta = \alpha\ R\ ;\ \eta$
⟨*proof*⟩

lemma *lambda-eta-alpha* [*simp*]: $\Lambda\ (\alpha\ (\alpha\ R\ ;\ \eta)) = \Lambda\ (\alpha\ R)$
⟨*proof*⟩

lemma *eta-lambda-idem* [*simp*]: $\alpha\ (\Lambda\ (\alpha\ R))\ ;\ \eta = \alpha\ R\ ;\ \eta$
⟨*proof*⟩

lemma *Grph-eta* [*simp*]: $\mathit{Grph}\ (\lambda x.\ \{x\}) = \eta$
⟨*proof*⟩

lemma *Grph-epsiloff* [*simp*]: $\mathit{Grph}\ (\lambda x.\ \{x\})\ ;\ \mathit{epsiloff} = \mathit{Id}$
⟨*proof*⟩

lemma *Image-epsiloff* [*simp*]: $\mathit{Image}\ \mathit{epsiloff} \circ (\lambda x.\ \{x\}) = \mathit{id}$
⟨*proof*⟩

2.4 Subset relation

definition *Omega* :: ($'a\ set$, $'a\ set$) *rel* (Ω) **where**
$\Omega = \mathit{epsilon} \setminus \mathit{epsilon}$

lemma *Omega-set*: $\Omega = \{(A,B).\ A \subseteq B\}$
⟨*proof*⟩

lemma *conv-Omega*: $\mathit{Omega}^{\smile} = \mathit{epsiloff}\ /\!\!/\ \mathit{epsiloff}$
⟨*proof*⟩

lemma *epsilon-eta-Omega* [*simp*]: $\eta\ ;\ \Omega = \mathit{epsilon}$
⟨*proof*⟩

lemma *epsiloff-eta-Omega* [*simp*]: $\Omega^{\smile}\ ;\ \eta^{\smile} = \mathit{epsiloff}$
⟨*proof*⟩

lemma *epsilon-Omega* [*simp*]: $\mathit{epsilon}\ ;\ \Omega = \mathit{epsilon}$
⟨*proof*⟩

lemma *conv-Omega-epsiloff* [*simp*]: $\Omega^{\smile}\ ;\ \mathit{epsiloff} = \mathit{epsiloff}$
⟨*proof*⟩

lemma *Lambda-conv* [*simp*]: $(\Lambda\ R)^{\smile} = \mathit{epsilon} \div R^{\smile}$
⟨*proof*⟩

lemma *Lambda-Omega*: $\Lambda\ R\ ;\ \Omega = R^{\smile}\ \backslash\ epsilon$
$\langle proof \rangle$

lemma *syq-epsiloff-prop* [*simp*]: $\Omega^{\smile}\ ;\ (epsilon \div R) = epsiloff\ /\!\!/\ R^{\smile}$
$\langle proof \rangle$

lemma *pow-semicomm*: $((P,Q) \in \mathcal{P}\ R\ ;\ \Omega) = (\Delta\ P\ ;\ R \subseteq R\ ;\ \Delta\ Q)$
$\langle proof \rangle$

2.5 Complementation relation

definition *Compl* :: $('a\ set,'a\ set)\ rel\ (\mathcal{C})$ **where**
$\mathcal{C} = epsilon \div -epsilon$

lemma *Compl-set*: $\mathcal{C} = \{(A,-A)\ |A.\ True\}$
$\langle proof \rangle$

lemma *Compl-Compl* [*simp*]: $\mathcal{C}\ ;\ \mathcal{C} = Id$
$\langle proof \rangle$

lemma *Compl-def-var*: $\mathcal{C} = \Lambda\ (-epsiloff)$
$\langle proof \rangle$

lemma *converse-Compl* [*simp*]: $\mathcal{C}^{\smile} = \mathcal{C}$
$\langle proof \rangle$

lemma *det-Compl*: *deterministic* $\mathcal{C}$
$\langle proof \rangle$

lemma *bij-Compl*: *bijective* $\mathcal{C}$
$\langle proof \rangle$

lemma *Compl-compl-epsiloff* [*simp*]: $\mathcal{C}\ ;\ -epsiloff = epsiloff$
$\langle proof \rangle$

lemma *Compl-epsiloff* [*simp*]: $\mathcal{C}\ ;\ epsiloff = -epsiloff$
$\langle proof \rangle$

lemma *compl-epsilon-Compl* [*simp*]: $-epsilon\ ;\ \mathcal{C} = epsilon$
$\langle proof \rangle$

lemma *epsilon-Compl* [*simp*]: $epsilon\ ;\ \mathcal{C} = -epsilon$
$\langle proof \rangle$

lemma *Lambda-Compl-var*: $\Lambda\ R\ ;\ \mathcal{C} = R^{\smile} \div -epsilon$
$\langle proof \rangle$

lemma *Lambda-Compl*: $\Lambda\ R\ ;\ \mathcal{C} = \Lambda\ (-R)$

⟨*proof*⟩

2.6 Kleisli lifting and Kleisli composition

definition *klift* :: $('a,'b\ set)\ rel \Rightarrow ('a\ set,'b\ set)\ rel$ ($-_{\mathcal{P}}$ [*1000*] *999*) **where**
$(R)_{\mathcal{P}} = \mathcal{P}\ (\alpha\ R)$

definition *kcomp* :: $('a,'b\ set)\ rel \Rightarrow ('b,'c\ set)\ rel \Rightarrow ('a,'c\ set)\ rel$ (**infixl** $\cdot_{\mathcal{P}}$ *70*) **where**
$R \cdot_{\mathcal{P}} S = R\ ;\ (S)_{\mathcal{P}}$

lemma *klift-var*: $(R)_{\mathcal{P}} = \Lambda\ (\mathit{epsiloff}\ ;\ R\ ;\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *klift-set*: $(R)_{\mathcal{P}} = \{(A,B).\ B = \bigcup(\mathit{Image}\ R\ A)\}$
⟨*proof*⟩

lemma *klift-set-var*: $(R)_{\mathcal{P}} = \{(A,B).\ B = \bigcup\{C.\ \exists\, a \in A.(a,C) \in R\}\}$
⟨*proof*⟩

lemma *klift-mu*: $(R)_{\mathcal{P}} = \mathcal{P}\ R\ ;\ \mu$
⟨*proof*⟩

lemma *klift-empty*: $(\{\},A) \in (R)_{\mathcal{P}} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *klift-ext1*: $(R\ ;\ (S)_{\mathcal{P}})_{\mathcal{P}} = (R)_{\mathcal{P}}\ ;\ (S)_{\mathcal{P}}$
⟨*proof*⟩

lemma *klift-ext2*: $\mathit{deterministic}\ R \Longrightarrow \eta\ ;\ (R)_{\mathcal{P}} = R$
⟨*proof*⟩

lemma *klift-ext3* [*simp*]: $(\eta)_{\mathcal{P}} = \mathit{Id}$
⟨*proof*⟩

lemma *pow-klift* [*simp*]: $(R\ ;\ \eta)_{\mathcal{P}} = \mathcal{P}\ R$
⟨*proof*⟩

lemma *mu-klift* [*simp*]: $(\mathit{Id})_{\mathcal{P}} = \mu$
⟨*proof*⟩

lemma *kcomp-var*: $R \cdot_{\mathcal{P}} S = R\ ;\ \mathcal{P}\ S\ ;\ \mu$
⟨*proof*⟩

lemma *kcomp-assoc*: $R \cdot_{\mathcal{P}} (S \cdot_{\mathcal{P}} T) = (R \cdot_{\mathcal{P}} S) \cdot_{\mathcal{P}} T$
⟨*proof*⟩

lemma *kcomp-oner*: $R \cdot_{\mathcal{P}} \eta = R$
⟨*proof*⟩

lemma *kcomp-onel*: $deterministic\ R \Longrightarrow \eta \cdot_{\mathcal{P}} R = R$
⟨*proof*⟩

2.7 Relational box

definition *rbox* :: $('a,'b)\ rel \Rightarrow ('b\ set,\ 'a\ set)\ rel$ **where**
$rbox\ R = \Lambda\ (epsiloff\ /\!\!/\ R)$

lemma *rbox-set*: $rbox\ R = \{(Q,P).\ P = \{a.\ \forall\, b.\ (a,b) \in R \longrightarrow b \in Q\}\}$
⟨*proof*⟩

lemma *rbox-exp*: $((Q,P) \in (rbox\ (R::('a,'b)\ rel))) = (P = -\{a.\ \exists\, b.\ (a,b) \in R\ \wedge$ $b \in -Q\})$
⟨*proof*⟩

lemma *rbox-subset*: $rbox\ R\ ;\ \Omega^{\smile} = \{(Q,P).\ P \subseteq \{a.\ \forall\, b.\ (a,b) \in R \longrightarrow b \in Q\}\}$
⟨*proof*⟩

lemma *rbox-semicomm*: $(Q,P) \in rbox\ R\ ;\ \Omega^{\smile} = (\Delta\ P\ ;\ R \subseteq R\ ;\ \Delta\ Q)$
⟨*proof*⟩

lemma *rbox-semicomm-var*: $(Q,P) \in rbox\ R\ ;\ \Omega^{\smile} = (\Delta\ P \subseteq (R\ ;\ \Delta\ Q)\ /\!\!/\ R)$
⟨*proof*⟩

lemma *rbox-omega*: $rbox\ epsiloff = \Lambda\ (\Omega^{\smile})$
⟨*proof*⟩

lemma *Omega-rbox*: $\Omega = (\alpha\ (rbox\ epsiloff))^{\smile}$
⟨*proof*⟩

lemma *pow-rbox*: $((Q,P) \in rbox\ R\ ;\ \Omega^{\smile}) = ((P,Q) \in \mathcal{P}\ R\ ;\ \Omega)$
⟨*proof*⟩

lemma *rbox-pow-Compl*: $rbox\ R = \mathcal{C}\ ;\ \mathcal{P}\ (R^{\smile})\ ;\ \mathcal{C}$
⟨*proof*⟩

lemma *pow-rbox-Compl*: $\mathcal{P}\ R = \mathcal{C}\ ;\ rbox\ (R^{\smile})\ ;\ \mathcal{C}$
⟨*proof*⟩

lemma *pow-conjugation*: $\mathcal{C}\ ;\ (\mathcal{P}\ (R^{\smile})\ ;\ \Omega)^{\smile} = \mathcal{P}\ R\ ;\ \mathcal{C}\ ;\ \Omega^{\smile}$
⟨*proof*⟩

lemma *pow-rbox-eq*: $rbox\ R\ ;\ \Omega^{\smile} = (\mathcal{P}\ R\ ;\ \Omega)^{\smile}$
⟨*proof*⟩

end

3 Basic Properties of Multirelations

theory *Multirelations-Basics*

imports *Power-Allegories-Properties*

begin

This theory extends a previous AFP entry for multirelations with one single objects to proper multirelations in Rel.

3.1 Peleg composition, parallel composition (inner union) and units

type-synonym $('a,'b)\ mrel = ('a,'b\ set)\ rel$

definition *s-prod* :: $('a,'b)\ mrel \Rightarrow ('b,'c)\ mrel \Rightarrow ('a,'c)\ mrel$ (**infixl** $\cdot$ *75*)
where
$R \cdot S = \{(a,A).\ (\exists B.\ (a,B) \in R \wedge (\exists f.\ (\forall b \in B.\ (b,f\ b) \in S) \wedge A = \bigcup (f\ `\ B)))\}$

definition *s-id* :: $('a,'a)\ mrel\ (1_\sigma)$ **where**
$1_\sigma = (\bigcup a.\ \{(a,\{a\})\})$

definition *p-prod* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel$ (**infixl** $\|$ *70*)
where
$R \| S = \{(a,A).\ (\exists B\ C.\ A = B \cup C \wedge (a,B) \in R \wedge (a,C) \in S)\}$

definition *p-id* :: $('a,'b)\ mrel\ (1_\pi)$ **where**
$1_\pi = (\bigcup a.\ \{(a,\{\})\})$

definition U :: $('a,'b)\ mrel$ **where**
$U = \{(a,A)\ |a\ A.\ True\}$

abbreviation $NC \equiv U - 1_\pi$

named-theorems *mr-simp*
declare *s-prod-def* [*mr-simp*] *p-prod-def* [*mr-simp*] *s-id-def* [*mr-simp*] *p-id-def* [*mr-simp*] *U-def* [*mr-simp*]

lemma *s-prod-idl* [*simp*]: $1_\sigma \cdot R = R$
⟨*proof*⟩

lemma *s-prod-idr* [*simp*]: $R \cdot 1_\sigma = R$
⟨*proof*⟩

lemma *p-prod-ild* [*simp*]: $1_\pi \| R = R$
⟨*proof*⟩

lemma *c-prod-idr* [*simp*]: $R \parallel 1_\pi = R$
⟨*proof*⟩

lemma *cl7* [*simp*]: $1_\sigma \parallel 1_\sigma = 1_\sigma$
⟨*proof*⟩

lemma *p-prod-assoc*: $R \parallel S \parallel T = R \parallel (S \parallel T)$
⟨*proof*⟩

lemma *p-prod-comm*: $R \parallel S = S \parallel R$
⟨*proof*⟩

lemma *subidem-par*: $R \subseteq R \parallel R$
⟨*proof*⟩

lemma *meet-le-par*: $R \cap S \subseteq R \parallel S$
⟨*proof*⟩

lemma *s-prod-distr*: $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$
⟨*proof*⟩

lemma *s-prod-sup-distr*: $(\bigcup X) \cdot S = (\bigcup R \in X.\ R \cdot S)$
⟨*proof*⟩

lemma *s-prod-subdistl*: $R \cdot S \cup R \cdot T \subseteq R \cdot (S \cup T)$
⟨*proof*⟩

lemma *s-prod-sup-subdistl*: $X \neq \{\} \Longrightarrow (\bigcup S \in X.\ R \cdot S) \subseteq R \cdot \bigcup X$
⟨*proof*⟩

lemma *s-prod-isol*: $R \subseteq S \Longrightarrow R \cdot T \subseteq S \cdot T$
⟨*proof*⟩

lemma *s-prod-isor*: $R \subseteq S \Longrightarrow T \cdot R \subseteq T \cdot S$
⟨*proof*⟩

lemma *s-prod-zerol* [*simp*]: $\{\} \cdot R = \{\}$
⟨*proof*⟩

lemma *s-prod-wzeror*: $R \cdot \{\} \subseteq R$
⟨*proof*⟩

lemma *p-prod-zeror* [*simp*]: $R \parallel \{\} = \{\}$
⟨*proof*⟩

lemma *s-prod-p-idl* [*simp*]: $1_\pi \cdot R = 1_\pi$
⟨*proof*⟩

lemma *p-id-st*: $R \cdot 1_\pi = \{(a,\{\})\ |a.\ \exists B.\ (a,B) \in R\}$

⟨*proof*⟩

lemma *c6*: $R \cdot 1_\pi \subseteq 1_\pi$
⟨*proof*⟩

lemma *p-prod-distl*: $R \parallel (S \cup T) = R \parallel S \cup R \parallel T$
⟨*proof*⟩

lemma *p-prod-sup-distl*: $R \parallel (\bigcup X) = (\bigcup S \in X.\ R \parallel S)$
⟨*proof*⟩

lemma *p-prod-isol*: $R \subseteq S \Longrightarrow R \parallel T \subseteq S \parallel T$
⟨*proof*⟩

lemma *p-prod-isor*: $R \subseteq S \Longrightarrow T \parallel R \subseteq T \parallel S$
⟨*proof*⟩

lemma *s-prod-assoc1*: $(R \cdot S) \cdot T \subseteq R \cdot (S \cdot T)$
⟨*proof*⟩

lemma *seq-conc-subdistr*: $(R \parallel S) \cdot T \subseteq R \cdot T \parallel S \cdot T$
⟨*proof*⟩

lemma *U-U* [*simp*]: $U \cdot U = U$
⟨*proof*⟩

lemma *U-par-idem* [*simp*]: $U \parallel U = U$
⟨*proof*⟩

lemma *p-id-NC*: $R - 1_\pi = R \cap NC$
⟨*proof*⟩

lemma *NC-NC* [*simp*]: $NC \cdot NC = NC$
⟨*proof*⟩

lemma *nc-par-idem* [*simp*]: $NC \parallel NC = NC$
⟨*proof*⟩

lemma *cl4*:
assumes $T \parallel T \subseteq T$
shows $R \cdot T \parallel S \cdot T \subseteq (R \parallel S) \cdot T$
⟨*proof*⟩

lemma *cl3*: $R \cdot (S \parallel T) \subseteq R \cdot S \parallel R \cdot T$
⟨*proof*⟩

lemma *p-id-assoc1*: $(1_\pi \cdot R) \cdot S = 1_\pi \cdot (R \cdot S)$
⟨*proof*⟩

lemma *p-id-assoc2*: $(R \cdot 1_\pi) \cdot T = R \cdot (1_\pi \cdot T)$
⟨*proof*⟩

lemma *cl1* [*simp*]: $R \cdot 1_\pi \cup R \cdot NC = R \cdot U$
⟨*proof*⟩

lemma *tarski-aux*:
assumes $R - 1_\pi \neq \{\}$
and $(a,A) \in NC$
shows $(a,A) \in NC \cdot ((R - 1_\pi) \cdot NC)$
⟨*proof*⟩

lemma *tarski*:
assumes $R - 1_\pi \neq \{\}$
shows $NC \cdot ((R - 1_\pi) \cdot NC) = NC$
⟨*proof*⟩

lemma *tarski-var*:
assumes $R \cap NC \neq \{\}$
shows $NC \cdot ((R \cap NC) \cdot NC) = NC$
⟨*proof*⟩

lemma *s-le-nc*: $1_\sigma \subseteq NC$
⟨*proof*⟩

lemma *U-nc* [*simp*]: $U \cdot NC = U$
⟨*proof*⟩

lemma *x-y-split* [*simp*]: $(R \cap NC) \cdot S \cup R \cdot \{\} = R \cdot S$
⟨*proof*⟩

lemma *c-nc-comp1* [*simp*]: $1_\pi \cup NC = U$
⟨*proof*⟩

3.2 Tests

lemma *s-id-st*: $R \cap 1_\sigma = \{(a,\{a\}) \mid a.\ (a,\{a\}) \in R\}$
⟨*proof*⟩

lemma *subid-aux2*:
assumes $(a,A) \in R \cap 1_\sigma$
shows $A = \{a\}$
⟨*proof*⟩

lemma *s-prod-test-aux1*:
assumes $(a,A) \in R \cdot (P \cap 1_\sigma)$
shows $((a,A) \in R \wedge (\forall a \in A.\ (a,\{a\}) \in (P \cap 1_\sigma)))$
⟨*proof*⟩

lemma *s-prod-test-aux2*:
  assumes $(a,A) \in R$
  and $\forall a \in A.\ (a,\{a\}) \in S$
  shows $(a,A) \in R \cdot S$
  ⟨*proof*⟩

lemma *s-prod-test*: $(a,A) \in R \cdot (P \cap 1_\sigma) \longleftrightarrow (a,A) \in R \wedge (\forall a \in A.\ (a,\{a\}) \in (P \cap 1_\sigma))$
  ⟨*proof*⟩

lemma *s-prod-test-var*: $R \cdot (P \cap 1_\sigma) = \{(a,A).\ (a,A) \in R \wedge (\forall a \in A.\ (a,\{a\}) \in (P \cap 1_\sigma))\}$
  ⟨*proof*⟩

lemma *test-s-prod-aux1*:
  assumes $(a,A) \in (P \cap 1_\sigma) \cdot R$
  shows $(a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$
  ⟨*proof*⟩

lemma *test-s-prod-aux2*:
  assumes $(a,A) \in R$
  and $(a,\{a\}) \in P$
  shows $(a,A) \in P \cdot R$
  ⟨*proof*⟩

lemma *test-s-prod*: $(a,A) \in (P \cap 1_\sigma) \cdot R \longleftrightarrow (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$
  ⟨*proof*⟩

lemma *test-s-prod-var*: $(P \cap 1_\sigma) \cdot R = \{(a,A).\ (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R\}$
  ⟨*proof*⟩

lemma *test-assoc1*: $(R \cdot (P \cap 1_\sigma)) \cdot S = R \cdot ((P \cap 1_\sigma) \cdot S)$
  ⟨*proof*⟩

lemma *test-assoc2*: $((P \cap 1_\sigma) \cdot R) \cdot S = (P \cap 1_\sigma) \cdot (R \cdot S)$
  ⟨*proof*⟩

lemma *test-assoc3*: $(R \cdot S) \cdot (P \cap 1_\sigma) = R \cdot (S \cdot (P \cap 1_\sigma))$
⟨*proof*⟩

lemma *s-distl-test*: $(P \cap 1_\sigma) \cdot (S \cup T) = (P \cap 1_\sigma) \cdot S \cup (P \cap 1_\sigma) \cdot T$
  ⟨*proof*⟩

lemma *s-distl-sup-test*: $(P \cap 1_\sigma) \cdot \bigcup X = (\bigcup S \in X.\ (P \cap 1_\sigma) \cdot S)$
  ⟨*proof*⟩

lemma *subid-par-idem* [*simp*]: $(P \cap 1_\sigma) \parallel (P \cap 1_\sigma) = (P \cap 1_\sigma)$
  ⟨*proof*⟩

lemma *seq-conc-subdistrl*: $(P \cap 1_\sigma) \cdot (S \parallel T) = ((P \cap 1_\sigma) \cdot S) \parallel ((P \cap 1_\sigma) \cdot T)$
⟨*proof*⟩

lemma *test-s-prod-is-meet* [*simp*]: $(P \cap 1_\sigma) \cdot (Q \cap 1_\sigma) = P \cap Q \cap 1_\sigma$
⟨*proof*⟩

lemma *test-p-prod-is-meet* [*simp*]: $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cap (Q \cap 1_\sigma)$
⟨*proof*⟩

lemma *test-multipliciativer*: $(P \cap Q \cap 1_\sigma) \cdot T = ((P \cap 1_\sigma) \cdot T) \cap ((Q \cap 1_\sigma) \cdot T)$
⟨*proof*⟩

lemma *cl9* [*simp*]: $(R \cap 1_\sigma) \cdot 1_\pi \parallel 1_\sigma = R \cap 1_\sigma$
⟨*proof*⟩

lemma *s-subid-closed* [*simp*]: $R \cap NC \cap 1_\sigma = R \cap 1_\sigma$
⟨*proof*⟩

lemma *sub-id-le-nc*: $R \cap 1_\sigma \subseteq NC$
⟨*proof*⟩

lemma *x-y-prop*: $1_\sigma \cap ((R \cap NC) \cdot S) = 1_\sigma \cap R \cdot S$
⟨*proof*⟩

lemma *s-nc-U*: $1_\sigma \cap R \cdot NC = 1_\sigma \cap R \cdot U$
⟨*proof*⟩

lemma *sid-le-nc-var*: $1_\sigma \cap R \subseteq 1_\sigma \cap (R \parallel NC)$
⟨*proof*⟩

lemma *s-nc-par-U*: $1_\sigma \cap (R \parallel NC) = 1_\sigma \cap (R \parallel U)$
⟨*proof*⟩

lemma *s-id-par-s-prod*: $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cdot (Q \cap 1_\sigma)$
⟨*proof*⟩

3.3 Parallel subidentities

lemma *p-id-zero-st*: $R \cap 1_\pi = \{(a,\{\}) \mid a.\ (a,\{\}) \in R\}$
⟨*proof*⟩

lemma *p-subid-iff*: $R \subseteq 1_\pi \longleftrightarrow R \cdot 1_\pi = R$
⟨*proof*⟩

lemma *p-subid-iff-var*: $R \subseteq 1_\pi \longleftrightarrow R \cdot \{\} = R$
⟨*proof*⟩

lemma *term-par-idem* [*simp*]: $(R \cap 1_\pi) \parallel (R \cap 1_\pi) = (R \cap 1_\pi)$

⟨*proof*⟩

lemma *c1* [*simp*]: $R \cdot 1_\pi \parallel R = R$
⟨*proof*⟩

lemma *p-id-zero*: $R \cap 1_\pi = R \cdot \{\}$
⟨*proof*⟩

lemma *cl5*: $(R \cdot S) \cdot (T \cdot \{\}) = R \cdot (S \cdot (T \cdot \{\}))$
⟨*proof*⟩

lemma *c4*: $(R \cdot S) \cdot 1_\pi = R \cdot (S \cdot 1_\pi)$
⟨*proof*⟩

lemma *c3*: $(R \parallel S) \cdot 1_\pi = R \cdot 1_\pi \parallel S \cdot 1_\pi$
⟨*proof*⟩

lemma *p-id-idem* [*simp*]: $(R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$
⟨*proof*⟩

lemma *x-c-par-idem* [*simp*]: $R \cdot 1_\pi \parallel R \cdot 1_\pi = R \cdot 1_\pi$
⟨*proof*⟩

lemma *x-zero-le-c*: $R \cdot \{\} \subseteq 1_\pi$
⟨*proof*⟩

lemma *p-subid-lb1*: $R \cdot \{\} \parallel S \cdot \{\} \subseteq R \cdot \{\}$
⟨*proof*⟩

lemma *p-subid-lb2*: $R \cdot \{\} \parallel S \cdot \{\} \subseteq S \cdot \{\}$
⟨*proof*⟩

lemma *p-subid-idem* [*simp*]: $R \cdot \{\} \parallel R \cdot \{\} = R \cdot \{\}$
⟨*proof*⟩

lemma *p-subid-glb*: $T \cdot \{\} \subseteq R \cdot \{\} \Longrightarrow T \cdot \{\} \subseteq S \cdot \{\} \Longrightarrow T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$
⟨*proof*⟩

lemma *p-subid-glb-iff*: $T \cdot \{\} \subseteq R \cdot \{\} \wedge T \cdot \{\} \subseteq S \cdot \{\} \longleftrightarrow T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$
⟨*proof*⟩

lemma *x-c-glb*: $(T::('a,'b)\ mrel) \cdot 1_\pi \subseteq (R::('a,'b)\ mrel) \cdot 1_\pi \Longrightarrow T \cdot 1_\pi \subseteq (S::('a,'b)\ mrel) \cdot 1_\pi \Longrightarrow T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$
⟨*proof*⟩

lemma *x-c-lb1*: $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq R \cdot 1_\pi$
⟨*proof*⟩

lemma *x-c-lb2*: $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq S \cdot 1_\pi$
⟨*proof*⟩

lemma *x-c-glb-iff*: $(T::('a,'b)\ mrel) \cdot 1_\pi \subseteq (R::('a,'b)\ mrel) \cdot 1_\pi \wedge T \cdot 1_\pi \subseteq (S::('a,'b)\ mrel) \cdot 1_\pi \longleftrightarrow T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$
⟨*proof*⟩

lemma *nc-iff1*: $R \subseteq NC \longleftrightarrow R \cap 1_\pi = \{\}$
⟨*proof*⟩

lemma *nc-iff2*: $R \subseteq NC \longleftrightarrow R \cdot \{\} = \{\}$
⟨*proof*⟩

lemma *zero-assoc3*: $(R \cdot S) \cdot \{\} = R \cdot (S \cdot \{\})$
⟨*proof*⟩

lemma *x-zero-interr*: $R \cdot \{\} \parallel S \cdot \{\} = (R \parallel S) \cdot \{\}$
⟨*proof*⟩

lemma *p-subid-interr*: $R \cdot T \cdot 1_\pi \parallel S \cdot T \cdot 1_\pi = (R \parallel S) \cdot T \cdot 1_\pi$
⟨*proof*⟩

lemma *cl2* [*simp*]: $1_\pi \cap (R \cup NC) = R \cdot \{\}$
⟨*proof*⟩

lemma *cl6* [*simp*]: $R \cdot \{\} \cdot S = R \cdot \{\}$
⟨*proof*⟩

lemma *cl11* [*simp*]: $(R \cap NC) \cdot 1_\pi \parallel NC = (R \cap NC) \cdot NC$
⟨*proof*⟩

lemma *x-split* [*simp*]: $(R \cap NC) \cup (R \cap 1_\pi) = R$
⟨*proof*⟩

lemma *x-split-var* [*simp*]: $(R \cap NC) \cup R \cdot \{\} = R$
⟨*proof*⟩

lemma *s-x-c* [*simp*]: $1_\sigma \cap R \cdot 1_\pi = \{\}$
⟨*proof*⟩

lemma *s-x-zero* [*simp*]: $1_\sigma \cap R \cdot \{\} = \{\}$
⟨*proof*⟩

lemma *c-nc* [*simp*]: $R \cdot 1_\pi \cap NC = \{\}$
⟨*proof*⟩

lemma *zero-nc* [*simp*]: $R \cdot \{\} \cap NC = \{\}$
⟨*proof*⟩

lemma *nc-zero* [*simp*]: $(R \cap NC) \cdot \{\} = \{\}$
⟨*proof*⟩

lemma *c-def* [*simp*]: $U \cdot \{\} = 1_\pi$
⟨*proof*⟩

lemma *U-c* [*simp*]: $U \cdot 1_\pi = 1_\pi$
⟨*proof*⟩

lemma *nc-c* [*simp*]: $NC \cdot 1_\pi = 1_\pi$
⟨*proof*⟩

lemma *nc-U* [*simp*]: $NC \cdot U = U$
⟨*proof*⟩

lemma *x-c-nc-split* [*simp*]: $((R \cap NC) \cdot NC) \cup (R \cdot \{\} \parallel NC) = (R \cdot 1_\pi) \parallel NC$
⟨*proof*⟩

lemma *x-c-U-split* [*simp*]: $R \cdot U \cup (R \cdot \{\} \parallel U) = R \cdot 1_\pi \parallel U$
⟨*proof*⟩

lemma *p-subid-par-eq-meet* [*simp*]: $R \cdot \{\} \parallel S \cdot \{\} = R \cdot \{\} \cap S \cdot \{\}$
⟨*proof*⟩

lemma *p-subid-par-eq-meet-var* [*simp*]: $R \cdot 1_\pi \parallel S \cdot 1_\pi = R \cdot 1_\pi \cap S \cdot 1_\pi$
⟨*proof*⟩

lemma *x-zero-add-closed*: $R \cdot \{\} \cup S \cdot \{\} = (R \cup S) \cdot \{\}$
⟨*proof*⟩

lemma *x-zero-meet-closed*: $R \cdot \{\} \cap S \cdot \{\} = (R \cap S) \cdot \{\}$
⟨*proof*⟩

lemma *scomp-univalent-pres*: *univalent* $R \Longrightarrow$ *univalent* $S \Longrightarrow$ *univalent* $(R \cdot S)$
⟨*proof*⟩

lemma *univalent s-id*
⟨*proof*⟩

lemma *det-peleg*: *deterministic* $R \Longrightarrow$ *deterministic* $S \Longrightarrow$ *deterministic* $(R \cdot S)$
⟨*proof*⟩

lemma *deterministic-sid*: *deterministic* 1_σ
⟨*proof*⟩

3.4 Domain

definition *Dom* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'a)$ *mrel* **where**

$Dom\ R = \{(a,\{a\})\ |a.\ \exists B.\ (a,B) \in R\}$

named-theorems *mrd-simp*
declare *mr-simp* [*mrd-simp*] *Dom-def* [*mrd-simp*]

lemma *d-def-expl*: $Dom\ R = R \cdot 1_\pi \parallel 1_\sigma$
⟨*proof*⟩

lemma *s-subid-iff2*: $(R \cap 1_\sigma = R) = (Dom\ R = R)$
⟨*proof*⟩

lemma *cl8-var*: $Dom\ R \cdot S = R \cdot 1_\pi \parallel S$
⟨*proof*⟩

lemma *cl8* [*simp*]: $R \cdot 1_\pi \parallel 1_\sigma \cdot S = R \cdot 1_\pi \parallel S$
⟨*proof*⟩

lemma *cl10-var*: $Dom\ (R - 1_\pi) = 1_\sigma \cap ((R - 1_\pi) \cdot NC)$
⟨*proof*⟩

lemma *c10*: $(R \cap NC) \cdot 1_\pi \parallel 1_\sigma = 1_\sigma \cap ((R \cap NC) \cdot NC)$
⟨*proof*⟩

lemma *cl9-var* [*simp*]: $Dom\ (R \cap 1_\sigma) = R \cap 1_\sigma$
⟨*proof*⟩

lemma *d-s-id* [*simp*]: $Dom\ R \cap 1_\sigma = Dom\ R$
⟨*proof*⟩

lemma *d-s-id-ax*: $Dom\ R \subseteq 1_\sigma$
⟨*proof*⟩

lemma *d-assoc1*: $Dom\ R \cdot (S \cdot T) = (Dom\ R \cdot S) \cdot T$
⟨*proof*⟩

lemma *d-meet-distr-var*: $(Dom\ R \cap Dom\ S) \cdot T = Dom\ R \cdot T \cap Dom\ S \cdot T$
⟨*proof*⟩

lemma *d-idem* [*simp*]: $Dom\ (Dom\ R) = Dom\ R$
⟨*proof*⟩

lemma *cd-2-var*: $Dom\ (R \cdot 1_\pi) \cdot S = R \cdot 1_\pi \parallel S$
⟨*proof*⟩

lemma *dc-prop1* [*simp*]: $Dom\ R \cdot 1_\pi = R \cdot 1_\pi$
⟨*proof*⟩

lemma *dc-prop2* [*simp*]: $Dom\ (R \cdot 1_\pi) = Dom\ R$
⟨*proof*⟩

lemma *ds-prop* [*simp*]: $Dom\ R \parallel 1_\sigma = Dom\ R$
⟨*proof*⟩

lemma *dc* [*simp*]: $Dom\ 1_\pi = 1_\sigma$
⟨*proof*⟩

lemma *cd-iso* [*simp*]: $Dom\ (R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$
⟨*proof*⟩

lemma *dc-iso* [*simp*]: $Dom\ (Dom\ R \cdot 1_\pi) = Dom\ R$
⟨*proof*⟩

lemma *d-s-id-inter* [*simp*]: $Dom\ R \cdot Dom\ S = Dom\ R \cap Dom\ S$
⟨*proof*⟩

lemma *d-conc6*: $Dom\ (R \parallel S) = Dom\ R \parallel Dom\ S$
⟨*proof*⟩

lemma *d-conc-inter* [*simp*]: $Dom\ R \parallel Dom\ S = Dom\ R \cap Dom\ S$
⟨*proof*⟩

lemma *d-conc-s-prod-ax*: $Dom\ R \parallel Dom\ S = Dom\ R \cdot Dom\ S$
⟨*proof*⟩

lemma *d-rest-ax* [*simp*]: $Dom\ R \cdot R = R$
⟨*proof*⟩

lemma *d-loc-ax* [*simp*]: $Dom\ (R \cdot Dom\ S) = Dom\ (R \cdot S)$
⟨*proof*⟩

lemma *assoc-p-subid*: $(R \cdot S) \cdot (T \cdot 1_\pi) = R \cdot (S \cdot (T \cdot 1_\pi))$
⟨*proof*⟩

lemma *d-exp-ax* [*simp*]: $Dom\ (Dom\ R \cdot S) = Dom\ R \cdot Dom\ S$
⟨*proof*⟩

lemma *d-comm-ax*: $Dom\ R \cdot Dom\ S = Dom\ S \cdot Dom\ R$
⟨*proof*⟩

lemma *d-s-id-prop* [*simp*]: $Dom\ 1_\sigma = 1_\sigma$
⟨*proof*⟩

lemma *d-s-prod-closed* [*simp*]: $Dom\ (Dom\ R \cdot Dom\ S) = Dom\ R \cdot Dom\ S$
⟨*proof*⟩

lemma *d-p-prod-closed* [*simp*]: $Dom\ (Dom\ R \parallel Dom\ S) = Dom\ R \parallel Dom\ S$
⟨*proof*⟩

lemma *d-idem2* [*simp*]: $Dom\ R \cdot Dom\ R = Dom\ R$
⟨*proof*⟩

lemma *d-assoc*: $(Dom\ R \cdot Dom\ S) \cdot Dom\ T = Dom\ R \cdot (Dom\ S \cdot Dom\ T)$
⟨*proof*⟩

lemma *iso-1* [*simp*]: $Dom\ R \cdot 1_\pi \parallel 1_\sigma = Dom\ R$
⟨*proof*⟩

lemma *d-idem-par* [*simp*]: $Dom\ R \parallel Dom\ R = Dom\ R$
⟨*proof*⟩

lemma *d-inter-r*: $Dom\ R \cdot (S \parallel T) = Dom\ R \cdot S \parallel Dom\ R \cdot T$
⟨*proof*⟩

lemma *d-add-ax*: $Dom\ (R \cup S) = Dom\ R \cup Dom\ S$
⟨*proof*⟩

lemma *d-sup-add*: $Dom\ (\bigcup X) = (\bigcup R \in X.\ Dom\ R)$
⟨*proof*⟩

lemma *d-distl*: $Dom\ R \cdot (S \cup T) = Dom\ R \cdot S \cup Dom\ R \cdot T$
⟨*proof*⟩

lemma *d-sup-distl*: $Dom\ R \cdot \bigcup X = (\bigcup S \in X.\ Dom\ R \cdot S)$
⟨*proof*⟩

lemma *d-zero-ax* [*simp*]: $Dom\ \{\} = \{\}$
⟨*proof*⟩

lemma *d-absorb1* [*simp*]: $Dom\ R \cup Dom\ R \cdot Dom\ S = Dom\ R$
⟨*proof*⟩

lemma *d-absorb2* [*simp*]: $Dom\ R \cdot (Dom\ R \cup Dom\ S) = Dom\ R$
⟨*proof*⟩

lemma *d-dist1*: $Dom\ R \cdot (Dom\ S \cup Dom\ T) = Dom\ R \cdot Dom\ S \cup Dom\ R \cdot Dom\ T$
⟨*proof*⟩

lemma *d-dist2*: $Dom\ R \cup (Dom\ S \cdot Dom\ T) = (Dom\ R \cup Dom\ S) \cdot (Dom\ R \cup Dom\ T)$
⟨*proof*⟩

lemma *d-add-prod-closed* [*simp*]: $Dom\ (Dom\ R \cup Dom\ S) = Dom\ R \cup Dom\ S$
⟨*proof*⟩

lemma *x-zero-prop*: $R \cdot \{\} \parallel S = Dom\ (R \cdot \{\}) \cdot S$
⟨*proof*⟩

lemma *cda-add-ax*: $Dom\ ((R \cup S) \cdot T) = Dom\ (R \cdot T) \cup Dom\ (S \cdot T)$
⟨*proof*⟩

lemma *d-x-zero*: $Dom\ (R \cdot \{\}) = R \cdot \{\} \parallel 1_\sigma$
⟨*proof*⟩

lemma *cda-ax2*:
assumes $(R \parallel S) \cdot Dom\ T = R \cdot Dom\ T \parallel S \cdot Dom\ T$
shows $Dom\ ((R \parallel S) \cdot T) = Dom\ (R \cdot T) \cdot Dom\ (S \cdot T)$
⟨*proof*⟩

lemma *d-lb1*: $Dom\ R \cdot Dom\ S \subseteq Dom\ R$
⟨*proof*⟩

lemma *d-lb2*: $Dom\ R \cdot Dom\ S \subseteq Dom\ S$
⟨*proof*⟩

lemma *d-glb*: $Dom\ T \subseteq Dom\ R \wedge Dom\ T \subseteq Dom\ S \Longrightarrow Dom\ T \subseteq Dom\ R \cdot Dom\ S$
⟨*proof*⟩

lemma *d-glb-iff*: $Dom\ T \subseteq Dom\ R \wedge Dom\ T \subseteq Dom\ S \longleftrightarrow Dom\ T \subseteq Dom\ R \cdot Dom\ S$
⟨*proof*⟩

lemma *d-interr*: $R \cdot Dom\ P \parallel S \cdot Dom\ P = (R \parallel S) \cdot Dom\ P$
⟨*proof*⟩

lemma *cl10-d*: $Dom\ (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot NC$
⟨*proof*⟩

lemma *cl11-d* [*simp*]: $Dom\ (R \cap NC) \cdot NC = (R \cap NC) \cdot NC$
⟨*proof*⟩

lemma *cl10-d-var1*: $Dom\ (R \cap NC) = 1_\sigma \cap R \cdot NC$
⟨*proof*⟩

lemma *cl10-d-var2*: $Dom\ (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot U$
⟨*proof*⟩

lemma *cl10-d-var3*: $Dom\ (R \cap NC) = 1_\sigma \cap R \cdot U$
⟨*proof*⟩

lemma *d-U* [*simp*]: $Dom\ U = 1_\sigma$
⟨*proof*⟩

lemma *d-nc* [*simp*]: $Dom\ NC = 1_\sigma$
⟨*proof*⟩

lemma *alt-d-def-nc-nc*: $Dom\ (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel NC)$
⟨*proof*⟩

lemma *alt-d-def-nc-U*: $Dom\ (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel U)$
⟨*proof*⟩

lemma *d-def-split* [*simp*]: $Dom\ (R \cap NC) \cup Dom\ (R \cdot \{\}) = Dom\ R$
⟨*proof*⟩

lemma *d-def-split-var* [*simp*]: $Dom\ (R \cap NC) \cup ((R \cdot \{\}) \parallel 1_\sigma) = Dom\ R$
⟨*proof*⟩

lemma *ax7* [*simp*]: $(1_\sigma \cap R \cdot U) \cup (R \cdot \{\} \parallel 1_\sigma) = Dom\ R$
⟨*proof*⟩

lemma *dom12-d*: $Dom\ R = 1_\sigma \cap (R \cdot 1_\pi \parallel NC)$
⟨*proof*⟩

lemma *dom12-d-U*: $Dom\ R = 1_\sigma \cap (R \cdot 1_\pi \parallel U)$
⟨*proof*⟩

lemma *dom-def-var*: $Dom\ R = (R \cdot U \cap 1_\pi) \parallel 1_\sigma$
⟨*proof*⟩

lemma *ax5-d* [*simp*]: $Dom\ (R \cap NC) \cdot U = (R \cap NC) \cdot U$
⟨*proof*⟩

lemma *ax5-0* [*simp*]: $Dom\ (R \cdot \{\}) \cdot U = R \cdot \{\} \parallel U$
⟨*proof*⟩

lemma *x-c-U-split2*: $Dom\ R \cdot NC = (R \cap NC) \cdot NC \cup (R \cdot \{\} \parallel NC)$
⟨*proof*⟩

lemma *x-c-U-split3*: $Dom\ R \cdot U = (R \cap NC) \cdot U \cup (R \cdot \{\} \parallel U)$
⟨*proof*⟩

lemma *x-c-U-split-d*: $Dom\ R \cdot U = R \cdot U \cup (R \cdot \{\} \parallel U)$
⟨*proof*⟩

lemma *x-U-prop2*: $R \cdot NC = Dom\ (R \cap NC) \cdot NC \cup R \cdot \{\}$
⟨*proof*⟩

lemma *x-U-prop3*: $R \cdot U = Dom\ (R \cap NC) \cdot U \cup R \cdot \{\}$
⟨*proof*⟩

lemma *d-x-nc* [*simp*]: $Dom\ (R \cdot NC) = Dom\ R$
⟨*proof*⟩

lemma *d-x-U* [*simp*]: $Dom\ (R \cdot U) = Dom\ R$
⟨*proof*⟩

lemma *d-llp1*: $Dom\ R \subseteq Dom\ S \Longrightarrow R \subseteq Dom\ S \cdot R$
⟨*proof*⟩

lemma *d-llp2*: $R \subseteq Dom\ S \cdot R \Longrightarrow Dom\ R \subseteq Dom\ S$
⟨*proof*⟩

lemma *demod1*: $Dom\ (R \cdot S) \subseteq Dom\ T \Longrightarrow R \cdot Dom\ S \subseteq Dom\ T \cdot R$
⟨*proof*⟩

lemma *demod2*: $R \cdot Dom\ S \subseteq Dom\ T \cdot R \Longrightarrow Dom\ (R \cdot S) \subseteq Dom\ T$
⟨*proof*⟩

lemma *d-meet-closed* [*simp*]: $Dom\ (Dom\ x \cap Dom\ y) = Dom\ x \cap Dom\ y$
⟨*proof*⟩

lemma *d-add-var*: $Dom\ P \cdot R \cup Dom\ Q \cdot R = Dom\ (P \cup Q) \cdot R$
⟨*proof*⟩

lemma *d-interr-U*: $Dom\ x \cdot U \parallel Dom\ y \cdot U = Dom\ (x \parallel y) \cdot U$
⟨*proof*⟩

lemma *d-meet*: $Dom\ x \cdot z \cap Dom\ y \cdot z = (Dom\ x \cap Dom\ y) \cdot z$
⟨*proof*⟩

lemma *cs-hom-meet*: $Dom\ (x \cdot 1_\pi \cap y \cdot 1_\pi) = Dom\ (x \cdot 1_\pi) \cap Dom\ (y \cdot 1_\pi)$
⟨*proof*⟩

lemma *iso3* [*simp*]: $Dom\ (Dom\ x \cdot U) = Dom\ x$
⟨*proof*⟩

lemma *iso4* [*simp*]: $Dom\ (x \cdot 1_\pi \parallel U) \cdot U = x \cdot 1_\pi \parallel U$
⟨*proof*⟩

lemma *iso3-sharp* [*simp*]: $Dom\ (Dom\ (x \cap NC) \cdot NC) = Dom\ (x \cap NC)$
⟨*proof*⟩

lemma *iso4-sharp* [*simp*]: $Dom\ ((x \cap NC) \cdot NC) \cdot NC = (x \cap NC) \cdot NC$
⟨*proof*⟩

3.5 Vectors

lemma *vec-iff1*:
assumes $\forall a.\ (\exists A.\ (a,A) \in R) \longrightarrow (\forall A.\ (a,A) \in R)$
shows $R \cdot 1_\pi \parallel U = R$
⟨*proof*⟩

lemma *vec-iff2*:
assumes $R \cdot 1_\pi \parallel U = R$
shows $(\forall a.\ (\exists A.\ (a,A) \in R) \longrightarrow (\forall A.\ (a,A) \in R))$
⟨*proof*⟩

lemma *vec-iff*: $(\forall a.\ (\exists A.\ (a,A) \in R) \longrightarrow (\forall A.\ (a,A) \in R)) \longleftrightarrow R \cdot 1_\pi \parallel U = R$
⟨*proof*⟩

lemma *U-par-zero* [*simp*]: $\{\} \cdot R \parallel U = \{\}$
⟨*proof*⟩

lemma *U-par-s-id* [*simp*]: $1_\sigma \cdot 1_\pi \parallel U = U$
⟨*proof*⟩

lemma *U-par-p-id* [*simp*]: $1_\pi \cdot 1_\pi \parallel U = U$
⟨*proof*⟩

lemma *U-par-nc* [*simp*]: $NC \cdot 1_\pi \parallel U = U$
⟨*proof*⟩

3.6 Up-closure and Parikh composition

definition *s-prod-pa* :: $('a,'b)\ mrel \Rightarrow ('b,'c)\ mrel \Rightarrow ('a,'c)\ mrel$ (**infixl** $\otimes$ *75*)
where
$R \otimes S = \{(a,A).\ (\exists B.\ (a,B) \in R \wedge (\forall b \in B.\ (b,A) \in S))\}$

lemma *U-par-st*: $(a,A) \in R \parallel U \longleftrightarrow (\exists B.\ B \subseteq A \wedge (a,B) \in R)$
⟨*proof*⟩

lemma *p-id-U*: $R \parallel U = \{(a,B).\ \exists A.\ (a,A) \in R \wedge A \subseteq B\}$
⟨*proof*⟩

lemma *ucl-iff*: $(\forall a\ A\ B.\ (a,A) \in R \wedge A \subseteq B \longrightarrow (a,B) \in R) \longleftrightarrow R \parallel U = R$
⟨*proof*⟩

lemma *upclosed-ext*: $R \subseteq R \parallel U$
⟨*proof*⟩

lemma *onelem*: $R \cdot S \parallel U \subseteq R \otimes (S \parallel U)$
⟨*proof*⟩

lemma *twolem*: $R \otimes (S \parallel U) \subseteq R \cdot S \parallel U$
⟨*proof*⟩

lemma *pe-pa-sim*: $R \cdot S \parallel U = R \otimes (S \parallel U)$
⟨*proof*⟩

lemma *pe-pa-sim-var*: $(R \parallel U) \cdot (S \parallel U) \parallel U = (R \parallel U) \otimes (S \parallel U)$
⟨*proof*⟩

lemma *pa-assoc1*: $((R \parallel U) \otimes (S \parallel U)) \otimes (T \parallel U) \subseteq (R \parallel U) \otimes ((S \parallel U) \otimes (T \parallel U))$
⟨*proof*⟩

lemma *up-closed-par-is-meet*: $(R \parallel U) \parallel (S \parallel U) = (R \parallel U) \cap (S \parallel U)$
⟨*proof*⟩

lemma *U-nc-par* [*simp*]: $NC \parallel U = NC$
⟨*proof*⟩

lemma *uc-par-meet*: $(R \parallel U) \cap (S \parallel U) = R \parallel U \parallel S \parallel U$
⟨*proof*⟩

lemma *uc-unc* [*simp*]: $R \parallel U \parallel R \parallel U = R \parallel U$
⟨*proof*⟩

lemma *uc-interr*: $(R \parallel S) \cdot (T \parallel U) = R \cdot (T \parallel U) \parallel S \cdot (T \parallel U)$
⟨*proof*⟩

lemma *iso5* [*simp*]: $(R \cdot 1_\pi \parallel U) \cdot 1_\pi = R \cdot 1_\pi$
⟨*proof*⟩

lemma *iso6* [*simp*]: $(R \cdot 1_\pi \parallel U) \cdot 1_\pi \parallel U = R \cdot 1_\pi \parallel U$
⟨*proof*⟩

lemma *sv-hom-par*: $(R \parallel S) \cdot U = R \cdot U \parallel S \cdot U$
⟨*proof*⟩

lemma *vs-hom-meet*: $Dom\ ((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) = Dom\ (R \cdot 1_\pi \parallel U) \cap Dom\ (S \cdot 1_\pi \parallel U)$
⟨*proof*⟩

lemma *cv-hom-meet*: $(R \cdot 1_\pi \cap S \cdot 1_\pi) \parallel U = (R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)$
⟨*proof*⟩

lemma *cv-hom-par* [*simp*]: $R \parallel U \parallel S \parallel U = (R \parallel S) \parallel U$
⟨*proof*⟩

lemma *vc-hom-meet*: $((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cap ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$
⟨*proof*⟩

lemma *vc-hom-seq*: $((R \cdot 1_\pi \parallel U) \cdot (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cdot ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$
⟨*proof*⟩

3.7 Nonterminal and terminal multirelations

definition *tau* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* (τ) **where**
$\tau\ R = R \cdot \{\}$

definition *nu* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* (ν) **where**
$\nu\ R = R \cap NC$

lemma *nc-s* [*simp*]: $\nu\ 1_\sigma = 1_\sigma$
$\langle proof \rangle$

lemma *nc-scomp-closed*: $\nu\ R \cdot \nu\ S \subseteq NC$
$\langle proof \rangle$

lemma *nc-scomp-closed-alt* [*simp*]: $\nu\ (\nu\ R \cdot \nu\ S) = \nu\ R \cdot \nu\ S$
$\langle proof \rangle$

lemma *nc-ccomp-closed*: $\nu\ R \parallel \nu\ S \subseteq NC$
$\langle proof \rangle$

lemma *nc-ccomp-closed-alt* [*simp*]: $\nu\ (R \parallel \nu\ S) = R \parallel \nu\ S$
$\langle proof \rangle$

lemma *tarski-prod*: $(\nu\ R \cdot NC) \cdot (\nu\ S \cdot NC) = (\mathit{if}\ \nu\ S = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ \nu\ R \cdot NC)$
$\langle proof \rangle$

lemma *nc-prod-aux* [*simp*]: $(\nu\ R \cdot NC) \cdot NC = \nu\ R \cdot NC$
$\langle proof \rangle$

lemma *nc-vec-add-closed*: $(\nu\ R \cdot NC \cup \nu\ S \cdot NC) \cdot NC = \nu\ R \cdot NC \cup \nu\ S \cdot NC$
$\langle proof \rangle$

lemma *nc-vec-par-is-meet*: $\nu\ R \cdot NC \parallel \nu\ S \cdot NC = \nu\ R \cdot NC \cap \nu\ S \cdot NC$
$\langle proof \rangle$

lemma *nc-vec-meet-closed*: $(\nu\ R \cdot NC \cap \nu\ S \cdot NC) \cdot NC = \nu\ R \cdot NC \cap \nu\ S \cdot NC$
$\langle proof \rangle$

lemma *nc-vec-par-closed*: $(\nu\ R \cdot NC \parallel \nu\ S \cdot NC) \cdot NC = \nu\ R \cdot NC \parallel \nu\ S \cdot NC$
$\langle proof \rangle$

lemma *nc-vec-seq-closed*: $((\nu\ R \cdot NC) \cdot (\nu\ S \cdot NC)) \cdot NC = (\nu\ R \cdot NC) \cdot (\nu\ S \cdot NC)$
$\langle proof \rangle$

lemma *iso5-sharp* [*simp*]: $(\nu\ R \cdot 1_\pi \parallel NC) \cdot 1_\pi = \nu\ R \cdot 1_\pi$
$\langle proof \rangle$

lemma *iso6-sharp* [*simp*]: $(\nu\ R \cdot NC \cdot 1_\pi) \parallel NC = \nu\ R \cdot NC$

$\langle proof \rangle$

lemma *nsv-hom-par*: $(R \parallel S) \cdot NC = R \cdot NC \parallel S \cdot NC$
$\langle proof \rangle$

lemma *nvs-hom-meet*: $Dom\ (\nu\ R \cdot NC \cap \nu\ S \cdot NC) = Dom\ (\nu\ R \cdot NC) \cap Dom\ (\nu\ S \cdot NC)$
$\langle proof \rangle$

lemma *ncv-hom-meet*: $R \cdot 1_\pi \cap S \cdot 1_\pi \parallel NC = (R \cdot 1_\pi \parallel NC) \cap (S \cdot 1_\pi \parallel NC)$
$\langle proof \rangle$

lemma *ncv-hom-par*: $(R \parallel S) \parallel NC = R \parallel NC \parallel S \parallel NC$
$\langle proof \rangle$

lemma *nvc-hom-meet*: $(\nu\ R \cdot NC \cap \nu\ S \cdot NC) \cdot 1_\pi = (\nu\ R \cdot NC) \cdot 1_\pi \cap (\nu\ S \cdot NC) \cdot 1_\pi$
$\langle proof \rangle$

lemma *tau-int*: $\tau\ R \leq R$
$\langle proof \rangle$

lemma *nu-int*: $\nu\ R \leq R$
$\langle proof \rangle$

lemma *tau-ret* [*simp*]: $\tau\ (\tau\ R) = \tau\ R$
$\langle proof \rangle$

lemma *nu-ret* [*simp*]: $\nu\ (\nu\ R) = \nu\ R$
$\langle proof \rangle$

lemma *tau-iso*: $R \leq S \Longrightarrow \tau\ R \leq \tau\ S$
$\langle proof \rangle$

lemma *nu-iso*: $R \leq S \Longrightarrow \nu\ R \leq \nu\ S$
$\langle proof \rangle$

lemma *tau-zero* [*simp*]: $\tau\ \{\} = \{\}$
$\langle proof \rangle$

lemma *nu-zero* [*simp*]: $\nu\ \{\} = \{\}$
$\langle proof \rangle$

lemma *tau-s* [*simp*]: $\tau\ 1_\sigma = \{\}$
$\langle proof \rangle$

lemma *tau-c* [*simp*]: $\tau\ 1_\pi = 1_\pi$
$\langle proof \rangle$

lemma *nu-c* [*simp*]: ν $1_\pi = \{\}$
⟨*proof*⟩

lemma *tau-nc* [*simp*]: τ $NC = \{\}$
⟨*proof*⟩

lemma *nu-nc* [*simp*]: ν $NC = NC$
⟨*proof*⟩

lemma *tau-U* [*simp*]: τ $U = 1_\pi$
⟨*proof*⟩

lemma *nu-U* [*simp*]: ν $U = NC$
⟨*proof*⟩

lemma *tau-add* [*simp*]: τ $(R \cup S) = \tau$ $R \cup \tau$ S
⟨*proof*⟩

lemma *nu-add* [*simp*]: ν $(R \cup S) = \nu$ $R \cup \nu$ S
⟨*proof*⟩

lemma *tau-meet* [*simp*]: τ $(R \cap S) = \tau$ $R \cap \tau$ S
⟨*proof*⟩

lemma *nu-meet* [*simp*]: ν $(R \cap S) = \nu$ $R \cap \nu$ S
⟨*proof*⟩

lemma *tau-seq*: τ $(R \cdot S) = \tau$ $R \cup \nu$ $R \cdot \tau$ S
⟨*proof*⟩

lemma *tau-par* [*simp*]: τ $(R \parallel S) = \tau$ $R \parallel \tau$ S
⟨*proof*⟩

lemma *nu-par-aux1*: $R \parallel \tau$ $S = Dom$ $(\tau$ $S) \cdot R$
⟨*proof*⟩

lemma *nu-par-aux3* [*simp*]: ν $(\nu$ $R \parallel \tau$ $S) = \nu$ $R \parallel \tau$ S
⟨*proof*⟩

lemma *nu-par-aux4* [*simp*]: ν $(\tau$ $R \parallel \tau$ $S) = \{\}$
⟨*proof*⟩

lemma *nu-par*: ν $(R \parallel S) = Dom$ $(\tau$ $R) \cdot \nu$ $S \cup Dom$ $(\tau$ $S) \cdot \nu$ $R \cup (\nu$ $R \parallel \nu$ $S)$
⟨*proof*⟩

lemma *sprod-tau-nu*: $R \cdot S = \tau$ $R \cup \nu$ $R \cdot S$
⟨*proof*⟩

lemma *pprod-tau-nu*: $R \parallel S = (\nu$ $R \parallel \nu$ $S) \cup Dom$ $(\tau$ $R) \cdot \nu$ $S \cup Dom$ $(\tau$ $S) \cdot \nu$

$R \cup (\tau\ R \parallel \tau\ S)$
⟨*proof*⟩

lemma *tau-idem* [*simp*]: $\tau\ R \cdot \tau\ R = \tau\ R$
⟨*proof*⟩

lemma *tau-interr*: $(R \parallel S) \cdot \tau\ T = R \cdot \tau\ T \parallel S \cdot \tau\ T$
⟨*proof*⟩

lemma *tau-le-c*: $\tau\ R \leq 1_\pi$
⟨*proof*⟩

lemma *c-le-tauc*: $1_\pi \leq \tau\ 1_\pi$
⟨*proof*⟩

lemma *x-alpha-tau* [*simp*]: $\nu\ R \cup \tau\ R = R$
⟨*proof*⟩

lemma *alpha-tau-zero* [*simp*]: $\nu\ (\tau\ R) = \{\}$
⟨*proof*⟩

lemma *tau-alpha-zero* [*simp*]: $\tau\ (\nu\ R) = \{\}$
⟨*proof*⟩

lemma *sprod-tau-nu-var* [*simp*]: $\nu\ (\nu\ R \cdot S) = \nu\ (R \cdot S)$
⟨*proof*⟩

lemma *tau-s-prod* [*simp*]: $\tau\ (R \cdot S) = R \cdot \tau\ S$
⟨*proof*⟩

lemma *alpha-fp*: $\nu\ R = R \longleftrightarrow R \cdot \{\} = \{\}$
⟨*proof*⟩

lemma *p-prod-tau-alpha*: $R \parallel S = (R \parallel \nu\ S) \cup (\nu\ R \parallel S) \cup (\tau\ R \parallel \tau\ S)$
⟨*proof*⟩

lemma *p-prod-tau-alpha-var*: $R \parallel S = (R \parallel \nu\ S) \cup (\nu\ R \parallel S) \cup \tau\ (R \parallel S)$
⟨*proof*⟩

lemma *alpha-par*: $\nu\ (R \parallel S) = (\nu\ R \parallel S) \cup (R \parallel \nu\ S)$
⟨*proof*⟩

lemma *alpha-tau* [*simp*]: $\nu\ (R \cdot \tau\ S) = \{\}$
⟨*proof*⟩

lemma *nu-par-prop*: $\nu\ R = R \Longrightarrow \nu\ (R \parallel S) = R \parallel S$
⟨*proof*⟩

lemma *tau-seq-prop*: $\tau\ R = R \Longrightarrow R \cdot S = R$

$\langle proof \rangle$

lemma *tau-seq-prop2*: $\tau\ R = R \Longrightarrow \tau\ (R \cdot S) = R \cdot S$
$\langle proof \rangle$

lemma *d-nu*: $\nu\ (Dom\ R \cdot S) = Dom\ R \cdot \nu\ S$
$\langle proof \rangle$

lemma *nu-ideal1*: $\nu\ R = R \Longrightarrow S \leq R \Longrightarrow \nu\ S = S$
$\langle proof \rangle$

lemma *tau-ideal1*: $\tau\ R = R \Longrightarrow S \leq R \Longrightarrow \tau\ S = S$
$\langle proof \rangle$

lemma *nu-ideal2*: $\nu\ R = R \Longrightarrow \nu\ S = S \Longrightarrow \nu\ (R \cup S) = R \cup S$
$\langle proof \rangle$

lemma *tau-ideal2*: $\tau\ R = R \Longrightarrow \tau\ S = S \Longrightarrow \tau\ (R \cup S) = R \cup S$
$\langle proof \rangle$

lemma *tau-add-precong*: $\tau\ R \leq \tau\ S \Longrightarrow \tau\ (R \cup T) \leq \tau\ (S \cup T)$
$\langle proof \rangle$

lemma *tau-meet-precong*: $\tau\ R \leq \tau\ S \Longrightarrow \tau\ (R \cap T) \leq \tau\ (S \cap T)$
$\langle proof \rangle$

lemma *tau-par-precong*: $\tau\ R \leq \tau\ S \Longrightarrow \tau\ (R \parallel T) \leq \tau\ (S \parallel T)$
$\langle proof \rangle$

lemma *tau-seq-precongl*: $\tau\ R \leq \tau\ S \Longrightarrow \tau\ (T \cdot R) \leq \tau\ (T \cdot S)$
$\langle proof \rangle$

lemma *nu-add-precong*: $\nu\ R \leq \nu\ S \Longrightarrow \nu\ (R \cup T) \leq \nu\ (S \cup T)$
$\langle proof \rangle$

lemma *nu-meet-precong*: $\nu\ R \leq \nu\ S \Longrightarrow \nu\ (R \cap T) \leq \nu\ (S \cap T)$
$\langle proof \rangle$

lemma *nu-seq-precongr*: $\nu\ R \leq \nu\ S \Longrightarrow \nu\ (R \cdot T) \leq \nu\ (S \cdot T)$
$\langle proof \rangle$

definition
$tcg\ R\ S = (\tau\ R \leq \tau\ S \wedge \tau\ S \leq \tau\ R)$

definition
$ncg\ R\ S = (\nu\ R \leq \nu\ S \wedge \nu\ S \leq \nu\ R)$

lemma *tcg-refl*: $tcg\ R\ R$
$\langle proof \rangle$

lemma *tcg-trans*: $tcg\ R\ S \Longrightarrow tcg\ S\ T \Longrightarrow tcg\ R\ T$
⟨*proof*⟩

lemma *tcg-sym*: $tcg\ R\ S \Longrightarrow tcg\ S\ R$
⟨*proof*⟩

lemma *ncg-refl*: $ncg\ R\ R$
⟨*proof*⟩

lemma *ncg-trans*: $ncg\ R\ S \Longrightarrow ncg\ S\ T \Longrightarrow ncg\ R\ T$
⟨*proof*⟩

lemma *ncg-sym*: $ncg\ R\ S \Longrightarrow ncg\ S\ R$
⟨*proof*⟩

lemma *tcg-alt*: $tcg\ R\ S = (\tau\ R = \tau\ S)$
⟨*proof*⟩

lemma *ncg-alt*: $ncg\ R\ S = (\nu\ R = \nu\ S)$
⟨*proof*⟩

lemma *tcg-add*: $\tau\ R = \tau\ S \Longrightarrow \tau\ (R \cup T) = \tau\ (S \cup T)$
⟨*proof*⟩

lemma *tcg-meet*: $\tau\ R = \tau\ S \Longrightarrow \tau\ (R \cap T) = \tau\ (S \cap T)$
⟨*proof*⟩

lemma *tcg-par*: $\tau\ R = \tau\ S \Longrightarrow \tau\ (R \parallel T) = \tau\ (S \parallel T)$
⟨*proof*⟩

lemma *tcg-seql*: $\tau\ R = \tau\ S \Longrightarrow \tau\ (T \cdot R) = \tau\ (T \cdot S)$
⟨*proof*⟩

lemma *ncg-add*: $\nu\ R = \nu\ S \Longrightarrow \nu\ (R \cup T) = \nu\ (S \cup T)$
⟨*proof*⟩

lemma *ncg-meet*: $\nu\ R = \nu\ S \Longrightarrow \nu\ (R \cap T) = \nu\ (S \cap T)$
⟨*proof*⟩

lemma *ncg-seqr*: $\nu\ R = \nu\ S \Longrightarrow \nu\ (R \cdot T) = \nu\ (S \cdot T)$
⟨*proof*⟩

3.8 Powers

primrec *p-power* :: $('a,'a)\ mrel \Rightarrow nat \Rightarrow ('a,'a)\ mrel$ **where**
p-power $R\ 0 \quad = 1_\sigma$ |
p-power $R\ (Suc\ n) = R \cdot$ *p-power* $R\ n$

primrec *power-rd* :: $('a,'a)$ *mrel* $\Rightarrow$ *nat* $\Rightarrow$ $('a,'a)$ *mrel* **where**
power-rd R 0 $= \{\}$ |
power-rd R (Suc n) $= 1_\sigma \cup R \cdot$ *power-rd R n*

primrec *power-sq* :: $('a,'a)$ *mrel* $\Rightarrow$ *nat* $\Rightarrow$ $('a,'a)$ *mrel* **where**
power-sq R 0 $= 1_\sigma$ |
power-sq R (Suc n) $= 1_\sigma \cup R \cdot$ *power-sq R n*

lemma *power-rd-chain*: *power-rd R n* $\leq$ *power-rd R* $(n + 1)$
⟨*proof*⟩

lemma *power-sq-chain*: *power-sq R n* $\leq$ *power-sq R* $(n + 1)$
⟨*proof*⟩

lemma *pow-chain*: *p-power* $(1_\sigma \cup R)$ $n \leq$ *p-power* $(1_\sigma \cup R)$ $(n + 1)$
⟨*proof*⟩

lemma *pow-prop*: *p-power* $(1_\sigma \cup R)$ $(n + 1) = 1_\sigma \cup R \cdot$ *p-power* $(1_\sigma \cup R)$ n
⟨*proof*⟩

lemma *power-rd-le-sq*: *power-rd R n* $\leq$ *power-sq R n*
⟨*proof*⟩

lemma *power-sq-le-rd*: *power-sq R n* $\leq$ *power-rd R (Suc n)*
⟨*proof*⟩

lemma *power-sq-power*: *power-sq R n* $=$ *p-power* $(1_\sigma \cup R)$ n
⟨*proof*⟩

3.9 Star

lemma *iso-prop*: *mono* $(\lambda X.\ S \cup R \cdot X)$
⟨*proof*⟩

lemma *gfp-lfp-prop*: *gfp* $(\lambda X.\ R \cdot X) \cup$ *lfp* $(\lambda X.\ S \cup R \cdot X) \subseteq$ *gfp* $(\lambda X.\ S \cup R \cdot X)$
⟨*proof*⟩

definition *star* :: $('a,'a)$ *mrel* $\Rightarrow$ $('a,'a)$ *mrel* **where**
star R $=$ *lfp* $(\lambda X.$ *s-id* $\cup R \cdot X)$

lemma *star-unfold*: $1_\sigma \cup R \cdot$ *star R* $\leq$ *star R*
⟨*proof*⟩

lemma *star-induct*: $1_\sigma \cup R \cdot S \leq S \Longrightarrow$ *star R* $\leq S$
⟨*proof*⟩

lemma *star-refl*: $1_\sigma \leq$ *star R*
⟨*proof*⟩

lemma *star-unfold-part*: $R \cdot star\ R \leq star\ R$
⟨*proof*⟩

lemma *star-ext-aux*: $R \leq R \cdot star\ R$
⟨*proof*⟩

lemma *star-ext*: $R \leq star\ R$
⟨*proof*⟩

lemma *star-co-trans*: $star\ R \leq star\ R \cdot star\ R$
⟨*proof*⟩

lemma *star-iso*: $R \leq S \Longrightarrow star\ R \leq star\ S$
⟨*proof*⟩

lemma *star-unfold-eq* [*simp*]: $1_\sigma \cup R \cdot star\ R = star\ R$
⟨*proof*⟩

lemma *nu-star1*:
assumes $\bigwedge(R::('a,'a)\ mrel)\ (S::('a,'a)\ mrel)\ (T::('a,'a)\ mrel).\ R \cdot (S \cdot T) = (R \cdot S) \cdot T$
shows $star\ (R::('a,'a)\ mrel) \leq star\ (\nu\ R) \cdot (1_\sigma \cup \tau\ R)$
⟨*proof*⟩

lemma *nu-star2*:
assumes $\bigwedge(R::('a,'a)\ mrel).\ star\ R \cdot star\ R \leq star\ R$
shows $star\ (\nu\ (R::('a,'a)\ mrel)) \cdot (1_\sigma \cup \tau\ R) \leq star\ R$
⟨*proof*⟩

lemma *nu-star*:
assumes $\bigwedge(R::('a,'a)\ mrel).\ star\ R \cdot star\ R \leq star\ R$
and $\bigwedge(R::('a,'a)\ mrel)\ (S::('a,'a)\ mrel)\ (T::('a,'a)\ mrel).\ R \cdot (S \cdot T) = (R \cdot S) \cdot T$
shows $star\ (\nu\ (R::('a,'a)\ mrel)) \cdot (1_\sigma \cup \tau\ R) = star\ R$
⟨*proof*⟩

lemma *tau-star*: $star\ (\tau\ R) = 1_\sigma \cup \tau\ R$
⟨*proof*⟩

lemma *tau-star-var*:
assumes $\bigwedge(R::('a,'a)\ mrel)\ (S::('a,'a)\ mrel)\ (T::('a,'a)\ mrel).\ R \cdot (S \cdot T) = (R \cdot S) \cdot T$
and $\bigwedge(R::('a,'a)\ mrel).\ star\ R \cdot star\ R \leq star\ R$
shows $\tau\ (star\ (R::('a,'a)\ mrel)) = star\ (\nu\ R) \cdot \tau\ R$
⟨*proof*⟩

lemma *nu-star-sub*: $star\ (\nu\ R) \leq \nu\ (star\ R)$
⟨*proof*⟩

lemma *nu-star-nu* [*simp*]: ν (*star* (ν *R*)) = *star* (ν *R*)
⟨*proof*⟩

lemma *nu-star-tau* [*simp*]: ν (*star* (τ *R*)) = 1_σ
⟨*proof*⟩

lemma *tau-star-tau* [*simp*]: τ (*star* (τ *R*)) = τ *R*
⟨*proof*⟩

lemma *tau-star-nu* [*simp*]: τ (*star* (ν *R*)) = {}
⟨*proof*⟩

lemma *d-star-unfold* [*simp*]: *Dom S* ∪ *Dom* (*R* · *Dom* (*star R* · *S*)) = *Dom* (*star R* · *S*)
⟨*proof*⟩

lemma *d-star-sim1*:
assumes $\bigwedge$*R S T*. *Dom* (*T*::('*a*,'*b*) *mrel*) ∪ (*R*::('*a*,'*a*) *mrel*) · (*S*::('*a*,'*a*) *mrel*) ≤ *S* ⟹ *star R* · *Dom T* ≤ *S*
shows (*R*::('*a*,'*a*) *mrel*) · *Dom* (*T*::('*a*,'*b*) *mrel*) ≤ *Dom T* · (*S*::('*a*,'*a*) *mrel*) ⟹ *star R* · *Dom T* ≤ *Dom T* · *star S*
⟨*proof*⟩

lemma *d-star-induct*:
assumes $\bigwedge$*R S T*. *Dom* (*T*::('*a*,'*b*) *mrel*) ∪ (*R*::('*a*,'*a*) *mrel*) · (*S*::('*a*,'*a*) *mrel*) ≤ *S* ⟹ *star R* · *Dom T* ≤ *S*
shows *Dom* ((*R*::('*a*,'*a*) *mrel*) · (*S*::('*a*,'*a*) *mrel*)) ≤ *Dom S* ⟹ *Dom* (*star R* · *S*) ≤ *Dom S*
⟨*proof*⟩

3.10 Omega

definition *omega* :: ('*a*,'*a*) *mrel* ⇒ ('*a*,'*a*) *mrel* **where**
omega R ≡ *gfp* (λ*X*. *R* · *X*)

lemma *om-unfold*: *omega R* ≤ *R* · *omega R*
⟨*proof*⟩

lemma *om-coinduct*: *S* ≤ *R* · *S* ⟹ *S* ≤ *omega R*
⟨*proof*⟩

lemma *om-unfold-eq* [*simp*]: *R* · *omega R* = *omega R*
⟨*proof*⟩

lemma *om-iso*: *R* ≤ *S* ⟹ *omega R* ≤ *omega S*
⟨*proof*⟩

lemma *zero-om* [*simp*]: *omega* {} = {}

⟨*proof*⟩

lemma *s-id-om* [*simp*]: $omega\ 1_\sigma = U$
⟨*proof*⟩

lemma *p-id-om* [*simp*]: $omega\ 1_\pi = 1_\pi$
⟨*proof*⟩

lemma *nc-om* [*simp*]: $omega\ NC = U$
⟨*proof*⟩

lemma *U-om* [*simp*]: $omega\ U = U$
⟨*proof*⟩

lemma *tau-om1*: $\tau\ R \leq \tau\ (omega\ R)$
⟨*proof*⟩

lemma *tau-om2* [*simp*]: $omega\ (\tau\ R) = \tau\ R$
⟨*proof*⟩

lemma *tau-om3*: $omega\ (\tau\ R) \leq \tau\ (omega\ R)$
⟨*proof*⟩

lemma *om-nu-tau*: $omega\ (\nu\ R) \cup star\ (\nu\ R) \cdot \tau\ R \leq omega\ R$
⟨*proof*⟩

end

4 Multirelational Properties of Power Allegories

theory *Power-Allegories-Multirelations*

imports *Multirelations-Basics*

begin

We start with random little properties.

lemma *eta-s-id*: $\eta = s\text{-}id$
⟨*proof*⟩

lemma *Lambda-empty* [*simp*]: $\Lambda\ \{\} = p\text{-}id$
⟨*proof*⟩

lemma *alpha-pid* [*simp*]: $\alpha\ p\text{-}id = \{\}$
⟨*proof*⟩

4.1 Peleg lifting

definition *plift* :: $('a,'b)\ mrel \Rightarrow ('a\ set,'b\ set)\ rel\ (\text{-}_*\ [1000]\ 999)$ **where**

$R_* = \{(A,B).\ \exists f.\ (\forall a \in A.\ (a,f(a)) \in R) \land B = \bigcup (f \,\text{`}\, A)\}$

lemma *pcomp-plift*: $R \cdot S = R \ ; S_*$
$\langle proof \rangle$

lemma *det-plift-klift*: *deterministic* $R \Longrightarrow R_* = (R)_{\mathcal{P}}$
$\langle proof \rangle$

lemma *plift-ext2* [*simp*]: $\eta \ ; R_* = R$
$\langle proof \rangle$

lemma *pliftext-3* [*simp*]: $\eta_* = Id$
$\langle proof \rangle$

lemma *d-dom-plift*: $(Dom\ R)_* = dom\ (R_*)$
$\langle proof \rangle$

lemma *d-pid-plift*: $(Dom\ R)_* \subseteq Id$
$\langle proof \rangle$

lemma *d-plift-sub*: $A \subseteq B \Longrightarrow (B,B) \in (Dom\ R)_* \Longrightarrow (A,A) \in (Dom\ R)_*$
$\langle proof \rangle$

lemma *plift-empty*: $(\{\},A) \in R_* \longleftrightarrow A = \{\}$
$\langle proof \rangle$

lemma *univ-plift-klift*:
assumes *univalent* R
shows $R_* = (Dom\ R)_* \ ; (R)_{\mathcal{P}}$
$\langle proof \rangle$

lemma *plift-ext1*:
assumes *univalent* f
shows $(R \ ; f_*)_* = R_* \ ; f_*$
$\langle proof \rangle$

lemma *plift-assoc-univ*: *univalent* $f \Longrightarrow (R \cdot S) \cdot f = R \cdot (S \cdot f)$
$\langle proof \rangle$

lemma *Lambda-funct*: $\Lambda\ (R \ ; S) = \Lambda\ R \cdot \Lambda\ S$
$\langle proof \rangle$

lemma *eta-funct*: $R \ ; S \ ; \eta = (R \ ; \eta) \cdot (S \ ; \eta)$
$\langle proof \rangle$

lemma *alpha-funct-det*: *deterministic* $R \Longrightarrow$ *deterministic* $S \Longrightarrow \alpha\ (R \cdot S) = \alpha$
$R \ ; \alpha\ S$
$\langle proof \rangle$

lemma *pcomp-det*: *deterministic* $S \Longrightarrow R \cdot S = R \mathbin{;} (S)_{\mathcal{P}}$
⟨*proof*⟩

lemma *pcomp-det2*: *deterministic* $R \Longrightarrow$ *deterministic* $S \Longrightarrow (R \cdot S)_{\mathcal{P}} = (R)_{\mathcal{P}} \mathbin{;} (S)_{\mathcal{P}}$
⟨*proof*⟩

lemma *pcomp-alpha*: $\alpha\ (R \cdot S) = R \mathbin{;} \alpha\ ((S)_*)$
⟨*proof*⟩

4.2 Fusion and fission

definition *fus* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* **where**
fus $R = \Lambda\ (\alpha\ R)$

definition *fis* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* **where**
fis $R = \alpha\ R \mathbin{;} \eta$

lemma *fus-set*: *fus* $R = \{(a,B)\ |a\ B.\ B = \bigcup(\mathit{Image}\ R\ \{a\})\}$
⟨*proof*⟩

lemma *fis-set*: *fis* $R = \{(a,\{b\})\ |a\ b.\ b \in \bigcup(\mathit{Image}\ R\ \{a\})\}$
⟨*proof*⟩

lemma *fis-det-comp*: *deterministic* $R \Longrightarrow$ *deterministic* $S \Longrightarrow$ *fis* $(R \cdot S) =$ *fis* $R \cdot$ *fis* S
⟨*proof*⟩

lemma *fis-fix-det*: *deterministic* $R = ($*fus* $R = R)$
⟨*proof*⟩

4.3 Galois connections for multirelations

lemma *sub-subh*: $R \subseteq S \Longrightarrow R \subseteq S \mathbin{;} (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *alpha-Lambda-galois*: $(\alpha\ R \subseteq S) = (R \subseteq \Lambda\ S \mathbin{;} (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff}))$
⟨*proof*⟩

lemma *alpha-Lambda-galois-set*: $(\alpha\ R \subseteq S) = (R \subseteq \{(a,A).\ \exists B.\ (a,B) \in \Lambda\ S \wedge A \subseteq B\})$
⟨*proof*⟩

lemma *epsiloff-eta-lres*: $\mathit{epsiloff} \mathbin{;} \eta \subseteq \mathit{epsiloff}\ /\!/\ \mathit{epsiloff}$
⟨*proof*⟩

lemma *eta-alpha-galois*: $(R \mathbin{;} \eta \subseteq S \mathbin{;} (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})) = (R \subseteq \alpha\ S)$
⟨*proof*⟩

lemma *eta-alpha-galois-set*: $(R ; \eta \subseteq \{(a,A).\ \exists B.\ (a,B) \in S \wedge A \subseteq B\}) = (R \subseteq \alpha\ S)$
⟨*proof*⟩

lemma *Lambda-iso*: $R \subseteq S \Longrightarrow \Lambda\ R \subseteq \Lambda\ S ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *eta-iso*: $R \subseteq S \Longrightarrow R ; \eta \subseteq S ; \eta ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *alpha-iso*: $R \subseteq S ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff}) \Longrightarrow \alpha\ R \subseteq \alpha\ S$
⟨*proof*⟩

lemma *Lambda-canc-dcl*: $R \subseteq \Lambda\ (\alpha\ R) ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *eta-canc-dcl*: $\alpha\ R ; \eta \subseteq R ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *alpha-surj*: *surj* α
⟨*proof*⟩

lemma *Lambda-inj*: *inj* Λ
⟨*proof*⟩

lemma *eta-inj*: *inj* $(\lambda x.\ x ; \eta)$
⟨*proof*⟩

lemma *fus-least-odet*:
assumes $\Lambda\ (\alpha\ S) = S$
and $R \subseteq S ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
shows $\Lambda\ (\alpha\ R) \subseteq S ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *fis-greatest-idet*:
assumes $\alpha\ S ; \eta = S$
and $S \subseteq R ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
shows $S \subseteq \alpha\ R ; \eta ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})$
⟨*proof*⟩

lemma *fis-fus-galois*: $(\alpha\ R ; \eta \subseteq S ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff})) = (R \subseteq \Lambda\ (\alpha\ S) ; (\mathit{epsiloff}\ /\!/\ \mathit{epsiloff}))$
⟨*proof*⟩

4.4 Properties of alpha, fission and fusion

lemma *alpha-lax*: $\alpha\ (R \cdot S) \subseteq \alpha\ R ; \alpha\ S$
⟨*proof*⟩

lemma *alpha-down* [*simp*]: $\alpha\ (R\ ;\ \Omega^{\smile}) = \alpha\ R$
⟨*proof*⟩

lemma *fis-fis* [*simp*]: $\mathit{fis} \circ \mathit{fis} = \mathit{fis}$
⟨*proof*⟩

lemma *fus-fus* [*simp*]: $\mathit{fus} \circ \mathit{fus} = \mathit{fus}$
⟨*proof*⟩

lemma *fis-fus* [*simp*]: $\mathit{fis} \circ \mathit{fus} = \mathit{fis}$
⟨*proof*⟩

lemma *fus-fis* [*simp*]: $\mathit{fus} \circ \mathit{fis} = \mathit{fus}$
⟨*proof*⟩

lemma *fis-alpha*: $\mathit{fis}\ R \cdot S = \alpha\ R\ ;\ S$
⟨*proof*⟩

lemma *fis-lax*: $\mathit{fis}\ (R \cdot S) \subseteq \mathit{fis}\ R \cdot \mathit{fis}\ S$
⟨*proof*⟩

lemma *klift-fus*: $(R)_{\mathcal{P}} = \mathit{fus}\ (\mathit{epsiloff}\ ;\ R)$
⟨*proof*⟩

lemma *fus-eta-klift*: $\mathit{fus}\ R = \eta\ ;\ (R)_{\mathcal{P}}$
⟨*proof*⟩

lemma *fus-Lambda-mu*: $\mathit{fus}\ R = \Lambda\ R\ ;\ \mu$
⟨*proof*⟩

4.5 Properties of fusion, fission, nu and tau

lemma *alpha-tau* [*simp*]: $\alpha\ (\tau\ R) = \{\}$
⟨*proof*⟩

lemma *alpha-nu* [*simp*]: $\alpha\ (\nu\ R) = \alpha\ R$
⟨*proof*⟩

lemma *nu-fis* [*simp*]: $\nu\ (\mathit{fis}\ R) = \mathit{fis}\ R$
⟨*proof*⟩

lemma *nu-fis-var*: $\nu\ (\mathit{fis}\ R) = \mathit{fis}\ (\nu\ R)$
⟨*proof*⟩

lemma *tau-fis* [*simp*]: $\tau\ (\mathit{fis}\ R) = \{\}$
⟨*proof*⟩

Properties of tests and domain

lemma *subid-plift*: $(P \cap \eta)_* = \{(A,A) | A.\ \forall a \in A.\ (a,\{a\}) \in (P \cap \eta)\}$
⟨*proof*⟩

lemma *U-subid*: R ; $(P \cap \eta)_* = R \cap U$; $(P \cap \eta)_*$
⟨*proof*⟩

lemma *subid-plift-down*: U ; $(P \cap \eta)_*$; $\Omega^\smile = U$; $(P \cap \eta)_*$
⟨*proof*⟩

lemma *nu-subid-plift*: ν $(R$; $(P \cap \eta)_*) = \nu$ R ; $(P \cap \eta)_*$
⟨*proof*⟩

lemma *dom-fis1*: *dom* (*fis* R) = *dom* (α R)
⟨*proof*⟩

lemma *dom-fis2*: *dom* (*fis* R) = *dom* (α (ν R))
⟨*proof*⟩

lemma *dom-fis3*: *dom* (*fis* R) = *dom* (ν R)
⟨*proof*⟩

lemma *dom-fis4*: *dom* (*fis* R) = *dom* (ν (*fus* R))
⟨*proof*⟩

lemma *dom-alpha*: *dom* (α R ; $(P \cap \eta)$) = *dom* (ν (R ; $\Omega^\smile$) ; $(P \cap \eta)_*$)
⟨*proof*⟩

4.6 Box and diamond

definition *box* :: $('a, 'b)$ *mrel* $\Rightarrow$ $('b$ *set*, $'a$ *set*) *rel* **where**
box R = *rbox* (α R)

definition *dia* :: $('a, 'b)$ *mrel* $\Rightarrow$ $('b$ *set*, $'a$ *set*) *rel* **where**
dia $R = \mathcal{P}$ $((\alpha\ R)^\smile)$

lemma *box-set*: *box* $R = \{(B, A).\ A = \{a.\ \forall C.\ (a, C) \in R \longrightarrow C \subseteq B\}\}$
⟨*proof*⟩

lemma *dia-set*: *dia* $R = \{(B, A).\ A = \{a.\ \exists C.\ (a, C) \in R \wedge C \cap B \neq \{\}\}\}$
⟨*proof*⟩

lemma *box-Omega*: *box* $R = \Lambda$ $(\Omega^\smile /\!\!/ R)$
⟨*proof*⟩

end
theory *Multirelations*

imports *Power-Allegories-Multirelations*

begin

lemma *nonempty-set-card*:
  assumes *finite S*
  shows $S \neq \{\} \longleftrightarrow card\ S \geq 1$
  ⟨*proof*⟩

no-notation *one-class.one* (*1*)
no-notation *times-class.times* (**infixl** $*$ *70*)

no-notation *rel-fdia* ((|-⟩-) [*61*,*81*] *82*)
no-notation *rel-bdia* ((⟨-|-) [*61*,*81*] *82*)
no-notation *rel-fbox* ((|-]-) [*61*,*81*] *82*)
no-notation *rel-bbox* (([-|-) [*61*,*81*] *82*)

declare *s-prod-pa-def* [*mr-simp*]

notation *s-prod* (**infixl** $*$ *70*)
notation *s-id* (*1*)

lemma *sp-oi-subdist*:
  $(P \cap Q) * (R \cap S) \subseteq P * R$
  ⟨*proof*⟩

lemma *sp-oi-subdist-2*:
  $(P \cap Q) * (R \cap S) \subseteq (P * R) \cap (Q * S)$
  ⟨*proof*⟩

5 Inner Structure

5.1 Inner union, inner intersection and inner complement

abbreviation *inner-union* (**infixl** ∪∪ *65*)
  where *inner-union* ≡ *p-prod*

definition *inner-intersection* :: $('a,'b)$ *mrel* ⇒ $('a,'b)$ *mrel* ⇒ $('a,'b)$ *mrel* (**infixl** ∩∩ *65*) **where**
  $R \cap\cap S \equiv \{ (a,B) .\ \exists\, C\ D .\ B = C \cap D \wedge (a,C) \in R \wedge (a,D) \in S \}$

definition *inner-complement* :: $('a,'b)$ *mrel* ⇒ $('a,'b)$ *mrel* (∼ - [*80*] *80*) **where**
  $\sim R \equiv \{ (a,B) .\ (a,-B) \in R \}$

abbreviation *iu-unit* ($1_{\cup\cup}$)
  where $1_{\cup\cup} \equiv$ *p-id*

definition *ii-unit* :: $('a,'a)$ *mrel* ($1_{\cap\cap}$)
  where $1_{\cap\cap} \equiv \{ (a,UNIV) \mid a .\ True \}$

declare *inner-intersection-def* [*mr-simp*] *inner-complement-def* [*mr-simp*]
ii-unit-def [*mr-simp*]

lemma *iu-assoc*:
$(R \cup\cup S) \cup\cup T = R \cup\cup (S \cup\cup T)$
⟨*proof*⟩

lemma *iu-commute*:
$R \cup\cup S = S \cup\cup R$
⟨*proof*⟩

lemma *iu-unit*:
$R \cup\cup 1_{\cup\cup} = R$
⟨*proof*⟩

lemma *ii-assoc*:
$(R \cap\cap S) \cap\cap T = R \cap\cap (S \cap\cap T)$
⟨*proof*⟩

lemma *ii-commute*:
$R \cap\cap S = S \cap\cap R$
⟨*proof*⟩

lemma *ii-unit* [*simp*]:
$R \cap\cap 1_{\cap\cap} = R$
⟨*proof*⟩

lemma *pa-ic*:
$\sim(R \otimes \sim S) = R \otimes S$
⟨*proof*⟩

lemma *ic-involutive* [*simp*]:
$\sim\sim R = R$
⟨*proof*⟩

lemma *ic-injective*:
$\sim R = \sim S \Longrightarrow R = S$
⟨*proof*⟩

lemma *ic-antidist-iu*:
$\sim(R \cup\cup S) = \sim R \cap\cap \sim S$
⟨*proof*⟩

lemma *ic-antidist-ii*:
$\sim(R \cap\cap S) = \sim R \cup\cup \sim S$
⟨*proof*⟩

lemma *ic-iu-unit* [*simp*]:
$\sim 1_{\cup\cup} = 1_{\cap\cap}$
⟨*proof*⟩

lemma *ic-ii-unit* [*simp*]:

$\sim 1_{\cap\cap} = 1_{\cup\cup}$
⟨*proof*⟩

lemma *ii-unit-split-iu* [*simp*]:
$1 \cup\cup \sim 1 = 1_{\cap\cap}$
⟨*proof*⟩

lemma *aux-1*:
$B = \{a\} \cap D \Longrightarrow -D = \{a\} \Longrightarrow B = \{\}$
⟨*proof*⟩

lemma *iu-unit-split-ii* [*simp*]:
$1 \cap\cap \sim 1 = 1_{\cup\cup}$
⟨*proof*⟩

lemma *iu-right-dist-ou*:
$(R \cup S) \cup\cup T = (R \cup\cup T) \cup (S \cup\cup T)$
⟨*proof*⟩

lemma *ii-right-dist-ou*:
$(R \cup S) \cap\cap T = (R \cap\cap T) \cup (S \cap\cap T)$
⟨*proof*⟩

lemma *iu-left-isotone*:
$R \subseteq S \Longrightarrow R \cup\cup T \subseteq S \cup\cup T$
⟨*proof*⟩

lemma *iu-right-isotone*:
$R \subseteq S \Longrightarrow T \cup\cup R \subseteq T \cup\cup S$
⟨*proof*⟩

lemma *iu-isotone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow R \cup\cup P \subseteq S \cup\cup Q$
⟨*proof*⟩

lemma *ii-left-isotone*:
$R \subseteq S \Longrightarrow R \cap\cap T \subseteq S \cap\cap T$
⟨*proof*⟩

lemma *ii-right-isotone*:
$R \subseteq S \Longrightarrow T \cap\cap R \subseteq T \cap\cap S$
⟨*proof*⟩

lemma *ii-isotone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow R \cap\cap P \subseteq S \cap\cap Q$
⟨*proof*⟩

lemma *iu-right-subdist-ii*:
$(R \cap\cap S) \cup\cup T \subseteq (R \cup\cup T) \cap\cap (S \cup\cup T)$

$\langle proof \rangle$

lemma *ii-right-subdist-iu*:
$(R \cup\cup S) \cap\cap T \subseteq (R \cap\cap T) \cup\cup (S \cap\cap T)$
$\langle proof \rangle$

lemma *ic-isotone*:
$R \subseteq S \Longrightarrow \sim R \subseteq \sim S$
$\langle proof \rangle$

lemma *ic-bot* [*simp*]:
$\sim\{\} = \{\}$
$\langle proof \rangle$

lemma *ic-top* [*simp*]:
$\sim U = U$
$\langle proof \rangle$

lemma *ic-dist-ou*:
$\sim(R \cup S) = \sim R \cup \sim S$
$\langle proof \rangle$

lemma *ic-dist-oi*:
$\sim(R \cap S) = \sim R \cap \sim S$
$\langle proof \rangle$

lemma *ic-dist-oc*:
$\sim -R = -(\sim R)$
$\langle proof \rangle$

lemma *ii-sub-idempotent*:
$R \subseteq R \cap\cap R$
$\langle proof \rangle$

definition *inner-Union* :: $('i \Rightarrow ('a,'b)\ mrel) \Rightarrow 'i\ set \Rightarrow ('a,'b)\ mrel$ ($\bigcup\bigcup$-|- [*80*,*80*] *80*) **where**
$\bigcup\bigcup X|I \equiv \{\ (a,B)\ .\ \exists f\ .\ B = (\bigcup i{\in}I\ .\ f\ i) \wedge (\forall i{\in}I\ .\ (a,f\ i) \in X\ i)\ \}$

definition *inner-Intersection* :: $('i \Rightarrow ('a,'b)\ mrel) \Rightarrow 'i\ set \Rightarrow ('a,'b)\ mrel$ ($\bigcap\bigcap$-|- [*80*,*80*] *80*) **where**
$\bigcap\bigcap X|I \equiv \{\ (a,B)\ .\ \exists f\ .\ B = (\bigcap i{\in}I\ .\ f\ i) \wedge (\forall i{\in}I\ .\ (a,f\ i) \in X\ i)\ \}$

declare *inner-Union-def* [*mr-simp*] *inner-Intersection-def* [*mr-simp*]

lemma *iU-empty*:
$\bigcup\bigcup X|\{\} = 1_{\cup\cup}$
$\langle proof \rangle$

lemma *iI-empty*:

$\bigcap\bigcap X|\{\} = \mathit{1}_{\cap\cap}$
⟨proof⟩

lemma *ic-antidist-iU*:
$\sim\bigcup\bigcup X|I = \bigcap\bigcap(\mathit{inner\text{-}complement}\ o\ X)|I$
⟨proof⟩

lemma *ic-antidist-iI*:
$\sim\bigcap\bigcap X|I = \bigcup\bigcup(\mathit{inner\text{-}complement}\ o\ X)|I$
⟨proof⟩

lemma *iu-right-dist-oU*:
$\bigcup X \cup\cup\ T = (\bigcup R{\in}X\ .\ R \cup\cup\ T)$
⟨proof⟩

lemma *ii-right-dist-oU*:
$\bigcup X \cap\cap\ T = (\bigcup R{\in}X\ .\ R \cap\cap\ T)$
⟨proof⟩

lemma *iu-right-subdist-iI*:
$\bigcap\bigcap X|I \cup\cup\ T \subseteq \bigcap\bigcap(\lambda i\ .\ X\ i \cup\cup\ T)|I$
⟨proof⟩

lemma *ii-right-subdist-iU*:
$\bigcup\bigcup X|I \cap\cap\ T \subseteq \bigcup\bigcup(\lambda i\ .\ X\ i \cap\cap\ T)|I$
⟨proof⟩

lemma *ic-dist-oU*:
$\sim\bigcup X = \bigcup(\mathit{inner\text{-}complement}\ `\ X)$
⟨proof⟩

lemma *ic-dist-oI*:
$\sim\bigcap X = \bigcap(\mathit{inner\text{-}complement}\ `\ X)$
⟨proof⟩

lemma *sp-left-subdist-iU*:
$R * (\bigcup\bigcup X|I) \subseteq \bigcup\bigcup(\lambda i\ .\ R * X\ i)|I$
⟨proof⟩

lemma *sp-right-subdist-iU*:
$(\bigcup\bigcup X|I) * R \subseteq \bigcup\bigcup(\lambda i\ .\ X\ i * R)|I$
⟨proof⟩

lemma *sp-right-dist-iU*:
assumes $\forall\, J{::}'a\ set\ .\ J \neq \{\} \longrightarrow (\bigcup\bigcup(\lambda j\ .\ R)|J) \subseteq R$
shows $(\bigcup\bigcup X|I) * R = \bigcup\bigcup(\lambda i\ .\ X\ i * R)|(I{::}'a\ set)$
⟨proof⟩

5.2 Dual

abbreviation *dual* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $(\text{-}^d\ [100]\ 100)$
where $R^d \equiv \sim -R$

lemma *dual*:
$R^d = \{\ (a,B)\ .\ (a,-B) \notin R\ \}$
$\langle proof \rangle$

declare *dual* [*mr-simp*]

lemma *dual-antitone*:
$R \subseteq S \Longrightarrow S^d \subseteq R^d$
$\langle proof \rangle$

lemma *ic-oc-dual*:
$\sim R = -R^d$
$\langle proof \rangle$

lemma *dual-involutive* [*simp*]:
$R^{dd} = R$
$\langle proof \rangle$

lemma *dual-antidist-ou*:
$(R \cup S)^d = R^d \cap S^d$
$\langle proof \rangle$

lemma *dual-antidist-oi*:
$(R \cap S)^d = R^d \cup S^d$
$\langle proof \rangle$

lemma *dual-dist-oc*:
$(-R)^d = -R^d$
$\langle proof \rangle$

lemma *dual-dist-ic*:
$(\sim R)^d = \sim R^d$
$\langle proof \rangle$

lemma *dual-antidist-oU*:
$(\bigcup X)^d = \bigcap (dual\ `\ X)$
$\langle proof \rangle$

lemma *dual-antidist-oI*:
$(\bigcap X)^d = \bigcup (dual\ `\ X)$
$\langle proof \rangle$

5.3 Co-composition

definition *co-prod* :: $('a,'b)$ *mrel* $\Rightarrow$ $('b,'c)$ *mrel* $\Rightarrow$ $('a,'c)$ *mrel* (**infixl** $\odot$ *70*)
where
$R \odot S \equiv \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f\, b) \in S) \wedge C = \bigcap\{ f\, b \mid b . b \in B \}) \}$

lemma *co-prod-im*:
$R \odot S = \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f\, b) \in S) \wedge C = \bigcap((\lambda x . f\, x)\ `\ B)) \}$
⟨*proof*⟩

lemma *co-prod-iff*:
$(a,C) \in (R \odot S) \longleftrightarrow (\exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f\, b) \in S) \wedge C = \bigcap\{ f\, b \mid b . b \in B \}))$
⟨*proof*⟩

declare *co-prod-im* [*mr-simp*]

lemma *co-prod*:
$R \odot S = \sim(R * \sim S)$
⟨*proof*⟩

lemma *cp-left-isotone*:
$R \subseteq S \Longrightarrow R \odot T \subseteq S \odot T$
⟨*proof*⟩

lemma *cp-right-isotone*:
$R \subseteq S \Longrightarrow T \odot R \subseteq T \odot S$
⟨*proof*⟩

lemma *cp-isotone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow R \odot P \subseteq S \odot Q$
⟨*proof*⟩

lemma *ic-dist-cp*:
$\sim(R \odot S) = R * \sim S$
⟨*proof*⟩

lemma *ic-dist-sp*:
$\sim(R * S) = R \odot \sim S$
⟨*proof*⟩

lemma *ic-cp-ic-unit*:
$\sim R = R \odot \sim 1$
⟨*proof*⟩

lemma *cp-left-zero* [*simp*]:
$\{\} \odot R = \{\}$
⟨*proof*⟩

lemma *cp-left-unit* [*simp*]:
$1 \odot R = R$
⟨*proof*⟩

lemma *cp-ic-unit* [*simp*]:
$\sim 1 \odot \sim 1 = 1$
⟨*proof*⟩

lemma *cp-right-dist-ou*:
$(R \cup S) \odot T = (R \odot T) \cup (S \odot T)$
⟨*proof*⟩

lemma *cp-left-iu-unit* [*simp*]:
$1_{\cup\cup} \odot R = 1_{\cap\cap}$
⟨*proof*⟩

lemma *cp-right-ii-unit*:
$R \odot 1_{\cap\cap} \subseteq R \cup\cup \sim R$
⟨*proof*⟩

lemma *sp-right-iu-unit*:
$R * 1_{\cup\cup} \subseteq R \cap\cap \sim R$
⟨*proof*⟩

lemma *cp-left-subdist-ii*:
$R \odot (S \cap\cap T) \subseteq (R \odot S) \cap\cap (R \odot T)$
⟨*proof*⟩

lemma *cp-right-subantidist-iu*:
$(R \cup\cup S) \odot T \subseteq (R \odot T) \cap\cap (S \odot T)$
⟨*proof*⟩

lemma *cp-right-antidist-iu*:
assumes $T \cap\cap T \subseteq T$
shows $(R \cup\cup S) \odot T = (R \odot T) \cap\cap (S \odot T)$
⟨*proof*⟩

lemma *cp-right-dist-oU*:
$\bigcup X \odot T = (\bigcup R{\in}X . R \odot T)$
⟨*proof*⟩

lemma *cp-left-subdist-iI*:
$R \odot (\bigcap\bigcap X|I) \subseteq \bigcap\bigcap(\lambda i . R \odot X\ i)|I$
⟨*proof*⟩

lemma *cp-right-subantidist-iU*:
$(\bigcup\bigcup X|I) \odot R \subseteq \bigcap\bigcap(\lambda i . X\ i \odot R)|I$
⟨*proof*⟩

lemma *cp-right-antidist-iU*:
  assumes $\forall\, J{::}'a\ set\ .\ J \neq \{\} \longrightarrow (\bigcap\bigcap(\lambda j\ .\ R)|J) \subseteq R$
  shows $(\bigcup\bigcup X|I) \odot R = \bigcap\bigcap(\lambda i\ .\ X\ i \odot R)|(I{::}'a\ set)$
$\langle proof \rangle$

5.4 Inner order

definition *inner-order-iu* :: $'a \times 'b\ set \Rightarrow 'a \times 'b\ set \Rightarrow bool$ (**infix** $\preceq_{\cup\cup}$ *50*)
where
  $x \preceq_{\cup\cup} y \equiv fst\ x = fst\ y \wedge snd\ x \subseteq snd\ y$

definition *inner-order-ii* :: $'a \times 'b\ set \Rightarrow 'a \times 'b\ set \Rightarrow bool$ (**infix** $\preceq_{\cap\cap}$ *50*)
where
  $x \preceq_{\cap\cap} y \equiv fst\ x = fst\ y \wedge snd\ x \supseteq snd\ y$

lemma *inner-order-dual*:
  $x \preceq_{\cup\cup} y \longleftrightarrow y \preceq_{\cap\cap} x$
  $\langle proof \rangle$

interpretation *inner-order-iu*: *order* $(\preceq_{\cup\cup})\ \lambda x\ y\ .\ x \preceq_{\cup\cup} y \wedge x \neq y$
  $\langle proof \rangle$

5.5 Up-closure, down-closure and convex-closure

abbreviation *up* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel$ (-$\uparrow$ [*100*] *100*)
  where $R\uparrow \equiv R \cup\cup U$

abbreviation *down* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel$ (-$\downarrow$ [*100*] *100*)
  where $R\downarrow \equiv R \cap\cap U$

abbreviation *convex* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel$ (-$\updownarrow$ [*100*] *100*)
  where $R\updownarrow \equiv R\uparrow \cap R\downarrow$

lemma *up*:
  $R\uparrow = \{\ (a,B)\ .\ \exists\, C\ .\ (a,C) \in R \wedge C \subseteq B\ \}$
  $\langle proof \rangle$

lemma *down*:
  $R\downarrow = \{\ (a,B)\ .\ \exists\, C\ .\ (a,C) \in R \wedge B \subseteq C\ \}$
  $\langle proof \rangle$

lemma *convex*:
  $R\updownarrow = \{\ (a,B)\ .\ \exists\, C\ D\ .\ (a,C) \in R \wedge (a,D) \in R \wedge C \subseteq B \wedge B \subseteq D\ \}$
  $\langle proof \rangle$

declare *up* [*mr-simp*] *down* [*mr-simp*] *convex* [*mr-simp*]

lemma *ic-up*:
  $\sim(R\uparrow) = (\sim R)\downarrow$

$\langle proof \rangle$

lemma *ic-down*:
$\sim(R\downarrow) = (\sim R)\uparrow$
$\langle proof \rangle$

lemma *ic-convex*:
$\sim(R\updownarrow) = (\sim R)\updownarrow$
$\langle proof \rangle$

lemma *up-isotone*:
$R \subseteq S \Longrightarrow R\uparrow \subseteq S\uparrow$
$\langle proof \rangle$

lemma *up-increasing*:
$R \subseteq R\uparrow$
$\langle proof \rangle$

lemma *up-idempotent* [*simp*]:
$R\uparrow\uparrow = R\uparrow$
$\langle proof \rangle$

lemma *up-dist-ou*:
$(R \cup S)\uparrow = R\uparrow \cup S\uparrow$
$\langle proof \rangle$

lemma *up-dist-iu*:
$(R \cup\cup S)\uparrow = R\uparrow \cup\cup S\uparrow$
$\langle proof \rangle$

lemma *up-dist-ii*:
$(R \cap\cap S)\uparrow = R\uparrow \cap\cap S\uparrow$
$\langle proof \rangle$

lemma *down-isotone*:
$R \subseteq S \Longrightarrow R\downarrow \subseteq S\downarrow$
$\langle proof \rangle$

lemma *down-increasing*:
$R \subseteq R\downarrow$
$\langle proof \rangle$

lemma *down-idempotent* [*simp*]:
$R\downarrow\downarrow = R\downarrow$
$\langle proof \rangle$

lemma *down-dist-ou*:
$(R \cup S)\downarrow = R\downarrow \cup S\downarrow$
$\langle proof \rangle$

lemma *down-dist-iu*:
$(R \cup\cup S)\downarrow = R\downarrow \cup\cup S\downarrow$
$\langle proof \rangle$

lemma *down-dist-ii*:
$(R \cap\cap S)\downarrow = R\downarrow \cap\cap S\downarrow$
$\langle proof \rangle$

lemma *convex-isotone*:
$R \subseteq S \Longrightarrow R\updownarrow \subseteq S\updownarrow$
$\langle proof \rangle$

lemma *convex-increasing*:
$R \subseteq R\updownarrow$
$\langle proof \rangle$

lemma *convex-idempotent* [*simp*]:
$R\updownarrow\updownarrow = R\updownarrow$
$\langle proof \rangle$

lemma *down-sp*:
$R\downarrow = R * (1_{\cup\cup} \cup 1)$
$\langle proof \rangle$

lemma *up-cp*:
$R\uparrow = \sim R \odot (1_{\cap\cap} \cup \sim 1)$
$\langle proof \rangle$

lemma *down-dist-sp*:
$(R * S)\downarrow = R * S\downarrow$
$\langle proof \rangle$

lemma *up-dist-cp*:
$(R \odot S)\uparrow = R \odot S\uparrow$
$\langle proof \rangle$

lemma *iu-up-oi*:
$R\uparrow \cup\cup S\uparrow = R\uparrow \cap S\uparrow$
$\langle proof \rangle$

lemma *ii-down-oi*:
$R\downarrow \cap\cap S\downarrow = R\downarrow \cap S\downarrow$
$\langle proof \rangle$

lemma *down-dist-ii-oi*:
$R\downarrow \cap S\downarrow = (R \cap\cap S)\downarrow$
$\langle proof \rangle$

lemma *up-dist-iu-oi*:
$R\uparrow \cap S\uparrow = (R \cup\cup S)\uparrow$
⟨*proof*⟩

lemma *oi-down-sub-up*:
$R\downarrow \cap S\uparrow \subseteq (R\downarrow \cap S)\uparrow$
⟨*proof*⟩

lemma *oi-down-up*:
$R\downarrow \cap S = \{\} \Longrightarrow R \cap S\uparrow = \{\}$
⟨*proof*⟩

lemma *oi-down-up-iff*:
$R\downarrow \cap S = \{\} \longleftrightarrow R \cap S\uparrow = \{\}$
⟨*proof*⟩

lemma *down-double-complement-up*:
$R\downarrow \subseteq S \longleftrightarrow R \subseteq -((-S)\uparrow)$
⟨*proof*⟩

lemma *up-double-complement-down*:
$R\uparrow \subseteq S \longleftrightarrow R \subseteq -((-S)\downarrow)$
⟨*proof*⟩

lemma *below-up-oi-down*:
$R \subseteq R\uparrow \cap R\downarrow$
⟨*proof*⟩

lemma *cp-pa-sim*:
$(R \odot S)\downarrow = R \otimes S\downarrow$
⟨*proof*⟩

lemma *domain-up-down-conjugate*:
$(R\uparrow \cap S) * \mathit{1}_{\cup\cup} = (R \cap S\downarrow) * \mathit{1}_{\cup\cup}$
⟨*proof*⟩

lemma *down-below-sp-top*:
$R\downarrow \subseteq R * U$
⟨*proof*⟩

lemma *down-oi-up-closed*:
assumes $Q\uparrow = Q$
shows $R\downarrow \cap Q \subseteq (R \cap Q)\downarrow$
⟨*proof*⟩

lemma *up-dist-oU*:
$(\bigcup X)\uparrow = \bigcup(up \text{ ` } X)$
⟨*proof*⟩

lemma *up-dist-iU*:
assumes $I \neq \{\}$
shows $(\bigcup\bigcup X|I)\uparrow = \bigcup\bigcup (up\ o\ X)|I$
⟨*proof*⟩

lemma *up-dist-iI*:
$(\bigcap\bigcap X|I)\uparrow = \bigcap\bigcap (up\ o\ X)|I$
⟨*proof*⟩

lemma *down-dist-oU*:
$(\bigcup X)\downarrow = \bigcup (down\ `\ X)$
⟨*proof*⟩

lemma *down-dist-iU*:
$(\bigcup\bigcup X|I)\downarrow = \bigcup\bigcup (down\ o\ X)|I$
⟨*proof*⟩

lemma *down-dist-iI*:
assumes $I \neq \{\}$
shows $(\bigcap\bigcap X|I)\downarrow = \bigcap\bigcap (down\ o\ X)|I$
⟨*proof*⟩

lemma *iU-up-oI*:
assumes $I \neq \{\}$
shows $\bigcup\bigcup (up\ o\ X)|I = \bigcap (up\ `\ X\ `\ I)$
⟨*proof*⟩

lemma *iI-down-oI*:
assumes $I \neq \{\}$
shows $\bigcap\bigcap (down\ o\ X)|I = \bigcap (down\ `\ X\ `\ I)$
⟨*proof*⟩

lemma *down-dist-iI-oI*:
$\bigcap (down\ `\ X\ `\ I) = (\bigcap\bigcap X|I)\downarrow$
⟨*proof*⟩

lemma *up-dist-iU-oI*:
$\bigcap (up\ `\ X\ `\ I) = (\bigcup\bigcup X|I)\uparrow$
⟨*proof*⟩

lemma *iu-up*:
$(R \cup\cup R)\uparrow = R\uparrow$
⟨*proof*⟩

lemma *ii-down*:
$(R \cap\cap R)\downarrow = R\downarrow$
⟨*proof*⟩

lemma *iU-up*:
assumes $I \neq \{\}$
shows $(\bigcup\bigcup(\lambda j\ .\ R)|I)\uparrow = R\uparrow$
$\langle proof \rangle$

lemma *iI-down*:
assumes $I \neq \{\}$
shows $(\bigcap\bigcap(\lambda j\ .\ R)|I)\downarrow = R\downarrow$
$\langle proof \rangle$

lemma *iu-unit-up*:
$1_{\cup\cup}\uparrow = U$
$\langle proof \rangle$

lemma *iu-unit-down*:
$1_{\cup\cup}\downarrow = 1_{\cup\cup}$
$\langle proof \rangle$

lemma *iu-unit-convex*:
$1_{\cup\cup}\updownarrow = 1_{\cup\cup}$
$\langle proof \rangle$

lemma *ii-unit-up*:
$1_{\cap\cap}\uparrow = 1_{\cap\cap}$
$\langle proof \rangle$

lemma *ii-unit-down*:
$1_{\cap\cap}\downarrow = U$
$\langle proof \rangle$

lemma *ii-unit-convex*:
$1_{\cap\cap}\updownarrow = 1_{\cap\cap}$
$\langle proof \rangle$

lemma *sp-unit-down*:
$1\downarrow = 1 \cup 1_{\cup\cup}$
$\langle proof \rangle$

lemma *sp-unit-convex*:
$1\updownarrow = 1$
$\langle proof \rangle$

lemma *top-up*:
$U\uparrow = U$
$\langle proof \rangle$

lemma *top-down*:
$U\downarrow = U$
$\langle proof \rangle$

lemma *top-convex*:
$U\updownarrow = U$
⟨*proof*⟩

lemma *bot-up*:
$\{\}\uparrow = \{\}$
⟨*proof*⟩

lemma *bot-down*:
$\{\}\downarrow = \{\}$
⟨*proof*⟩

lemma *bot-convex*:
$\{\}\updownarrow = \{\}$
⟨*proof*⟩

lemma *down-oi-up-convex*:
$(R\downarrow \cap S\uparrow)\updownarrow = R\downarrow \cap S\uparrow$
⟨*proof*⟩

lemma *convex-iff-down-oi-up*:
$Q = Q\updownarrow \longleftrightarrow (\exists R\ S\ .\ Q = R\downarrow \cap S\uparrow)$
⟨*proof*⟩

lemma *convex-closed-oI*:
$(\bigcap R{\in}X\ .\ R\updownarrow)\updownarrow = (\bigcap R{\in}X\ .\ R\updownarrow)$
⟨*proof*⟩

lemma *convex-closed-oi*:
$(R\updownarrow \cap S\updownarrow)\updownarrow = R\updownarrow \cap S\updownarrow$
⟨*proof*⟩

lemma
$(R\updownarrow \cup\cup S\updownarrow)\updownarrow = R\updownarrow \cup\cup S\updownarrow$
nitpick[*expect=genuine*,*card=1,3*]
⟨*proof*⟩

6 Powerdomain Preorders

abbreviation *lower-less-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $\sqsubseteq\downarrow$ *50*)
where
$R \sqsubseteq\downarrow S \equiv R \subseteq S\downarrow$

abbreviation *upper-less-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $\sqsubseteq\uparrow$ *50*)
where
$R \sqsubseteq\uparrow S \equiv S \subseteq R\uparrow$

abbreviation *convex-less-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $\sqsubseteq\updownarrow$ *50*)

where
 $R \sqsubseteq\updownarrow S \equiv R \sqsubseteq\downarrow S \wedge R \sqsubseteq\uparrow S$

abbreviation *Convex-less-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $\sqsubseteq\Updownarrow$ *50*) **where**
 $R \sqsubseteq\Updownarrow S \equiv R \subseteq S\updownarrow$

lemma *lower-less-eq*:
 $R \sqsubseteq\downarrow S \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow (\exists\, C\ .\ (a,C) \in S \wedge B \subseteq C))$
 ⟨*proof*⟩

lemma *upper-less-eq*:
 $R \sqsubseteq\uparrow S \longleftrightarrow (\forall\, a\ C\ .\ (a,C) \in S \longrightarrow (\exists\, B\ .\ (a,B) \in R \wedge B \subseteq C))$
 ⟨*proof*⟩

lemma *Convex-less-eq*:
 $R \sqsubseteq\Updownarrow S \longleftrightarrow (\forall\, a\ C\ .\ (a,C) \in R \longrightarrow (\exists\, B\ D\ .\ (a,B) \in S \wedge (a,D) \in S \wedge B \subseteq C \wedge C \subseteq D))$
 ⟨*proof*⟩

lemma *Convex-lower-upper*:
 $R \sqsubseteq\Updownarrow S \longleftrightarrow R \sqsubseteq\downarrow S \wedge S \sqsubseteq\uparrow R$
 ⟨*proof*⟩

lemma *lower-reflexive*:
 $R \sqsubseteq\downarrow R$
 ⟨*proof*⟩

lemma *upper-reflexive*:
 $R \sqsubseteq\uparrow R$
 ⟨*proof*⟩

lemma *convex-reflexive*:
 $R \sqsubseteq\updownarrow R$
 ⟨*proof*⟩

lemma *Convex-reflexive*:
 $R \sqsubseteq\Updownarrow R$
 ⟨*proof*⟩

lemma *lower-transitive*:
 $R \sqsubseteq\downarrow S \Longrightarrow S \sqsubseteq\downarrow T \Longrightarrow R \sqsubseteq\downarrow T$
 ⟨*proof*⟩

lemma *upper-transitive*:
 $R \sqsubseteq\uparrow S \Longrightarrow S \sqsubseteq\uparrow T \Longrightarrow R \sqsubseteq\uparrow T$
 ⟨*proof*⟩

lemma *convex-transitive*:

$R \sqsubseteq\updownarrow S \Longrightarrow S \sqsubseteq\updownarrow T \Longrightarrow R \sqsubseteq\updownarrow T$
⟨*proof*⟩

lemma *Convex-transitive*:
$R \sqsubseteq\Updownarrow S \Longrightarrow S \sqsubseteq\Updownarrow T \Longrightarrow R \sqsubseteq\Updownarrow T$
⟨*proof*⟩

lemma *bot-lower-least*:
$\{\} \sqsubseteq\downarrow R$
⟨*proof*⟩

lemma *top-upper-least*:
$U \sqsubseteq\uparrow R$
⟨*proof*⟩

lemma *bot-Convex-least*:
$\{\} \sqsubseteq\Updownarrow R$
⟨*proof*⟩

lemma *top-lower-greatest*:
$R \sqsubseteq\downarrow U$
⟨*proof*⟩

lemma *bot-upper-greatest*:
$R \sqsubseteq\uparrow \{\}$
⟨*proof*⟩

lemma *top-Convex-greatest*:
$R \sqsubseteq\Updownarrow U$
⟨*proof*⟩

lemma *lower-iu-increasing*:
$R \sqsubseteq\downarrow R \cup\cup R$
⟨*proof*⟩

lemma *upper-iu-increasing*:
$R \sqsubseteq\uparrow R \cup\cup S$
⟨*proof*⟩

lemma *convex-iu-increasing*:
$R \sqsubseteq\updownarrow R \cup\cup R$
⟨*proof*⟩

lemma *Convex-iu-increasing*:
$R \sqsubseteq\Updownarrow R \cup\cup R$
⟨*proof*⟩

lemma *lower-ii-decreasing*:
$R \cap\cap S \sqsubseteq\downarrow R$

$\langle proof \rangle$

lemma *upper-ii-decreasing*:
$R \cap\cap R \sqsubseteq\uparrow R$
$\langle proof \rangle$

lemma *convex-ii-decreasing*:
$R \cap\cap R \sqsubseteq\updownarrow R$
$\langle proof \rangle$

lemma *Convex-ii-increasing*:
$R \sqsubseteq\Updownarrow R \cap\cap R$
$\langle proof \rangle$

lemma *iu-lower-left-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R \cup\cup T \sqsubseteq\downarrow S \cup\cup T$
$\langle proof \rangle$

lemma *iu-upper-left-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R \cup\cup T \sqsubseteq\uparrow S \cup\cup T$
$\langle proof \rangle$

lemma *iu-convex-left-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R \cup\cup T \sqsubseteq\updownarrow S \cup\cup T$
$\langle proof \rangle$

lemma *iu-Convex-left-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R \cup\cup T \sqsubseteq\Updownarrow S \cup\cup T$
$\langle proof \rangle$

lemma *iu-lower-right-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow T \cup\cup R \sqsubseteq\downarrow T \cup\cup S$
$\langle proof \rangle$

lemma *iu-upper-right-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow T \cup\cup R \sqsubseteq\uparrow T \cup\cup S$
$\langle proof \rangle$

lemma *iu-convex-right-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow T \cup\cup R \sqsubseteq\updownarrow T \cup\cup S$
$\langle proof \rangle$

lemma *iu-Convex-right-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow T \cup\cup R \sqsubseteq\Updownarrow T \cup\cup S$
$\langle proof \rangle$

lemma *iu-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow P \sqsubseteq\downarrow Q \Longrightarrow R \cup\cup P \sqsubseteq\downarrow S \cup\cup Q$
$\langle proof \rangle$

lemma *iu-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow P \sqsubseteq\uparrow Q \Longrightarrow R \cup\cup P \sqsubseteq\uparrow S \cup\cup Q$
$\langle proof \rangle$

lemma *iu-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow P \sqsubseteq\updownarrow Q \Longrightarrow R \cup\cup P \sqsubseteq\updownarrow S \cup\cup Q$
$\langle proof \rangle$

lemma *iu-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow P \sqsubseteq\Updownarrow Q \Longrightarrow R \cup\cup P \sqsubseteq\Updownarrow S \cup\cup Q$
$\langle proof \rangle$

lemma *ii-lower-left-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R \cap\cap T \sqsubseteq\downarrow S \cap\cap T$
$\langle proof \rangle$

lemma *ii-upper-left-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R \cap\cap T \sqsubseteq\uparrow S \cap\cap T$
$\langle proof \rangle$

lemma *ii-convex-left-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R \cap\cap T \sqsubseteq\updownarrow S \cap\cap T$
$\langle proof \rangle$

lemma *ii-Convex-left-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R \cap\cap T \sqsubseteq\Updownarrow S \cap\cap T$
$\langle proof \rangle$

lemma *ii-lower-right-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow T \cap\cap R \sqsubseteq\downarrow T \cap\cap S$
$\langle proof \rangle$

lemma *ii-upper-right-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow T \cap\cap R \sqsubseteq\uparrow T \cap\cap S$
$\langle proof \rangle$

lemma *ii-convex-right-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow T \cap\cap R \sqsubseteq\updownarrow T \cap\cap S$
$\langle proof \rangle$

lemma *ii-Convex-right-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow T \cap\cap R \sqsubseteq\Updownarrow T \cap\cap S$
$\langle proof \rangle$

lemma *ii-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow P \sqsubseteq\downarrow Q \Longrightarrow R \cap\cap P \sqsubseteq\downarrow S \cap\cap Q$
$\langle proof \rangle$

lemma *ii-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow P \sqsubseteq\uparrow Q \Longrightarrow R \cap\cap P \sqsubseteq\uparrow S \cap\cap Q$
⟨*proof*⟩

lemma *ii-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow P \sqsubseteq\updownarrow Q \Longrightarrow R \cap\cap P \sqsubseteq\updownarrow S \cap\cap Q$
⟨*proof*⟩

lemma *ii-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow P \sqsubseteq\Updownarrow Q \Longrightarrow R \cap\cap P \sqsubseteq\Updownarrow S \cap\cap Q$
⟨*proof*⟩

lemma *ou-lower-left-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R \cup T \sqsubseteq\downarrow S \cup T$
⟨*proof*⟩

lemma *ou-upper-left-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R \cup T \sqsubseteq\uparrow S \cup T$
⟨*proof*⟩

lemma *ou-convex-left-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R \cup T \sqsubseteq\updownarrow S \cup T$
⟨*proof*⟩

lemma *ou-Convex-left-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R \cup T \sqsubseteq\Updownarrow S \cup T$
⟨*proof*⟩

lemma *ou-lower-right-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow T \cup R \sqsubseteq\downarrow T \cup S$
⟨*proof*⟩

lemma *ou-upper-right-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow T \cup R \sqsubseteq\uparrow T \cup S$
⟨*proof*⟩

lemma *ou-convex-right-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow T \cup R \sqsubseteq\updownarrow T \cup S$
⟨*proof*⟩

lemma *ou-Convex-right-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow T \cup R \sqsubseteq\Updownarrow T \cup S$
⟨*proof*⟩

lemma *ou-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow P \sqsubseteq\downarrow Q \Longrightarrow R \cup P \sqsubseteq\downarrow S \cup Q$
⟨*proof*⟩

lemma *ou-upper-isotone*:

$R \sqsubseteq\uparrow S \Longrightarrow P \sqsubseteq\uparrow Q \Longrightarrow R \cup P \sqsubseteq\uparrow S \cup Q$
⟨*proof*⟩

lemma *ou-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow P \sqsubseteq\updownarrow Q \Longrightarrow R \cup P \sqsubseteq\updownarrow S \cup Q$
⟨*proof*⟩

lemma *ou-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow P \sqsubseteq\Updownarrow Q \Longrightarrow R \cup P \sqsubseteq\Updownarrow S \cup Q$
⟨*proof*⟩

lemma *sp-lower-left-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow T * R \sqsubseteq\downarrow T * S$
⟨*proof*⟩

lemma *sp-upper-left-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow T * R \sqsubseteq\uparrow T * S$
⟨*proof*⟩

lemma *sp-convex-left-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow T * R \sqsubseteq\updownarrow T * S$
⟨*proof*⟩

lemma *sp-Convex-left-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow T * R \sqsubseteq\Updownarrow T * S$
⟨*proof*⟩

lemma *cp-lower-left-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow T \odot R \sqsubseteq\downarrow T \odot S$
⟨*proof*⟩

lemma *cp-upper-left-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow T \odot R \sqsubseteq\uparrow T \odot S$
⟨*proof*⟩

lemma *cp-convex-left-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow T \odot R \sqsubseteq\updownarrow T \odot S$
⟨*proof*⟩

lemma *cp-Convex-left-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow T \odot R \sqsubseteq\Updownarrow T \odot S$
⟨*proof*⟩

lemma *lower-ic-upper*:
$R \sqsubseteq\downarrow S \longleftrightarrow {\sim}S \sqsubseteq\uparrow {\sim}R$
⟨*proof*⟩

lemma *upper-ic-lower*:
$R \sqsubseteq\uparrow S \longleftrightarrow {\sim}S \sqsubseteq\downarrow {\sim}R$

$\langle proof \rangle$

lemma *convex-ic*:
$R \sqsubseteq\updownarrow S \longleftrightarrow \sim S \sqsubseteq\updownarrow \sim R$
$\langle proof \rangle$

lemma *Convex-ic*:
$R \sqsubseteq\Updownarrow S \longleftrightarrow \sim R \sqsubseteq\Updownarrow \sim S$
$\langle proof \rangle$

lemma *up-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R\uparrow \sqsubseteq\downarrow S\uparrow$
$\langle proof \rangle$

lemma *up-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R\uparrow \sqsubseteq\uparrow S\uparrow$
$\langle proof \rangle$

lemma *up-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R\uparrow \sqsubseteq\updownarrow S\uparrow$
$\langle proof \rangle$

lemma *up-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R\uparrow \sqsubseteq\Updownarrow S\uparrow$
$\langle proof \rangle$

lemma *down-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R\downarrow \sqsubseteq\downarrow S\downarrow$
$\langle proof \rangle$

lemma *down-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R\downarrow \sqsubseteq\uparrow S\downarrow$
$\langle proof \rangle$

lemma *down-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R\downarrow \sqsubseteq\updownarrow S\downarrow$
$\langle proof \rangle$

lemma *down-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R\downarrow \sqsubseteq\Updownarrow S\downarrow$
$\langle proof \rangle$

lemma *convex-lower-isotone*:
$R \sqsubseteq\downarrow S \Longrightarrow R\updownarrow \sqsubseteq\downarrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow R\updownarrow \sqsubseteq\uparrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-convex-isotone*:
$R \sqsubseteq\updownarrow S \Longrightarrow R\updownarrow \sqsubseteq\updownarrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-Convex-isotone*:
$R \sqsubseteq\Updownarrow S \Longrightarrow R\updownarrow \sqsubseteq\Updownarrow S\updownarrow$
$\langle proof \rangle$

lemma *subset-lower*:
$R \subseteq S \Longrightarrow R \sqsubseteq\downarrow S$
$\langle proof \rangle$

lemma *subset-upper*:
$R \subseteq S \Longrightarrow S \sqsubseteq\uparrow R$
$\langle proof \rangle$

lemma *subset-Convex*:
$R \subseteq S \Longrightarrow R \sqsubseteq\Updownarrow S$
$\langle proof \rangle$

lemma *oi-subset-lower-left-isotone*:
$R \subseteq S \Longrightarrow R \cap T \sqsubseteq\downarrow S \cap T$
$\langle proof \rangle$

lemma *oi-subset-upper-left-antitone*:
$R \subseteq S \Longrightarrow S \cap T \sqsubseteq\uparrow R \cap T$
$\langle proof \rangle$

lemma *oi-subset-Convex-left-isotone*:
$R \subseteq S \Longrightarrow R \cap T \sqsubseteq\Updownarrow S \cap T$
$\langle proof \rangle$

lemma *oi-subset-lower-right-isotone*:
$R \subseteq S \Longrightarrow T \cap R \sqsubseteq\downarrow T \cap S$
$\langle proof \rangle$

lemma *oi-subset-upper-right-antitone*:
$R \subseteq S \Longrightarrow T \cap S \sqsubseteq\uparrow T \cap R$
$\langle proof \rangle$

lemma *oi-subset-Convex-right-isotone*:
$R \subseteq S \Longrightarrow T \cap R \sqsubseteq\Updownarrow T \cap S$
$\langle proof \rangle$

lemma *oi-subset-lower-isotone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow R \cap P \sqsubseteq\downarrow S \cap Q$
$\langle proof \rangle$

lemma *oi-subset-upper-antitone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow S \cap Q \sqsubseteq\uparrow R \cap P$
$\langle proof \rangle$

lemma *oi-subset-Convex-isotone*:
$R \subseteq S \Longrightarrow P \subseteq Q \Longrightarrow R \cap P \sqsubseteq\Updownarrow S \cap Q$
$\langle proof \rangle$

lemma *sp-iu-unit-lower*:
$R * 1_{\cup\cup} \sqsubseteq\downarrow R$
$\langle proof \rangle$

lemma *cp-ii-unit-upper*:
$R \sqsubseteq\uparrow R \odot 1_{\cap\cap}$
$\langle proof \rangle$

lemma *lower-ii-down*:
$R \sqsubseteq\downarrow S \longleftrightarrow R\downarrow = (R \cap\cap S)\downarrow$
$\langle proof \rangle$

lemma *lower-ii-lower-bound*:
$R \sqsubseteq\downarrow S \longleftrightarrow R \subseteq R \cap\cap S$
$\langle proof \rangle$

lemma *upper-ii-up*:
$R \sqsubseteq\uparrow S \longleftrightarrow S\uparrow = (R \cup\cup S)\uparrow$
$\langle proof \rangle$

lemma *upper-ii-upper-bound*:
$R \sqsubseteq\uparrow S \longleftrightarrow S \subseteq R \cup\cup S$
$\langle proof \rangle$

lemma
$R \sqsubseteq\downarrow S \longleftrightarrow R = R \cap\cap S$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma
$R \sqsubseteq\uparrow S \longleftrightarrow S = R \cup\cup S$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma *convex-oi-Convex-iu*:
$R\updownarrow \cap S\updownarrow \sqsubseteq\Updownarrow R \cup\cup S$
$\langle proof \rangle$

lemma *convex-oi-Convex-ii*:
$R\updownarrow \cap S\updownarrow \sqsubseteq\Updownarrow R \cap\cap S$
$\langle proof \rangle$

lemma *convex-oi-iu-ii*:
$R\updownarrow \cap S\updownarrow = (R \cup\cup S)\uparrow \cap (R \cap\cap S)\downarrow$
⟨*proof*⟩

lemma *ii-lower-iu*:
$R \cap\cap S \sqsubseteq\downarrow R \cup\cup S$
⟨*proof*⟩

lemma *ii-upper-iu*:
$R \cap\cap S \sqsubseteq\uparrow R \cup\cup S$
⟨*proof*⟩

lemma *ii-convex-iu*:
$R \cap\cap S \sqsubseteq\updownarrow R \cup\cup S$
⟨*proof*⟩

lemma *convex-oi-iu-ii-convex*:
$R\updownarrow \cap S\updownarrow = (R \cup\cup S)\updownarrow \cap (R \cap\cap S)\updownarrow$
⟨*proof*⟩

6.1 Functional properties of multirelations

lemma *id-one-converse*:
$Id = 1 ; 1^{\smile}$
⟨*proof*⟩

lemma *dom-explicit*:
$Dom\ R = R ; U \cap 1$
⟨*proof*⟩

lemma *dom-explicit-2*:
$Dom\ R = R ; top \cap 1$
⟨*proof*⟩

lemma *total-dom*:
$total\ R \longleftrightarrow Dom\ R = 1$
⟨*proof*⟩

lemma *total-eq*:
$total\ R \longleftrightarrow 1_{\cup\cup} = R * 1_{\cup\cup}$
⟨*proof*⟩

lemma *domain-pointwise*:
$x \in R * 1_{\cup\cup} \longleftrightarrow (\exists\, a\ B .\ (a,B) \in R \wedge x = (a,\{\}))$
⟨*proof*⟩

card only works for finite sets

lemma *univalent-2*:

$univalent\ R \longleftrightarrow (\forall\, a\ .\ finite\ \{\ B\ .\ (a,B) \in R\ \} \wedge card\ \{\ B\ .\ (a,B) \in R\ \} \leq one\text{-}class.one)$
⟨*proof*⟩

lemma *univalent-3*:
$univalent\ R \longleftrightarrow (\forall\, S\ .\ R * 1_{\cup\cup} = S * 1_{\cup\cup} \wedge S \subseteq R \longrightarrow S = R)$
⟨*proof*⟩

lemma *total-2*:
$total\ R \longleftrightarrow (\forall\, a\ .\ \{\ B\ .\ (a,B) \in R\ \} \neq \{\})$
⟨*proof*⟩

lemma *total-3*:
$total\ R \longleftrightarrow (\forall\, a\ .\ finite\ \{\ B\ .\ (a,B) \in R\ \} \longrightarrow card\ \{\ B\ .\ (a,B) \in R\ \} \geq one\text{-}class.one)$
⟨*proof*⟩

lemma *total-4*: $total\ R \longleftrightarrow 1_{\cup\cup} \subseteq R * 1_{\cup\cup}$
⟨*proof*⟩

lemma *deterministic-2*:
$deterministic\ R \longleftrightarrow (\forall\, a\ .\ card\ \{\ B\ .\ (a,B) \in R\ \} = one\text{-}class.one)$
⟨*proof*⟩

lemma *univalent-convex*:
assumes *univalent S*
shows $S = S\updownarrow$
⟨*proof*⟩

lemma *univalent-iu-idempotent*:
assumes *univalent S*
shows $S = S \cup\cup S$
⟨*proof*⟩

lemma *univalent-ii-idempotent*:
assumes *univalent S*
shows $S = S \cap\cap S$
⟨*proof*⟩

lemma *univalent-down-iu-idempotent*:
assumes *univalent S*
shows $S = S\downarrow \cup\cup S$
⟨*proof*⟩

lemma *univalent-up-ii-idempotent*:
assumes *univalent S*
shows $S = S\uparrow \cap\cap S$
⟨*proof*⟩

lemma *univalent-convex-iu-idempotent*:
  assumes *univalent* S
  shows $S = S\updownarrow \cup\cup\ S$
  ⟨*proof*⟩

lemma *univalent-convex-ii-idempotent*:
  assumes *univalent* S
  shows $S = S\updownarrow \cap\cap\ S$
  ⟨*proof*⟩

lemma *univalent-iu-closed*:
  univalent $R \Longrightarrow$ *univalent* $S \Longrightarrow$ *univalent* $(R \cup\cup\ S)$
  ⟨*proof*⟩

lemma *univalent-ii-closed*:
  univalent $R \Longrightarrow$ *univalent* $S \Longrightarrow$ *univalent* $(R \cap\cap\ S)$
  ⟨*proof*⟩

lemma *total-lower*:
  total $R \longleftrightarrow 1_{\cup\cup} \sqsubseteq\downarrow R$
  ⟨*proof*⟩

lemma *total-upper*:
  total $R \longleftrightarrow R \sqsubseteq\uparrow 1_{\cap\cap}$
  ⟨*proof*⟩

lemma *total-lower-iu*:
  assumes *total* T
  shows $R \sqsubseteq\downarrow R \cup\cup\ T$
  ⟨*proof*⟩

lemma *total-upper-ii*:
  assumes *total* T
  shows $R \cap\cap\ T \sqsubseteq\uparrow R$
  ⟨*proof*⟩

lemma *total-univalent-lower-iu*:
  assumes *total* T
    and *univalent* S
    and $T \sqsubseteq\downarrow S$
  shows $T \cup\cup\ S = S$
⟨*proof*⟩

lemma *total-iu-closed*:
  total $R \Longrightarrow$ *total* $S \Longrightarrow$ *total* $(R \cup\cup\ S)$
  ⟨*proof*⟩

lemma *total-ii-closed*:
  total $R \Longrightarrow$ *total* $S \Longrightarrow$ *total* $(R \cap\cap\ S)$

⟨*proof*⟩

lemma *deterministic-lower*:
  assumes *deterministic* V
  shows $R \sqsubseteq\downarrow V \longleftrightarrow (\forall\, a\ B\ C\ .\ (a,B) \in R \wedge (a,C) \in V \longrightarrow B \subseteq C)$
⟨*proof*⟩

lemma *deterministic-upper*:
  assumes *deterministic* V
  shows $V \sqsubseteq\uparrow R \longleftrightarrow (\forall\, a\ B\ C\ .\ (a,B) \in R \wedge (a,C) \in V \longrightarrow C \subseteq B)$
⟨*proof*⟩

lemma *deterministic-iu-closed*:
  deterministic $R \Longrightarrow$ *deterministic* $S \Longrightarrow$ *deterministic* $(R \cup\cup S)$
  ⟨*proof*⟩

lemma *deterministic-ii-closed*:
  deterministic $R \Longrightarrow$ *deterministic* $S \Longrightarrow$ *deterministic* $(R \cap\cap S)$
  ⟨*proof*⟩

lemma *total-univalent-lower-implies-upper*:
  assumes *total* T
    and *univalent* S
    and $T \sqsubseteq\downarrow S$
  shows $T \sqsubseteq\uparrow S$
  ⟨*proof*⟩

lemma *total-univalent-lower-implies-convex*:
  assumes *total* T
    and *univalent* S
    and $T \sqsubseteq\downarrow S$
  shows $T \sqsubseteq\updownarrow S$
  ⟨*proof*⟩

lemma *total-univalent-upper-implies-lower*:
  assumes *total* T
    and *univalent* S
    and $S \sqsubseteq\uparrow T$
  shows $S \sqsubseteq\downarrow T$
⟨*proof*⟩

lemma *total-univalent-upper-implies-convex*:
  assumes *total* T
    and *univalent* S
    and $S \sqsubseteq\uparrow T$
  shows $S \sqsubseteq\updownarrow T$
  ⟨*proof*⟩

lemma *deterministic-lower-upper*:

assumes *deterministic* T
and *deterministic* S
shows $S \sqsubseteq\downarrow T \longleftrightarrow S \sqsubseteq\uparrow T$
⟨*proof*⟩

lemma *deterministic-lower-convex*:
assumes *deterministic* T
and *deterministic* S
shows $S \sqsubseteq\downarrow T \longleftrightarrow S \sqsubseteq\updownarrow T$
⟨*proof*⟩

lemma *deterministic-upper-convex*:
assumes *deterministic* T
and *deterministic* S
shows $S \sqsubseteq\uparrow T \longleftrightarrow S \sqsubseteq\updownarrow T$
⟨*proof*⟩

lemma *total-down-sp-sp-down*:
assumes *total* T
shows $R\downarrow * T \subseteq R * T\downarrow$
⟨*proof*⟩

lemma *total-down-sp-semi-commute*:
total $T \Longrightarrow R\downarrow * T \subseteq (R * T)\downarrow$
⟨*proof*⟩

lemma *total-down-dist-sp*:
total $T \Longrightarrow (R * T)\downarrow = R\downarrow * T\downarrow$
⟨*proof*⟩

lemma *univalent-ic-closed*:
univalent $R \longleftrightarrow$ *univalent* $(\sim R)$
⟨*proof*⟩

lemma *total-ic-closed*:
total $R \longleftrightarrow$ *total* $(\sim R)$
⟨*proof*⟩

lemma *deterministic-ic-closed*:
deterministic $R \longleftrightarrow$ *deterministic* $(\sim R)$
⟨*proof*⟩

lemma *iu-unit-deterministic*:
deterministic $(1_{\cup\cup})$
⟨*proof*⟩

lemma *ii-unit-deterministic*:
deterministic $(1_{\cap\cap})$
⟨*proof*⟩

lemma *univalent-upper-iu*:
  assumes *univalent* R
  shows $(R \sqsubseteq\uparrow S) \longleftrightarrow (R \cup\cup S = S)$
$\langle proof \rangle$

lemma *univalent-lower-ii*:
  assumes *univalent* S
  shows $(R \sqsubseteq\downarrow S) = (R \cap\cap S = R)$
  $\langle proof \rangle$

6.2 Equivalences induced by powerdomain preorders

abbreviation *lower-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $=\downarrow$ *50*)
where
  $R =\downarrow S \equiv R \sqsubseteq\downarrow S \wedge S \sqsubseteq\downarrow R$

abbreviation *upper-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $=\uparrow$ *50*)
where
  $R =\uparrow S \equiv R \sqsubseteq\uparrow S \wedge S \sqsubseteq\uparrow R$

abbreviation *convex-eq* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* $\Rightarrow$ *bool* (**infixl** $=\updownarrow$ *50*)
where
  $R =\updownarrow S \equiv R \sqsubseteq\updownarrow S \wedge S \sqsubseteq\updownarrow R$

lemma *Convex-eq*:
  $R =\updownarrow S \equiv R \sqsubseteq\Updownarrow S \wedge S \sqsubseteq\Updownarrow R$
  $\langle proof \rangle$

lemma *convex-lower-upper*:
  $R =\updownarrow S \longleftrightarrow R =\downarrow S \wedge R =\uparrow S$
  $\langle proof \rangle$

lemma *lower-eq-down*:
  $R =\downarrow S \longleftrightarrow R\downarrow = S\downarrow$
  $\langle proof \rangle$

lemma *upper-eq-up*:
  $R =\uparrow S \longleftrightarrow R\uparrow = S\uparrow$
  $\langle proof \rangle$

lemma *convex-eq-convex*:
  $R =\updownarrow S \longleftrightarrow R\updownarrow = S\updownarrow$
  $\langle proof \rangle$

lemma *lower-eq*:
  $R =\downarrow S \longleftrightarrow (\forall a\ B\ .\ (\exists C\ .\ (a,C) \in R \wedge B \subseteq C) \longleftrightarrow (\exists C\ .\ (a,C) \in S \wedge B \subseteq C))$
  $\langle proof \rangle$

lemma *upper-eq*:
$R =\uparrow S \longleftrightarrow (\forall a\ C\ .\ (\exists B\ .\ (a,B) \in R \wedge B \subseteq C) \longleftrightarrow (\exists B\ .\ (a,B) \in S \wedge B \subseteq C))$
⟨*proof*⟩

lemma *lower-eq-reflexive*:
$R =\downarrow R$
⟨*proof*⟩

lemma *upper-eq-reflexive*:
$R =\uparrow R$
⟨*proof*⟩

lemma *convex-eq-reflexive*:
$R =\updownarrow R$
⟨*proof*⟩

lemma *lower-eq-symmetric*:
$R =\downarrow S \Longrightarrow S =\downarrow R$
⟨*proof*⟩

lemma *upper-eq-symmetric*:
$R =\uparrow S \Longrightarrow S =\uparrow R$
⟨*proof*⟩

lemma *convex-eq-symmetric*:
$R =\updownarrow S \Longrightarrow S =\updownarrow R$
⟨*proof*⟩

lemma *lower-eq-transitive*:
$R =\downarrow S \Longrightarrow S =\downarrow T \Longrightarrow R =\downarrow T$
⟨*proof*⟩

lemma *upper-eq-transitive*:
$R =\uparrow S \Longrightarrow S =\uparrow T \Longrightarrow R =\uparrow T$
⟨*proof*⟩

lemma *convex-eq-transitive*:
$R =\updownarrow S \Longrightarrow S =\updownarrow T \Longrightarrow R =\updownarrow T$
⟨*proof*⟩

lemma *ou-lower-eq-left-congruence*:
$R =\downarrow S \Longrightarrow R \cup T =\downarrow S \cup T$
⟨*proof*⟩

lemma *ou-upper-eq-left-congruence*:
$R =\uparrow S \Longrightarrow R \cup T =\uparrow S \cup T$
⟨*proof*⟩

lemma *ou-convex-eq-left-congruence*:
$R =\updownarrow S \Longrightarrow R \cup T =\updownarrow S \cup T$
⟨*proof*⟩

lemma *ou-lower-eq-right-congruence*:
$R =\downarrow S \Longrightarrow T \cup R =\downarrow T \cup S$
⟨*proof*⟩

lemma *ou-upper-eq-right-congruence*:
$R =\uparrow S \Longrightarrow T \cup R =\uparrow T \cup S$
⟨*proof*⟩

lemma *ou-convex-eq-right-congruence*:
$R =\updownarrow S \Longrightarrow T \cup R =\updownarrow T \cup S$
⟨*proof*⟩

lemma *ou-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow P =\downarrow Q \Longrightarrow R \cup P =\downarrow S \cup Q$
⟨*proof*⟩

lemma *ou-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow P =\uparrow Q \Longrightarrow R \cup P =\uparrow S \cup Q$
⟨*proof*⟩

lemma *ou-convex-eq-congruence*:
$R =\updownarrow S \Longrightarrow P =\updownarrow Q \Longrightarrow R \cup P =\updownarrow S \cup Q$
⟨*proof*⟩

lemma *iu-lower-eq-left-congruence*:
$R =\downarrow S \Longrightarrow R \cup\!\!\cup T =\downarrow S \cup\!\!\cup T$
⟨*proof*⟩

lemma *iu-upper-eq-left-congruence*:
$R =\uparrow S \Longrightarrow R \cup\!\!\cup T =\uparrow S \cup\!\!\cup T$
⟨*proof*⟩

lemma *iu-convex-eq-left-congruence*:
$R =\updownarrow S \Longrightarrow R \cup\!\!\cup T =\updownarrow S \cup\!\!\cup T$
⟨*proof*⟩

lemma *iu-lower-eq-right-congruence*:
$R =\downarrow S \Longrightarrow T \cup\!\!\cup R =\downarrow T \cup\!\!\cup S$
⟨*proof*⟩

lemma *iu-upper-eq-right-congruence*:
$R =\uparrow S \Longrightarrow T \cup\!\!\cup R =\uparrow T \cup\!\!\cup S$
⟨*proof*⟩

lemma *iu-convex-eq-right-congruence*:
$R =\updownarrow S \Longrightarrow T \cup\cup R =\updownarrow T \cup\cup S$
⟨*proof*⟩

lemma *iu-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow P =\downarrow Q \Longrightarrow R \cup\cup P =\downarrow S \cup\cup Q$
⟨*proof*⟩

lemma *iu-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow P =\uparrow Q \Longrightarrow R \cup\cup P =\uparrow S \cup\cup Q$
⟨*proof*⟩

lemma *iu-convex-eq-congruence*:
$R =\updownarrow S \Longrightarrow P =\updownarrow Q \Longrightarrow R \cup\cup P =\updownarrow S \cup\cup Q$
⟨*proof*⟩

lemma *ii-lower-eq-left-congruence*:
$R =\downarrow S \Longrightarrow R \cap\cap T =\downarrow S \cap\cap T$
⟨*proof*⟩

lemma *ii-upper-eq-left-congruence*:
$R =\uparrow S \Longrightarrow R \cap\cap T =\uparrow S \cap\cap T$
⟨*proof*⟩

lemma *ii-convex-eq-left-congruence*:
$R =\updownarrow S \Longrightarrow R \cap\cap T =\updownarrow S \cap\cap T$
⟨*proof*⟩

lemma *ii-lower-eq-right-congruence*:
$R =\downarrow S \Longrightarrow T \cap\cap R =\downarrow T \cap\cap S$
⟨*proof*⟩

lemma *ii-upper-eq-right-congruence*:
$R =\uparrow S \Longrightarrow T \cap\cap R =\uparrow T \cap\cap S$
⟨*proof*⟩

lemma *ii-convex-eq-right-congruence*:
$R =\updownarrow S \Longrightarrow T \cap\cap R =\updownarrow T \cap\cap S$
⟨*proof*⟩

lemma *ii-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow P =\downarrow Q \Longrightarrow R \cap\cap P =\downarrow S \cap\cap Q$
⟨*proof*⟩

lemma *ii-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow P =\uparrow Q \Longrightarrow R \cap\cap P =\uparrow S \cap\cap Q$
⟨*proof*⟩

lemma *ii-convex-eq-congruence*:

$R =\updownarrow S \Longrightarrow P =\updownarrow Q \Longrightarrow R \cap\cap P =\updownarrow S \cap\cap Q$
⟨*proof*⟩

lemma *sp-lower-eq-left-congruence*:
$R =\downarrow S \Longrightarrow T * R =\downarrow T * S$
⟨*proof*⟩

lemma *sp-upper-eq-left-congruence*:
$R =\uparrow S \Longrightarrow T * R =\uparrow T * S$
⟨*proof*⟩

lemma *sp-convex-eq-left-congruence*:
$R =\updownarrow S \Longrightarrow T * R =\updownarrow T * S$
⟨*proof*⟩

lemma *cp-lower-eq-left-congruence*:
$R =\downarrow S \Longrightarrow T \odot R =\downarrow T \odot S$
⟨*proof*⟩

lemma *cp-upper-eq-left-congruence*:
$R =\uparrow S \Longrightarrow T \odot R =\uparrow T \odot S$
⟨*proof*⟩

lemma *cp-convex-eq-left-congruence*:
$R =\updownarrow S \Longrightarrow T \odot R =\updownarrow T \odot S$
⟨*proof*⟩

lemma *lower-eq-ic-upper*:
$R =\downarrow S \longleftrightarrow \sim R =\uparrow \sim S$
⟨*proof*⟩

lemma *upper-eq-ic-lower*:
$R =\uparrow S \longleftrightarrow \sim R =\downarrow \sim S$
⟨*proof*⟩

lemma *convex-eq-ic-lower*:
$R =\updownarrow S \longleftrightarrow \sim R =\updownarrow \sim S$
⟨*proof*⟩

lemma *up-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow R\uparrow =\downarrow S\uparrow$
⟨*proof*⟩

lemma *up-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow R\uparrow =\uparrow S\uparrow$
⟨*proof*⟩

lemma *up-convex-eq-congruence*:
$R =\updownarrow S \Longrightarrow R\uparrow =\updownarrow S\uparrow$

$\langle proof \rangle$

lemma *down-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow R\downarrow =\downarrow S\downarrow$
$\langle proof \rangle$

lemma *down-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow R\downarrow =\uparrow S\downarrow$
$\langle proof \rangle$

lemma *down-convex-eq-congruence*:
$R =\updownarrow S \Longrightarrow R\downarrow =\updownarrow S\downarrow$
$\langle proof \rangle$

lemma *convex-lower-eq-congruence*:
$R =\downarrow S \Longrightarrow R\updownarrow =\downarrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-upper-eq-congruence*:
$R =\uparrow S \Longrightarrow R\updownarrow =\uparrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-convex-eq-congruence*:
$R =\updownarrow S \Longrightarrow R\updownarrow =\updownarrow S\updownarrow$
$\langle proof \rangle$

lemma *univalent-lower-eq-subset*:
assumes *univalent* S
and $S =\downarrow R$
shows $S \subseteq R$
$\langle proof \rangle$

lemma *univalent-lower-eq*:
assumes *univalent* R
and *univalent* S
and $R =\downarrow S$
shows $R = S$
$\langle proof \rangle$

lemma *univalent-lower-eq-iff*:
assumes *univalent* R
and *univalent* S
shows $(R =\downarrow S) \longleftrightarrow (R = S)$
$\langle proof \rangle$

lemma *univalent-upper-eq-subset*:
assumes *univalent* S
and $S =\uparrow R$
shows $S \subseteq R$

⟨*proof*⟩

lemma *univalent-upper-eq*:
  assumes *univalent* R
    and *univalent* S
    and $R =\uparrow S$
  shows $R = S$
  ⟨*proof*⟩

lemma *univalent-upper-eq-iff*:
  assumes *univalent* R
    and *univalent* S
  shows $(R =\uparrow S) \longleftrightarrow (R = S)$
  ⟨*proof*⟩

lemma *univalent-convex-eq-iff*:
  assumes *univalent* R
    and *univalent* S
  shows $(R =\updownarrow S) \longleftrightarrow (R = S)$
  ⟨*proof*⟩

lemma *total-univalent-upper-ii*:
  assumes *total* T
    and *univalent* S
    and $S \sqsubseteq\uparrow T$
  shows $T \cap\cap S = S$
  ⟨*proof*⟩

lemma *lower-eq-down-closed*:
  $R =\downarrow R\downarrow$
  ⟨*proof*⟩

lemma *upper-eq-up-closed*:
  $R =\uparrow R\uparrow$
  ⟨*proof*⟩

lemma *convex-eq-up-closed*:
  $R =\updownarrow R\updownarrow$
  ⟨*proof*⟩

lemma *lower-join*:
  $(\forall P\ .\ Q \sqsubseteq\downarrow P \longleftrightarrow R \sqsubseteq\downarrow P \wedge S \sqsubseteq\downarrow P) \longleftrightarrow Q =\downarrow R \cup S$
  ⟨*proof*⟩

lemma *lower-meet*:
  $(\forall P\ .\ P \sqsubseteq\downarrow Q \longleftrightarrow P \sqsubseteq\downarrow R \wedge P \sqsubseteq\downarrow S) \longleftrightarrow Q =\downarrow R \cap\cap S$
  ⟨*proof*⟩

lemma *upper-join*:

$(\forall P\ .\ Q \sqsubseteq\uparrow P \longleftrightarrow R \sqsubseteq\uparrow P \wedge S \sqsubseteq\uparrow P) \longleftrightarrow Q =\uparrow R \cup\cup S$
⟨*proof*⟩

lemma *upper-meet*:
$(\forall P\ .\ P \sqsubseteq\uparrow Q \longleftrightarrow P \sqsubseteq\uparrow R \wedge P \sqsubseteq\uparrow S) \longleftrightarrow Q =\uparrow R \cup S$
⟨*proof*⟩

lemma *lower-ii-idempotent*:
$R \cap\cap R =\downarrow R$
⟨*proof*⟩

lemma *upper-iu-idempotent*:
$R \cup\cup R =\uparrow R$
⟨*proof*⟩

lemma *lower-iI-idempotent*:
$I \neq \{\} \Longrightarrow (\bigcap\bigcap (\lambda j\ .\ R)|I) =\downarrow R$
⟨*proof*⟩

lemma *upper-iU-idempotent*:
$I \neq \{\} \Longrightarrow (\bigcup\bigcup (\lambda j\ .\ R)|I) =\uparrow R$
⟨*proof*⟩

lemma *down-closed-intersection-closed*:
$R = R\downarrow \Longrightarrow \forall I\ .\ I \neq \{\} \longrightarrow (\bigcap\bigcap (\lambda j\ .\ R)|I) \subseteq R$
⟨*proof*⟩

lemma *up-closed-union-closed*:
$R = R\uparrow \Longrightarrow \forall I\ .\ I \neq \{\} \longrightarrow (\bigcup\bigcup (\lambda j\ .\ R)|I) \subseteq R$
⟨*proof*⟩

lemma *ou-down-lower-eq-ou*:
$R\downarrow \cup S\downarrow =\downarrow R \cup S$
⟨*proof*⟩

lemma *oi-down-lower-eq-ii*:
$R\downarrow \cap S\downarrow =\downarrow R \cap\cap S$
⟨*proof*⟩

lemma *ou-up-upper-eq-ou*:
$R\uparrow \cup S\uparrow =\uparrow R \cup S$
⟨*proof*⟩

lemma *oi-up-upper-eq-iu*:
$R\uparrow \cap S\uparrow =\uparrow R \cup\cup S$
⟨*proof*⟩

lemma *oU-down-lower-eq-oU*:
$(\bigcup R{\in}X\ .\ R\downarrow) =\downarrow \bigcup X$

$\langle proof \rangle$

lemma *oI-down-lower-eq-iI*:
$(\bigcap i{\in}I\ .\ X\ i{\downarrow}) ={\downarrow} \bigcap\bigcap X|I$
$\langle proof \rangle$

lemma *oU-up-upper-eq-oU*:
$(\bigcup R{\in}X\ .\ R{\uparrow}) ={\uparrow} \bigcup X$
$\langle proof \rangle$

lemma *oI-up-upper-eq-iI*:
$(\bigcap i{\in}I\ .\ X\ i{\uparrow}) ={\uparrow} \bigcup\bigcup X|I$
$\langle proof \rangle$

lemma *down-order-lower*:
$R{\downarrow} \subseteq S{\downarrow} \longleftrightarrow R \sqsubseteq{\downarrow}\ S$
$\langle proof \rangle$

lemma *up-order-upper*:
$R{\uparrow} \subseteq S{\uparrow} \longleftrightarrow S \sqsubseteq{\uparrow}\ R$
$\langle proof \rangle$

lemma *convex-order-lower-upper*:
$R{\updownarrow} \subseteq S{\updownarrow} \longleftrightarrow R \sqsubseteq{\downarrow}\ S \wedge S \sqsubseteq{\uparrow}\ R$
$\langle proof \rangle$

lemma *convex-order-Convex*:
$R{\updownarrow} \subseteq S{\updownarrow} \longleftrightarrow R \sqsubseteq{\Updownarrow}\ S$
$\langle proof \rangle$

6.3 Further results for convex-closure

lemma *convex-down*:
$R{\updownarrow}{\downarrow} = R{\downarrow}$
$\langle proof \rangle$

lemma *convex-up*:
$R{\updownarrow}{\uparrow} = R{\uparrow}$
$\langle proof \rangle$

lemma *iu-dist-oi-convex*:
assumes $R = R{\updownarrow}$
and $S = S{\updownarrow}$
and $T = T{\updownarrow}$
shows $(R \cap S) \cup\!\cup\ T = (R \cup\!\cup\ T) \cap (S \cup\!\cup\ T)$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma *ii-dist-oi-convex*:

assumes $R = R\updownarrow$
and $S = S\updownarrow$
and $T = T\updownarrow$
shows $(R \cap S) \cap\cap\ T = (R \cap\cap\ T) \cap (S \cap\cap\ T)$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma *oI-up-closed*:
assumes $\forall R{\in}X .\ R\uparrow = R$
shows $(\bigcap X)\uparrow = \bigcap X$
$\langle proof \rangle$

lemma *oI-down-closed*:
assumes $\forall R{\in}X .\ R\downarrow = R$
shows $(\bigcap X)\downarrow = \bigcap X$
$\langle proof \rangle$

lemma *oI-convex-closed*:
assumes $\forall R{\in}X .\ R\updownarrow = R$
shows $(\bigcap X)\updownarrow = \bigcap X$
$\langle proof \rangle$

lemma *up-dist-Union*:
$(\bigcup X)\uparrow = \bigcup \{\ R\uparrow \mid R .\ R \in X\ \}$
$\langle proof \rangle$

lemma *down-dist-Union*:
$(\bigcup X)\downarrow = \bigcup \{\ R\downarrow \mid R .\ R \in X\ \}$
$\langle proof \rangle$

lemma *convex-dist-Union*:
$(\bigcup X)\updownarrow = \bigcup \{\ R\updownarrow \mid R .\ R \in X\ \}$
nitpick[*expect=genuine,card=1,2*]
$\langle proof \rangle$

lemma *up-dist-Inter*:
$(\bigcap X)\uparrow = \bigcap \{\ R\uparrow \mid R .\ R \in X\ \}$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma *down-dist-Inter*:
$(\bigcap X)\downarrow = \bigcap \{\ R\downarrow \mid R .\ R \in X\ \}$
nitpick[*expect=genuine,card=1*]
$\langle proof \rangle$

lemma *convex-dist-Inter*:
$(\bigcap X)\updownarrow = \bigcap \{\ R\updownarrow \mid R .\ R \in X\ \}$
nitpick[*expect=genuine,card=1,2*]
$\langle proof \rangle$

lemma *Inter-convex-closed*:
$(\bigcap X)\updownarrow = \bigcap X$
nitpick[*expect=genuine,card=1,2*]
$\langle proof \rangle$

abbreviation *convex-iu* (**infixl** $\cup\cup\updownarrow$ *70*)
where $R \cup\cup\updownarrow S \equiv (R \cup\cup S)\updownarrow$

lemma *convex-iu*:
$R \cup\cup\updownarrow S = (R\downarrow \cup\cup S\downarrow) \cap R\uparrow \cap S\uparrow$
$\langle proof \rangle$

lemma *convex-iu-sub*:
$R\updownarrow \cup\cup S \subseteq R \cup\cup\updownarrow S$
$\langle proof \rangle$

lemma *convex-iu-convex-left*:
$R \cup\cup\updownarrow S = R\updownarrow \cup\cup\updownarrow S$
$\langle proof \rangle$

lemma *convex-iu-convex-right*:
$R \cup\cup\updownarrow S = R \cup\cup\updownarrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-iu-convex*:
$R \cup\cup\updownarrow S = R\updownarrow \cup\cup\updownarrow S\updownarrow$
$\langle proof \rangle$

lemma *convex-iu-assoc*:
$(R \cup\cup\updownarrow S) \cup\cup\updownarrow T = R \cup\cup\updownarrow (S \cup\cup\updownarrow T)$
$\langle proof \rangle$

lemma *convex-iu-comm*:
$R \cup\cup\updownarrow S = S \cup\cup\updownarrow R$
$\langle proof \rangle$

lemma *convex-iu-unit*:
$R = R\updownarrow \Longrightarrow R \cup\cup\updownarrow 1_{\cup\cup} = R$
$\langle proof \rangle$

abbreviation *convex-ii* (**infixl** $\cap\cap\updownarrow$ *70*)
where $R \cap\cap\updownarrow S \equiv (R \cap\cap S)\updownarrow$

lemma *convex-ii*:
$R \cap\cap\updownarrow S = (R\uparrow \cap\cap S\uparrow) \cap R\downarrow \cap S\downarrow$
$\langle proof \rangle$

lemma *convex-ii-sub*:

$R\updownarrow \cap\cap S \subseteq R \cap\cap\updownarrow S$
⟨*proof*⟩

lemma *convex-ii-convex-left*:
$R \cap\cap\updownarrow S = R\updownarrow \cap\cap\updownarrow S$
⟨*proof*⟩

lemma *convex-ii-convex-right*:
$R \cap\cap\updownarrow S = R \cap\cap\updownarrow S\updownarrow$
⟨*proof*⟩

lemma *convex-ii-convex*:
$R \cap\cap\updownarrow S = R\updownarrow \cap\cap\updownarrow S\updownarrow$
⟨*proof*⟩

lemma *convex-ii-assoc*:
$(R \cap\cap\updownarrow S) \cap\cap\updownarrow T = R \cap\cap\updownarrow (S \cap\cap\updownarrow T)$
⟨*proof*⟩

lemma *convex-ii-comm*:
$R \cap\cap\updownarrow S = S \cap\cap\updownarrow R$
⟨*proof*⟩

lemma *convex-ii-unit*:
$R = R\updownarrow \Longrightarrow R \cap\cap\updownarrow 1_{\cap\cap} = R$
⟨*proof*⟩

lemma *convex-iu-ic*:
$\sim(R \cup\cup\updownarrow S) = \sim R \cap\cap\updownarrow \sim S$
⟨*proof*⟩

lemma *convex-ii-ic*:
$\sim(R \cap\cap\updownarrow S) = \sim R \cup\cup\updownarrow \sim S$
⟨*proof*⟩

abbreviation *convex-sup* :: $('a,'b)$ *mrel set* $\Rightarrow$ $('a,'b)$ *mrel* $(\bigcup\updownarrow)$ **where**
$\bigcup\updownarrow X \equiv (\bigcup X)\updownarrow$

lemma *convex-sup-convex*:
$\bigcup\updownarrow X = (\bigcup\updownarrow X)\updownarrow$
⟨*proof*⟩

lemma *convex-sup-inter*:
$\bigcup\updownarrow X = \bigcap\{\ Y\ .\ Y = Y\updownarrow \wedge \bigcup X \subseteq Y\ \}$
⟨*proof*⟩

lemma *convex-iu-dist-convex-sup*:
$\bigcup\updownarrow X \cup\cup\updownarrow S = \bigcup\updownarrow\{\ R \cup\cup\updownarrow S \mid R\ .\ R \in X\ \}$
⟨*proof*⟩

lemma *convex-ii-dist-convex-sup*:
$\bigcup \updownarrow X \cap\cap\updownarrow S = \bigcup\updownarrow\{ R \cap\cap\updownarrow S \mid R . R \in X \}$
⟨*proof*⟩

lemma *convex-dist-sup*:
$(\bigcup X)\updownarrow = \bigcup\updownarrow\{ R\updownarrow \mid R . R \in X \}$
⟨*proof*⟩

7 Fusion and Fission

7.1 Atoms and co-atoms

definition *atoms* :: $('a,'b)$ *mrel* $(A_{\cup\cup})$
where $A_{\cup\cup} \equiv \{ (a,\{b\}) \mid a\ b\ .\ True \}$

definition *co-atoms* :: $('a,'b)$ *mrel* $(A_{\cap\cap})$
where $A_{\cap\cap} \equiv \{ (a, UNIV - \{b\}) \mid a\ b\ .\ True \}$

declare *atoms-def* [*mr-simp*] *co-atoms-def* [*mr-simp*]

lemma *atoms-solution*:
$A_{\cup\cup}\uparrow = -1_{\cup\cup}$
⟨*proof*⟩

lemma *atoms-least-solution*:
assumes $R\uparrow = -1_{\cup\cup}$
shows $A_{\cup\cup} \subseteq R$
⟨*proof*⟩

lemma *ic-atoms*:
$\sim A_{\cup\cup} = A_{\cap\cap}$
⟨*proof*⟩

lemma *ic-co-atoms*:
$\sim A_{\cap\cap} = A_{\cup\cup}$
⟨*proof*⟩

lemma *co-atoms-solution*:
$A_{\cap\cap}\downarrow = -1_{\cap\cap}$
⟨*proof*⟩

lemma *co-atoms-least-solution*:
assumes $R\downarrow = -1_{\cap\cap}$
shows $A_{\cap\cap} \subseteq R$
⟨*proof*⟩

lemma *iu-unit-atoms-disjoint*:
$1_{\cup\cup} \cap A_{\cup\cup} = \{\}$

$\langle proof \rangle$

lemma *ii-unit-co-atoms-disjoint*:
$1_{\cap\cap} \cap A_{\cap\cap} = \{\}$
$\langle proof \rangle$

lemma *atoms-sp-idempotent*:
$A_{\cup\cup} * A_{\cup\cup} = A_{\cup\cup}$
$\langle proof \rangle$

lemma *atoms-sp-cp*:
$(R \cap A_{\cup\cup}) * S = (R \cap A_{\cup\cup}) \odot S$
$\langle proof \rangle$

7.2 Inner-functional properties

abbreviation *inner-univalent* :: $('a,'b)$ *mrel* $\Rightarrow$ *bool* **where**
inner-univalent $R \equiv R \subseteq 1_{\cup\cup} \cup A_{\cup\cup}$

abbreviation *inner-total* :: $('a,'b)$ *mrel* $\Rightarrow$ *bool* **where**
inner-total $R \equiv R \subseteq -1_{\cup\cup}$

abbreviation *inner-deterministic* :: $('a,'b)$ *mrel* $\Rightarrow$ *bool* **where**
inner-deterministic $R \equiv$ *inner-total* $R \wedge$ *inner-univalent* R

lemma *inner-deterministic-atoms*:
inner-deterministic $R \longleftrightarrow R \subseteq A_{\cup\cup}$
$\langle proof \rangle$

lemma *inner-univalent*:
inner-univalent $R \longleftrightarrow (\forall\, a\ b\ c\ B\ .\ (a,B) \in R \wedge b \in B \wedge c \in B \longrightarrow b = c)$
$\langle proof \rangle$

lemma *inner-univalent-2*:
inner-univalent $R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow$ *finite* $B \wedge$ *card* $B \leq$
one-class.one$)$
$\langle proof \rangle$

lemma *inner-total*:
inner-total $R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow (\exists\, b\ .\ b \in B))$
$\langle proof \rangle$

lemma *inner-total-2*:
inner-total $R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow B \neq \{\})$
$\langle proof \rangle$

lemma *inner-total-3*:
inner-total $R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \wedge$ *finite* $B \longrightarrow$ *card* $B \geq$ *one-class.one*$)$
$\langle proof \rangle$

lemma *inner-deterministic*:
$\mathit{inner\text{-}deterministic}\ R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow (\exists!b\ .\ b \in B))$
⟨*proof*⟩

lemma *inner-deterministic-2*:
$\mathit{inner\text{-}deterministic}\ R \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in R \longrightarrow \mathit{card}\ B = \mathit{one\text{-}class.one})$
⟨*proof*⟩

lemma *inner-deterministic-sp-unit*:
$\mathit{inner\text{-}deterministic}\ \mathit{1}$
⟨*proof*⟩

lemma *inner-univalent-down*:
assumes $\mathit{inner\text{-}univalent}\ S$
shows $S\!\downarrow\ \subseteq S \cup \mathit{1}_{\cup\cup}$
⟨*proof*⟩

lemma *inner-deterministic-lower-eq*:
assumes $\mathit{inner\text{-}deterministic}\ V$
and $\mathit{inner\text{-}deterministic}\ W$
and $V =\!\downarrow W$
shows $V = W$
⟨*proof*⟩

lemma *inner-total-down-closed*:
$\mathit{inner\text{-}total}\ T \Longrightarrow R \subseteq T \Longrightarrow \mathit{inner\text{-}total}\ R$
⟨*proof*⟩

lemma *inner-univalent-down-closed*:
$\mathit{inner\text{-}univalent}\ T \Longrightarrow R \subseteq T \Longrightarrow \mathit{inner\text{-}univalent}\ R$
⟨*proof*⟩

lemma *inner-deterministic-down-closed*:
$\mathit{inner\text{-}deterministic}\ T \Longrightarrow R \subseteq T \Longrightarrow \mathit{inner\text{-}deterministic}\ R$
⟨*proof*⟩

lemma *inner-univalent-convex*:
assumes $\mathit{inner\text{-}univalent}\ R$
shows $R = R\updownarrow$
⟨*proof*⟩

lemma *inner-deterministic-alt-closure*:
$\mathit{inner\text{-}deterministic}\ R = (R\ O\ \mathit{converse}\ \mathit{1}\ O\ \mathit{1} = R)$
⟨*proof*⟩

lemma *inner-deterministic-s-id-conv-epsiloff*:
$\mathit{inner\text{-}deterministic}\ R \Longrightarrow R\ O\ \mathit{converse}\ \mathit{s\text{-}id} = R\ O\ \mathit{epsiloff}$
⟨*proof*⟩

lemma *inner-deterministic-lower-iff*:
  assumes *inner-deterministic* R
    and *inner-deterministic* S
  shows $(R \sqsubseteq\!\downarrow S) \longleftrightarrow (R \subseteq S)$
  $\langle proof \rangle$

lemma *inner-deterministic-upper-iff*:
  assumes *inner-deterministic* R
    and *inner-deterministic* S
  shows $(R \sqsubseteq\!\uparrow S) \longleftrightarrow (S \subseteq R)$
  $\langle proof \rangle$

lemma *inner-deterministic-lower-eq-iff*:
  assumes *inner-deterministic* R
    and *inner-deterministic* S
  shows $(R =\!\downarrow S) \longleftrightarrow (R = S)$
  $\langle proof \rangle$

lemma *inner-deterministic-upper-eq-iff*:
  assumes *inner-deterministic* R
    and *inner-deterministic* S
  shows $(R =\!\uparrow S) \longleftrightarrow (R = S)$
  $\langle proof \rangle$

lemma *inner-deterministic-convex-eq-iff*:
  assumes *inner-deterministic* R
    and *inner-deterministic* S
  shows $(R =\!\updownarrow S) \longleftrightarrow (R = S)$
  $\langle proof \rangle$

lemma
  inner-univalent $R \Longrightarrow$ *inner-univalent* $S \Longrightarrow$ *inner-univalent* $(R \cup\!\cup S)$
  nitpick[*expect=genuine,card=1,2*]
  $\langle proof \rangle$

lemma *inner-univalent-ii-closed*:
  inner-univalent $R \Longrightarrow$ *inner-univalent* $S \Longrightarrow$ *inner-univalent* $(R \cap\!\cap S)$
  $\langle proof \rangle$

lemma *inner-total-iu-closed*:
  inner-total $R \Longrightarrow$ *inner-total* $S \Longrightarrow$ *inner-total* $(R \cup\!\cup S)$
  $\langle proof \rangle$

lemma
  inner-total $R \Longrightarrow$ *inner-total* $S \Longrightarrow$ *inner-total* $(R \cap\!\cap S)$
  nitpick[*expect=genuine,card=1,2*]
  $\langle proof \rangle$

7.3 Fusion

lemma *fusion-set*:
$fus\ R \equiv \{\ (a,B)\ .\ B = \bigcup\{\ C\ .\ (a,C) \in R\ \}\ \}$
⟨*proof*⟩

declare *fusion-set* [*mr-simp*]

lemma *fusion-lower-increasing*:
$R \sqsubseteq\downarrow fus\ R$
⟨*proof*⟩

lemma *fusion-deterministic*:
$deterministic\ (fus\ R)$
⟨*proof*⟩

lemma *fusion-least*:
assumes $R \sqsubseteq\downarrow S$
and $deterministic\ S$
shows $fus\ R \sqsubseteq\downarrow S$
⟨*proof*⟩

lemma *fusion-unique*:
assumes $\forall R\ .\ R \sqsubseteq\downarrow f\ R$
and $\forall R\ .\ deterministic\ (f\ R)$
and $\forall R\ S\ .\ R \sqsubseteq\downarrow S \wedge deterministic\ S \longrightarrow f\ R \sqsubseteq\downarrow S$
shows $f\ T = fus\ T$
⟨*proof*⟩

lemma *fusion-down-char*:
$(fus\ R)\downarrow = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow)$
⟨*proof*⟩

lemma *fusion-up-char*:
$(fus\ R)\uparrow = -((\sim(R\downarrow) \cap A_{\cap\cap})\downarrow)$
⟨*proof*⟩

lemma *fusion-up-char-2*:
$(fus\ R)\uparrow = -(((R\downarrow \cap A_{\cup\cup}) * \sim 1)\downarrow)$
⟨*proof*⟩

lemma *fusion-char*:
$fus\ R = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow) \cap -((\sim(R\downarrow) \cap A_{\cap\cap})\downarrow)$
⟨*proof*⟩

lemma *fusion-char-2*:
$fus\ R = -((-(R\downarrow) \cap A_{\cup\cup})\uparrow) \cap -(((R\downarrow \cap A_{\cup\cup}) * \sim 1)\downarrow)$
⟨*proof*⟩

lemma *fusion-lower-isotone*:

$R \sqsubseteq\!\downarrow S \Longrightarrow fus\ R \sqsubseteq\!\downarrow fus\ S$
⟨*proof*⟩

lemma *fusion-iu-idempotent*:
$fus\ R \cup\!\cup\ fus\ R = fus\ R$
⟨*proof*⟩

lemma *fusion-down*:
$fus\ R = fus\ (R\downarrow)$
⟨*proof*⟩

lemma *fusion-iu-total*:
$total\ T \Longrightarrow T \cup\!\cup\ fus\ T = fus\ T$
⟨*proof*⟩

lemma *fusion-deterministic-fixpoint*:
$deterministic\ R \longleftrightarrow R = fus\ R$
⟨*proof*⟩

abbreviation *non-empty* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel\ (ne$ - $[100]\ 100)$
where $ne\ R \equiv R \cap -1_{\cup\cup}$

lemma *non-empty*:
$ne\ R = \{\ (a,B) \mid a\ B\ .\ (a,B) \in R \wedge B \neq \{\}\ \}$
⟨*proof*⟩

lemma *ne-equality*:
$ne\ R = R \longleftrightarrow R \subseteq -1_{\cup\cup}$
⟨*proof*⟩

lemma *ne-dist-ou*:
$ne\ (R \cup S) = ne\ R \cup ne\ S$
⟨*proof*⟩

lemma *ne-down-idempotent*:
$ne\ ((ne\ (R\downarrow))\downarrow) = ne\ (R\downarrow)$
⟨*proof*⟩

lemma *ne-up*:
$(ne\ R)\uparrow = ne\ R * 1\uparrow$
⟨*proof*⟩

lemma *ne-dist-down-sp*:
$ne\ (R\downarrow * S) = ne\ (R\downarrow) * ne\ S$
⟨*proof*⟩

lemma *total-ne-down-dist-sp*:
$total\ T \Longrightarrow ne\ ((R * T)\downarrow) = ne\ (R\downarrow) * ne\ (T\downarrow)$
⟨*proof*⟩

lemma *inner-univalent-char*:
$\textit{inner-univalent } S \longleftrightarrow (\forall R \,.\, \textit{fus } R = \textit{fus } S \wedge R \sqsubseteq\!\downarrow S \longrightarrow \textit{ne } R = \textit{ne } S)$
⟨*proof*⟩

lemma *ne-dist-oU*:
$\textit{ne } (\bigcup X) = \bigcup(\textit{non-empty ` } X)$
⟨*proof*⟩

7.4 Fission

lemma *fission-set*:
$\textit{fis } R = \{ (a,\{b\}) \mid a\ b \,.\, \exists B \,.\, (a,B) \in R \wedge b \in B \}$
⟨*proof*⟩

declare *fission-set* [*mr-simp*]

lemma *fission-var*:
$\textit{fis } R = R\!\downarrow \cap\, A_{\cup\cup}$
⟨*proof*⟩

lemma *fission-lower-decreasing*:
$\textit{fis } R \sqsubseteq\!\downarrow R$
⟨*proof*⟩

lemma *fission-inner-deterministic*:
$\textit{inner-deterministic } (\textit{fis } R)$
⟨*proof*⟩

lemma *fission-greatest*:
assumes $S \sqsubseteq\!\downarrow R$
and $\textit{inner-deterministic } S$
shows $S \sqsubseteq\!\downarrow \textit{fis } R$
⟨*proof*⟩

lemma *fission-unique*:
assumes $\forall R \,.\, f\ R \sqsubseteq\!\downarrow R$
and $\forall R \,.\, \textit{inner-deterministic } (f\ R)$
and $\forall R\ S \,.\, S \sqsubseteq\!\downarrow R \wedge \textit{inner-deterministic } S \longrightarrow S \sqsubseteq\!\downarrow f\ R$
shows $f\ T = \textit{fis } T$
⟨*proof*⟩

lemma *fission-lower-isotone*:
$R \sqsubseteq\!\downarrow S \Longrightarrow \textit{fis } R \sqsubseteq\!\downarrow \textit{fis } S$
⟨*proof*⟩

lemma *fission-idempotent*:
$\textit{fis } (\textit{fis } R) = \textit{fis } R$
⟨*proof*⟩

lemma *fission-top*:
$\mathit{fis}\ U = A_{\cup\cup}$
⟨*proof*⟩

lemma *fission-down*:
$\mathit{fis}\ R = \mathit{fis}\ (R\downarrow)$
⟨*proof*⟩

lemma *fission-ne-fixpoint*:
$\mathit{fis}\ R = \mathit{ne}\ (\mathit{fis}\ R)$
⟨*proof*⟩

lemma *fission-down-ne-fixpoint*:
$\mathit{fis}\ R = \mathit{ne}\ ((\mathit{fis}\ R)\downarrow)$
⟨*proof*⟩

lemma *fission-inner-deterministic-fixpoint*:
$\mathit{inner\text{-}deterministic}\ R \longleftrightarrow R = \mathit{fis}\ R$
⟨*proof*⟩

lemma *fission-sp-subdist*:
$\mathit{fis}\ (R * S) \subseteq \mathit{fis}\ R * \mathit{fis}\ S$
⟨*proof*⟩

lemma *fission-sp-total-dist*:
assumes $\mathit{total}\ T$
shows $\mathit{fis}\ (R * T) = \mathit{fis}\ R * \mathit{fis}\ T$
⟨*proof*⟩

lemma *fission-dist-ou*:
$\mathit{fis}\ (R \cup S) = \mathit{fis}\ R \cup \mathit{fis}\ S$
⟨*proof*⟩

lemma *fission-sp-iu-unit*:
$\mathit{fis}\ (R * \mathit{1}_{\cup\cup}) = \{\}$
⟨*proof*⟩

lemma *fission-fusion-lower-decreasing*:
$\mathit{fis}\ (\mathit{fus}\ R) \sqsubseteq\downarrow R$
⟨*proof*⟩

lemma *fusion-fission-lower-increasing*:
$R \sqsubseteq\downarrow \mathit{fus}\ (\mathit{fis}\ R)$
⟨*proof*⟩

lemma *fission-fusion-galois*:
$\mathit{fis}\ R \sqsubseteq\downarrow S \longleftrightarrow R \sqsubseteq\downarrow \mathit{fus}\ S$
⟨*proof*⟩

lemma *fission-fusion*:
  $fis\ (fus\ R) = fis\ R$
  ⟨*proof*⟩

lemma *fusion-fission*:
  $fus\ (fis\ R) = fus\ R$
  ⟨*proof*⟩

lemma *same-fusion-fission-lower*:
  $fus\ R = fus\ S \Longrightarrow fis\ R \sqsubseteq\downarrow S$
  ⟨*proof*⟩

lemma *fission-below-ne-down-fusion*:
  $fis\ R \subseteq ne\ ((fus\ R)\downarrow)$
  ⟨*proof*⟩

lemma *ne-fusion-fission*:
  $(ne\ ((fus\ R)\downarrow))\uparrow = (fis\ R)\uparrow$
  ⟨*proof*⟩

lemma *fission-up-ne-down-up*:
  $(fis\ R)\uparrow = (ne\ (R\downarrow))\uparrow$
  ⟨*proof*⟩

lemma *fusion-idempotent*:
  $fus\ (fus\ R) = fus\ R$
  ⟨*proof*⟩

lemma *fission-dist-oU*:
  $fis\ (\bigcup X) = \bigcup (fis\ `\ X)$
  ⟨*proof*⟩

7.5 Co-fusion and co-fission

definition *co-fusion* :: $('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel\ (\bigsqcap\bigsqcap\ \text{-}\ [80]\ 80)$ **where**
  $\bigsqcap\bigsqcap R \equiv \{\ (a,B)\ .\ B = \bigcap\{\ C\ .\ (a,C) \in R\ \}\ \}$

declare *co-fusion-def* [*mr-simp*]

lemma *co-fusion-upper-decreasing*:
  $\bigsqcap\bigsqcap R \sqsubseteq\uparrow R$
  ⟨*proof*⟩

lemma *co-fusion-deterministic*:
  $deterministic\ (\bigsqcap\bigsqcap R)$
  ⟨*proof*⟩

lemma *co-fusion-greatest*:

assumes $S \sqsubseteq\uparrow R$
and *deterministic* S
shows $S \sqsubseteq\uparrow \bigsqcap\bigsqcap R$
⟨*proof*⟩

lemma *co-fusion-unique*:
assumes $\forall R . f\ R \sqsubseteq\uparrow R$
and $\forall R$. *deterministic* $(f\ R)$
and $\forall R\ S\ .\ S \sqsubseteq\uparrow R \wedge$ *deterministic* $S \longrightarrow S \sqsubseteq\uparrow f\ R$
shows $f\ T = \bigsqcap\bigsqcap T$
⟨*proof*⟩

lemma *co-fusion-up-char*:
$(\bigsqcap\bigsqcap R)\uparrow = -((-(R\uparrow) \cap A_{\cap\cap})\downarrow)$
⟨*proof*⟩

lemma *co-fusion-down-char*:
$(\bigsqcap\bigsqcap R)\downarrow = -((\sim(R\uparrow) \cap A_{\cup\cup})\uparrow)$
⟨*proof*⟩

lemma *co-fusion-down-char-2*:
$(\bigsqcap\bigsqcap R)\downarrow = -(((R\uparrow \cap A_{\cap\cap}) \odot \sim 1)\uparrow)$
⟨*proof*⟩

lemma *co-fusion-char*:
$\bigsqcap\bigsqcap R = -((-(R\uparrow) \cap A_{\cap\cap})\downarrow) \cap -((\sim(R\uparrow) \cap A_{\cup\cup})\uparrow)$
⟨*proof*⟩

lemma *co-fusion-char-2*:
$\bigsqcap\bigsqcap R = -((-(R\uparrow) \cap A_{\cap\cap})\downarrow) \cap -(((R\uparrow \cap A_{\cap\cap}) \odot \sim 1)\uparrow)$
⟨*proof*⟩

lemma *co-fusion-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow \bigsqcap\bigsqcap R \sqsubseteq\uparrow \bigsqcap\bigsqcap S$
⟨*proof*⟩

lemma *co-fusion-ii-idempotent*:
$\bigsqcap\bigsqcap R \cap\cap \bigsqcap\bigsqcap R = \bigsqcap\bigsqcap R$
⟨*proof*⟩

lemma *co-fusion-up*:
$\bigsqcap\bigsqcap R = \bigsqcap\bigsqcap (R\uparrow)$
⟨*proof*⟩

lemma *co-fusion-ii-total*:
total $T \Longrightarrow T \cap\cap \bigsqcap\bigsqcap T = \bigsqcap\bigsqcap T$
⟨*proof*⟩

lemma *co-fusion-deterministic-fixpoint*:

deterministic $R \longleftrightarrow R = \bigsqcap \bigsqcap R$
⟨*proof*⟩

abbreviation *co-fission* :: $('a,'b)$ *mrel* $\Rightarrow$ $('a,'b)$ *mrel* ($at_{\cap\cap}$ - [*80*] *80*) **where**
$at_{\cap\cap}\ R \equiv R\uparrow \cap A_{\cap\cap}$

lemma *co-fission*:
$at_{\cap\cap}\ R = \{\ (a,B) \mid a\ B\ .\ (\exists\, b\ .\ -B = \{b\}) \wedge (\exists\, C\ .\ (a,C) \in R \wedge C \subseteq B)\ \}$
⟨*proof*⟩

declare *co-fission* [*mr-simp*]

lemma *co-fission-upper-increasing*:
$R \sqsubseteq\uparrow at_{\cap\cap}\ R$
⟨*proof*⟩

lemma *co-fission-ic-inner-deterministic*:
inner-deterministic $(\sim at_{\cap\cap}\ R)$
⟨*proof*⟩

lemma *co-fission-least*:
assumes $R \sqsubseteq\uparrow S$
and *inner-deterministic* $(\sim S)$
shows $at_{\cap\cap}\ R \sqsubseteq\uparrow S$
⟨*proof*⟩

lemma *co-fission-unique*:
assumes $\forall\, R\ .\ R \sqsubseteq\uparrow f\ R$
and $\forall\, R$. *inner-deterministic* $(\sim f\ R)$
and $\forall\, R\ S\ .\ R \sqsubseteq\uparrow S \wedge$ *inner-deterministic* $(\sim S) \longrightarrow f\ R \sqsubseteq\uparrow S$
shows $f\ T = at_{\cap\cap}\ T$
⟨*proof*⟩

lemma *co-fission-upper-isotone*:
$R \sqsubseteq\uparrow S \Longrightarrow at_{\cap\cap}\ R \sqsubseteq\uparrow at_{\cap\cap}\ S$
⟨*proof*⟩

lemma *co-fission-idempotent*:
$at_{\cap\cap}\ (at_{\cap\cap}\ R) = at_{\cap\cap}\ R$
⟨*proof*⟩

lemma *co-fission-top*:
$at_{\cap\cap}\ U = A_{\cap\cap}$
⟨*proof*⟩

lemma *co-fission-up*:
$at_{\cap\cap}\ R = at_{\cap\cap}\ (R\uparrow)$
⟨*proof*⟩

lemma *co-fission-ic-inner-deterministic-fixpoint*:
$inner\text{-}deterministic\ (\sim R) \longleftrightarrow R = at_{\cap\cap}\ R$
⟨*proof*⟩

lemma *co-fusion-co-fission-upper-decreasing*:
$\prod\prod(at_{\cap\cap}\ R) \sqsubseteq\uparrow R$
⟨*proof*⟩

lemma *co-fission-co-fusion-upper-increasing*:
$R \sqsubseteq\uparrow at_{\cap\cap}\ (\prod\prod R)$
⟨*proof*⟩

lemma *co-fusion-co-fission-galois*:
$\prod\prod R \sqsubseteq\uparrow S \longleftrightarrow R \sqsubseteq\uparrow at_{\cap\cap}\ S$
⟨*proof*⟩

lemma *co-fission-co-fusion*:
$at_{\cap\cap}\ (\prod\prod R) = at_{\cap\cap}\ R$
⟨*proof*⟩

lemma *co-fusion-co-fission*:
$\prod\prod(at_{\cap\cap}\ R) = \prod\prod R$
⟨*proof*⟩

lemma *same-co-fusion-co-fission-upper*:
$\prod\prod R = \prod\prod S \Longrightarrow S \sqsubseteq\uparrow at_{\cap\cap}\ R$
⟨*proof*⟩

lemma *co-fusion-idempotent*:
$\prod\prod(\prod\prod R) = \prod\prod R$
⟨*proof*⟩

8 Modalities

8.1 Tests

abbreviation *test* :: $('a,'a)\ mrel \Rightarrow bool$ **where**
$test\ R \equiv R \subseteq 1$

lemma *test*:
$test\ R \longleftrightarrow (\forall\ a\ B\ .\ (a,B) \in R \longrightarrow B = \{a\})$
⟨*proof*⟩

lemma *test-fix*: $test\ R \equiv R \cap 1_\sigma = R$
⟨*proof*⟩

lemma *test-ou-closed*:
$test\ p \Longrightarrow test\ q \Longrightarrow test\ (p \cup q)$
⟨*proof*⟩

lemma *test-oi-closed*:
$test\ p \Longrightarrow test\ (p \cap q)$
⟨*proof*⟩

abbreviation *test-complement* :: $('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel$ ($\wr$ - [*80*] *80*) **where**
$\wr\ R \equiv -R \cap 1$

lemma *test-complement-closed*:
$test\ (\wr\ p)$
⟨*proof*⟩

lemma *test-double-complement*:
$test\ p \longleftrightarrow p = \wr\ \wr\ p$
⟨*proof*⟩

lemma *test-complement*:
$(a,\{a\}) \in \wr\ p \longleftrightarrow \neg\ (a,\{a\}) \in p$
⟨*proof*⟩

declare *test-complement* [*mr-simp*]

lemma *test-complement-antitone*:
assumes *test p*
shows $p \subseteq q \longleftrightarrow \wr\ q \subseteq \wr\ p$
⟨*proof*⟩

lemma *test-complement-huntington*:
$test\ p \Longrightarrow p = \wr\ (\wr\ p \cup \wr\ q) \cup \wr\ (\wr\ p \cup q)$
⟨*proof*⟩

abbreviation *test-implication* :: $('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel$
(**infixl** $\rightarrow$ *65*) **where**
$p \rightarrow q \equiv \wr\ p \cup q$

lemma *test-implication-closed*:
$test\ q \Longrightarrow test\ (p \rightarrow q)$
⟨*proof*⟩

lemma *test-implication*:
$(a,\{a\}) \in p \rightarrow q \longleftrightarrow ((a,\{a\}) \in p \longrightarrow (a,\{a\}) \in q)$
⟨*proof*⟩

declare *test-implication* [*mr-simp*]

lemma *test-implication-left-antitone*:
assumes *test p*
shows $p \subseteq r \Longrightarrow r \rightarrow q \subseteq p \rightarrow q$
⟨*proof*⟩

lemma *test-implication-right-isotone*:
assumes *test* p
shows $q \subseteq r \Longrightarrow p \rightarrow q \subseteq p \rightarrow r$
⟨*proof*⟩

lemma *test-sp-idempotent*:
test $p \Longrightarrow p * p = p$
⟨*proof*⟩

lemma *test-sp*:
assumes *test* p
shows $p * R = (p * U) \cap R$
⟨*proof*⟩

lemma *sp-test*:
test $p \Longrightarrow R * p = R \cap (U * p)$
⟨*proof*⟩

lemma *sp-test-dist-oi*:
test $p \Longrightarrow (R \cap S) * p = (R * p) \cap (S * p)$
⟨*proof*⟩

lemma *sp-test-dist-oi-left*:
test $p \Longrightarrow (R \cap S) * p = (R * p) \cap S$
⟨*proof*⟩

lemma *sp-test-dist-oi-right*:
test $p \Longrightarrow (R \cap S) * p = R \cap (S * p)$
⟨*proof*⟩

lemma *sp-test-sp-oi-left*:
test $p \Longrightarrow (R \cap (U * p)) * T = R * p * T$
⟨*proof*⟩

lemma *sp-test-sp-oi-right*:
test $p \Longrightarrow R * ((p * U) \cap T) = R * p * T$
⟨*proof*⟩

lemma *test-sp-ne*:
test $p \Longrightarrow p * ne\ R = ne\ (p * R)$
⟨*proof*⟩

lemma *ne-sp-test*:
test $p \Longrightarrow ne\ R * p = ne\ (R * p)$
⟨*proof*⟩

lemma *top-sp-test-down-closed*:
assumes *test* p

shows $U * p = (U * p)\downarrow$
$\langle proof \rangle$

lemma *oc-top-sp-test-up-closed*:
$test\ p \Longrightarrow -(U * p) = (-(U * p))\uparrow$
$\langle proof \rangle$

lemma *top-sp-test*:
$test\ p \Longrightarrow (a,B) \in U * p \longleftrightarrow (\forall\, b{\in}B\ .\ (b,\{b\}) \in p)$
$\langle proof \rangle$

lemma *oc-top-sp-test*:
$test\ p \Longrightarrow (a,B) \in -(U * p) \longleftrightarrow (\exists\, b{\in}B\ .\ (b,\{b\}) \notin p)$
$\langle proof \rangle$

declare *top-sp-test* [*mr-simp*] *oc-top-sp-test* [*mr-simp*]

lemma *oc-top-sp-test-0*:
$-1_{\cup\cup} * \wr\ p = ne\ (U * \wr\ p)$
$\langle proof \rangle$

lemma *oc-top-sp-test-1*:
assumes $test\ p$
shows $-(U * p) = (ne\ (U * \wr\ p))\uparrow$
$\langle proof \rangle$

lemma *oc-top-sp-test-2*:
$test\ p \Longrightarrow -(U * p) = (-1_{\cup\cup} * \wr\ p)\uparrow$
$\langle proof \rangle$

lemma *split-sp-test*:
assumes $test\ p$
shows $R = (R * p) \cup (ne\ R \cap (ne\ (R\downarrow * \wr\ p))\uparrow)$
$\langle proof \rangle$

lemma *top-sp-test-down-iff-1*:
assumes $test\ p$
shows $R \subseteq U * p \longleftrightarrow R\downarrow \subseteq U * p$
$\langle proof \rangle$

lemma *test-ne*:
$test\ p \Longrightarrow ne\ p = p$
$\langle proof \rangle$

lemma *ne-test-up*:
$test\ p \Longrightarrow ne\ (p\uparrow) = p\uparrow$
$\langle proof \rangle$

lemma *ne-sp-test-up*:

$test\ p \Longrightarrow (ne\ (R * p))\uparrow = ne\ R * p\uparrow$
⟨proof⟩

lemma *ne-down-sp-test-up*:
$test\ p \Longrightarrow ne\ (R\downarrow * p\uparrow) = ne\ (R\downarrow) * p\uparrow$
⟨proof⟩

lemma *test-up-sp*:
$test\ p \Longrightarrow p\uparrow = p * 1\uparrow$
⟨proof⟩

lemma *top-test-oi-top-complement*:
$test\ p \Longrightarrow (U * p) \cap (U * \wr\ p) = 1_{\cup\cup}$
⟨proof⟩

lemma *sp-test-oi-complement*:
$test\ p \Longrightarrow (R * p) \cap (R * \wr\ p) = R \cap 1_{\cup\cup}$
⟨proof⟩

lemma *ne-top-sp-test-complement*:
assumes $test\ p$
shows $ne\ (U * p) * \wr\ p = \{\}$
⟨proof⟩

lemma *complement-test-sp-top*:
assumes $test\ p$
shows $-(p * U) = \wr\ p * U$
⟨proof⟩

lemma *top-sp-test-shunt*:
assumes $test\ p$
shows $R \subseteq U * p \longrightarrow R * \wr\ p \subseteq 1_{\cup\cup}$
⟨proof⟩

lemma *top-sp-test-down-iff-2*:
assumes $test\ p$
shows $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow * \wr\ p \subseteq 1_{\cup\cup}$
⟨proof⟩

lemma *top-sp-test-down-iff-3*:
$R\downarrow * \wr\ p \subseteq 1_{\cup\cup} \longleftrightarrow ne\ (R\downarrow) * \wr\ p \subseteq \{\}$
⟨proof⟩

lemma *top-sp-test-down-iff-4*:
assumes $test\ p$
shows $R\downarrow \cap (U * \wr\ p) \subseteq 1_{\cup\cup} \longleftrightarrow R\downarrow \subseteq 1_{\cup\cup} \cup (U * p)$
⟨proof⟩

lemma *top-sp-test-down-iff-5*:

assumes *test p*
shows $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow \subseteq 1_{\cup\cup} \cup (U * p)$
⟨*proof*⟩

lemma *iu-test-sp-left-zero*:
assumes $q \subseteq 1_{\cup\cup}$
shows $q * R = q$
⟨*proof*⟩

lemma *test-iu-test-split*:
$t \subseteq 1 \cup 1_{\cup\cup} \longleftrightarrow (\exists p\ q\ .\ p \subseteq 1 \wedge q \subseteq 1_{\cup\cup} \wedge t = p \cup q)$
⟨*proof*⟩

lemma *test-iu-test-sp-assoc-1*:
$t \subseteq 1 \cup 1_{\cup\cup} \Longrightarrow t * (R * S) = (t * R) * S$
⟨*proof*⟩

lemma *test-iu-test-sp-assoc-2*:
$t \subseteq 1_{\cup\cup} \Longrightarrow R * (t * S) = (R * t) * S$
⟨*proof*⟩

lemma *test-iu-test-sp-assoc-3*:
assumes $t \subseteq 1 \cup 1_{\cup\cup}$
shows $R * (t * S) = (R * t) * S$
⟨*proof*⟩

lemma *test-iu-test-sp-assoc-4*:
$t \subseteq 1_{\cup\cup} \Longrightarrow R * (S * t) = (R * S) * t$
⟨*proof*⟩

lemma *test-iu-test-sp-assoc-5*:
assumes $t \subseteq 1 \cup 1_{\cup\cup}$
shows $R * (S * t) = (R * S) * t$
⟨*proof*⟩

lemma *inner-deterministic-sp-assoc*:
assumes *inner-univalent t*
shows $t * (R * S) = (t * R) * S$
⟨*proof*⟩

lemma *iu-unit-below-top-sp-test*:
$1_{\cup\cup} \subseteq U * R$
⟨*proof*⟩

lemma *ne-oi-complement-top-sp-test-1*:
$ne\ (R \cap -(U * S)) = R \cap -(U * S)$
⟨*proof*⟩

lemma *ne-oi-complement-top-sp-test-2*:

$ne\ R \cap -(U * S) = R \cap -(U * S)$
⟨*proof*⟩

lemma *schroeder-test*:
assumes $test\ p$
shows $R * p \subseteq S \longleftrightarrow -S * p \subseteq -R$
⟨*proof*⟩

lemma *complement-test-sp-test*:
$test\ p \Longrightarrow -p * p \subseteq -1$
⟨*proof*⟩

lemma *test-sp-commute*:
$test\ p \Longrightarrow test\ q \Longrightarrow p * q = q * p$
⟨*proof*⟩

lemma *test-shunting*:
assumes $test\ p$
and $test\ q$
and $test\ r$
shows $p * q \subseteq r \longleftrightarrow p \subseteq r \cup \wr\ q$
⟨*proof*⟩

lemma *test-sp-is-iu*:
$test\ p \Longrightarrow test\ q \Longrightarrow p * q = p \cup\cup\ q$
⟨*proof*⟩

lemma *test-set*:
$test\ p \Longrightarrow p = \{\ (a,\{a\})\ |\ a\ .\ (a,\{a\}) \in p\ \}$
⟨*proof*⟩

lemma *test-sp-is-ii*:
assumes $test\ p$
and $test\ q$
shows $p * q = p \cap\cap\ q$
⟨*proof*⟩

lemma *test-galois-1*:
assumes $test\ p$
and $test\ q$
shows $p * q \subseteq r \longleftrightarrow q \subseteq p \rightarrow r$
⟨*proof*⟩

lemma *test-sp-shunting*:
assumes $test\ p$
shows $\wr\ p * R \subseteq \{\} \longleftrightarrow R \subseteq p * R$
⟨*proof*⟩

lemma *test-oU-closed*:

$\forall\, p{\in}X\ .\ test\ p \Longrightarrow test\ (\bigcup X)$
⟨*proof*⟩

lemma *test-oI-closed*:
$\exists\, p{\in}X\ .\ test\ p \Longrightarrow test\ (\bigcap X)$
⟨*proof*⟩

lemma *sp-test-dist-oI*:
assumes *test p*
and $X \neq \{\}$
shows $(\bigcap X) * p = (\bigcap R{\in}X\ .\ R * p)$
⟨*proof*⟩

lemma *test-iU-is-iI*:
assumes $\forall\, i{\in}I\ .\ test\ (X\ i)$
and $I \neq \{\}$
shows $\bigcup\bigcup X|I = \bigcap\bigcap X|I$
⟨*proof*⟩

lemma *test-iU-is-oI*:
assumes $\forall\, i{\in}I\ .\ test\ (X\ i)$
and $I \neq \{\}$
shows $\bigcup\bigcup X|I = \bigcap(X\ `\ I)$
⟨*proof*⟩

8.2 Domain and antidomain

declare *Dom-def* [*mr-simp*]

abbreviation *aDom* :: $('a,'b)\ mrel \Rightarrow ('a,'a)\ mrel$ **where**
$aDom\ R \equiv \wr\ Dom\ R$

lemma *ad-set*: $aDom\ R = \{(a,\{a\})|\ a.\ \neg(\exists\, A.\ (a,A) \in R)\}$
⟨*proof*⟩

lemma *d-test*:
test (*Dom R*)
⟨*proof*⟩

lemma *ad-test*:
test (*aDom R*)
⟨*proof*⟩

lemma *ad-expl*:
$aDom\ R = -((R * 1_{\cup\cup}) \cup\cup\ 1) \cap 1$
⟨*proof*⟩

lemma *ad-expl-2*:
$aDom\ (R{::}('a,'b)\ mrel) = -((R * (1_{\cup\cup}{::}('b,'a)\ mrel))\uparrow) \cap (1{::}('a,'a)\ mrel)$

⟨*proof*⟩

lemma *aDom*:
$aDom\ R = \{\ (a,\{a\})\ |\ a\ .\ \neg(\exists\, B\ .\ (a,B) \in R)\ \}$
⟨*proof*⟩

declare *aDom* [*mr-simp*]

lemma *d-down-oi-up-1*:
$Dom\ (R{\downarrow} \cap S) = Dom\ (R \cap S{\uparrow})$
⟨*proof*⟩

lemma *d-down-oi-up-2*:
$Dom\ (R{\downarrow} \cap S) = Dom\ (R{\downarrow} \cap S{\uparrow})$
⟨*proof*⟩

lemma *d-ne-down-dp-complement-test*:
assumes *test p*
shows $Dom\ (R \cap -(U * p)) = Dom\ (ne\ (R{\downarrow}) * \wr\ p)$
⟨*proof*⟩

lemma *d-strict*:
$R = \{\} \longleftrightarrow Dom\ R = \{\}$
⟨*proof*⟩

lemma *d-sp-strict*:
$R * S = \{\} \longleftrightarrow R * Dom\ S = \{\}$
⟨*proof*⟩

lemma *d-complement-ad*:
$Dom\ R = \wr\ aDom\ R$
⟨*proof*⟩

lemma *down-sp-below-iu-unit*:
$R{\downarrow} * S \subseteq 1_{\cup\cup} \longleftrightarrow R \subseteq U * aDom\ (ne\ S)$
⟨*proof*⟩

lemma *ad-sp-bot*:
$aDom\ R * R = \{\}$
⟨*proof*⟩

lemma *sp-top-d*:
$R * U \subseteq Dom\ R * U$
⟨*proof*⟩

lemma *d-sp-top*:
$Dom\ (R * U) = Dom\ R$
⟨*proof*⟩

lemma *d-down*:
$Dom\ (R\downarrow) = Dom\ R$
⟨*proof*⟩

lemma *d-up*:
$Dom\ (R\uparrow) = Dom\ R$
⟨*proof*⟩

lemma *d-isotone*:
$R \subseteq S \Longrightarrow Dom\ R \subseteq Dom\ S$
⟨*proof*⟩

lemma *ad-antitone*:
$R \subseteq S \Longrightarrow aDom\ S \subseteq aDom\ R$
⟨*proof*⟩

lemma *d-dist-ou*:
$Dom\ (R \cup S) = Dom\ R \cup Dom\ S$
⟨*proof*⟩

lemma *d-dist-iu*:
$Dom\ (R \cup\cup S) = Dom\ R * Dom\ S$
⟨*proof*⟩

lemma *d-dist-ii*:
$Dom\ (R \cap\cap S) = Dom\ R * Dom\ S$
⟨*proof*⟩

lemma *d-loc*:
$Dom\ (R * Dom\ S) = Dom\ (R * S)$
⟨*proof*⟩

lemma *ad-loc*:
$aDom\ (R * Dom\ S) = aDom\ (R * S)$
⟨*proof*⟩

lemma *d-ne-down*:
$Dom\ (ne\ (R\downarrow)) = Dom\ (ne\ R)$
⟨*proof*⟩

lemma *ne-sp-iu-unit-up*:
$ne\ R = R \Longrightarrow (R * 1_{\cup\cup})\uparrow = R * U$
⟨*proof*⟩

lemma *ne-d-expl*:
$ne\ R = R \Longrightarrow Dom\ R = R * U \cap 1$
⟨*proof*⟩

lemma *ne-a-expl*:

$ne\ R = R \Longrightarrow aDom\ R = -(R * U) \cap 1$
$\langle proof \rangle$

lemma *d-dist-oU*:
$Dom\ (\bigcup X) = \bigcup (Dom\ `\ X)$
$\langle proof \rangle$

lemma *d-dist-iU-iI*:
$Dom\ (\bigcup\bigcup X|I) = Dom\ (\bigcap\bigcap X|I)$
$\langle proof \rangle$

lemma *d-dist-iU-oI*:
assumes $I \neq \{\}$
shows $Dom\ (\bigcup\bigcup X|I) = \bigcap (Dom\ `\ X\ `\ I)$
$\langle proof \rangle$

8.3 Left residual

definition *sp-lres* :: $('a,'c)\ mrel \Rightarrow ('b,'c)\ mrel \Rightarrow ('a,'b)\ mrel$ (**infixl** $\oslash$ *65*)
where
$Q \oslash R \equiv \{\ (a,B)\ .\ \forall f\ .\ (\forall b \in B\ .\ (b,f\ b) \in R) \longrightarrow (a,\bigcup\{\ f\ b\ |\ b\ .\ b \in B\ \}) \in Q\ \}$

declare *sp-lres-def* [*mr-simp*]

lemma *sp-lres-galois*:
$S * R \subseteq Q \longleftrightarrow S \subseteq Q \oslash R$
$\langle proof \rangle$

lemma *sp-lres-expl*:
$Q \oslash R = \bigcup\{\ S\ .\ S * R \subseteq Q\ \}$
$\langle proof \rangle$

lemma *bot-sp-lres-d*:
$\{\} \oslash R = \{\} \oslash Dom\ R$
$\langle proof \rangle$

lemma *bot-sp-lres-expl*:
$\{\} \oslash R = -(U * Dom\ R)$
$\langle proof \rangle$

lemma *sp-lres-sp-below*:
$(Q \oslash R) * R \subseteq Q$
$\langle proof \rangle$

lemma *sp-lres-left-isotone*:
$Q \subseteq S \Longrightarrow Q \oslash R \subseteq S \oslash R$
$\langle proof \rangle$

lemma *sp-lres-right-antitone*:
$S \subseteq R \Longrightarrow Q \oslash R \subseteq Q \oslash S$
$\langle proof \rangle$

lemma *sp-lres-down-closed-1*:
$Q\downarrow \oslash R = Q\downarrow \oslash R\downarrow$
$\langle proof \rangle$

lemma *sp-lres-down-closed-2*:
assumes $R\downarrow = R$
and *total* T
shows $(R \oslash T)\downarrow = R \oslash T$
$\langle proof \rangle$

lemma *down-sp-sp*:
$R\downarrow * S = R * (1_{\cup\cup} \cup S)$
$\langle proof \rangle$

lemma *iu-unit-sp-lres-iu-unit-ou*:
$U * aDom\ (ne\ R) = 1_{\cup\cup} \oslash (1_{\cup\cup} \cup R)$
$\langle proof \rangle$

lemma *bot-sl-below-complement-d*:
$\{\} \oslash R \subseteq -\ Dom\ R$
$\langle proof \rangle$

lemma *sp-unit-oi-bot-sp-lres*:
$1 \cap -\ Dom\ R = 1 \cap (\{\} \oslash R)$
$\langle proof \rangle$

lemma *ad-explicit-d*:
$aDom\ R = -(U * Dom\ R) \cap 1$
$\langle proof \rangle$

lemma *top-test-sp-lres-total-expl-1*:
assumes *test* p
shows $\forall S\ .\ S\downarrow \subseteq (U * p) \oslash R \longleftrightarrow S \subseteq U * aDom\ (R \cap -(U * p))$
$\langle proof \rangle$

lemma *top-test-sp-lres-total-expl-2*:
assumes *test* p
and *total* T
shows $(U * p) \oslash T = U * aDom\ (T \cap -(U * p))$
$\langle proof \rangle$

lemma *top-test-sp-lres-total-expl-3*:
assumes *test* p
shows $((U * p) \oslash R) \cap 1 = aDom\ (R \cap -(U * p))$
$\langle proof \rangle$

lemma *top-test-sp-lres-total-expl-4*:
assumes *test p*
shows $aDom\ (ne\ (R\downarrow) * \wr\ p) = ((U * p) \oslash R) \cap 1$
⟨*proof*⟩

lemma *oi-complement-top-sp-test-top-1*:
assumes *test p*
shows $(R \cap -(U * p)) * U = (R\downarrow \cap -(U * p)) * U$
⟨*proof*⟩

lemma *oi-complement-top-sp-test-top-2*:
assumes *test p*
shows $(R\downarrow \cap -(U * p)) * U = ne\ (R\downarrow) * \wr\ p * U$
⟨*proof*⟩

lemma *oi-complement-top-sp-test-top-3*:
assumes *test p*
shows $(R\downarrow \cap -(U * p)) * U = ne\ (R\downarrow) * -(p * U)$
⟨*proof*⟩

lemma *split-sp-test-2*:
$test\ p \Longrightarrow R \subseteq R * p \cup ne\ (R\downarrow) * (\wr\ p)\uparrow$
⟨*proof*⟩

lemma *split-sp-test-3*:
$test\ p \Longrightarrow R \subseteq R * p \cup R\downarrow * (\wr\ p)\uparrow$
⟨*proof*⟩

lemma *split-sp-test-4*:
assumes *test p*
and *test q*
shows $R * (p \cup q) \subseteq R * p \cup ne\ (R\downarrow) * q\uparrow$
⟨*proof*⟩

lemma *split-sp-test-5*:
assumes *test p*
and *test q*
shows $R * (p \cup q) \subseteq R * p \cup R\downarrow * q\uparrow$
⟨*proof*⟩

lemma *split-sp-test-6*:
assumes *test p*
and *test q*
shows $Dom\ (R * (p \cup q)) \subseteq Dom\ (R * p \cup ne\ (R\downarrow) * q)$
⟨*proof*⟩

lemma *split-sp-test-7*:
assumes *test p*

and *test q*
shows $Dom\ (ne\ (R\downarrow) * (p \cup q)) = Dom\ (ne\ (R\downarrow) * p \cup ne\ (R\downarrow) * q)$
⟨*proof*⟩

lemma *test-sp-left-dist-iu-1*:
$test\ p \Longrightarrow p * (R \cup\cup S) = p * R \cup\cup S$
⟨*proof*⟩

lemma *test-sp-left-dist-iu-2*:
$test\ p \Longrightarrow p * (R \cup\cup S) = R \cup\cup p * S$
⟨*proof*⟩

lemma *d-sp-below-iu-down*:
$Dom\ R * S \subseteq (R \cup\cup S)\downarrow$
⟨*proof*⟩

lemma *d-sp-ne-down-below-ne-iu-down*:
$Dom\ R * ne\ (S\downarrow) \subseteq ne\ ((R \cup\cup S)\downarrow)$
⟨*proof*⟩

lemma *top-test*:
$test\ p \Longrightarrow U * p = \{\ (a,B)\ .\ (\forall\, b{\in}B\ .\ (b,\{b\}) \in p)\ \}$
⟨*proof*⟩

lemma *iu-oi-complement-top-test-ou-up*:
$test\ p \Longrightarrow (R \cup\cup S) \cap -(U * p) \subseteq ((R \cup S) \cap -(U * p))\uparrow$
⟨*proof*⟩

lemma *d-ne-iu-down-sp-test-ou*:
assumes *test p*
shows $Dom\ (ne\ ((R \cup\cup S)\downarrow) * p) \subseteq Dom\ ((ne\ (R\downarrow) \cup ne\ (S\downarrow)) * p)$
⟨*proof*⟩

lemma *test-sp-left-dist-iU*:
assumes *test p*
and $I \neq \{\}$
shows $p * (\bigcup\bigcup X|I) = \bigcup\bigcup (\lambda i\ .\ p * X\ i)|I$
⟨*proof*⟩

8.4 Modal operations

definition *adia* :: $('a,'b)\ mrel \Rightarrow ('b,'b)\ mrel \Rightarrow ('a,'a)\ mrel$ (| - ⟩ - [*50*,*90*] *95*)
where
$|R\rangle p \equiv \{\ (a,\{a\})\ |\ a\ .\ \exists\, B\ .\ (a,B) \in R \wedge (\forall\, b{\in}B\ .\ (b,\{b\}) \in p)\ \}$

definition *abox* :: $('a,'b)\ mrel \Rightarrow ('b,'b)\ mrel \Rightarrow ('a,'a)\ mrel$ (| -] - [*50*,*90*] *95*)
where
$|R]p \equiv \{\ (a,\{a\})\ |\ a\ .\ \forall\, B\ .\ (a,B) \in R \longrightarrow (\forall\, b{\in}B\ .\ (b,\{b\}) \in p)\ \}$

definition *edia* :: $('a,'b)$ *mrel* $\Rightarrow$ $('b,'b)$ *mrel* $\Rightarrow$ $('a,'a)$ *mrel* $(|\text{ - }\rangle\rangle\text{ - }[50,90]\ 95)$
where
  $|R\rangle\rangle p \equiv \{\ (a,\{a\})\ |\ a\ .\ \exists\, B\ .\ (a,B) \in R \wedge (\exists\, b{\in}B\ .\ (b,\{b\}) \in p)\ \}$

definition *ebox* :: $('a,'b)$ *mrel* $\Rightarrow$ $('b,'b)$ *mrel* $\Rightarrow$ $('a,'a)$ *mrel* $(|\text{ - }]]\text{ - }[50,90]\ 95)$
where
  $|R]]p \equiv \{\ (a,\{a\})\ |\ a\ .\ \forall\, B\ .\ (a,B) \in R \longrightarrow (\exists\, b{\in}B\ .\ (b,\{b\}) \in p)\ \}$

declare *adia-def* [*mr-simp*] *abox-def* [*mr-simp*] *edia-def* [*mr-simp*] *ebox-def* [*mr-simp*]

lemma *adia*:
  assumes *test p*
  shows $|R\rangle p = Dom\ (R * p)$
⟨*proof*⟩

lemma *abox-1*:
  assumes *test p*
  shows $|R]p = aDom\ (R \cap -(U * p))$
⟨*proof*⟩

lemma *abox*:
  assumes *test p*
  shows $|R]p = aDom\ (ne\ (R\!\downarrow) * \wr\ p)$
  ⟨*proof*⟩

lemma *edia-1*:
  assumes *test p*
  shows $|R\rangle\rangle p = Dom\ (R \cap -(U * \wr\ p))$
⟨*proof*⟩

lemma *edia*:
  assumes *test p*
  shows $|R\rangle\rangle p = Dom\ (ne\ (R\!\downarrow) * p)$
  ⟨*proof*⟩

lemma *ebox*:
  assumes *test p*
  shows $|R]]p = aDom\ (R * \wr\ p)$
⟨*proof*⟩

lemma *abox-2*:
  assumes *test p*
  shows $|R]p = -((R \cap -(U * p)) * U) \cap 1$
  ⟨*proof*⟩

lemma *abox-3*:
  assumes *test p*
  shows $|R]p = -(ne\ (R\!\downarrow) * \wr\ p * U) \cap 1$

⟨*proof*⟩

lemma *abox-4*:
  assumes *test p*
  shows $|R]p = ((U * p) \oslash R) \cap 1$
  ⟨*proof*⟩

lemma *abox-ebox*:
  assumes *test p*
  shows $|R]p = |ne\ (R\downarrow)]]p$
  ⟨*proof*⟩

lemma *abox-edia*:
  assumes *test p*
  shows $|R]p = \wr\ |R\rangle\rangle(\wr\ p)$
  ⟨*proof*⟩

lemma *abox-adia*:
  assumes *test p*
  shows $|R]p = \wr\ |ne\ (R\downarrow)\rangle(\wr\ p)$
  ⟨*proof*⟩

lemma *edia-adia*:
  assumes *test p*
  shows $|R\rangle\rangle p = |ne\ (R\downarrow)\rangle p$
  ⟨*proof*⟩

lemma *edia-abox*:
  assumes *test p*
  shows $|R\rangle\rangle p = \wr\ |R](\wr\ p)$
  ⟨*proof*⟩

lemma *edia-ebox*:
  assumes *test p*
  shows $|R\rangle\rangle p = \wr\ |ne\ (R\downarrow)]](\wr\ p)$
  ⟨*proof*⟩

lemma *abox-ne-down*:
  assumes *test p*
  shows $|R]p = |ne\ (R\downarrow)]p$
  ⟨*proof*⟩

lemma *edia-ne-down*:
  assumes *test p*
  shows $|R\rangle\rangle p = |ne\ (R\downarrow)\rangle\rangle p$
  ⟨*proof*⟩

lemma *adia-up*:
  assumes *test p*

shows $|R\rangle p = |R\uparrow\rangle p$
$\langle proof \rangle$

lemma *ebox-up*:
assumes *test p*
shows $|R]]p = |R\uparrow]]p$
$\langle proof \rangle$

lemma *adia-ebox*:
assumes *test p*
shows $|R\rangle p = \wr\ |R]](\wr\ p)$
$\langle proof \rangle$

lemma *ebox-adia*:
assumes *test p*
shows $|R]]p = \wr\ |R\rangle(\wr\ p)$
$\langle proof \rangle$

lemma *abox-down*:
assumes *test p*
shows $|R]p = |R\downarrow]p$
$\langle proof \rangle$

lemma *edia-down*:
assumes *test p*
shows $|R\rangle\rangle p = |R\downarrow\rangle\rangle p$
$\langle proof \rangle$

lemma *fusion-oi-complement-top-test-up*:
$\mathit{test}\ p \Longrightarrow \mathit{fus}\ R \cap -(U * p) \subseteq (R \cap -(U * p))\uparrow$
$\langle proof \rangle$

lemma *adia-left-isotone*:
$\mathit{test}\ p \Longrightarrow R \subseteq S \Longrightarrow |R\rangle p \subseteq |S\rangle p$
$\langle proof \rangle$

lemma *adia-right-isotone*:
$\mathit{test}\ p \Longrightarrow \mathit{test}\ q \Longrightarrow p \subseteq q \Longrightarrow |R\rangle p \subseteq |R\rangle q$
$\langle proof \rangle$

lemma *abox-left-antitone*:
$\mathit{test}\ p \Longrightarrow R \subseteq S \Longrightarrow |S]p \subseteq |R]p$
$\langle proof \rangle$

lemma *abox-right-isotone*:
$\mathit{test}\ p \Longrightarrow \mathit{test}\ q \Longrightarrow p \subseteq q \Longrightarrow |R]p \subseteq |R]q$
$\langle proof \rangle$

lemma *edia-left-isotone*:

test $p \Longrightarrow R \subseteq S \Longrightarrow |R\rangle\rangle p \subseteq |S\rangle\rangle p$
⟨*proof*⟩

lemma *edia-right-isotone*:
test $p \Longrightarrow$ *test* $q \Longrightarrow p \subseteq q \Longrightarrow |R\rangle\rangle p \subseteq |R\rangle\rangle q$
⟨*proof*⟩

lemma *ebox-left-antitone*:
test $p \Longrightarrow R \subseteq S \Longrightarrow |S]]p \subseteq |R]]p$
⟨*proof*⟩

lemma *ebox-right-isotone*:
test $p \Longrightarrow$ *test* $q \Longrightarrow p \subseteq q \Longrightarrow |R]]p \subseteq |R]]q$
⟨*proof*⟩

lemma *edia-fusion*:
assumes *test* p
shows $|R\rangle\rangle p = |\mathit{fus}\ R\rangle\rangle p$
⟨*proof*⟩

lemma *abox-fusion*:
assumes *test* p
shows $|R]p = |\mathit{fus}\ R]p$
⟨*proof*⟩

lemma *abox-fission*:
assumes *test* p
shows $|R]p = |\mathit{fis}\ R]p$
⟨*proof*⟩

lemma *edia-fission*:
assumes *test* p
shows $|R\rangle\rangle p = |\mathit{fis}\ R\rangle\rangle p$
⟨*proof*⟩

lemma *fission-below*:
$\mathit{fis}\ R \subseteq S \longleftrightarrow (\forall\, a\ b\ B\ .\ (a,B) \in R \wedge b \in B \longrightarrow (a,\{b\}) \in S)$
⟨*proof*⟩

lemma *below-fission-up*:
$S \subseteq (\mathit{fis}\ R)\uparrow \longleftrightarrow (\forall\, a\ B\ .\ (a,B) \in S \longrightarrow (\exists\, C\ .\ (a,C) \in R \wedge C \cap B \neq \{\}))$
⟨*proof*⟩

lemma *ebox-below-abox*:
assumes *test* p
and $\mathit{fis}\ R \subseteq S$
shows $|S]]p \subseteq |R]p$
⟨*proof*⟩

lemma *abox-below-ebox*:
  assumes *test p*
    and $S \subseteq (\mathit{fis}\ R)\uparrow$
  shows $|R]p \subseteq |S]]p$
  ⟨*proof*⟩

lemma *abox-eq-ebox*:
  assumes *test p*
    and $\mathit{fis}\ R \subseteq S$
    and $S \subseteq (\mathit{fis}\ R)\uparrow$
  shows $|R]p = |S]]p$
  ⟨*proof*⟩

lemma *abox-eq-ebox-sufficient*:
  $S = \mathit{fis}\ R \vee S = \mathit{ne}\ (R\downarrow) \vee S = (\mathit{ne}\ (R\downarrow))\uparrow \longrightarrow \mathit{fis}\ R \subseteq S \wedge S \subseteq (\mathit{fis}\ R)\uparrow$
  ⟨*proof*⟩

lemma *ebox-fission-abox*:
  $\mathit{test}\ p \Longrightarrow |R]p = |\mathit{fis}\ R]]p$
  ⟨*proof*⟩

lemma *ebox-down-ne-up-abox*:
  $\mathit{test}\ p \Longrightarrow |R]p = |(\mathit{ne}\ (R\downarrow))\uparrow]]p$
  ⟨*proof*⟩

lemma *same-fusion*:
  assumes $\mathit{fis}\ R \sqsubseteq\downarrow S$
    and $S \sqsubseteq\downarrow \mathit{fus}\ R$
  shows $\mathit{fis}\ R = \mathit{fis}\ S$
  ⟨*proof*⟩

lemma *same-abox*:
  assumes $\mathit{fis}\ R \sqsubseteq\downarrow S$
    and $S \sqsubseteq\downarrow \mathit{fus}\ R$
    and *test p*
  shows $|R]p = |S]p$
  ⟨*proof*⟩

lemma *abox-ebox-inner-deterministic*:
  assumes *test p*
    and *inner-deterministic R*
  shows $|R]p = |R]]p$
  ⟨*proof*⟩

lemma *adia-edia-inner-deterministic*:
  assumes *test p*
    and *inner-deterministic R*
  shows $|R\rangle p = |R\rangle\rangle p$
  ⟨*proof*⟩

lemma *abox-adia-deterministic*:
  assumes *test p*
    and *deterministic R*
  shows $|R]p = |R\rangle p$
$\langle proof \rangle$

lemma *ebox-edia-deterministic*:
  assumes *test p*
    and *deterministic R*
  shows $|R]]p = |R\rangle\rangle p$
  $\langle proof \rangle$

lemma *abox-ebox-fusion*:
  assumes *test p*
  shows $|fis\ R]p = |fis\ R]]p$
  $\langle proof \rangle$

lemma *abox-fission-edia-fusion*:
  assumes *test p*
  shows $|fis\ R]p = |fus\ R\rangle p$
  $\langle proof \rangle$

lemma *abox-adia-fusion*:
  assumes *test p*
  shows $|fus\ R]p = |fus\ R\rangle p$
  $\langle proof \rangle$

8.5 Goldblatt's axioms without star

lemma *abox-sp-unit*:
  $|R]1 = 1$
  $\langle proof \rangle$

lemma *ou-unit-abox*:
  $test\ p \Longrightarrow |\{\}]p = 1$
  $\langle proof \rangle$

lemma *ou-unit-test-implication*:
  $test\ p \Longrightarrow \{\} \rightarrow p = 1$
  $\langle proof \rangle$

lemma *sp-unit-abox*:
  $test\ p \Longrightarrow |1]p = p$
  $\langle proof \rangle$

lemma *sp-unit-test-implication*:
  $test\ p \Longrightarrow 1 \rightarrow p = p$
  $\langle proof \rangle$

lemma *test-abox-ebox*:
$test\ p \Longrightarrow test\ q \Longrightarrow |q]p = |q]]p$
⟨*proof*⟩

lemma *test-abox*:
$test\ p \Longrightarrow test\ q \Longrightarrow |q]p = q \rightarrow p$
⟨*proof*⟩

lemma *abox-ou-adia-sp-unit*:
assumes $test\ p$
shows $|R]p \cup |R\rangle 1 = 1$
⟨*proof*⟩

lemma *d-test-sp*:
$test\ p \Longrightarrow Dom\ (p * R) = p * Dom\ R$
⟨*proof*⟩

lemma *ad-test-sp*:
$test\ p \Longrightarrow aDom\ (p * R) = \wr\ p \cup aDom\ R$
⟨*proof*⟩

lemma *adia-test-sp*:
$test\ p \Longrightarrow test\ q \Longrightarrow |p * R\rangle q = p * |R\rangle q$
⟨*proof*⟩

lemma *ebox-test-sp*:
$test\ p \Longrightarrow test\ q \Longrightarrow |p * R]]q = \wr\ p \cup |R]]q$
⟨*proof*⟩

lemma *abox-test-sp*:
assumes $test\ p$
and $test\ q$
shows $|p * R]q = \wr\ p \cup |R]q$
⟨*proof*⟩

lemma *abox-test-sp-2*:
$test\ p \Longrightarrow test\ q \Longrightarrow p \cup |R]q = |\wr\ p * R]q$
⟨*proof*⟩

lemma *abox-test-sp-3*:
$test\ p \Longrightarrow test\ q \Longrightarrow p \rightarrow |R]q = |p * R]q$
⟨*proof*⟩

lemma *fission-sp-dist*:
$fis\ (R * S) = fis\ (R * Dom\ S) * fis\ S$
⟨*proof*⟩

lemma *abox-test*:

$test\ p \Longrightarrow test\ (\ |R]p\ )$
$\langle proof \rangle$

lemma *adia-test*:
$test\ p \Longrightarrow test\ (\ |R\rangle p\ )$
$\langle proof \rangle$

lemma *ebox-test*:
$test\ p \Longrightarrow test\ (\ |R]]p\ )$
$\langle proof \rangle$

lemma *edia-test*:
$test\ p \Longrightarrow test\ (\ |R\rangle\rangle p\ )$
$\langle proof \rangle$

lemma *abox-sp*:
assumes $test\ p$
and $test\ q$
shows $|R](p * q) = |R]p * |R]q$
$\langle proof \rangle$

lemma *adia-ou-below-ne-down*:
assumes $test\ p$
shows $|R\rangle(p \cup \wr\ q) \subseteq |R\rangle p \cup |ne\ (R\downarrow)\rangle(\wr\ q)$
$\langle proof \rangle$

lemma *abox-adia-mp*:
assumes $test\ p$
and $test\ q$
shows $|R\rangle(p \rightarrow q) * |R]p \subseteq |R\rangle q$
$\langle proof \rangle$

lemma *adia-abox-mp*:
assumes $test\ p$
and $test\ q$
shows $|R\rangle p * |R](p \rightarrow q) \subseteq |R\rangle q$
$\langle proof \rangle$

lemma *abox-implication-adia*:
assumes $test\ p$
and $test\ q$
shows $|R](p \rightarrow q) \subseteq |R\rangle p \rightarrow |R\rangle q$
$\langle proof \rangle$

lemma *abox-adia-implication*:
assumes $test\ p$
and $test\ q$
shows $|R]p \subseteq |R\rangle q \rightarrow |R\rangle(p * q)$
$\langle proof \rangle$

lemma *abox-mp*:
  assumes *test p*
    and *test q*
  shows $|R]p * |R](p \to q) \subseteq |R]q$
  ⟨*proof*⟩

lemma *abox-implication*:
  assumes *test p*
    and *test q*
  shows $|R](p \to q) \subseteq |R]p \to |R]q$
  ⟨*proof*⟩

lemma *ebox-left-dist-ou*:
  assumes *test p*
  shows $|R \cup S]]p = |R]]p * |S]]p$
  ⟨*proof*⟩

lemma *abox-left-dist-ou*:
  assumes *test p*
  shows $|R \cup S]p = |R]p * |S]p$
  ⟨*proof*⟩

lemma *adia-left-dist-ou*:
  assumes *test p*
  shows $|R \cup S\rangle p = |R\rangle p \cup |S\rangle p$
  ⟨*proof*⟩

lemma *edia-left-dist-ou*:
  assumes *test p*
  shows $|R \cup S\rangle\rangle p = |R\rangle\rangle p \cup |S\rangle\rangle p$
  ⟨*proof*⟩

lemma *abox-dist-iu-1*:
  assumes *test p*
  shows $|R \cup\!\cup S]p = |Dom\ R * ne\ (S\downarrow)]]p * |Dom\ S * ne\ (R\downarrow)]]p$
⟨*proof*⟩

lemma *abox-dist-iu-2*:
  assumes *test p*
  shows $|R \cup\!\cup S]p = |Dom\ R * S]p * |Dom\ S * R]p$
⟨*proof*⟩

lemma *abox-dist-iu-3*:
  assumes *test p*
  shows $|R \cup\!\cup S]p = (\ |R\rangle 1 \to |S]p\ ) * (\ |S\rangle 1 \to |R]p\ )$
  ⟨*proof*⟩

lemma *abox-adia-sp-one-set*:

$|R]|S\rangle 1 = \{ (a,\{a\}) \mid a . \forall B . (a,B) \in R \longrightarrow (\forall b \in B . \exists D . (b,D) \in S) \}$
⟨*proof*⟩

lemma *abox-abox-set*:
$|R]|S]p = \{ (a,\{a\}) \mid a . \forall B . (a,B) \in R \longrightarrow (\forall C . (\exists b \in B . (b,C) \in S) \longrightarrow (\forall c \in C . (c,\{c\}) \in p)) \}$
⟨*proof*⟩

lemma *sp-abox-set*:
$|R * S]p = \{ (a,\{a\}) \mid a . \forall B . (a,B) \in R \longrightarrow (\forall C . (\exists f . (\forall b \in B . (b,f\ b) \in S) \wedge C = \bigcup \{ f\ b \mid b . b \in B \}) \longrightarrow (\forall c \in C . (c,\{c\}) \in p)) \}$
⟨*proof*⟩

lemma *abox-sp-1*:
assumes *test p*
shows $|R]|S\rangle 1 * |R * S]p \subseteq |R]|S]p$
⟨*proof*⟩

lemma *abox-sp-2*:
assumes *test p*
shows $|R]|S]p = |R\downarrow * S]p$
⟨*proof*⟩

lemma *abox-sp-3*:
assumes *test p*
shows $|R]|S]p \subseteq |R * S]p$
⟨*proof*⟩

lemma *abox-sp-4*:
assumes *test p*
shows $|R * S]p \subseteq |R]|S\rangle 1 \rightarrow |R]|S]p$
⟨*proof*⟩

lemma *abox-sp-5*:
assumes *test p*
shows $|R]|S\rangle 1 * |R * S]p = |R]|S\rangle 1 * |R]|S]p$
⟨*proof*⟩

lemma *abox-sp-6*:
assumes *test p*
shows $|R]|S\rangle 1 \rightarrow |R * S]p = |R]|S\rangle 1 \rightarrow |R]|S]p$
⟨*proof*⟩

lemma *abox-sp-7*:
assumes *test p*
and *total S*
shows $|R * S]p = |R]|S]p$
⟨*proof*⟩

lemma *adia-sp-associative*:
  assumes *test p*
  shows $|Q * (R * S)\rangle p = |(Q * R) * S\rangle p$
$\langle proof \rangle$

lemma *ebox-sp-associative*:
  assumes *test p*
  shows $|Q * (R * S)]]p = |(Q * R) * S]]p$
  $\langle proof \rangle$

lemma *edia-sp-associative*:
  assumes *test p*
  shows $|Q * (R * S)\rangle\rangle p = |(Q * R) * S\rangle\rangle p$
$\langle proof \rangle$

lemma *abox-sp-associative*:
  assumes *test p*
  shows $|Q * (R * S)]p = |(Q * R) * S]p$
  $\langle proof \rangle$

lemma *abox-oI*:
  assumes $X \neq \{\}$
  shows $|R]\bigcap X = (\bigcap p{\in}X . |R]p)$
  $\langle proof \rangle$

lemma *ebox-left-dist-oU*:
  assumes $X \neq \{\}$
  shows $|\bigcup X]]p = (\bigcap R{\in}X . |R]]p)$
  $\langle proof \rangle$

lemma *abox-left-dist-oU*:
  assumes $X \neq \{\}$
  shows $|\bigcup X]p = (\bigcap R{\in}X . |R]p)$
  $\langle proof \rangle$

lemma *adia-left-dist-oU*:
  $|\bigcup X\rangle p = (\bigcup R{\in}X . |R\rangle p)$
  $\langle proof \rangle$

lemma *edia-left-dist-oU*:
  $|\bigcup X\rangle\rangle p = (\bigcup R{\in}X . |R\rangle\rangle p)$
  $\langle proof \rangle$

8.6 Goldblatt's axioms with star

no-notation *rtrancl* ((-*) [*1000*] *999*)
notation *star* (-* [*1000*] *999*)

lemma *star-induct-1*:

assumes $1 \subseteq X$
and $R * X \subseteq X$
shows $R^* \subseteq X$
$\langle proof \rangle$

lemma *star-induct*:
assumes $S \subseteq 1 \cup 1_{\cup\cup}$
and $S \subseteq X$
and $R * X \subseteq X$
shows $R^* * S \subseteq X$
$\langle proof \rangle$

lemma *star-total*:
$total\ (R^*)$
$\langle proof \rangle$

lemma *star-down*:
$R^*\downarrow = (R\downarrow)^* \cup 1_{\cup\cup}$
$\langle proof \rangle$

lemma *ne-star-down*:
$ne\ (R^*\downarrow) = ne\ ((R\downarrow)^*)$
$\langle proof \rangle$

lemma *ne-down-star*:
$ne\ ((R\downarrow)^*) = (ne\ (R\downarrow))^*$
$\langle proof \rangle$

lemma *abox-star-unfold*:
$test\ p \Longrightarrow |R^*]p = p * |R]|R^*]p$
$\langle proof \rangle$

lemma *star-sp-test-commute*:
assumes $S \subseteq 1 \cup 1_{\cup\cup}$
and $Q * S \subseteq S * R$
shows $Q^* * S \subseteq S * R^*$
$\langle proof \rangle$

lemma *adia-star-induct*:
assumes $test\ p$
shows $|R\rangle p \subseteq p \longleftrightarrow |R^*\rangle p \subseteq p$
$\langle proof \rangle$

lemma *ebox-star-induct*:
assumes $test\ p$
shows $p \subseteq |R]]p \longleftrightarrow p \subseteq |R^*]]p$
$\langle proof \rangle$

lemma *abox-star-induct*:

assumes *test p*
shows $p \subseteq |R]p \longleftrightarrow p \subseteq |R^*]p$
⟨*proof*⟩

lemma *edia-star-induct*:
assumes *test p*
shows $|R\rangle\rangle p \subseteq p \longleftrightarrow |R^*\rangle\rangle p \subseteq p$
⟨*proof*⟩

lemma *abox-star-induct-1*:
assumes *test p*
and *test q*
and $q \subseteq p * |R]q$
shows $q \subseteq |R^*]p$
⟨*proof*⟩

lemma *adia-star-induct-1*:
assumes *test p*
and *test q*
and $p \cup |R\rangle q \subseteq q$
shows $|R^*\rangle p \subseteq q$
⟨*proof*⟩

lemma *abox-segerberg*:
assumes *test p*
shows $|R^*](p \to |R]p) \subseteq p \to |R^*]p$
⟨*proof*⟩

lemma *abox-segerberg-adia*:
assumes *test p*
shows $|R^*](\ |R\rangle p \to p\ ) \subseteq |R^*\rangle p \to p$
⟨*proof*⟩

lemma *s-p-id-sp*:
$(s\text{-}id \cup p\text{-}id) * R = R \cup p\text{-}id$
⟨*proof*⟩

8.7 Propositional Hoare logic

abbreviation *hoare* :: $('a,'a)\ mrel \Rightarrow ('a,'b)\ mrel \Rightarrow ('b,'b)\ mrel \Rightarrow bool$ (-⦃-⦄-[50,60,50] 95)
where $p⦃R⦄q \equiv p \subseteq |R]q$

abbreviation *if-then-else* :: $('a,'a)\ mrel \Rightarrow ('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel \Rightarrow ('a,'b)\ mrel$
where $if\text{-}then\text{-}else\ p\ R\ S \equiv p * R \cup \wr\ p * S$

abbreviation *while-do* :: $('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel$
where $while\text{-}do\ p\ R \equiv (p * R)^* * \wr\ p$

lemma *hoare-skip*:
  assumes *test p*
    shows $p\{\!|1|\!\}p$
  ⟨*proof*⟩

lemma *hoare-cons*:
  assumes *test s*
      and $r \subseteq p$
      and $q \subseteq s$
      and $p\{\!|R|\!\}q$
    shows $r\{\!|R|\!\}s$
  ⟨*proof*⟩

lemma *hoare-seq*:
  assumes *test q*
      and *test r*
      and $p\{\!|R|\!\}q$
      and $q\{\!|S|\!\}r$
    shows $p\{\!|R{*}S|\!\}r$
⟨*proof*⟩

lemma *hoare-if*:
  assumes *test p*
      and *test q*
      and *test r*
      and $(p{*}q)\{\!|R|\!\}r$
      and $((\wr\ p){*}q)\{\!|S|\!\}r$
  shows $q\{\!|\textit{if-then-else}\ p\ R\ S|\!\}r$
  ⟨*proof*⟩

lemma *hoare-while*:
  assumes *test p*
      and *test q*
      and $(p{*}q)\{\!|R|\!\}q$
    shows $q\{\!|\textit{while-do}\ p\ R|\!\}(q{*}(\wr\ p))$
⟨*proof*⟩

lemma *hoare-par*:
  assumes *test q*
      and $p\{\!|R|\!\}q$
      and $p\{\!|S|\!\}q$
    shows $p\{\!|R \cup\!\cup S|\!\}q$
⟨*proof*⟩

9 Counterexamples

locale *counterexamples*
begin

lemma *counter-01*:
$\neg$ ((U::($'a$,$'b$) *mrel*) $*$ $-$((U::($'b$,$'c$) *mrel*) $*$ (R::($'c$,$'d$) *mrel*)) $\subseteq$ $-$(U $*$ R))
$\langle proof \rangle$

abbreviation *a-1* $\equiv$ *finite-1*.a_1

lemma *counter-02*:
$\exists$ R::(*Enum.finite-1*,*Enum.finite-1*) *mrel* . $\exists$ p . $\neg$ (*test* p $\longrightarrow$ (R $\cap$ $-$(U $*$ p)) $*$ U = R $*$ $-$(p $*$ U))
$\langle proof \rangle$

lemma *counter-03*:
$\exists$ R::(*Enum.finite-1*,*Enum.finite-1*) *mrel* . $\exists$ p . $\neg$ (*test* p $\longrightarrow$ (R $\cap$ $-$(U $*$ p)) $*$ $1_{\cup\cup}$ = R $*$ ($-$(p $*$ U) $\cap$ $1_{\cup\cup}$))
$\langle proof \rangle$

abbreviation *b-1* $\equiv$ *finite-2*.a_1
abbreviation *b-2* $\equiv$ *finite-2*.a_2
abbreviation *b-1-0* $\equiv$ (*b-1*,{})
abbreviation *b-1-1* $\equiv$ (*b-1*,{*b-1*})
abbreviation *b-1-2* $\equiv$ (*b-1*,{*b-2*})
abbreviation *b-1-3* $\equiv$ (*b-1*,{*b-1*,*b-2*})
abbreviation *b-2-0* $\equiv$ (*b-2*,{})
abbreviation *b-2-1* $\equiv$ (*b-2*,{*b-1*})
abbreviation *b-2-2* $\equiv$ (*b-2*,{*b-2*})
abbreviation *b-2-3* $\equiv$ (*b-2*,{*b-1*,*b-2*})

lemma *counter-04*:
$\exists$ R::(*Enum.finite-2*,*Enum.finite-2*) *mrel* . $\exists$ p q . $\neg$ (*test* p $\longrightarrow$ *test* q $\longrightarrow$ $|R$ $*$ $p]q$ = $|R]|p]q$)
$\langle proof \rangle$

lemma *counter-05*:
$\neg$ ($\exists$ f . $\forall$ R p . *test* p $\longrightarrow$ $|R\rangle p$ = $|f$ $R]p$)
$\langle proof \rangle$

lemma *counter-06*:
$\neg$ ($\exists$ f . $\forall$ R p . *test* p $\longrightarrow$ $|R]]p$ = $|f$ $R]p$)
$\langle proof \rangle$

lemma *counter-07*:
$\neg$ ($\exists$ f . *mono* f $\wedge$ ($\forall$ R . *fus* R = *lfp* (λX . f R X)))
$\langle proof \rangle$

abbreviation *c-1* $\equiv$ *finite-3*.a_1
abbreviation *c-2* $\equiv$ *finite-3*.a_2
abbreviation *c-3* $\equiv$ *finite-3*.a_3

lemma *counter-08*:
$\neg\ (\sim(1::(Enum.finite\text{-}3,Enum.finite\text{-}3)\ mrel) * \sim 1 \in \{1,\ \sim 1\})$
⟨*proof*⟩

lemma *counter-09*:
$\neg\ (\sim(1::(Enum.finite\text{-}3,Enum.finite\text{-}3)\ mrel) \odot 1 \in \{1,\ \sim 1\})$
⟨*proof*⟩

lemma *ex-2-cases*:
$\exists\, b.\ b = b\text{-}1 \vee b = b\text{-}2$
⟨*proof*⟩

lemma *all-2-cases*:
$(\forall\, b.\ b = b\text{-}2 \wedge b = b\text{-}1) = False$
⟨*proof*⟩

lemma *impl-2-cases*:
$\bigcup\{\ X\ .\ \exists\, b.\ (b = b\text{-}1 \longrightarrow X = Y) \wedge (b = b\text{-}2 \longrightarrow X = Z)\} = Y \cup Z$
⟨*proof*⟩

lemma *ex-2-set-cases*:
$(\exists\, B::Enum.finite\text{-}2\ set\ .\ P\ B) \longleftrightarrow P\ \{\} \vee P\ \{b\text{-}1\} \vee P\ \{b\text{-}2\} \vee P\ \{b\text{-}1,b\text{-}2\}$
⟨*proof*⟩

abbreviation $B\text{-}0 \equiv \{\}::Enum.finite\text{-}2\ set$
abbreviation $B\text{-}1 \equiv \{b\text{-}1\}$
abbreviation $B\text{-}2 \equiv \{b\text{-}2\}$
abbreviation $B\text{-}3 \equiv \{b\text{-}1,b\text{-}2\}$
abbreviation $mkf\ x\ y \equiv \lambda z\ .\ if\ z = b\text{-}1\ then\ x\ else\ y$

lemma *mkf*:
$f = mkf\ (f\ b\text{-}1)\ (f\ b\text{-}2)$
⟨*proof*⟩

lemma *mkf2*:
$f\ b\text{-}1 = X \wedge f\ b\text{-}2 = Y \Longrightarrow f = mkf\ X\ Y$
⟨*proof*⟩

lemma *ex-2-mrel-cases*:
$(\exists\, f::Enum.finite\text{-}2 \Rightarrow Enum.finite\text{-}2\ set\ .\ P\ f) \longleftrightarrow$
$P\ (mkf\ B\text{-}0\ B\text{-}0) \vee P\ (mkf\ B\text{-}0\ B\text{-}1) \vee P\ (mkf\ B\text{-}0\ B\text{-}2) \vee P\ (mkf\ B\text{-}0\ B\text{-}3) \vee$
$P\ (mkf\ B\text{-}1\ B\text{-}0) \vee P\ (mkf\ B\text{-}1\ B\text{-}1) \vee P\ (mkf\ B\text{-}1\ B\text{-}2) \vee P\ (mkf\ B\text{-}1\ B\text{-}3) \vee$
$P\ (mkf\ B\text{-}2\ B\text{-}0) \vee P\ (mkf\ B\text{-}2\ B\text{-}1) \vee P\ (mkf\ B\text{-}2\ B\text{-}2) \vee P\ (mkf\ B\text{-}2\ B\text{-}3) \vee$
$P\ (mkf\ B\text{-}3\ B\text{-}0) \vee P\ (mkf\ B\text{-}3\ B\text{-}1) \vee P\ (mkf\ B\text{-}3\ B\text{-}2) \vee P\ (mkf\ B\text{-}3\ B\text{-}3)$
⟨*proof*⟩

lemma *counter-10*:
$\exists\, R::(Enum.finite\text{-}2,Enum.finite\text{-}2)\ mrel\ .\ \neg\ (U::(Enum.finite\text{-}2,Enum.finite\text{-}2)\ mrel) * (U * R) \subseteq U * R$

$\langle proof \rangle$

lemma *counter-11*:
$\exists (R{::}(Enum.finite\text{-}2,Enum.finite\text{-}2)\ mrel)\ (s{::}(Enum.finite\text{-}2,Enum.finite\text{-}2)\ mrel)\ (t{::}(Enum.finite\text{-}2,Enum.finite\text{-}2)\ mrel)\ .\ \neg\ (inner\text{-}univalent\ s \wedge inner\text{-}univalent\ t \longrightarrow R * (s * t) = (R * s) * t)$
$\langle proof \rangle$

lemma *counter-12*:
$\neg(\exists S\ .\ 1_{\cup\cup} \odot S = 1_{\cup\cup})$
$\langle proof \rangle$

lemma *counter-13*:
$\neg(\exists S\ .\ \forall R\ .\ R \odot S = R)$
$\langle proof \rangle$
end

end

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