

Inner Structure, Determinism and Modal Algebra of Multirelations

Walter Guttmann and Georg Struth

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Abstract

Binary multirelations form a model of alternating nondeterminism useful for analysing games, interactions of computing systems with their environments or abstract interpretations of probabilistic programs. We investigate this alternating structure in a relational language based on power allegories extended with specific operations on multirelations. We develop algebras of modal operators over multirelations, related to concurrent dynamic logics, in this language.

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The theories formally verify results in [3, 1, 2]. See these papers for further details and related work.

The basic algebra of homogeneous binary multirelations is formalised in [4]. The present theories consider heterogeneous binary multirelations, which may have different source and target sets. While homogeneous multirelations arise as a special case where source and target sets coincide, we do not attempt to generalise the algebras of [4] to the heterogeneous case but study new concepts instead. Thus the present theories and [4] are complementary. A unification of the two approaches based on category theory is possible future work.

Algebraic structures for multirelations with Parikh composition are formalised in [5].

1 Properties of Binary Relations

theory *Relational-Properties*

imports *Main*

begin

no-notation *set-difference* (**infixl** \setminus 65)

This is a general-purpose theory for enrichments of Rel, which is still quite basic, but helpful for developing properties of multirelations.

notation *relcomp* (**infixl** $\langle ; \rangle$ 75)

notation *converse* ($\langle \smile \rangle$ [1000] 999)

type-synonym ($\langle 'a, 'b \rangle$ *rel* = ($\langle 'a \times 'b \rangle$ *set*)

lemma *modular-law*: $R ; S \cap T \subseteq (R \cap T ; S \smile) ; S$

by *blast*

lemma *compl-conv*: $\neg(R \smile) = (\neg R) \smile$

by *fastforce*

definition *top* :: ($\langle 'a, 'b \rangle$ *rel*) **where**

top = $\{(a, b) \mid a \text{ b. True}\}$

abbreviation *neg* $R \equiv Id \cap \neg R$

1.1 Univalence, totality, determinism, and related properties

definition *univalent* :: ($\langle 'a, 'b \rangle$ *rel*) \Rightarrow *bool* **where**

univalent $R = (R^\smile ; R \subseteq Id)$

definition *total* :: ('a,'b) rel \Rightarrow bool **where**
total $R = (Id \subseteq R ; R^\smile)$

definition *injective* :: ('a,'b) rel \Rightarrow bool **where**
injective $R = (R ; R^\smile \subseteq Id)$

definition *surjective* :: ('a,'b) rel \Rightarrow bool **where**
surjective $R = (Id \subseteq R^\smile ; R)$

definition *deterministic* :: ('a,'b) rel \Rightarrow bool **where**
deterministic $R = (\text{univalent } R \wedge \text{total } R)$

definition *bijective* :: ('a,'b) rel \Rightarrow bool **where**
bijective $R = (\text{injective } R \wedge \text{surjective } R)$

lemma *univalent-set*: *univalent* $R = (\forall a b c. (a,b) \in R \wedge (a,c) \in R \longrightarrow b = c)$
unfolding *univalent-def converse-unfold Id-def* **by** *force*

[Univalent relations feature as single-valued relations in Main.](#)

lemma *univ-single-valued*: *univalent* $R = \text{single-valued } R$
unfolding *univalent-set single-valued-def* **by** *auto*

lemma *total-set*: *total* $R = (\forall a. \exists b. (a,b) \in R)$
unfolding *total-def converse-unfold Id-def* **by** *force*

lemma *total-var*: *total* $R = (R ; \text{top} = \text{top})$
unfolding *total-set top-def* **by** *force*

lemma *deterministic-set*: *deterministic* $R = (\forall a. \exists ! B. (a,B) \in R)$
unfolding *deterministic-def univalent-set total-set* **by** *force*

lemma *deterministic-var1*: *deterministic* $R = (R ; -Id = -R)$
unfolding *deterministic-set Id-def* **by** *force*

lemma *deterministic-var2*: *deterministic* $R = (\forall S. R ; -S = -(R ; S))$
unfolding *deterministic-set* **by** (*standard, force, blast*)

lemma *inj-univ*: *injective* $R = \text{univalent } (R^\smile)$
by (*simp add: injective-def univalent-def*)

lemma *injective-set*: *injective* $S = (\forall a b c. (a,c) \in S \wedge (b,c) \in S \longrightarrow a = b)$
by (*meson converseD converseI inj-univ univalent-set*)

lemma *surj-tot*: *surjective* $R = \text{total } (R^\smile)$
by (*simp add: surjective-def total-def*)

lemma *surjective-set*: *surjective* $S = (\forall b. \exists a. (a,b) \in S)$

by (*simp add: surj-tot total-set*)

lemma *surj-var*: *surjective* $R = (R^\sim ; \text{top} = \text{top})$
by (*simp add: surj-tot total-var*)

lemma *bij-det*: *bijection* $R = \text{deterministic} (R^\sim)$
by (*simp add: bijective-def deterministic-def inj-univ surj-tot*)

lemma *univ-relcomp*: *univalent* $R \implies \text{univalent } S \implies \text{univalent} (R ; S)$
by (*simp add: single-valued-relcomp univ-single-valued*)

lemma *tot-relcomp*: *total* $R \implies \text{total } S \implies \text{total} (R ; S)$
by (*meson relcomp.simps total-set*)

lemma *det-relcomp*: *deterministic* $R \implies \text{deterministic } S \implies \text{deterministic} (R ; S)$
by (*simp add: deterministic-def tot-relcomp univ-relcomp*)

lemma *inj-relcomp*: *injective* $R \implies \text{injective } S \implies \text{injective} (R ; S)$
by (*simp add: converse-relcomp inj-univ univ-relcomp*)

lemma *surj-relcomp*: *surjective* $R \implies \text{surjective } S \implies \text{surjective} (R ; S)$
by (*simp add: converse-relcomp surj-tot tot-relcomp*)

lemma *bij-relcomp*: *bijection* $R \implies \text{bijection } S \implies \text{bijection} (R ; S)$
by (*simp add: bijective-def inj-relcomp surj-relcomp*)

lemma *det-Id*: *deterministic* Id
by (*simp add: deterministic-var1*)

lemma *bij-Id*: *bijection* Id
by (*simp add: bij-det det-Id*)

lemma *tot-top*: *total* top
by (*simp add: top-def total-set*)

lemma *tot-surj*: *surjective* top
by (*simp add: surjective-set top-def*)

lemma *det-meet-distl*: *univalent* $R \implies R ; (S \cap T) = R ; S \cap R ; T$
unfolding *univalent-set relcomp-def relcompp-apply* **by** *force*

lemma *inj-meet-distr*: *injective* $T \implies (R \cap S) ; T = R ; T \cap S ; T$
unfolding *injective-def converse-def Id-def relcomp.simps* **by** *force*

lemma *univ-modular*: *univalent* $S \implies R ; S \cap T = (R \cap T ; S^\sim) ; S$
unfolding *univalent-set converse-unfold relcomp.simps* **by** *force*

1.2 Inverse image and the diagonal and graph functors

definition $Invim :: ('a, 'b) \text{ rel} \Rightarrow 'b \text{ set} \Rightarrow 'a \text{ set}$ **where**

$$Invim R = Image (R^\smile)$$

definition $Delta :: 'a \text{ set} \Rightarrow ('a, 'a) \text{ rel}$ ($\langle \Delta \rangle$) **where**

$$\Delta P = \{(p, p) \mid p. p \in P\}$$

definition $Grph :: ('a \Rightarrow 'b) \Rightarrow ('a, 'b) \text{ rel}$ **where**

$$Grph f = \{(x, y). y = f x\}$$

lemma $Image\text{-}Grph$ [*simp*]: $Image \circ Grph = image$

unfolding $Image\text{-}def$ $Grph\text{-}def$ $image\text{-}def$ **by** *auto*

1.3 Relational domain, codomain and modalities

Domain and codomain (range) maps have been defined in Main, but they return sets instead of relations.

definition $dom :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$ **where**

$$dom R = Id \cap R ; R^\smile$$

definition $cod :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$ **where**

$$cod R = dom (R^\smile)$$

definition $rel\text{-}fdia :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$ ($\langle (|-) \rangle$) [61,81] 82)

where

$$|R\rangle Q = dom (R ; dom Q)$$

definition $rel\text{-}bdia :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$ ($\langle \langle (|-) \rangle \rangle$) [61,81] 82)

where

$$rel\text{-}bdia R = rel\text{-}fdia (R^\smile)$$

definition $rel\text{-}fbox :: ('a, 'b) \text{ rel} \Rightarrow ('b, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel}$ ($\langle (|-) \rangle$) [61,81] 82)

where

$$|R\rangle Q = neg (dom (R ; neg (dom Q)))$$

definition $rel\text{-}bbox :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'a) \text{ rel} \Rightarrow ('b, 'b) \text{ rel}$ ($\langle \langle (|-) \rangle \rangle$) [61,81] 82)

where

$$rel\text{-}bbox R = rel\text{-}fbox (R^\smile)$$

lemma $rel\text{-}bdia\text{-}def\text{-}var$: $rel\text{-}bdia = rel\text{-}fdia \circ converse$

unfolding $rel\text{-}fdia\text{-}def$ $fun\text{-}eq\text{-}iff$ $comp\text{-}def$ $rel\text{-}bdia\text{-}def$ **by** *simp*

lemma $dom\text{-}set$: $dom R = \{(a, a) \mid a. \exists b. (a, b) \in R\}$

unfolding $dom\text{-}def$ $Id\text{-}def$ $converse\text{-}unfold$ $relcomp\text{-}unfold$ **by** *force*

lemma $dom\text{-}Domain$: $dom = \Delta \circ Domain$

unfolding $fun\text{-}eq\text{-}iff$ $dom\text{-}set$ $Delta\text{-}def$ $Domain\text{-}def$ **by** *clarsimp blast*

lemma *cod-set*: $\text{cod } R = \{(b,b) \mid b. \exists a. (a,b) \in R\}$
by (*smt* (*verit*) *Collect-cong cod-def converseD converseI dom-set*)

lemma *cod-Range*: $\text{cod} = \Delta \circ \text{Range}$
unfolding *fun-eq-iff cod-set Delta-def Range-def* **by** *clarsimp blast*

lemma *rel-fdia-set*: $|R\rangle Q = \{(a,a) \mid a. \exists b. (a,b) \in R \wedge (b,b) \in \text{dom } Q\}$
unfolding *rel-fdia-def dom-set relcomp-unfold* **by** *force*

lemma *rel-bdia-set*: $\langle R| P = \{(b,b) \mid b. \exists a. (a,b) \in R \wedge (a,a) \in \text{dom } P\}$
by (*smt* (*verit*, *best*) *Collect-cong converseD converseI rel-bdia-def rel-fdia-set*)

lemma *rel-fbox-set*: $|R] Q = \{(a,a) \mid a. \forall b. (a,b) \in R \longrightarrow (b,b) \in \text{dom } Q\}$
unfolding *rel-fbox-def dom-set relcomp-unfold* **by** *force*

lemma *rel-bbox-set*: $[R| P = \{(b,b) \mid b. \forall a. (a,b) \in R \longrightarrow (a,a) \in \text{dom } P\}$
by (*smt* (*verit*) *Collect-cong converseD converseI rel-bbox-def rel-fbox-set*)

lemma *dom-alt-def*: $\text{dom } R = \text{Id} \cap R ; \text{top}$
unfolding *dom-set top-def Id-def* **by** *force*

lemma *dom-gla*: $(\text{dom } R \subseteq \text{Id} \cap S) = (R \subseteq (\text{Id} \cap S) ; R)$
unfolding *dom-set Id-def relcomp-unfold* **by** *blast*

lemma *dom-gla-top*: $(\text{dom } R \subseteq \text{Id} \cap S) = (R \subseteq (\text{Id} \cap S) ; \text{top})$
unfolding *dom-set Id-def top-def relcomp-unfold* **by** *blast*

lemma *dom-subid*: $(\text{dom } R = R) = (R = \text{Id} \cap R)$
by (*metis* (*no-types*, *lifting*) *Id-O-R R-O-Id dom-alt-def dom-gla-top equalityD1 inf.absorb-iff2 inf.commute inf.idem le-inf-iff relcomp-mono*)

lemma *dom-cod*: $(\text{dom } R = R) = (\text{cod } R = R)$
by (*metis* *dom-def cod-def converse-Id converse-Int converse-converse converse-relcomp*)

lemma *dom-top*: $R ; \text{top} = \text{dom } R ; \text{top}$
unfolding *dom-set top-def* **by** *blast*

lemma *top-dom*: $\text{dom } R = \text{dom } (R ; \text{top})$
unfolding *dom-def top-def* **by** *blast*

lemma *cod-top*: $\text{cod } R = \text{Id} \cap \text{top} ; R$
by (*metis* *dom-def cod-def converse-Id converse-Int converse-converse converse-relcomp dom-alt-def surj-var tot-surj*)

lemma *dom-conv* [*simp*]: $(\text{dom } R)^\smile = \text{dom } R$
by (*simp* *add: dom-def converse-Int converse-relcomp*)

lemma *total-dom*: $\text{total } R = (\text{dom } R = \text{Id})$

by (*metis dom-def inf.orderE inf-le2 total-def*)

lemma *surj-cod*: *surjective* $R = (\text{cod } R = \text{Id})$
 by (*simp add: cod-def surj-tot total-dom*)

lemma *fdia-demod*: $(\langle R \rangle P \subseteq \text{dom } Q) = (R ; \text{dom } P \subseteq \text{dom } Q ; R)$
unfolding *rel-fdia-set dom-set relcomp-unfold* **by** *force*

lemma *bbox-demod*: $(\text{dom } P \subseteq [R] Q) = (R ; \text{dom } P \subseteq \text{dom } Q ; R)$
unfolding *rel-bbox-set dom-def* **by** *force*

lemma *bdia-demod*: $(\langle R \rangle P \subseteq \text{dom } Q) = (\text{dom } P ; R \subseteq R ; \text{dom } Q)$
proof –

have $(\langle R \rangle P \subseteq \text{dom } Q) = (\langle R^\sim \rangle P \subseteq \text{dom } Q)$

by (*simp add: rel-bdia-def*)

also have $\dots = ((R^\sim) ; \text{dom } P \subseteq \text{dom } Q ; (R^\sim))$

by (*simp add: fdia-demod*)

also have $\dots = (((\text{dom } P ; R)^\sim) \subseteq ((R ; \text{dom } Q)^\sim))$

by (*simp add: converse-relcomp*)

also have $\dots = (\text{dom } P ; R \subseteq R ; \text{dom } Q)$

by *simp*

finally show *?thesis*.

qed

lemma *fbox-demod*: $(\text{dom } P \subseteq [R] Q) = (\text{dom } P ; R \subseteq R ; \text{dom } Q)$
unfolding *rel-fbox-set dom-set relcomp-unfold* **by** *force*

lemma *fdia-demod-top*: $(\langle R \rangle P \subseteq \text{dom } Q) = (R ; \text{dom } P ; \text{top} \subseteq \text{dom } Q ; \text{top})$
 by (*metis dom-def dom-gla-top rel-fdia-def top-dom*)

lemma *bbox-demod-top*: $(\text{dom } P \subseteq [R] Q) = (R ; \text{dom } P ; \text{top} \subseteq \text{dom } Q ; \text{top})$
unfolding *rel-bbox-def rel-fbox-def dom-def top-def* **by** *force*

lemma *fdia-bbox-galois*: $(\langle R \rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq [R] Q)$
 by (*meson bbox-demod-top fdia-demod-top*)

lemma *bdia-fbox-galois*: $(\langle R \rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq [R] Q)$
 by (*simp add: fdia-bbox-galois rel-bbox-def rel-bdia-def*)

lemma *fdia-bdia-conjugation*: $(\langle R \rangle P \subseteq \text{neg } (\text{dom } Q)) = (\langle R \rangle Q \subseteq \text{neg } (\text{dom } P))$
unfolding *rel-fdia-set rel-bdia-set dom-set* **by** *force*

lemma *bfox-bbox-conjugation*: $(\text{neg } (\text{dom } Q) \subseteq [R] P) = (\text{neg } (\text{dom } P) \subseteq [R] Q)$
unfolding *rel-fbox-set rel-bbox-set dom-set* **by** *clarsimp blast*

1.4 Residuation

definition *lres* :: $(a, c) \text{ rel} \Rightarrow (b, c) \text{ rel} \Rightarrow (a, b) \text{ rel}$ (**infixl** $\langle // \rangle$ 75)
 where $R // S = \{(a, b). \forall c. (b, c) \in S \longrightarrow (a, c) \in R\}$

definition $rres :: ('c, 'a) rel \Rightarrow ('c, 'b) rel \Rightarrow ('a, 'b) rel$ (**infixl** $\langle \backslash \rangle$ 75)
where $R \setminus S = \{(b, a). \forall c. (c, b) \in R \longrightarrow (c, a) \in S\}$

lemma $rres\text{-}lres\text{-}conv$: $R \setminus S = (S^\smile \parallel R^\smile)^\smile$
unfolding $rres\text{-}def$ $lres\text{-}def$ **by** $clarsimp$ $fastforce$

lemma $lres\text{-}galois$: $(R ; S \subseteq T) = (R \subseteq T \parallel S)$
unfolding $lres\text{-}def$ **by** $blast$

lemma $rres\text{-}galois$: $(R ; S \subseteq T) = (S \subseteq R \setminus T)$
by ($metis$ $converse\text{-}converse$ $converse\text{-}mono$ $converse\text{-}relcomp$ $lres\text{-}galois$ $rres\text{-}lres\text{-}conv$)

lemma $lres\text{-}compl$: $R \parallel S = -(-R ; S^\smile)$
unfolding $lres\text{-}def$ $converse\text{-}unfold$ **by** $force$

lemma $rres\text{-}compl$: $R \setminus S = -(R^\smile ; -S)$
unfolding $rres\text{-}def$ $converse\text{-}unfold$ **by** $force$

lemma $lres\text{-}simp$ [$simp$]: $(R \parallel R) ; R = R$
by ($metis$ $Id\text{-}O\text{-}R$ $lres\text{-}galois$ $relcomp\text{-}mono$ $subsetI$ $subsetI$ $subset\text{-}antisym$)

lemma $rres\text{-}simp$ [$simp$]: $R ; (R \setminus R) = R$
by ($metis$ $converse\text{-}converse$ $converse\text{-}relcomp$ $lres\text{-}simp$ $rres\text{-}lres\text{-}conv$)

lemma $lres\text{-}curry$: $R \parallel (T ; S) = (R \parallel S) \parallel T$
by ($metis$ ($no\text{-}types$, $opaque\text{-}lifting$) $O\text{-}assoc$ $dual\text{-}order.refl$ $lres\text{-}galois$ $subset\text{-}antisym$)

lemma $rres\text{-}curry$: $(R ; S) \setminus T = S \setminus (R \setminus T)$
by ($simp$ add : $converse\text{-}relcomp$ $lres\text{-}curry$ $rres\text{-}lres\text{-}conv$)

lemma $lres\text{-}Id$: $Id \subseteq R \parallel R$
unfolding $lres\text{-}def$ $Id\text{-}def$ **by** $force$

lemma $det\text{-}lres$: $deterministic R \Longrightarrow (R ; S) \parallel S = R ; (S \parallel S)$
by ($metis$ ($no\text{-}types$, $lifting$) $O\text{-}assoc$ $deterministic\text{-}var2$ $lres\text{-}compl$)

lemma $det\text{-}rres$: $deterministic (R^\smile) \Longrightarrow S \setminus (S ; R) = (S \setminus S) ; R$
by ($simp$ add : $converse\text{-}relcomp$ $det\text{-}lres$ $rres\text{-}lres\text{-}conv$)

lemma $rres\text{-}bij$: $bijjective S \Longrightarrow (R \setminus T) ; S = R \setminus (T ; S)$
unfolding $bijjective\text{-}def$ $injective\text{-}set$ $surjective\text{-}set$ $relcomp\text{-}unfold$ $cod\text{-}def$ $Id\text{-}def$ $rres\text{-}def$ **by** $clarsimp$ $blast$

lemma $lres\text{-}bij$: $bijjective S \Longrightarrow (R \parallel T^\smile) ; S = R \parallel (T ; S)^\smile$
unfolding $bijjective\text{-}def$ $injective\text{-}set$ $surjective\text{-}set$ $relcomp\text{-}unfold$ $cod\text{-}def$ $Id\text{-}def$ $lres\text{-}def$ $converse\text{-}unfold$ **by** $blast$

lemma *dom-rres-top*: $(\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top})) = (\text{dom } P ; \text{top} \subseteq R \setminus (\text{dom } Q ; \text{top}))$

unfolding *dom-def top-def rres-def relcomp-unfold Id-def converse-unfold* **by** *clarsimp blast*

lemma *dom-rres-top-var*: $(\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top})) = (P ; \text{top} \subseteq R \setminus (Q ; \text{top}))$

by *(metis dom-rres-top dom-top)*

lemma *fdia-rres-top*: $(|R\rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq R \setminus (\text{dom } Q ; \text{top}))$

by *(metis dom-alt-def dom-gla-top rel-fdia-def rres-galois)*

lemma *fdia-rres-top-var*: $(|R\rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq R \setminus (Q ; \text{top}))$

by *(metis dom-top fdia-rres-top)*

lemma *dom-galois-var2*: $(|R\rangle (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q) ; \text{top}))$

by *(metis dom-subid fdia-rres-top-var inf-sup-aci(4))*

lemma *rres-top*: $R \setminus (\text{dom } Q ; \text{top}) ; \text{top} = R \setminus (\text{dom } Q ; \text{top})$

unfolding *rres-def top-def dom-def relcomp-unfold* **by** *clarsimp*

lemma *ddd-var*: $(|R\rangle P \subseteq \text{dom } Q) = (\text{dom } P \subseteq \text{dom } ((R \setminus (\text{dom } Q ; \text{top})) ; \text{top}))$

unfolding *rel-fdia-def dom-def rres-def top-def relcomp-unfold Id-def* **by** *force*

lemma *wlp-prop*: $\text{dom } ((R \setminus (\text{dom } Q ; \text{top})) ; \text{top}) = \text{neg } (\text{cod } (\text{neg } (\text{dom } Q) ; R))$

unfolding *rres-def Id-def cod-def dom-def top-def relcomp-unfold* **by** *fastforce*

lemma *wlp-prop-var*: $\text{dom } ((R \setminus (\text{dom } Q ; \text{top}))) = \text{neg } (\text{cod } ((\text{neg } (\text{dom } Q)) ; R))$

by *(metis rres-top wlp-prop)*

lemma *dom-demod*: $(|R\rangle (Id \cap P) \subseteq Id \cap Q) = (R ; (Id \cap P) \subseteq (Id \cap Q) ; R)$

proof

assume $|R\rangle (Id \cap P) \subseteq Id \cap Q$

hence $R ; (Id \cap P) \subseteq (Id \cap Q) ; R ; (Id \cap P)$

by *(metis O-assoc dom-gla dom-subid inf.absorb-iff2 inf-le1 rel-fdia-def)*

thus $R ; (Id \cap P) \subseteq (Id \cap Q) ; R$

by *auto*

next

assume $R ; (Id \cap P) \subseteq (Id \cap Q) ; R$

hence $|R\rangle (Id \cap P) \subseteq \text{dom } ((Id \cap Q) ; R)$

by *(metis (no-types, lifting) dom-subid dom-top fdia-demod-top inf.absorb-iff2 inf-le1 relcomp-distrib2 sup.order-iff)*

hence $|R\rangle (Id \cap P) \subseteq (Id \cap Q) \cap \text{dom } R$

unfolding *dom-def Id-def* **by** *blast*

thus $|R\rangle (Id \cap P) \subseteq Id \cap Q$

by *blast*

qed

lemma *fdia-bbox-galois-var*: $(|R| (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq Id \cap - \text{cod} ((Id \cap - Q); R))$
unfolding *rel-fdia-def dom-def cod-def Id-def* **by** *blast*

lemma *dom-demod-var2*: $(|R| (Id \cap P) \subseteq Id \cap Q) = (Id \cap P \subseteq R \setminus ((Id \cap Q); R))$
by (*meson dom-demod rres-galois*)

1.5 Symmetric quotient

definition *syq* :: $('c, 'a) \text{ rel} \Rightarrow ('c, 'b) \text{ rel} \Rightarrow ('a, 'b) \text{ rel}$ (**infixl** $\dot{\div}$ 75)
where $R \dot{\div} S = (R \setminus S) \cap (R^\smile // S^\smile)$

lemma *syq-set*: $R \dot{\div} S = \{(a, b). \forall c. (c, a) \in R \iff (c, b) \in S\}$
unfolding *syq-def relcomp-unfold lres-def rres-def* **by** *force*

lemma *converse-syq* [*simp*]: $(R \dot{\div} S)^\smile = S \dot{\div} R$
unfolding *syq-def converse-def rres-def lres-def* **by** *blast*

lemma *syq-compl*: $R \dot{\div} S = - (R^\smile ; -S) \cap - (- (R^\smile) ; S)$
by (*simp add: lres-compl rres-compl syq-def*)

lemma *syq-compl2* [*simp*]: $-R \dot{\div} -S = R \dot{\div} S$
unfolding *syq-compl* **by** *blast*

lemma *syq-expand1*: $R ; (R \dot{\div} S) = S \cap (top ; (R \dot{\div} S))$
unfolding *syq-set top-def relcomp-unfold* **by** *force*

lemma *syq-expand2*: $(R \dot{\div} S) ; S^\smile = R^\smile \cap ((R \dot{\div} S) ; top)$
unfolding *syq-set top-def relcomp-unfold* **by** *force*

lemma *syq-comp1*: $(R \dot{\div} S) ; (S \dot{\div} T) = (R \dot{\div} T) \cap (top ; (S \dot{\div} T))$
unfolding *syq-set top-def relcomp-unfold* **by** *fastforce*

lemma *syq-comp2*: $(R \dot{\div} S) ; (S \dot{\div} T) = (R \dot{\div} T) \cap ((R \dot{\div} S) ; top)$
unfolding *syq-set top-def relcomp-unfold* **by** *fastforce*

lemma *syq-bij*: *bijective* $T \implies (R \dot{\div} S) ; T = R \dot{\div} (S ; T)$
by (*simp add: bijective-def inj-meet-distr lres-bij rres-bij syq-def*)

end

2 Properties of Power Allegories

theory *Power-Allegories-Properties*

imports *Relational-Properties*

begin

2.1 Power transpose, epsilon, epsilonoff

definition $\Lambda R :: ('a, 'b) \text{ rel} \Rightarrow ('a, 'b \text{ set}) \text{ rel} (\langle \Lambda \rangle)$ **where**
 $\Lambda R = \{(a, B) \mid a \in B. B = \{b. (a, b) \in R\}\}$

definition $\epsilon :: ('a, 'a \text{ set}) \text{ rel}$ **where**
 $\epsilon = \{(a, A). a \in A\}$

definition $\epsilon\text{off} = \{(A, a). a \in A\}$

definition $\alpha :: ('a, 'b \text{ set}) \text{ rel} \Rightarrow ('a, 'b) \text{ rel} (\langle \alpha \rangle)$ **where**
 $\alpha R = R ; \epsilon\text{off}$

[alpha](#) can be seen as a relational approximation of a multirelation. The next lemma provides a relational definition of Λ .

lemma $\Lambda R = R^\smile \div \epsilon$
unfolding Λ -def syq-set ϵ -def **by** *blast*

lemma $\epsilon\text{off} = \epsilon^\smile$
unfolding ϵoff -def ϵ -def *converse-unfold* **by** *simp*

lemma $\alpha R = \{(a, b) \mid a \in b. b \in \bigcup \{B. (a, B) \in R\}\}$
unfolding α -def ϵoff -def **by** *force*

lemma $\alpha (R ; S) = R ; \alpha S$
by (*simp add: O-assoc alpha-def*)

lemma $f = \Lambda R \Longrightarrow R = \alpha f$
unfolding Λ -def α -set **by** *simp*

lemma $f = \Lambda R \Longrightarrow R = \alpha f \Longrightarrow f = \Lambda R$
unfolding Λ -def α -set *deterministic-set*
apply *safe*
apply *force*
by (*clarsimp, smt (verit, best) mem-Collect-eq set-eq-iff*)

lemma $\Lambda R = \alpha f \Longrightarrow f = \Lambda R$
assumes *deterministic f*
shows $(R = \alpha f) = (f = \Lambda R)$
by (*meson Lambda-epsilonoff-up1 Lambda-epsilonoff-up2 assms*)

lemma $\Lambda R = \alpha f \Longrightarrow f = \Lambda R$
unfolding Λ -def *deterministic-set* **by** *simp*

lemma $\Lambda (\alpha f) = f$
using Λ -def ϵoff -up2 **by** *blast*

lemma *alpha-Lambda-canc* [*simp*]: $\alpha (\Lambda R) = R$
using *Lambda-epsiloff-up1* **by** *blast*

lemma *alpha-cancel*:
assumes *deterministic f*
and *deterministic g*
shows $\alpha f = \alpha g \implies f = g$
by (*metis Lambda-epsiloff-up2 assms*)

lemma *Lambda-fusion*:
assumes *deterministic f*
shows $\Lambda (f ; R) = f ; \Lambda R$
proof –
have *h: deterministic (f ; ΛR)*
by (*simp add: assms det-lambda det-relcomp*)
have $f ; R = \alpha (f ; \Lambda R)$
by *simp*
also have $\dots = f ; \alpha (\Lambda R)$
by *simp*
thus *?thesis*
by (*simp add: alpha-cancel det-lambda h*)
qed

lemma *Lambda-fusion-var*: $\Lambda (\Lambda R ; S) = \Lambda R ; \Lambda S$
by (*simp add: Lambda-fusion det-lambda*)

lemma *Lambda-epsiloff* [*simp*]: $\Lambda \text{epsiloff} = \text{Id}$
proof –
have $\Lambda \text{epsiloff} = \Lambda (\text{Id} ; \text{epsiloff})$
by *simp*
also have $\dots = \text{Id}$
by (*metis Lambda-epsiloff-up alpha-def det-Id*)
finally show *?thesis*.
qed

lemma *alpha-epsiloff* [*simp*]: $\alpha \text{Id} = \text{epsiloff}$
by (*simp add: alpha-def*)

lemma *alpha-Sup-pres*: $\alpha (\bigcup \mathcal{R}) = (\bigcup R \in \mathcal{R}. \alpha R)$
unfolding *alpha-def* **by** *force*

lemma *alpha-ord-pres*: $R \subseteq S \implies \alpha R \subseteq \alpha S$
unfolding *alpha-def* **by** *force*

lemma *alpha-inf-pres*: $\alpha \{(a,A). \exists B C. A = B \cap C \wedge (a,B) \in R \wedge (a,C) \in S\}$
 $= \alpha R \cap \alpha S$
unfolding *alpha-set* **by** *blast*

2.2 Relational image functor

definition $pow :: ('a, 'b) rel \Rightarrow ('a\ set, 'b\ set) rel (\langle \mathcal{P} \rangle)$ **where**
 $\mathcal{P}\ R = \Lambda\ (epsilon\ off ; R)$

lemma $pow\ set: \mathcal{P}\ R = \{(A,B). B = Image\ R\ A\}$
unfolding $pow\ def\ epsilon\ off\ def\ Lambda\ def\ relcomp\ def\ Image\ def$ **by** $force$

lemma $pow\ set\ var: \mathcal{P}\ R = \{(A,B). B = \{b. \exists a \in A. (a,b) \in R\}\}$
unfolding $pow\ set\ Image\ def$ **by** $simp$

lemma $pow\ converse\ set: \mathcal{P}\ (R^\sim) = \{(Q,P). P = \{a. \exists b. (a,b) \in R \wedge b \in Q\}\}$
unfolding $pow\ set\ Image\ def$ **by** $force$

lemma $det\ pow: deterministic\ (\mathcal{P}\ R)$
unfolding $pow\ set\ deterministic\ set\ Image\ def$ **by** $simp$

lemma $Lambda\ pow: \Lambda\ (R ; S) = \Lambda\ R ; \mathcal{P}\ S$

proof –

have $\Lambda\ R ; \mathcal{P}\ S = \Lambda\ R ; \Lambda\ (epsilon\ off ; S)$
by $(simp\ add: pow\ def)$
also have $\dots = \Lambda\ (\Lambda\ R ; epsilon\ off ; S)$
by $(simp\ add: Lambda\ fusion\ var\ O\ assoc)$
also have $\dots = \Lambda\ (R ; S)$
by $(metis\ alpha\ Lambda\ canc\ alpha\ def)$
finally show $?thesis..$

qed

lemma $pow\ func1 [simp]: \mathcal{P}\ Id = Id$
by $(simp\ add: pow\ def)$

lemma $pow\ func2: \mathcal{P}\ (R ; S) = \mathcal{P}\ R ; \mathcal{P}\ S$
by $(metis\ Lambda\ pow\ pow\ def\ O\ assoc)$

lemma $Grph\ Image [simp]: Grph \circ Image = \mathcal{P}$
apply $(simp\ add: fun\ eq\ iff)$
unfolding $pow\ def\ Grph\ def\ Image\ def\ Lambda\ def\ epsilon\ off\ def$ **by** $blast$

lemma $lambda\ alpha\ idem [simp]: \Lambda\ (\alpha\ (\Lambda\ (\alpha\ R))) = \Lambda\ (\alpha\ R)$
by $simp$

2.3 Unit and multiplication of powerset monad

definition $eta :: ('a, 'a\ set) rel (\langle \eta \rangle)$ **where**
 $\eta = \Lambda\ Id$

definition $mu :: ('a\ set\ set, 'a\ set) rel (\langle \mu \rangle)$ **where**
 $\mu = pow\ epsilon\ off$

lemma $eta\ set: \eta = \{(a, \{a\}) \mid a. True\}$

unfolding *eta-def Lambda-def Id-def* **by** *simp*

lemma *alpha-eta* [*simp*]: $\alpha \eta = Id$
by (*simp add: eta-def*)

lemma *det-eta: deterministic η*
unfolding *deterministic-set eta-set* **by** *simp*

lemma *mu-set*: $\mu = \{(A,B). B = \{b. \exists C. C \in A \wedge b \in C\}\}$
unfolding *mu-def pow-def Lambda-def epsiloff-def* **by** *force*

lemma *det-mu: deterministic μ*
unfolding *deterministic-set mu-set* **by** *simp*

lemma *Lambda-eta*:
assumes *deterministic R*
shows $\Lambda R = R ; \eta$
proof –
have $\Lambda R = \Lambda (R ; Id)$
by *simp*
also have $\dots = R ; \Lambda Id$
using *Lambda-fusion assms* **by** *blast*
also have $\dots = R ; \eta$
by (*simp add: eta-def*)
finally show *?thesis.*
qed

lemma *eta-nat-trans*:
assumes *deterministic R*
shows $\eta ; \mathcal{P} R = R ; \eta$
proof –
have $\eta ; \mathcal{P} R = \Lambda Id ; \mathcal{P} R$
by (*simp add: eta-def*)
also have $\dots = \Lambda (Id ; R)$
using *Lambda-pow* **by** *blast*
also have $\dots = \Lambda R$
by *simp*
also have $\dots = R ; \eta$
by (*simp add: Lambda-eta assms*)
finally show *?thesis.*
qed

lemma *mu-nat-trans*:
assumes *deterministic R*
shows $\mathcal{P} (\mathcal{P} R) ; \mu = \mu ; \mathcal{P} R$
by (*metis pow-def alpha-Lambda-canc alpha-def mu-def pow-func2*)

The standard axioms for the powerset monad are derivable.

lemma *pow-monad1* [*simp*]: $\mathcal{P} \mu ; \mu = \mu ; \mu$

by (*metis pow-def alpha-Lambda-canc alpha-def mu-def pow-func2*)

lemma *pow-monad2* [*simp*]: $\mathcal{P} \eta ; \mu = Id$
by (*metis alpha-Lambda-canc alpha-def eta-def mu-def pow-func1 pow-func2*)

lemma *pow-monad3* [*simp*]: $\eta ; \mu = Id$
by (*metis Lambda-epsiloff Lambda-pow alpha-def alpha-epsiloff eta-def mu-def*)

lemma *Lambda-mu*:
assumes *deterministic R*
shows $\Lambda(R) ; \mu = R$
proof –
have $\Lambda R ; \mu = R ; \eta ; \mu$
by (*simp add: Lambda-eta assms*)
also have $\dots = R$
by (*simp add: O-assoc*)
finally show *?thesis*.
qed

lemma *pow-Lambda-mu* [*simp*]: $\mathcal{P} (\Lambda R) ; \mu = \mathcal{P} R$
by (*metis alpha-Lambda-canc alpha-def mu-def pow-func2*)

lemma *lambda-alpha-mu*: $\Lambda (\alpha R) = \Lambda R ; \mu$
by (*simp add: Lambda-pow alpha-def mu-def*)

lemma *alpha-eta-pow* [*simp*]: $\alpha (\eta ; \mathcal{P} R) = R$
proof –
have $\alpha (\eta ; \mathcal{P} R) = \alpha (\Lambda Id ; \mathcal{P} R)$
by (*simp add: eta-def*)
also have $\dots = \alpha (\Lambda (Id ; R))$
by (*metis Lambda-pow*)
also have $\dots = R$
by *simp*
finally show *?thesis*.
qed

lemma *eta-pow-Lambda* [*simp*]: $\eta ; \mathcal{P} R = \Lambda R$
by (*metis Id-O-R Lambda-pow eta-def*)

lemma *pow-prop1*: $\mathcal{P} R \subseteq S \implies R \subseteq \alpha (\eta ; S)$
by (*metis alpha-eta-pow alpha-ord-pres relcomp-distrib subset-Un-eq*)

lemma *pow-prop-2*: $R \subseteq \mathcal{P} S \implies \alpha (\eta ; R) \subseteq S$
by (*metis alpha-eta-pow alpha-ord-pres relcomp-distrib subset-Un-eq*)

lemma *pow-prop*: $R = \mathcal{P} S \implies \alpha (\eta ; R) = S$
using *alpha-eta-pow* **by** *blast*

lemma *alpha-eta-id* [*simp*]: $\alpha (R ; \eta) = R$

by *simp*

lemma *eta-alpha-idem* [*simp*]: $\alpha (\alpha R ; \eta) ; \eta = \alpha R ; \eta$
by *simp*

lemma *lambda-eta-alpha* [*simp*]: $\Lambda (\alpha (\alpha R ; \eta)) = \Lambda (\alpha R)$
by *simp*

lemma *eta-lambda-idem* [*simp*]: $\alpha (\Lambda (\alpha R)) ; \eta = \alpha R ; \eta$
by *simp*

lemma *Grph-eta* [*simp*]: $Grph (\lambda x. \{x\}) = \eta$
unfolding *Grph-def eta-def Lambda-def Id-def* by *force*

lemma *Grph-epsiloff* [*simp*]: $Grph (\lambda x. \{x\}) ; \text{epsiloff} = Id$
by (*metis Grph-eta alpha-def alpha-eta*)

lemma *Image-epsiloff* [*simp*]: $Image \text{epsiloff} \circ (\lambda x. \{x\}) = id$
unfolding *Image-def epsiloff-def id-def* by *force*

2.4 Subset relation

definition *Omega* :: ('a set, 'a set) rel ($\langle \Omega \rangle$) **where**
 $\Omega = \text{epsilon} \setminus \text{epsilon}$

lemma *Omega-set*: $\Omega = \{(A,B). A \subseteq B\}$
unfolding *Omega-def rres-def epsilon-def* by *force*

lemma *conv-Omega*: $\Omega^\smile = \text{epsiloff} \parallel \text{epsiloff}$
by (*simp add: Omega-def epsiloff-epsilon rres-lres-conv*)

lemma *epsilon-eta-Omega* [*simp*]: $\eta ; \Omega = \text{epsilon}$
unfolding *eta-set Omega-set epsilon-def* by *force*

lemma *epsiloff-eta-Omega* [*simp*]: $\Omega^\smile ; \eta^\smile = \text{epsiloff}$
by (*metis converse-relcomp epsiloff-epsilon epsilon-eta-Omega*)

lemma *epsilon-Omega* [*simp*]: $\text{epsilon} ; \Omega = \text{epsilon}$
by (*simp add: Omega-def*)

lemma *conv-Omega-epsiloff* [*simp*]: $\Omega^\smile ; \text{epsiloff} = \text{epsiloff}$
by (*simp add: conv-Omega*)

lemma *Lambda-conv* [*simp*]: $(\Lambda R)^\smile = \text{epsilon} \div R^\smile$
by (*simp add: Lambda-syq*)

lemma *Lambda-Omega*: $\Lambda R ; \Omega = R^\smile \setminus \text{epsilon}$
proof –
have $\Lambda R ; \Omega = \Lambda R ; (\text{epsilon} \setminus \text{epsilon})$

by (*simp add: Omega-def*)
 also have $\dots = \Lambda R ; -(epsilon\text{off} ; -epsilon)$
 by (*simp add: epsilonoff-epsilon rres-compl*)
 also have $\dots = -(\Lambda R ; epsilon\text{off} ; -epsilon)$
 by (*metis O-assoc det-lambda deterministic-var2*)
 also have $\dots = - (R ; -epsilon)$
 by (*metis alpha-Lambda-canc alpha-def*)
 also have $\dots = R^\sim \setminus epsilon$
 by (*simp add: rres-compl*)
 finally show *?thesis*.
 qed

lemma *syq-epsilonoff-prop* [*simp*]: $\Omega^\sim ; (epsilon \div R) = epsilon\text{off} // R^\sim$
 by (*metis Lambda-Omega Lambda-syq converse-converse converse-relcomp*
converse-syq epsilonoff-epsilon rres-lres-conv)

lemma *pow-semicom*: $((P, Q) \in \mathcal{P} R ; \Omega) = (\Delta P ; R \subseteq R ; \Delta Q)$
unfolding *pow-set Image-def Omega-def rres-def epsilon-def Delta-def* by *blast*

2.5 Complementation relation

definition *Compl* :: ('a set, 'a set) rel ($\langle C \rangle$) **where**
 $C = epsilon \div -epsilon$

lemma *Compl-set*: $C = \{(A, -A) \mid A. \text{True}\}$
unfolding *Compl-def syq-set epsilon-def* by *force*

lemma *Compl-Compl* [*simp*]: $C ; C = Id$
 by (*metis Compl-def Lambda-syq*
boolean-algebra-class.boolean-algebra.double-compl converse-converse converse-syq
det-lambda deterministic-def set-eq-subset syq-compl2 total-def univalent-def)

lemma *Compl-def-var*: $C = \Lambda (-epsilon\text{off})$
 by (*metis Compl-def Lambda-syq*
boolean-algebra-class.boolean-algebra.double-compl compl-conv converse-converse
epsilonoff-epsilon syq-compl2)

lemma *converse-Compl* [*simp*]: $C^\sim = C$
 by (*metis Compl-def converse-syq double-complement syq-compl2*)

lemma *det-Compl*: *deterministic C*
 by (*simp add: Compl-def-var det-lambda*)

lemma *bij-Compl*: *bijjective C*
 by (*simp add: bij-det det-Compl*)

lemma *Compl-compl-epsilonoff* [*simp*]: $C ; -epsilon\text{off} = epsilon\text{off}$
 by (*metis Compl-Compl Compl-def-var alpha-Lambda-canc alpha-epsilonoff*
alpha-relcomp)

lemma *Compl-epsiloff* [*simp*]: $\mathcal{C} ; \text{epsiloff} = -\text{epsiloff}$
by (*smt* (*z3*) *Compl-def-var alpha-Lambda-canc alpha-def*)

lemma *compl-epsilon-Compl* [*simp*]: $-\text{epsilon} ; \mathcal{C} = \text{epsilon}$
by (*metis* *Compl-compl-epsiloff compl-conv converse-Compl converse-converse converse-relcomp epsiloff-epsilon*)

lemma *epsilon-Compl* [*simp*]: $\text{epsilon} ; \mathcal{C} = -\text{epsilon}$
by (*metis* *Compl-epsiloff compl-conv converse-Compl converse-converse converse-relcomp epsiloff-epsilon*)

lemma *Lambda-Compl-var*: $\Lambda R ; \mathcal{C} = R^\sim \div -\text{epsilon}$
by (*simp* *add: Lambda-syq bij-det det-Compl syq-bij*)

lemma *Lambda-Compl*: $\Lambda R ; \mathcal{C} = \Lambda (-R)$

proof –

have $\Lambda R ; \mathcal{C} = \Lambda R ; \Lambda (-\text{epsiloff})$
by (*simp* *add: Compl-def-var*)
also have $\dots = \Lambda (\Lambda R ; -\text{epsiloff})$
by (*simp* *add: Lambda-fusion-var*)
also have $\dots = \Lambda (-\Lambda R ; \text{epsiloff})$
by (*metis* *det-lambda deterministic-var2*)
also have $\dots = \Lambda (-R)$
by (*metis* *alpha-Lambda-canc alpha-def*)
finally show *?thesis*.

qed

2.6 Kleisli lifting and Kleisli composition

definition *klift* :: $(\text{'a}, \text{'b set}) \text{rel} \Rightarrow (\text{'a set}, \text{'b set}) \text{rel}$ ($\langle \cdot \rangle_{\mathcal{P}}$ [1000] 999) **where**
 $(R)_{\mathcal{P}} = \mathcal{P} (\alpha R)$

definition *kcomp* :: $(\text{'a}, \text{'b set}) \text{rel} \Rightarrow (\text{'b}, \text{'c set}) \text{rel} \Rightarrow (\text{'a}, \text{'c set}) \text{rel}$ (**infixl** $\langle \cdot \rangle_{\mathcal{P}}$ 70) **where**
 $R \cdot_{\mathcal{P}} S = R ; (S)_{\mathcal{P}}$

lemma *klift-var*: $(R)_{\mathcal{P}} = \Lambda (\text{epsiloff} ; R ; \text{epsiloff})$
by (*simp* *add: pow-def O-assoc alpha-def klift-def*)

lemma *klift-set*: $(R)_{\mathcal{P}} = \{(A, B). B = \bigcup (\text{Image } R A)\}$
unfolding *klift-def Image-def pow-set alpha-set* **by** *force*

lemma *klift-set-var*: $(R)_{\mathcal{P}} = \{(A, B). B = \bigcup \{C. \exists a \in A. (a, C) \in R\}\}$
unfolding *klift-set Image-def* **by** *auto*

lemma *klift-mu*: $(R)_{\mathcal{P}} = \mathcal{P} R ; \mu$

proof –

have $(R)_{\mathcal{P}} = \mathcal{P} (R ; \text{epsiloff})$

by (*simp add: alpha-def klift-def*)
 also have $\dots = \mathcal{P} R ; \mathcal{P} \text{ epsilonoff}$
 by (*simp add: pow-func2*)
 also have $\dots = \mathcal{P} R ; \mu$
 by (*simp add: mu-def*)
 finally show *?thesis*.
 qed

lemma *klift-empty*: $(\{\}, A) \in (R)_{\mathcal{P}} \iff A = \{\}$
 using *klift-set* by *auto*

lemma *klift-ext1*: $(R ; (S)_{\mathcal{P}})_{\mathcal{P}} = (R)_{\mathcal{P}} ; (S)_{\mathcal{P}}$
 by (*metis (no-types, opaque-lifting) Lambda-epsilonoff-up1 Lambda-fusion-var O-assoc alpha-def klift-var*)

lemma *klift-ext2*: *deterministic* $R \implies \eta ; (R)_{\mathcal{P}} = R$
 by (*metis Id-O-R Lambda-alpha-canc Lambda-pow eta-def klift-def*)

lemma *klift-ext3* [*simp*]: $(\eta)_{\mathcal{P}} = \text{Id}$
 by (*simp add: klift-def*)

lemma *pow-klift* [*simp*]: $(R ; \eta)_{\mathcal{P}} = \mathcal{P} R$
 by (*simp add: klift-def*)

lemma *mu-klift* [*simp*]: $(\text{Id})_{\mathcal{P}} = \mu$
 by (*simp add: klift-def mu-def*)

lemma *kcomp-var*: $R \cdot_{\mathcal{P}} S = R ; \mathcal{P} S ; \mu$
 by (*simp add: O-assoc kcomp-def klift-mu*)

lemma *kcomp-assoc*: $R \cdot_{\mathcal{P}} (S \cdot_{\mathcal{P}} T) = (R \cdot_{\mathcal{P}} S) \cdot_{\mathcal{P}} T$

proof –

have $R \cdot_{\mathcal{P}} (S \cdot_{\mathcal{P}} T) = R ; (S ; (T)_{\mathcal{P}})_{\mathcal{P}}$

by (*simp add: kcomp-def*)

also have $\dots = R ; ((S)_{\mathcal{P}} ; (T)_{\mathcal{P}})$

by (*simp add: klift-ext1*)

also have $\dots = (R \cdot_{\mathcal{P}} S) \cdot_{\mathcal{P}} T$

by (*simp add: O-assoc kcomp-def*)

finally show *?thesis*.

qed

lemma *kcomp-oner*: $R \cdot_{\mathcal{P}} \eta = R$
 by (*simp add: kcomp-def*)

lemma *kcomp-onel*: *deterministic* $R \implies \eta \cdot_{\mathcal{P}} R = R$
 by (*simp add: kcomp-def klift-ext2*)

2.7 Relational box

definition $rbox :: ('a, 'b) rel \Rightarrow ('b\ set, 'a\ set) rel$ **where**
 $rbox\ R = \Lambda\ (epsilon\ off\ \parallel\ R)$

lemma $rbox\ set$: $rbox\ R = \{(Q, P). P = \{a. \forall b. (a, b) \in R \longrightarrow b \in Q\}\}$
unfolding $rbox\ def\ Lambda\ def\ lres\ def\ epsilon\ off\ def$
by *force*

lemma $rbox\ exp$: $((Q, P) \in (rbox\ (R :: ('a, 'b) rel))) = (P = -\{a. \exists b. (a, b) \in R \wedge b \in -Q\})$
by (*smt (z3) Collect-cong Collect-neg-eq ComplD ComplI case-prodD case-prodI mem-Collect-eq rbox-set*)

lemma $rbox\ subset$: $rbox\ R ; \Omega^\sim = \{(Q, P). P \subseteq \{a. \forall b. (a, b) \in R \longrightarrow b \in Q\}\}$
unfolding $rbox\ set\ Omega\ set$ **by** *blast*

lemma $rbox\ semicomm$: $(Q, P) \in rbox\ R ; \Omega^\sim = (\Delta\ P ; R \subseteq R ; \Delta\ Q)$
unfolding $rbox\ subset\ Delta\ def$ **by** *blast*

lemma $rbox\ semicomm\ var$: $(Q, P) \in rbox\ R ; \Omega^\sim = (\Delta\ P \subseteq (R ; \Delta\ Q) \parallel R)$
by (*simp add: lres-galois rbox-semicomm*)

lemma $rbox\ omega$: $rbox\ epsilon\ off = \Lambda\ (\Omega^\sim)$
by (*simp add: conv-Omega rbox-def*)

lemma $Omega\ rbox$: $\Omega = (\alpha\ (rbox\ epsilon\ off))^\sim$
by (*simp add: rbox-omega*)

lemma $pow\ rbox$: $((Q, P) \in rbox\ R ; \Omega^\sim) = ((P, Q) \in \mathcal{P}\ R ; \Omega)$

proof –

have $(Q, P) \in rbox\ R ; \Omega^\sim = (\Delta\ P ; R \subseteq R ; \Delta\ Q)$
by (*simp add: rbox-semicomm*)
also have $\dots = ((P, Q) \in \mathcal{P}\ R ; \Omega)$
by (*simp add: pow-semicomm*)
finally show *?thesis*.

qed

lemma $rbox\ pow\ Compl$: $rbox\ R = \mathcal{C} ; \mathcal{P}\ (R^\sim) ; \mathcal{C}$

proof –

have $\mathcal{C} ; \mathcal{P}\ (R^\sim) ; \mathcal{C} = \Lambda\ (-epsilon\ off) ; \mathcal{P}\ (R^\sim) ; \mathcal{C}$
by (*simp add: Compl-def-var*)
also have $\dots = \Lambda\ (-epsilon\ off ; R^\sim) ; \mathcal{C}$
by (*simp add: Lambda-pow*)
also have $\dots = \Lambda\ (-(-epsilon\ off ; R^\sim))$
by (*simp add: Lambda-Compl*)
also have $\dots = \Lambda\ (epsilon\ off\ \parallel\ R)$
by (*simp add: lres-compl*)
also have $\dots = rbox\ R$
by (*simp add: rbox-def*)

finally show *?thesis..*
qed

lemma *pow-rbox-Compl*: $\mathcal{P} R = \mathcal{C} ; \text{rbox } (R^\smile) ; \mathcal{C}$
by (*metis Compl-Compl Id-O-R O-assoc R-O-Id converse-converse rbox-pow-Compl*)

lemma *pow-conjugation*: $\mathcal{C} ; (\mathcal{P} (R^\smile) ; \Omega)^\smile = \mathcal{P} R ; \mathcal{C} ; \Omega^\smile$
proof –

have $\mathcal{P} R ; \mathcal{C} ; \Omega^\smile = \Lambda (-(\text{epsiloff} ; R)) ; -(- \text{epsiloff} ; \text{epsilon})$
by (*simp add: Lambda-Compl pow-def conv-Omega epsiloff-epsilon lres-compl*)
also have $\dots = -(\Lambda (-(\text{epsiloff} ; R)) ; - \text{epsiloff} ; \text{epsilon})$
by (*metis (no-types, opaque-lifting) alpha-Lambda-canc alpha-def converse-converse det-lambda det-lres deterministic-var2 epsiloff-epsilon lres-compl*)
also have $\dots = -(\Lambda (-(\text{epsiloff} ; R)) ; \mathcal{C} ; \text{epsiloff} ; \text{epsilon})$
by (*metis Compl-epsiloff alpha-def alpha-relcomp*)
also have $\dots = -(\Lambda (\text{epsiloff} ; R) ; \text{epsiloff} ; \text{epsilon})$
by (*simp add: Lambda-Compl*)
also have $\dots = -(\text{epsiloff} ; R ; \text{epsilon})$
by (*metis Lambda-epsiloff-up1 alpha-def*)
also have $\dots = -(\mathcal{C} ; -\text{epsiloff} ; (\text{epsiloff} ; R^\smile)^\smile)$
by (*metis Compl-compl-epsiloff O-assoc converse-converse converse-relcomp epsiloff-epsilon*)
also have $\dots = \mathcal{C} ; (\text{epsiloff} \parallel (\text{epsiloff} ; R^\smile))$
by (*metis (no-types, opaque-lifting) Compl-def-var Lambda-Compl Lambda-fusion-var pow-def alpha-Lambda-canc boolean-algebra-class.boolean-algebra.double-compl converse-converse pow-rbox-Compl rbox-def*)
also have $\dots = \mathcal{C} ; (\mathcal{P} (R^\smile) ; \Omega)^\smile$
by (*simp add: Lambda-Omega pow-def epsiloff-epsilon rres-lres-conv*)
finally show *?thesis..*
qed

lemma *pow-rbox-eq*: $\text{rbox } R ; \Omega^\smile = (\mathcal{P} R ; \Omega)^\smile$
by (*metis (no-types, lifting) Compl-Compl O-assoc R-O-Id converse-converse converse-relcomp pow-conjugation rbox-pow-Compl*)

end

3 Basic Properties of Multirelations

theory *Multirelations-Basics*

imports *Power-Allegories-Properties*

begin

This theory extends a previous AFP entry for multirelations with one

single objects to proper multirelations in Rel.

3.1 Peleg composition, parallel composition (inner union) and units

type-synonym $(\text{'a}, \text{'b}) \text{ mrel} = (\text{'a}, \text{'b} \text{ set}) \text{ rel}$

definition $s\text{-prod} :: (\text{'a}, \text{'b}) \text{ mrel} \Rightarrow (\text{'b}, \text{'c}) \text{ mrel} \Rightarrow (\text{'a}, \text{'c}) \text{ mrel}$ (**infixl** $\langle \cdot \rangle$ 75)
where

$R \cdot S = \{(a, A). (\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge A = \bigcup (f \cdot B)))\}$

definition $s\text{-id} :: (\text{'a}, \text{'a}) \text{ mrel}$ ($\langle 1_\sigma \rangle$) **where**

$1_\sigma = (\bigcup a. \{(a, \{a\})\})$

definition $p\text{-prod} :: (\text{'a}, \text{'b}) \text{ mrel} \Rightarrow (\text{'a}, \text{'b}) \text{ mrel} \Rightarrow (\text{'a}, \text{'b}) \text{ mrel}$ (**infixl** $\langle \parallel \rangle$ 70)
where

$R \parallel S = \{(a, A). (\exists B C. A = B \cup C \wedge (a, B) \in R \wedge (a, C) \in S)\}$

definition $p\text{-id} :: (\text{'a}, \text{'b}) \text{ mrel}$ ($\langle 1_\pi \rangle$) **where**

$1_\pi = (\bigcup a. \{(a, \{\})\})$

definition $U :: (\text{'a}, \text{'b}) \text{ mrel}$ **where**

$U = \{(a, A) \mid a A. \text{True}\}$

abbreviation $NC \equiv U - 1_\pi$

named-theorems $mr\text{-simp}$

declare $s\text{-prod-def}$ [$mr\text{-simp}$] $p\text{-prod-def}$ [$mr\text{-simp}$] $s\text{-id-def}$ [$mr\text{-simp}$] $p\text{-id-def}$ [$mr\text{-simp}$] $U\text{-def}$ [$mr\text{-simp}$]

lemma $s\text{-prod-idl}$ [$simp$]: $1_\sigma \cdot R = R$

by ($auto \text{ simp: } mr\text{-simp}$)

lemma $s\text{-prod-idr}$ [$simp$]: $R \cdot 1_\sigma = R$

by ($auto \text{ simp: } mr\text{-simp}$) ($metis \text{ UN-singleton}$)

lemma $p\text{-prod-ild}$ [$simp$]: $1_\pi \parallel R = R$

by ($simp \text{ add: } mr\text{-simp}$)

lemma $c\text{-prod-idr}$ [$simp$]: $R \parallel 1_\pi = R$

by ($simp \text{ add: } mr\text{-simp}$)

lemma $cl7$ [$simp$]: $1_\sigma \parallel 1_\sigma = 1_\sigma$

by ($auto \text{ simp: } mr\text{-simp}$)

lemma $p\text{-prod-assoc}$: $R \parallel S \parallel T = R \parallel (S \parallel T)$

apply ($rule \text{ set-eqI}$, $clarsimp \text{ simp: } mr\text{-simp}$)

by ($metis \text{ (no-types, lifting) sup-assoc}$)

lemma *p-prod-comm*: $R \parallel S = S \parallel R$

by (*auto simp: mr-simp*)

lemma *subidem-par*: $R \subseteq R \parallel R$

by (*auto simp: mr-simp*)

lemma *meet-le-par*: $R \cap S \subseteq R \parallel S$

by (*auto simp: mr-simp*)

lemma *s-prod-distr*: $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$

by (*auto simp: mr-simp*)

lemma *s-prod-sup-distr*: $(\bigcup X) \cdot S = (\bigcup R \in X. R \cdot S)$

by (*auto simp: mr-simp*)

lemma *s-prod-subdistl*: $R \cdot S \cup R \cdot T \subseteq R \cdot (S \cup T)$

by (*auto simp: mr-simp*)

lemma *s-prod-sup-subdistl*: $X \neq \{\} \implies (\bigcup S \in X. R \cdot S) \subseteq R \cdot \bigcup X$

by (*simp add: mr-simp blast*)

lemma *s-prod-isol*: $R \subseteq S \implies R \cdot T \subseteq S \cdot T$

by (*metis s-prod-distr sup.order-iff*)

lemma *s-prod-isor*: $R \subseteq S \implies T \cdot R \subseteq T \cdot S$

by (*metis le-supE s-prod-subdistl sup.absorb-iff1*)

lemma *s-prod-zeroL* [*simp*]: $\{\} \cdot R = \{\}$

by (*force simp: mr-simp*)

lemma *s-prod-wzeror*: $R \cdot \{\} \subseteq R$

by (*force simp: mr-simp*)

lemma *p-prod-zeroR* [*simp*]: $R \parallel \{\} = \{\}$

by (*simp add: mr-simp*)

lemma *s-prod-p-idl* [*simp*]: $1_\pi \cdot R = 1_\pi$

by (*force simp: mr-simp*)

lemma *p-id-st*: $R \cdot 1_\pi = \{(a, \{\}) \mid a. \exists B. (a, B) \in R\}$

by (*force simp: mr-simp*)

lemma *c6*: $R \cdot 1_\pi \subseteq 1_\pi$

by (*clarsimp simp: mr-simp*)

lemma *p-prod-distl*: $R \parallel (S \cup T) = R \parallel S \cup R \parallel T$

by (*fastforce simp: mr-simp*)

lemma *p-prod-sup-distl*: $R \parallel (\bigcup X) = (\bigcup S \in X. R \parallel S)$
by (*fastforce simp: mr-simp*)

lemma *p-prod-isol*: $R \subseteq S \implies R \parallel T \subseteq S \parallel T$
by (*metis p-prod-comm p-prod-distl sup.orderE sup.orderI*)

lemma *p-prod-isor*: $R \subseteq S \implies T \parallel R \subseteq T \parallel S$
by (*simp add: p-prod-comm p-prod-isol*)

lemma *s-prod-assoc1*: $(R \cdot S) \cdot T \subseteq R \cdot (S \cdot T)$
by (*clarsimp simp: mr-simp metis*)

lemma *seq-conc-subdistr*: $(R \parallel S) \cdot T \subseteq R \cdot T \parallel S \cdot T$
by (*clarsimp simp: mr-simp UnI1 UnI2*) *blast*

lemma *U-U [simp]*: $U \cdot U = U$
by (*simp add: mr-simp*) *blast*

lemma *U-par-idem [simp]*: $U \parallel U = U$
by (*simp add: U-def equalityI subidem-par*)

lemma *p-id-NC*: $R - 1_\pi = R \cap NC$
by (*simp add: Diff-eq U-def*)

lemma *NC-NC [simp]*: $NC \cdot NC = NC$
by (*rule set-eqI, clarsimp simp: mr-simp, metis SUP-bot-conv(2) UN-constant insert-not-empty*)

lemma *nc-par-idem [simp]*: $NC \parallel NC = NC$
by (*force simp: mr-simp*)

lemma *cl4*:

assumes $T \parallel T \subseteq T$

shows $R \cdot T \parallel S \cdot T \subseteq (R \parallel S) \cdot T$

proof *clarify*

fix $a A$

assume $(a, A) \in R \cdot T \parallel S \cdot T$

hence $\exists B C. A = B \cup C \wedge (\exists D. (a, D) \in R \wedge (\exists f. (\forall d \in D. (d, f d) \in T) \wedge B = \bigcup ((\lambda x. f x) ' D))) \wedge (\exists E. (a, E) \in S \wedge (\exists g. (\forall e \in E. (e, g e) \in T) \wedge C = \bigcup ((\lambda x. g x) ' E)))$

by (*simp add: mr-simp*)

hence $\exists D E. (a, D \cup E) \in R \parallel S \wedge (\exists f g. (\forall d \in D. (d, f d) \in T) \wedge (\forall e \in E. (e, g e) \in T) \wedge A = (\bigcup ((\lambda x. f x) ' D)) \cup (\bigcup ((\lambda x. g x) ' E)))$

by (*auto simp: mr-simp*)

hence $\exists D E. (a, D \cup E) \in R \parallel S \wedge (\exists f g. (\forall d \in D - E. (d, f d) \in T) \wedge (\forall e \in E - D. (e, g e) \in T) \wedge (\forall x \in D \cap E. (x, f x) \in T \wedge (x, g x) \in T) \wedge A = (\bigcup ((\lambda x. f x) ' (D - E))) \cup (\bigcup ((\lambda x. g x) ' (E - D))) \cup (\bigcup ((\lambda y. f y \cup g y) ' (D \cap E))))$

by *auto blast*

hence $\exists D E. (a, D \cup E) \in R \parallel S \wedge (\exists f g. (\forall d \in D - E. (d, f d) \in T) \wedge (\forall e \in$

$E-D. (e, g e) \in T \wedge (\forall x \in D \cap E. (x, f x \cup g x) \in T) \wedge A = (\bigcup ((\lambda x. f x) ' (D-E))) \cup (\bigcup ((\lambda x. g x) ' (E-D))) \cup (\bigcup ((\lambda y. f y \cup g y) ' (D \cap E)))$
apply *clarify*
subgoal for $D E f g$
apply (*rule exI[of - D]*)
apply (*rule exI[of - E]*)
apply *clarify*
apply (*rule exI[of - f]*)
apply (*rule exI[of - g]*)
using *assms* **by** (*auto simp: p-prod-def, blast*)
done
hence $\exists D E. (a, D \cup E) \in R \parallel S \wedge (\exists h. (\forall d \in D-E. (d, h d) \in T) \wedge (\forall e \in E-D. (e, h e) \in T) \wedge (\forall x \in D \cap E. (x, h x) \in T) \wedge A = (\bigcup ((\lambda x. h x) ' (D-E))) \cup (\bigcup ((\lambda x. h x) ' (E-D))) \cup (\bigcup ((\lambda y. h y) ' (D \cap E))))$
apply *clarify*
subgoal for $D E f g$
apply (*rule exI[of - D]*)
apply (*rule exI[of - E]*)
apply *clarify*
apply (*rule exI[of - \lambda x. if x \in (D - E) then f x else (if x \in D \cap E then (f x \cup g x) else g x)]*)
by *auto*
done
hence $(\exists B. (a, B) \in R \parallel S \wedge (\exists h. (\forall b \in B. (b, h b) \in T) \wedge A = \bigcup((\lambda x. h x) ' B)))$
by *clarsimp blast*
thus $(a, A) \in (R \parallel S) \cdot T$
by (*simp add: mr-simp*)
qed

lemma *cl3*: $R \cdot (S \parallel T) \subseteq R \cdot S \parallel R \cdot T$

proof *clarify*

fix $a A$
assume $(a, A) \in R \cdot (S \parallel T)$
hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. \exists C D. f b = C \cup D \wedge (b, C) \in S \wedge (b, D) \in T) \wedge A = \bigcup((\lambda x. f x) ' B))$
by (*clarsimp simp: mr-simp*)
hence $\exists B. (a, B) \in R \wedge (\exists f g h. (\forall b \in B. f b = g b \cup h b \wedge (b, g b) \in S \wedge (b, h b) \in T) \wedge A = \bigcup((\lambda x. f x) ' B))$
by (*clarsimp simp: bchoice, metis*)
hence $\exists B. (a, B) \in R \wedge (\exists g h. (\forall b \in B. (b, g b) \in S \wedge (b, h b) \in T) \wedge A = (\bigcup ((\lambda x. g x) ' B)) \cup (\bigcup ((\lambda x. h x) ' B)))$
by *blast*
hence $\exists C D. (\exists B. (a, B) \in R \wedge (\exists g. (\forall b \in B. (b, g b) \in S) \wedge C = \bigcup ((\lambda x. g x) ' B))) \wedge (\exists B. (a, B) \in R \wedge (\exists h. (\forall b \in B. (b, h b) \in T) \wedge D = \bigcup ((\lambda x. h x) ' B))) \wedge A = C \cup D$
by *blast*
thus $(a, A) \in R \cdot S \parallel R \cdot T$
by (*auto simp: mr-simp*)

qed

lemma *p-id-assoc1*: $(1_\pi \cdot R) \cdot S = 1_\pi \cdot (R \cdot S)$
by *simp*

lemma *p-id-assoc2*: $(R \cdot 1_\pi) \cdot T = R \cdot (1_\pi \cdot T)$
by (*rule set-eqI, clarsimp simp: mr-simp*) *fastforce*

lemma *cl1* [*simp*]: $R \cdot 1_\pi \cup R \cdot NC = R \cdot U$
by (*rule set-eqI, clarsimp simp: mr-simp,metis UN-constant UN-empty*)

lemma *tarski-aux*:
assumes $R - 1_\pi \neq \{\}$
and $(a,A) \in NC$
shows $(a,A) \in NC \cdot ((R - 1_\pi) \cdot NC)$
using *assms apply* (*clarsimp simp: mr-simp*)
by (*metis UN-constant insert-not-empty singletonD*)

lemma *tarski*:
assumes $R - 1_\pi \neq \{\}$
shows $NC \cdot ((R - 1_\pi) \cdot NC) = NC$
by *standard* (*simp add: U-def p-id-def s-prod-def, force, metis assms tarski-aux*
subrelI)

lemma *tarski-var*:
assumes $R \cap NC \neq \{\}$
shows $NC \cdot ((R \cap NC) \cdot NC) = NC$
by (*metis assms p-id-NC tarski*)

lemma *s-le-nc*: $1_\sigma \subseteq NC$
by (*auto simp: mr-simp*)

lemma *U-nc* [*simp*]: $U \cdot NC = U$
by (*metis NC-NC cl1 s-prod-distr s-prod-idl s-prod-p-idl*)

lemma *x-y-split* [*simp*]: $(R \cap NC) \cdot S \cup R \cdot \{\} = R \cdot S$
by (*auto simp: mr-simp*)

lemma *c-nc-comp1* [*simp*]: $1_\pi \cup NC = U$
using *cl1 s-prod-idl* **by** *blast*

3.2 Tests

lemma *s-id-st*: $R \cap 1_\sigma = \{(a,\{a\}) \mid a. (a,\{a\}) \in R\}$
by (*force simp: mr-simp*)

lemma *subid-aux2*:
assumes $(a,A) \in R \cap 1_\sigma$
shows $A = \{a\}$

using *assms* **by** (*auto simp: mr-simp*)

lemma *s-prod-test-aux1*:
assumes $(a,A) \in R \cdot (P \cap 1_\sigma)$
shows $((a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma)))$
using *assms* **by** (*auto simp: mr-simp*)

lemma *s-prod-test-aux2*:
assumes $(a,A) \in R$
and $\forall a \in A. (a,\{a\}) \in S$
shows $(a,A) \in R \cdot S$
using *assms* **by** (*fastforce simp: mr-simp*)

lemma *s-prod-test*: $(a,A) \in R \cdot (P \cap 1_\sigma) \longleftrightarrow (a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma))$
by (*meson s-prod-test-aux1 s-prod-test-aux2*)

lemma *s-prod-test-var*: $R \cdot (P \cap 1_\sigma) = \{(a,A). (a,A) \in R \wedge (\forall a \in A. (a,\{a\}) \in (P \cap 1_\sigma))\}$
apply (*rule antisym*)
by (*fastforce simp: mr-simp*)+

lemma *test-s-prod-aux1*:
assumes $(a,A) \in (P \cap 1_\sigma) \cdot R$
shows $(a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$
using *assms* **by** (*auto simp: mr-simp*)

lemma *test-s-prod-aux2*:
assumes $(a,A) \in R$
and $(a,\{a\}) \in P$
shows $(a,A) \in P \cdot R$
using *assms s-prod-def* **by** *fastforce*

lemma *test-s-prod*: $(a,A) \in (P \cap 1_\sigma) \cdot R \longleftrightarrow (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R$
by (*meson test-s-prod-aux1 test-s-prod-aux2*)

lemma *test-s-prod-var*: $(P \cap 1_\sigma) \cdot R = \{(a,A). (a,\{a\}) \in (P \cap 1_\sigma) \wedge (a,A) \in R\}$
by (*simp add: set-eq-iff test-s-prod*)

lemma *test-assoc1*: $(R \cdot (P \cap 1_\sigma)) \cdot S = R \cdot ((P \cap 1_\sigma) \cdot S)$
apply (*rule antisym*)
apply (*simp add: s-prod-assoc1*)
apply (*clarsimp simp: mr-simp*)
by (*metis UN-singleton*)

lemma *test-assoc2*: $((P \cap 1_\sigma) \cdot R) \cdot S = (P \cap 1_\sigma) \cdot (R \cdot S)$
apply (*rule antisym*)
apply (*simp add: s-prod-assoc1*)
by (*fastforce simp: mr-simp s-prod-assoc1*)

lemma *test-assoc3*: $(R \cdot S) \cdot (P \cap 1_\sigma) = R \cdot (S \cdot (P \cap 1_\sigma))$
proof (*rule antisym*)
show $(R \cdot S) \cdot (P \cap 1_\sigma) \subseteq R \cdot (S \cdot (P \cap 1_\sigma))$
by (*simp add: s-prod-assoc1*)
show $R \cdot (S \cdot (P \cap 1_\sigma)) \subseteq (R \cdot S) \cdot (P \cap 1_\sigma)$
proof *clarify*
fix $a A$
assume *hyp1*: $(a, A) \in R \cdot (S \cdot (P \cap 1_\sigma))$
hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S \cdot (P \cap 1_\sigma))) \wedge A = \bigcup((\lambda x. f x) \text{ ` } B)$
by (*simp add: s-prod-test s-prod-def*)
hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S \wedge (\forall a \in f b. (a, \{a\}) \in (P \cap 1_\sigma)))) \wedge A = \bigcup((\lambda x. f x) \text{ ` } B)$
by (*simp add: s-prod-test*)
hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge (\forall a \in \bigcup((\lambda x. f x) \text{ ` } B). (a, \{a\}) \in (P \cap 1_\sigma))) \wedge A = \bigcup((\lambda x. f x) \text{ ` } B)$
by *auto*
hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge (\forall a \in A. (a, \{a\}) \in (P \cap 1_\sigma))) \wedge A = \bigcup((\lambda x. f x) \text{ ` } B)$
by *auto blast*
hence $(a, A) \in R \cdot S \wedge (\forall a \in A. (a, \{a\}) \in (P \cap 1_\sigma))$
by (*auto simp: mr-simp*)
thus $(a, A) \in (R \cdot S) \cdot (P \cap 1_\sigma)$
by (*simp add: s-prod-test*)
qed
qed

lemma *s-distl-test*: $(P \cap 1_\sigma) \cdot (S \cup T) = (P \cap 1_\sigma) \cdot S \cup (P \cap 1_\sigma) \cdot T$
by (*fastforce simp: mr-simp*)

lemma *s-distl-sup-test*: $(P \cap 1_\sigma) \cdot \bigcup X = (\bigcup S \in X. (P \cap 1_\sigma) \cdot S)$
by (*auto simp: mr-simp*)

lemma *subid-par-idem* [*simp*]: $(P \cap 1_\sigma) \parallel (P \cap 1_\sigma) = (P \cap 1_\sigma)$
by (*auto simp: mr-simp*)

lemma *seq-conc-subdistrl*: $(P \cap 1_\sigma) \cdot (S \parallel T) = ((P \cap 1_\sigma) \cdot S) \parallel ((P \cap 1_\sigma) \cdot T)$
apply (*rule antisym*)
apply (*simp add: cl3*)
by (*fastforce simp: mr-simp*)

lemma *test-s-prod-is-meet* [*simp*]: $(P \cap 1_\sigma) \cdot (Q \cap 1_\sigma) = P \cap Q \cap 1_\sigma$
by (*auto simp: mr-simp*)

lemma *test-p-prod-is-meet* [*simp*]: $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cap (Q \cap 1_\sigma)$
by (*auto simp: mr-simp*)

lemma *test-multipliativer*: $(P \cap Q \cap 1_\sigma) \cdot T = ((P \cap 1_\sigma) \cdot T) \cap ((Q \cap 1_\sigma) \cdot T)$

T)
by (*auto simp: mr-simp*)

lemma *cl9* [*simp*]: $(R \cap 1_\sigma) \cdot 1_\pi \parallel 1_\sigma = R \cap 1_\sigma$
by (*auto simp: mr-simp*)

lemma *s-subid-closed* [*simp*]: $R \cap NC \cap 1_\sigma = R \cap 1_\sigma$
using *s-le-nc* **by** *auto*

lemma *sub-id-le-nc*: $R \cap 1_\sigma \subseteq NC$
by (*simp add: le-infI2 s-le-nc*)

lemma *x-y-prop*: $1_\sigma \cap ((R \cap NC) \cdot S) = 1_\sigma \cap R \cdot S$
by (*auto simp: mr-simp*)

lemma *s-nc-U*: $1_\sigma \cap R \cdot NC = 1_\sigma \cap R \cdot U$
by (*rule set-eqI, clarsimp simp: mr-simp,metis SUP-constant UN-empty insert-not-empty*)

lemma *sid-le-nc-var*: $1_\sigma \cap R \subseteq 1_\sigma \cap (R \parallel NC)$
using *meet-le-par s-le-nc* **by** *fastforce*

lemma *s-nc-par-U*: $1_\sigma \cap (R \parallel NC) = 1_\sigma \cap (R \parallel U)$
by (*metis c-nc-comp1 c-prod-idr inf-sup-distrib1 le-iff-sup p-prod-distl sid-le-nc-var*)

lemma *s-id-par-s-prod*: $(P \cap 1_\sigma) \parallel (Q \cap 1_\sigma) = (P \cap 1_\sigma) \cdot (Q \cap 1_\sigma)$
by *force*

3.3 Parallel subidentities

lemma *p-id-zero-st*: $R \cap 1_\pi = \{(a, \{\}) \mid a. (a, \{\}) \in R\}$
by (*auto simp: mr-simp*)

lemma *p-subid-iff*: $R \subseteq 1_\pi \iff R \cdot 1_\pi = R$
by (*clarsimp simp: mr-simp, safe, simp-all*) *blast+*

lemma *p-subid-iff-var*: $R \subseteq 1_\pi \iff R \cdot \{\} = R$
by (*clarsimp simp: mr-simp, safe, simp-all*) *blast+*

lemma *term-par-idem* [*simp*]: $(R \cap 1_\pi) \parallel (R \cap 1_\pi) = (R \cap 1_\pi)$
by (*metis Int-Un-eq(4) c-prod-idr p-prod-distl subidem-par subset-Un-eq*)

lemma *c1* [*simp*]: $R \cdot 1_\pi \parallel R = R$
apply (*rule set-eqI, clarsimp simp: mr-simp*)
by (*metis (no-types, lifting) SUP-bot SUP-bot-conv(2) sup-bot-left*)

lemma *p-id-zero*: $R \cap 1_\pi = R \cdot \{\}$
by (*auto simp: mr-simp*)

lemma *cl5*: $(R \cdot S) \cdot (T \cdot \{\}) = R \cdot (S \cdot (T \cdot \{\}))$
proof (*rule antisym*)
show $(R \cdot S) \cdot (T \cdot \{\}) \subseteq R \cdot (S \cdot (T \cdot \{\}))$
by (*metis s-prod-assoc1*)
show $R \cdot (S \cdot (T \cdot \{\})) \subseteq (R \cdot S) \cdot (T \cdot \{\})$
proof *clarify*
fix $a::'a$ **and** $A::'f$ *set*
assume $(a,A) \in R \cdot (S \cdot (T \cdot \{\}))$
hence $\exists B. (a,B) \in R \wedge (\exists f. (\forall b \in B. (\exists C. (b,C) \in S \wedge (\exists g. (\forall x \in C. (x,g x) \in T \cdot \{\}) \wedge f b = \bigcup((\lambda x. g x) ' C)))) \wedge A = \bigcup((\lambda x. f x) ' B))$
by (*clarsimp simp: mr-simp*)
hence $\exists B. (a,B) \in R \wedge (\exists f. (\forall b \in B. (\exists C. (b,C) \in S \wedge (\forall x \in C. (x,\{x\}) \in T \cdot \{\}) \wedge f b = \{x\})) \wedge A = \bigcup((\lambda x. f x) ' B))$
by (*clarsimp simp: mr-simp fastforce*)
hence $\exists B. (a,B) \in R \wedge (\forall b \in B. (\exists C. (b,C) \in S \wedge (\forall x \in C. (x,\{x\}) \in T \cdot \{\}))) \wedge A = \{x\}$
by *fastforce*
hence $\exists B. (a,B) \in R \wedge (\exists f. (\forall b \in B. (b,f b) \in S \wedge (\forall x \in f b. (x,\{x\}) \in T \cdot \{\}))) \wedge A = \{x\}$
by (*metis (mono-tags)*)
hence $\exists B. (a,B) \in R \wedge (\exists f. (\forall b \in B. (b,f b) \in S) \wedge (\forall x \in \bigcup((\lambda x. f x) ' B). (x,\{x\}) \in T \cdot \{\})) \wedge A = \{x\}$
by (*metis UN-E*)
hence $\exists C B. (a,B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge C = \bigcup((\lambda x. f x) ' B) \wedge (\forall x \in C. (x,\{x\}) \in T \cdot \{\})) \wedge A = \{x\}$
by *metis*
hence $\exists C. (a,C) \in R \cdot S \wedge (\forall x \in C. (x,\{x\}) \in T \cdot \{\}) \wedge A = \{x\}$
by (*auto simp: mr-simp*)
thus $(a,A) \in (R \cdot S) \cdot (T \cdot \{\})$
by (*clarsimp simp: mr-simp blast*)
qed
qed

lemma *c4*: $(R \cdot S) \cdot 1_\pi = R \cdot (S \cdot 1_\pi)$
proof –
have $(R \cdot S) \cdot 1_\pi = \{(a,\{x\}) \mid a. \exists B. (a,B) \in R \cdot S\}$
by (*simp add: p-id-st*)
also have $\dots = R \cdot \{(a,\{x\}) \mid a. \exists B. (a,B) \in S\}$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *blast*
apply *clarsimp*
by *metis*
also have $\dots = R \cdot (S \cdot 1_\pi)$
by (*simp add: p-id-st*)
finally show *?thesis*.
qed

lemma *c3*: $(R \parallel S) \cdot 1_\pi = R \cdot 1_\pi \parallel S \cdot 1_\pi$
by (*simp add: Orderings.order-eq-iff cl4 seq-conc-subdistr*)

lemma *p-id-idem* [*simp*]: $(R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$
by (*simp add: c4*)

lemma *x-c-par-idem* [*simp*]: $R \cdot 1_\pi \parallel R \cdot 1_\pi = R \cdot 1_\pi$
by (*metis c1 p-id-idem*)

lemma *x-zero-le-c*: $R \cdot \{\} \subseteq 1_\pi$
by (*simp add: c4 p-subid-iff*)

lemma *p-subid-lb1*: $R \cdot \{\} \parallel S \cdot \{\} \subseteq R \cdot \{\}$
by (*metis c-prod-idr p-prod-isor x-zero-le-c*)

lemma *p-subid-lb2*: $R \cdot \{\} \parallel S \cdot \{\} \subseteq S \cdot \{\}$
using *p-prod-comm p-subid-lb1* **by** *blast*

lemma *p-subid-idem* [*simp*]: $R \cdot \{\} \parallel R \cdot \{\} = R \cdot \{\}$
by (*simp add: p-subid-lb1 subidem-par subset-antisym*)

lemma *p-subid-glb*: $T \cdot \{\} \subseteq R \cdot \{\} \implies T \cdot \{\} \subseteq S \cdot \{\} \implies T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$
by (*auto simp: mr-simp*)

lemma *p-subid-glb-iff*: $T \cdot \{\} \subseteq R \cdot \{\} \wedge T \cdot \{\} \subseteq S \cdot \{\} \iff T \cdot \{\} \subseteq (R \cdot \{\}) \parallel (S \cdot \{\})$
by (*auto simp: mr-simp*)

lemma *x-c-glb*: $(T::('a,'b) mrel) \cdot 1_\pi \subseteq (R::('a,'b) mrel) \cdot 1_\pi \implies T \cdot 1_\pi \subseteq (S::('a,'b) mrel) \cdot 1_\pi \implies T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$
by (*smt (verit, best) order-subst1 p-id-idem p-prod-isor p-prod-isor s-prod-isor x-c-par-idem*)

lemma *x-c-lb1*: $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq R \cdot 1_\pi$
by (*metis c6 c-prod-idr p-prod-isor*)

lemma *x-c-lb2*: $R \cdot 1_\pi \parallel S \cdot 1_\pi \subseteq S \cdot 1_\pi$
using *p-prod-comm x-c-lb1* **by** *blast*

lemma *x-c-glb-iff*: $(T::('a,'b) mrel) \cdot 1_\pi \subseteq (R::('a,'b) mrel) \cdot 1_\pi \wedge T \cdot 1_\pi \subseteq (S::('a,'b) mrel) \cdot 1_\pi \iff T \cdot 1_\pi \subseteq (R \cdot 1_\pi) \parallel (S \cdot 1_\pi)$
by (*meson subset-trans x-c-glb x-c-lb1 x-c-lb2*)

lemma *nc-iff1*: $R \subseteq NC \iff R \cap 1_\pi = \{\}$
by (*metis (no-types, lifting) Diff-Diff-Int Int-Diff Int-absorb diff-shunt-var p-id-NC*)

lemma *nc-iff2*: $R \subseteq NC \iff R \cdot \{\} = \{\}$

by (*metis c4 nc-iff1 p-id-zero s-prod-zerol*)

lemma *zero-assoc3*: $(R \cdot S) \cdot \{\} = R \cdot (S \cdot \{\})$
by (*metis cl5 s-prod-zerol*)

lemma *x-zero-interr*: $R \cdot \{\} \parallel S \cdot \{\} = (R \parallel S) \cdot \{\}$
by (*clarsimp simp: mr-simp*) *blast*

lemma *p-subid-interr*: $R \cdot T \cdot 1_\pi \parallel S \cdot T \cdot 1_\pi = (R \parallel S) \cdot T \cdot 1_\pi$

proof –

have $R \cdot T \cdot 1_\pi \parallel S \cdot T \cdot 1_\pi = (R \cdot \{(a, \{\}) \mid a. \exists B. (a, B) \in T\}) \parallel (S \cdot \{(a, \{\}) \mid a. \exists B. (a, B) \in T\})$

by (*metis c4 p-id-st*)

also have $\dots = (R \parallel S) \cdot \{(a, \{\}) \mid a. \exists B. (a, B) \in T\}$

apply (*rule antisym*)

apply (*metis cl4 dual-order.refl p-id-st x-c-par-idem*)

by (*simp add: seq-conc-subdistr*)

also have $\dots = (R \parallel S) \cdot T \cdot 1_\pi$

by (*metis c4 p-id-st*)

finally show *?thesis*.

qed

lemma *cl2 [simp]*: $1_\pi \cap (R \cup NC) = R \cdot \{\}$

by (*metis Diff-disjoint Int-Un-distrib inf-commute p-id-zero sup-bot.right-neutral*)

lemma *cl6 [simp]*: $R \cdot \{\} \cdot S = R \cdot \{\}$

by (*metis c4 p-id-assoc2 s-prod-p-idl s-prod-zerol*)

lemma *cl11 [simp]*: $(R \cap NC) \cdot 1_\pi \parallel NC = (R \cap NC) \cdot NC$

apply (*rule antisym*)

apply (*clarsimp simp: mr-simp*)

apply (*metis UN-constant*)

apply (*clarsimp simp: mr-simp*)

by (*metis UN-empty2 UN-insert Un-empty-left equalsOI insert-absorb sup-bot-right*)

lemma *x-split [simp]*: $(R \cap NC) \cup (R \cap 1_\pi) = R$

by (*metis Un-Diff-Int p-id-NC*)

lemma *x-split-var [simp]*: $(R \cap NC) \cup R \cdot \{\} = R$

by (*metis p-id-zero x-split*)

lemma *s-x-c [simp]*: $1_\sigma \cap R \cdot 1_\pi = \{\}$

using *c6 s-le-nc* by *fastforce*

lemma *s-x-zero [simp]*: $1_\sigma \cap R \cdot \{\} = \{\}$

using *cl6 s-x-c* by *blast*

lemma *c-nc* [*simp*]: $R \cdot 1_\pi \cap NC = \{\}$
using *c6* **by** *blast*

lemma *zero-nc* [*simp*]: $R \cdot \{\} \cap NC = \{\}$
using *x-zero-le-c* **by** *fastforce*

lemma *nc-zero* [*simp*]: $(R \cap NC) \cdot \{\} = \{\}$
using *nc-iff2* **by** *auto*

lemma *c-def* [*simp*]: $U \cdot \{\} = 1_\pi$
by (*metis c-nc-comp1 cl2 cl6 inf-commute p-id-zero s-prod-p-idl*)

lemma *U-c* [*simp*]: $U \cdot 1_\pi = 1_\pi$
by (*metis U-U c-def zero-assoc3*)

lemma *nc-c* [*simp*]: $NC \cdot 1_\pi = 1_\pi$
by (*auto simp: mr-simp*)

lemma *nc-U* [*simp*]: $NC \cdot U = U$
using *NC-NC c-nc-comp1 cl1 nc-c* **by** *blast*

lemma *x-c-nc-split* [*simp*]: $((R \cap NC) \cdot NC) \cup (R \cdot \{\} \parallel NC) = (R \cdot 1_\pi) \parallel NC$
by (*metis cl11 p-prod-comm p-prod-distl x-y-split*)

lemma *x-c-U-split* [*simp*]: $R \cdot U \cup (R \cdot \{\} \parallel U) = R \cdot 1_\pi \parallel U$
apply (*rule set-eqI, clarsimp simp: mr-simp*)
by (*metis SUP-constant UN-extend-simps(2)*)

lemma *p-subid-par-eq-meet* [*simp*]: $R \cdot \{\} \parallel S \cdot \{\} = R \cdot \{\} \cap S \cdot \{\}$
by (*auto simp: mr-simp*)

lemma *p-subid-par-eq-meet-var* [*simp*]: $R \cdot 1_\pi \parallel S \cdot 1_\pi = R \cdot 1_\pi \cap S \cdot 1_\pi$
by (*metis c-def p-subid-par-eq-meet zero-assoc3*)

lemma *x-zero-add-closed*: $R \cdot \{\} \cup S \cdot \{\} = (R \cup S) \cdot \{\}$
by (*simp add: s-prod-distr*)

lemma *x-zero-meet-closed*: $R \cdot \{\} \cap S \cdot \{\} = (R \cap S) \cdot \{\}$
by (*force simp: mr-simp*)

lemma *scomp-univalent-pres*: *univalent* $R \implies$ *univalent* $S \implies$ *univalent* $(R \cdot S)$
unfolding *univalent-set s-prod-def*
apply *clarsimp*
by (*metis Sup.SUP-cong*)

lemma *univalent s-id*
unfolding *univalent-set s-id-def* **by** *simp*

lemma *det-peleg*: *deterministic* $R \implies$ *deterministic* $S \implies$ *deterministic* $(R \cdot S)$

unfolding *deterministic-set s-prod-def*
apply *clarsimp*
apply *safe*
apply *metis*
apply (*metis UN-I*)
by (*metis UN-I*)

lemma *deterministic-sid: deterministic 1_σ*
unfolding *deterministic-set s-id-def* **by** *simp*

3.4 Domain

definition *Dom :: ('a,'b) mrel ⇒ ('a,'a) mrel* **where**
 $Dom R = \{(a, \{a\}) \mid a. \exists B. (a, B) \in R\}$

named-theorems *mrd-simp*
declare *mr-simp* [*mrd-simp*] *Dom-def* [*mrd-simp*]

lemma *d-def-expl: Dom R = R · 1_π || 1_σ*
by (*force simp: mrd-simp set-eqI*)

lemma *s-subid-iff2: (R ∩ 1_σ = R) = (Dom R = R)*
by (*metis c6 cl9 d-def-expl inf.order-iff p-prod-comm p-prod-ild p-prod-isor*)

lemma *cl8-var: Dom R · S = R · 1_π || S*
apply (*rule antisym*)
apply (*metis d-def-expl p-id-assoc2 s-prod-idl s-prod-p-idl seq-conc-subdistr*)
by (*force simp: mrd-simp*)

lemma *cl8* [*simp*]: $R · 1_π || 1_σ · S = R · 1_π || S$
by *simp*

lemma *cl10-var: Dom (R - 1_π) = 1_σ ∩ ((R - 1_π) · NC)*
apply (*rule set-eqI, clarsimp simp: mrd-simp*)
apply *safe*
apply (*metis SUP-constant insert-not-empty*)
by *blast*

lemma *cl10: (R ∩ NC) · 1_π || 1_σ = 1_σ ∩ ((R ∩ NC) · NC)*
by (*metis Int-Diff cl10-var d-def-expl*)

lemma *cl9-var* [*simp*]: $Dom (R ∩ 1_σ) = R ∩ 1_σ$
by (*simp add: d-def-expl*)

lemma *d-s-id* [*simp*]: $Dom R ∩ 1_σ = Dom R$
by (*metis cl8-var d-def-expl p-prod-comm p-prod-ild s-subid-iff2*)

lemma *d-s-id-ax: Dom R ⊆ 1_σ*
by (*simp add: le-iff-inf*)

lemma *d-assoc1*: $Dom R \cdot (S \cdot T) = (Dom R \cdot S) \cdot T$
by (*metis d-s-id test-assoc2*)

lemma *d-meet-distr-var*: $(Dom R \cap Dom S) \cdot T = Dom R \cdot T \cap Dom S \cdot T$
by (*metis (no-types, lifting) d-s-id inf-assoc test-multiplicativer*)

lemma *d-idem* [*simp*]: $Dom (Dom R) = Dom R$
by (*meson d-s-id s-subid-iff2*)

lemma *cd-2-var*: $Dom (R \cdot 1_\pi) \cdot S = R \cdot 1_\pi \parallel S$
by (*simp add: cl8-var p-id-assoc2*)

lemma *dc-prop1* [*simp*]: $Dom R \cdot 1_\pi = R \cdot 1_\pi$
by (*simp add: cl8-var*)

lemma *dc-prop2* [*simp*]: $Dom (R \cdot 1_\pi) = Dom R$
by (*simp add: d-def-expl p-id-assoc2*)

lemma *ds-prop* [*simp*]: $Dom R \parallel 1_\sigma = Dom R$
by (*simp add: p-prod-assoc d-def-expl*)

lemma *dc* [*simp*]: $Dom 1_\pi = 1_\sigma$
by (*simp add: d-def-expl*)

lemma *cd-iso* [*simp*]: $Dom (R \cdot 1_\pi) \cdot 1_\pi = R \cdot 1_\pi$
by *simp*

lemma *dc-iso* [*simp*]: $Dom (Dom R \cdot 1_\pi) = Dom R$
by *simp*

lemma *d-s-id-inter* [*simp*]: $Dom R \cdot Dom S = Dom R \cap Dom S$
by (*metis d-s-id inf-assoc test-s-prod-is-meet*)

lemma *d-conc6*: $Dom (R \parallel S) = Dom R \parallel Dom S$
by (*metis (no-types, lifting) c3 d-def-expl ds-prop p-prod-assoc p-prod-comm*)

lemma *d-conc-inter* [*simp*]: $Dom R \parallel Dom S = Dom R \cap Dom S$
by (*metis d-s-id test-p-prod-is-meet*)

lemma *d-conc-s-prod-ax*: $Dom R \parallel Dom S = Dom R \cdot Dom S$
by *simp*

lemma *d-rest-ax* [*simp*]: $Dom R \cdot R = R$
by (*simp add: cl8-var*)

lemma *d-loc-ax* [*simp*]: $Dom (R \cdot Dom S) = Dom (R \cdot S)$
by (*metis c4 dc-prop1 dc-prop2*)

lemma *assoc-p-subid*: $(R \cdot S) \cdot (T \cdot 1_\pi) = R \cdot (S \cdot (T \cdot 1_\pi))$
by (*smt* (*verit*, *del-insts*) *c4 cd-iso d-idem d-loc-ax p-id-idem s-subid-iff2 test-assoc3*)

lemma *d-exp-ax* [*simp*]: $\text{Dom } (Dom R \cdot S) = Dom R \cdot Dom S$
by (*metis d-conc6 d-conc-s-prod-ax d-idem d-loc-ax*)

lemma *d-comm-ax*: $Dom R \cdot Dom S = Dom S \cdot Dom R$
by *force*

lemma *d-s-id-prop* [*simp*]: $Dom 1_\sigma = 1_\sigma$
by (*simp add: d-def-expl*)

lemma *d-s-prod-closed* [*simp*]: $Dom (Dom R \cdot Dom S) = Dom R \cdot Dom S$
using *d-exp-ax d-loc-ax* **by** *blast*

lemma *d-p-prod-closed* [*simp*]: $Dom (Dom R \parallel Dom S) = Dom R \parallel Dom S$
using *d-s-prod-closed* **by** *auto*

lemma *d-idem2* [*simp*]: $Dom R \cdot Dom R = Dom R$
by (*metis d-exp-ax d-rest-ax*)

lemma *d-assoc*: $(Dom R \cdot Dom S) \cdot Dom T = Dom R \cdot (Dom S \cdot Dom T)$
using *d-assoc1* **by** *blast*

lemma *iso-1* [*simp*]: $Dom R \cdot 1_\pi \parallel 1_\sigma = Dom R$
using *d-def-expl* **by** *force*

lemma *d-idem-par* [*simp*]: $Dom R \parallel Dom R = Dom R$
by (*simp add: d-conc-s-prod-ax*)

lemma *d-inter-r*: $Dom R \cdot (S \parallel T) = Dom R \cdot S \parallel Dom R \cdot T$
by (*metis d-s-id seq-conc-subdistrl*)

lemma *d-add-ax*: $Dom (R \cup S) = Dom R \cup Dom S$
by (*simp add: d-def-expl p-prod-comm p-prod-distl s-prod-distr*)

lemma *d-sup-add*: $Dom (\bigcup X) = (\bigcup R \in X. Dom R)$
by (*auto simp: mrd-simp*)

lemma *d-distl*: $Dom R \cdot (S \cup T) = Dom R \cdot S \cup Dom R \cdot T$
by (*metis d-s-id s-distl-test*)

lemma *d-sup-distl*: $Dom R \cdot \bigcup X = (\bigcup S \in X. Dom R \cdot S)$
by (*metis d-s-id s-distl-sup-test*)

lemma *d-zero-ax* [*simp*]: $Dom \{\} = \{\}$
by (*simp add: d-def-expl p-prod-comm*)

lemma *d-absorb1* [*simp*]: $\text{Dom } R \cup \text{Dom } R \cdot \text{Dom } S = \text{Dom } R$
by *simp*

lemma *d-absorb2* [*simp*]: $\text{Dom } R \cdot (\text{Dom } R \cup \text{Dom } S) = \text{Dom } R$
by (*metis d-absorb1 d-idem2 d-s-id s-distl-test*)

lemma *d-dist1*: $\text{Dom } R \cdot (\text{Dom } S \cup \text{Dom } T) = \text{Dom } R \cdot \text{Dom } S \cup \text{Dom } R \cdot \text{Dom } T$
by (*simp add: cl8-var p-prod-distl*)

lemma *d-dist2*: $\text{Dom } R \cup (\text{Dom } S \cdot \text{Dom } T) = (\text{Dom } R \cup \text{Dom } S) \cdot (\text{Dom } R \cup \text{Dom } T)$
by (*smt (verit) boolean-algebra.disj-conj-distrib d-add-ax d-s-id-inter dc-prop2*)

lemma *d-add-prod-closed* [*simp*]: $\text{Dom } (\text{Dom } R \cup \text{Dom } S) = \text{Dom } R \cup \text{Dom } S$
by (*simp add: d-add-ax*)

lemma *x-zero-prop*: $R \cdot \{\} \parallel S = \text{Dom } (R \cdot \{\}) \cdot S$
by (*simp add: cl8-var*)

lemma *cda-add-ax*: $\text{Dom } ((R \cup S) \cdot T) = \text{Dom } (R \cdot T) \cup \text{Dom } (S \cdot T)$
by (*simp add: d-add-ax s-prod-distr*)

lemma *d-x-zero*: $\text{Dom } (R \cdot \{\}) = R \cdot \{\} \parallel 1_\sigma$
by (*simp add: d-def-expl*)

lemma *cda-ax2*:
assumes $(R \parallel S) \cdot \text{Dom } T = R \cdot \text{Dom } T \parallel S \cdot \text{Dom } T$
shows $\text{Dom } ((R \parallel S) \cdot T) = \text{Dom } (R \cdot T) \cdot \text{Dom } (S \cdot T)$
by (*metis assms d-conc6 d-conc-s-prod-ax d-loc-ax*)

lemma *d-lb1*: $\text{Dom } R \cdot \text{Dom } S \subseteq \text{Dom } R$
using *d-absorb1* **by** *blast*

lemma *d-lb2*: $\text{Dom } R \cdot \text{Dom } S \subseteq \text{Dom } S$
using *d-comm-ax d-lb1* **by** *blast*

lemma *d-glb*: $\text{Dom } T \subseteq \text{Dom } R \wedge \text{Dom } T \subseteq \text{Dom } S \implies \text{Dom } T \subseteq \text{Dom } R \cdot \text{Dom } S$
by *simp*

lemma *d-glb-iff*: $\text{Dom } T \subseteq \text{Dom } R \wedge \text{Dom } T \subseteq \text{Dom } S \iff \text{Dom } T \subseteq \text{Dom } R \cdot \text{Dom } S$
by *force*

lemma *d-interr*: $R \cdot \text{Dom } P \parallel S \cdot \text{Dom } P = (R \parallel S) \cdot \text{Dom } P$
by (*simp add: cl4 seq-conc-subdistr subset-antisym*)

lemma *cl10-d*: $\text{Dom } (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot NC$

by (*simp add: c10 d-def-expl*)

lemma *cl11-d* [*simp*]: $\text{Dom } (R \cap NC) \cdot NC = (R \cap NC) \cdot NC$
by (*simp add: cl8-var*)

lemma *cl10-d-var1*: $\text{Dom } (R \cap NC) = 1_\sigma \cap R \cdot NC$
by (*simp add: cl10-d x-y-prop*)

lemma *cl10-d-var2*: $\text{Dom } (R \cap NC) = 1_\sigma \cap (R \cap NC) \cdot U$
by (*simp add: cl10-d-var1 s-nc-U x-y-prop*)

lemma *cl10-d-var3*: $\text{Dom } (R \cap NC) = 1_\sigma \cap R \cdot U$
by (*simp add: cl10-d-var1 s-nc-U*)

lemma *d-U* [*simp*]: $\text{Dom } U = 1_\sigma$
by (*metis U-c dc dc-prop2*)

lemma *d-nc* [*simp*]: $\text{Dom } NC = 1_\sigma$
by (*metis dc dc-prop2 nc-c*)

lemma *alt-d-def-nc-nc*: $\text{Dom } (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel NC)$
using *c10 cl11-d cl8-var d-def-expl* **by** *blast*

lemma *alt-d-def-nc-U*: $\text{Dom } (R \cap NC) = 1_\sigma \cap (((R \cap NC) \cdot 1_\pi) \parallel U)$
using *alt-d-def-nc-nc s-nc-par-U* **by** *blast*

lemma *d-def-split* [*simp*]: $\text{Dom } (R \cap NC) \cup \text{Dom } (R \cdot \{\}) = \text{Dom } R$
by (*metis d-add-ax d-loc-ax d-zero-ax p-id-zero x-split*)

lemma *d-def-split-var* [*simp*]: $\text{Dom } (R \cap NC) \cup ((R \cdot \{\}) \parallel 1_\sigma) = \text{Dom } R$
using *d-def-split d-x-zero* **by** *blast*

lemma *ax7* [*simp*]: $(1_\sigma \cap R \cdot U) \cup (R \cdot \{\}) \parallel 1_\sigma = \text{Dom } R$
using *cl10-d-var3 d-def-split d-x-zero* **by** *blast*

lemma *dom12-d*: $\text{Dom } R = 1_\sigma \cap (R \cdot 1_\pi \parallel NC)$
by (*metis cl10-d-var3 cl8-var d-idem d-s-id inf.orderE s-nc-par-U sub-id-le-nc*)

lemma *dom12-d-U*: $\text{Dom } R = 1_\sigma \cap (R \cdot 1_\pi \parallel U)$
by (*simp add: dom12-d s-nc-par-U*)

lemma *dom-def-var*: $\text{Dom } R = (R \cdot U \cap 1_\pi) \parallel 1_\sigma$
by (*simp add: d-def-expl p-id-zero zero-assoc3*)

lemma *ax5-d* [*simp*]: $\text{Dom } (R \cap NC) \cdot U = (R \cap NC) \cdot U$
by (*metis cl1 cl11-d dc-prop1*)

lemma *ax5-0* [*simp*]: $\text{Dom } (R \cdot \{\}) \cdot U = R \cdot \{\} \parallel U$
using *x-zero-prop* **by** *auto*

lemma *x-c-U-split2*: $Dom R \cdot NC = (R \cap NC) \cdot NC \cup (R \cdot \{\} \parallel NC)$
by (*simp add: cl8-var*)

lemma *x-c-U-split3*: $Dom R \cdot U = (R \cap NC) \cdot U \cup (R \cdot \{\} \parallel U)$
by (*metis ax5-0 ax5-d d-def-split s-prod-distr*)

lemma *x-c-U-split-d*: $Dom R \cdot U = R \cdot U \cup (R \cdot \{\} \parallel U)$
by (*simp add: cl8-var*)

lemma *x-U-prop2*: $R \cdot NC = Dom (R \cap NC) \cdot NC \cup R \cdot \{\}$
by *simp*

lemma *x-U-prop3*: $R \cdot U = Dom (R \cap NC) \cdot U \cup R \cdot \{\}$
by (*metis ax5-d x-y-split*)

lemma *d-x-nc* [*simp*]: $Dom (R \cdot NC) = Dom R$
by (*metis d-loc-ax d-nc dc dc-prop2*)

lemma *d-x-U* [*simp*]: $Dom (R \cdot U) = Dom R$
by (*metis d-U d-loc-ax dc dc-prop2*)

lemma *d-llp1*: $Dom R \subseteq Dom S \implies R \subseteq Dom S \cdot R$
by (*metis d-rest-ax s-prod-isol*)

lemma *d-llp2*: $R \subseteq Dom S \cdot R \implies Dom R \subseteq Dom S$
by (*metis d-assoc1 d-exp-ax d-meet-distr-var d-rest-ax d-s-id-inter inf.absorb-iff2*)

lemma *demod1*: $Dom (R \cdot S) \subseteq Dom T \implies R \cdot Dom S \subseteq Dom T \cdot R$

proof –

assume *h*: $Dom (R \cdot S) \subseteq Dom T$

have $R \cdot Dom S = Dom (R \cdot Dom S) \cdot (R \cdot Dom S)$

using *d-rest-ax* **by** *blast*

also have $\dots \subseteq Dom T \cdot (R \cdot Dom S)$

by (*metis d-loc-ax h s-prod-distr subset-Un-eq*)

also have $\dots \subseteq Dom T \cdot R$

by (*metis d-s-id-ax s-prod-idr s-prod-isol*)

finally show $R \cdot Dom S \subseteq Dom T \cdot R$.

qed

lemma *demod2*: $R \cdot Dom S \subseteq Dom T \cdot R \implies Dom (R \cdot S) \subseteq Dom T$

proof –

assume *h*: $R \cdot Dom S \subseteq Dom T \cdot R$

have $Dom (R \cdot S) = Dom (R \cdot Dom S)$

by *auto*

also have $\dots \subseteq Dom (Dom T \cdot R)$

by (*metis d-add-ax h subset-Un-eq*)

also have $\dots = Dom T \cdot Dom R$

by *simp*
 also have $\dots \subseteq \text{Dom } T$
 by (*simp add: d-lb1*)
 finally show $\text{Dom } (R \cdot S) \subseteq \text{Dom } T$.
 qed

lemma *d-meet-closed* [*simp*]: $\text{Dom } (\text{Dom } x \cap \text{Dom } y) = \text{Dom } x \cap \text{Dom } y$
 by (*metis d-s-id inf-assoc s-subid-iff2*)

lemma *d-add-var*: $\text{Dom } P \cdot R \cup \text{Dom } Q \cdot R = \text{Dom } (P \cup Q) \cdot R$
 by (*simp add: d-add-ax s-prod-distr*)

lemma *d-interr-U*: $\text{Dom } x \cdot U \parallel \text{Dom } y \cdot U = \text{Dom } (x \parallel y) \cdot U$
 by (*metis U-nc U-par-idem cl4 d-conc6 seq-conc-subdistr subset-antisym*)

lemma *d-meet*: $\text{Dom } x \cdot z \cap \text{Dom } y \cdot z = (\text{Dom } x \cap \text{Dom } y) \cdot z$
 by (*simp add: d-meet-distr-var*)

lemma *cs-hom-meet*: $\text{Dom } (x \cdot 1_\pi \cap y \cdot 1_\pi) = \text{Dom } (x \cdot 1_\pi) \cap \text{Dom } (y \cdot 1_\pi)$
 by (*metis d-conc6 d-conc-inter dc-prop2 p-subid-par-eq-meet-var*)

lemma *iso3* [*simp*]: $\text{Dom } (\text{Dom } x \cdot U) = \text{Dom } x$
 by *simp*

lemma *iso4* [*simp*]: $\text{Dom } (x \cdot 1_\pi \parallel U) \cdot U = x \cdot 1_\pi \parallel U$
 by (*metis cl8-var iso3*)

lemma *iso3-sharp* [*simp*]: $\text{Dom } (\text{Dom } (x \cap NC) \cdot NC) = \text{Dom } (x \cap NC)$
 by *simp*

lemma *iso4-sharp* [*simp*]: $\text{Dom } ((x \cap NC) \cdot NC) \cdot NC = (x \cap NC) \cdot NC$
 by *simp*

3.5 Vectors

lemma *vec-iff1*:
 assumes $\forall a. (\exists A. (a, A) \in R) \longrightarrow (\forall A. (a, A) \in R)$
 shows $R \cdot 1_\pi \parallel U = R$
 using *assms* by (*auto simp: mr-simp*)

lemma *vec-iff2*:
 assumes $R \cdot 1_\pi \parallel U = R$
 shows $(\forall a. (\exists A. (a, A) \in R) \longrightarrow (\forall A. (a, A) \in R))$
 using *assms* apply (*clarsimp simp: mr-simp*)
 by (*smt (z3) SUP-bot case-prod-conv mem-Collect-eq sup-bot-left*)

lemma *vec-iff*: $(\forall a. (\exists A. (a, A) \in R) \longrightarrow (\forall A. (a, A) \in R)) \longleftrightarrow R \cdot 1_\pi \parallel U = R$
 by (*metis vec-iff1 vec-iff2*)

lemma *U-par-zero* [*simp*]: $\{\} \cdot R \parallel U = \{\}$
by (*simp add: p-prod-comm*)

lemma *U-par-s-id* [*simp*]: $1_\sigma \cdot 1_\pi \parallel U = U$
by *auto*

lemma *U-par-p-id* [*simp*]: $1_\pi \cdot 1_\pi \parallel U = U$
by *auto*

lemma *U-par-nc* [*simp*]: $NC \cdot 1_\pi \parallel U = U$
by *auto*

3.6 Up-closure and Parikh composition

definition *s-prod-pa* :: $('a, 'b) \text{ mrel} \Rightarrow ('b, 'c) \text{ mrel} \Rightarrow ('a, 'c) \text{ mrel}$ (**infixl** $\langle \otimes \rangle$ 75)

where

$$R \otimes S = \{(a, A). (\exists B. (a, B) \in R \wedge (\forall b \in B. (b, A) \in S))\}$$

lemma *U-par-st*: $(a, A) \in R \parallel U \longleftrightarrow (\exists B. B \subseteq A \wedge (a, B) \in R)$
by (*auto simp: mr-simp*)

lemma *p-id-U*: $R \parallel U = \{(a, B). \exists A. (a, A) \in R \wedge A \subseteq B\}$
by (*clarsimp simp: mr-simp*) *blast*

lemma *ucl-iff*: $(\forall a A B. (a, A) \in R \wedge A \subseteq B \longrightarrow (a, B) \in R) \longleftrightarrow R \parallel U = R$
by (*clarsimp simp: mr-simp*) *blast*

lemma *upclosed-ext*: $R \subseteq R \parallel U$
by (*clarsimp simp: mr-simp*) *blast*

lemma *onelem*: $R \cdot S \parallel U \subseteq R \otimes (S \parallel U)$
by (*auto simp: s-prod-def p-prod-def U-def s-prod-pa-def*)

lemma *twolem*: $R \otimes (S \parallel U) \subseteq R \cdot S \parallel U$

proof *clarify*

fix $a A$

assume $(a, A) \in R \otimes (S \parallel U)$

hence $\exists B. (a, B) \in R \wedge (\forall b \in B. (b, A) \in S \parallel U)$

by (*auto simp: s-prod-pa-def*)

hence $\exists B. (a, B) \in R \wedge (\forall b \in B. \exists C. C \subseteq A \wedge (b, C) \in S)$

by (*clarsimp simp: mr-simp*) *blast*

hence $\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. f b \subseteq A \wedge (b, f b) \in S))$

by *metis*

hence $\exists C. C \subseteq A \wedge (\exists B. (a, B) \in R \wedge (\exists f. (\forall b \in B. (b, f b) \in S) \wedge C = \bigcup ((\lambda x. f x) \text{ ` } B)))$

by *clarsimp* *blast*

hence $\exists C. C \subseteq A \wedge (a, C) \in R \cdot S$

by (*clarsimp simp: mr-simp*)

thus $(a, A) \in (R \cdot S) \parallel U$

by (*clarsimp simp: mr-simp*) *blast*
qed

lemma *pe-pa-sim*: $R \cdot S \parallel U = R \otimes (S \parallel U)$
by (*metis antisym onelem twolem*)

lemma *pe-pa-sim-var*: $(R \parallel U) \cdot (S \parallel U) \parallel U = (R \parallel U) \otimes (S \parallel U)$
apply (*rule order.antisym*)
by (*simp add: p-prod-assoc pe-pa-sim*)+

lemma *pa-assoc1*: $((R \parallel U) \otimes (S \parallel U)) \otimes (T \parallel U) \subseteq (R \parallel U) \otimes ((S \parallel U) \otimes (T \parallel U))$
by (*clarsimp simp: p-prod-def s-prod-pa-def U-def, metis*)

lemma *up-closed-par-is-meet*: $(R \parallel U) \parallel (S \parallel U) = (R \parallel U) \cap (S \parallel U)$
by (*auto simp: mr-simp*)

lemma *U-nc-par* [*simp*]: $NC \parallel U = NC$
by (*metis c-nc-comp1 c-prod-idr nc-par-idem p-prod-distl sup-idem*)

lemma *uc-par-meet*: $(R \parallel U) \cap (S \parallel U) = R \parallel U \parallel S \parallel U$
using *p-prod-assoc up-closed-par-is-meet* **by** *blast*

lemma *uc-unc* [*simp*]: $R \parallel U \parallel R \parallel U = R \parallel U$
using *uc-par-meet* **by** *auto*

lemma *uc-interr*: $(R \parallel S) \cdot (T \parallel U) = R \cdot (T \parallel U) \parallel S \cdot (T \parallel U)$
by (*simp add: Orderings.order-eq-iff cl4 seq-conc-subdistr up-closed-par-is-meet*)

lemma *iso5* [*simp*]: $(R \cdot 1_\pi \parallel U) \cdot 1_\pi = R \cdot 1_\pi$
by (*simp add: c3*)

lemma *iso6* [*simp*]: $(R \cdot 1_\pi \parallel U) \cdot 1_\pi \parallel U = R \cdot 1_\pi \parallel U$
by *simp*

lemma *sv-hom-par*: $(R \parallel S) \cdot U = R \cdot U \parallel S \cdot U$
by (*metis U-par-idem uc-interr*)

lemma *vs-hom-meet*: $Dom ((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) = Dom (R \cdot 1_\pi \parallel U) \cap Dom (S \cdot 1_\pi \parallel U)$
by (*metis d-conc6 d-conc-inter dc-prop2 iso5 up-closed-par-is-meet*)

lemma *cv-hom-meet*: $(R \cdot 1_\pi \cap S \cdot 1_\pi) \parallel U = (R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)$
by (*metis U-par-idem p-prod-assoc p-prod-comm p-subid-par-eq-meet-var uc-par-meet*)

lemma *cv-hom-par* [*simp*]: $R \parallel U \parallel S \parallel U = (R \parallel S) \parallel U$
by (*metis U-par-idem p-prod-assoc p-prod-comm*)

lemma *vc-hom-meet*: $((R \cdot 1_\pi \parallel U) \cap (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cap ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$

by (*metis c4 cl8-var cv-hom-meet iso5 p-subid-par-eq-meet-var*)

lemma *vc-hom-seq*: $((R \cdot 1_\pi \parallel U) \cdot (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = ((R \cdot 1_\pi \parallel U) \cdot 1_\pi) \cdot ((S \cdot 1_\pi \parallel U) \cdot 1_\pi)$

proof –

have $((R \cdot 1_\pi \parallel U) \cdot (S \cdot 1_\pi \parallel U)) \cdot 1_\pi = (R \cdot 1_\pi \parallel U) \cdot (S \cdot 1_\pi)$

by (*metis c4 iso5*)

also have $\dots = R \cdot 1_\pi \parallel U \cdot (S \cdot 1_\pi)$

by (*metis cl8-var d-assoc1*)

also have $\dots = R \cdot 1_\pi \parallel (NC \cdot (S \cdot 1_\pi) \cup 1_\pi \cdot (S \cdot 1_\pi))$

by (*metis c-nc-comp1 s-prod-distr sup-commute*)

also have $\dots = R \cdot 1_\pi \parallel 1_\pi$

by (*metis Un-absorb1 c4 c6 s-prod-p-idl*)

thus *?thesis*

by (*simp add: assoc-p-subid calculation*)

qed

3.7 Nonterminal and terminal multirelations

definition *tau* :: $(a, b) \text{ mrel} \Rightarrow (a, b) \text{ mrel } (\langle \tau \rangle)$ **where**

$\tau R = R \cdot \{\}$

definition *nu* :: $(a, b) \text{ mrel} \Rightarrow (a, b) \text{ mrel } (\langle \nu \rangle)$ **where**

$\nu R = R \cap NC$

lemma *nc-s [simp]*: $\nu 1_\sigma = 1_\sigma$

using *nu-def s-le-nc* **by** *auto*

lemma *nc-scomp-closed*: $\nu R \cdot \nu S \subseteq NC$

by (*simp add: nu-def nc-iff1 p-id-zero zero-assoc3*)

lemma *nc-scomp-closed-alt [simp]*: $\nu (\nu R \cdot \nu S) = \nu R \cdot \nu S$

by (*metis inf.orderE nc-scomp-closed nu-def*)

lemma *nc-ccomp-closed*: $\nu R \parallel \nu S \subseteq NC$

unfolding *nu-def* **by** (*clarsimp simp: mr-simp*)

lemma *nc-ccomp-closed-alt [simp]*: $\nu (R \parallel \nu S) = R \parallel \nu S$

unfolding *nu-def* **by** (*clarsimp simp: mr-simp*) *blast*

lemma *tarski-prod*: $(\nu R \cdot NC) \cdot (\nu S \cdot NC) = (\text{if } \nu S = \{\} \text{ then } \{\} \text{ else } \nu R \cdot NC)$

proof (*cases* $\nu S = \{\}$)

case *True*

show *?thesis*

by (*metis True nc-zero nu-def p-id-NC s-prod-zero zero-assoc3*)

next

case *False*
hence $a: NC \cdot (\nu S \cdot NC) = NC$
unfolding *nu-def* **by** (*metis p-id-NC tarski*)
have $(\nu R \cdot NC) \cdot (\nu S \cdot NC) = (Dom (\nu R) \cdot NC) \cdot (\nu S \cdot NC)$
by (*simp add: nu-def*)
also have $\dots = Dom (\nu R) \cdot (NC \cdot (\nu S \cdot NC))$
using *d-assoc1* **by** *blast*
also have $\dots = Dom (\nu R) \cdot NC$
by (*simp add: a*)
also have $\dots = \nu R \cdot NC$
by (*simp add: nu-def*)
finally have $(\nu R \cdot NC) \cdot (\nu S \cdot NC) = \nu R \cdot NC.$
thus *?thesis*
using *False* **by** *auto*
qed

lemma *nc-prod-aux* [*simp*]: $(\nu R \cdot NC) \cdot NC = \nu R \cdot NC$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *clarsimp*
apply (*smt (verit) SUP-bot-conv(1) ex-in-conv*)
apply *clarsimp*
by (*smt (verit, best) SUP-bot-conv(2) UNIV-I UN-constant*)

lemma *nc-vec-add-closed*: $(\nu R \cdot NC \cup \nu S \cdot NC) \cdot NC = \nu R \cdot NC \cup \nu S \cdot NC$
by (*simp add: s-prod-distr*)

lemma *nc-vec-par-is-meet*: $\nu R \cdot NC \parallel \nu S \cdot NC = \nu R \cdot NC \cap \nu S \cdot NC$
by (*metis (no-types, lifting) U-nc-par cl11 nu-def p-prod-assoc*
up-closed-par-is-meet)

lemma *nc-vec-meet-closed*: $(\nu R \cdot NC \cap \nu S \cdot NC) \cdot NC = \nu R \cdot NC \cap \nu S \cdot NC$
apply (*clarsimp simp: nu-def mr-simp*)
apply *safe*
apply (*metis SUP-const UN-I ex-in-conv*)
apply (*clarsimp, smt (verit) SUP-bot-conv(2) ex-in-conv*)
by (*clarsimp, smt (verit, ccfv-threshold) SUP-bot-conv(1) SUP-const UNIV-I*
all-not-in-conv)

lemma *nc-vec-par-closed*: $(\nu R \cdot NC \parallel \nu S \cdot NC) \cdot NC = \nu R \cdot NC \parallel \nu S \cdot NC$
by (*metis U-nc-par nc-prod-aux uc-interr*)

lemma *nc-vec-seq-closed*: $((\nu R \cdot NC) \cdot (\nu S \cdot NC)) \cdot NC = (\nu R \cdot NC) \cdot (\nu S \cdot NC)$

proof (*cases* $\nu S = \{\}$)
case *True* **thus** *?thesis*
by *simp*
next
case *False* **thus** *?thesis*

by (*simp add: tarski-prod*)
qed

lemma iso5-sharp [*simp*]: $(\nu R \cdot 1_\pi \parallel NC) \cdot 1_\pi = \nu R \cdot 1_\pi$
 by (*simp add: c3*)

lemma iso6-sharp [*simp*]: $(\nu R \cdot NC \cdot 1_\pi) \parallel NC = \nu R \cdot NC$
 by (*simp add: c4 nu-def*)

lemma nsv-hom-par: $(R \parallel S) \cdot NC = R \cdot NC \parallel S \cdot NC$
 by (*simp add: cl4 seq-conc-subdistr subset-antisym*)

lemma nvs-hom-meet: $Dom (\nu R \cdot NC \cap \nu S \cdot NC) = Dom (\nu R \cdot NC) \cap Dom (\nu S \cdot NC)$
 by (*rule antisym*) (*fastforce simp: nu-def mrd-simp*)+

lemma ncv-hom-meet: $R \cdot 1_\pi \cap S \cdot 1_\pi \parallel NC = (R \cdot 1_\pi \parallel NC) \cap (S \cdot 1_\pi \parallel NC)$
 by (*metis c4 cl8-var d-exp-ax d-meet d-s-id-inter p-subid-par-eq-meet-var*)

lemma ncv-hom-par: $(R \parallel S) \parallel NC = R \parallel NC \parallel S \parallel NC$
 by (*metis nc-par-idem p-prod-assoc p-prod-comm*)

lemma nvc-hom-meet: $(\nu R \cdot NC \cap \nu S \cdot NC) \cdot 1_\pi = (\nu R \cdot NC) \cdot 1_\pi \cap (\nu S \cdot NC) \cdot 1_\pi$
 by (*rule antisym*) (*fastforce simp: nu-def mr-simp*)+

lemma tau-int: $\tau R \leq R$
 using *p-id-zero tau-def* **by auto**

lemma nu-int: $\nu R \leq R$
 by (*simp add: nu-def*)

lemma tau-ret [*simp*]: $\tau (\tau R) = \tau R$
 by (*simp add: tau-def*)

lemma nu-ret [*simp*]: $\nu (\nu R) = \nu R$
 by (*simp add: nu-def*)

lemma tau-iso: $R \leq S \implies \tau R \leq \tau S$
 by (*simp add: inf.order-iff tau-def x-zero-meet-closed*)

lemma nu-iso: $R \leq S \implies \nu R \leq \nu S$
 by (*metis Int-mono nu-def order-refl*)

lemma tau-zero [*simp*]: $\tau \{\} = \{\}$
 by (*simp add: tau-def*)

lemma nu-zero [*simp*]: $\nu \{\} = \{\}$
 using *nu-def* **by auto**

lemma *tau-s* [*simp*]: $\tau 1_\sigma = \{\}$
by (*simp add: tau-def*)

lemma *tau-c* [*simp*]: $\tau 1_\pi = 1_\pi$
by (*simp add: tau-def*)

lemma *nu-c* [*simp*]: $\nu 1_\pi = \{\}$
by (*simp add: nu-def*)

lemma *tau-nc* [*simp*]: $\tau NC = \{\}$
by (*metis nc-iff2 order-refl tau-def*)

lemma *nu-nc* [*simp*]: $\nu NC = NC$
using *nu-def* **by** *auto*

lemma *tau-U* [*simp*]: $\tau U = 1_\pi$
by (*simp add: tau-def*)

lemma *nu-U* [*simp*]: $\nu U = NC$
by (*simp add: Diff-eq nu-def*)

lemma *tau-add* [*simp*]: $\tau (R \cup S) = \tau R \cup \tau S$
by (*simp add: tau-def x-zero-add-closed*)

lemma *nu-add* [*simp*]: $\nu (R \cup S) = \nu R \cup \nu S$
by (*simp add: Int-Un-distrib2 nu-def*)

lemma *tau-meet* [*simp*]: $\tau (R \cap S) = \tau R \cap \tau S$
by (*simp add: tau-def x-zero-meet-closed*)

lemma *nu-meet* [*simp*]: $\nu (R \cap S) = \nu R \cap \nu S$
by (*simp add: inf-commute inf-left-commute nu-def*)

lemma *tau-seq*: $\tau (R \cdot S) = \tau R \cup \nu R \cdot \tau S$
unfolding *nu-def tau-def*
by (*metis sup-commute x-y-split zero-assoc3*)

lemma *tau-par* [*simp*]: $\tau (R \parallel S) = \tau R \parallel \tau S$
by (*metis U-par-zero tau-def uc-interr*)

lemma *nu-par-aux1*: $R \parallel \tau S = \text{Dom} (\tau S) \cdot R$
by (*metis p-prod-comm tau-def x-zero-prop*)

lemma *nu-par-aux3* [*simp*]: $\nu (\nu R \parallel \tau S) = \nu R \parallel \tau S$
by (*metis nc-ccomp-closed-alt p-prod-comm*)

lemma *nu-par-aux4* [*simp*]: $\nu (\tau R \parallel \tau S) = \{\}$
by (*metis nu-def tau-def tau-par zero-nc*)

lemma *nu-par*: $\nu (R \parallel S) = \text{Dom } (\tau R) \cdot \nu S \cup \text{Dom } (\tau S) \cdot \nu R \cup (\nu R \parallel \nu S)$
apply (*rule antisym*)
apply (*fastforce simp: mrd-simp nu-def tau-def*)
by (*auto simp: mrd-simp nu-def tau-def*)

lemma *sprod-tau-nu*: $R \cdot S = \tau R \cup \nu R \cdot S$
by (*metis nu-def sup-commute tau-def x-y-split*)

lemma *pprod-tau-nu*: $R \parallel S = (\nu R \parallel \nu S) \cup \text{Dom } (\tau R) \cdot \nu S \cup \text{Dom } (\tau S) \cdot \nu R \cup (\tau R \parallel \tau S)$
by (*smt (verit) nu-def nu-par sup-assoc sup-left-commute tau-def tau-par x-split-var*)

lemma *tau-idem* [*simp*]: $\tau R \cdot \tau R = \tau R$
by (*simp add: tau-def*)

lemma *tau-interr*: $(R \parallel S) \cdot \tau T = R \cdot \tau T \parallel S \cdot \tau T$
by (*simp add: tau-def cl4 seq-conc-subdistr subset-antisym*)

lemma *tau-le-c*: $\tau R \leq 1_\pi$
by (*simp add: tau-def x-zero-le-c*)

lemma *c-le-tauc*: $1_\pi \leq \tau 1_\pi$
by *simp*

lemma *x-alpha-tau* [*simp*]: $\nu R \cup \tau R = R$
by (*simp add: nu-def tau-def*)

lemma *alpha-tau-zero* [*simp*]: $\nu (\tau R) = \{\}$
by (*simp add: nu-def tau-def*)

lemma *tau-alpha-zero* [*simp*]: $\tau (\nu R) = \{\}$
by (*simp add: nu-def tau-def*)

lemma *sprod-tau-nu-var* [*simp*]: $\nu (\nu R \cdot S) = \nu (R \cdot S)$
by (*metis nu-add nu-def nu-ret x-y-split zero-nc*)

lemma *tau-s-prod* [*simp*]: $\tau (R \cdot S) = R \cdot \tau S$
by (*simp add: tau-def zero-assoc3*)

lemma *alpha-fp*: $\nu R = R \longleftrightarrow R \cdot \{\} = \{\}$
by (*metis inf.orderE nc-iff2 nc-zero nu-def*)

lemma *p-prod-tau-alpha*: $R \parallel S = (R \parallel \nu S) \cup (\nu R \parallel S) \cup (\tau R \parallel \tau S)$
by (*smt (verit) p-prod-comm p-prod-distl sup.idem sup-assoc sup-commute x-alpha-tau*)

lemma *p-prod-tau-alpha-var*: $R \parallel S = (R \parallel \nu S) \cup (\nu R \parallel S) \cup \tau (R \parallel S)$

using *p-prod-tau-alpha tau-par* **by** *blast*

lemma *alpha-par*: $\nu (R \parallel S) = (\nu R \parallel S) \cup (R \parallel \nu S)$
by (*metis alpha-tau-zero nc-ccomp-closed-alt nu-add p-prod-comm p-prod-tau-alpha sup-bot-right tau-par*)

lemma *alpha-tau [simp]*: $\nu (R \cdot \tau S) = \{\}$
by (*metis alpha-tau-zero tau-s-prod*)

lemma *nu-par-prop*: $\nu R = R \implies \nu (R \parallel S) = R \parallel S$
by (*metis nc-ccomp-closed-alt p-prod-comm*)

lemma *tau-seq-prop*: $\tau R = R \implies R \cdot S = R$
by (*metis cl6 tau-def*)

lemma *tau-seq-prop2*: $\tau R = R \implies \tau (R \cdot S) = R \cdot S$
by (*metis cl6 tau-def*)

lemma *d-nu*: $\nu (Dom R \cdot S) = Dom R \cdot \nu S$
by (*auto simp: nu-def mrd-simp*)

lemma *nu-ideal1*: $\nu R = R \implies S \leq R \implies \nu S = S$
unfolding *nu-def* **by** *blast*

lemma *tau-ideal1*: $\tau R = R \implies S \leq R \implies \tau S = S$
by (*metis dual-order.trans p-subid-iff-var tau-def*)

lemma *nu-ideal2*: $\nu R = R \implies \nu S = S \implies \nu (R \cup S) = R \cup S$
by *simp*

lemma *tau-ideal2*: $\tau R = R \implies \tau S = S \implies \tau (R \cup S) = R \cup S$
by *simp*

lemma *tau-add-precong*: $\tau R \leq \tau S \implies \tau (R \cup T) \leq \tau (S \cup T)$
by *auto*

lemma *tau-meet-precong*: $\tau R \leq \tau S \implies \tau (R \cap T) \leq \tau (S \cap T)$
by *force*

lemma *tau-par-precong*: $\tau R \leq \tau S \implies \tau (R \parallel T) \leq \tau (S \parallel T)$
by (*simp add: p-prod-isol*)

lemma *tau-seq-precongl*: $\tau R \leq \tau S \implies \tau (T \cdot R) \leq \tau (T \cdot S)$
by (*simp add: s-prod-isor*)

lemma *nu-add-precong*: $\nu R \leq \nu S \implies \nu (R \cup T) \leq \nu (S \cup T)$
by *auto*

lemma *nu-meet-precong*: $\nu R \leq \nu S \implies \nu (R \cap T) \leq \nu (S \cap T)$

by *force*

lemma *nu-seq-precongr*: $\nu R \leq \nu S \implies \nu (R \cdot T) \leq \nu (S \cdot T)$
by (*metis nu-iso s-prod-isol spro-d-tau-nu-var*)

definition

$tcg R S = (\tau R \leq \tau S \wedge \tau S \leq \tau R)$

definition

$ncg R S = (\nu R \leq \nu S \wedge \nu S \leq \nu R)$

lemma *tcg-refl*: $tcg R R$

by (*simp add: tcg-def*)

lemma *tcg-trans*: $tcg R S \implies tcg S T \implies tcg R T$

by (*meson subset-trans tcg-def*)

lemma *tcg-sym*: $tcg R S \implies tcg S R$

by (*simp add: tcg-def*)

lemma *ncg-refl*: $ncg R R$

using *ncg-def* **by** *blast*

lemma *ncg-trans*: $ncg R S \implies ncg S T \implies ncg R T$

by (*meson ncg-def order-trans*)

lemma *ncg-sym*: $ncg R S \implies ncg S R$

by (*simp add: ncg-def*)

lemma *tcg-alt*: $tcg R S = (\tau R = \tau S)$

using *tcg-def* **by** *auto*

lemma *ncg-alt*: $ncg R S = (\nu R = \nu S)$

by (*simp add: Orderings.order-eq-iff ncg-def*)

lemma *tcg-add*: $\tau R = \tau S \implies \tau (R \cup T) = \tau (S \cup T)$

by *simp*

lemma *tcg-meet*: $\tau R = \tau S \implies \tau (R \cap T) = \tau (S \cap T)$

by *simp*

lemma *tcg-par*: $\tau R = \tau S \implies \tau (R \parallel T) = \tau (S \parallel T)$

by *simp*

lemma *tcg-seq1*: $\tau R = \tau S \implies \tau (T \cdot R) = \tau (T \cdot S)$

by *simp*

lemma *ncg-add*: $\nu R = \nu S \implies \nu (R \cup T) = \nu (S \cup T)$

by *simp*

lemma *ncg-meet*: $\nu R = \nu S \implies \nu (R \cap T) = \nu (S \cap T)$
by *simp*

lemma *ncg-seqr*: $\nu R = \nu S \implies \nu (R \cdot T) = \nu (S \cdot T)$
by (*metis sprod-tau-nu-var*)

3.8 Powers

primrec *p-power* :: $('a, 'a) \text{ mrel} \Rightarrow \text{nat} \Rightarrow ('a, 'a) \text{ mrel}$ **where**
p-power $R \ 0 = 1_\sigma$ |
p-power $R \ (\text{Suc } n) = R \cdot \text{p-power } R \ n$

primrec *power-rd* :: $('a, 'a) \text{ mrel} \Rightarrow \text{nat} \Rightarrow ('a, 'a) \text{ mrel}$ **where**
power-rd $R \ 0 = \{\}$ |
power-rd $R \ (\text{Suc } n) = 1_\sigma \cup R \cdot \text{power-rd } R \ n$

primrec *power-sq* :: $('a, 'a) \text{ mrel} \Rightarrow \text{nat} \Rightarrow ('a, 'a) \text{ mrel}$ **where**
power-sq $R \ 0 = 1_\sigma$ |
power-sq $R \ (\text{Suc } n) = 1_\sigma \cup R \cdot \text{power-sq } R \ n$

lemma *power-rd-chain*: $\text{power-rd } R \ n \leq \text{power-rd } R \ (n + 1)$
apply (*induct n*)
apply *simp*
by (*smt (verit, best) Suc-eq-plus1 Un-subset-iff le-iff-sup power-rd.simps(2) s-prod-subdistl subsetI*)

lemma *power-sq-chain*: $\text{power-sq } R \ n \leq \text{power-sq } R \ (n + 1)$
apply (*induct n*)
apply *clarsimp*
by (*smt (verit, ccfv-SIG) UnCI Un-subset-iff add.commute le-iff-sup plus-1-eq-Suc power-sq.simps(2) s-prod-subdistl subsetI*)

lemma *pow-chain*: $\text{p-power } (1_\sigma \cup R) \ n \leq \text{p-power } (1_\sigma \cup R) \ (n + 1)$
apply (*induct n*)
apply *simp*
by (*simp add: s-prod-isor*)

lemma *pow-prop*: $\text{p-power } (1_\sigma \cup R) \ (n + 1) = 1_\sigma \cup R \cdot \text{p-power } (1_\sigma \cup R) \ n$
apply (*induct n*)
apply *simp*
by (*smt (verit, best) add.commute p-power.simps(2) plus-1-eq-Suc s-prod-distr s-prod-idl s-prod-subdistl subset-antisym sup.commute sup.left-commute sup.right-idem sup-ge1*)

lemma *power-rd-le-sq*: $\text{power-rd } R \ n \leq \text{power-sq } R \ n$
apply (*induct n*)
apply *simp*
by (*smt (verit, best) UnCI UnE le-iff-sup power-rd.simps(2) power-sq.simps(2)*)

s-prod-subdistl subsetI)

lemma *power-sq-le-rd*: $\text{power-sq } R \ n \leq \text{power-rd } R \ (\text{Suc } n)$
apply (*induct n*)
apply *simp*
by (*smt (verit, del-insts) UnCI UnE power-rd.simps(2) power-sq.simps(2) s-prod-subdistl subsetI sup.absorb-iff1*)

lemma *power-sq-power*: $\text{power-sq } R \ n = \text{p-power } (1_\sigma \cup R) \ n$
apply (*induct n*)
apply *simp*
using *pow-prop* **by** *auto*

3.9 Star

lemma *iso-prop*: $\text{mono } (\lambda X. S \cup R \cdot X)$
by (*rule monoI, (clarsimp simp: mr-simp), blast*)

lemma *gfp-lfp-prop*: $\text{gfp } (\lambda X. R \cdot X) \cup \text{lfp } (\lambda X. S \cup R \cdot X) \subseteq \text{gfp } (\lambda X. S \cup R \cdot X)$
by (*simp add: lfp-le-gfp gfp-mono iso-prop*)

definition *star* :: $(\text{'a}, \text{'a}) \text{ mrel} \Rightarrow (\text{'a}, \text{'a}) \text{ mrel}$ **where**
star $R = \text{lfp } (\lambda X. s\text{-id} \cup R \cdot X)$

lemma *star-unfold*: $1_\sigma \cup R \cdot \text{star } R \leq \text{star } R$
unfolding *star-def* **using** *iso-prop lfp-unfold* **by** *blast*

lemma *star-induct*: $1_\sigma \cup R \cdot S \leq S \implies \text{star } R \leq S$
unfolding *star-def* **by** (*meson lfp-lowerbound*)

lemma *star-refl*: $1_\sigma \leq \text{star } R$
using *star-unfold* **by** *auto*

lemma *star-unfold-part*: $R \cdot \text{star } R \leq \text{star } R$
using *star-unfold* **by** *auto*

lemma *star-ext-aux*: $R \leq R \cdot \text{star } R$
by (*metis s-prod-idr s-prod-isor star-refl*)

lemma *star-ext*: $R \leq \text{star } R$
using *star-ext-aux star-unfold-part* **by** *blast*

lemma *star-co-trans*: $\text{star } R \leq \text{star } R \cdot \text{star } R$
by (*metis s-prod-idr s-prod-isor star-refl*)

lemma *star-iso*: $R \leq S \implies \text{star } R \leq \text{star } S$
by (*metis (no-types, lifting) le-sup-iff s-prod-distr star-induct star-refl star-unfold-part subset-Un-eq*)

lemma *star-unfold-eq* [*simp*]: $1_\sigma \cup R \cdot \text{star } R = \text{star } R$

by (*metis iso-prop lfp-unfold star-def*)

lemma *nu-star1*:

assumes $\bigwedge(R::('a,'a) \text{ mrel}) (S::('a,'a) \text{ mrel}) (T::('a,'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$

shows $\text{star } (R::('a,'a) \text{ mrel}) \leq \text{star } (\nu R) \cdot (1_\sigma \cup \tau R)$

by (*smt (verit, ccfv-threshold) assms s-prod-distr s-prod-idl sprod-tau-nu star-induct star-unfold-eq subsetI sup-assoc*)

lemma *nu-star2*:

assumes $\bigwedge(R::('a,'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$

shows $\text{star } (\nu (R::('a,'a) \text{ mrel})) \cdot (1_\sigma \cup \tau R) \leq \text{star } R$

by (*smt (verit) assms le-sup-iff nu-int s-prod-isol s-prod-isor star-ext star-refl star-iso subset-trans tau-int*)

lemma *nu-star*:

assumes $\bigwedge(R::('a,'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$

and $\bigwedge(R::('a,'a) \text{ mrel}) (S::('a,'a) \text{ mrel}) (T::('a,'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$

shows $\text{star } (\nu (R::('a,'a) \text{ mrel})) \cdot (1_\sigma \cup \tau R) = \text{star } R$

using *assms nu-star1 nu-star2 by blast*

lemma *tau-star*: $\text{star } (\tau R) = 1_\sigma \cup \tau R$

by (*metis cl6 tau-def star-unfold-eq*)

lemma *tau-star-var*:

assumes $\bigwedge(R::('a,'a) \text{ mrel}) (S::('a,'a) \text{ mrel}) (T::('a,'a) \text{ mrel}). R \cdot (S \cdot T) = (R \cdot S) \cdot T$

and $\bigwedge(R::('a,'a) \text{ mrel}). \text{star } R \cdot \text{star } R \leq \text{star } R$

shows $\tau (\text{star } (R::('a,'a) \text{ mrel})) = \text{star } (\nu R) \cdot \tau R$

by (*metis (mono-tags, lifting) assms nu-star s-prod-distr s-prod-zeroI sup-bot-left tau-def tau-s*)

lemma *nu-star-sub*: $\text{star } (\nu R) \leq \nu (\text{star } R)$

proof –

have *a*: $1_\sigma \subseteq \text{star } R$

by (*simp add: star-refl*)

have *b*: $(R \cap NC) \cdot (\text{star } R \cap NC) \subseteq \text{star } R$

by (*metis nu-def nu-int s-prod-isol s-prod-isor star-unfold-part subset-trans*)

have *c*: $1_\sigma \subseteq NC$

by (*simp add: s-le-nc*)

have $(R \cap NC) \cdot (\text{star } R \cap NC) \subseteq NC$

by (*metis nc-scomp-closed nu-def*)

thus *?thesis*

by (*metis Un-least a b c le-infI nu-def star-induct*)

qed

lemma *nu-star-nu* [*simp*]: $\nu (\text{star } (\nu R)) = \text{star } (\nu R)$
using *nu-int nu-star-sub* **by** *fastforce*

lemma *nu-star-tau* [*simp*]: $\nu (\text{star } (\tau R)) = 1_\sigma$
using *tau-star* **by** (*metis alpha-tau-zero nu-add tau-s x-alpha-tau*)

lemma *tau-star-tau* [*simp*]: $\tau (\text{star } (\tau R)) = \tau R$
by (*simp add: tau-star*)

lemma *tau-star-nu* [*simp*]: $\tau (\text{star } (\nu R)) = \{\}$
using *alpha-fp tau-def nu-star-nu* **by** *metis*

lemma *d-star-unfold* [*simp*]: $\text{Dom } S \cup \text{Dom } (R \cdot \text{Dom } (\text{star } R \cdot S)) = \text{Dom } (\text{star } R \cdot S)$

proof –

have $\text{Dom } S \cup \text{Dom } (R \cdot \text{Dom } (\text{star } R \cdot S)) = \text{Dom } S \cup \text{Dom } (R \cdot (\text{star } R \cdot \text{Dom } S))$

by (*metis d-loc-ax*)

also have $\dots = \text{Dom } (1_\sigma \cdot \text{Dom } S \cup (R \cdot (\text{star } R \cdot \text{Dom } S)))$

by (*simp add: d-add-ax*)

also have $\dots = \text{Dom } (1_\sigma \cdot \text{Dom } S \cup (R \cdot \text{star } R) \cdot \text{Dom } S)$

by (*metis d-comm-ax d-s-id-inter d-s-id-prop s-prod-idl test-assoc3*)

moreover have $\dots = \text{Dom } ((1_\sigma \cup R \cdot \text{star } R) \cdot \text{Dom } S)$

using *s-prod-distr* **by** *metis*

ultimately show *?thesis*

by *simp*

qed

lemma *d-star-sim1*:

assumes $\bigwedge R S T. \text{Dom } (T::('a,'b) \text{ mrel}) \cup (R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel}) \leq S \implies \text{star } R \cdot \text{Dom } T \leq S$

shows $(R::('a,'a) \text{ mrel}) \cdot \text{Dom } (T::('a,'b) \text{ mrel}) \leq \text{Dom } T \cdot (S::('a,'a) \text{ mrel}) \implies \text{star } R \cdot \text{Dom } T \leq \text{Dom } T \cdot \text{star } S$

proof –

fix $R S::('a,'a) \text{ mrel}$ **and** $T::('a,'b) \text{ mrel}$

assume $a: R \cdot \text{Dom } T \leq \text{Dom } T \cdot S$

hence $(R \cdot \text{Dom } T) \cdot \text{star } S \leq (\text{Dom } T \cdot S) \cdot \text{star } S$

by (*simp add: s-prod-isol*)

hence $b: R \cdot (\text{Dom } T \cdot \text{star } S) \leq \text{Dom } T \cdot (S \cdot \text{star } S)$

by (*metis d-assoc1 d-s-id-ax inf.order-iff test-assoc1*)

hence $R \cdot (\text{Dom } T \cdot \text{star } S) \leq \text{Dom } T \cdot \text{star } S$

by (*meson order-trans s-prod-isor star-unfold-part*)

hence $\text{Dom } T \cup R \cdot (\text{Dom } T \cdot \text{star } S) \leq \text{Dom } T \cdot \text{star } S$

by (*metis le-supI s-prod-idr s-prod-isor star-refl*)

thus $\text{star } R \cdot \text{Dom } T \leq \text{Dom } T \cdot \text{star } S$

using *assms* **by** *presburger*

qed

lemma *d-star-induct*:

assumes $\bigwedge R S T. \text{Dom } (T::('a,'b) \text{ mrel}) \cup (R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel})$
 $\leq S \implies \text{star } R \cdot \text{Dom } T \leq S$
shows $\text{Dom } ((R::('a,'a) \text{ mrel}) \cdot (S::('a,'a) \text{ mrel})) \leq \text{Dom } S \implies \text{Dom } (\text{star } R \cdot S) \leq \text{Dom } S$
by (*metis assms d-star-sim1 dc-prop2 demod1 demod2*)

3.10 Omega

definition $\text{omega} :: ('a,'a) \text{ mrel} \Rightarrow ('a,'a) \text{ mrel}$ **where**
 $\text{omega } R \equiv \text{gfp } (\lambda X. R \cdot X)$

lemma *om-unfold*: $\text{omega } R \leq R \cdot \text{omega } R$
unfolding *omega-def*
by (*metis (no-types, lifting) gfp-least gfp-upperbound order-trans s-prod-isol*)

lemma *om-coinduct*: $S \leq R \cdot S \implies S \leq \text{omega } R$
unfolding *omega-def* **by** (*simp add: gfp-upperbound omega-def*)

lemma *om-unfold-eq* [*simp*]: $R \cdot \text{omega } R = \text{omega } R$
by (*rule antisym*) (*auto simp: om-coinduct om-unfold s-prod-isol*)

lemma *om-iso*: $R \leq S \implies \text{omega } R \leq \text{omega } S$
by (*metis om-coinduct s-prod-isol om-unfold-eq*)

lemma *zero-om* [*simp*]: $\text{omega } \{\} = \{\}$
using *om-unfold-eq s-prod-zero* **by** *blast*

lemma *s-id-om* [*simp*]: $\text{omega } 1_\sigma = U$
by (*simp add: U-def eq-iff om-coinduct*)

lemma *p-id-om* [*simp*]: $\text{omega } 1_\pi = 1_\pi$
using *om-unfold-eq s-prod-p-idl* **by** *blast*

lemma *nc-om* [*simp*]: $\text{omega } NC = U$
by (*metis dual-order.refl nc-U om-coinduct s-id-om s-prod-idl subset-antisym*)

lemma *U-om* [*simp*]: $\text{omega } U = U$
by (*metis U-U dual-order.refl om-coinduct s-id-om s-prod-idl subset-antisym*)

lemma *tau-om1*: $\tau R \leq \tau (\text{omega } R)$
by (*metis om-unfold-eq order-refl sup.boundedE tau-seq*)

lemma *tau-om2* [*simp*]: $\text{omega } (\tau R) = \tau R$
by (*metis cl6 om-unfold-eq tau-def*)

lemma *tau-om3*: $\text{omega } (\tau R) \leq \tau (\text{omega } R)$
by (*simp add: tau-om1*)

lemma *om-nu-tau*: $\text{omega } (\nu R) \cup \text{star } (\nu R) \cdot \tau R \leq \text{omega } R$

```

proof –
  have  $\text{omega } (\nu R) \cup \text{star } (\nu R) \cdot \tau R = \text{omega } (\nu R) \cup (1_\sigma \cup \nu R \cdot \text{star } (\nu R)) \cdot \tau R$ 
    by auto
  also have  $\dots = \text{omega } (\nu R) \cup \tau R \cup \nu R \cdot \text{star } (\nu R) \cdot \tau R$ 
    using s-prod-distr s-prod-idl by blast
  also have  $\dots = \tau R \cup \nu R \cdot \text{omega } (\nu R) \cup \nu R \cdot (\text{star } (\nu R) \cdot \tau R)$ 
    by (simp add: cl5 sup-commute tau-def)
  also have  $\dots \leq \tau R \cup \nu R \cdot (\text{omega } (\nu R) \cup \text{star } (\nu R) \cdot \tau R)$ 
    by (smt (verit) Un-subset-iff s-prod-isor sup.cobounded2 sup.coboundedI2 sup-commute)
  also have  $\dots = R \cdot (\text{omega } (\nu R) \cup \text{star } (\nu R) \cdot \tau R)$ 
    using sprod-tau-nu by blast
  finally show ?thesis
    using om-coinduct by blast
qed

end

```

4 Multirelational Properties of Power Allegories

theory *Power-Allegories-Multirelations*

imports *Multirelations-Basics*

begin

We start with random little properties.

lemma *eta-s-id*: $\eta = s\text{-id}$
unfolding *s-id-def eta-set* **by** *force*

lemma *Lambda-empty* [*simp*]: $\Lambda \{\} = p\text{-id}$
unfolding *Lambda-def p-id-def* **by** *blast*

lemma *alpha-pid* [*simp*]: $\alpha p\text{-id} = \{\}$
unfolding *alpha-def epsiloff-def p-id-def* **by** *force*

4.1 Peleg lifting

definition *plift* :: $(\text{'a}, \text{'b}) \text{ mrel} \Rightarrow (\text{'a set}, \text{'b set}) \text{ rel } (\langle _ \rangle [1000] 999)$ **where**
 $R_* = \{(A, B). \exists f. (\forall a \in A. (a, f(a)) \in R) \wedge B = \bigcup (f \text{ ` } A)\}$

lemma *pcomp-plift*: $R \cdot S = R ; S_*$
unfolding *s-prod-def plift-def relcomp-unfold* **by** *simp*

lemma *det-plift-klift*: *deterministic* $R \Longrightarrow R_* = (R)_\mathcal{P}$
unfolding *deterministic-set plift-def klift-set-var*
apply (*simp add: set-eq-iff*)
apply *safe*

by *metis+*

lemma *plift-ext2* [*simp*]: $\eta ; R_* = R$
 by (*metis eta-s-id pcomp-plift s-prod-idl*)

lemma *plift-ext-3* [*simp*]: $\eta_* = Id$
 by (*simp add: det-eta det-plift-klift*)

lemma *d-dom-plift*: $(Dom R)_* = dom (R_*)$
unfolding *Dom-def dom-set plift-def*
apply *clarsimp*
apply *safe*
apply (*metis (full-types) UN-singleton image-cong*)
by (*metis UN-singleton*)

lemma *d-pid-plift*: $(Dom R)_* \subseteq Id$
 by (*metis d-dom-plift d-idem dom-subid inf.absorb-iff2*)

lemma *d-plift-sub*: $A \subseteq B \implies (B, B) \in (Dom R)_* \implies (A, A) \in (Dom R)_*$
 by (*smt (z3) Pair-inject UN-singleton Dom-def mem-Collect-eq plift-def split-conv subsetD*)

lemma *plift-empty*: $(\{\}, A) \in R_* \iff A = \{\}$
 using *plift-def* by *auto*

lemma *univ-plift-klift*:
assumes *univalent R*
shows $R_* = (Dom R)_* ; (R)_{\mathcal{P}}$
proof –
have $\forall A B . (A, B) \in R_* \iff (A, B) \in (Dom R)_* ; (R)_{\mathcal{P}}$
proof (*intro allI, rule iffI*)
fix $A B$
assume $1: (A, B) \in R_*$
hence $(A, B) \in (R)_{\mathcal{P}}$
unfolding *klift-set-var*
apply *clarsimp*
by (*smt (z3) Collect-cong Pair-inject Union-eq assms case-prodE image-iff mem-Collect-eq plift-def univalent-set*)
thus $(A, B) \in (Dom R)_* ; (R)_{\mathcal{P}}$
using 1 *d-dom-plift dom-set* by *fastforce*

next
fix $A B$
assume $(A, B) \in (Dom R)_* ; (R)_{\mathcal{P}}$
from *this* **obtain** C **where** $2: (A, C) \in (Dom R)_* \wedge (C, B) \in (R)_{\mathcal{P}}$
by *auto*
from *this* **obtain** f **where** $3: (\forall a \in A. (a, f(a)) \in Dom R) \wedge C = \bigcup (f \text{ ` } A)$
by (*smt (verit) Pair-inject case-prodE mem-Collect-eq plift-def*)
hence $\forall a \in A . \exists D . (a, D) \in R$
by (*simp add: Dom-def*)

from this obtain g where $4: \forall a \in A . (a, g(a)) \in R$
by *metis*
hence $\forall a \in A . f(a) = \{a\}$
using 3 by (*simp add: Dom-def*)
hence $A = C$
using 2 3 by (*smt (verit) Pair-inject UN-singleton case-prodE image-cong*
mem-Collect-eq plift-def)
hence $(A, B) \in (R)_{\mathcal{P}}$
using 2 by *simp*
thus $(A, B) \in R_*$
unfolding *plift-def klift-set-var*
apply *clarsimp*
apply (*rule exI[where ?x=g]*)
using 4 by (*smt (verit, ccfv-SIG) Collect-cong assms image-def*
univalent-set)
qed
thus $R_* = (Dom R)_* ; (R)_{\mathcal{P}}$
by *force*
qed

lemma *plift-ext1*:
assumes *univalent f*
shows $(R ; f_*)_* = R_* ; f_*$
proof –
have $\forall A B . (A, B) \in (R ; (Dom f)_* ; (f)_{\mathcal{P}})_* \longleftrightarrow (A, B) \in R_* ; (Dom f)_* ; (f)_{\mathcal{P}}$
proof (*intro allI, rule iffI*)
fix $A B$
assume 1: $(A, B) \in (R ; (Dom f)_* ; (f)_{\mathcal{P}})_*$
from this obtain g where 2: $(\forall a \in A . (a, g(a)) \in R ; (Dom f)_* ; (f)_{\mathcal{P}}) \wedge B$
 $= \bigcup (g \text{ ` } A)$
by (*smt (verit) Pair-inject case-prodE mem-Collect-eq plift-def*)
hence $\forall a \in A . \exists C D . (a, C) \in R \wedge (C, D) \in (Dom f)_* \wedge (D, g(a)) \in (f)_{\mathcal{P}}$
by (*meson relcompEpair*)
hence $\forall a \in A . \exists C . (a, C) \in R \wedge (C, C) \in (Dom f)_* \wedge (C, g(a)) \in (f)_{\mathcal{P}}$
using *d-pid-plift* by *fastforce*
from this obtain h where 3: $\forall a \in A . (a, h a) \in R \wedge (h a, h a) \in (Dom f)_*$
 $\wedge (h a, g(a)) \in (f)_{\mathcal{P}}$
by *metis*
let $?h = \bigcup (h \text{ ` } A)$
have 4: $(A, ?h) \in R_*$
using 3 *plift-def* by *fastforce*
have 5: $(?h, ?h) \in (Dom f)_*$
using 3
unfolding *plift-def Dom-def*
apply *clarsimp*
by (*metis UN-extend-simps(9) UN-singleton*)
have $(?h, B) \in (f)_{\mathcal{P}}$
using 2 3
unfolding *klift-set*

```

    by auto
  thus  $(A,B) \in R_* ; (Dom f)_* ; (f)_{\mathcal{P}}$ 
    using 4 5 by blast
next
fix A B
assume  $(A,B) \in R_* ; (Dom f)_* ; (f)_{\mathcal{P}}$ 
from this obtain C D where 6:  $(A,C) \in R_* \wedge (C,D) \in (Dom f)_* \wedge (D,B) \in (f)_{\mathcal{P}}$ 
  by blast
hence 7:  $C = D$ 
  using d-pid-pltft by auto
from 6 obtain g where 8:  $(\forall a \in A. (a,g(a)) \in R) \wedge C = \bigcup (g \text{ `` } A)$ 
  by (smt (verit) Pair-inject case-prodE mem-Collect-eq pltft-def)
hence 9:  $\forall a \in A. (g(a),g(a)) \in (Dom f)_*$ 
  using 6 7 by (metis d-pltft-sub UN-iff subsetI)
let ?h =  $\lambda a . \bigcup (f \text{ `` } g a)$ 
have  $\forall a \in A. (g(a),?h a) \in (f)_{\mathcal{P}}$ 
  using 6 by (simp add: klift-set)
hence 10:  $\forall a \in A. (a,?h a) \in R ; (Dom f)_* ; (f)_{\mathcal{P}}$ 
  using 8 9 by blast
have  $B = \bigcup (f \text{ `` } D)$ 
  using 6 klift-set by fastforce
hence 11:  $B = (\bigcup a \in A . ?h a)$ 
  using 7 8 image-empty by blast
show  $(A,B) \in (R ; (Dom f)_* ; (f)_{\mathcal{P}})_*$ 
  apply (subst pltft-def)
  apply clarsimp
  apply (rule exI[where ?x=?h])
  using 10 11 by simp
qed
hence  $(R ; (Dom f)_* ; (f)_{\mathcal{P}})_* = R_* ; (Dom f)_* ; (f)_{\mathcal{P}}$ 
  by force
thus ?thesis
  by (metis (no-types, opaque-lifting) assms univ-pltft-klift O-assoc)
qed

```

lemma *pltft-assoc-univ*: $univalent f \implies (R \cdot S) \cdot f = R \cdot (S \cdot f)$
 by (simp add: pcomp-pltft O-assoc pltft-ext1)

lemma *Lambda-funct*: $\Lambda (R ; S) = \Lambda R \cdot \Lambda S$
 by (simp add: Lambda-pow det-lambda det-pltft-klift klift-def pcomp-pltft)

lemma *eta-funct*: $R ; S ; \eta = (R ; \eta) \cdot (S ; \eta)$

proof –
 have $(R ; \eta) \cdot (S ; \eta) = R ; \eta ; (S ; \eta)_*$
 by (simp add: pcomp-pltft)
 also have $\dots = R ; (\eta \cdot (S ; \eta))$
 by (simp add: O-assoc pcomp-pltft)
 also have $\dots = R ; S ; \eta$

by (*simp add: O-assoc pcomp-plift*)
finally show *?thesis..*
qed

lemma *alpha-funct-det: deterministic R \implies deterministic S \implies $\alpha (R \cdot S) = \alpha R ; \alpha S$*
by (*metis Lambda-epsiloff-up2 Lambda-funct alpha-Lambda-canc*)

lemma *pcomp-det: deterministic S \implies $R \cdot S = R ; (S)_{\mathcal{P}}$*
by (*simp add: det-plift-klift pcomp-plift*)

lemma *pcomp-det2: deterministic R \implies deterministic S \implies $(R \cdot S)_{\mathcal{P}} = (R)_{\mathcal{P}} ; (S)_{\mathcal{P}}$*
by (*simp add: klift-ext1 pcomp-det*)

lemma *pcomp-alpha: $\alpha (R \cdot S) = R ; \alpha ((S)_*)$*
by (*simp add: pcomp-plift*)

4.2 Fusion and fission

definition *fus :: ('a,'b) mrel \Rightarrow ('a,'b) mrel* **where**
fus R = $\Lambda (\alpha R)$

definition *fis :: ('a,'b) mrel \Rightarrow ('a,'b) mrel* **where**
fis R = $\alpha R ; \eta$

lemma *fus-set: fus R = $\{(a,B) \mid a \in B. B = \bigcup (\text{Image } R \ \{a\})\}$*
unfolding *fus-def Lambda-def alpha-set* **by** *force*

lemma *fis-set: fis R = $\{(a,\{b\}) \mid a \in b. b \in \bigcup (\text{Image } R \ \{a\})\}$*
unfolding *fis-def alpha-set eta-set relcomp-unfold* **by** *force*

lemma *fis-det-comp: deterministic R \implies deterministic S \implies $\text{fis } (R \cdot S) = \text{fis } R \cdot \text{fis } S$*
by (*simp add: alpha-funct-det eta-funct fis-def*)

lemma *fis-fix-det: deterministic R = (fus R = R)*
by (*metis Lambda-alpha-canc det-lambda fus-def*)

4.3 Galois connections for multirelations

lemma *sub-subh: $R \subseteq S \implies R \subseteq S ; (\text{epsiloff} \ // \ \text{epsiloff})$*
by (*metis R-O-Id alpha-def alpha-epsiloff lres-galois order-refl relcomp-mono*)

lemma *alpha-Lambda-galois: $(\alpha R \subseteq S) = (R \subseteq \Lambda S ; (\text{epsiloff} \ // \ \text{epsiloff}))$*

proof –

have *a: $(\alpha R \subseteq S) = (R \subseteq S \ // \ \text{epsiloff})$*

by (*simp add: alpha-def lres-galois*)

have *S // epsiloff = $(\Lambda S ; \text{epsiloff}) \ // \ \text{epsiloff}$*

by (*metis alpha-Lambda-canc alpha-def*)

also have $\dots = \Lambda S ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*simp add: det-lambda det-lres*)
finally have $S \parallel \text{epsiloff} = \Lambda S ; (\text{epsiloff} \parallel \text{epsiloff})$
 \cdot
thus *?thesis*
using *a* **by** *presburger*
qed

lemma *alpha-Lambda-galois-set*: $(\alpha R \subseteq S) = (R \subseteq \{(a,A). \exists B. (a,B) \in \Lambda S \wedge A \subseteq B\})$
unfolding *alpha-set Lambda-def* **by** *blast*

lemma *epsiloff-eta-lres*: $\text{epsiloff} ; \eta \subseteq \text{epsiloff} \parallel \text{epsiloff}$
proof –
have $\text{epsiloff} ; \eta ; \text{epsiloff} = \text{epsiloff} ; \alpha (\Lambda \text{Id})$
by (*simp add: O-assoc alpha-def eta-def*)
also have $\dots = \text{epsiloff}$
by *simp*
finally have $\text{epsiloff} ; \eta ; \text{epsiloff} = \text{epsiloff}$.
thus *?thesis*
by (*smt (verit) lres-galois order-refl*)
qed

lemma *eta-alpha-galois*: $(R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})) = (R \subseteq \alpha S)$
proof
assume $R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$
hence $R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff}) ; \text{epsiloff}$
by (*metis R-O-Id alpha-def alpha-eta alpha-ord-pres alpha-relcomp*)
thus $R \subseteq \alpha S$
by (*simp add: O-assoc alpha-def*)
next
assume $R \subseteq \alpha S$
hence $R ; \eta \subseteq \alpha S ; \eta$
by (*simp add: relcomp-mono*)
hence $R ; \eta \subseteq S ; \text{epsiloff} ; \eta$
by (*simp add: alpha-def*)
thus $R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$
using *epsiloff-eta-lres in-mono* **by** *fastforce*
qed

lemma *eta-alpha-galois-set*: $(R ; \eta \subseteq \{(a,A). \exists B. (a,B) \in S \wedge A \subseteq B\}) = (R \subseteq \alpha S)$
unfolding *eta-set alpha-set* **by** *auto*

lemma *Lambda-iso*: $R \subseteq S \implies \Lambda R \subseteq \Lambda S ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*metis alpha-Lambda-canc alpha-Lambda-galois*)

lemma *eta-iso*: $R \subseteq S \implies R ; \eta \subseteq S ; \eta ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*simp add: eta-alpha-galois*)

lemma *alpha-iso*: $R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff}) \implies \alpha R \subseteq \alpha S$
by (*metis (no-types, lifting) alpha-def alpha-ord-pres alpha-relcomp conv-Omega conv-Omega-epsiloff*)

lemma *Lambda-canc-dcl*: $R \subseteq \Lambda (\alpha R) ; (\text{epsiloff} \parallel \text{epsiloff})$
using *alpha-Lambda-galois* **by** *blast*

lemma *eta-canc-dcl*: $\alpha R ; \eta \subseteq R ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*simp add: eta-alpha-galois*)

lemma *alpha-surj*: *surj* α
using *alpha-Lambda-canc* **by** *blast*

lemma *Lambda-inj*: *inj* Λ
by (*metis alpha-Lambda-canc injI*)

lemma *eta-inj*: *inj* $(\lambda x. x ; \eta)$
by (*metis alpha-eta-id injI*)

lemma *fus-least-odet*:
assumes $\Lambda (\alpha S) = S$
and $R \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$
shows $\Lambda (\alpha R) \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})$
proof –
have $\alpha R \subseteq \alpha S$
by (*simp add: alpha-iso assms(2)*)
hence $\Lambda (\alpha R) \subseteq \Lambda (\alpha S) ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*simp add: Lambda-iso*)
thus *?thesis*
using *assms(1)* **by** *auto*
qed

lemma *fis-greatest-idet*:
assumes $\alpha S ; \eta = S$
and $S \subseteq R ; (\text{epsiloff} \parallel \text{epsiloff})$
shows $S \subseteq \alpha R ; \eta ; (\text{epsiloff} \parallel \text{epsiloff})$
proof –
have $\alpha S \subseteq \alpha R$
by (*simp add: alpha-iso assms(2)*)
hence $\alpha S ; \eta \subseteq \alpha R ; \eta ; (\text{epsiloff} \parallel \text{epsiloff})$
by (*simp add: eta-iso*)
thus *?thesis*
using *assms(1)* **by** *auto*
qed

lemma *fis-fus-galois*: $(\alpha R ; \eta \subseteq S ; (\text{epsiloff} \parallel \text{epsiloff})) = (R \subseteq \Lambda (\alpha S) ; (\text{epsiloff} \parallel \text{epsiloff}))$
by (*simp add: alpha-Lambda-galois eta-alpha-galois*)

4.4 Properties of alpha, fission and fusion

lemma *alpha-lax*: $\alpha (R \cdot S) \subseteq \alpha R ; \alpha S$
unfolding *alpha-def s-prod-def epsiloff-def relcomp-unfold* **by** *blast*

lemma *alpha-down* [*simp*]: $\alpha (R ; \Omega^\smile) = \alpha R$
by (*metis alpha-def alpha-relcomp conv-Omega-epsiloff*)

lemma *fis-fis* [*simp*]: $fis \circ fis = fis$
unfolding *fun-eq-iff fis-def* **by** *simp*

lemma *fus-fus* [*simp*]: $fus \circ fus = fus$
unfolding *fun-eq-iff fus-def* **by** *simp*

lemma *fis-fus* [*simp*]: $fis \circ fus = fis$
unfolding *fun-eq-iff fus-def fis-def* **by** *simp*

lemma *fus-fis* [*simp*]: $fus \circ fis = fus$
unfolding *fun-eq-iff fus-def fis-def* **by** *simp*

lemma *fis-alpha*: $fis R \cdot S = \alpha R ; S$
by (*simp add: O-assoc fis-def pcomp-plit*)

lemma *fis-lax*: $fis (R \cdot S) \subseteq fis R \cdot fis S$
by (*metis fis-def alpha-lax eta-funct relcomp-mono subsetI*)

lemma *klift-fus*: $(R)_{\mathcal{P}} = fus (epsiloff ; R)$
by (*simp add: alpha-def fus-def klift-var*)

lemma *fus-eta-klift*: $fus R = \eta ; (R)_{\mathcal{P}}$
by (*metis Id-O-R Lambda-pow eta-def fus-def klift-def*)

lemma *fus-Lambda-mu*: $fus R = \Lambda R ; \mu$
by (*simp add: fus-def lambda-alpha-mu*)

4.5 Properties of fusion, fission, nu and tau

lemma *alpha-tau* [*simp*]: $\alpha (\tau R) = \{\}$
by (*metis alpha-ord-pres alpha-pid subset-empty tau-le-c*)

lemma *alpha-nu* [*simp*]: $\alpha (\nu R) = \alpha R$
unfolding *alpha-def nu-def epsiloff-def U-def p-id-def relcomp-unfold* **by** *force*

lemma *nu-fis* [*simp*]: $\nu (fis R) = fis R$
by (*metis alpha-fp empty-iff equalsOI fis-alpha relcompE*)

lemma *nu-fis-var*: $\nu (fis R) = fis (\nu R)$
by (*metis alpha-nu fis-def nu-fis*)

lemma *tau-fis* [*simp*]: $\tau (fis R) = \{\}$

by (*metis nu-fis tau-alpha-zero*)

Properties of tests and domain

lemma *subid-pltft*: $(P \cap \eta)_* = \{(A,A) \mid A. \forall a \in A. (a, \{a\}) \in (P \cap \eta)\}$
unfolding *pltft-def eta-set* **by** *safe auto*

lemma *U-subid*: $R ; (P \cap \eta)_* = R \cap U ; (P \cap \eta)_*$
unfolding *pltft-def U-def eta-set relcomp-unfold*
apply *safe*
apply *force*
apply *blast*
by *force*

lemma *subid-pltft-down*: $U ; (P \cap \eta)_* ; \Omega^\smile = U ; (P \cap \eta)_*$
unfolding *U-def relcomp-unfold pltft-def Omega-set eta-set converse-def*
apply *safe*
apply *clarsimp*
apply (*metis IntE UN-singleton inf.orderE*)
by *blast*

lemma *nu-subid-pltft*: $\nu (R ; (P \cap \eta)_*) = \nu R ; (P \cap \eta)_*$
unfolding *nu-def relcomp-unfold pltft-def U-def p-id-def eta-set* **by** *safe fastforce+*

lemma *dom-fis1*: $\text{dom} (fis R) = \text{dom} (\alpha R)$
unfolding *dom-set fis-set alpha-set* **by** *blast*

lemma *dom-fis2*: $\text{dom} (fis R) = \text{dom} (\alpha (\nu R))$
by (*simp add: dom-fis1*)

lemma *dom-fis3*: $\text{dom} (fis R) = \text{dom} (\nu R)$
unfolding *dom-set fis-set nu-def U-def p-id-def* **by** *safe fastforce+*

lemma *dom-fis4*: $\text{dom} (fis R) = \text{dom} (\nu (fus R))$
by (*metis comp-eq-dest-lhs dom-fis3 fis-fus*)

lemma *dom-alpha*: $\text{dom} (\alpha R ; (P \cap \eta)) = \text{dom} (\nu (R ; \Omega^\smile) ; (P \cap \eta)_*)$
unfolding *dom-set alpha-set eta-set Omega-set pltft-def nu-def relcomp-unfold*
U-def p-id-def
apply *safe*
apply (*clarsimp, metis empty-iff empty-subsetI insert-subset singletonD*
singletonI UN-singleton)
by *clarsimp fastforce*

4.6 Box and diamond

definition *box* :: $('a, 'b) \text{ mrel} \Rightarrow ('b \text{ set}, 'a \text{ set}) \text{ rel}$ **where**
 $\text{box } R = \text{rbox } (\alpha R)$

definition *dia* :: $('a, 'b) \text{ mrel} \Rightarrow ('b \text{ set}, 'a \text{ set}) \text{ rel}$ **where**

$dia\ R = \mathcal{P}((\alpha\ R)^\smile)$

lemma *box-set*: $box\ R = \{(B, A). A = \{a. \forall C. (a, C) \in R \longrightarrow C \subseteq B\}\}$
unfolding *box-def rbox-set alpha-set* **by** *force*

lemma *dia-set*: $dia\ R = \{(B, A). A = \{a. \exists C. (a, C) \in R \wedge C \cap B \neq \{\}\}\}$
unfolding *dia-def pow-set Image-def alpha-set converse-def* **by** *force*

lemma *box-Omega*: $box\ R = \Lambda(\Omega^\smile \parallel R)$
unfolding *box-set Lambda-def lres-def Omega-set* **by** *auto*

end
theory *Multirelations*

imports *Power-Allegories-Multirelations*

begin

lemma *nonempty-set-card*:
assumes *finite S*
shows $S \neq \{\} \longleftrightarrow card\ S \geq 1$
using *assms card-0-eq* **by** *fastforce*

no-notation *one-class.one* ($\langle 1 \rangle$)
no-notation *times-class.times* (**infixl** $\langle * \rangle$ 70)

no-notation *rel-fdia* ($\langle (| -) \rangle$ [61,81] 82)
no-notation *rel-bdia* ($\langle \langle - | - \rangle \rangle$ [61,81] 82)
no-notation *rel-fbox* ($\langle (| -) \rangle$ [61,81] 82)
no-notation *rel-bbox* ($\langle \langle - | - \rangle \rangle$ [61,81] 82)

declare *s-prod-pa-def* [*mr-simp*]

notation *s-prod* (**infixl** $\langle * \rangle$ 70)
notation *s-id* ($\langle 1 \rangle$)

lemma *sp-oi-subdist*:
 $(P \cap Q) * (R \cap S) \subseteq P * R$
unfolding *s-prod-def* **by** *blast*

lemma *sp-oi-subdist-2*:
 $(P \cap Q) * (R \cap S) \subseteq (P * R) \cap (Q * S)$
unfolding *s-prod-def* **by** *blast*

5 Inner Structure

5.1 Inner union, inner intersection and inner complement

abbreviation *inner-union* (**infixl** $\langle \cup \cup \rangle$ 65)

where *inner-union* \equiv *p-prod*

definition *inner-intersection* $:: ('a,'b) \text{ mrel} \Rightarrow ('a,'b) \text{ mrel} \Rightarrow ('a,'b) \text{ mrel}$ (**infixl** $\langle \cap \rangle$ 65) **where**

$R \cap S \equiv \{ (a,B) . \exists C D . B = C \cap D \wedge (a,C) \in R \wedge (a,D) \in S \}$

definition *inner-complement* $:: ('a,'b) \text{ mrel} \Rightarrow ('a,'b) \text{ mrel}$ ($\langle \sim \rightarrow$ [80] 80) **where**

$\sim R \equiv \{ (a,B) . (a,-B) \in R \}$

abbreviation *iu-unit* ($\langle 1_{\cup\cup} \rangle$)

where $1_{\cup\cup} \equiv$ *p-id*

definition *ii-unit* $:: ('a,'a) \text{ mrel}$ ($\langle 1_{\cap\cap} \rangle$)

where $1_{\cap\cap} \equiv \{ (a,UNIV) \mid a . \text{True} \}$

declare *inner-intersection-def* [*mr-simp*] *inner-complement-def* [*mr-simp*]

ii-unit-def [*mr-simp*]

lemma *iu-assoc*:

$(R \cup\cup S) \cup\cup T = R \cup\cup (S \cup\cup T)$

by (*simp add: p-prod-assoc*)

lemma *iu-commute*:

$R \cup\cup S = S \cup\cup R$

by (*simp add: p-prod-comm*)

lemma *iu-unit*:

$R \cup\cup 1_{\cup\cup} = R$

by *simp*

lemma *ii-assoc*:

$(R \cap\cap S) \cap\cap T = R \cap\cap (S \cap\cap T)$

apply (*clarsimp simp: mr-simp*)

by (*metis (no-types, opaque-lifting) semilattice-inf-class.inf-assoc*)

lemma *ii-commute*:

$R \cap\cap S = S \cap\cap R$

by (*auto simp: mr-simp*)

lemma *ii-unit* [*simp*]:

$R \cap\cap 1_{\cap\cap} = R$

by (*simp add: mr-simp*)

lemma *pa-ic*:

$\sim(R \otimes \sim S) = R \otimes S$

by (*clarsimp simp: mr-simp*)

lemma *ic-involutive* [*simp*]:

$\sim\sim R = R$

by (*simp add: mr-simp*)

lemma *ic-injective*:

$$\sim R = \sim S \implies R = S$$

by (*metis ic-involutive*)

lemma *ic-antidist-iu*:

$$\sim(R \cup\cup S) = \sim R \cap\cap \sim S$$

apply (*clarsimp simp: mr-simp*)

by (*metis (no-types, opaque-lifting) compl-sup double-compl*)

lemma *ic-antidist-ii*:

$$\sim(R \cap\cap S) = \sim R \cup\cup \sim S$$

by (*metis ic-antidist-iu ic-involutive*)

lemma *ic-iu-unit [simp]*:

$$\sim 1_{\cup\cup} = 1_{\cap\cap}$$

unfolding *ii-unit-def p-id-def inner-complement-def* by *force*

lemma *ic-ii-unit [simp]*:

$$\sim 1_{\cap\cap} = 1_{\cup\cup}$$

by (*metis ic-involutive ic-iu-unit*)

lemma *ii-unit-split-iu [simp]*:

$$1 \cup\cup \sim 1 = 1_{\cap\cap}$$

by (*force simp: mr-simp*)

lemma *aux-1*:

$$B = \{a\} \cap D \implies \neg D = \{a\} \implies B = \{\}$$

by *auto*

lemma *iu-unit-split-ii [simp]*:

$$1 \cap\cap \sim 1 = 1_{\cup\cup}$$

by (*metis ic-antidist-iu ic-ii-unit ic-involutive ii-commute ii-unit-split-iu*)

lemma *iu-right-dist-ou*:

$$(R \cup S) \cup\cup T = (R \cup\cup T) \cup (S \cup\cup T)$$

unfolding *p-prod-def* by *auto*

lemma *ii-right-dist-ou*:

$$(R \cup S) \cap\cap T = (R \cap\cap T) \cup (S \cap\cap T)$$

by (*auto simp: mr-simp*)

lemma *iu-left-isotone*:

$$R \subseteq S \implies R \cup\cup T \subseteq S \cup\cup T$$

by (*metis iu-right-dist-ou subset-Un-eq*)

lemma *iu-right-isotone*:

$$R \subseteq S \implies T \cup\cup R \subseteq T \cup\cup S$$

by (*simp add: iu-commute iu-left-isotone*)

lemma *iu-isotone*:

$R \subseteq S \implies P \subseteq Q \implies R \cup P \subseteq S \cup Q$

by (*meson dual-order.trans iu-left-isotone iu-right-isotone*)

lemma *ii-left-isotone*:

$R \subseteq S \implies R \cap T \subseteq S \cap T$

by (*metis ii-right-dist-ou subset-Un-eq*)

lemma *ii-right-isotone*:

$R \subseteq S \implies T \cap R \subseteq T \cap S$

by (*simp add: ii-commute ii-left-isotone*)

lemma *ii-isotone*:

$R \subseteq S \implies P \subseteq Q \implies R \cap P \subseteq S \cap Q$

by (*meson ii-left-isotone ii-right-isotone order-trans*)

lemma *iu-right-subdist-ii*:

$(R \cap S) \cup T \subseteq (R \cup T) \cap (S \cup T)$

apply (*clarsimp simp: mr-simp*)

by (*metis sup-inf-distrib2*)

lemma *ii-right-subdist-iu*:

$(R \cup S) \cap T \subseteq (R \cap T) \cup (S \cap T)$

apply (*clarsimp simp: mr-simp*)

by (*metis inf-sup-distrib2*)

lemma *ic-isotone*:

$R \subseteq S \implies \sim R \subseteq \sim S$

by (*simp add: inner-complement-def subset-eq*)

lemma *ic-bot* [*simp*]:

$\sim\{\} = \{\}$

by (*simp add: mr-simp*)

lemma *ic-top* [*simp*]:

$\sim U = U$

by (*auto simp: mr-simp*)

lemma *ic-dist-ou*:

$\sim(R \cup S) = \sim R \cup \sim S$

by (*auto simp: mr-simp*)

lemma *ic-dist-oi*:

$\sim(R \cap S) = \sim R \cap \sim S$

by (*auto simp: mr-simp*)

lemma *ic-dist-oc*:

$\sim -R = -(\sim R)$
by (*auto simp: mr-simp*)

lemma *ii-sub-idempotent*:

$R \subseteq R \cap \cap R$

unfolding *inner-intersection-def* **by** *force*

definition *inner-Union* :: $('i \Rightarrow ('a, 'b) \text{ mrel}) \Rightarrow 'i \text{ set} \Rightarrow ('a, 'b) \text{ mrel} \langle \cup \cup - \rangle$
 $[80, 80] \ 80$ **where**

$\cup \cup X|I \equiv \{ (a, B) . \exists f . B = (\cup_{i \in I} . f \ i) \wedge (\forall i \in I . (a, f \ i) \in X \ i) \}$

definition *inner-Intersection* :: $('i \Rightarrow ('a, 'b) \text{ mrel}) \Rightarrow 'i \text{ set} \Rightarrow ('a, 'b) \text{ mrel} \langle \cap \cap - \rangle$
 $[80, 80] \ 80$ **where**

$\cap \cap X|I \equiv \{ (a, B) . \exists f . B = (\cap_{i \in I} . f \ i) \wedge (\forall i \in I . (a, f \ i) \in X \ i) \}$

declare *inner-Union-def* [*mr-simp*] *inner-Intersection-def* [*mr-simp*]

lemma *iU-empty*:

$\cup \cup X|\{\} = 1_{\cup \cup}$

by (*auto simp: mr-simp*)

lemma *iI-empty*:

$\cap \cap X|\{\} = 1_{\cap \cap}$

by (*auto simp: mr-simp*)

lemma *ic-antidist-iU*:

$\sim \cup \cup X|I = \cap \cap (\text{inner-complement } o \ X)|I$

apply (*rule antisym*)

apply (*clarsimp simp: mr-simp*)

apply (*metis (mono-tags, lifting) Compl-UN double-compl*)

by (*clarsimp simp: mr-simp*) *blast*

lemma *ic-antidist-iI*:

$\sim \cap \cap X|I = \cup \cup (\text{inner-complement } o \ X)|I$

apply (*rule antisym*)

apply (*clarsimp simp: mr-simp*)

apply (*metis Compl-INT double-complement*)

by (*clarsimp simp: mr-simp*) *blast*

lemma *iu-right-dist-oU*:

$\cup X \cup \cup T = (\cup R \in X . R \cup \cup T)$

by (*clarsimp simp: mr-simp*) *blast*

lemma *ii-right-dist-oU*:

$\cup X \cap \cap T = (\cup R \in X . R \cap \cap T)$

by (*clarsimp simp: mr-simp*) *blast*

lemma *iu-right-subdist-iI*:

$\cap \cap X|I \cup \cup T \subseteq \cap \cap (\lambda i . X \ i \cup \cup T)|I$

apply (*clarsimp simp: mr-simp*)
by (*metis INT-simps(6)*)

lemma *ii-right-subdist-iU*:
 $\bigcup\bigcup X|I \cap T \subseteq \bigcup\bigcup(\lambda i . X i \cap T)|I$
by (*clarsimp simp: mr-simp, metis UN-extend-simps(4)*)

lemma *ic-dist-oU*:
 $\sim\bigcup X = \bigcup(\text{inner-complement } ' X)$
by (*auto simp: mr-simp*)

lemma *ic-dist-oI*:
 $\sim\bigcap X = \bigcap(\text{inner-complement } ' X)$
by (*auto simp: mr-simp*)

lemma *sp-left-subdist-iU*:
 $R * (\bigcup\bigcup X|I) \subseteq \bigcup\bigcup(\lambda i . R * X i)|I$
apply (*clarsimp simp: mr-simp*)
subgoal for a B f proof –
assume 1: $(a,B) \in R$
assume $\forall b \in B . \exists g . f b = \bigcup(g ' I) \wedge (\forall i \in I . (b,g i) \in X i)$
from this obtain g where 2: $\forall b \in B . f b = \bigcup(g b ' I) \wedge (\forall i \in I . (b,g b i) \in X i)$
by *metis*
hence 3: $\bigcup(f ' B) = (\bigcup b \in B . \bigcup(g b ' I))$
by (*meson SUP-cong*)
let ?h = $\lambda i . \bigcup b \in B . g b i$
have $\bigcup(f ' B) = \bigcup(?h ' I) \wedge (\forall i \in I . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in X i) \wedge ?h i = \bigcup(f ' B)))$
using 1 2 3 **by** (*metis SUP-commute*)
thus ?thesis
by *auto*
qed
done

lemma *sp-right-subdist-iU*:
 $(\bigcup\bigcup X|I) * R \subseteq \bigcup\bigcup(\lambda i . X i * R)|I$
by (*clarsimp simp: mr-simp, blast*)

lemma *sp-right-dist-iU*:
assumes $\forall J::'a \text{ set} . J \neq \{\} \longrightarrow (\bigcup\bigcup(\lambda j . R)|J) \subseteq R$
shows $(\bigcup\bigcup X|I) * R = \bigcup\bigcup(\lambda i . X i * R)|(I::'a \text{ set})$
apply (*rule antisym*)
using *sp-right-subdist-iU* **apply** *blast*
apply (*clarsimp simp: mr-simp*)
subgoal for a f proof –
assume $\forall i \in I . \exists B . (a,B) \in X i \wedge (\exists g . (\forall b \in B . (b,g b) \in R) \wedge f i = \bigcup(g ' B))$
from this obtain B g where 1: $\forall i \in I . (a,B i) \in X i \wedge (\forall b \in B i . (b,g i b) \in$

```

R)  $\wedge f i = \bigcup (g i \text{ ' } B i)$ 
  by metis
  let  $?B = \bigcup (B \text{ ' } I)$ 
  let  $?g = \lambda b . \bigcup \{ g i b \mid i . i \in I \wedge b \in B i \}$ 
  have  $\forall b \in ?B . (b, ?g b) \in R$ 
  proof (rule ballI)
    fix b
    let  $?I = \{ i \mid i . i \in I \wedge b \in B i \}$ 
    assume  $? : b \in \bigcup (B \text{ ' } I)$ 
    have  $(b, ?g b) \in \bigcup \bigcup (\lambda j . R) | ?I$ 
      apply (clarsimp simp: mr-simp)
      apply (rule exI[of -  $\lambda i . g i b$ ])
      using 1 by blast
    thus  $(b, ?g b) \in R$ 
      using ? by (smt (verit) assms UN-E empty-Collect-eq subset-iff)
  qed
  hence  $?B = \bigcup (B \text{ ' } I) \wedge (\forall i \in I . (a, B i) \in X i) \wedge (\forall b \in ?B . (b, ?g b) \in R) \wedge$ 
 $\bigcup (f \text{ ' } I) = \bigcup (?g \text{ ' } ?B)$ 
  using 1 by auto
  thus  $\exists B . (\exists f . B = \bigcup (f \text{ ' } I) \wedge (\forall i \in I . (a, f i) \in X i)) \wedge (\exists g . (\forall b \in B . (b, g$ 
 $b) \in R) \wedge \bigcup (f \text{ ' } I) = \bigcup (g \text{ ' } B))$ 
  by (metis (no-types, lifting))
  qed
done

```

5.2 Dual

abbreviation *dual* :: $(\text{'a}, \text{'b}) \text{ mrel} \Rightarrow (\text{'a}, \text{'b}) \text{ mrel} (\text{'}^d, [100] 100)$
 where $R^d \equiv \sim -R$

lemma *dual*:
 $R^d = \{ (a, B) . (a, -B) \notin R \}$
 by (simp add: inner-complement-def)

declare *dual* [*mr-simp*]

lemma *dual-antitone*:
 $R \subseteq S \Longrightarrow S^d \subseteq R^d$
 by (simp add: ic-isotone)

lemma *ic-oc-dual*:
 $\sim R = -R^d$
 by (simp add: ic-dist-oc)

lemma *dual-involutive* [*simp*]:
 $R^{dd} = R$
 by (simp add: ic-dist-oc)

lemma *dual-antidist-ou*:

$(R \cup S)^d = R^d \cap S^d$
by (*simp add: ic-dist-oi*)

lemma *dual-antidist-oi*:
 $(R \cap S)^d = R^d \cup S^d$
by (*simp add: ic-dist-ou*)

lemma *dual-dist-oc*:
 $(-R)^d = -R^d$
by (*fact ic-dist-oc*)

lemma *dual-dist-ic*:
 $(\sim R)^d = \sim R^d$
by (*simp add: ic-dist-oc*)

lemma *dual-antidist-oU*:
 $(\bigcup X)^d = \bigcap (\text{dual } ' X)$
by (*simp add: ic-dist-oI uminus-Sup*)

lemma *dual-antidist-oI*:
 $(\bigcap X)^d = \bigcup (\text{dual } ' X)$
by (*simp add: ic-dist-oU uminus-Inf*)

5.3 Co-composition

definition *co-prod* :: ('a,'b) mrel \Rightarrow ('b,'c) mrel \Rightarrow ('a,'c) mrel (**infixl** $\langle \odot \rangle$ 70)
where

$R \odot S \equiv \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap \{ f b \mid b . b \in B \}) \}$

lemma *co-prod-im*:
 $R \odot S = \{ (a,C) . \exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap ((\lambda x . f x) ' B)) \}$
by (*auto simp: co-prod-def*)

lemma *co-prod-iff*:
 $(a,C) \in (R \odot S) \iff (\exists B . (a,B) \in R \wedge (\exists f . (\forall b \in B . (b,f b) \in S) \wedge C = \bigcap \{ f b \mid b . b \in B \}))$
by (*unfold co-prod-im, auto*)

declare *co-prod-im* [*mr-simp*]

lemma *co-prod*:
 $R \odot S = \sim(R * \sim S)$
apply (*clarsimp simp: mr-simp*)
by (*smt (verit) Collect-cong Compl-INT Compl-UN case-prodI2 double-complement old.prod.case*)

lemma *cp-left-isotone*:

$R \subseteq S \implies R \odot T \subseteq S \odot T$
by (*simp add: co-prod ic-isotone s-prod-isol*)

lemma *cp-right-isotone*:
 $R \subseteq S \implies T \odot R \subseteq T \odot S$
by (*smt (verit) co-prod-iff in-mono subrelI*)

lemma *cp-isotone*:
 $R \subseteq S \implies P \subseteq Q \implies R \odot P \subseteq S \odot Q$
by (*meson cp-left-isotone cp-right-isotone order-trans*)

lemma *ic-dist-cp*:
 $\sim(R \odot S) = R * \sim S$
by (*simp add: co-prod*)

lemma *ic-dist-sp*:
 $\sim(R * S) = R \odot \sim S$
by (*simp add: co-prod*)

lemma *ic-cp-ic-unit*:
 $\sim R = R \odot \sim 1$
by (*simp add: co-prod*)

lemma *cp-left-zero* [*simp*]:
 $\{\} \odot R = \{\}$
by (*simp add: co-prod-im*)

lemma *cp-left-unit* [*simp*]:
 $1 \odot R = R$
by (*simp add: co-prod*)

lemma *cp-ic-unit* [*simp*]:
 $\sim 1 \odot \sim 1 = 1$
using *ic-cp-ic-unit ic-involutive* **by** *blast*

lemma *cp-right-dist-ou*:
 $(R \cup S) \odot T = (R \odot T) \cup (S \odot T)$
by (*simp add: co-prod ic-dist-ou s-prod-distr*)

lemma *cp-left-iu-unit* [*simp*]:
 $1_{\cup\cup} \odot R = 1_{\cap\cap}$
by (*simp add: co-prod*)

lemma *cp-right-ii-unit*:
 $R \odot 1_{\cap\cap} \subseteq R \cup\cup \sim R$
apply (*clarsimp simp: mr-simp*)
by (*metis double-compl sup-compl-top*)

lemma *sp-right-iu-unit*:

$R * 1_{\cup\cup} \subseteq R \cap \cap \sim R$
apply (*clarsimp simp: mr-simp*)
by (*metis Compl-disjoint double-complement*)

lemma *cp-left-subdist-ii*:
 $R \odot (S \cap \cap T) \subseteq (R \odot S) \cap \cap (R \odot T)$
by (*metis cl3 co-prod ic-antidist-ii ic-antidist-iu ic-isotone*)

lemma *cp-right-subantidist-iu*:
 $(R \cup\cup S) \odot T \subseteq (R \odot T) \cap \cap (S \odot T)$
by (*metis co-prod ic-antidist-iu ic-isotone seq-conc-subdistr*)

lemma *cp-right-antidist-iu*:
assumes $T \cap \cap T \subseteq T$
shows $(R \cup\cup S) \odot T = (R \odot T) \cap \cap (S \odot T)$
by (*smt (verit) assms cl4 co-prod cp-right-subantidist-iu ic-antidist-ii ic-involutive ic-isotone subset-antisym*)

lemma *cp-right-dist-oU*:
 $\bigcup X \odot T = \bigcup_{R \in X} R \odot T$
by (*auto simp: mr-simp*)

lemma *cp-left-subdist-iI*:
 $R \odot (\bigcap \bigcap X|I) \subseteq \bigcap \bigcap (\lambda i . R \odot X i)|I$
proof –
have $R \odot (\bigcap \bigcap X|I) = \sim(R * (\bigcup \bigcup (\text{inner-complement } o X)|I))$
by (*simp add: co-prod ic-antidist-iI*)
also have $\dots \subseteq \sim(\bigcup \bigcup (\lambda i . R * \sim(X i)|I))$
apply (*rule ic-isotone*)
using *sp-left-subdist-iU* **by force**
also have $\dots = \bigcap \bigcap (\lambda i . R \odot X i)|I$
apply (*subst ic-antidist-iU*)
by (*metis co-prod comp-apply*)
finally show *?thesis*

qed

lemma *cp-right-subantidist-iU*:
 $(\bigcup \bigcup X|I) \odot R \subseteq \bigcap \bigcap (\lambda i . X i \odot R)|I$
proof –
have $(\bigcup \bigcup X|I) \odot R = \sim((\bigcup \bigcup X|I) * \sim R)$
by (*simp add: co-prod*)
also have $\dots \subseteq \sim((\bigcup \bigcup (\lambda i . X i * \sim R)|I))$
by (*simp add: ic-isotone sp-right-subdist-iU*)
also have $\dots = \bigcap \bigcap (\lambda i . X i \odot R)|I$
apply (*subst ic-antidist-iU*)
by (*metis co-prod comp-apply*)
finally show *?thesis*

qed

lemma *cp-right-antidist-iU*:

assumes $\forall J::'a \text{ set} . J \neq \{\} \longrightarrow (\bigcap (\lambda j . R)|J) \subseteq R$

shows $(\bigcup X|I) \odot R = \bigcap (\lambda i . X i \odot R)|(I::'a \text{ set})$

proof –

have $1: \bigwedge J . \bigcup (\lambda j . \sim R)|J = \sim \bigcap (\lambda j . R)|J$

apply (*subst ic-antidist-iI*)

by (*metis comp-apply*)

have $(\bigcup X|I) \odot R = \sim((\bigcup X|I) * \sim R)$

by (*simp add: co-prod*)

also have $\dots = \sim((\bigcup (\lambda i . X i * \sim R)|I))$

by (*simp add: 1 assms sp-right-dist-iU ic-isotone*)

also have $\dots = \bigcap (\lambda i . X i \odot R)|I$

apply (*subst ic-antidist-iU*)

by (*metis co-prod comp-apply*)

finally show *?thesis*

qed

5.4 Inner order

definition *inner-order-iu* :: $'a \times 'b \text{ set} \Rightarrow 'a \times 'b \text{ set} \Rightarrow \text{bool}$ (**infix** $\langle \preceq_{\cup\cup} \rangle$ 50)

where

$x \preceq_{\cup\cup} y \equiv \text{fst } x = \text{fst } y \wedge \text{snd } x \subseteq \text{snd } y$

definition *inner-order-ii* :: $'a \times 'b \text{ set} \Rightarrow 'a \times 'b \text{ set} \Rightarrow \text{bool}$ (**infix** $\langle \preceq_{\cap\cap} \rangle$ 50)

where

$x \preceq_{\cap\cap} y \equiv \text{fst } x = \text{fst } y \wedge \text{snd } x \supseteq \text{snd } y$

lemma *inner-order-dual*:

$x \preceq_{\cup\cup} y \longleftrightarrow y \preceq_{\cap\cap} x$

by (*metis inner-order-ii-def inner-order-iu-def*)

interpretation *inner-order-iu*: *order* ($\preceq_{\cup\cup}$) $\lambda x y . x \preceq_{\cup\cup} y \wedge x \neq y$

by (*unfold-locales, auto simp add: inner-order-iu-def*)

5.5 Up-closure, down-closure and convex-closure

abbreviation *up* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$ ($\langle \uparrow \rangle$ [100] 100)

where $R\uparrow \equiv R \cup\cup U$

abbreviation *down* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$ ($\langle \downarrow \rangle$ [100] 100)

where $R\downarrow \equiv R \cap\cap U$

abbreviation *convex* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$ ($\langle \updownarrow \rangle$ [100] 100)

where $R\updownarrow \equiv R\uparrow \cap R\downarrow$

lemma *up*:

$R\uparrow = \{ (a, B) . \exists C . (a, C) \in R \wedge C \subseteq B \}$

by (*simp add: p-id-U*)

lemma *down*:
 $R\downarrow = \{ (a,B) . \exists C . (a,C) \in R \wedge B \subseteq C \}$
by (*auto simp: mr-simp*)

lemma *convex*:
 $R\updownarrow = \{ (a,B) . \exists C D . (a,C) \in R \wedge (a,D) \in R \wedge C \subseteq B \wedge B \subseteq D \}$
by (*auto simp: mr-simp*)

declare *up* [*mr-simp*] *down* [*mr-simp*] *convex* [*mr-simp*]

lemma *ic-up*:
 $\sim(R\uparrow) = (\sim R)\downarrow$
by (*simp add: ic-antidist-ii*)

lemma *ic-down*:
 $\sim(R\downarrow) = (\sim R)\uparrow$
by (*simp add: ic-antidist-ii*)

lemma *ic-convex*:
 $\sim(R\updownarrow) = (\sim R)\updownarrow$
by (*simp add: ic-dist-oi ic-down ic-up inf-commute*)

lemma *up-isotone*:
 $R \subseteq S \implies R\uparrow \subseteq S\uparrow$
by (*fact iu-left-isotone*)

lemma *up-increasing*:
 $R \subseteq R\uparrow$
by (*simp add: upclosed-ext*)

lemma *up-idempotent* [*simp*]:
 $R\uparrow\uparrow = R\uparrow$
by (*simp add: iu-assoc*)

lemma *up-dist-ou*:
 $(R \cup S)\uparrow = R\uparrow \cup S\uparrow$
by (*simp add: iu-right-dist-ou*)

lemma *up-dist-iu*:
 $(R \cup\cup S)\uparrow = R\uparrow \cup\cup S\uparrow$
using *cv-hom-par p-prod-assoc* **by** *blast*

lemma *up-dist-ii*:
 $(R \cap\cap S)\uparrow = R\uparrow \cap\cap S\uparrow$
proof (*rule antisym*)
show $(R \cap\cap S)\uparrow \subseteq R\uparrow \cap\cap S\uparrow$
by (*simp add: iu-right-subdist-ii*)

next
have $\bigwedge a B C D E . (a,B) \in R \implies (a,C) \in S \implies \exists F . (\exists G . (B \cup D) \cap (C \cup E) = F \cup G) \wedge (\exists H I . F = H \cap I \wedge (a,H) \in R \wedge (a,I) \in S)$
proof –
fix $a B C D E$
assume 1: $(a,B) \in R$
assume 2: $(a,C) \in S$
let $?F = B \cap C$
let $?G = (B \cap E) \cup (D \cap C) \cup (D \cap E)$
have $(B \cup D) \cap (C \cup E) = ?F \cup ?G$
by *auto*
thus $\exists F . (\exists G . (B \cup D) \cap (C \cup E) = F \cup G) \wedge (\exists H I . F = H \cap I \wedge (a,H) \in R \wedge (a,I) \in S)$
using 1 2 **by** *auto*
qed
thus $R \uparrow \cap \cap S \uparrow \subseteq (R \cap \cap S) \uparrow$
by (*clarsimp simp: mr-simp*)
qed

lemma *down-isotone*:
 $R \subseteq S \implies R \downarrow \subseteq S \downarrow$
by (*fact ii-left-isotone*)

lemma *down-increasing*:
 $R \subseteq R \downarrow$
by (*metis ic-involutive ic-isotone ic-up up-increasing*)

lemma *down-idempotent [simp]*:
 $R \downarrow \downarrow = R \downarrow$
by (*simp add: ic-down ic-injective*)

lemma *down-dist-ou*:
 $(R \cup S) \downarrow = R \downarrow \cup S \downarrow$
by (*fact ii-right-dist-ou*)

lemma *down-dist-iu*:
 $(R \cup \cup S) \downarrow = R \downarrow \cup \cup S \downarrow$
by (*simp add: ic-antidist-ii ic-antidist-iu ic-injective up-dist-ii*)

lemma *down-dist-ii*:
 $(R \cap \cap S) \downarrow = R \downarrow \cap \cap S \downarrow$
by (*metis down-idempotent ii-assoc ii-commute*)

lemma *convex-isotone*:
 $R \subseteq S \implies R \uparrow \subseteq S \uparrow$
by (*meson Int-mono down-isotone up-isotone*)

lemma *convex-increasing*:
 $R \subseteq R \uparrow$

by (simp add: down-increasing up-increasing)

lemma convex-idempotent [simp]:

$$R\downarrow\downarrow = R\downarrow$$

by (smt (verit, ccfv-threshold) U-par-idem convex-increasing convex-isotone ic-top ic-up ii-assoc iu-assoc le-inf-iff subsetI subset-antisym)

lemma down-sp:

$$R\downarrow = R * (1_{\cup\cup} \cup 1)$$

proof –

have $\forall a B . (\exists C . (\exists D . B = C \cap D) \wedge (a, C) \in R) \longleftrightarrow (\exists C . (a, C) \in R \wedge (\exists f . (\forall c \in C . f c = \{\} \vee f c = \{c\}) \wedge B = (\bigcup_{c \in C} . f c)))$

proof (intro allI, rule iffI)

fix a B

assume $\exists C . (\exists D . B = C \cap D) \wedge (a, C) \in R$

from this obtain C where 1: $\exists D . B = C \cap D$ and 2: $(a, C) \in R$

by auto

let $?f = \lambda c . \text{if } c \in B \text{ then } \{c\} \text{ else } \{\}$

have $(\bigcup_{c \in C} . ?f c) = (\bigcup_{c \in B} . ?f c) \cup (\bigcup_{c \in C \cap -B} . ?f c)$

using 1 by blast

hence 3: $B = (\bigcup_{c \in C} . ?f c)$

by auto

have $\forall c \in C . ?f c = \{\} \vee ?f c = \{c\}$

by auto

thus $\exists C . (a, C) \in R \wedge (\exists f . (\forall c \in C . f c = \{\} \vee f c = \{c\}) \wedge B = (\bigcup_{c \in C} . f c))$

using 2 3 by smt

next

fix a B

assume $\exists C . (a, C) \in R \wedge (\exists f . (\forall c \in C . f c = \{\} \vee f c = \{c\}) \wedge B = (\bigcup_{c \in C} . f c))$

from this obtain C where 4: $(a, C) \in R$ and $\exists f . (\forall c \in C . f c = \{\} \vee f c = \{c\}) \wedge B = (\bigcup_{c \in C} . f c)$

by auto

hence $B \subseteq C$

by auto

thus $\exists C . (\exists D . B = C \cap D) \wedge (a, C) \in R$

using 4 by auto

qed

thus ?thesis

by (clarsimp simp: mr-simp)

qed

lemma up-cp:

$$R\uparrow = \sim R \odot (1_{\cap\cap} \cup \sim 1)$$

by (metis co-prod down-sp ic-dist-ou ic-ii-unit ic-involutive ic-up)

lemma down-dist-sp:

$$(R * S)\downarrow = R * S\downarrow$$

proof (*rule antisym*)
show $(R * S)\downarrow \subseteq R * S\downarrow$
by (*simp add: down-sp s-prod-assoc1*)
next
have $\bigwedge a B f . (a, B) \in R \implies \forall b \in B . \exists C . (\exists D . f b = C \cap D) \wedge (b, C) \in S$
 $\implies \exists E . (\exists F . (\bigcup b \in B . f b) = E \cap F) \wedge (\exists B . (a, B) \in R \wedge (\exists g . (\forall b \in B . (b, g b) \in S) \wedge E = (\bigcup b \in B . g b)))$
proof –
fix $a B f$
assume $1: (a, B) \in R$
assume $\forall b \in B . \exists C . (\exists D . f b = C \cap D) \wedge (b, C) \in S$
hence $\exists g . \forall b \in B . (\exists D . f b = g b \cap D) \wedge (b, g b) \in S$
by *metis*
from this obtain g **where** $2: \forall b \in B . (\exists D . f b = g b \cap D) \wedge (b, g b) \in S$
by *auto*
hence $(\bigcup b \in B . f b) \subseteq (\bigcup b \in B . g b)$
by *blast*
thus $\exists E . (\exists F . (\bigcup b \in B . f b) = E \cap F) \wedge (\exists B . (a, B) \in R \wedge (\exists g . (\forall b \in B . (b, g b) \in S) \wedge E = (\bigcup b \in B . g b)))$
using $1\ 2$ **by** (*metis semilattice-inf-class.inf.absorb-iff2*)
qed
thus $R * S\downarrow \subseteq (R * S)\downarrow$
by (*clarsimp simp: mr-simp*)
qed

lemma *up-dist-cp*:
 $(R \odot S)\uparrow = R \odot S\uparrow$
by (*metis co-prod down-dist-sp ic-down ic-up*)

lemma *iu-up-oi*:
 $R\uparrow \cup\cup S\uparrow = R\uparrow \cap S\uparrow$
by (*fact up-closed-par-is-meet*)

lemma *ii-down-oi*:
 $R\downarrow \cap\cap S\downarrow = R\downarrow \cap S\downarrow$
by (*metis ic-antidist-ii ic-dist-oi ic-down ic-involutive up-closed-par-is-meet*)

lemma *down-dist-ii-oi*:
 $R\downarrow \cap S\downarrow = (R \cap\cap S)\downarrow$
by (*simp add: down-dist-ii ii-down-oi*)

lemma *up-dist-iu-oi*:
 $R\uparrow \cap S\uparrow = (R \cup\cup S)\uparrow$
by (*simp add: up-closed-par-is-meet up-dist-iu*)

lemma *oi-down-sub-up*:
 $R\downarrow \cap S\uparrow \subseteq (R\downarrow \cap S)\uparrow$
by (*auto simp: mr-simp*)

lemma *oi-down-up*:

$$R\downarrow \cap S = \{\} \implies R \cap S\uparrow = \{\}$$

by (*metis (no-types, lifting) cp-left-zero down-increasing ic-bot inf.orderE inf-assoc inf-bot-right oi-down-sub-up up-cp*)

lemma *oi-down-up-iff*:

$$R\downarrow \cap S = \{\} \iff R \cap S\uparrow = \{\}$$

proof (*rule iffI*)

$$\text{show } R\downarrow \cap S = \{\} \implies R \cap S\uparrow = \{\}$$

by (*simp add: oi-down-up*)

next

$$\text{assume } 1: R \cap S\uparrow = \{\}$$

$$\text{have } (\sim S)\downarrow = \sim(S\uparrow)$$

by (*metis (no-types) ic-down ic-involutive*)

$$\text{hence } \sim(R\downarrow \cap S) = \{\}$$

using 1 **by** (*metis Int-commute ic-bot ic-dist-oi ic-down oi-down-up*)

$$\text{thus } R\downarrow \cap S = \{\}$$

by (*metis (no-types) ic-bot ic-involutive*)

qed

lemma *down-double-complement-up*:

$$R\downarrow \subseteq S \iff R \subseteq -((-S)\uparrow)$$

by (*metis disjoint-eq-subset-Compl double-compl oi-down-up-iff*)

lemma *up-double-complement-down*:

$$R\uparrow \subseteq S \iff R \subseteq -((-S)\downarrow)$$

by (*metis Compl-subset-Compl-iff double-compl down-double-complement-up*)

lemma *below-up-oi-down*:

$$R \subseteq R\uparrow \cap R\downarrow$$

by (*fact convex-increasing*)

lemma *cp-pa-sim*:

$$(R \odot S)\downarrow = R \otimes S\downarrow$$

by (*metis co-prod ic-involutive ic-up pa-ic pe-pa-sim*)

lemma *domain-up-down-conjugate*:

$$(R\uparrow \cap S) * 1_{\cup\cup} = (R \cap S\downarrow) * 1_{\cup\cup}$$

apply (*rule set-eqI, clarsimp simp: mr-simp*)

by (*smt (verit, del-Insts) Int-Un-eq(1) SUP-bot SUP-bot-conv(1) Un-Int-eq(1)*)

lemma *down-below-sp-top*:

$$R\downarrow \subseteq R * U$$

apply (*clarsimp simp: mr-simp*)

by (*metis Int-Union UN-constant image-empty inf-commute*)

lemma *down-oi-up-closed*:

$$\text{assumes } Q\uparrow = Q$$

$$\text{shows } R\downarrow \cap Q \subseteq (R \cap Q)\downarrow$$

using *assms* **apply** (*clarsimp simp: mr-simp*)
by (*metis (no-types, lifting) assms inf.cobounded1 ucl-iff*)

lemma *up-dist-oU*:
 $(\bigcup X)\uparrow = \bigcup (up \text{ ` } X)$
by (*simp add: iu-right-dist-oU*)

lemma *up-dist-iU*:
assumes $I \neq \{\}$
shows $(\bigcup \bigcup X|I)\uparrow = \bigcup \bigcup (up \ o \ X)|I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*metis UN-simps(2) assms*)
apply (*clarsimp simp: mr-simp*)
subgoal for $a \ f$
proof –
fix $a \ f$
assume $\forall i \in I . \exists B . (\exists C . f \ i = B \cup C) \wedge (a, B) \in X \ i$
from *this* **obtain** g **where** $\forall i \in I . (\exists C . f \ i = g \ i \cup C) \wedge (a, g \ i) \in X \ i$
by *metis*
hence $(\exists C . \bigcup (f \text{ ` } I) = \bigcup (g \text{ ` } I) \cup C) \wedge (\bigcup (g \text{ ` } I) = \bigcup (g \text{ ` } I) \wedge (\forall i \in I . (a, g \ i) \in X \ i))$
by *auto*
thus $\exists B . (\exists C . \bigcup (f \text{ ` } I) = B \cup C) \wedge (\exists f . B = \bigcup (f \text{ ` } I) \wedge (\forall i \in I . (a, f \ i) \in X \ i))$
by *auto*
qed
done

lemma *up-dist-iI*:
 $(\bigcap \bigcap X|I)\uparrow = \bigcap \bigcap (up \ o \ X)|I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*smt (z3) INT-simps(10) sup-Inf sup-commute*)
apply (*clarsimp simp: mr-simp*)
subgoal for $a \ f$
proof –
assume $\forall i \in I . \exists B . (\exists C . f \ i = B \cup C) \wedge (a, B) \in X \ i$
from *this* **obtain** g **where** $\forall i \in I . (\exists C . f \ i = g \ i \cup C) \wedge (a, g \ i) \in X \ i$
by *metis*
hence $(\exists C . \bigcap (f \text{ ` } I) = \bigcap (g \text{ ` } I) \cup C) \wedge (\bigcap (g \text{ ` } I) = \bigcap (g \text{ ` } I) \wedge (\forall i \in I . (a, g \ i) \in X \ i))$
by *auto*
thus $\exists B . (\exists C . \bigcap (f \text{ ` } I) = B \cup C) \wedge (\exists f . B = \bigcap (f \text{ ` } I) \wedge (\forall i \in I . (a, f \ i) \in X \ i))$
by *auto*
qed
done

lemma *down-dist-oU*:

$(\bigcup X)\downarrow = \bigcup(\text{down } \cdot X)$
by (*simp add: ii-right-dist-oU*)

lemma *down-dist-iU*:

$(\bigcup\bigcup X|I)\downarrow = \bigcup\bigcup(\text{down } o X)|I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*metis UN-extend-simps(4)*)
apply (*clarsimp simp: mr-simp*)
subgoal for $a f$
proof –
assume $\forall i \in I . \exists B . (\exists C . f i = B \cap C) \wedge (a, B) \in X i$
from this obtain g **where** $\forall i \in I . (\exists C . f i = g i \cap C) \wedge (a, g i) \in X i$
by *metis*
hence $(\exists C . \bigcup(f \cdot I) = \bigcup(g \cdot I) \cap C) \wedge (\bigcup(g \cdot I) = \bigcup(g \cdot I) \wedge (\forall i \in I . (a, g i) \in X i))$
by *auto*
thus $\exists B . (\exists C . \bigcup(f \cdot I) = B \cap C) \wedge (\exists f . B = \bigcup(f \cdot I) \wedge (\forall i \in I . (a, f i) \in X i))$
by *auto*
qed
done

lemma *down-dist-iI*:

assumes $I \neq \{\}$
shows $(\bigcap\bigcap X|I)\downarrow = \bigcap\bigcap(\text{down } o X)|I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*smt (verit, del-insts) INF-const INT-absorb Int-commute assms semilattice-inf-class.inf-left-commute*)
apply (*clarsimp simp: mr-simp*)
subgoal for $a f$
proof –
assume $\forall i \in I . \exists B . (\exists C . f i = B \cap C) \wedge (a, B) \in X i$
from this obtain g **where** $\forall i \in I . (\exists C . f i = g i \cap C) \wedge (a, g i) \in X i$
by *metis*
hence $(\exists C . \bigcap(f \cdot I) = \bigcap(g \cdot I) \cap C) \wedge (\bigcap(g \cdot I) = \bigcap(g \cdot I) \wedge (\forall i \in I . (a, g i) \in X i))$
by *auto*
thus $\exists B . (\exists C . \bigcap(f \cdot I) = B \cap C) \wedge (\exists f . B = \bigcap(f \cdot I) \wedge (\forall i \in I . (a, f i) \in X i))$
by *auto*
qed
done

lemma *iU-up-oI*:

assumes $I \neq \{\}$

shows $\bigcup\bigcup(\text{up } o \ X)|I = \bigcap(\text{up } \acute{X} \acute{I})$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*metis UN-absorb sup-assoc*)
apply (*clarsimp simp: mr-simp*)
by (*metis UN-constant assms*)

lemma *iI-down-oI*:
assumes $I \neq \{\}$
shows $\bigcap\bigcap(\text{down } o \ X)|I = \bigcap(\text{down } \acute{X} \acute{I})$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*metis INF-absorb Int-assoc*)
apply (*clarsimp simp: mr-simp*)
using *INF-eq-const assms* **by** *auto*

lemma *down-dist-iI-oI*:
 $\bigcap(\text{down } \acute{X} \acute{I}) = (\bigcap\bigcap X|I)\downarrow$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*metis INF-const INF-greatest INT-absorb empty-iff*
semilattice-inf-class.inf.absorb-iff2 semilattice-inf-class.le-inf-iff)
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *up-dist-iU-oI*:
 $\bigcap(\text{up } \acute{X} \acute{I}) = (\bigcup\bigcup X|I)\uparrow$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
subgoal for a D proof –
assume $\forall i \in I . \exists B . (\exists C . D = B \cup C) \wedge (a, B) \in X \ i$
from this obtain f where 1: $\forall i \in I . (\exists C . D = f \ i \cup C) \wedge (a, f \ i) \in X \ i$
by *metis*
hence $\exists C . D = \bigcup(f \ \acute{I}) \cup C$
by *auto*
thus *?thesis*
using 1 **by** *auto*
qed
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *iu-up*:
 $(R \cup\cup R)\uparrow = R\uparrow$
using *up-dist-iu-oi* **by** *auto*

lemma *ii-down*:
 $(R \cap\cap R)\downarrow = R\downarrow$
using *down-dist-ii-oi* **by** *blast*

lemma *iU-up*:
assumes $I \neq \{\}$
shows $(\bigcup\bigcup(\lambda j . R)|I)\uparrow = R\uparrow$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
using *assms* **apply** *blast*
apply (*clarsimp simp: mr-simp*)
by (*metis UN-constant assms*)

lemma *iI-down*:
assumes $I \neq \{\}$
shows $(\bigcap\bigcap(\lambda j . R)|I)\downarrow = R\downarrow$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
using *assms* **apply** *blast*
apply (*clarsimp simp: mr-simp*)
by (*metis INF-const assms*)

lemma *iu-unit-up*:
 $1_{\cup\cup}\uparrow = U$
by (*simp add: iu-commute*)

lemma *iu-unit-down*:
 $1_{\cup\cup}\downarrow = 1_{\cup\cup}$
by (*simp add: down-sp*)

lemma *iu-unit-convex*:
 $1_{\cup\cup}\updownarrow = 1_{\cup\cup}$
by (*simp add: iu-unit-down p-id-zero*)

lemma *ii-unit-up*:
 $1_{\cap\cap}\uparrow = 1_{\cap\cap}$
by (*simp add: up-cp*)

lemma *ii-unit-down*:
 $1_{\cap\cap}\downarrow = U$
using *ii-commute ii-unit* **by** *blast*

lemma *ii-unit-convex*:
 $1_{\cap\cap}\updownarrow = 1_{\cap\cap}$
using *down-increasing ii-unit-up* **by** *blast*

lemma *sp-unit-down*:
 $1\downarrow = 1 \cup 1_{\cup\cup}$
by (*simp add: down-sp inf-sup-aci(5)*)

lemma *sp-unit-convex*:
 $1\updownarrow = 1$
unfolding *convex s-id-def* **by** *force*

lemma *top-up*:

$$U\uparrow = U$$

by *simp*

lemma *top-down*:

$$U\downarrow = U$$

by (*metis U-par-idem ic-top ic-up*)

lemma *top-convex*:

$$U\downarrow\uparrow = U$$

by (*simp add: top-down*)

lemma *bot-up*:

$$\{\}\uparrow = \{\}$$

by (*simp add: p-prod-comm*)

lemma *bot-down*:

$$\{\}\downarrow = \{\}$$

using *oi-down-up-iff* **by** *fastforce*

lemma *bot-convex*:

$$\{\}\downarrow\uparrow = \{\}$$

by (*simp add: bot-down*)

lemma *down-oi-up-convex*:

$$(R\downarrow \cap S\uparrow)\downarrow = R\downarrow \cap S\uparrow$$

unfolding *up down convex* **by** *blast*

lemma *convex-iff-down-oi-up*:

$$Q = Q\downarrow\uparrow \iff (\exists R S . Q = R\downarrow \cap S\uparrow)$$

using *down-oi-up-convex* **by** *blast*

lemma *convex-closed-oI*:

$$(\bigcap R \in X . R\downarrow\uparrow)\downarrow = (\bigcap R \in X . R\downarrow)$$

apply (*rule antisym*)

apply (*clarsimp simp: mr-simp*)

apply (*smt (verit, best) semilattice-inf-class.inf-commute*

semilattice-inf-class.inf-left-commute sup-commute sup-left-commute)

by (*meson convex-increasing*)

lemma *convex-closed-oi*:

$$(R\downarrow \cap S\downarrow)\downarrow = R\downarrow \cap S\downarrow$$

using *convex-closed-oI*[*of {R,S}*] **by** *simp*

lemma

$$(R\downarrow \cup S\downarrow)\downarrow = R\downarrow \cup S\downarrow$$

nitpick[*expect=genuine,card=1,3*]

oops

6 Powerdomain Preorders

abbreviation *lower-less-eq* :: ('a,'b) mrel \Rightarrow ('a,'b) mrel \Rightarrow bool (**infixl** $\sqsubseteq\downarrow$ 50)
where
 $R \sqsubseteq\downarrow S \equiv R \subseteq S\downarrow$

abbreviation *upper-less-eq* :: ('a,'b) mrel \Rightarrow ('a,'b) mrel \Rightarrow bool (**infixl** $\sqsubseteq\uparrow$ 50) **where**
 $R \sqsubseteq\uparrow S \equiv S \subseteq R\uparrow$

abbreviation *convex-less-eq* :: ('a,'b) mrel \Rightarrow ('a,'b) mrel \Rightarrow bool (**infixl** $\sqsubseteq\updownarrow$ 50) **where**
 $R \sqsubseteq\updownarrow S \equiv R \sqsubseteq\downarrow S \wedge R \sqsubseteq\uparrow S$

abbreviation *Convex-less-eq* :: ('a,'b) mrel \Rightarrow ('a,'b) mrel \Rightarrow bool (**infixl** $\sqsubseteq\updownarrow$ 50) **where**
 $R \sqsubseteq\updownarrow S \equiv R \subseteq S\updownarrow$

lemma *lower-less-eq*:

$R \sqsubseteq\downarrow S \iff (\forall a B . (a,B) \in R \longrightarrow (\exists C . (a,C) \in S \wedge B \subseteq C))$

apply (*clarsimp simp: mr-simp*)

apply *safe*

apply *blast*

by (*metis inf.absorb-iff2*)

lemma *upper-less-eq*:

$R \sqsubseteq\uparrow S \iff (\forall a C . (a,C) \in S \longrightarrow (\exists B . (a,B) \in R \wedge B \subseteq C))$

by (*meson U-par-st subrelI subsetD*)

lemma *Convex-less-eq*:

$R \sqsubseteq\updownarrow S \iff (\forall a C . (a,C) \in R \longrightarrow (\exists B D . (a,B) \in S \wedge (a,D) \in S \wedge B \subseteq C \wedge C \subseteq D))$

by (*meson lower-less-eq semilattice-inf-class.le-inf-iff upper-less-eq*)

lemma *Convex-lower-upper*:

$R \sqsubseteq\updownarrow S \iff R \sqsubseteq\downarrow S \wedge S \sqsubseteq\uparrow R$

by *auto*

lemma *lower-reflexive*:

$R \sqsubseteq\downarrow R$

by (*fact down-increasing*)

lemma *upper-reflexive*:

$R \sqsubseteq\uparrow R$

by (*fact up-increasing*)

lemma *convex-reflexive*:

$R \sqsubseteq\updownarrow R$

by (*simp add: lower-reflexive upper-reflexive*)

lemma *Convex-reflexive:*

$$R \sqsubseteq\Downarrow R$$

by (*fact convex-increasing*)

lemma *lower-transitive:*

$$R \sqsubseteq\Downarrow S \implies S \sqsubseteq\Downarrow T \implies R \sqsubseteq\Downarrow T$$

using *down-idempotent down-isotone* **by** *blast*

lemma *upper-transitive:*

$$R \sqsubseteq\Uparrow S \implies S \sqsubseteq\Uparrow T \implies R \sqsubseteq\Uparrow T$$

using *up-idempotent up-isotone* **by** *blast*

lemma *convex-transitive:*

$$R \sqsubseteq\Downarrow S \implies S \sqsubseteq\Downarrow T \implies R \sqsubseteq\Downarrow T$$

by (*meson lower-transitive upper-transitive*)

lemma *Convex-transitive:*

$$R \sqsubseteq\Downarrow S \implies S \sqsubseteq\Downarrow T \implies R \sqsubseteq\Downarrow T$$

by (*metis le-inf-iff lower-transitive upper-transitive*)

lemma *bot-lower-least:*

$$\{\} \sqsubseteq\Downarrow R$$

by *simp*

lemma *top-upper-least:*

$$U \sqsubseteq\Uparrow R$$

by (*metis U-par-idem iu-assoc le-inf-iff up-dist-iu-oi upper-reflexive*)

lemma *bot-Convex-least:*

$$\{\} \sqsubseteq\Downarrow R$$

by *simp*

lemma *top-lower-greatest:*

$$R \sqsubseteq\Downarrow U$$

using *U-par-idem top-down top-upper-least* **by** *blast*

lemma *bot-upper-greatest:*

$$R \sqsubseteq\Uparrow \{\}$$

by *simp*

lemma *top-Convex-greatest:*

$$R \sqsubseteq\Downarrow U$$

using *U-par-idem top-down top-upper-least* **by** *auto*

lemma *lower-iu-increasing:*

$$R \sqsubseteq\Downarrow R \cup\cup R$$

by (*meson dual-order.trans lower-reflexive subidem-par*)

lemma *upper-ii-increasing*:
 $R \sqsubseteq\uparrow R \sqcup\cup S$
using *p-prod-isor top-upper-least by auto*

lemma *convex-ii-increasing*:
 $R \sqsubseteq\updownarrow R \sqcup\cup R$
by (*simp add: lower-ii-increasing upper-ii-increasing*)

lemma *Convex-ii-increasing*:
 $R \sqsubseteq\updownarrow R \sqcup\cup R$
by (*simp add: ii-up lower-ii-increasing upper-reflexive*)

lemma *lower-ii-decreasing*:
 $R \sqcap\cap S \sqsubseteq\downarrow R$
by (*metis ii-right-isotone top-down top-lower-greatest*)

lemma *upper-ii-decreasing*:
 $R \sqcap\cap R \sqsubseteq\uparrow R$
using *convex-reflexive ii-sub-idempotent by fastforce*

lemma *convex-ii-decreasing*:
 $R \sqcap\cap R \sqsubseteq\updownarrow R$
by (*simp add: lower-ii-decreasing upper-ii-decreasing*)

lemma *Convex-ii-increasing*:
 $R \sqsubseteq\updownarrow R \sqcap\cap R$
by (*simp add: ii-down lower-reflexive upper-ii-decreasing*)

lemma *iu-lower-left-isotone*:
 $R \sqsubseteq\downarrow S \implies R \sqcup\cup T \sqsubseteq\downarrow S \sqcup\cup T$
by (*simp add: down-dist-ii iu-isotone lower-reflexive*)

lemma *iu-upper-left-isotone*:
 $R \sqsubseteq\uparrow S \implies R \sqcup\cup T \sqsubseteq\uparrow S \sqcup\cup T$
by (*metis (no-types, lifting) iu-assoc iu-commute iu-left-isotone*)

lemma *iu-convex-left-isotone*:
 $R \sqsubseteq\updownarrow S \implies R \sqcup\cup T \sqsubseteq\updownarrow S \sqcup\cup T$
by (*simp add: iu-lower-left-isotone iu-upper-left-isotone*)

lemma *iu-Convex-left-isotone*:
 $R \sqsubseteq\updownarrow S \implies R \sqcup\cup T \sqsubseteq\updownarrow S \sqcup\cup T$
by (*simp add: iu-lower-left-isotone iu-upper-left-isotone*)

lemma *iu-lower-right-isotone*:
 $R \sqsubseteq\downarrow S \implies T \sqcup\cup R \sqsubseteq\downarrow T \sqcup\cup S$
by (*simp add: iu-commute iu-lower-left-isotone*)

lemma *iu-upper-right-isotone*:

$R \sqsubseteq\uparrow S \implies T \sqcup\cup R \sqsubseteq\uparrow T \sqcup\cup S$
by (*simp add: iu-assoc iu-right-isotone*)

lemma *iu-convex-right-isotone*:
 $R \sqsubseteq\updownarrow S \implies T \sqcup\cup R \sqsubseteq\updownarrow T \sqcup\cup S$
by (*simp add: iu-lower-right-isotone iu-upper-right-isotone*)

lemma *iu-Convex-right-isotone*:
 $R \sqsubseteq\updownarrow S \implies T \sqcup\cup R \sqsubseteq\updownarrow T \sqcup\cup S$
by (*simp add: iu-lower-right-isotone iu-upper-right-isotone*)

lemma *iu-lower-isotone*:
 $R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcup\cup P \sqsubseteq\downarrow S \sqcup\cup Q$
by (*simp add: down-dist-iu iu-isotone*)

lemma *iu-upper-isotone*:
 $R \sqsubseteq\uparrow S \implies P \sqsubseteq\uparrow Q \implies R \sqcup\cup P \sqsubseteq\uparrow S \sqcup\cup Q$
by (*simp add: iu-isotone up-dist-iu*)

lemma *iu-convex-isotone*:
 $R \sqsubseteq\updownarrow S \implies P \sqsubseteq\updownarrow Q \implies R \sqcup\cup P \sqsubseteq\updownarrow S \sqcup\cup Q$
by (*simp add: iu-lower-isotone iu-upper-isotone*)

lemma *iu-Convex-isotone*:
 $R \sqsubseteq\updownarrow S \implies P \sqsubseteq\updownarrow Q \implies R \sqcup\cup P \sqsubseteq\updownarrow S \sqcup\cup Q$
by (*simp add: down-dist-iu iu-isotone up-dist-iu*)

lemma *ii-lower-left-isotone*:
 $R \sqsubseteq\downarrow S \implies R \sqcap\cap T \sqsubseteq\downarrow S \sqcap\cap T$
by (*simp add: down-dist-ii ii-isotone lower-reflexive*)

lemma *ii-upper-left-isotone*:
 $R \sqsubseteq\uparrow S \implies R \sqcap\cap T \sqsubseteq\uparrow S \sqcap\cap T$
by (*simp add: ii-isotone up-dist-ii upper-reflexive*)

lemma *ii-convex-left-isotone*:
 $R \sqsubseteq\updownarrow S \implies R \sqcap\cap T \sqsubseteq\updownarrow S \sqcap\cap T$
by (*simp add: ii-lower-left-isotone ii-upper-left-isotone*)

lemma *ii-Convex-left-isotone*:
 $R \sqsubseteq\updownarrow S \implies R \sqcap\cap T \sqsubseteq\updownarrow S \sqcap\cap T$
by (*simp add: ii-lower-left-isotone ii-upper-left-isotone*)

lemma *ii-lower-right-isotone*:
 $R \sqsubseteq\downarrow S \implies T \sqcap\cap R \sqsubseteq\downarrow T \sqcap\cap S$
by (*simp add: ii-assoc ii-right-isotone*)

lemma *ii-upper-right-isotone*:
 $R \sqsubseteq\uparrow S \implies T \sqcap\cap R \sqsubseteq\uparrow T \sqcap\cap S$

by (simp add: ii-commute ii-upper-left-isotone)

lemma *ii-convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \sqcap R \sqsubseteq\downarrow T \sqcap S$$

by (simp add: ii-lower-right-isotone ii-upper-right-isotone)

lemma *ii-Convex-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \sqcap R \sqsubseteq\downarrow T \sqcap S$$

by (simp add: ii-lower-right-isotone ii-upper-right-isotone)

lemma *ii-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap P \sqsubseteq\downarrow S \sqcap Q$$

by (simp add: down-dist-ii ii-isotone)

lemma *ii-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies P \sqsubseteq\uparrow Q \implies R \sqcap P \sqsubseteq\uparrow S \sqcap Q$$

by (simp add: ii-isotone up-dist-ii)

lemma *ii-convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap P \sqsubseteq\downarrow S \sqcap Q$$

by (simp add: ii-lower-isotone ii-upper-isotone)

lemma *ii-Convex-isotone*:

$$R \sqsubseteq\downarrow S \implies P \sqsubseteq\downarrow Q \implies R \sqcap P \sqsubseteq\downarrow S \sqcap Q$$

by (simp add: ii-lower-isotone ii-upper-isotone)

lemma *ou-lower-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

by (meson le-sup-iff lower-reflexive lower-transitive)

lemma *ou-upper-left-isotone*:

$$R \sqsubseteq\uparrow S \implies R \cup T \sqsubseteq\uparrow S \cup T$$

by (metis Un-subset-iff sup.coboundedI1 up-dist-ou upclosed-ext)

lemma *ou-convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

by (meson ou-lower-left-isotone ou-upper-left-isotone)

lemma *ou-Convex-left-isotone*:

$$R \sqsubseteq\downarrow S \implies R \cup T \sqsubseteq\downarrow S \cup T$$

by (meson le-inf-iff ou-lower-left-isotone ou-upper-left-isotone)

lemma *ou-lower-right-isotone*:

$$R \sqsubseteq\downarrow S \implies T \cup R \sqsubseteq\downarrow T \cup S$$

by (metis Un-commute ou-lower-left-isotone)

lemma *ou-upper-right-isotone*:

$$R \sqsubseteq\uparrow S \implies T \cup R \sqsubseteq\uparrow T \cup S$$

by (metis Un-commute ou-upper-left-isotone)

lemma *ou-convex-right-isotone*:

$$R \sqsubseteq\Downarrow S \implies T \cup R \sqsubseteq\Downarrow T \cup S$$

by (*meson ou-lower-right-isotone ou-upper-right-isotone*)

lemma *ou-Convex-right-isotone*:

$$R \sqsubseteq\Downarrow S \implies T \cup R \sqsubseteq\Downarrow T \cup S$$

by (*metis Un-commute ou-Convex-left-isotone*)

lemma *ou-lower-isotone*:

$$R \sqsubseteq\Downarrow S \implies P \sqsubseteq\Downarrow Q \implies R \cup P \sqsubseteq\Downarrow S \cup Q$$

using *down-dist-ou* **by** *blast*

lemma *ou-upper-isotone*:

$$R \sqsubseteq\Uparrow S \implies P \sqsubseteq\Uparrow Q \implies R \cup P \sqsubseteq\Uparrow S \cup Q$$

by (*simp add: iu-right-dist-ou sup.coboundedI1 sup.coboundedI2*)

lemma *ou-convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies P \sqsubseteq\Downarrow Q \implies R \cup P \sqsubseteq\Downarrow S \cup Q$$

by (*meson ou-lower-isotone ou-upper-isotone*)

lemma *ou-Convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies P \sqsubseteq\Downarrow Q \implies R \cup P \sqsubseteq\Downarrow S \cup Q$$

by (*metis le-inf-iff ou-lower-isotone ou-upper-isotone*)

lemma *sp-lower-left-isotone*:

$$R \sqsubseteq\Downarrow S \implies T * R \sqsubseteq\Downarrow T * S$$

by (*simp add: down-dist-sp s-prod-isor*)

lemma *sp-upper-left-isotone*:

$$R \sqsubseteq\Uparrow S \implies T * R \sqsubseteq\Uparrow T * S$$

by (*meson cl3 dual-order.trans s-prod-isor upper-iu-increasing*)

lemma *sp-convex-left-isotone*:

$$R \sqsubseteq\Downarrow S \implies T * R \sqsubseteq\Downarrow T * S$$

by (*simp add: sp-lower-left-isotone sp-upper-left-isotone*)

lemma *sp-Convex-left-isotone*:

$$R \sqsubseteq\Downarrow S \implies T * R \sqsubseteq\Downarrow T * S$$

by (*simp add: sp-lower-left-isotone sp-upper-left-isotone*)

lemma *cp-lower-left-isotone*:

$$R \sqsubseteq\Downarrow S \implies T \odot R \sqsubseteq\Downarrow T \odot S$$

by (*smt (verit) co-prod ic-antidist-ii ic-antidist-iu ic-isotone ic-top sp-upper-left-isotone*)

lemma *cp-upper-left-isotone*:

$$R \sqsubseteq\Uparrow S \implies T \odot R \sqsubseteq\Uparrow T \odot S$$

by (*simp add: cp-right-isotone up-dist-cp*)

lemma *cp-convex-left-isotone*:

$$R \sqsubseteq\uparrow S \implies T \odot R \sqsubseteq\uparrow T \odot S$$

by (*simp add: cp-lower-left-isotone cp-upper-left-isotone*)

lemma *cp-Convex-left-isotone*:

$$R \sqsubseteq\uparrow S \implies T \odot R \sqsubseteq\uparrow T \odot S$$

by (*simp add: cp-lower-left-isotone cp-upper-left-isotone*)

lemma *lower-ic-upper*:

$$R \sqsubseteq\downarrow S \longleftrightarrow \sim S \sqsubseteq\uparrow \sim R$$

by (*metis ic-down ic-involutive ic-isotone*)

lemma *upper-ic-lower*:

$$R \sqsubseteq\uparrow S \longleftrightarrow \sim S \sqsubseteq\downarrow \sim R$$

by (*simp add: lower-ic-upper*)

lemma *convex-ic*:

$$R \sqsubseteq\uparrow S \longleftrightarrow \sim S \sqsubseteq\uparrow \sim R$$

by (*meson lower-ic-upper upper-ic-lower*)

lemma *Convex-ic*:

$$R \sqsubseteq\uparrow S \longleftrightarrow \sim R \sqsubseteq\uparrow \sim S$$

by (*metis le-inf-iff lower-ic-upper upper-ic-lower*)

lemma *up-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies R\uparrow \sqsubseteq\downarrow S\uparrow$$

by (*fact iu-lower-left-isotone*)

lemma *up-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies R\uparrow \sqsubseteq\uparrow S\uparrow$$

by (*fact iu-left-isotone*)

lemma *up-convex-isotone*:

$$R \sqsubseteq\uparrow S \implies R\uparrow \sqsubseteq\uparrow S\uparrow$$

by (*fact iu-convex-left-isotone*)

lemma *up-Convex-isotone*:

$$R \sqsubseteq\uparrow S \implies R\uparrow \sqsubseteq\uparrow S\uparrow$$

by (*fact iu-Convex-left-isotone*)

lemma *down-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies R\downarrow \sqsubseteq\downarrow S\downarrow$$

by (*fact down-isotone*)

lemma *down-upper-isotone*:

$$R \sqsubseteq\uparrow S \implies R\downarrow \sqsubseteq\uparrow S\downarrow$$

by (*fact ii-upper-left-isotone*)

lemma *down-convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

by (*fact ii-convex-left-isotone*)

lemma *down-Convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies R\downarrow \sqsubseteq\Downarrow S\downarrow$$

by (*fact ii-Convex-left-isotone*)

lemma *convex-lower-isotone*:

$$R \sqsubseteq\Downarrow S \implies R\Downarrow \sqsubseteq\Downarrow S\Downarrow$$

by (*metis convex-idempotent convex-increasing le-inf-iff lower-transitive*)

lemma *convex-upper-isotone*:

$$R \sqsubseteq\Uparrow S \implies R\Downarrow \sqsubseteq\Uparrow S\Downarrow$$

by (*simp add: convex-lower-isotone ic-convex upper-ic-lower*)

lemma *convex-convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies R\Downarrow \sqsubseteq\Downarrow S\Downarrow$$

by (*simp add: convex-lower-isotone convex-upper-isotone*)

lemma *convex-Convex-isotone*:

$$R \sqsubseteq\Downarrow S \implies R\Downarrow \sqsubseteq\Downarrow S\Downarrow$$

by (*fact convex-isotone*)

lemma *subset-lower*:

$$R \subseteq S \implies R \sqsubseteq\Downarrow S$$

using *lower-reflexive* **by** *auto*

lemma *subset-upper*:

$$R \subseteq S \implies S \sqsubseteq\Uparrow R$$

using *upper-reflexive* **by** *blast*

lemma *subset-Convex*:

$$R \subseteq S \implies R \sqsubseteq\Downarrow S$$

by (*simp add: subset-lower subset-upper*)

lemma *oi-subset-lower-left-isotone*:

$$R \subseteq S \implies R \cap T \sqsubseteq\Downarrow S \cap T$$

using *lower-reflexive* **by** *fastforce*

lemma *oi-subset-upper-left-antitone*:

$$R \subseteq S \implies S \cap T \sqsubseteq\Uparrow R \cap T$$

using *upper-reflexive* **by** *force*

lemma *oi-subset-Convex-left-isotone*:

$$R \subseteq S \implies R \cap T \sqsubseteq\Downarrow S \cap T$$

by (*simp add: oi-subset-lower-left-isotone oi-subset-upper-left-antitone*)

lemma *oi-subset-lower-right-isotone*:

$R \subseteq S \implies T \cap R \sqsubseteq \downarrow T \cap S$
by (*simp add: oi-subset-lower-left-isotone semilattice-inf-class.inf-commute*)

lemma *oi-subset-upper-right-antitone*:
 $R \subseteq S \implies T \cap S \sqsubseteq \uparrow T \cap R$
by (*simp add: oi-subset-upper-left-antitone semilattice-inf-class.inf-commute*)

lemma *oi-subset-Convex-right-isotone*:
 $R \subseteq S \implies T \cap R \sqsubseteq \updownarrow T \cap S$
using *oi-subset-Convex-left-isotone* **by** *blast*

lemma *oi-subset-lower-isotone*:
 $R \subseteq S \implies P \subseteq Q \implies R \cap P \sqsubseteq \downarrow S \cap Q$
by (*meson Int-mono subset-lower*)

lemma *oi-subset-upper-antitone*:
 $R \subseteq S \implies P \subseteq Q \implies S \cap Q \sqsubseteq \uparrow R \cap P$
by (*meson Int-mono subset-upper*)

lemma *oi-subset-Convex-isotone*:
 $R \subseteq S \implies P \subseteq Q \implies R \cap P \sqsubseteq \updownarrow S \cap Q$
by (*simp add: oi-subset-lower-isotone oi-subset-upper-antitone*)

lemma *sp-iu-unit-lower*:
 $R * 1_{\cup\cup} \sqsubseteq \downarrow R$
using *lower-ii-decreasing sp-right-iu-unit* **by** *blast*

lemma *cp-ii-unit-upper*:
 $R \sqsubseteq \uparrow R \odot 1_{\cap\cap}$
by (*meson cp-right-ii-unit in-mono subsetI upper-iu-increasing*)

lemma *lower-ii-down*:
 $R \sqsubseteq \downarrow S \iff R \downarrow = (R \cap\cap S) \downarrow$
apply *safe*
apply (*metis down-dist-ii-oi inf.orderE lower-ii-decreasing lower-transitive*)
using *ii-assoc lower-ii-decreasing* **apply** *blast*
by (*metis IntE down-dist-ii-oi lower-reflexive subset-eq*)

lemma *lower-ii-lower-bound*:
 $R \sqsubseteq \downarrow S \iff R \subseteq R \cap\cap S$
by (*clarsimp simp: mr-simp*) *blast*

lemma *upper-ii-up*:
 $R \sqsubseteq \uparrow S \iff S \uparrow = (R \cup\cup S) \uparrow$
by (*metis inf.absorb-iff2 up-dist-iu-oi upclosed-ext upper-iu-increasing upper-transitive*)

lemma *upper-ii-upper-bound*:
 $R \sqsubseteq \uparrow S \iff S \subseteq R \cup\cup S$

by (*clarsimp simp: mr-simp*) *blast*

lemma

$$R \sqsubseteq\downarrow S \iff R = R \cap\cap S$$

nitpick[*expect=genuine,card=1*]

oops

lemma

$$R \sqsubseteq\uparrow S \iff S = R \cup\cup S$$

nitpick[*expect=genuine,card=1*]

oops

lemma *convex-oi-Convex-iu:*

$$R\uparrow \cap S\uparrow \sqsubseteq\uparrow R \cup\cup S$$

by (*meson inf-le1 inf-le2 iu-Convex-isotone order-trans subidem-par*)

lemma *convex-oi-Convex-ii:*

$$R\uparrow \cap S\uparrow \sqsubseteq\uparrow R \cap\cap S$$

by (*meson ii-Convex-isotone ii-sub-idempotent inf-le1 inf-le2 order-trans*)

lemma *convex-oi-iu-ii:*

$$R\uparrow \cap S\uparrow = (R \cup\cup S)\uparrow \cap (R \cap\cap S)\downarrow$$

by (*metis down-dist-ii-oi inf-assoc inf-left-commute up-dist-iu-oi*)

lemma *ii-lower-iu:*

$$R \cap\cap S \sqsubseteq\downarrow R \cup\cup S$$

apply (*clarsimp simp: mr-simp*)

by (*metis Un-Int-eq(2) inf-left-commute*)

lemma *ii-upper-iu:*

$$R \cap\cap S \sqsubseteq\uparrow R \cup\cup S$$

by (*simp add: ic-antidist-ii ic-antidist-iu ii-lower-iu upper-ic-lower*)

lemma *ii-convex-iu:*

$$R \cap\cap S \sqsubseteq\uparrow R \cup\cup S$$

by (*simp add: ii-lower-iu ii-upper-iu*)

lemma *convex-oi-iu-ii-convex:*

$$R\uparrow \cap S\uparrow = (R \cup\cup S)\uparrow \cap (R \cap\cap S)\uparrow$$

by (*metis convex-oi-iu-ii ii-lower-iu ii-upper-iu inf.commute lower-ii-down upper-ii-up*)

6.1 Functional properties of multirelations

lemma *id-one-converse:*

$$Id = 1 ; 1^\sim$$

unfolding *Id-def converse-def relcomp-unfold s-id-def* **by** *force*

lemma *dom-explicit:*

$Dom R = R ; U \cap 1$
by (*clarsimp simp: mr-simp Dom-def*) *blast*

lemma *dom-explicit-2*:
 $Dom R = R ; top \cap 1$
apply (*clarsimp simp: mr-simp Dom-def*)
apply *safe*
apply (*simp add: relcomp.relcompI top-def*)
apply *blast*
by *blast*

lemma *total-dom*:
 $total R \longleftrightarrow Dom R = 1$
unfolding *total-def dom-explicit-2*
apply (*rule iffI*)
using *top-def* **apply** *fastforce*
by (*metis Int-subset-iff dom-def dom-gla-top dom-top id-one-converse inf.idem inf-le1*)

lemma *total-eq*:
 $total R \longleftrightarrow 1_{\cup\cup} = R * 1_{\cup\cup}$
by (*metis total-dom U-c cd-iso dc dc-prop2*)

lemma *domain-pointwise*:
 $x \in R * 1_{\cup\cup} \longleftrightarrow (\exists a B . (a,B) \in R \wedge x = (a,\{\}))$
by (*smt mem-Collect-eq p-id-st*)

card only works for finite sets

lemma *univalent-2*:
 $univalent R \longleftrightarrow (\forall a . finite \{ B . (a,B) \in R \} \wedge card \{ B . (a,B) \in R \} \leq one-class.one)$
proof
assume *1: univalent R*
show $\forall a . finite \{ B . (a,B) \in R \} \wedge card \{ B . (a,B) \in R \} \leq one-class.one$
proof
fix *a*
let $?B = \{ B . (a,B) \in R \}$
show $finite ?B \wedge card ?B \leq one-class.one$
proof (*rule conjI*)
show *2: finite ?B*
proof (*rule ccontr*)
assume *3: infinite ?B*
from *this* **obtain** *B* **where** *4: (a,B) \in R*
using *not-finite-existsD* **by** *auto*
have $?B = \{B\}$
proof
show $?B \subseteq \{B\}$
using *1 4* **by** (*metis (no-types, lifting) univalent-set insertCI mem-Collect-eq subsetI*)

```

next
  show  $\{B\} \subseteq ?B$ 
    using 4 by simp
  qed
  thus False
    using 3 by auto
  qed
show  $\text{card } ?B \leq \text{one-class.one}$ 
proof (rule ccontr)
  assume 5:  $\neg \text{card } ?B \leq \text{one-class.one}$ 
  from this obtain B where 6:  $(a, B) \in R$ 
  by fastforce
  hence  $\text{card } (?B - \{B\}) \geq \text{one-class.one}$ 
  using 2 5 by auto
  from this obtain C where  $(a, C) \in R \wedge B \neq C$ 
  using 5 by (metis (no-types, lifting) CollectD One-nat-def
card.insert-remove card-Diff-singleton-if card.empty card-mono empty-iff
finite.emptyI finite.insertI insert-iff subsetI)
  thus False
    using 1 6 by (meson univalent-set)
  qed
  qed
  qed
next
  assume 5:  $\forall a . \text{finite } \{ B . (a, B) \in R \} \wedge \text{card } \{ B . (a, B) \in R \} \leq$ 
   $\text{one-class.one}$ 
  have  $\forall a B C . (a, B) \in R \wedge (a, C) \in R \longrightarrow B = C$ 
  proof (intro allI, rule impI)
    fix a B C
    let  $?B = \{ B . (a, B) \in R \}$ 
    have 6:  $\text{finite } ?B$ 
    using 5 by simp
    assume  $(a, B) \in R \wedge (a, C) \in R$ 
    hence  $\{B, C\} \subseteq ?B$ 
    by simp
    hence  $\text{card } \{B, C\} \leq \text{one-class.one}$ 
    using 5 6 by (meson card-mono le-trans)
    thus  $B = C$ 
    by (metis One-nat-def card.empty card-insert-disjoint empty-iff finite.emptyI
finite.insertI insert-absorb lessI not-le singleton-insert-inj-eq)
  qed
  thus univalent R
    by (simp add: univalent-set)
  qed

lemma univalent-3:
  univalent R  $\longleftrightarrow (\forall S . R * 1_{UU} = S * 1_{UU} \wedge S \subseteq R \longrightarrow S = R)$ 
proof
  assume 1:  $\forall S . R * 1_{UU} = S * 1_{UU} \wedge S \subseteq R \longrightarrow S = R$ 

```

```

have  $\forall a B C . (a,B) \in R \wedge (a,C) \in R \longrightarrow B = C$ 
proof (intro allI, rule impI)
  fix a B C
  assume 2:  $(a,B) \in R \wedge (a,C) \in R$ 
  show  $B = C$ 
  proof (rule ccontr)
    assume 3:  $B \neq C$ 
    let ?S =  $R - \{ (a,C) \}$ 
    have 4:  $R * 1_{UU} = ?S * 1_{UU}$ 
    proof
      show  $R * 1_{UU} \subseteq ?S * 1_{UU}$ 
      proof
        fix x: 'a × 'f set
        assume x  $\in R * 1_{UU}$ 
        from this obtain b D where  $(b,D) \in R \wedge x = (b,\{\})$ 
        by (meson domain-pointwise)
        thus x  $\in ?S * 1_{UU}$ 
        using 2 3 by (metis domain-pointwise Pair-inject insertE insert-Diff)
      qed
    next
      show  $?S * 1_{UU} \subseteq R * 1_{UU}$ 
      by (simp add: s-prod-isol)
    qed
  have ?S  $\neq R$ 
  using 2 by blast
  thus False
  using 1 4 by blast
  qed
  qed
  thus univalent R
  by (simp add: univalent-set)
next
assume 5: univalent R
show  $\forall S . R * 1_{UU} = S * 1_{UU} \wedge S \subseteq R \longrightarrow S = R$ 
proof
  fix S
  show  $R * 1_{UU} = S * 1_{UU} \wedge S \subseteq R \longrightarrow S = R$ 
  proof
    assume 6:  $R * 1_{UU} = S * 1_{UU} \wedge S \subseteq R$ 
    have  $R \subseteq S$ 
    proof
      fix x
      assume 7:  $x \in R$ 
      from this obtain a B where 8:  $x = (a,B)$ 
      by fastforce
      show x  $\in S$ 
      proof (cases  $\exists C . C \neq B \wedge (a,C) \in S$ )
        case True
        thus ?thesis
      
```

```

      using 5 6 7 8 by (metis subsetD univalent-set)
    next
      case False
      thus ?thesis
        using 6 7 8 by (metis (no-types, lifting) domain-pointwise prod.inject)
      qed
    qed
  thus  $S = R$ 
    using 6 by simp
  qed
qed

```

lemma total-2:
 $total\ R \longleftrightarrow (\forall a . \{ B . (a,B) \in R \} \neq \{\})$
 by (simp add: total-set)

lemma total-3:
 $total\ R \longleftrightarrow (\forall a . finite\ \{ B . (a,B) \in R \} \longrightarrow card\ \{ B . (a,B) \in R \} \geq one-class.one)$
 by (metis finite.emptyI nonempty-set-card total-2)

lemma total-4: $total\ R \longleftrightarrow 1_{\cup\cup} \subseteq R * 1_{\cup\cup}$
 by (simp add: c6 order-antisym-conv total-eq)

lemma deterministic-2:
 $deterministic\ R \longleftrightarrow (\forall a . card\ \{ B . (a,B) \in R \} = one-class.one)$
 apply (rule iffI)
 apply (metis One-nat-def bot-nat-0.extremum-unique deterministic-def
 le-simps(2) less-Suc-eq nonempty-set-card total-2 univalent-2)
 by (metis card-1-singletonE deterministic-def finite.emptyI finite-insert
 order.refl total-3 univalent-2)

lemma univalent-convex:
 assumes univalent S
 shows $S = S \downarrow$
 apply (rule antisym)
 apply (simp add: lower-reflexive upper-reflexive)
 apply (clarsimp simp: mr-simp)
 by (metis assms lattice-class.sup-inf-absorb sup-left-idem univalent-set)

lemma univalent-ii-idempotent:
 assumes univalent S
 shows $S = S \cup\cup S$
 apply (rule antisym)
 apply (meson convex-reflexive upper-ii-upper-bound)
 apply (clarsimp simp: mr-simp)
 by (metis assms sup.idem univalent-set)

lemma *univalent-ii-idempotent*:

assumes *univalent S*

shows $S = S \cap \cap S$

apply (*rule antisym*)

apply (*simp add: ii-sub-idempotent*)

apply (*clarsimp simp: mr-simp*)

by (*metis assms semilattice-inf-class.inf.idem univalent-set*)

lemma *univalent-down-iu-idempotent*:

assumes *univalent S*

shows $S = S \downarrow \cup \cup S$

apply (*rule antisym*)

apply (*meson convex-reflexive subset-upper upper-ii-upper-bound*)

apply (*clarsimp simp: mr-simp*)

by (*metis assms lattice-class.sup-inf-absorb sup-commute univalent-set*)

lemma *univalent-up-ii-idempotent*:

assumes *univalent S*

shows $S = S \uparrow \cap \cap S$

apply (*rule antisym*)

apply (*metis assms ii-left-isotone univalent-ii-idempotent upclosed-ext*)

apply (*clarsimp simp: mr-simp*)

by (*metis Int-commute assms lattice-class.inf-sup-absorb univalent-set*)

lemma *univalent-convex-iu-idempotent*:

assumes *univalent S*

shows $S = S \updownarrow \cup \cup S$

by (*metis assms univalent-convex univalent-iu-idempotent*)

lemma *univalent-convex-ii-idempotent*:

assumes *univalent S*

shows $S = S \updownarrow \cap \cap S$

by (*metis assms univalent-convex univalent-ii-idempotent*)

lemma *univalent-iu-closed*:

univalent R \implies univalent S \implies univalent (R $\cup \cup$ S)

by (*smt (verit, best) case-prodD mem-Collect-eq p-prod-def univalent-set*)

lemma *univalent-ii-closed*:

univalent R \implies univalent S \implies univalent (R $\cap \cap$ S)

by (*smt (verit, ccfv-SIG) CollectD Pair-inject case-prodE inner-intersection-def univalent-set*)

lemma *total-lower*:

total R \iff $1_{\cup \cup} \sqsubseteq \downarrow R$

unfolding *lower-less-eq*

by (*simp add: p-id-def total-set*)

lemma *total-upper*:

$total\ R \longleftrightarrow R \sqsubseteq\uparrow\ 1 \cap\cap$
unfolding *upper-less-eq*
by (*simp add: ii-unit-def total-set*)

lemma *total-lower-iu*:
assumes *total T*
shows $R \sqsubseteq\downarrow R \cup\cup T$
by (*metis assms iu-lower-right-isotone iu-unit total-lower*)

lemma *total-upper-ii*:
assumes *total T*
shows $R \cap\cap T \sqsubseteq\uparrow R$
by (*smt (verit, ccfv-threshold) U-par-idem assms iu-assoc iu-commute lower-ii-lower-bound total-lower-iu up-dist-ii upper-ii-up*)

lemma *total-univalent-lower-iu*:
assumes *total T*
and *univalent S*
and $T \sqsubseteq\downarrow S$
shows $T \cup\cup S = S$
proof –
have 1: $\forall a. \exists B. (a, B) \in T$
by (*meson assms(1) total-set*)
have 2: $\forall a\ B\ C. (a, B) \in S \wedge (a, C) \in S \longrightarrow B = C$
by (*meson assms(2) univalent-set*)
hence 3: $T \cup\cup S \subseteq S$
by (*metis assms(2,3) iu-left-isotone univalent-down-iu-idempotent*)
hence $S \subseteq T \cup\cup S$
apply (*clarsimp simp: mr-simp*)
using 1 2 **by** (*metis (mono-tags, lifting) CollectI Un-iff case-prodI subset-Un-eq*)
thus *?thesis*
using 3 **by** (*simp add: subset-antisym*)
qed

lemma *total-iu-closed*:
 $total\ R \implies total\ S \implies total\ (R \cup\cup S)$
by (*meson lower-transitive total-lower total-lower-iu*)

lemma *total-ii-closed*:
 $total\ R \implies total\ S \implies total\ (R \cap\cap S)$
by (*metis down-dist-ii-oi le-inf-iff total-lower*)

lemma *deterministic-lower*:
assumes *deterministic V*
shows $R \sqsubseteq\downarrow V \longleftrightarrow (\forall a\ B\ C. (a, B) \in R \wedge (a, C) \in V \longrightarrow B \subseteq C)$
proof –
have $R \sqsubseteq\downarrow V \longleftrightarrow (\forall a\ B. (a, B) \in R \longrightarrow (\exists C. (a, C) \in V \wedge B \subseteq C))$
by (*simp add: lower-less-eq*)

also have ... $\longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow (\forall C . (a,C) \in V \longrightarrow B \subseteq C))$
 by (*metis assms deterministic-set*)
finally show ?thesis
 by *blast*
qed

lemma *deterministic-upper*:
 assumes *deterministic V*
 shows $V \sqsubseteq\uparrow R \longleftrightarrow (\forall a B C . (a,B) \in R \wedge (a,C) \in V \longrightarrow C \subseteq B)$
proof –
 have $V \sqsubseteq\uparrow R \longleftrightarrow (\forall a C . (a,C) \in R \longrightarrow (\exists B . (a,B) \in V \wedge B \subseteq C))$
 by (*simp add: upper-less-eq*)
 also have ... $\longleftrightarrow (\forall a C . (a,C) \in R \longrightarrow (\forall B . (a,B) \in V \longrightarrow B \subseteq C))$
 by (*metis assms deterministic-set*)
finally show ?thesis
 by *blast*
qed

lemma *deterministic-iu-closed*:
deterministic R \implies *deterministic S* \implies *deterministic (R $\cup\cup$ S)*
 by (*simp add: deterministic-def univalent-iu-closed total-iu-closed*)

lemma *deterministic-ii-closed*:
deterministic R \implies *deterministic S* \implies *deterministic (R $\cap\cap$ S)*
 by (*simp add: deterministic-def univalent-ii-closed total-ii-closed*)

lemma *total-univalent-lower-implies-upper*:
 assumes *total T*
 and *univalent S*
 and $T \sqsubseteq\downarrow S$
 shows $T \sqsubseteq\uparrow S$
 by (*simp add: assms total-univalent-lower-iu upper-ii-upper-bound*)

lemma *total-univalent-lower-implies-convex*:
 assumes *total T*
 and *univalent S*
 and $T \sqsubseteq\downarrow S$
 shows $T \sqsubseteq\updownarrow S$
 by (*simp add: assms total-univalent-lower-implies-upper*)

lemma *total-univalent-upper-implies-lower*:
 assumes *total T*
 and *univalent S*
 and $S \sqsubseteq\uparrow T$
 shows $S \sqsubseteq\downarrow T$
proof (*clarsimp simp: mr-simp*)
 fix $a B$
 assume 1: $(a,B) \in S$
 from *this* obtain C where 2: $(a,C) \in T$

by (*meson assms(1) total-set*)
 hence $(a, C) \in S \uparrow$
 using *assms(3)* by *auto*
 from *this* obtain D where $\exists: (a, D) \in S \wedge D \subseteq C$
 using \exists by (*meson assms(3) upper-less-eq*)
 hence $D = B$
 using 1 by (*meson assms(2) univalent-set*)
 thus $\exists C . (\exists D . B = C \cap D) \wedge (a, C) \in T$
 using \exists by (*metis Int-absorb1*)
 qed

lemma *total-univalent-upper-implies-convex*:
 assumes *total T*
 and *univalent S*
 and $S \sqsubseteq \uparrow T$
 shows $S \sqsubseteq \downarrow T$
 by (*simp add: assms total-univalent-upper-implies-lower*)

lemma *deterministic-lower-upper*:
 assumes *deterministic T*
 and *deterministic S*
 shows $S \sqsubseteq \downarrow T \longleftrightarrow S \sqsubseteq \uparrow T$
 by (*meson assms deterministic-def total-univalent-lower-implies-convex total-univalent-upper-implies-lower*)

lemma *deterministic-lower-convex*:
 assumes *deterministic T*
 and *deterministic S*
 shows $S \sqsubseteq \downarrow T \longleftrightarrow S \sqsubseteq \downarrow T$
 by (*simp add: assms deterministic-lower-upper*)

lemma *deterministic-upper-convex*:
 assumes *deterministic T*
 and *deterministic S*
 shows $S \sqsubseteq \uparrow T \longleftrightarrow S \sqsubseteq \uparrow T$
 by (*simp add: assms deterministic-lower-upper*)

lemma *total-down-sp-sp-down*:
 assumes *total T*
 shows $R \downarrow * T \subseteq R * T \downarrow$

proof –
 have $R \downarrow * T \subseteq R * ((1_{\cup\cup} \cup 1) * T)$
 by (*simp add: down-sp s-prod-assoc1*)
 also have $\dots = R * (1_{\cup\cup} \cup T * 1)$
 by (*simp add: s-prod-distr*)
 also have $\dots = R * (T * 1_{\cup\cup} \cup T * 1)$
 by (*metis assms c6 order-antisym-conv total-4*)
 also have $\dots \subseteq R * (T * (1_{\cup\cup} \cup 1))$
 by (*metis down-sp le-supI s-prod-isor sp-iu-unit-lower sup-ge2*)

also have $\dots = R * T \downarrow$
by (*simp add: down-sp*)
finally show *?thesis*
by *simp*
qed

lemma *total-down-sp-semi-commute*:
 $total\ T \implies R \downarrow * T \subseteq (R * T) \downarrow$
by (*simp add: down-dist-sp total-down-sp-sp-down*)

lemma *total-down-dist-sp*:
 $total\ T \implies (R * T) \downarrow = R \downarrow * T \downarrow$
by (*smt (verit, best) down-dist-sp equalityI ii-assoc ii-isotone lower-reflexive s-prod-isol top-down total-down-sp-semi-commute*)

lemma *univalent-ic-closed*:
 $univalent\ R \longleftrightarrow univalent\ (\sim R)$
apply (*unfold univalent-set*)
apply (*clarsimp simp: mr-simp*)
by (*metis double-compl*)

lemma *total-ic-closed*:
 $total\ R \longleftrightarrow total\ (\sim R)$
by (*metis total-dom d-def-expl domain-up-down-conjugate equalityI ic-down ic-top ic-up ii-commute inf.orderE lower-ic-upper top-down top-lower-greatest total-lower total-upper-ii*)

lemma *deterministic-ic-closed*:
 $deterministic\ R \longleftrightarrow deterministic\ (\sim R)$
by (*meson deterministic-def total-ic-closed univalent-ic-closed*)

lemma *iu-unit-deterministic*:
 $deterministic\ (1 \cup \cup)$
by (*metis Lambda-empty det-lambda*)

lemma *ii-unit-deterministic*:
 $deterministic\ (1 \cap \cap)$
using *deterministic-ic-closed iu-unit-deterministic by force*

lemma *univalent-upper-iu*:
assumes *univalent R*
shows $(R \sqsubseteq \uparrow S) \longleftrightarrow (R \cup \cup S = S)$
proof –
have *1: R $\cup \cup S = S \implies R \sqsubseteq \uparrow S$*
using *upper-iu-increasing by blast*
have *2: R $\sqsubseteq \uparrow S \implies S \subseteq R \cup \cup S$*
by (*simp add: upper-ii-upper-bound*)
have $R \sqsubseteq \uparrow S \implies R \cup \cup S \subseteq S$
apply (*clarsimp simp: mr-simp*)

by (*smt* (*verit*) *Ball-Collect* *assms* *case-prodD* *le-iff-sup* *subset-refl*
sup.bounded-iff *univalent-set*)
thus *?thesis*
using 1 2 **by** *blast*
qed

lemma *univalent-lower-ii*:
assumes *univalent* *S*
shows $(R \sqsubseteq\downarrow S) = (R \cap\cap S = R)$
apply (*clarsimp* *simp*: *mr-simp*)
apply *safe*
apply (*smt* (*z3*) *CollectD* *CollectI* *Collect-cong* *Int-iff* *assms* *case-prodD*
inf-set-def *subsetD* *univalent-set*)
apply *blast*
by (*smt* (*verit*, *ccfv-threshold*) *CollectD* *Pair-inject* *case-prodE* *inf-commute*)

6.2 Equivalences induced by powerdomain preorders

abbreviation *lower-eq* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$ (**infixl** $\langle =\downarrow \rangle$ 50)
where
 $R =\downarrow S \equiv R \sqsubseteq\downarrow S \wedge S \sqsubseteq\downarrow R$

abbreviation *upper-eq* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$ (**infixl** $\langle =\uparrow \rangle$ 50)
where
 $R =\uparrow S \equiv R \sqsubseteq\uparrow S \wedge S \sqsubseteq\uparrow R$

abbreviation *convex-eq* :: $('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow \text{bool}$ (**infixl** $\langle =\updownarrow \rangle$ 50)
where
 $R =\updownarrow S \equiv R \sqsubseteq\updownarrow S \wedge S \sqsubseteq\updownarrow R$

lemma *Convex-eq*:
 $R =\updownarrow S \equiv R \sqsubseteq\updownarrow S \wedge S \sqsubseteq\updownarrow R$
by (*smt* (*z3*) *semilattice-inf-class.le-inf-iff*)

lemma *convex-lower-upper*:
 $R =\updownarrow S \longleftrightarrow R =\downarrow S \wedge R =\uparrow S$
by *auto*

lemma *lower-eq-down*:
 $R =\downarrow S \longleftrightarrow R\downarrow = S\downarrow$
using *down-idempotent* *down-lower-isotone* *lower-reflexive* **by** *blast*

lemma *upper-eq-up*:
 $R =\uparrow S \longleftrightarrow R\uparrow = S\uparrow$
by (*metis* *p-prod-comm* *upclosed-ext* *upper-ii-up*)

lemma *convex-eq-convex*:
 $R =\updownarrow S \longleftrightarrow R\updownarrow = S\updownarrow$
by (*metis* *Convex-lower-upper* *lower-eq-down* *upper-eq-up*)

lemma *lower-eq*:

$R =\downarrow S \iff (\forall a B . (\exists C . (a,C) \in R \wedge B \subseteq C) \iff (\exists C . (a,C) \in S \wedge B \subseteq C))$

by (*meson lower-less-eq order-refl order-trans*)

lemma *upper-eq*:

$R =\uparrow S \iff (\forall a C . (\exists B . (a,B) \in R \wedge B \subseteq C) \iff (\exists B . (a,B) \in S \wedge B \subseteq C))$

by (*meson order-refl order-trans upper-less-eq*)

lemma *lower-eq-reflexive*:

$R =\downarrow R$

by (*simp add: lower-reflexive*)

lemma *upper-eq-reflexive*:

$R =\uparrow R$

by (*simp add: upper-reflexive*)

lemma *convex-eq-reflexive*:

$R =\Downarrow R$

by (*simp add: lower-reflexive upper-reflexive*)

lemma *lower-eq-symmetric*:

$R =\downarrow S \implies S =\downarrow R$

by *simp*

lemma *upper-eq-symmetric*:

$R =\uparrow S \implies S =\uparrow R$

by *simp*

lemma *convex-eq-symmetric*:

$R =\Downarrow S \implies S =\Downarrow R$

by *simp*

lemma *lower-eq-transitive*:

$R =\downarrow S \implies S =\downarrow T \implies R =\downarrow T$

using *lower-transitive by auto*

lemma *upper-eq-transitive*:

$R =\uparrow S \implies S =\uparrow T \implies R =\uparrow T$

using *upper-transitive by auto*

lemma *convex-eq-transitive*:

$R =\Downarrow S \implies S =\Downarrow T \implies R =\Downarrow T$

by (*meson lower-transitive upper-transitive*)

lemma *ou-lower-eq-left-congruence*:

$R =\downarrow S \implies R \cup T =\downarrow S \cup T$

using *ou-lower-left-isotone* **by** *blast*

lemma *ou-upper-eq-left-congruence*:

$$R =\uparrow S \implies R \cup T =\uparrow S \cup T$$

using *ou-upper-left-isotone* **by** *blast*

lemma *ou-convex-eq-left-congruence*:

$$R =\Downarrow S \implies R \cup T =\Downarrow S \cup T$$

by (*meson ou-lower-left-isotone ou-upper-left-isotone*)

lemma *ou-lower-eq-right-congruence*:

$$R =\downarrow S \implies T \cup R =\downarrow T \cup S$$

using *ou-lower-right-isotone* **by** *blast*

lemma *ou-upper-eq-right-congruence*:

$$R =\uparrow S \implies T \cup R =\uparrow T \cup S$$

using *ou-upper-right-isotone* **by** *blast*

lemma *ou-convex-eq-right-congruence*:

$$R =\Downarrow S \implies T \cup R =\Downarrow T \cup S$$

by (*meson ou-lower-right-isotone ou-upper-right-isotone*)

lemma *ou-lower-eq-congruence*:

$$R =\downarrow S \implies P =\downarrow Q \implies R \cup P =\downarrow S \cup Q$$

using *ou-lower-isotone* **by** *blast*

lemma *ou-upper-eq-congruence*:

$$R =\uparrow S \implies P =\uparrow Q \implies R \cup P =\uparrow S \cup Q$$

using *ou-upper-isotone* **by** *blast*

lemma *ou-convex-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \cup P =\Downarrow S \cup Q$$

by (*meson ou-lower-isotone ou-upper-isotone*)

lemma *iu-lower-eq-left-congruence*:

$$R =\downarrow S \implies R \cup\cup T =\downarrow S \cup\cup T$$

using *iu-lower-left-isotone* **by** *blast*

lemma *iu-upper-eq-left-congruence*:

$$R =\uparrow S \implies R \cup\cup T =\uparrow S \cup\cup T$$

using *iu-upper-left-isotone* **by** *blast*

lemma *iu-convex-eq-left-congruence*:

$$R =\Downarrow S \implies R \cup\cup T =\Downarrow S \cup\cup T$$

by (*simp add: iu-lower-left-isotone iu-upper-left-isotone*)

lemma *iu-lower-eq-right-congruence*:

$$R =\downarrow S \implies T \cup\cup R =\downarrow T \cup\cup S$$

using *iu-lower-right-isotone* **by** *blast*

lemma *iu-upper-eq-right-congruence*:

$R =\uparrow S \implies T \cup\cup R =\uparrow T \cup\cup S$
using *iu-upper-right-isotone* **by** *blast*

lemma *iu-convex-eq-right-congruence*:

$R =\Downarrow S \implies T \cup\cup R =\Downarrow T \cup\cup S$
by (*simp add: iu-lower-right-isotone iu-upper-right-isotone*)

lemma *iu-lower-eq-congruence*:

$R =\downarrow S \implies P =\downarrow Q \implies R \cup\cup P =\downarrow S \cup\cup Q$
using *iu-lower-isotone* **by** *blast*

lemma *iu-upper-eq-congruence*:

$R =\uparrow S \implies P =\uparrow Q \implies R \cup\cup P =\uparrow S \cup\cup Q$
using *iu-upper-isotone* **by** *blast*

lemma *iu-convex-eq-congruence*:

$R =\Downarrow S \implies P =\Downarrow Q \implies R \cup\cup P =\Downarrow S \cup\cup Q$
by (*simp add: iu-lower-isotone iu-upper-isotone*)

lemma *ii-lower-eq-left-congruence*:

$R =\downarrow S \implies R \cap\cap T =\downarrow S \cap\cap T$
using *ii-lower-left-isotone* **by** *blast*

lemma *ii-upper-eq-left-congruence*:

$R =\uparrow S \implies R \cap\cap T =\uparrow S \cap\cap T$
using *ii-upper-left-isotone* **by** *blast*

lemma *ii-convex-eq-left-congruence*:

$R =\Downarrow S \implies R \cap\cap T =\Downarrow S \cap\cap T$
by (*simp add: ii-lower-left-isotone ii-upper-left-isotone*)

lemma *ii-lower-eq-right-congruence*:

$R =\downarrow S \implies T \cap\cap R =\downarrow T \cap\cap S$
using *ii-lower-right-isotone* **by** *blast*

lemma *ii-upper-eq-right-congruence*:

$R =\uparrow S \implies T \cap\cap R =\uparrow T \cap\cap S$
using *ii-upper-right-isotone* **by** *blast*

lemma *ii-convex-eq-right-congruence*:

$R =\Downarrow S \implies T \cap\cap R =\Downarrow T \cap\cap S$
by (*simp add: ii-lower-right-isotone ii-upper-right-isotone*)

lemma *ii-lower-eq-congruence*:

$R =\downarrow S \implies P =\downarrow Q \implies R \cap\cap P =\downarrow S \cap\cap Q$
using *ii-lower-isotone* **by** *blast*

lemma *ii-upper-eq-congruence*:

$$R =\uparrow S \implies P =\uparrow Q \implies R \sqcap P =\uparrow S \sqcap Q$$

using *ii-upper-isotone* **by** *blast*

lemma *ii-convex-eq-congruence*:

$$R =\Downarrow S \implies P =\Downarrow Q \implies R \sqcap P =\Downarrow S \sqcap Q$$

by (*simp add: ii-lower-isotone ii-upper-isotone*)

lemma *sp-lower-eq-left-congruence*:

$$R =\downarrow S \implies T * R =\downarrow T * S$$

by (*simp add: sp-lower-left-isotone*)

lemma *sp-upper-eq-left-congruence*:

$$R =\uparrow S \implies T * R =\uparrow T * S$$

by (*simp add: sp-upper-left-isotone*)

lemma *sp-convex-eq-left-congruence*:

$$R =\Downarrow S \implies T * R =\Downarrow T * S$$

by (*simp add: sp-lower-left-isotone sp-upper-left-isotone*)

lemma *cp-lower-eq-left-congruence*:

$$R =\downarrow S \implies T \odot R =\downarrow T \odot S$$

by (*simp add: cp-lower-left-isotone*)

lemma *cp-upper-eq-left-congruence*:

$$R =\uparrow S \implies T \odot R =\uparrow T \odot S$$

by (*simp add: cp-upper-left-isotone*)

lemma *cp-convex-eq-left-congruence*:

$$R =\Downarrow S \implies T \odot R =\Downarrow T \odot S$$

by (*simp add: cp-lower-left-isotone cp-upper-left-isotone*)

lemma *lower-eq-ic-upper*:

$$R =\downarrow S \iff \sim R =\uparrow \sim S$$

using *lower-ic-upper* **by** *auto*

lemma *upper-eq-ic-lower*:

$$R =\uparrow S \iff \sim R =\downarrow \sim S$$

using *upper-ic-lower* **by** *auto*

lemma *convex-eq-ic-lower*:

$$R =\Downarrow S \iff \sim R =\Downarrow \sim S$$

by (*meson lower-ic-upper upper-ic-lower*)

lemma *up-lower-eq-congruence*:

$$R =\downarrow S \implies R\uparrow =\downarrow S\uparrow$$

by (*fact iu-lower-eq-left-congruence*)

lemma *up-upper-eq-congruence*:

$R = \uparrow S \implies R \uparrow = \uparrow S \uparrow$
by (*fact iu-upper-eq-left-congruence*)

lemma *up-convex-eq-congruence*:
 $R = \Downarrow S \implies R \uparrow = \Downarrow S \uparrow$
by (*fact iu-convex-eq-left-congruence*)

lemma *down-lower-eq-congruence*:
 $R = \downarrow S \implies R \downarrow = \downarrow S \downarrow$
by (*fact ii-lower-eq-left-congruence*)

lemma *down-upper-eq-congruence*:
 $R = \uparrow S \implies R \downarrow = \uparrow S \downarrow$
by (*fact ii-upper-eq-left-congruence*)

lemma *down-convex-eq-congruence*:
 $R = \Downarrow S \implies R \downarrow = \Downarrow S \downarrow$
by (*fact ii-convex-eq-left-congruence*)

lemma *convex-lower-eq-congruence*:
 $R = \downarrow S \implies R \Downarrow = \downarrow S \Downarrow$
by (*simp add: convex-lower-isotone*)

lemma *convex-upper-eq-congruence*:
 $R = \uparrow S \implies R \Downarrow = \uparrow S \Downarrow$
by (*simp add: convex-upper-isotone*)

lemma *convex-convex-eq-congruence*:
 $R = \Downarrow S \implies R \Downarrow = \Downarrow S \Downarrow$
by (*simp add: convex-lower-isotone convex-upper-isotone*)

lemma *univalent-lower-eq-subset*:

assumes *univalent S*

and $S = \downarrow R$

shows $S \subseteq R$

proof –

have 1: $\forall a B C. (a, B) \in S \wedge (a, C) \in S \longrightarrow B = C$

using *assms(1)* **by** (*simp add: univalent-set*)

have $\forall a B. (\exists A. (a, A) \in S \wedge B \subseteq A) = (\exists A. (a, A) \in R \wedge B \subseteq A)$

by (*meson assms(2) lower-eq*)

hence $\forall a B. (a, B) \in S \longrightarrow (a, B) \in R$

using 1 **by** (*smt (verit, del-insts) assms(2) lower-less-eq subset-antisym*)

thus *?thesis*

by (*simp add: subset-iff*)

qed

lemma *univalent-lower-eq*:

assumes *univalent R*

and *univalent S*

and $R = \downarrow S$
shows $R = S$
by (*meson assms subset-antisym univalent-lower-eq-subset*)

lemma *univalent-lower-eq-iff*:
assumes *univalent R*
and *univalent S*
shows $(R = \downarrow S) \longleftrightarrow (R = S)$
using *assms lower-reflexive univalent-lower-eq* **by** *auto*

lemma *univalent-upper-eq-subset*:
assumes *univalent S*
and $S = \uparrow R$
shows $S \subseteq R$

proof –
have $1: \forall a B C. (a, B) \in S \wedge (a, C) \in S \longrightarrow B = C$
using *assms(1)* **by** (*simp add: univalent-set*)
have $\forall a B. (\exists A. (a, A) \in S \wedge A \subseteq B) = (\exists A. (a, A) \in R \wedge A \subseteq B)$
by (*meson assms(2) upper-eq*)
hence $\forall a B. (a, B) \in S \longrightarrow (a, B) \in R$
using 1 **by** (*smt (verit) order-refl subset-antisym*)
thus *?thesis*
by (*simp add: subset-iff*)

qed

lemma *univalent-upper-eq*:
assumes *univalent R*
and *univalent S*
and $R = \uparrow S$
shows $R = S$
by (*meson assms subset-antisym univalent-upper-eq-subset*)

lemma *univalent-upper-eq-iff*:
assumes *univalent R*
and *univalent S*
shows $(R = \uparrow S) \longleftrightarrow (R = S)$
using *assms univalent-upper-eq upclosed-ext* **by** *blast*

lemma *univalent-convex-eq-iff*:
assumes *univalent R*
and *univalent S*
shows $(R = \updownarrow S) \longleftrightarrow (R = S)$
by (*metis assms univalent-lower-eq-iff univalent-upper-eq-iff*)

lemma *total-univalent-upper-ii*:
assumes *total T*
and *univalent S*
and $S \sqsubseteq \uparrow T$
shows $T \cap \cap S = S$

apply (*rule antisym*)
apply (*metis assms(2,3) ii-left-isotone univalent-up-ii-idempotent*)
by (*metis assms ii-commute lower-ii-lower-bound*
total-univalent-upper-implies-lower)

lemma *lower-eq-down-closed*:
 $R =\downarrow R\downarrow$
by (*simp add: subset-lower*)

lemma *upper-eq-up-closed*:
 $R =\uparrow R\uparrow$
by (*simp add: subset-upper*)

lemma *convex-eq-up-closed*:
 $R =\Downarrow R\Downarrow$
by (*simp add: subset-lower subset-upper*)

lemma *lower-join*:
 $(\forall P . Q \sqsubseteq\downarrow P \longleftrightarrow R \sqsubseteq\downarrow P \wedge S \sqsubseteq\downarrow P) \longleftrightarrow Q =\downarrow R \cup S$
by (*meson Un-subset-iff lower-reflexive lower-transitive*)

lemma *lower-meet*:
 $(\forall P . P \sqsubseteq\downarrow Q \longleftrightarrow P \sqsubseteq\downarrow R \wedge P \sqsubseteq\downarrow S) \longleftrightarrow Q =\downarrow R \cap S$
by (*metis (no-types, lifting) down-dist-ii-oi le-inf-iff lower-eq-down*
lower-reflexive)

lemma *upper-join*:
 $(\forall P . Q \sqsubseteq\uparrow P \longleftrightarrow R \sqsubseteq\uparrow P \wedge S \sqsubseteq\uparrow P) \longleftrightarrow Q =\uparrow R \cup\cup S$
by (*metis (no-types, lifting) convex-increasing le-inf-iff up-dist-iu-oi*
upper-eq-up)

lemma *upper-meet*:
 $(\forall P . P \sqsubseteq\uparrow Q \longleftrightarrow P \sqsubseteq\uparrow R \wedge P \sqsubseteq\uparrow S) \longleftrightarrow Q =\uparrow R \cup S$
by (*meson Un-subset-iff upper-reflexive upper-transitive*)

lemma *lower-ii-idempotent*:
 $R \cap\cap R =\downarrow R$
using *ii-down lower-reflexive by blast*

lemma *upper-iu-idempotent*:
 $R \cup\cup R =\uparrow R$
using *iu-up upper-reflexive by auto*

lemma *lower-iI-idempotent*:
 $I \neq \{\} \Longrightarrow (\bigcap\bigcap(\lambda j . R)|I) =\downarrow R$
by (*metis iI-down lower-eq-down*)

lemma *upper-iU-idempotent*:
 $I \neq \{\} \Longrightarrow (\bigcup\bigcup(\lambda j . R)|I) =\uparrow R$

by (*metis iU-up upper-eq-up*)

lemma *down-closed-intersection-closed*:

$R = R\downarrow \implies \forall I . I \neq \{\} \longrightarrow (\bigcap (\lambda j . R)|I) \subseteq R$
by (*metis lower-iI-idempotent*)

lemma *up-closed-union-closed*:

$R = R\uparrow \implies \forall I . I \neq \{\} \longrightarrow (\bigcup (\lambda j . R)|I) \subseteq R$
by (*metis upper-iU-idempotent*)

lemma *ou-down-lower-eq-ou*:

$R\downarrow \cup S\downarrow =\downarrow R \cup S$
using *down-dist-ou lower-eq-down-closed* by *blast*

lemma *oi-down-lower-eq-ii*:

$R\downarrow \cap S\downarrow =\downarrow R \cap S$
by (*simp add: down-dist-ii-oi lower-reflexive*)

lemma *ou-up-upper-eq-ou*:

$R\uparrow \cup S\uparrow =\uparrow R \cup S$
by (*metis ou-upper-isotone up-idempotent upper-reflexive*)

lemma *oi-up-upper-eq-iu*:

$R\uparrow \cap S\uparrow =\uparrow R \cup S$
by (*simp add: up-dist-iu-oi upper-reflexive*)

lemma *oU-down-lower-eq-oU*:

$(\bigcup R \in X . R\downarrow) =\downarrow \bigcup X$
by (*metis down-dist-oU lower-eq-down-closed*)

lemma *oI-down-lower-eq-iI*:

$(\bigcap i \in I . X i\downarrow) =\downarrow \bigcap X|I$
apply *safe*
using *down-dist-iI-oI* apply *fastforce*
by (*metis (no-types, lifting) down-dist-iI-oI image-cong image-image lower-eq-down subsetD*)

lemma *oU-up-upper-eq-oU*:

$(\bigcup R \in X . R\uparrow) =\uparrow \bigcup X$
by (*metis up-dist-oU upper-eq-up-closed*)

lemma *oI-up-upper-eq-iI*:

$(\bigcap i \in I . X i\uparrow) =\uparrow \bigcup X|I$
by (*smt (z3) INT-extend-simps(10) Sup.SUP-cong U-par-idem p-prod-assoc p-prod-comm top-upper-least up-dist-iU-oI upper-ii-upper-bound*)

lemma *down-order-lower*:

$R\downarrow \subseteq S\downarrow \iff R \sqsubseteq\downarrow S$
by (*meson lower-eq-down-closed lower-transitive*)

lemma *up-order-upper*:
 $R\uparrow \subseteq S\uparrow \iff S \sqsubseteq\uparrow R$
by (*meson upper-eq-up-closed upper-transitive*)

lemma *convex-order-lower-upper*:
 $R\updownarrow \subseteq S\updownarrow \iff R \sqsubseteq\downarrow S \wedge S \sqsubseteq\uparrow R$
by (*meson convex-eq-up-closed le-inf-iff lower-transitive upper-transitive*)

lemma *convex-order-Convex*:
 $R\updownarrow \subseteq S\updownarrow \iff R \sqsubseteq\updownarrow S$
by (*meson Convex-lower-upper convex-order-lower-upper*)

6.3 Further results for convex-closure

lemma *convex-down*:
 $R\updownarrow\downarrow = R\downarrow$
by (*meson convex-eq-up-closed lower-eq-down*)

lemma *convex-up*:
 $R\updownarrow\uparrow = R\uparrow$
by (*meson convex-eq-up-closed upper-eq-up*)

lemma *iu-dist-oi-convex*:
assumes $R = R\updownarrow$
and $S = S\updownarrow$
and $T = T\updownarrow$
shows $(R \cap S) \cup T = (R \cup T) \cap (S \cup T)$
nitpick[*expect=genuine,card=1*]
oops

lemma *ii-dist-oi-convex*:
assumes $R = R\updownarrow$
and $S = S\updownarrow$
and $T = T\updownarrow$
shows $(R \cap S) \cap T = (R \cap T) \cap (S \cap T)$
nitpick[*expect=genuine,card=1*]
oops

lemma *oI-up-closed*:
assumes $\forall R \in X . R\uparrow = R$
shows $(\bigcap X)\uparrow = \bigcap X$
by (*meson Inter-iff assms ucl-iff*)

lemma *oI-down-closed*:
assumes $\forall R \in X . R\downarrow = R$
shows $(\bigcap X)\downarrow = \bigcap X$
by (*metis (no-types, opaque-lifting) antisym assms down-isotone dual-order.refl le-Inf-iff lower-reflexive*)

lemma *oI-convex-closed*:
assumes $\forall R \in X . R \downarrow = R$
shows $(\bigcap X) \downarrow = \bigcap X$
by (*metis (no-types, lifting) Inf-lower Inter-greatest antisym assms convex-increasing convex-isotone*)

lemma *up-dist-Union*:
 $(\bigcup X) \uparrow = \bigcup \{ R \uparrow \mid R . R \in X \}$
by (*simp add: Setcompr-eq-image up-dist-oU*)

lemma *down-dist-Union*:
 $(\bigcup X) \downarrow = \bigcup \{ R \downarrow \mid R . R \in X \}$
by (*simp add: Setcompr-eq-image down-dist-oU*)

lemma *convex-dist-Union*:
 $(\bigcup X) \downarrow = \bigcup \{ R \downarrow \mid R . R \in X \}$
nitpick[*expect=genuine,card=1,2*]
oops

lemma *up-dist-Inter*:
 $(\bigcap X) \uparrow = \bigcap \{ R \uparrow \mid R . R \in X \}$
nitpick[*expect=genuine,card=1*]
oops

lemma *down-dist-Inter*:
 $(\bigcap X) \downarrow = \bigcap \{ R \downarrow \mid R . R \in X \}$
nitpick[*expect=genuine,card=1*]
oops

lemma *convex-dist-Inter*:
 $(\bigcap X) \downarrow = \bigcap \{ R \downarrow \mid R . R \in X \}$
nitpick[*expect=genuine,card=1,2*]
oops

lemma *Inter-convex-closed*:
 $(\bigcap X) \downarrow = \bigcap X$
nitpick[*expect=genuine,card=1,2*]
oops

abbreviation *convex-iu* (**infixl** $\langle \cup \cup \downarrow \rangle$ 70)
where $R \cup \cup \downarrow S \equiv (R \cup \cup S) \downarrow$

lemma *convex-iu*:
 $R \cup \cup \downarrow S = (R \downarrow \cup \cup S \downarrow) \cap R \uparrow \cap S \uparrow$
by (*metis Int-assoc Int-commute down-dist-iu up-dist-iu-oi*)

lemma *convex-iu-sub*:
 $R \downarrow \cup \cup S \subseteq R \cup \cup \downarrow S$

by (meson equalityE iu-Convex-left-isotone)

lemma convex-iu-convex-left:

$$R \cup\cup\downarrow S = R\downarrow \cup\cup\downarrow S$$

by (simp add: convex-down convex-iu convex-up)

lemma convex-iu-convex-right:

$$R \cup\cup\downarrow S = R \cup\cup\downarrow S\downarrow$$

by (simp add: convex-down convex-iu convex-up)

lemma convex-iu-convex:

$$R \cup\cup\downarrow S = R\downarrow \cup\cup\downarrow S\downarrow$$

by (simp add: convex-down convex-iu convex-up)

lemma convex-iu-assoc:

$$(R \cup\cup\downarrow S) \cup\cup\downarrow T = R \cup\cup\downarrow (S \cup\cup\downarrow T)$$

by (smt convex-iu-convex-left convex-iu-convex-right p-prod-assoc)

lemma convex-iu-comm:

$$R \cup\cup\downarrow S = S \cup\cup\downarrow R$$

by (simp add: p-prod-comm)

lemma convex-iu-unit:

$$R = R\downarrow \implies R \cup\cup\downarrow 1_{\cup\cup} = R$$

by simp

abbreviation convex-ii (infixl $\langle \cap\cap\downarrow \rangle$ 70)

where $R \cap\cap\downarrow S \equiv (R \cap\cap S)\downarrow$

lemma convex-ii:

$$R \cap\cap\downarrow S = (R\uparrow \cap\cap S\uparrow) \cap R\downarrow \cap S\downarrow$$

by (simp add: down-dist-ii-oi inf-assoc up-dist-ii)

lemma convex-ii-sub:

$$R\downarrow \cap\cap S \subseteq R \cap\cap\downarrow S$$

by (meson equalityE ii-Convex-left-isotone)

lemma convex-ii-convex-left:

$$R \cap\cap\downarrow S = R\downarrow \cap\cap\downarrow S$$

by (simp add: convex-down convex-ii convex-up)

lemma convex-ii-convex-right:

$$R \cap\cap\downarrow S = R \cap\cap\downarrow S\downarrow$$

by (simp add: convex-down convex-ii convex-up)

lemma convex-ii-convex:

$$R \cap\cap\downarrow S = R\downarrow \cap\cap\downarrow S\downarrow$$

by (simp add: convex-down convex-ii convex-up)

lemma *convex-ii-assoc*:

$$(R \text{ nri } S) \text{ nri } T = R \text{ nri } (S \text{ nri } T)$$

by (*smt convex-ii-convex-left convex-ii-convex-right ii-assoc*)

lemma *convex-ii-comm*:

$$R \text{ nri } S = S \text{ nri } R$$

by (*simp add: ii-commute*)

lemma *convex-ii-unit*:

$$R = R \downarrow \implies R \text{ nri } 1_{\text{ri}} = R$$

by *simp*

lemma *convex-iu-ic*:

$$\sim(R \text{ iu } S) = \sim R \text{ nri } \sim S$$

by (*simp add: ic-antidist-iu ic-convex*)

lemma *convex-ii-ic*:

$$\sim(R \text{ nri } S) = \sim R \text{ iu } \sim S$$

by (*simp add: ic-antidist-ii ic-convex*)

abbreviation *convex-sup* :: ('a,'b) mrel set \Rightarrow ('a,'b) mrel ($\cup \downarrow$) **where**

$$\cup \downarrow X \equiv (\cup X) \downarrow$$

lemma *convex-sup-convex*:

$$\cup \downarrow X = (\cup \downarrow X) \downarrow$$

by *simp*

lemma *convex-sup-inter*:

$$\cup \downarrow X = \bigcap \{ Y . Y = Y \downarrow \wedge \cup X \subseteq Y \}$$

apply (*rule antisym*)

apply (*smt (verit) Inf-greatest convex-isotone mem-Collect-eq*)

by (*smt (verit, del-Insts) Inter-lower convex-down convex-increasing convex-up mem-Collect-eq*)

lemma *convex-iu-dist-convex-sup*:

$$\cup \downarrow X \text{ iu } S = \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$$

proof –

have $\cup \downarrow X \text{ iu } S = \cup X \text{ iu } S$

using *convex-iu-convex-left* **by** *blast*

also have $\dots = \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$

by (*simp add: Setcompr-eq-image iu-right-dist-oU*)

also have $\dots = \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$

proof (*rule antisym*)

show $\cup \downarrow \{ R \text{ iu } S \mid R . R \in X \} \subseteq \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$

by (*smt (verit, cfv-SIG) Convex-transitive Sup-mono dual-order.refl mem-Collect-eq subset-Convex*)

have $\forall R \in X . R \text{ iu } S \subseteq \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$

by (*smt (verit, cfv-SIG) Union-upper convex-isotone mem-Collect-eq*)

thus $\cup \downarrow \{ R \text{ iu } S \mid R . R \in X \} \subseteq \cup \downarrow \{ R \text{ iu } S \mid R . R \in X \}$

```

    apply (subst convex-order-Convex)
    apply (rule Union-least)
    by blast
  qed
  finally show ?thesis
.
qed

lemma convex-ii-dist-convex-sup:
   $\bigcup\uparrow X \text{ nn}\downarrow S = \bigcup\uparrow\{ R \text{ nn}\downarrow S \mid R . R \in X \}$ 
proof -
  have  $\bigcup\uparrow X \text{ nn}\downarrow S = \bigcup X \text{ nn}\downarrow S$ 
    using convex-ii-convex-left by blast
  also have ... =  $\bigcup\uparrow\{ R \text{ nn} S \mid R . R \in X \}$ 
    by (simp add: Setcompr-eq-image ii-right-dist-oU)
  also have ... =  $\bigcup\uparrow\{ R \text{ nn}\downarrow S \mid R . R \in X \}$ 
  proof (rule antisym)
    show  $\bigcup\uparrow\{ R \text{ nn} S \mid R . R \in X \} \subseteq \bigcup\uparrow\{ R \text{ nn}\downarrow S \mid R . R \in X \}$ 
      by (smt (verit, ccfv-SIG) Convex-transitive Sup-mono dual-order.refl
mem-Collect-eq subset-Convex)
    have  $\forall R \in X . R \text{ nn}\downarrow S \subseteq \bigcup\uparrow\{ R \text{ nn} S \mid R . R \in X \}$ 
      by (smt (verit, ccfv-SIG) Union-upper convex-isotone mem-Collect-eq)
    thus  $\bigcup\uparrow\{ R \text{ nn}\downarrow S \mid R . R \in X \} \subseteq \bigcup\uparrow\{ R \text{ nn} S \mid R . R \in X \}$ 
      apply (subst convex-order-Convex)
      apply (rule Union-least)
      by blast
  qed
  finally show ?thesis
.
qed

lemma convex-dist-sup:
   $(\bigcup X)\downarrow = \bigcup\uparrow\{ R\downarrow \mid R . R \in X \}$ 
proof (rule antisym)
  have  $\forall R \in X . R \subseteq \bigcup\uparrow\{ R\downarrow \mid R . R \in X \}$ 
    by (metis (mono-tags, lifting) Sup-upper2 convex-increasing mem-Collect-eq
subset-Convex)
  thus  $(\bigcup X)\downarrow \subseteq \bigcup\uparrow\{ R\downarrow \mid R . R \in X \}$ 
    by (meson Sup-least convex-order-Convex)
  have  $\forall R \in X . R\downarrow \subseteq (\bigcup X)\downarrow$ 
    by (meson Union-upper convex-isotone)
  thus  $\bigcup\uparrow\{ R\downarrow \mid R . R \in X \} \subseteq (\bigcup X)\downarrow$ 
    apply (subst convex-order-Convex)
    apply (rule Union-least)
    by blast
qed

```

7 Fusion and Fission

7.1 Atoms and co-atoms

definition *atoms* :: ('a,'b) mrel (⋈A_{UU})
where $A_{UU} \equiv \{ (a, \{b\}) \mid a \ b . \text{True} \}$

definition *co-atoms* :: ('a,'b) mrel (⋈A_{∩∩})
where $A_{\cap\cap} \equiv \{ (a, \text{UNIV} - \{b\}) \mid a \ b . \text{True} \}$

declare *atoms-def* [mr-simp] *co-atoms-def* [mr-simp]

lemma *atoms-solution*:

$A_{UU}\uparrow = -1_{UU}$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply (*clarsimp simp: mr-simp*)
by (*metis equals0I insert-is-Un mk-disjoint-insert*)

lemma *atoms-least-solution*:

assumes $R\uparrow = -1_{UU}$
shows $A_{UU} \subseteq R$

proof

fix $x :: 'a \times 'b$ set
assume $1: x \in A_{UU}$
from *this* **obtain** $a \ b$ **where** $2: x = (a, \{b\})$
by (*smt CollectD atoms-def*)
have $3: x \in R\uparrow$
using 1 *assms atoms-solution upper-reflexive* **by** *fastforce*
have $(a, \{b\}) \notin R\uparrow$
by (*metis ComplD IntE U-c U-par-idem assms domain-pointwise p-prod-assoc up-dist-iu-oi*)
thus $x \in R$
using $2 \ 3$ **by** (*smt (verit) U-par-st subsetD subset-singletonD upclosed-ext*)
qed

lemma *ic-atoms*:

$\sim A_{UU} = A_{\cap\cap}$
apply (*clarsimp simp: mr-simp*)
by *fastforce*

lemma *ic-co-atoms*:

$\sim A_{\cap\cap} = A_{UU}$
by (*metis ic-atoms ic-involutive*)

lemma *co-atoms-solution*:

$A_{\cap\cap}\downarrow = -1_{\cap\cap}$
by (*metis atoms-solution ic-atoms ic-dist-oc ic-iu-unit ic-up*)

lemma *co-atoms-least-solution*:

assumes $R \downarrow = -1_{\cap\cap}$
shows $A_{\cap\cap} \subseteq R$
by (*metis assms atoms-least-solution ic-atoms ic-dist-oc ic-down ic-ii-unit ic-involutive ic-isotone*)

lemma *iu-unit-atoms-disjoint*:

$1_{\cup\cup} \cap A_{\cup\cup} = \{\}$
by (*metis Compl-disjoint atoms-solution iu-unit-down oi-down-up-iff*)

lemma *ii-unit-co-atoms-disjoint*:

$1_{\cap\cap} \cap A_{\cap\cap} = \{\}$
using *co-atoms-solution lower-reflexive* **by** *fastforce*

lemma *atoms-sp-idempotent*:

$A_{\cup\cup} * A_{\cup\cup} = A_{\cup\cup}$
by (*auto simp: mr-simp*)

lemma *atoms-sp-cp*:

$(R \cap A_{\cup\cup}) * S = (R \cap A_{\cup\cup}) \odot S$
by (*auto simp: mr-simp*)

7.2 Inner-functional properties

abbreviation *inner-univalent* :: ('a,'b) mrel \Rightarrow bool **where**

inner-univalent $R \equiv R \subseteq 1_{\cup\cup} \cup A_{\cup\cup}$

abbreviation *inner-total* :: ('a,'b) mrel \Rightarrow bool **where**

inner-total $R \equiv R \subseteq -1_{\cup\cup}$

abbreviation *inner-deterministic* :: ('a,'b) mrel \Rightarrow bool **where**

inner-deterministic $R \equiv \text{inner-total } R \wedge \text{inner-univalent } R$

lemma *inner-deterministic-atoms*:

inner-deterministic $R \longleftrightarrow R \subseteq A_{\cup\cup}$
using *atoms-solution upper-reflexive* **by** *fastforce*

lemma *inner-univalent*:

inner-univalent $R \longleftrightarrow (\forall a b c B . (a,B) \in R \wedge b \in B \wedge c \in B \longrightarrow b = c)$
apply (*clarsimp simp: mr-simp, safe*)
apply *blast*
by (*smt (z3) UNIV-I UN-iff equals0I insertI1 insert-absorb singleton-insert-inj-eq' subsetI*)

lemma *inner-univalent-2*:

inner-univalent $R \longleftrightarrow (\forall a B . (a,B) \in R \longrightarrow \text{finite } B \wedge \text{card } B \leq \text{one-class.one})$
apply (*rule iffI*)
apply (*metis card-eq-0-iff finite.emptyI inner-univalent is-singletonI' is-singleton-altdef linear nonempty-set-card*)

by (*metis all-not-in-conv card-1-singletonE eq-iff inner-univalent nonempty-set-card singletonD*)

lemma *inner-total*:

inner-total $R \iff (\forall a B . (a,B) \in R \longrightarrow (\exists b . b \in B))$

apply (*rule iffI*)

apply (*smt (verit, del-insts) Collect-empty-eq all-not-in-conv disjoint-eq-subset-Compl p-id-zero-st*)

by (*smt (verit) Collect-empty-eq disjoint-eq-subset-Compl ex-in-conv p-id-zero-st*)

lemma *inner-total-2*:

inner-total $R \iff (\forall a B . (a,B) \in R \longrightarrow B \neq \{\})$

by (*meson all-not-in-conv inner-total*)

lemma *inner-total-3*:

inner-total $R \iff (\forall a B . (a,B) \in R \wedge \text{finite } B \longrightarrow \text{card } B \geq \text{one-class.one})$

by (*metis finite.emptyI inner-total-2 nonempty-set-card*)

lemma *inner-deterministic*:

inner-deterministic $R \iff (\forall a B . (a,B) \in R \longrightarrow (\exists! b . b \in B))$

by (*metis (no-types, lifting) inner-total inner-univalent*)

lemma *inner-deterministic-2*:

inner-deterministic $R \iff (\forall a B . (a,B) \in R \longrightarrow \text{card } B = \text{one-class.one})$

by (*metis card-1-singletonE eq-iff finite.emptyI finite-insert inner-total-3 inner-univalent-2*)

lemma *inner-deterministic-sp-unit*:

inner-deterministic 1

by (*simp add: inner-deterministic s-id-def*)

lemma *inner-univalent-down*:

assumes *inner-univalent* S

shows $S \downarrow \subseteq S \cup 1_{\cup\cup}$

using *assms* **by** (*auto simp: mr-simp*)

lemma *inner-deterministic-lower-eq*:

assumes *inner-deterministic* V

and *inner-deterministic* W

and $V = \downarrow W$

shows $V = W$

using *assms* *inner-univalent-down* **by** *blast*

lemma *inner-total-down-closed*:

inner-total $T \implies R \subseteq T \implies \text{inner-total } R$

by *simp*

lemma *inner-univalent-down-closed*:

inner-univalent $T \implies R \subseteq T \implies \textit{inner-univalent } R$
by *simp*

lemma *inner-deterministic-down-closed*:
inner-deterministic $T \implies R \subseteq T \implies \textit{inner-deterministic } R$
by *blast*

lemma *inner-univalent-conver*:
assumes *inner-univalent* R
shows $R = R\downarrow$
apply (*rule antisym*)
using *convex-increasing* **apply** *blast*
apply (*clarsimp simp: mr-simp*)
by (*smt (verit) Un-Int-eq(2) Un-Int-eq(3) assms boolean-algebra-cancel.sup0 disjoint-iff inner-univalent semilattice-inf-class.inf.orderE subsetI*)

lemma *inner-deterministic-alt-closure*:
inner-deterministic $R = (R \textit{ O } \textit{converse } 1 \textit{ O } 1 = R)$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *force*
by *blast+*

lemma *inner-deterministic-s-id-conv-epsiloff*:
inner-deterministic $R \implies R \textit{ O } \textit{converse } s\text{-id} = R \textit{ O } \textit{epsiloff}$
apply (*clarsimp simp: mr-simp*)
unfolding *epsiloff-def*
by *blast*

lemma *inner-deterministic-lower-iff*:
assumes *inner-deterministic* R
and *inner-deterministic* S
shows $(R \sqsubseteq\downarrow S) \longleftrightarrow (R \subseteq S)$
apply *standard*
apply (*smt (verit, ccfv-threshold) Un-commute assms disjoint-eq-subset-Compl inf.orderE inf.orderI inf-commute inf-sup-distrib2 inner-univalent-down sup.orderE sup-bot-left*)
by (*simp add: subset-lower*)

lemma *inner-deterministic-upper-iff*:
assumes *inner-deterministic* R
and *inner-deterministic* S
shows $(R \sqsubseteq\uparrow S) \longleftrightarrow (S \subseteq R)$
apply *standard*
apply (*clarsimp simp: mr-simp*)
using *inner-deterministic* **apply** (*smt (verit, del-insts) Ball-Collect Un-subset-iff assms case-prodD subsetD subsetI subset-antisym*)
by (*simp add: subset-upper*)

lemma *inner-deterministic-lower-eq-iff*:
assumes *inner-deterministic R*
and *inner-deterministic S*
shows $(R = \downarrow S) \longleftrightarrow (R = S)$
by (*meson assms inner-deterministic-lower-eq lower-reflexive*)

lemma *inner-deterministic-upper-eq-iff*:
assumes *inner-deterministic R*
and *inner-deterministic S*
shows $(R = \uparrow S) \longleftrightarrow (R = S)$
by (*simp add: antisym assms inner-deterministic-upper-iff*)

lemma *inner-deterministic-convex-eq-iff*:
assumes *inner-deterministic R*
and *inner-deterministic S*
shows $(R = \downarrow\uparrow S) \longleftrightarrow (R = S)$
by (*metis assms inner-deterministic-lower-eq-iff inner-deterministic-upper-eq-iff*)

lemma
inner-univalent R \implies inner-univalent S \implies inner-univalent (R $\cup\cup$ S)
nitpick[*expect=genuine,card=1,2*]
oops

lemma *inner-univalent-ii-closed*:
inner-univalent R \implies inner-univalent S \implies inner-univalent (R $\cap\cap$ S)
by (*metis (no-types, lifting) Un-subset-iff convex-reflexive down-dist-ii-oi inner-univalent-down inner-univalent-down-closed le-inf-iff subsetI*)

lemma *inner-total-iu-closed*:
inner-total R \implies inner-total S \implies inner-total (R $\cup\cup$ S)
by (*metis U-par-idem U-par-p-id atoms-solution c-prod-idr iu-upper-isotone s-prod-p-idl top-upper-least*)

lemma
inner-total R \implies inner-total S \implies inner-total (R $\cap\cap$ S)
nitpick[*expect=genuine,card=1,2*]
oops

7.3 Fusion

lemma *fusion-set*:
fus R \equiv { (a,B) . B = \bigcup { C . (a,C) \in R } }
unfolding *fus-set Image-singleton*
by (*smt (verit) Collect-cong Pair-inject case-prodE case-prodI2*)

declare *fusion-set* [*mr-simp*]

lemma *fusion-lower-increasing*:

```

R  $\sqsubseteq\downarrow$  fus R
apply (clarsimp simp: mr-simp)
by blast

lemma fusion-deterministic:
  deterministic (fus R)
by (simp add: deterministic-set fusion-set)

lemma fusion-least:
  assumes R  $\sqsubseteq\downarrow$  S
  and deterministic S
  shows fus R  $\sqsubseteq\downarrow$  S
proof (clarsimp simp: mr-simp)
  fix a::'a
  from assms(2) obtain C::'b set where 1: (a,C)  $\in$  S
  by (meson deterministic-set)
  hence  $\bigcup\{ B . (a,B) \in R \} \subseteq C$ 
  using assms deterministic-lower by (smt (verit, del-insts) Sup-le-iff
  mem-Collect-eq)
  thus  $\exists C . (\exists D . \bigcup\{ B . (a,B) \in R \} = C \cap D) \wedge (a,C) \in S$ 
  using 1 by blast
qed

lemma fusion-unique:
  assumes  $\forall R . R \sqsubseteq\downarrow f R$ 
  and  $\forall R .$  deterministic (f R)
  and  $\forall R S . R \sqsubseteq\downarrow S \wedge$  deterministic S  $\longrightarrow$  f R  $\sqsubseteq\downarrow$  S
  shows f T = fus T
  apply (rule univalent-lower-eq)
  using assms(2) deterministic-def apply blast
  using deterministic-def fusion-deterministic apply blast
  by (simp add: assms fusion-deterministic fusion-least fusion-lower-increasing)

lemma fusion-down-char:
  (fus R) $\downarrow$  =  $\neg((\neg(R\downarrow) \cap A_{\cup\cup})\uparrow)$ 
proof
  show (fus R) $\downarrow$   $\subseteq$   $\neg((\neg(R\downarrow) \cap A_{\cup\cup})\uparrow)$ 
  apply (clarsimp simp: mr-simp)
  by blast
next
  show  $\neg((\neg(R\downarrow) \cap A_{\cup\cup})\uparrow) \subseteq$  (fus R) $\downarrow$ 
  proof (clarsimp simp: mr-simp)
  fix a A
  assume 1:  $\forall B . (\forall C . A \neq B \cup C) \vee (\exists C . (\exists D . B = C \cap D) \wedge (a,C) \in$ 
  R)  $\vee (\forall b . B \neq \{b\})$ 
  have  $A \subseteq \bigcup\{ C . (a,C) \in R \}$ 
  proof
  fix x
  assume x  $\in$  A

```

from this obtain C **where** $x \in C \wedge (a, C) \in R$
using 1 **by** (*metis IntD1 insert-Diff insert-is-Un singletonI*)
thus $x \in \bigcup \{ C . (a, C) \in R \}$
by *blast*
qed
thus $\exists D . A = \bigcup \{ C . (a, C) \in R \} \cap D$
by *auto*
qed
qed

lemma *fusion-up-char*:
 $(fus R) \uparrow = -((\sim(R \downarrow) \cap A_{\cap \cap}) \downarrow)$
proof
show $(fus R) \uparrow \subseteq -((\sim(R \downarrow) \cap A_{\cap \cap}) \downarrow)$
apply (*clarsimp simp: mr-simp*)
by *blast*
next
show $-((\sim(R \downarrow) \cap A_{\cap \cap}) \downarrow) \subseteq (fus R) \uparrow$
proof (*clarsimp simp: mr-simp*)
fix $a A$
assume 1: $\forall C . (\forall D . A \neq C \cap D) \vee (\forall B . (\forall D . - C \neq B \cap D) \vee (a, B) \notin R) \vee (\forall b . C \neq UNIV - \{b\})$
have $\bigcup \{ C . (a, C) \in R \} \subseteq A$
proof
fix x
assume $x \in \bigcup \{ C . (a, C) \in R \}$
from this obtain C **where** $x \in C \wedge (a, C) \in R$
by *blast*
thus $x \in A$
using 1 **by** (*metis Compl-eq-Diff-UNIV Diff-Diff-Int Diff-cancel Diff-eq Diff-insert-absorb Int-commute double-complement insert-Diff insert-inter-insert*)
qed
thus $\exists C . A = \bigcup \{ C . (a, C) \in R \} \cup C$
by *auto*
qed
qed

lemma *fusion-up-char-2*:
 $(fus R) \uparrow = -(((R \downarrow \cap A_{\cup \cup}) * \sim I) \downarrow)$
by (*simp add: atoms-sp-cp co-prod fusion-up-char ic-atoms ic-dist-oi*)

lemma *fusion-char*:
 $fus R = -((- (R \downarrow) \cap A_{\cup \cup}) \uparrow) \cap -((\sim(R \downarrow) \cap A_{\cap \cap}) \downarrow)$
by (*metis deterministic-def fusion-deterministic fusion-down-char fusion-up-char inf-commute univalent-convex*)

lemma *fusion-char-2*:
 $fus R = -((- (R \downarrow) \cap A_{\cup \cup}) \uparrow) \cap -(((R \downarrow \cap A_{\cup \cup}) * \sim I) \downarrow)$
using *fusion-char fusion-up-char fusion-up-char-2* **by** *blast*

lemma *fusion-lower-isotone*:

$$R \sqsubseteq\downarrow S \implies \text{fus } R \sqsubseteq\downarrow \text{fus } S$$

by (*meson fusion-deterministic fusion-least fusion-lower-increasing lower-transitive*)

lemma *fusion-iu-idempotent*:

$$\text{fus } R \cup\cup \text{fus } R = \text{fus } R$$

using *deterministic-def fusion-deterministic univalent-iu-idempotent* **by** *blast*

lemma *fusion-down*:

$$\text{fus } R = \text{fus } (R\downarrow)$$

by (*simp add: fusion-char*)

lemma *fusion-iu-total*:

$$\text{total } T \implies T \cup\cup \text{fus } T = \text{fus } T$$

by (*meson deterministic-def fusion-deterministic fusion-lower-increasing total-univalent-lower-iu*)

lemma *fusion-deterministic-fixpoint*:

$$\text{deterministic } R \longleftrightarrow R = \text{fus } R$$

by (*metis deterministic-def fusion-deterministic fusion-iu-total fusion-least lower-reflexive p-prod-comm total-univalent-lower-iu*)

abbreviation *non-empty* :: ('a,'b) mrel \Rightarrow ('a,'b) mrel (*⟨ne -> [100] 100*)

where $ne \ R \equiv R \cap -1 \cup\cup$

lemma *non-empty*:

$$ne \ R = \{ (a,B) \mid a \ B . (a,B) \in R \wedge B \neq \{\} \}$$

by (*auto simp: mr-simp*)

lemma *ne-equality*:

$$ne \ R = R \longleftrightarrow R \subseteq -1 \cup\cup$$

by *blast*

lemma *ne-dist-ou*:

$$ne \ (R \cup S) = ne \ R \cup ne \ S$$

by (*fact inf-sup-distrib2*)

lemma *ne-down-idempotent*:

$$ne \ ((ne \ (R\downarrow))\downarrow) = ne \ (R\downarrow)$$

by (*auto simp: mr-simp*)

lemma *ne-up*:

$$(ne \ R)\uparrow = ne \ R * 1\uparrow$$

proof

$$\text{show } (ne \ R)\uparrow \subseteq ne \ R * 1\uparrow$$

apply (*clarsimp simp: mr-simp*)

by (*metis UN-constant Un-insert-left insert-absorb*)

next

show $ne R * 1\uparrow \subseteq (ne R)\uparrow$

proof (*clarsimp simp: mr-simp*)

fix $a B f$

assume $1: (a, B) \in R$ **and** $2: B \neq \{\}$ **and** $\forall b \in B. \exists C. f b = insert b C$

hence $B \subseteq (\bigcup x \in B. f x)$

by *blast*

thus $\exists D. (\exists C. (\bigcup x \in B. f x) = D \cup C) \wedge (a, D) \in R \wedge D \neq \{\}$

using $1\ 2$ **by** *blast*

qed

qed

lemma *ne-dist-down-sp:*

$ne (R\downarrow * S) = ne (R\downarrow) * ne S$

proof (*rule antisym*)

show $ne (R\downarrow * S) \subseteq ne (R\downarrow) * ne S$

proof (*clarsimp simp: mr-simp*)

fix $a C f D x$

assume $1: (a, C) \in R$ **and** $2: \forall b \in C \cap D. (b, f b) \in S$ **and** $3: f x \neq \{\}$ **and** $4: x \in C$ **and** $5: x \in D$

let $?B = \{ b \in C \cap D. f b \neq \{\} \}$

have $6: \exists C'. (\exists D'. ?B = C' \cap D) \wedge (a, C') \in R$

using 1 **by** *auto*

have $7: ?B \neq \{\}$

using $3\ 4\ 5$ **by** *auto*

have $8: \forall b \in ?B. (b, f b) \in S \wedge f b \neq \{\}$

using 2 **by** *auto*

have $(\bigcup x \in C \cap D. f x) = (\bigcup x \in ?B. f x)$

by *auto*

thus $\exists B. (\exists C'. (\exists D'. B = C' \cap D) \wedge (a, C') \in R) \wedge B \neq \{\} \wedge (\exists g. (\forall b \in B. (b, g b) \in S \wedge g b \neq \{\})) \wedge (\bigcup x \in C \cap D. f x) = (\bigcup x \in B. g x)$

using $6\ 7\ 8$ **by** *blast*

qed

next

show $ne (R\downarrow) * ne S \subseteq ne (R\downarrow * S)$

by (*clarsimp simp: mr-simp*) *blast*

qed

lemma *total-ne-down-dist-sp:*

$total T \implies ne ((R * T)\downarrow) = ne (R\downarrow) * ne (T\downarrow)$

by (*simp add: ne-dist-down-sp total-down-dist-sp*)

lemma *inner-univalent-char:*

$inner-univalent S \iff (\forall R. fus R = fus S \wedge R \sqsubseteq\downarrow S \implies ne R = ne S)$

proof

assume $1: inner-univalent S$

show $\forall R. fus R = fus S \wedge R \sqsubseteq\downarrow S \implies ne R = ne S$

proof (*rule allI, rule impI*)

fix R

```

assume 2:  $\text{fus } R = \text{fus } S \wedge R \sqsubseteq \downarrow S$ 
show  $\text{ne } R = \text{ne } S$ 
proof (rule antisym)
  show  $\text{ne } R \subseteq \text{ne } S$ 
  proof
    fix  $x$ 
    assume 3:  $x \in \text{ne } R$ 
    from this obtain  $a B$  where 4:  $x = (a, B) \wedge x \in R \wedge B \neq \{\}$ 
      by (metis Int-iff Int-lower2 inner-total-2 surj-pair)
    from this obtain  $C$  where 5:  $B \subseteq C \wedge (a, C) \in S$ 
      using 2 by (meson lower-less-eq)
    from this obtain  $b$  where  $C = \{b\}$ 
      using 1 4 by (metis Un-empty inner-univalent is-singletonI'
is-singleton-the-elem subset-Un-eq)
    hence  $B = C$ 
      using 4 5 by blast
    thus  $x \in \text{ne } S$ 
      using 3 4 5 by blast
  qed
next
show  $\text{ne } S \subseteq \text{ne } R$ 
proof
  fix  $x$ 
  assume 6:  $x \in \text{ne } S$ 
  from this obtain  $a B$  where 7:  $x = (a, B) \wedge x \in S \wedge B \neq \{\}$ 
    by (metis Int-absorb Int-iff inner-total-2
semilattice-inf-class.inf.absorb-iff2 surj-pair)
  from this obtain  $b$  where 8:  $B = \{b\}$ 
    using 1 by (meson antisym card-1-singletonE inner-univalent-2
nonempty-set-card)
  from this obtain  $C$  where 9:  $b \in C \wedge (a, C) \in \text{fus } S$ 
    using 7 by (meson fusion-lower-increasing insert-subset lower-less-eq)
  hence  $(a, C) \in \text{fus } R$ 
    using 2 by simp
  hence  $C = \bigcup \{ D \mid (a, D) \in R \}$ 
    by (clarsimp simp: mr-simp)
  from this obtain  $D$  where 10:  $b \in D \wedge (a, D) \in R$ 
    using 9 by blast
  from this obtain  $E$  where 11:  $D \subseteq E \wedge (a, E) \in S$ 
    using 2 by (meson lower-less-eq)
  from this obtain  $c$  where  $E = \{c\}$ 
    using 1 10 by (metis antisym card-1-singletonE empty-iff
inner-univalent-2 nonempty-set-card subsetI)
  hence  $D = \{b\}$ 
    using 10 11 by blast
  thus  $x \in \text{ne } R$ 
    using 6 7 8 10 by blast
  qed
qed

```

```

qed
next
assume 12:  $\forall R . fus R = fus S \wedge R \sqsubseteq \downarrow S \longrightarrow ne R = ne S$ 
show inner-univalent S
proof (unfold inner-univalent, intro allI, rule impI)
  fix a b c B
  assume 13:  $(a, B) \in S \wedge b \in B \wedge c \in B$ 
  let ?R =  $\{ (f, \{e\}) \mid f e . \exists C . (f, C) \in S \wedge e \in C \}$ 
  have 14:  $fus ?R = fus S$ 
  apply (clarsimp simp: mr-simp)
  by blast
  have ?R  $\sqsubseteq \downarrow S$ 
  using lower-less-eq by fastforce
  hence  $ne ?R = ne S$ 
  using 12 14 by simp
  hence  $(a, B) \in ?R$ 
  using 13 by (smt (verit, del-insts) empty-iff mem-Collect-eq non-empty)
  thus  $b = c$ 
  using 13 by blast
qed
qed

```

```

lemma ne-dist-oU:
   $ne (\bigcup X) = \bigcup (non-empty \text{ ` } X)$ 
  by blast

```

7.4 Fission

```

lemma fission-set:
   $fis R = \{ (a, \{b\}) \mid a b . \exists B . (a, B) \in R \wedge b \in B \}$ 
  unfolding fis-set Image-singleton
  by simp

```

```

declare fission-set [mr-simp]

```

```

lemma fission-var:
   $fis R = R \downarrow \cap A_{\cup \cup}$ 
  apply (clarsimp simp: mr-simp)
  by blast

```

```

lemma fission-lower-decreasing:
   $fis R \sqsubseteq \downarrow R$ 
  by (simp add: fission-var)

```

```

lemma fission-inner-deterministic:
  inner-deterministic (fis R)
  by (simp add: fission-var inner-deterministic-atoms)

```

```

lemma fission-greatest:

```

assumes $S \sqsubseteq\downarrow R$
and *inner-deterministic* S
shows $S \sqsubseteq\downarrow \text{fis } R$
proof (*clarsimp simp: mr-simp*)
fix $a B$
assume $1: (a, B) \in S$
from *this* **obtain** b **where** $2: B = \{b\}$
using *assms(2)* **by** (*meson card-1-singletonE inner-deterministic-2*)
from 1 **obtain** C **where** $B \subseteq C \wedge (a, C) \in R$
using *assms(1)* **by** (*meson lower-less-eq*)
thus $\exists C . (\exists D . B = C \cap D) \wedge (\exists b . C = \{b\}) \wedge (\exists E . (a, E) \in R \wedge b \in E)$
using 2 **by** *auto*
qed

lemma *fission-unique*:
assumes $\forall R . f R \sqsubseteq\downarrow R$
and $\forall R . \text{inner-deterministic } (f R)$
and $\forall R S . S \sqsubseteq\downarrow R \wedge \text{inner-deterministic } S \longrightarrow S \sqsubseteq\downarrow f R$
shows $f T = \text{fis } T$
apply (*rule inner-deterministic-lower-eq*)
apply (*simp add: assms(2)*)
apply (*simp add: fission-inner-deterministic*)
by (*simp add: assms fission-greatest fission-inner-deterministic fission-lower-decreasing*)

lemma *fission-lower-isotone*:
 $R \sqsubseteq\downarrow S \implies \text{fis } R \sqsubseteq\downarrow \text{fis } S$
by (*meson fission-greatest fission-inner-deterministic fission-lower-decreasing lower-transitive*)

lemma *fission-idempotent*:
 $\text{fis } (\text{fis } R) = \text{fis } R$
by (*metis comp-apply fis-fis*)

lemma *fission-top*:
 $\text{fis } U = A_{\cup\cup}$
using *fission-var top-down top-upper-least* **by** *fastforce*

lemma *fission-down*:
 $\text{fis } R = \text{fis } (R\downarrow)$
by (*simp add: fission-var*)

lemma *fission-ne-fixpoint*:
 $\text{fis } R = \text{ne } (\text{fis } R)$
using *fission-inner-deterministic* **by** *blast*

lemma *fission-down-ne-fixpoint*:
 $\text{fis } R = \text{ne } ((\text{fis } R)\downarrow)$
by (*metis fission-inner-deterministic fission-ne-fixpoint fission-down*)

inner-univalent-char lower-ii-decreasing)

lemma *fission-inner-deterministic-fixpoint:*

inner-deterministic $R \longleftrightarrow R = \text{fis } R$

apply (*rule iffI*)

apply (*metis comp-eq-dest-lhs fission-lower-decreasing fission-ne-fixpoint fus-fis inner-univalent-char le-iff-inf*)

using *fission-inner-deterministic* **by** *auto*

lemma *fission-sp-subdist:*

$\text{fis } (R * S) \subseteq \text{fis } R * \text{fis } S$

proof

fix x

assume $x \in \text{fis } (R * S)$

from *this* **obtain** $a \ b \ B$ **where** $1: x = (a, \{b\}) \wedge (a, B) \in R * S \wedge b \in B$

by (*smt CollectD fission-set*)

from *this* **obtain** $C \ f$ **where** $2: (a, C) \in R \wedge (\forall c \in C . (c, f \ c) \in S) \wedge B = \bigcup \{ f \ c \mid c . c \in C \}$

by (*simp add: mr-simp*) **blast**

from *this* **obtain** c **where** $3: b \in f \ c \wedge c \in C$

using 1 **by** *blast*

let $?B = \{c\}$

let $?f = \lambda x . \{b\}$

have $4: (a, ?B) \in \text{fis } R$

using $2 \ 3$ *fission-set* **by** *blast*

have $5: \forall b \in ?B . (b, ?f \ b) \in \text{fis } S$

using $2 \ 3$ *fission-set* **by** *blast*

have $\{b\} = \bigcup \{ ?f \ b \mid b . b \in ?B \}$

by *simp*

hence $\exists f . (\forall b \in ?B . (b, f \ b) \in \text{fis } S) \wedge \{b\} = \bigcup \{ f \ b \mid b . b \in ?B \}$

using 5 **by** *auto*

hence $(a, \{b\}) \in \text{fis } R * \text{fis } S$

apply (*unfold s-prod-def*)

using 4 **by** *auto*

thus $x \in \text{fis } R * \text{fis } S$

using 1 **by** *simp*

qed

lemma *fission-sp-total-dist:*

assumes *total* T

shows $\text{fis } (R * T) = \text{fis } R * \text{fis } T$

by (*metis assms atoms-sp-idempotent fis-lax fission-var sp-oi-subdist-2 subset-antisym total-down-dist-sp*)

lemma *fission-dist-ou:*

$\text{fis } (R \cup S) = \text{fis } R \cup \text{fis } S$

by (*simp add: down-dist-ou fission-var inf-sup-distrib2*)

lemma *fission-sp-iu-unit:*

$\text{fis } (R * 1_{\cup\cup}) = \{\}$
by (*metis c-nc down-sp fission-lower-decreasing nu-def nu-fis nu-fis-var s-prod-zerol subset-empty*)

lemma *fission-fusion-lower-decreasing*:

$\text{fis } (\text{fus } R) \sqsubseteq\downarrow R$
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *fusion-fission-lower-increasing*:

$R \sqsubseteq\downarrow \text{fus } (\text{fis } R)$
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *fission-fusion-galois*:

$\text{fis } R \sqsubseteq\downarrow S \iff R \sqsubseteq\downarrow \text{fus } S$
apply (*rule iffI*)
apply (*meson fusion-fission-lower-increasing fusion-lower-isotone lower-transitive*)
by (*meson fission-fusion-lower-decreasing fission-lower-isotone lower-transitive*)

lemma *fission-fusion*:

$\text{fis } (\text{fus } R) = \text{fis } R$
by (*metis fission-fusion-lower-decreasing fission-idempotent fission-inner-deterministic fission-lower-isotone fusion-lower-increasing inner-deterministic-lower-eq*)

lemma *fusion-fission*:

$\text{fus } (\text{fis } R) = \text{fus } R$
by (*metis comp-def fus-fis*)

lemma *same-fusion-fission-lower*:

$\text{fus } R = \text{fus } S \implies \text{fis } R \sqsubseteq\downarrow S$
by (*metis fission-fusion-galois fusion-lower-increasing*)

lemma *fission-below-ne-down-fusion*:

$\text{fis } R \subseteq \text{ne } ((\text{fus } R)\downarrow)$
using *fission-fusion fission-inner-deterministic fission-lower-decreasing by blast*

lemma *ne-fusion-fission*:

$(\text{ne } ((\text{fus } R)\downarrow))\uparrow = (\text{fis } R)\uparrow$
by (*metis (mono-tags, lifting) atoms-solution fission-below-ne-down-fusion fission-fusion oi-down-sub-up subset-trans upper-eq-up upper-reflexive fission-var*)

lemma *fission-up-ne-down-up*:

$(\text{fis } R)\uparrow = (\text{ne } (R\downarrow))\uparrow$
by (*metis (mono-tags, lifting) atoms-solution fission-ne-fixpoint fission-top oi-down-sub-up semilattice-inf-class.inf-le2 semilattice-inf-class.inf-left-commute subset-trans upper-eq-up fission-var*)

lemma *fusion-idempotent*:
 $fus (fus R) = fus R$
by (*metis fission-fusion fusion-fission*)

lemma *fission-dist-oU*:
 $fis (\bigcup X) = \bigcup (fis \text{ ` } X)$
by (*metis (no-types, lifting) SUP-cong UN-simps(4) fission-var ii-right-dist-oU*)

7.5 Co-fusion and co-fission

definition *co-fusion* :: $('a, 'b) mrel \Rightarrow ('a, 'b) mrel (\text{ ` } \bigcap \bigcap \rightarrow [80] 80)$ **where**
 $\bigcap \bigcap R \equiv \{ (a, B) . B = \bigcap \{ C . (a, C) \in R \} \}$

declare *co-fusion-def* [*mr-simp*]

lemma *co-fusion-upper-decreasing*:
 $\bigcap \bigcap R \sqsubseteq \uparrow R$
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *co-fusion-deterministic*:
deterministic ($\bigcap \bigcap R$)
by (*simp add: deterministic-set co-fusion-def*)

lemma *co-fusion-greatest*:
assumes $S \sqsubseteq \uparrow R$
and *deterministic* S
shows $S \sqsubseteq \uparrow \bigcap \bigcap R$
proof (*clarsimp simp: mr-simp*)
fix a
from *assms(2)* **obtain** B **where** $1: (a, B) \in S$
by (*meson deterministic-set*)
hence $B \subseteq \bigcap \{ C . (a, C) \in R \}$
using *assms deterministic-upper* **by** (*smt (verit, ccfv-threshold) Inter-greatest mem-Collect-eq*)
thus $\exists B . (\exists D . \bigcap \{ C . (a, C) \in R \} = B \cup D) \wedge (a, B) \in S$
using 1 **by** *blast*
qed

lemma *co-fusion-unique*:
assumes $\forall R . f R \sqsubseteq \uparrow R$
and $\forall R . \text{deterministic } (f R)$
and $\forall R S . S \sqsubseteq \uparrow R \wedge \text{deterministic } S \longrightarrow S \sqsubseteq \uparrow f R$
shows $f T = \bigcap \bigcap T$
apply (*rule univalent-upper-eq*)
using *assms(2) deterministic-def* **apply** *blast*
using *co-fusion-deterministic deterministic-def* **apply** *blast*
by (*simp add: assms co-fusion-deterministic co-fusion-greatest*)

co-fusion-upper-decreasing)

lemma *co-fusion-up-char*:

$$(\prod \prod R)\uparrow = -((-(R\uparrow) \cap A_{\cap\cap})\downarrow)$$

proof

$$\text{show } (\prod \prod R)\uparrow \subseteq -((-(R\uparrow) \cap A_{\cap\cap})\downarrow)$$

apply (*clarsimp simp: mr-simp*)

by *blast*

next

$$\text{show } -((-(R\uparrow) \cap A_{\cap\cap})\downarrow) \subseteq (\prod \prod R)\uparrow$$

proof (*clarsimp simp: mr-simp*)

fix $a A$

assume $1: \forall B . (\forall C . A \neq B \cap C) \vee (\exists C . (\exists D . B = C \cup D) \wedge (a, C) \in R) \vee (\forall b . B \neq \text{UNIV} - \{b\})$

have $\bigcap \{ C . (a, C) \in R \} \subseteq A$

proof

fix x

assume $x \in \bigcap \{ C . (a, C) \in R \}$

hence $\forall C . A \neq (\text{UNIV} - \{x\}) \cap C$

using 1 **by** *blast*

thus $x \in A$

by *blast*

qed

thus $\exists D . A = \bigcap \{ C . (a, C) \in R \} \cup D$

by *auto*

qed

qed

lemma *co-fusion-down-char*:

$$(\prod \prod R)\downarrow = -((\sim(R\uparrow) \cap A_{\cup\cup})\uparrow)$$

proof

$$\text{show } (\prod \prod R)\downarrow \subseteq -((\sim(R\uparrow) \cap A_{\cup\cup})\uparrow)$$

apply (*clarsimp simp: mr-simp*)

by *blast*

next

$$\text{show } -((\sim(R\uparrow) \cap A_{\cup\cup})\uparrow) \subseteq (\prod \prod R)\downarrow$$

proof (*clarsimp simp: mr-simp*)

fix $a A$

assume $1: \forall C . (\forall D . A \neq C \cup D) \vee (\forall B . (\forall D . - C \neq B \cup D) \vee (a, B) \notin R) \vee (\forall b . C \neq \{b\})$

have $A \subseteq \bigcap \{ C . (a, C) \in R \}$

proof

fix x

assume $x \in A$

hence $\forall B . (\forall D . - \{x\} \neq B \cup D) \vee (a, B) \notin R$

using 1 **by** *blast*

thus $x \in \bigcap \{ C . (a, C) \in R \}$

by *blast*

qed

thus $\exists C . A = \bigcap \{ C . (a, C) \in R \} \cap C$
by *auto*
qed
qed

lemma *co-fusion-down-char-2*:
 $(\bigcap \bigcap R) \downarrow = -(((R \uparrow \cap A_{\cap\cap}) \odot \sim 1) \uparrow)$
by (*metis co-fusion-down-char ic-co-atoms ic-cp-ic-unit ic-dist-oi*)

lemma *co-fusion-char*:
 $\bigcap \bigcap R = -((-(R \uparrow) \cap A_{\cap\cap}) \downarrow) \cap -((\sim(R \uparrow) \cap A_{\cup\cup}) \uparrow)$
by (*metis deterministic-def co-fusion-deterministic co-fusion-down-char co-fusion-up-char univalent-convex*)

lemma *co-fusion-char-2*:
 $\bigcap \bigcap R = -((-(R \uparrow) \cap A_{\cap\cap}) \downarrow) \cap -(((R \uparrow \cap A_{\cap\cap}) \odot \sim 1) \uparrow)$
using *co-fusion-char co-fusion-down-char co-fusion-down-char-2* **by** *blast*

lemma *co-fusion-upper-isotone*:
 $R \sqsubseteq \uparrow S \implies \bigcap \bigcap R \sqsubseteq \uparrow \bigcap \bigcap S$
by (*meson co-fusion-deterministic co-fusion-greatest co-fusion-upper-decreasing upper-transitive*)

lemma *co-fusion-ii-idempotent*:
 $\bigcap \bigcap R \cap \bigcap \bigcap R = \bigcap \bigcap R$
by (*metis deterministic-def co-fusion-deterministic univalent-ii-idempotent*)

lemma *co-fusion-up*:
 $\bigcap \bigcap R = \bigcap \bigcap (R \uparrow)$
by (*simp add: co-fusion-char*)

lemma *co-fusion-ii-total*:
 $total\ T \implies T \cap \bigcap \bigcap T = \bigcap \bigcap T$
by (*meson co-fusion-deterministic co-fusion-upper-decreasing deterministic-def total-univalent-upper-ii*)

lemma *co-fusion-deterministic-fixpoint*:
deterministic $R \longleftrightarrow R = \bigcap \bigcap R$
apply (*rule iffI*)
apply (*metis deterministic-def co-fusion-deterministic co-fusion-greatest co-fusion-ii-total ii-commute total-univalent-upper-ii upper-reflexive*)
by (*metis co-fusion-deterministic*)

abbreviation *co-fission* :: $('a, 'b)\ mrel \implies ('a, 'b)\ mrel (\langle at_{\cap\cap} \rightarrow [80] 80 \rangle)$ **where**
 $at_{\cap\cap}\ R \equiv R \uparrow \cap A_{\cap\cap}$

lemma *co-fission*:
 $at_{\cap\cap}\ R = \{ (a, B) \mid a \in B . (\exists b . -B = \{b\}) \wedge (\exists C . (a, C) \in R \wedge C \subseteq B) \}$
apply (*clarsimp simp: mr-simp*)

by *blast*

declare *co-fission* [*mr-simp*]

lemma *co-fission-upper-increasing*:
 $R \sqsubseteq \uparrow at_{\cap\cap} R$
by (*fact semilattice-inf-class.inf-le1*)

lemma *co-fission-ic-inner-deterministic*:
inner-deterministic ($\sim at_{\cap\cap} R$)
by (*simp add: ic-co-atoms ic-dist-oi inner-deterministic-atoms*)

lemma *co-fission-least*:
assumes $R \sqsubseteq \uparrow S$
and *inner-deterministic* ($\sim S$)
shows $at_{\cap\cap} R \sqsubseteq \uparrow S$
proof (*clarsimp simp: mr-simp*)
fix $a B$
assume $1: (a, B) \in S$
hence $(a, -B) \in \sim S$
by (*simp add: inner-complement-def*)
from this obtain b **where** $2: -B = \{b\}$
using *assms(2)* **by** (*meson card-1-singletonE inner-deterministic-2*)
from 1 obtain C **where** $C \subseteq B \wedge (a, C) \in R$
using *assms(1)* **by** (*meson upper-less-eq*)
thus $\exists C . (\exists D . B = C \cup D) \wedge (\exists E . (\exists D . C = E \cup D) \wedge (a, E) \in R) \wedge (\exists b . C = UNIV - \{b\})$
using 2 **by** (*metis Compl-eq-Diff-UNIV double-compl subset-Un-eq sup.idem*)
qed

lemma *co-fission-unique*:
assumes $\forall R . R \sqsubseteq \uparrow f R$
and $\forall R .$ *inner-deterministic* ($\sim f R$)
and $\forall R S . R \sqsubseteq \uparrow S \wedge$ *inner-deterministic* ($\sim S$) $\longrightarrow f R \sqsubseteq \uparrow S$
shows $f T = at_{\cap\cap} T$
apply (*rule ic-injective*)
apply (*rule inner-deterministic-lower-eq*)
apply (*simp add: assms(2)*)
apply (*simp add: co-fission-ic-inner-deterministic*)
by (*meson assms co-fission-ic-inner-deterministic co-fission-least semilattice-inf-class.inf-le1 upper-ic-lower*)

lemma *co-fission-upper-isotone*:
 $R \sqsubseteq \uparrow S \implies at_{\cap\cap} R \sqsubseteq \uparrow at_{\cap\cap} S$
by (*simp add: oi-subset-upper-left-antitone upper-transitive*)

lemma *co-fission-idempotent*:
 $at_{\cap\cap} (at_{\cap\cap} R) = at_{\cap\cap} R$
by (*meson equalityI semilattice-inf-class.inf-le1 semilattice-inf-class.inf-le2*)

semilattice-inf-class.le-inf-iff upper-reflexive upper-transitive)

lemma *co-fission-top:*

$at_{\cap\cap} U = A_{\cap\cap}$

using *top-lower-greatest U-par-idem top-down by blast*

lemma *co-fission-up:*

$at_{\cap\cap} R = at_{\cap\cap} (R\uparrow)$

by *simp*

lemma *co-fission-ic-inner-deterministic-fixpoint:*

inner-deterministic $(\sim R) \longleftrightarrow R = at_{\cap\cap} R$

apply (*rule iffI*)

apply (*simp add: fission-var fission-inner-deterministic-fixpoint ic-antidist-iu ic-co-atoms ic-dist-oi ic-injective ic-up*)

by (*metis co-fission-ic-inner-deterministic*)

lemma *co-fusion-co-fission-upper-decreasing:*

$\prod\prod(at_{\cap\cap} R) \sqsubseteq\uparrow R$

proof (*clarsimp simp: mr-simp*)

fix $a B$

assume $1: (a, B) \in R$

have $\bigcap\{ D . (\exists E . (\exists F . D = E \cup F) \wedge (a, E) \in R) \wedge (\exists b . D = UNIV - \{b\}) \} \subseteq B$

proof

fix x

assume $2: x \in \bigcap\{ D . (\exists E . (\exists F . D = E \cup F) \wedge (a, E) \in R) \wedge (\exists b . D = UNIV - \{b\}) \}$

show $x \in B$

proof (*rule ccontr*)

let $?D = -\{x\}$

assume $3: x \notin B$

hence $B \subseteq ?D$

by *simp*

hence $\bigcap\{ D . (\exists E . (\exists F . D = E \cup F) \wedge (a, E) \in R) \wedge (\exists b . D = UNIV - \{b\}) \} \subseteq ?D$

using 1 **by** (*smt CollectI Compl-eq-Diff-UNIV Inf-lower subset-Un-eq*)

thus *False*

using 2 **by** *auto*

qed

qed

thus $\exists C . B = \bigcap\{ D . (\exists E . (\exists F . D = E \cup F) \wedge (a, E) \in R) \wedge (\exists b . D = UNIV - \{b\}) \} \cup C$

by *auto*

qed

lemma *co-fission-co-fusion-upper-increasing:*

$R \sqsubseteq\uparrow at_{\cap\cap} (\prod\prod R)$

proof (*clarsimp simp: mr-simp*)

fix $a b B$
assume $\bigcap \{ C . (a, C) \in R \} \cup B = UNIV - \{b\}$
hence $b \notin \bigcap \{ C . (a, C) \in R \}$
by *blast*
hence $\exists C . b \notin C \wedge (a, C) \in R$
by *blast*
thus $\exists C . (\exists D . UNIV - \{b\} = C \cup D) \wedge (a, C) \in R$
by *blast*
qed

lemma *co-fusion-co-fission-galois*:
 $\bigcap \bigcap R \sqsubseteq \uparrow S \iff R \sqsubseteq \uparrow at_{\bigcap} S$
apply (*rule iffI*)
apply (*meson co-fission-co-fusion-upper-increasing co-fission-upper-isotone upper-transitive*)
by (*meson co-fusion-co-fission-upper-decreasing co-fusion-upper-isotone upper-transitive*)

lemma *co-fission-co-fusion*:
 $at_{\bigcap} (\bigcap \bigcap R) = at_{\bigcap} R$
using *co-fission-co-fusion-upper-increasing co-fission-idempotent co-fission-upper-isotone co-fusion-upper-decreasing* **by** *blast*

lemma *co-fusion-co-fission*:
 $\bigcap \bigcap (at_{\bigcap} R) = \bigcap \bigcap R$
apply (*rule antisym*)
apply (*metis deterministic-def co-fission-co-fusion co-fission-co-fusion-upper-increasing co-fusion-co-fission-upper-decreasing co-fusion-deterministic co-fusion-upper-isotone univalent-upper-eq-subset*)
by (*metis deterministic-def co-fission-co-fusion co-fission-co-fusion-upper-increasing co-fusion-co-fission-upper-decreasing co-fusion-deterministic co-fusion-upper-isotone univalent-upper-eq-subset*)

lemma *same-co-fusion-co-fission-upper*:
 $\bigcap \bigcap R = \bigcap \bigcap S \implies S \sqsubseteq \uparrow at_{\bigcap} R$
by (*metis co-fusion-co-fission-galois co-fusion-upper-decreasing*)

lemma *co-fusion-idempotent*:
 $\bigcap \bigcap (\bigcap \bigcap R) = \bigcap \bigcap R$
by (*metis co-fission-co-fusion co-fusion-co-fission*)

8 Modalities

8.1 Tests

abbreviation $test :: ('a, 'a) mrel \Rightarrow bool$ **where**
 $test R \equiv R \subseteq 1$

lemma *test*:

$test\ R \longleftrightarrow (\forall a\ B . (a,B) \in R \longrightarrow B = \{a\})$
by (*force simp: s-id-def*)

lemma *test-fix*: $test\ R \equiv R \cap 1_\sigma = R$
by (*simp add: le-iff-inf*)

lemma *test-ou-closed*:
 $test\ p \Longrightarrow test\ q \Longrightarrow test\ (p \cup q)$
by (*fact sup-least*)

lemma *test-oi-closed*:
 $test\ p \Longrightarrow test\ (p \cap q)$
by *blast*

abbreviation *test-complement* :: $('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel\ (\lrcorner \rightarrow [80]\ 80)$
where
 $\lrcorner\ R \equiv -R \cap 1$

lemma *test-complement-closed*:
 $test\ (\lrcorner\ p)$
by *simp*

lemma *test-double-complement*:
 $test\ p \longleftrightarrow p = \lrcorner\ \lrcorner\ p$
by *blast*

lemma *test-complement*:
 $(a,\{a\}) \in \lrcorner\ p \longleftrightarrow \neg (a,\{a\}) \in p$
by (*simp add: s-id-def*)

declare *test-complement* [*mr-simp*]

lemma *test-complement-antitone*:
assumes $test\ p$
shows $p \subseteq q \longleftrightarrow \lrcorner\ q \subseteq \lrcorner\ p$
using *assms(1)* **by** *blast*

lemma *test-complement-huntington*:
 $test\ p \Longrightarrow p = \lrcorner\ (\lrcorner\ p \cup \lrcorner\ q) \cup \lrcorner\ (\lrcorner\ p \cup q)$
by *blast*

abbreviation *test-implication* :: $('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel \Rightarrow ('a,'a)\ mrel$
(infixl $\langle \rightarrow \rangle$ *65*) **where**
 $p \rightarrow q \equiv \lrcorner\ p \cup q$

lemma *test-implication-closed*:
 $test\ q \Longrightarrow test\ (p \rightarrow q)$
by *simp*

lemma *test-implication*:
 $(a, \{a\}) \in p \rightarrow q \iff ((a, \{a\}) \in p \longrightarrow (a, \{a\}) \in q)$
by (*simp add: s-id-def*)

declare *test-implication* [*mr-simp*]

lemma *test-implication-left-antitone*:
assumes *test p*
shows $p \subseteq r \implies r \rightarrow q \subseteq p \rightarrow q$
by *blast*

lemma *test-implication-right-isotone*:
assumes *test p*
shows $q \subseteq r \implies p \rightarrow q \subseteq p \rightarrow r$
by *blast*

lemma *test-sp-idempotent*:
 $test\ p \implies p * p = p$
by (*metis d-rest-ax inf.order-iff s-subid-iff2*)

lemma *test-sp*:
assumes *test p*
shows $p * R = (p * U) \cap R$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *blast*
using *assms subid-aux2* **by** *fastforce+*

lemma *sp-test*:
 $test\ p \implies R * p = R \cap (U * p)$
apply (*rule antisym*)
apply (*metis (no-types, lifting) U-par-idem inf.absorb-iff2 inf.idem le-inf-iff s-prod-idr sp-oi-subdist top-upper-least*)
using *test-fix* **by** (*smt IntE s-prod-test-aux1 s-prod-test-aux2 subrelI*)

lemma *sp-test-dist-oi*:
 $test\ p \implies (R \cap S) * p = (R * p) \cap (S * p)$
by (*smt Int-left-commute semilattice-inf-class.inf.assoc semilattice-inf-class.inf.right-idem sp-test*)

lemma *sp-test-dist-oi-left*:
 $test\ p \implies (R \cap S) * p = (R * p) \cap S$
by (*smt Int-commute semilattice-inf-class.inf.left-commute sp-test*)

lemma *sp-test-dist-oi-right*:
 $test\ p \implies (R \cap S) * p = R \cap (S * p)$
by (*metis semilattice-inf-class.inf.commute sp-test-dist-oi-left*)

lemma *sp-test-sp-oi-left*:

test $p \implies (R \cap (U * p)) * T = R * p * T$
by (*metis sp-test*)

lemma *sp-test-sp-oi-right*:

test $p \implies R * ((p * U) \cap T) = R * p * T$
by (*metis inf.orderE test-assoc1 test-sp*)

lemma *test-sp-ne*:

test $p \implies p * ne R = ne (p * R)$
by (*smt lattice-class.inf-sup-aci(1) lattice-class.inf-sup-aci(3) test-sp*)

lemma *ne-sp-test*:

test $p \implies ne R * p = ne (R * p)$
by (*fact sp-test-dist-oi-left*)

lemma *top-sp-test-down-closed*:

assumes *test p*

shows $U * p = (U * p) \downarrow$

proof –

have $1: p \cap 1_\sigma = p$

using *assms* **by** *blast*

hence $(U * p) \downarrow = \{(a,A). (a,A) \in U \wedge (\forall a \in A. (a,\{a\}) \in p)\} \cap U$

by (*smt (verit) Collect-cong case-prodI2 case-prod-conv s-prod-test-var*)

also have $\dots = \{(a,A). \forall a \in A. (a,\{a\}) \in p\} \cap U$

by (*smt (verit) Collect-cong ii-assoc lower-ii-down mem-Collect-eq split-cong subsetD top-down*)

also have $\dots = \{(a,A). (a,A) \in U \wedge (\forall a \in A. (a,\{a\}) \in p)\}$

by (*auto simp: mr-simp*)

also have $\dots = U * p$

using 1 **by** (*smt (verit) Collect-cong case-prodI2 case-prod-conv s-prod-test-var*)

finally show *?thesis*

by *blast*

qed

lemma *oc-top-sp-test-up-closed*:

test $p \implies -(U * p) = (-(U * p)) \uparrow$

by (*metis antisym convex-reflexive disjoint-eq-subset-Compl inf-compl-bot oi-down-up-iff semilattice-inf-class.inf commute top-sp-test-down-closed*)

lemma *top-sp-test*:

test $p \implies (a,B) \in U * p \iff (\forall b \in B. (b,\{b\}) \in p)$

using *test-fix* **by** (*metis IntE UNIV-I s-prod-test sp-test*)

lemma *oc-top-sp-test*:

test $p \implies (a,B) \in -(U * p) \iff (\exists b \in B. (b,\{b\}) \notin p)$

by (*simp add: top-sp-test*)

declare *top-sp-test* [*mr-simp*] *oc-top-sp-test* [*mr-simp*]

lemma *oc-top-sp-test-0*:
 $-1_{\cup\cup} * \wr p = ne (U * \wr p)$
by (*metis Int-lower1 semilattice-inf-class.inf commute sp-test*)

lemma *oc-top-sp-test-1*:
assumes *test p*
shows $-(U * p) = (ne (U * \wr p))^\uparrow$
proof (*rule antisym*)
show $-(U * p) \subseteq (ne (U * \wr p))^\uparrow$
proof
fix $x::'c \times 'a$ *set*
assume $1: x \in -(U * p)$
from this obtain $a B$ **where** $2: x = (a, B)$
by force
from this obtain c **where** $3: c \in B \wedge (c, \{c\}) \notin p$
using 1 **by** (*meson assms oc-top-sp-test*)
hence $4: (a, \{c\}) \in U * \wr p$
by (*metis singletonD test-complement test-complement-closed top-sp-test*)
have $(a, \{c\}) \in -1_{\cup\cup}$
using *oc-top-sp-test* **by** (*smt (verit, del-insts) ComplI Int-iff assms boolean-algebra.conj-cancel-left inf.coboundedI2 p-id-zero s-prod-test-aux1 singleton-iff*)
hence $(a, \{c\}) \in ne (U * \wr p)$
using 4 **by** *simp*
thus $x \in (ne (U * \wr p))^\uparrow$
using $2\ 3$ **by** (*metis (no-types, lifting) U-par-st singletonD subset-eq*)
qed

next
have $(U * p)^\downarrow = U * p$
using *assms top-sp-test-down-closed* **by** *auto*
also have $\dots \subseteq -(-1_{\cup\cup} * \wr p)$
by (*smt (verit) Compl-disjoint assms disjoint-eq-subset-Compl inf-commute oc-top-sp-test-0 p-id-zero s-prod-idl sp-test-dist-oi-right test-assoc1 test-double-complement*)
also have $\dots = -ne (U * \wr p)$
by (*simp add: oc-top-sp-test-0*)
finally have $U * p \subseteq -((ne (U * \wr p))^\uparrow)$
by (*simp add: down-double-complement-up*)
thus $(ne (U * \wr p))^\uparrow \subseteq -(U * p)$
by *auto*
qed

lemma *oc-top-sp-test-2*:
 $test\ p \implies -(U * p) = (-1_{\cup\cup} * \wr p)^\uparrow$
by (*simp add: oc-top-sp-test-1 oc-top-sp-test-0*)

lemma *split-sp-test*:
assumes *test p*

```

shows  $R = (R * p) \cup (ne R \cap (ne (R\downarrow * \wr p))\uparrow)$ 
proof (rule antisym)
show  $R \subseteq (R * p) \cup (ne R \cap (ne (R\downarrow * \wr p))\uparrow)$ 
proof
  fix x
  assume 1:  $x \in R$ 
  from this obtain a B where 2:  $x = (a, B)$ 
  by force
  show  $x \in (R * p) \cup (ne R \cap (ne (R\downarrow * \wr p))\uparrow)$ 
  proof (cases  $\forall b \in B. (b, \{b\}) \in p$ )
    case True
    hence  $(a, B) \in U * p$ 
    by (simp add: assms top-sp-test)
    thus ?thesis
    using 1 2 by (metis Int-iff UnCI assms sp-test)
  next
  case False
  from this obtain b where 3:  $\{b\} \subseteq B \wedge (b, \{b\}) \notin p$ 
  by auto
  hence  $(a, \{b\}) \in R\downarrow$ 
  using 1 2 down by fastforce
  hence  $(a, \{b\}) \in R\downarrow * \wr p$ 
  using 3 by (metis s-prod-test-aux2 singletonD test-complement)
  hence  $(a, \{b\}) \in ne (R\downarrow * \wr p)$ 
  by (simp add: non-empty)
  hence  $(a, B) \in (ne (R\downarrow * \wr p))\uparrow$ 
  using 3 by (meson U-par-st)
  thus ?thesis
  using 1 2 3 non-empty by auto
qed
qed
next
show  $(R * p) \cup (ne R \cap (ne (R\downarrow * \wr p))\uparrow) \subseteq R$ 
using assms sp-test by auto
qed

lemma top-sp-test-down-iff-1:
  assumes test p
  shows  $R \subseteq U * p \iff R\downarrow \subseteq U * p$ 
  by (smt (verit, del-insts) assms down-order-lower top-sp-test-down-closed)

lemma test-ne:
  test p  $\implies ne p = p$ 
  using inner-deterministic-sp-unit by blast

lemma ne-test-up:
  test p  $\implies ne (p\uparrow) = p\uparrow$ 
  by (metis atoms-solution ne-equality test-ne up-idempotent up-isotone)

```

lemma *ne-sp-test-up*:

test p \implies $(ne (R * p))\uparrow = ne R * p\uparrow$

using *test-fix* **by** (*smt ne-up sp-test-dist-oi-left test-assoc1 test-ne*)

lemma *ne-down-sp-test-up*:

test p \implies $ne (R\downarrow * p\uparrow) = ne (R\downarrow) * p\uparrow$

by (*simp add: ne-dist-down-sp ne-test-up*)

lemma *test-up-sp*:

test p \implies $p\uparrow = p * 1\uparrow$

by (*metis ne-up test-ne*)

lemma *top-test-oi-top-complement*:

test p \implies $(U * p) \cap (U * \imath p) = 1_{\cup\cup}$

by (*smt (verit) Compl-disjoint U-par-idem inf.absorb-iff2 inf-commute p-id-zero s-prod-idl sp-test-dist-oi-right test-assoc1 top-upper-least*)

lemma *sp-test-oi-complement*:

test p \implies $(R * p) \cap (R * \imath p) = R \cap 1_{\cup\cup}$

by (*smt semilattice-inf-class.inf-idem sp-test sp-test-dist-oi-left sp-test-dist-oi-right test-complement-closed top-test-oi-top-complement*)

lemma *ne-top-sp-test-complement*:

assumes *test p*

shows $ne (U * p) * \imath p = \{\}$

by (*metis Compl-disjoint Int-assoc assms oc-top-sp-test-0*

semilattice-inf-class.inf-le2 sp-test-dist-oi-right top-test-oi-top-complement)

lemma *complement-test-sp-top*:

assumes *test p*

shows $-(p * U) = \imath p * U$

proof –

have $-(p * U) = -\{(a,A). (a,\{a\}) \in p \wedge (a,A) \in U\}$

by (*metis (no-types, lifting) Collect-cong assms inf.orderE split-cong test-s-prod-var*)

also have $\dots = -\{(a,A). (a,\{a\}) \in p\}$

using *top-upper-least* **by** *auto*

also have $\dots = \{(a,A). (a,\{a\}) \notin p\}$

by *force*

also have $\dots = \{(a,A). (a,\{a\}) \in \imath p\}$

by (*meson test-complement*)

also have $\dots = \{(a,A). (a,\{a\}) \in \imath p \wedge (a,A) \in U\}$

using *U-par-idem top-upper-least* **by** *auto*

also have $\dots = \imath p * U$

by (*simp add: test-s-prod-var*)

finally show *?thesis*

qed

lemma *top-sp-test-shunt*:
assumes *test p*
shows $R \subseteq U * p \longrightarrow R * \wr p \subseteq 1_{\cup\cup}$
by (*metis assms inf.absorb-iff1 sp-test sp-test-dist-oi test-complement-closed top-test-oi-top-complement*)

lemma *top-sp-test-down-iff-2*:
assumes *test p*
shows $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow * \wr p \subseteq 1_{\cup\cup}$

proof
assume $R\downarrow \subseteq U * p$
thus $R\downarrow * \wr p \subseteq 1_{\cup\cup}$
using *assms top-sp-test-shunt* **by** *blast*
next

assume $1: R\downarrow * \wr p \subseteq 1_{\cup\cup}$
have $R \subseteq U * p$

proof
fix x
assume $x \in R$
from *this* **obtain** $a B$ **where** $2: x = (a, B) \wedge x \in R$
by *force*
have $\forall b \in B. (b, \{b\}) \in p$

proof
fix b
assume $3: b \in B$
have $(b, \{b\}) \notin \wr p$
proof
assume $4: (b, \{b\}) \in \wr p$
have $(a, \{b\}) \in R\downarrow$
using $2\ 3$ **down** **by** *fastforce*
hence $(a, \{b\}) \in R\downarrow * \wr p$
using 4 **by** (*simp add: s-prod-test*)
thus *False*
using 1 **by** (*metis Pair-inject domain-pointwise insert-not-empty*)

p-subid-iff)
qed
thus $(b, \{b\}) \in p$
by (*meson test-complement*)
qed
thus $x \in U * p$
using 2 **by** (*simp add: assms top-sp-test*)
qed
thus $R\downarrow \subseteq U * p$
using *assms top-sp-test-down-iff-1* **by** *blast*
qed

lemma *top-sp-test-down-iff-3*:
 $R\downarrow * \wr p \subseteq 1_{\cup\cup} \longleftrightarrow ne (R\downarrow) * \wr p \subseteq \{\}$
by (*simp add: disjoint-eq-subset-Compl ne-sp-test*)

lemma *top-sp-test-down-iff-4*:

assumes *test p*

shows $R\downarrow \cap (U * \wr p) \subseteq 1_{UU} \longleftrightarrow R\downarrow \subseteq 1_{UU} \cup (U * p)$

by (*metis assms lattice-class.sup-inf-absorb semilattice-inf-class.inf-le2 sp-test sup-commute top-sp-test-down-iff-2 top-test-oi-top-complement*)

lemma *top-sp-test-down-iff-5*:

assumes *test p*

shows $R\downarrow \subseteq U * p \longleftrightarrow R\downarrow \subseteq 1_{UU} \cup (U * p)$

by (*metis assms semilattice-inf-class.inf-le1 sup.absorb2 top-test-oi-top-complement*)

lemma *iu-test-sp-left-zero*:

assumes $q \subseteq 1_{UU}$

shows $q * R = q$

by (*metis assms p-id-assoc2 p-subid-iff s-prod-p-idl*)

lemma *test-iu-test-split*:

$t \subseteq 1 \cup 1_{UU} \longleftrightarrow (\exists p q . p \subseteq 1 \wedge q \subseteq 1_{UU} \wedge t = p \cup q)$

by (*meson subset-UnE sup.mono*)

lemma *test-iu-test-sp-assoc-1*:

$t \subseteq 1 \cup 1_{UU} \implies t * (R * S) = (t * R) * S$

unfolding *test-iu-test-split*

by (*smt (verit, ccfv-threshold) inf.orderE p-id-assoc2 p-subid-iff s-prod-distr s-prod-p-idl test-assoc2*)

lemma *test-iu-test-sp-assoc-2*:

$t \subseteq 1_{UU} \implies R * (t * S) = (R * t) * S$

proof –

assume $1: t \subseteq 1_{UU}$

have $R * (t * S) = R * (t * \{\})$

using 1 **by** (*metis iu-test-sp-left-zero p-id-assoc2 s-prod-p-idl*)

also have $\dots = (R * t) * \{\}$

by (*metis cl5 s-prod-idl*)

also have $\dots \subseteq (R * t) * S$

by (*simp add: s-prod-isor*)

finally have $R * (t * S) \subseteq (R * t) * S$

.

thus *?thesis*

by (*simp add: s-prod-assoc1 set-eq-subset*)

qed

lemma *test-iu-test-sp-assoc-3*:

assumes $t \subseteq 1 \cup 1_{UU}$

shows $R * (t * S) = (R * t) * S$

proof

let $?g = \lambda b . \text{if } (b, \{b\}) \in t \wedge (b, \{\}) \notin t \text{ then } \{b\} \text{ else } \{\}$

show $R * (t * S) \subseteq (R * t) * S$
proof
fix x
assume $x \in R * (t * S)$
from this obtain $a B C f$ **where** $1: x = (a, C) \wedge (a, B) \in R \wedge (\forall b \in B . (b, f b) \in t * S) \wedge C = \bigcup \{ f b \mid b . b \in B \}$
by (*simp add: mr-simp*) *blast*
hence $\forall b \in B . \exists D . (b, D) \in t \wedge (\exists g . (\forall e \in D . (e, g e) \in S) \wedge f b = \bigcup \{ g e \mid e . e \in D \})$
by (*simp add: mr-simp Setcompr-eq-image*)
hence $\exists Db . \forall b \in B . (b, Db b) \in t \wedge (\exists g . (\forall e \in Db b . (e, g e) \in S) \wedge f b = \bigcup \{ g e \mid e . e \in Db b \})$
by (*rule bchoice*)
from this obtain Db **where** $2: \forall b \in B . (b, Db b) \in t \wedge (\exists g . (\forall e \in Db b . (e, g e) \in S) \wedge f b = \bigcup \{ g e \mid e . e \in Db b \})$
by *auto*
let $?D = \bigcup \{ Db b \mid b . b \in B \}$
have $\forall b \in B . (b, Db b) \in t$
using 2 **by** *auto*
hence $3: \forall b \in B . Db b = \{b\} \vee Db b = \{\}$
using *assms* **by** (*metis Pair-inject Un-iff domain-pointwise inf.orderE p-subid-iff subid-aux2 test-iu-test-split*)
have $4: (a, ?D) \in R * t$
apply (*simp add: mr-simp*)
apply (*rule exI[where ?x=B]*)
apply (*rule conjI*)
using 1 **apply** *simp*
apply (*rule exI[where ?x=Db]*)
using 2 **by** *auto*
have $5: \forall b \in ?D . (b, f b) \in S$
proof
fix b
assume $b \in ?D$
hence $b \in B \wedge Db b = \{b\}$
using 3 **by** *auto*
thus $(b, f b) \in S$
using 2 **by** *force*
qed
have $6: C = \bigcup \{ f b \mid b . b \in ?D \}$
proof
show $C \subseteq \bigcup \{ f b \mid b . b \in ?D \}$
proof
fix y
assume $y \in C$
from this 1 obtain b **where** $7: b \in B \wedge y \in f b$
by *auto*
hence $Db b = \{b\}$
using 2 3 **by** *blast*
thus $y \in \bigcup \{ f b \mid b . b \in ?D \}$

```

      using 7 by blast
    qed
  next
  show  $\bigcup \{ f b \mid b . b \in ?D \} \subseteq C$ 
  proof
    fix y
    assume  $y \in \bigcup \{ f b \mid b . b \in ?D \}$ 
    from this obtain b where  $\delta: b \in ?D \wedge y \in f b$ 
    by auto
    hence  $b \in B$ 
    using 3 by auto
    thus  $y \in C$ 
    using 1 8 by auto
  qed
  qed
  have  $(a, C) \in (R * t) * S$ 
  using 4 5 6 apply (clarsimp simp: mr-simp) by blast
  thus  $x \in (R * t) * S$ 
  using 1 by simp
  qed
next
  show  $(R * t) * S \subseteq R * (t * S)$ 
  using s-prod-assoc1 by blast
  qed

lemma test-iu-test-sp-assoc-4:
   $t \subseteq 1_{\cup\cup} \implies R * (S * t) = (R * S) * t$ 
  by (metis cl5 iu-test-sp-left-zero)

lemma test-iu-test-sp-assoc-5:
  assumes  $t \subseteq 1 \cup 1_{\cup\cup}$ 
  shows  $R * (S * t) = (R * S) * t$ 
  proof
    show  $R * (S * t) \subseteq (R * S) * t$ 
    proof
      fix x
      assume  $x \in R * (S * t)$ 
      from this obtain a B C f where 1:  $x = (a, C) \wedge (a, B) \in R \wedge (\forall b \in B . (b, f b) \in S * t) \wedge C = \bigcup \{ f b \mid b . b \in B \}$ 
      by (clarsimp simp: mr-simp) blast
      hence  $\forall b \in B . \exists D . (b, D) \in S \wedge (\exists g . (\forall e \in D . (e, g e) \in t) \wedge f b = \bigcup \{ g e \mid e . e \in D \})$ 
      by (clarsimp simp: mr-simp Setcompr-eq-image)
      hence  $\exists Db . \forall b \in B . (b, Db b) \in S \wedge (\exists g . (\forall e \in Db b . (e, g e) \in t) \wedge f b = \bigcup \{ g e \mid e . e \in Db b \})$ 
      by (rule bchoice)
      from this obtain Db where 2:  $\forall b \in B . (b, Db b) \in S \wedge (\exists g . (\forall e \in Db b . (e, g e) \in t) \wedge f b = \bigcup \{ g e \mid e . e \in Db b \})$ 
      by auto
    end
  end

```

hence $\exists gb . \forall b \in B . (\forall e \in Db\ b . (e, gb\ b\ e) \in t) \wedge f\ b = \bigcup \{ gb\ b\ e \mid e . e \in Db\ b \}$
by (*auto intro: bchoice*)
from this obtain gb **where** $\exists : \forall b \in B . (\forall e \in Db\ b . (e, gb\ b\ e) \in t) \wedge f\ b = \bigcup \{ gb\ b\ e \mid e . e \in Db\ b \}$
by *auto*
let $?g = \lambda x . \bigcup \{ gb\ b\ x \mid b . b \in B \wedge x \in Db\ b \}$
have $4 : \forall b \in B . \forall e \in Db\ b . gb\ b\ e = \{ \} \vee gb\ b\ e = \{ e \}$
proof (*intro ballI*)
fix $b\ e$
assume $b \in B\ e \in Db\ b$
hence $(e, gb\ b\ e) \in t$
using \exists **by** *simp*
thus $gb\ b\ e = \{ \} \vee gb\ b\ e = \{ e \}$
by (*metis Un-iff assms domain-pointwise inf.orderE p-subid-iff prod.inject subid-aux2 test-iu-test-split*)
qed
hence $\forall e . ?g\ e \subseteq \{ e \}$
by *auto*
hence $5 : \forall e . ?g\ e = \{ \} \vee ?g\ e = \{ e \}$
by *auto*
let $?D = \bigcup \{ Db\ b \mid b . b \in B \}$
have $6 : (a, ?D) \in R * S$
apply (*unfold s-prod-def*)
using $1\ 2$ **by** *blast*
have $7 : \forall b \in ?D . (b, ?g\ b) \in t$
proof
fix e
assume $e \in ?D$
from this obtain b **where** $8 : b \in B \wedge e \in Db\ b$
by *auto*
show $(e, ?g\ e) \in t$
proof (*cases ?g\ e = \{ \}*)
case *True*
hence $gb\ b\ e = \{ \}$
using 8 **by** *auto*
thus *?thesis*
using $\exists\ 8\ True$ **by** *metis*
next
case *False*
hence $9 : ?g\ e = \{ e \}$
using 5 **by** *auto*
from this obtain bb **where** $10 : bb \in B \wedge e \in Db\ bb \wedge e \in gb\ bb\ e$
by *auto*
hence $gb\ bb\ e = \{ e \}$
using 4 **by** *auto*
thus *?thesis*
using $\exists\ 9\ 10$ **by** *force*
qed

```

qed
have 11:  $C = \bigcup \{ ?g\ e \mid e . e \in ?D \}$ 
proof
  show  $C \subseteq \bigcup \{ ?g\ e \mid e . e \in ?D \}$ 
  proof
    fix y
    assume  $y \in C$ 
    from this obtain b where 12:  $b \in B \wedge y \in f\ b$ 
      using 1 by auto
    from this 3 obtain e where  $e \in Db\ b \wedge y \in gb\ b\ e$ 
      by auto
    thus  $y \in \bigcup \{ ?g\ e \mid e . e \in ?D \}$ 
      using 4 12 by auto
  qed
next
show  $\bigcup \{ ?g\ e \mid e . e \in ?D \} \subseteq C$ 
proof
  fix y
  assume  $y \in \bigcup \{ ?g\ e \mid e . e \in ?D \}$ 
  from this obtain b e where 13:  $b \in B \wedge e \in Db\ b \wedge y \in gb\ b\ e$ 
    by auto
  hence  $y \in f\ b$ 
    using 3 by auto
  thus  $y \in C$ 
    using 1 13 by auto
qed
qed
have  $(a, C) \in (R * S) * t$ 
  apply (simp add: mr-simp)
  apply (rule exI[where ?x=?D])
  apply (rule conjI)
  using 6 apply (simp add: mr-simp)
  apply (rule exI[where ?x=?g])
  using 7 11 by auto
thus  $x \in (R * S) * t$ 
  using 1 by simp
qed
next
show  $(R * S) * t \subseteq R * (S * t)$ 
  using s-prod-assoc1 by blast
qed

lemma inner-deterministic-sp-assoc:
  assumes inner-univalent t
  shows  $t * (R * S) = (t * R) * S$ 
proof (rule antisym)
  show  $t * (R * S) \subseteq (t * R) * S$ 
  proof
    fix x

```

```

assume  $x \in t * (R * S)$ 
from this obtain  $a B D f$  where  $1: x = (a, D) \wedge (a, B) \in t \wedge (\forall b \in B . (b, f b) \in R * S) \wedge D = \bigcup \{ f b \mid b . b \in B \}$ 
  by (simp add: mr-simp) blast
have  $(a, D) \in (t * R) * S$ 
proof (cases  $(a, B) \in 1_{\cup\cup}$ )
  case True
    hence  $B = \{\}$ 
    by (metis Pair-inject domain-pointwise iu-test-sp-left-zero order-refl)
    hence  $D = \{\}$ 
    using  $1$  by fastforce
    hence  $(a, D) \in t * (R * \{\})$ 
    using  $1 \langle (B::'b::\text{type set}) = \{\} \rangle$  s-prod-def by fastforce
    hence  $(a, D) \in (t * R) * \{\}$ 
    by (metis cl5 s-prod-idl)
  thus ?thesis
    using s-prod-isor by auto
next
  case False
    hence  $(a, B) \in A_{\cup\cup}$ 
    using  $1$  assms by blast
    from this obtain  $b$  where  $2: B = \{b\}$ 
    by (smt atoms-def Pair-inject mem-Collect-eq)
    hence  $D = f b \wedge (b, f b) \in R * S$ 
    using  $1$  by auto
    from this obtain  $C g$  where  $3: (b, C) \in R \wedge (\forall c \in C . (c, g c) \in S) \wedge D = \bigcup \{ g c \mid c . c \in C \}$ 
    by (clarsimp simp: mr-simp) blast
    have  $(a, C) \in t * R$ 
    apply (simp add: mr-simp)
    apply (rule exI[where ?x=B])
    using  $1 2 3$  by auto
    thus ?thesis
    using  $3$  s-prod-def by blast
  qed
thus  $x \in (t * R) * S$ 
using  $1$  by auto
qed
next
show  $(t * R) * S \subseteq t * (R * S)$ 
using s-prod-assoc1 by blast
qed

lemma iu-unit-below-top-sp-test:
   $1_{\cup\cup} \subseteq U * R$ 
  by (clarsimp simp: mr-simp) force

lemma ne-oi-complement-top-sp-test-1:
   $ne (R \cap -(U * S)) = R \cap -(U * S)$ 

```

```

using iu-unit-below-top-sp-test by auto

lemma ne-oi-complement-top-sp-test-2:
  ne  $R \cap -(U * S) = R \cap -(U * S)$ 
using iu-unit-below-top-sp-test by auto

lemma schroeder-test:
  assumes test p
  shows  $R * p \subseteq S \longleftrightarrow -S * p \subseteq -R$ 
  by (metis assms disjoint-eq-subset-Compl double-compl
semilattice-inf-class.inf-commute sp-test-dist-oi-right)

lemma complement-test-sp-test:
  test p  $\implies -p * p \subseteq -1$ 
  by (simp add: schroeder-test)

lemma test-sp-commute:
  test p  $\implies$  test q  $\implies p * q = q * p$ 
  by (metis s-prod-idl sp-test-dist-oi-left sp-test-dist-oi-right test-fix)

lemma test-shunting:
  assumes test p
    and test q
    and test r
  shows  $p * q \subseteq r \longleftrightarrow p \subseteq r \cup \setminus q$ 
proof
  assume  $1: p * q \subseteq r$ 
  have  $p = p * q \cup p * \setminus q$ 
    by (smt (verit, del-insts) Int-Un-distrib assms(1,2) inf-sup-aci(1)
inf-sup-ord(2) le-iff-inf s-distl-test s-prod-idr sup-neg-inf)
  also have  $\dots \subseteq r \cup \setminus q$ 
    using  $1$  by (metis assms(1) inf-le2 sup.mono test-sp)
  finally show  $p \subseteq r \cup \setminus q$ 
  .
next
  assume  $p \subseteq r \cup \setminus q$ 
  hence  $p * q \subseteq p * r$ 
    by (smt (verit) assms boolean-algebra-class.boolean-algebra.double-compl
inf.orderE inf-le1 le-inf-iff schroeder-test sup-neg-inf test-double-complement
test-s-prod-is-meet test-sp-commute)
  also have  $\dots \subseteq r$ 
    using assms(1) test-sp by auto
  finally show  $p * q \subseteq r$ 
  .
qed

lemma test-sp-is-iu:
  test p  $\implies$  test q  $\implies p * q = p \cup \cup q$ 
  by (metis Int-assoc inf.right-idem p-prod-comm s-prod-idr test-p-prod-is-meet)

```

test-sp)

lemma *test-set*:

$test\ p \implies p = \{ (a, \{a\}) \mid a \cdot (a, \{a\}) \in p \}$
using *s-id-st* **by** *blast*

lemma *test-sp-is-ii*:

assumes *test p*
and *test q*

shows $p * q = p \cap \cap q$

proof –

have $p \cap q = \{ (a, \{a\}) \mid a \cdot (a, \{a\}) \in p \} \cap \{ (a, \{a\}) \mid a \cdot (a, \{a\}) \in q \}$
using *assms test-set* **by** *blast*

also have $\dots = \{ (a, \{a\}) \mid a \cdot (a, \{a\}) \in p \} \cap \cap \{ (a, \{a\}) \mid a \cdot (a, \{a\}) \in q \}$
apply (*rule antisym*)

apply (*clarsimp simp: mr-simp*)

apply (*rule Int-greatest*)

apply (*clarsimp simp: mr-simp*)

by (*clarsimp simp: mr-simp*)

also have $\dots = p \cap \cap q$

using *test-set[symmetric, OF assms(1)] test-set[symmetric, OF assms(2)]* **by**

simp

finally show $p * q = p \cap \cap q$

by (*metis assms inf.orderE test-oi-closed test-s-prod-is-meet*)

qed

lemma *test-galois-1*:

assumes *test p*

and *test q*

shows $p * q \subseteq r \iff q \subseteq p \rightarrow r$

by (*metis (no-types, lifting) Int-Un-eq(2) assms inf.orderE sup-neg-inf test-double-complement test-s-prod-is-meet test-sp-commute*)

lemma *test-sp-shunting*:

assumes *test p*

shows $\lambda p * R \subseteq \{\} \iff R \subseteq p * R$

apply (*rule iffI*)

apply (*metis (no-types, opaque-lifting) assms complement-test-sp-top disjoint-eq-subset-Compl double-compl semilattice-inf-class.inf.absorb-iff2 semilattice-inf-class.inf-commute semilattice-inf-class.inf-right-idem test-sp*)

by (*metis (no-types, opaque-lifting) assms complement-test-sp-top disjoint-eq-subset-Compl double-compl semilattice-inf-class.inf-commute semilattice-inf-class.inf-le1 subset-antisym test-sp*)

lemma *test-oU-closed*:

$\forall p \in X . test\ p \implies test\ (\bigcup X)$

by *blast*

lemma *test-oI-closed*:

$\exists p \in X . \text{test } p \implies \text{test } (\bigcap X)$
by *blast*

lemma *sp-test-dist-oI*:

assumes *test p*
and $X \neq \{\}$
shows $(\bigcap X) * p = (\bigcap R \in X . R * p)$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply *blast*
apply (*clarsimp simp: mr-simp*)
subgoal for $a \ C$ **proof** –
assume $1: \forall R \in X . \exists B . (a, B) \in R \wedge (\exists f . (\forall b \in B . (b, f \ b) \in p) \wedge C = \bigcup (f \ ' B))$
have $2: (\forall R \in X . (a, C) \in R \wedge (\forall b \in C . (b, \{b\}) \in p))$
proof
fix R
assume $R \in X$
from *this* **obtain** B **where** $3: (a, B) \in R \wedge (\exists f . (\forall b \in B . (b, f \ b) \in p) \wedge C = \bigcup (f \ ' B))$
using 1 **by** *force*
from *this* **obtain** f **where** $4: \forall b \in B . (b, f \ b) \in p$ **and** $5: C = \bigcup (f \ ' B)$
by *auto*
hence $6: \forall b \in B . f \ b = \{b\}$
using *assms(1)* **test** **by** *blast*
hence $7: C = B$
using 5 **by** *auto*
hence $(a, C) \in R$
using 3 **by** *auto*
thus $(a, C) \in R \wedge (\forall b \in C . (b, \{b\}) \in p)$
using $4 \ 6 \ 7$ **by** *fastforce*
qed
show *?thesis*
apply (*rule exI[of - C]*)
apply (*rule conjI*)
using 2 **apply** *simp*
apply (*rule exI[of - $\lambda x . \{x\}$]*)
apply (*rule conjI*)
using 2 *assms(2)* **apply** *blast*
by *simp*
qed
done

lemma *test-iU-is-iI*:

assumes $\forall i \in I . \text{test } (X \ i)$
and $I \neq \{\}$
shows $\bigcup \bigcup X | I = \bigcap \bigcap X | I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)

subgoal for $a f$ proof –
assume $1: \forall i \in I . (a, f i) \in X i$
hence $\forall i \in I . f i = \{a\}$
using $assms(1)$ **by** $(meson test)$
hence $\bigcup (f ' I) = \bigcap (f ' I) \wedge (\forall i \in I . (a, f i) \in X i)$
using $1 assms(2)$ **by** $auto$
thus $?thesis$
by $meson$
qed
apply $(clarsimp simp: mr-simp)$
by $(metis (no-types, opaque-lifting) INF-eq-const SUP-eq-const assms test)$

lemma $test-iU-is-oI$:
assumes $\forall i \in I . test (X i)$
and $I \neq \{\}$
shows $\bigcup \bigcup X | I = \bigcap (X ' I)$
apply $(rule antisym)$
apply $(clarsimp simp: mr-simp)$
apply $(metis (no-types, opaque-lifting) SUP-eq-const assms test)$
apply $(clarsimp simp: mr-simp)$
by $(metis UN-constant assms(2))$

8.2 Domain and antidomain

declare $Dom-def$ $[mr-simp]$

abbreviation $aDom :: ('a, 'b) mrel \Rightarrow ('a, 'a) mrel$ **where**
 $aDom R \equiv \iota Dom R$

lemma $ad-set$: $aDom R = \{(a, \{a\}) \mid a. \neg(\exists A. (a, A) \in R)\}$
by $(clarsimp simp: mr-simp) force$

lemma $d-test$:
 $test (Dom R)$
unfolding $Dom-def$ **using** $s-id-def$ **by** $fastforce$

lemma $ad-test$:
 $test (aDom R)$
by $simp$

lemma $ad-expl$:
 $aDom R = -((R * 1_{\cup\cup}) \cup\cup 1) \cap 1$
by $(simp add: d-def-expl)$

lemma $ad-expl-2$:
 $aDom (R :: ('a, 'b) mrel) = -((R * (1_{\cup\cup} :: ('b, 'a) mrel)) \uparrow) \cap (1 :: ('a, 'a) mrel)$
proof
have $-((R * 1_{\cup\cup}) \uparrow) \cap 1 = -((R * 1_{\cup\cup}) \cup\cup U) \cap 1$
by $simp$

also have $\dots \subseteq -((R * 1_{\cup\cup}) \cup\cup 1) \cap 1$
by (*metis c6 convex-increasing iu-commute iu-isotone iu-unit sp-unit-convex test-complement-antitone upper-iu-increasing*)
also have $\dots = aDom R$
by (*simp add: d-def-expl*)
finally show $-((R * 1_{\cup\cup})\uparrow) \cap 1 \subseteq aDom R$
using $\langle (R::('a::type \times 'b::type set) set) * 1_{\cup\cup})\uparrow \subseteq \wr (R * 1_{\cup\cup} \cup\cup 1) \rangle$ **by**
blast
have $aDom R = \{(a, \{a\}) \mid a. \neg(\exists B. (a, B) \in R)\}$
by (*simp add: ad-set*)
also have $\dots \subseteq \{(a, \{a\}) \mid a. (a, \{a\}) \notin (R * (p-id::('b, 'a) mrel))\uparrow\}$
by (*simp add: U-par-st domain-pointwise*)
also have $\dots = \{(a, \{a\}) \mid a. (a, \{a\}) \in -(R * (p-id::('b, 'a) mrel))\uparrow \cap 1\}$
using *test-complement* **by** *fastforce*
finally show $aDom R \subseteq -((R * (1_{\cup\cup}::('b, 'a) mrel))\uparrow) \cap (1::('a, 'a) mrel)$
by *blast*
qed

lemma *aDom*:
 $aDom R = \{ (a, \{a\}) \mid a. \neg(\exists B. (a, B) \in R) \}$
by (*simp add: ad-set*)

declare *aDom* [*mr-simp*]

lemma *d-down-oi-up-1*:
 $Dom (R\downarrow \cap S) = Dom (R \cap S\uparrow)$
by (*metis Int-commute d-def-expl domain-up-down-conjugate*)

lemma *d-down-oi-up-2*:
 $Dom (R\downarrow \cap S) = Dom (R\downarrow \cap S\uparrow)$
by (*simp add: d-down-oi-up-1*)

lemma *d-ne-down-dp-complement-test*:
assumes *test p*
shows $Dom (R \cap -(U * p)) = Dom (ne (R\downarrow) * \wr p)$
by (*simp add: asms d-down-oi-up-1 oc-top-sp-test-0 oc-top-sp-test-1 sp-test-dist-oi-right*)

lemma *d-strict*:
 $R = \{\} \longleftrightarrow Dom R = \{\}$
using *Dom-def* **by** *fastforce*

lemma *d-sp-strict*:
 $R * S = \{\} \longleftrightarrow R * Dom S = \{\}$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *metis*
by (*metis UN-singleton*)

lemma *d-complement-ad*:

$Dom R = \wr aDom R$

using *d-test* **by** *blast*

lemma *down-sp-below-iu-unit*:

$R\downarrow * S \subseteq 1_{UU} \iff R \subseteq U * aDom (ne S)$

proof –

have $R\downarrow * S \subseteq 1_{UU} \iff ne (R\downarrow * S) = \{\}$

by (*simp add: disjoint-eq-subset-Compl*)

also have $\dots \iff ne (R\downarrow) * ne S = \{\}$

by (*simp add: ne-dist-down-sp*)

also have $\dots \iff ne (R\downarrow) * Dom (ne S) = \{\}$

using *d-sp-strict* **by** *auto*

also have $\dots \iff ne (R\downarrow) * \wr aDom (ne S) = \{\}$

by (*metis d-complement-ad*)

also have $\dots \iff R \subseteq U * aDom (ne S)$

by (*metis top-sp-test-down-iff-1 top-sp-test-down-iff-2 top-sp-test-down-iff-3 ad-test subset-empty*)

finally show *?thesis*

qed

lemma *ad-sp-bot*:

$aDom R * R = \{\}$

by (*simp add: d-s-id-ax d-sp-strict inf-sup-aci(1) sp-test-dist-oi-left*)

lemma *sp-top-d*:

$R * U \subseteq Dom R * U$

by (*simp add: cl8-var iu-unit-up sp-upper-left-isotone*)

lemma *d-sp-top*:

$Dom (R * U) = Dom R$

by (*clarsimp simp: mr-simp*) *blast*

lemma *d-down*:

$Dom (R\downarrow) = Dom R$

by (*metis U-par-idem d-down-oi-up-1 inf.orderE top-down top-lower-greatest*)

lemma *d-up*:

$Dom (R\uparrow) = Dom R$

by (*metis Int-absorb1 U-par-idem d-down-oi-up-1 top-down top-upper-least*)

lemma *d-isotone*:

$R \subseteq S \implies Dom R \subseteq Dom S$

unfolding *Dom-def* **by** *blast*

lemma *ad-antitone*:

$R \subseteq S \implies aDom S \subseteq aDom R$

by (*simp add: Int-commute d-isotone semilattice-inf-class.le-infI2*)

lemma *d-dist-ou*:
 $Dom (R \cup S) = Dom R \cup Dom S$
unfolding *Dom-def* **by** *blast*

lemma *d-dist-iu*:
 $Dom (R \cup\cup S) = Dom R * Dom S$
by (*clarsimp simp: mr-simp*) *auto*

lemma *d-dist-ii*:
 $Dom (R \cap\cap S) = Dom R * Dom S$
by (*metis antisym-conv d-U d-dist-iu d-down d-isotone ii-convex-iu s-prod-idr*)

lemma *d-loc*:
 $Dom (R * Dom S) = Dom (R * S)$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply *metis*
by (*metis UN-singleton*)

lemma *ad-loc*:
 $aDom (R * Dom S) = aDom (R * S)$
by *simp*

lemma *d-ne-down*:
 $Dom (ne (R\downarrow)) = Dom (ne R)$
by (*metis atoms-solution d-down-oi-up-1 d-down-oi-up-2*)

lemma *ne-sp-iu-unit-up*:
 $ne R = R \implies (R * 1_{\cup\cup})\uparrow = R * U$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply (*metis (no-types, lifting) Compl-iff IntE Inter-iff UNIV-I UN-simps(2)*)
image-eqI singletonI
apply *clarsimp*
by (*metis SUP-bot sup-bot-left*)

lemma *ne-d-expl*:
 $ne R = R \implies Dom R = R * U \cap 1$
by (*metis cl8-var d-def-expl d-test ne-sp-iu-unit-up test-sp*)

lemma *ne-a-expl*:
 $ne R = R \implies aDom R = -(R * U) \cap 1$
by (*simp add: ad-expl-2 ne-sp-iu-unit-up*)

lemma *d-dist-oU*:
 $Dom (\bigcup X) = \bigcup (Dom \text{ ` } X)$
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *d-dist-iU-iI*:
 $Dom (\bigcup \bigcup X|I) = Dom (\bigcap \bigcap X|I)$
by (*clarsimp simp: mr-simp*)

lemma *d-dist-iU-oI*:
assumes $I \neq \{\}$
shows $Dom (\bigcup \bigcup X|I) = \bigcap (Dom \text{ ` } X \text{ ` } I)$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
apply *blast*
apply (*clarsimp simp: mr-simp*)
by (*meson all-not-in-conv assms*)

8.3 Left residual

definition *sp-lres* :: $('a,'c) \text{ mrel} \Rightarrow ('b,'c) \text{ mrel} \Rightarrow ('a,'b) \text{ mrel}$ (**infixl** $\langle \circ \rangle$ 65)

where

$Q \circ R \equiv \{ (a,B) . \forall f . (\forall b \in B . (b,f b) \in R) \longrightarrow (a, \bigcup \{ f b \mid b . b \in B \}) \in Q \}$

declare *sp-lres-def* [*mr-simp*]

lemma *sp-lres-galois*:

$S * R \subseteq Q \longleftrightarrow S \subseteq Q \circ R$

proof

assume $1: S * R \subseteq Q$

show $S \subseteq Q \circ R$

proof

fix x

assume $x \in S$

from this obtain $a B$ **where** $2: x = (a,B) \wedge (a,B) \in S$

by (*metis surj-pair*)

have $\forall f . (\forall b \in B . (b,f b) \in R) \longrightarrow (a, \bigcup \{ f b \mid b . b \in B \}) \in Q$

proof (*rule allI, rule impI*)

fix f

assume $\forall b \in B . (b,f b) \in R$

hence $(a, \bigcup \{ f b \mid b . b \in B \}) \in S * R$

apply (*unfold s-prod-def*)

using 2 **by** *auto*

thus $(a, \bigcup \{ f b \mid b . b \in B \}) \in Q$

using 1 **by** *auto*

qed

thus $x \in Q \circ R$

using 2 *sp-lres-def* **by** *auto*

qed

next

assume $3: S \subseteq Q \circ R$

have $(Q \circ R) * R \subseteq Q$

proof
fix x
assume $x \in (Q \otimes R) * R$
from this obtain $a D$ **where** $4: x = (a, D) \wedge (a, D) \in (Q \otimes R) * R$
by (*metis surj-pair*)
from this obtain C **where** $(a, C) \in Q \otimes R \wedge (\exists g . (\forall c \in C . (c, g c) \in R) \wedge$
 $D = \bigcup \{ g c \mid c . c \in C \})$
by (*simp add: mr-simp*) *blast*
thus $x \in Q$
apply (*unfold sp-lres-def*)
using 4 **by** *auto*
qed
thus $S * R \subseteq Q$
using 3 **by** (*meson dual-order.trans s-prod-isol*)
qed

lemma *sp-lres-expl*:
 $Q \otimes R = \bigcup \{ S . S * R \subseteq Q \}$
using *sp-lres-galois* **by** *blast*

lemma *bot-sp-lres-d*:
 $\{\} \otimes R = \{\} \otimes \text{Dom } R$
by (*metis d-sp-strict order-refl sp-lres-galois subset-antisym subset-empty*)

lemma *bot-sp-lres-expl*:
 $\{\} \otimes R = \neg(U * \text{Dom } R)$
apply (*rule antisym*)
apply (*metis d-sp-strict d-test disjoint-eq-subset-Compl order-refl sp-lres-galois*
sp-test subset-empty)
by (*metis Compl-disjoint2 bot-sp-lres-d d-test sp-lres-galois sp-test subset-empty*)

lemma *sp-lres-sp-below*:
 $(Q \otimes R) * R \subseteq Q$
by (*simp add: sp-lres-galois*)

lemma *sp-lres-left-isotone*:
 $Q \subseteq S \implies Q \otimes R \subseteq S \otimes R$
by (*meson dual-order.refl sp-lres-galois subset-trans*)

lemma *sp-lres-right-antitone*:
 $S \subseteq R \implies Q \otimes R \subseteq Q \otimes S$
by (*meson dual-order.trans s-prod-isor sp-lres-galois sp-lres-sp-below*)

lemma *sp-lres-down-closed-1*:
 $Q \downarrow \otimes R = Q \downarrow \otimes R \downarrow$

proof
show $Q \downarrow \otimes R \subseteq Q \downarrow \otimes R \downarrow$
by (*metis down-dist-sp down-idempotent down-isotone sp-lres-galois*
sp-lres-sp-below)

next
show $Q\downarrow \otimes R\downarrow \subseteq Q\downarrow \otimes R$
by (*simp add: lower-reflexive sp-lres-right-antitone*)
qed

lemma *sp-lres-down-closed-2*:
assumes $R\downarrow = R$
and *total T*
shows $(R \otimes T)\downarrow = R \otimes T$
proof –
have $(R \otimes T)\downarrow \subseteq R \otimes T$
by (*metis assms lower-transitive sp-lres-galois sp-lres-sp-below total-down-sp-semi-commute*)
thus *?thesis*
by (*simp add: lower-reflexive subset-antisym*)
qed

lemma *down-sp-sp*:
 $R\downarrow * S = R * (1_{\cup\cup} \cup S)$
proof –
have $R\downarrow * S = R * (1_{\cup\cup} \cup 1) * S$
by (*simp add: down-sp*)
also have $\dots = R * ((1_{\cup\cup} \cup 1) * S)$
by (*simp add: test-iu-test-sp-assoc-3*)
also have $\dots = R * (1_{\cup\cup} \cup S)$
apply (*clarsimp simp: mr-simp*)
apply *safe*
apply (*smt (z3) Sup-empty ccpo-Sup-singleton image-empty image-insert singletonI*)
by (*smt (verit, del-insts) Sup-empty all-not-in-conv ccpo-Sup-singleton image-insert image-is-empty singletonD*)
finally show *?thesis*
qed

lemma *iu-unit-sp-lres-iu-unit-ou*:
 $U * aDom (ne R) = 1_{\cup\cup} \otimes (1_{\cup\cup} \cup R)$
apply (*rule antisym*)
apply (*metis down-sp-sp sp-lres-galois down-sp-below-iu-unit order-refl*)
by (*metis down-sp-sp sp-lres-galois down-sp-below-iu-unit order-refl*)

lemma *bot-sl-below-complement-d*:
 $\{\} \otimes R \subseteq - Dom R$
by (*metis Compl-anti-mono bot-sp-lres-expl d-test dual-order.refl inf.order-iff sp-test test-s-prod-is-meet*)

lemma *sp-unit-oi-bot-sp-lres*:
 $1 \cap - Dom R = 1 \cap (\{\} \otimes R)$
by (*smt (verit, ccfv-SIG) ad-sp-bot boolean-algebra-cancel.inf1*)

*bot-sl-below-complement-d inf.orderE inf-bot-right inf-commute inf-le2
sp-lres-galois*)

lemma *ad-explicit-d:*

*aDom R = -(U * Dom R) ∩ 1*

by (*simp add: bot-sp-lres-expl lattice-class.inf-sup-aci(1) sp-unit-oi-bot-sp-lres*)

lemma *top-test-sp-lres-total-expl-1:*

assumes *test p*

shows $\forall S . S \downarrow \subseteq (U * p) \circlearrowleft R \iff S \subseteq U * aDom (R \cap -(U * p))$

proof

fix *S::('b,'c) mrel*

have $S \subseteq U * aDom (R \cap -(U * p)) \iff ne (S \downarrow) * Dom (R \cap -(U * p)) = \{\}$

by (*metis (no-types, lifting) d-complement-ad inf-le2 subset-empty*

top-sp-test-down-iff-1 top-sp-test-down-iff-2 top-sp-test-down-iff-3)

also have $\dots \iff ne (S \downarrow) * Dom (ne (R \downarrow) * \wr p) = \{\}$

by (*simp add: assms d-ne-down-dp-complement-test*)

also have $\dots \iff ne (S \downarrow) * (ne (R \downarrow) * \wr p) = \{\}$

using *d-sp-strict* **by** *auto*

also have $\dots \iff ne (S \downarrow) * ne (R \downarrow) * \wr p = \{\}$

by (*simp add: test-assoc3*)

also have $\dots \iff ne ((S \downarrow * R) \downarrow) * \wr p = \{\}$

by (*simp add: down-dist-sp ne-dist-down-sp*)

also have $\dots \iff S \downarrow * R \subseteq U * p$

by (*metis assms top-sp-test-down-iff-1 top-sp-test-down-iff-2*

top-sp-test-down-iff-3 subset-empty)

also have $\dots \iff S \downarrow \subseteq (U * p) \circlearrowleft R$

by (*simp add: sp-lres-galois*)

finally show $S \downarrow \subseteq (U * p) \circlearrowleft R \iff S \subseteq U * aDom (R \cap -(U * p))$

by *simp*

qed

lemma *top-test-sp-lres-total-expl-2:*

assumes *test p*

and *total T*

shows $(U * p) \circlearrowleft T = U * aDom (T \cap -(U * p))$

proof $-$

have $\forall S . S \subseteq (U * p) \circlearrowleft T \iff S \subseteq U * aDom (T \cap -(U * p))$

by (*smt assms lower-reflexive sp-lres-down-closed-2 subset-trans*

top-sp-test-down-closed top-test-sp-lres-total-expl-1)

thus *?thesis*

by *blast*

qed

lemma *top-test-sp-lres-total-expl-3:*

assumes *test p*

shows $((U * p) \circlearrowleft R) \cap 1 = aDom (R \cap -(U * p))$

proof

have $((U * p) \otimes R) \cap 1 \downarrow * R = ((U * p) \otimes R) \cap 1 * (1_{UU} \cup R)$
using *down-sp-sp by blast*
also have $\dots = (((U * p) \otimes R) \cap 1) * 1_{UU} \cup (((U * p) \otimes R) \cap 1) * R$
by (*simp add: s-distl-test*)
also have $\dots \subseteq 1_{UU} \cup (((U * p) \otimes R) \cap 1) * R$
using *c6 by auto*
also have $\dots \subseteq 1_{UU} \cup ((U * p) \otimes R) * R$
by (*metis Un-Int-eq(4) inf-le2 iu-unit-convex iu-unit-down sp-oi-subdist sup.mono*)
also have $\dots \subseteq 1_{UU} \cup U * p$
using *sp-lres-sp-below by auto*
also have $\dots = U * p$
by (*simp add: iu-unit-below-top-sp-test sup.absorb2*)
finally have $((U * p) \otimes R) \cap 1 \downarrow \subseteq (U * p) \otimes R$
by (*simp add: sp-lres-galois*)
hence $((U * p) \otimes R) \cap 1 \subseteq U * aDom (R \cap -(U * p))$
by (*metis assms top-test-sp-lres-total-expl-1*)
thus $((U * p) \otimes R) \cap 1 \subseteq aDom (R \cap -(U * p))$
by (*metis (no-types, lifting) inf.idem inf.orderE inf-commute sp-oi-subdist sp-test test-s-prod-is-meet*)
next
have $aDom (R \cap -(U * p)) = U * aDom (R \cap -(U * p)) \cap 1$
by (*metis (no-types, lifting) inf.absorb-iff2 inf.idem inf-commute inf-le2 sp-test test-s-prod-is-meet*)
thus $aDom (R \cap -(U * p)) \subseteq ((U * p) \otimes R) \cap 1$
by (*metis Int-mono ad-test assms order-refl top-sp-test-down-closed top-test-sp-lres-total-expl-1*)
qed

lemma *top-test-sp-lres-total-expl-4:*

assumes *test p*
shows $aDom (ne (R \downarrow) * \downarrow p) = ((U * p) \otimes R) \cap 1$
by (*simp add: assms d-ne-down-dp-complement-test top-test-sp-lres-total-expl-3*)

lemma *oi-complement-top-sp-test-top-1:*

assumes *test p*
shows $(R \cap -(U * p)) * U = (R \downarrow \cap -(U * p)) * U$
proof (*rule antisym*)
show $(R \cap -(U * p)) * U \subseteq (R \downarrow \cap -(U * p)) * U$
by (*metis (no-types, lifting) assms cl8-var d-down-oi-up-1 equalityD2 ne-oi-complement-top-sp-test-1 ne-sp-iu-unit-up oc-top-sp-test-up-closed*)
next
have $R \downarrow \cap -(U * p) \subseteq (R \cap -(U * p)) \downarrow$
by (*metis oc-top-sp-test-up-closed down-oi-up-closed assms*)
also have $\dots \subseteq (R \cap -(U * p)) * U$
by (*simp add: down-below-sp-top*)
finally show $(R \downarrow \cap -(U * p)) * U \subseteq (R \cap -(U * p)) * U$
by (*metis assms domain-up-down-conjugate inf-commute ne-oi-complement-top-sp-test-1 ne-sp-iu-unit-up oc-top-sp-test-up-closed*)

set-eq-subset)
qed

lemma *oi-complement-top-sp-test-top-2*:

assumes *test p*

shows $(R\downarrow \cap -(U * p)) * U = ne (R\downarrow) * \wr p * U$

proof (*rule antisym*)

have $R\downarrow \cap -(U * p) * U \subseteq Dom (R\downarrow \cap -(U * p)) * U$

using *sp-top-d* **by** *blast*

also have $\dots = Dom (ne (R\downarrow) * \wr p) * U$

by (*simp add: assms d-ne-down-dp-complement-test*)

also have $\dots = Dom (ne (R\downarrow * \wr p)) * U$

by (*simp add: ne-sp-test*)

also have $\dots = ne (R\downarrow * \wr p) * U$

by (*simp add: cl8-var ne-sp-iu-unit-up*)

also have $\dots = ne (R\downarrow) * \wr p * U$

by (*simp add: ne-sp-test*)

finally show $(R\downarrow \cap -(U * p)) * U \subseteq ne (R\downarrow) * \wr p * U$

.

next

have $ne (R\downarrow) * \wr p \subseteq -(U * p)$

by (*metis assms disjoint-eq-subset-Compl double-complement inf-compl-bot-right schroeder-test sp-test test-complement-closed top-test-oi-top-complement*)

thus $ne (R\downarrow) * \wr p * U \subseteq (R\downarrow \cap -(U * p)) * U$

by (*metis (no-types) assms cl8-var d-down-oi-up-1*)

d-ne-down-dp-complement-test ne-oi-complement-top-sp-test-1 ne-sp-iu-unit-up oc-top-sp-test-up-closed sp-top-d)

qed

lemma *oi-complement-top-sp-test-top-3*:

assumes *test p*

shows $(R\downarrow \cap -(U * p)) * U = ne (R\downarrow) * -(p * U)$

by (*simp add: assms complement-test-sp-top oi-complement-top-sp-test-top-2 test-assoc1*)

lemma *split-sp-test-2*:

test p $\implies R \subseteq R * p \cup ne (R\downarrow) * (\wr p)\uparrow$

proof –

assume *test p*

hence $R \subseteq R * p \cup (ne (R\downarrow) * \wr p)\uparrow$

by (*smt (verit, best) IntE UnCI UnE split-sp-test subsetI*)

thus $R \subseteq R * p \cup ne (R\downarrow) * (\wr p)\uparrow$

by (*simp add: ne-sp-test-up*)

qed

lemma *split-sp-test-3*:

test p $\implies R \subseteq R * p \cup R\downarrow * (\wr p)\uparrow$

by (*smt IntE UnCI UnE ne-dist-down-sp ne-sp-test-up ne-test-up split-sp-test*)

subsetI)

lemma *split-sp-test-4*:

assumes *test p*

and *test q*

shows $R * (p \cup q) \subseteq R * p \cup ne (R\downarrow) * q\uparrow$

proof –

have $1: (p \cup q) * p = p$

by (*metis Un-Int-eq(1) assms sp-test-dist-oi-left test-ou-closed test-sp-commute test-sp-idempotent*)

have $(R * (p \cup q))\downarrow * \wr p \subseteq R\downarrow * q$

proof –

have $(R * (p \cup q))\downarrow * \wr p = R * (p \cup q) * (1_{\cup\cup} \cup \wr p)$

using *down-sp-sp* **by** *blast*

also have $\dots = R * ((p \cup q) * 1_{\cup\cup} \cup (p \cup q) * \wr p)$

by (*smt (verit) assms inf.orderE s-distl-test test-assoc1 test-ou-closed*)

also have $\dots \subseteq R * (1_{\cup\cup} \cup (p \cup q) * \wr p)$

by (*meson c6 s-prod-isor subset-refl sup.mono*)

also have $\dots = R\downarrow * ((p \cup q) * \wr p)$

using *down-sp-sp* **by** *blast*

also have $\dots = R\downarrow * (p * \wr p \cup q * \wr p)$

by (*metis assms ii-right-dist-ou inf-commute inf-le1 test-ou-closed test-sp-is-ii*)

also have $\dots = R\downarrow * (q * \wr p)$

by (*metis Compl-disjoint assms(1) inf-commute inf-le1 s-prod-idl sp-test-dist-oi-left subset-Un-eq test-sp-commute*)

also have $\dots \subseteq R\downarrow * q$

by (*metis inf-le1 inf-le2 s-prod-isor sp-test*)

finally show *?thesis*

qed

hence $2: ne ((R * (p \cup q))\downarrow) * \wr p \subseteq ne (R\downarrow) * q$

by (*metis assms(2) inf-le2 ne-dist-ou ne-sp-test subset-Un-eq*)

have $R * (p \cup q) \subseteq R * (p \cup q) * p \cup ne ((R * (p \cup q))\downarrow) * (\wr p)\uparrow$

by (*simp add: assms split-sp-test-2*)

also have $\dots = R * p \cup ne ((R * (p \cup q))\downarrow) * (\wr p)\uparrow$

using 1 **by** (*metis assms(1) inf.orderE test-assoc3*)

also have $\dots \subseteq R * p \cup ne (R\downarrow) * q\uparrow$

using 2 **by** (*metis (no-types, lifting) Un-Int-eq(1) assms(2) inf-le2 iu-isotone ne-sp-test ne-sp-test-up sup.mono*)

finally show *?thesis*

qed

lemma *split-sp-test-5*:

assumes *test p*

and *test q*

shows $R * (p \cup q) \subseteq R * p \cup R\downarrow * q\uparrow$

proof –

have $R * (p \cup q) \subseteq R * p \cup ne (R\downarrow) * q\uparrow$
by (*simp add: assms split-sp-test-4*)
thus *?thesis*
by (*metis (no-types, lifting) assms(2) le-inf-iff ne-dist-down-sp ne-test-up sup-neg-inf*)
qed

lemma *split-sp-test-6*:
assumes *test p*
and *test q*
shows $Dom (R * (p \cup q)) \subseteq Dom (R * p \cup ne (R\downarrow) * q)$
proof –
have $Dom (R * (p \cup q)) \subseteq Dom (R * p \cup ne (R\downarrow) * q\uparrow)$
by (*simp add: assms d-isotone split-sp-test-4*)
also have $\dots = Dom (R * p \cup (ne (R\downarrow) * q)\uparrow)$
by (*simp add: assms(2) ne-sp-test ne-sp-test-up*)
also have $\dots \subseteq Dom (R * p \cup ne (R\downarrow) * q)$
by (*metis d-up subsetI up-dist-ou up-idempotent*)
finally show *?thesis*
qed

lemma *split-sp-test-7*:
assumes *test p*
and *test q*
shows $Dom (ne (R\downarrow) * (p \cup q)) = Dom (ne (R\downarrow) * p \cup ne (R\downarrow) * q)$
apply (*rule antisym*)
apply (*metis assms ne-down-idempotent split-sp-test-6*)
by (*smt (verit, ccfv-SIG) Un-Int-eq(1) Un-subset-iff assms(2) d-isotone inf.orderE inf-le1 le-inf-iff lower-eq-down ne-dist-ou sp-oi-subdist-2 subset-Un-eq sup.mono*)

lemma *test-sp-left-dist-iu-1*:
 $test\ p \implies p * (R \cup\cup S) = p * R \cup\cup S$
by (*metis cl8-var inf.orderE p-prod-assoc s-subid-iff2*)

lemma *test-sp-left-dist-iu-2*:
 $test\ p \implies p * (R \cup\cup S) = R \cup\cup p * S$
by (*metis iu-commute test-sp-left-dist-iu-1*)

lemma *d-sp-below-iu-down*:
 $Dom\ R * S \subseteq (R \cup\cup S)\downarrow$
by (*simp add: cl8-var iu-lower-left-isotone sp-iu-unit-lower*)

lemma *d-sp-ne-down-below-ne-iu-down*:
 $Dom\ R * ne (S\downarrow) \subseteq ne ((R \cup\cup S)\downarrow)$
proof –
have $Dom\ R * S\downarrow \subseteq (R \cup\cup S)\downarrow$
by (*simp add: cl8-var iu-lower-isotone sp-iu-unit-lower*)

hence $ne (Dom R * S\downarrow) \subseteq ne ((R \cup S)\downarrow)$
by *blast*
thus *?thesis*
by (*smt d-test test-sp-ne*)
qed

lemma *top-test*:

$test p \implies U * p = \{ (a,B) . (\forall b \in B . (b, \{b\}) \in p) \}$
apply (*unfold test*)
apply (*clarsimp simp: mr-simp*)
by *fastforce*

lemma *iu-oi-complement-top-test-ou-up*:

$test p \implies (R \cup S) \cap -(U * p) \subseteq ((R \cup S) \cap -(U * p))\uparrow$
apply (*unfold top-test*)
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *d-ne-iu-down-sp-test-ou*:

assumes *test p*
shows $Dom (ne ((R \cup S)\downarrow) * p) \subseteq Dom ((ne (R\downarrow) \cup ne (S\downarrow)) * p)$
proof –
have $Dom (ne ((R \cup S)\downarrow) * p) = Dom ((R \cup S) \cap -(U * \downarrow p))$
by (*metis assms d-ne-down-dp-complement-test test-double-complement*)
also have $\dots \subseteq Dom ((R \cup S) \cap -(U * \downarrow p))$
by (*metis iu-oi-complement-top-test-ou-up d-isotone d-up semilattice-inf-class.inf-le2*)
also have $\dots = Dom (ne ((R \cup S)\downarrow) * p)$
by (*metis assms d-ne-down-dp-complement-test test-double-complement*)
finally show *?thesis*
by (*simp add: down-dist-ou ne-dist-ou*)
qed

lemma *test-sp-left-dist-iU*:

assumes *test p*
and $I \neq \{\}$
shows $p * (\bigcup X|I) = \bigcup (\lambda i . p * X i)|I$
apply (*rule antisym*)
apply (*clarsimp simp: mr-simp*)
subgoal for $a B f$ **proof** –
assume $1: (a,B) \in p$
hence $2: B = \{a\}$
by (*metis assms(1) test*)
assume $\forall b \in B . \exists g . f b = \bigcup (g \text{ ' } I) \wedge (\forall i \in I . (b,g i) \in X i)$
from this obtain g **where** $3: f a = \bigcup (g \text{ ' } I) \wedge (\forall i \in I . (a,g i) \in X i)$
using 2 **by** *auto*
have $\bigcup (f \text{ ' } B) = \bigcup (g \text{ ' } I) \wedge (\forall i \in I . \exists B . (a,B) \in p \wedge (\exists f . (\forall b \in B . (b,f b) \in X i) \wedge g i = \bigcup (f \text{ ' } B)))$
apply (*rule conjI*)

```

    using 2 3 apply blast
    apply (rule ballI)
    apply (rule exI[of - B])
    apply (rule conjI)
    using 1 apply simp
    subgoal for i
      apply (rule exI[of -  $\lambda b . g i$ ])
      using 2 3 by blast
    done
  thus ?thesis
    by auto
qed
apply (clarsimp simp: mr-simp)
subgoal for a f proof -
  assume 4:  $\forall i \in I . \exists B . (a, B) \in p \wedge (\exists g . (\forall b \in B . (b, g b) \in X i) \wedge f i =$ 
 $\bigcup (g \text{ ` } B))$ 
  have  $(a, \{a\}) \in p \wedge (\exists g . (\forall b \in \{a\} . \exists f . g b = \bigcup (f \text{ ` } I) \wedge (\forall i \in I . (b, f i) \in X$ 
 $i)) \wedge \bigcup (f \text{ ` } I) = \bigcup (g \text{ ` } \{a\}))$ 
  apply (rule conjI)
  using 4 apply (metis assms equals0I test)
  apply (rule exI[of -  $\lambda a . \bigcup (f \text{ ` } I)$ ])
  apply clarsimp
  apply (rule exI[of - f])
  using 4 assms(1) test by fastforce
  thus ?thesis
    by auto
qed
done

```

8.4 Modal operations

definition *adia* :: $(a, b) \text{ mrel} \Rightarrow (b, b) \text{ mrel} \Rightarrow (a, a) \text{ mrel} (\langle | - \rangle \rightarrow [50, 90] \ 95)$
where

$$|R\rangle p \equiv \{ (a, \{a\}) \mid a . \exists B . (a, B) \in R \wedge (\forall b \in B . (b, \{b\}) \in p) \}$$

definition *abox* :: $(a, b) \text{ mrel} \Rightarrow (b, b) \text{ mrel} \Rightarrow (a, a) \text{ mrel} (\langle | - \rangle \rightarrow [50, 90] \ 95)$
where

$$|R\rangle p \equiv \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall b \in B . (b, \{b\}) \in p) \}$$

definition *edia* :: $(a, b) \text{ mrel} \Rightarrow (b, b) \text{ mrel} \Rightarrow (a, a) \text{ mrel} (\langle | - \rangle \rightarrow [50, 90] \ 95)$
where

$$|R\rangle\rangle p \equiv \{ (a, \{a\}) \mid a . \exists B . (a, B) \in R \wedge (\exists b \in B . (b, \{b\}) \in p) \}$$

definition *ebox* :: $(a, b) \text{ mrel} \Rightarrow (b, b) \text{ mrel} \Rightarrow (a, a) \text{ mrel} (\langle | - \rangle \rightarrow [50, 90] \ 95)$
where

$$|R\rangle\rangle p \equiv \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\exists b \in B . (b, \{b\}) \in p) \}$$

declare *adia-def* [*mr-simp*] *abox-def* [*mr-simp*] *edia-def* [*mr-simp*] *ebox-def* [*mr-simp*]

```

lemma adia:
  assumes test p
  shows  $|R\rangle p = \text{Dom} (R * p)$ 
proof
  show  $|R\rangle p \subseteq \text{Dom} (R * p)$ 
  proof
    fix x
    assume  $x \in |R\rangle p$ 
    from this obtain a B where 1:  $x = (a, \{a\}) \wedge (a, B) \in R \wedge (\forall b \in B . (b, \{b\}) \in p)$ 
    by (smt adia-def surj-pair mem-Collect-eq)
    have  $(a, B) \in R * p$ 
    apply (clarsimp simp: s-prod-def)
    apply (rule exI[where ?x=B])
    apply (rule conjI)
    using 1 apply simp
    apply (rule exI[where ?x= $\lambda x . \{x\}$ ])
    using 1 by auto
    thus  $x \in \text{Dom} (R * p)$ 
    using 1 Dom-def by auto
  qed
next
  show  $\text{Dom} (R * p) \subseteq |R\rangle p$ 
  proof
    fix x
    assume  $x \in \text{Dom} (R * p)$ 
    from this obtain a A where 2:  $x = (a, \{a\}) \wedge (a, A) \in R * p$ 
    by (smt Dom-def surj-pair mem-Collect-eq)
    from this obtain B f where 3:  $(a, B) \in R \wedge (\forall b \in B . (b, f b) \in p) \wedge A = \bigcup \{f b \mid b . b \in B\}$ 
    by (simp add: mr-simp) blast
    hence  $\forall b \in B . (b, \{b\}) \in p$ 
    using assms subid-aux2 by fastforce
    thus  $x \in |R\rangle p$ 
    using 2 3 adia-def by blast
  qed
qed

```

lemma abox-1:

```

  assumes test p
  shows  $|R\rangle p = a\text{Dom} (R \cap -(U * p))$ 
proof
  show  $|R\rangle p \subseteq a\text{Dom} (R \cap -(U * p))$ 
  proof
    fix x
    assume  $x \in |R\rangle p$ 
    from this obtain a where 1:  $x = (a, \{a\}) \wedge (\forall B . (a, B) \in R \longrightarrow (\forall b \in B . (b, \{b\}) \in p))$ 

```

```

    by (smt abox-def surj-pair mem-Collect-eq)
  have  $\neg(\exists B . (a,B) \in R \cap \neg(U * p))$ 
  proof
    assume  $\exists B . (a,B) \in R \cap \neg(U * p)$ 
    from this obtain B where  $(a,B) \in R \wedge (a,B) \notin U * p$ 
    by auto
    thus False
    using 1 by (metis (no-types, lifting) assms top-sp-test)
  qed
  thus  $x \in aDom (R \cap \neg(U * p))$ 
  using 1 aDom by blast
qed
next
show  $aDom (R \cap \neg(U * p)) \subseteq |R]p$ 
proof
  fix x
  assume  $x \in aDom (R \cap \neg(U * p))$ 
  from this obtain a where  $\exists a: x = (a, \{a\}) \wedge \neg(\exists B . (a,B) \in R \cap \neg(U * p))$ 
  by (smt aDom surj-pair mem-Collect-eq)
  hence  $\forall B . (a,B) \in R \longrightarrow (\forall b \in B . (b, \{b\}) \in p)$ 
  using assms by (metis (no-types, lifting) IntI oc-top-sp-test)
  thus  $x \in |R]p$ 
  using 2 abox-def by blast
qed
qed

lemma abox:
  assumes test p
  shows  $|R]p = aDom (ne (R\downarrow) * \wr p)$ 
  by (simp add: abox-1 assms d-ne-down-dp-complement-test)

lemma edia-1:
  assumes test p
  shows  $|R\rangle p = Dom (R \cap \neg(U * \wr p))$ 
  proof
    show  $|R\rangle p \subseteq Dom (R \cap \neg(U * \wr p))$ 
    proof
      fix x
      assume  $x \in |R\rangle p$ 
      from this obtain a b B where  $1: x = (a, \{a\}) \wedge (a,B) \in R \wedge b \in B \wedge (b, \{b\}) \in p$ 
      by (smt edia-def surj-pair mem-Collect-eq)
      hence  $(a,B) \in \neg(U * \wr p)$ 
      by (metis (no-types, lifting) lattice-class.inf-sup-ord(2) oc-top-sp-test test-complement)
      thus  $x \in Dom (R \cap \neg(U * \wr p))$ 
      using 1 Dom-def by auto
    qed
  next

```

show $Dom (R \cap -(U * \wr p)) \subseteq |R\rangle p$
proof
fix x
assume $x \in Dom (R \cap -(U * \wr p))$
from this obtain $a B$ **where** $2: x = (a, \{a\}) \wedge (a, B) \in R \wedge (a, B) \in -(U * \wr p)$
by (*smt Dom-def surj-pair mem-Collect-eq IntE*)
hence $\exists b \in B . (b, \{b\}) \in p$
by (*meson oc-top-sp-test test-complement test-complement-closed*)
thus $x \in |R\rangle p$
using 2 *edia-def* **by** *blast*
qed
qed

lemma *edia*:
assumes *test p*
shows $|R\rangle p = Dom (ne (R \downarrow) * p)$
by (*metis assms d-ne-down-dp-complement-test edia-1 test-double-complement*)

lemma *ebox*:
assumes *test p*
shows $|R]]p = aDom (R * \wr p)$
proof
show $|R]]p \subseteq aDom (R * \wr p)$
proof
fix x
assume $x \in |R]]p$
from this obtain a **where** $1: x = (a, \{a\}) \wedge (\forall B . (a, B) \in R \longrightarrow (\exists b \in B . (b, \{b\}) \in p))$
by (*smt ebox-def surj-pair mem-Collect-eq*)
hence $\neg(\exists B . (a, B) \in R * \wr p)$
by (*metis (no-types, lifting) s-prod-test test-complement*)
thus $x \in aDom (R * \wr p)$
using 1 *aDom* **by** *blast*
qed

next
show $aDom (R * \wr p) \subseteq |R]]p$
proof
fix x
assume $x \in aDom (R * \wr p)$
from this obtain a **where** $2: x = (a, \{a\}) \wedge \neg(\exists B . (a, B) \in R * \wr p)$
by (*smt aDom surj-pair mem-Collect-eq*)
have $\forall B . (a, B) \in R \longrightarrow (\exists b \in B . (b, \{b\}) \in p)$
proof (*rule allI, rule impI*)
fix B
assume $(a, B) \in R$
hence $(a, B) \notin U * \wr p$
using 2 **by** (*metis Int-iff Int-lower2 sp-test*)
thus $\exists b \in B . (b, \{b\}) \in p$

by (*meson test-complement test-complement-closed top-sp-test*)
qed
thus $x \in |R|]p$
using 2 *ebox-def* **by** *blast*
qed
qed

lemma *abox-2*:
assumes *test p*
shows $|R]p = -((R \cap -(U * p)) * U) \cap 1$
by (*simp add: abox-1 assms ne-a-expl ne-oi-complement-top-sp-test-1*)

lemma *abox-3*:
assumes *test p*
shows $|R]p = -(ne (R\downarrow) * \wr p * U) \cap 1$
by (*simp add: abox assms ne-a-expl ne-sp-test*)

lemma *abox-4*:
assumes *test p*
shows $|R]p = ((U * p) \circlearrowleft R) \cap 1$
by (*simp add: abox-1 assms top-test-sp-lres-total-expl-3*)

lemma *abox-ebox*:
assumes *test p*
shows $|R]p = |ne (R\downarrow)]]p$
by (*simp add: abox assms ebox*)

lemma *abox-edia*:
assumes *test p*
shows $|R]p = \wr |R\rangle(\wr p)$
by (*simp add: abox assms edia*)

lemma *abox-adia*:
assumes *test p*
shows $|R]p = \wr |ne (R\downarrow)\rangle(\wr p)$
by (*simp add: abox adia assms*)

lemma *edia-adia*:
assumes *test p*
shows $|R\rangle]p = |ne (R\downarrow)\rangle]p$
by (*simp add: adia assms edia*)

lemma *edia-abox*:
assumes *test p*
shows $|R\rangle]p = \wr |R](\wr p)$
by (*metis abox-1 assms d-complement-ad edia-1 semilattice-inf-class.inf.cobounded2*)

lemma *edia-ebox*:

assumes *test p*
shows $|R\rangle p = \iota |ne (R\downarrow)\rangle(\iota p)$
by (*simp add: abox assms ebox edia-abox*)

lemma *abox-ne-down:*
assumes *test p*
shows $|R]p = |ne (R\downarrow)]p$
by (*simp add: abox assms ne-down-idempotent*)

lemma *edia-ne-down:*
assumes *test p*
shows $|R\rangle p = |ne (R\downarrow)\rangle p$
by (*simp add: assms edia ne-down-idempotent*)

lemma *adia-up:*
assumes *test p*
shows $|R\rangle p = |R\uparrow\rangle p$
proof –
have $|R\uparrow\rangle p = Dom (R\uparrow \cap U * p)$
by (*metis adia assms iu-assoc iu-unit-up up-dist-iu-oi*)
also have $\dots = Dom (R \cap U * p)$
by (*metis assms d-def-expl domain-up-down-conjugate sp-test-dist-oi-right top-sp-test-down-closed*)
also have $\dots = |R\rangle p$
by (*metis adia assms inf.absorb-iff2 inf-commute top-down top-lower-greatest*)
finally show *?thesis*
by *simp*
qed

lemma *ebox-up:*
assumes *test p*
shows $|R]]p = |R\uparrow]]p$
by (*metis Int-commute adia adia-up assms ebox semilattice-inf-class.inf-le1*)

lemma *adia-ebox:*
assumes *test p*
shows $|R]p = \iota |R]](\iota p)$
by (*metis (no-types, lifting) adia assms d-complement-ad ebox test-double-complement*)

lemma *ebox-adia:*
assumes *test p*
shows $|R]]p = \iota |R](\iota p)$
by (*simp add: adia assms ebox*)

lemma *abox-down:*
assumes *test p*
shows $|R]p = |R\downarrow]p$
by (*simp add: abox assms*)

lemma *edia-down*:

assumes *test p*
shows $|R\rangle\rangle p = |R\downarrow\rangle\rangle p$
by (*simp add: assms edia*)

lemma *fusion-oi-complement-top-test-up*:

test p \implies $\text{fus } R \cap -(U * p) \subseteq (R \cap -(U * p))\uparrow$
apply (*unfold top-test*)
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *adia-left-isotone*:

test p $\implies R \subseteq S \implies |R\rangle p \subseteq |S\rangle p$
by (*metis adia d-isotone inf.absorb-iff1 sp-test-dist-oi*)

lemma *adia-right-isotone*:

test p \implies *test q* $\implies p \subseteq q \implies |R\rangle p \subseteq |R\rangle q$
by (*metis (no-types, opaque-lifting) adia d-isotone inf.orderE inf-commute inf-le1 sp-test test-assoc3 test-s-prod-is-meet*)

lemma *abox-left-antitone*:

test p $\implies R \subseteq S \implies |S\rangle p \subseteq |R\rangle p$
apply (*clarsimp simp: mr-simp*) **by** *force*

lemma *abox-right-isotone*:

test p \implies *test q* $\implies p \subseteq q \implies |R\rangle p \subseteq |R\rangle q$
by (*smt (verit, ccfv-threshold) IntE abox-def inf.orderE mem-Collect-eq subsetI*)

lemma *edia-left-isotone*:

test p $\implies R \subseteq S \implies |R\rangle\rangle p \subseteq |S\rangle\rangle p$
by (*metis Int-mono adia-left-isotone down-isotone edia-adia order-refl*)

lemma *edia-right-isotone*:

test p \implies *test q* $\implies p \subseteq q \implies |R\rangle\rangle p \subseteq |R\rangle\rangle q$
by (*simp add: adia-right-isotone edia-adia*)

lemma *ebox-left-antitone*:

test p $\implies R \subseteq S \implies |S\rangle\rangle p \subseteq |R\rangle\rangle p$
by (*metis (no-types, lifting) adia-ebox adia-left-isotone ebox-adia test-complement-antitone test-double-complement*)

lemma *ebox-right-isotone*:

test p \implies *test q* $\implies p \subseteq q \implies |R\rangle\rangle p \subseteq |R\rangle\rangle q$
by (*smt (verit, ccfv-SIG) adia-ebox adia-right-isotone ebox inf-le2 test-complement-antitone test-double-complement*)

lemma *edia-fusion*:

assumes *test p*

shows $|R\rangle\rangle p = |fus R\rangle\rangle p$
proof
have $|fus R\rangle\rangle p = Dom (fus R \cap -(U * \wr p))$
using *assms edia-1 by blast*
also have $\dots \subseteq Dom (R \cap -(U * \wr p))$
by (*metis fusion-oi-complement-top-test-up d-isotone d-up semilattice-inf-class.inf-le2*)
also have $\dots = |R\rangle\rangle p$
using *assms edia-1 by blast*
finally show $|fus R\rangle\rangle p \subseteq |R\rangle\rangle p$
next
have $|R\rangle\rangle p \subseteq |(fus R)\downarrow\rangle\rangle p$
by (*simp add: assms edia-left-isotone fusion-lower-increasing*)
thus $|R\rangle\rangle p \subseteq |fus R\rangle\rangle p$
using *assms edia-down by blast*
qed

lemma *abox-fusion*:
assumes *test p*
shows $|R]p = |fus R]p$
by (*metis Int-lower2 abox-edia assms edia-fusion*)

lemma *abox-fission*:
assumes *test p*
shows $|R]p = |fis R]p$
by (*metis assms abox-fusion fusion-fission*)

lemma *edia-fission*:
assumes *test p*
shows $|R\rangle\rangle p = |fis R\rangle\rangle p$
by (*metis assms edia-fusion fusion-fission*)

lemma *fission-below*:
 $fis R \subseteq S \iff (\forall a b B . (a,B) \in R \wedge b \in B \longrightarrow (a,\{b\}) \in S)$
apply *standard*
apply (*simp add: basic-trans-rules(31) fission-set*)
apply (*clarsimp simp: mr-simp*)
by *blast*

lemma *below-fission-up*:
 $S \subseteq (fis R)\uparrow \iff (\forall a B . (a,B) \in S \longrightarrow (\exists C . (a,C) \in R \wedge C \cap B \neq \{\}))$

proof
assume $S \subseteq (fis R)\uparrow$
thus $\forall a B . (a,B) \in S \longrightarrow (\exists C . (a,C) \in R \wedge C \cap B \neq \{\})$
apply (*clarsimp simp: mr-simp*)
by *fastforce*

next
assume $1: \forall a B . (a,B) \in S \longrightarrow (\exists C . (a,C) \in R \wedge C \cap B \neq \{\})$

show $S \subseteq (fis\ R)\uparrow$
proof
 fix x
 assume $x \in S$
 from this obtain $a\ B$ **where** $2: x = (a, B) \wedge (a, B) \in S$
 by (*metis surj-pair*)
 hence $\exists C . (a, C) \in R \wedge C \cap B \neq \{\}$
 using 1 by simp
 from this obtain $C\ b$ **where** $3: (a, C) \in R \wedge b \in C \wedge b \in B$
 by auto
 hence $(a, \{b\}) \in fis\ R$
 using fission-set by blast
 thus $x \in (fis\ R)\uparrow$
 using 2 3 U-par-st by fastforce
qed
qed

lemma *ebox-below-abox*:
 assumes *test p*
 and $fis\ R \subseteq S$
 shows $|S|]p \subseteq |R]p$
 by (*metis abox-ebox abox-fission assms ebox-left-antitone fission-down-ne-fixpoint*)

lemma *abox-below-ebox*:
 assumes *test p*
 and $S \subseteq (fis\ R)\uparrow$
 shows $|R]p \subseteq |S|]p$
 by (*metis abox-ebox abox-fission assms ebox-left-antitone ebox-up fission-down-ne-fixpoint*)

lemma *abox-eq-ebox*:
 assumes *test p*
 and $fis\ R \subseteq S$
 and $S \subseteq (fis\ R)\uparrow$
 shows $|R]p = |S|]p$
 by (*simp add: abox-below-ebox assms ebox-below-abox subset-antisym*)

lemma *abox-eq-ebox-sufficient*:
 $S = fis\ R \vee S = ne\ (R\downarrow) \vee S = (ne\ (R\downarrow))\uparrow \longrightarrow fis\ R \subseteq S \wedge S \subseteq (fis\ R)\uparrow$
 apply (*unfold imp-disjL*)
 apply (*intro conjI*)
 apply (*simp add: convex-reflexive*)
 apply (*simp add: fission-inner-deterministic fission-up-ne-down-up oi-subset-upper-right-antitone same-fusion-fission-lower*)
 by (*metis convex-reflexive fission-up-ne-down-up order-refl*)

lemma *ebox-fission-abox*:
 test p $\implies |R]p = |fis\ R|]p$

by (*metis abox abox-fission ebox fission-down-ne-fixpoint*)

lemma *ebox-down-ne-up-abox*:

test $p \implies |R]p = |(ne (R\downarrow))\uparrow]p$
using *abox-ebox ebox-up* by *blast*

lemma *same-fusion*:

assumes $fis\ R \sqsubseteq\downarrow S$
and $S \sqsubseteq\downarrow fus\ R$
shows $fis\ R = fis\ S$
by (*metis assms fission-down fission-fusion fission-fusion-galois subset-antisym*)

lemma *same-abox*:

assumes $fis\ R \sqsubseteq\downarrow S$
and $S \sqsubseteq\downarrow fus\ R$
and *test* p
shows $|R]p = |S]p$
by (*metis assms ebox-fission-abox same-fusion*)

lemma *abox-ebox-inner-deterministic*:

assumes *test* p
and *inner-deterministic* R
shows $|R]p = |R]]p$
apply (*rule abox-eq-ebox*)
apply (*simp add: assms(1)*)
using *assms(2) fission-inner-deterministic-fixpoint* apply *blast*
by (*metis assms(2) convex-reflexive fission-inner-deterministic-fixpoint*)

lemma *adia-edia-inner-deterministic*:

assumes *test* p
and *inner-deterministic* R
shows $|R\rangle p = |R\rangle\rangle p$
by (*metis assms edia-adia fission-down-ne-fixpoint fission-inner-deterministic-fixpoint*)

lemma *abox-adia-deterministic*:

assumes *test* p
and *deterministic* R
shows $|R]p = |R\rangle p$
proof
show $|R]p \subseteq |R\rangle p$
proof
fix x
assume $x \in |R]p$
from *this* obtain a where $1: x = (a, \{a\}) \wedge (\forall B. (a, B) \in R \longrightarrow (\forall b \in B. (b, \{b\}) \in p))$
using *abox-def* by *force*
from *assms(2)* obtain B where $(a, B) \in R$
by (*meson deterministic-set*)

```

    thus  $x \in |R\rangle p$ 
      using 1 adia-def by fastforce
  qed
next
  show  $|R\rangle p \subseteq |R]p$ 
  proof
    fix  $x$ 
    assume  $x \in |R\rangle p$ 
    from this obtain  $a B$  where 2:  $x = (a, \{a\}) \wedge (a, B) \in R \wedge (\forall b \in B . (b, \{b\}) \in p)$ 
    by (smt adia-def mem-Collect-eq)
    have  $\forall C . (a, C) \in R \longrightarrow (\forall b \in C . (b, \{b\}) \in p)$ 
    proof (rule allI, rule impI)
      fix  $C$ 
      assume  $(a, C) \in R$ 
      hence  $B = C$ 
      using 2 by (metis assms(2) deterministic-set)
      thus  $\forall b \in C . (b, \{b\}) \in p$ 
      using 2 by simp
    qed
    thus  $x \in |R]p$ 
    using 2 abox-def by blast
  qed
qed

```

lemma *ebox-edia-deterministic*:
 assumes *test p*
 and *deterministic R*
 shows $|R]]p = |R\rangle p$
 by (*simp add: assms abox-adia-deterministic ebox-edia edia-abox*)

lemma *abox-ebox-fusion*:
 assumes *test p*
 shows $|fis R]p = |fis R]]p$
 by (*metis abox-fission assms ebox-fission-abox*)

lemma *abox-fission-edia-fusion*:
 assumes *test p*
 shows $|fis R]p = |fus R\rangle p$
 by (*simp add: abox-adia-deterministic abox-fusion assms fusion-deterministic fusion-fission*)

lemma *abox-adia-fusion*:
 assumes *test p*
 shows $|fus R]p = |fus R\rangle p$
 by (*simp add: abox-adia-deterministic assms fusion-deterministic*)

8.5 Goldblatt's axioms without star

lemma *abox-sp-unit*:

$|R]1 = 1$

apply (*clarsimp simp: mr-simp*) **by** *force*

lemma *ou-unit-abox*:

$test\ p \implies |\{\}\}p = 1$

by (*metis abox-1 abox-sp-unit disjoint-eq-subset-Compl empty-subsetI inf.absorb-iff2 test-complement-closed*)

lemma *ou-unit-test-implication*:

$test\ p \implies \{\} \rightarrow p = 1$

by *blast*

lemma *sp-unit-abox*:

$test\ p \implies |1]p = p$

by (*smt (verit) Int-left-commute abox-1 c1 cl8-var convex-reflexive d-ne-down-dp-complement-test fission-down-ne-fixpoint fission-inner-deterministic-fixpoint inf.absorb-iff2 inf-commute inner-deterministic-sp-unit s-subid-iff2 test-double-complement test-sp*)

lemma *sp-unit-test-implication*:

$test\ p \implies 1 \rightarrow p = p$

by *simp*

lemma *test-abox-ebox*:

$test\ p \implies test\ q \implies |q]p = |q]]p$

apply (*rule antisym*)

apply (*metis abox-ebox-inner-deterministic dual-order.trans inner-deterministic-sp-unit subset-refl*)

by (*metis abox-ebox-inner-deterministic dual-order.eq-iff inner-deterministic-sp-unit inner-univalent-down-closed ne-equality test-ne*)

lemma *test-abox*:

$test\ p \implies test\ q \implies |q]p = q \rightarrow p$

by (*smt Int-commute Int-lower2 abox cl9-var compl-sup d-complement-ad d-ne-down-dp-complement-test lattice-class.inf-sup-aci(2) sp-unit-abox test-ou-closed*)

lemma *abox-ou-adia-sp-unit*:

assumes *test p*

shows $|R]p \cup |R]1 = 1$

apply (*rule antisym*)

apply (*simp add: assms abox adia-ebox*)

by (*clarsimp simp: mr-simp*)

lemma *d-test-sp*:

$test\ p \implies Dom\ (p * R) = p * Dom\ R$

by (*simp add: c4 d-def-expl test-sp-left-dist-iu-1*)

lemma *ad-test-sp*:

test p \implies $aDom (p * R) = \lambda p \cup aDom R$

by (*metis (no-types, opaque-lifting) Int-commute boolean-algebra.conj-disj-distrib boolean-algebra.de-Morgan-conj d-s-id-inter d-test-sp s-subid-iff2 test-fix*)

lemma *adia-test-sp*:

test p \implies *test q* \implies $|p * R\rangle q = p * |R\rangle q$

by (*metis (no-types, lifting) adia d-test-sp test-assoc3 test-double-complement*)

lemma *ebox-test-sp*:

test p \implies *test q* \implies $|p * R]\!q = \lambda p \cup |R]\!q$

by (*simp add: ad-test-sp ebox test-assoc3*)

lemma *abox-test-sp*:

assumes *test p*

and *test q*

shows $|p * R]\!q = \lambda p \cup |R]\!q$

proof –

have $|p * R]\!q = aDom ((p * R) \cap -(U * q))$

by (*simp add: abox-1 assms(2)*)

also have $\dots = aDom (p * (R \cap -(U * q)))$

by (*metis Int-assoc assms(1) test-sp*)

also have $\dots = \lambda p \cup |R]\!q$

by (*simp add: abox-1 ad-test-sp assms*)

finally show *?thesis*

qed

lemma *abox-test-sp-2*:

test p \implies *test q* \implies $p \cup |R]\!q = |\lambda p * R]\!q$

by (*simp add: abox-test-sp test-double-complement*)

lemma *abox-test-sp-3*:

test p \implies *test q* \implies $p \rightarrow |R]\!q = |p * R]\!q$

by (*simp add: abox-test-sp*)

lemma *fission-sp-dist*:

$fis (R * S) = fis (R * Dom S) * fis S$

proof –

have $S = Dom S * (S \cup aDom S * 1_{\cup\cup})$

by (*auto simp: mr-simp*)

hence $fis (R * S) = fis (R * Dom S * (S \cup aDom S * 1_{\cup\cup}))$

by (*metis d-s-id-ax sp-test-sp-oi-right test-sp*)

also have $\dots = fis (R * Dom S) * fis (S \cup aDom S * 1_{\cup\cup})$

apply (*rule fission-sp-total-dist*)

by (*smt (verit) total-dom Compl-disjoint ad-sp-bot ad-test-sp c6 compl-inf-bot d-complement-ad d-dist-ou inf-le2 iu-unit-down subset-Un-eq sup-ge2 sup-inf-absorb total-lower*)

also have ... = $\text{fis } (R * \text{Dom } S) * (\text{fis } S \cup \text{fis } (a\text{Dom } S * 1_{\cup\cup}))$
by (*simp add: fission-dist-ou*)
also have ... = $\text{fis } (R * \text{Dom } S) * \text{fis } S$
by (*simp add: fission-sp-iu-unit*)
finally show ?thesis

•
qed

lemma *abox-test*:
 $\text{test } p \implies \text{test } (|R]p)$
by (*simp add: abox*)

lemma *adia-test*:
 $\text{test } p \implies \text{test } (|R\rangle p)$
by (*simp add: adia d-test*)

lemma *ebox-test*:
 $\text{test } p \implies \text{test } (|R]]p)$
by (*simp add: ebox*)

lemma *edia-test*:
 $\text{test } p \implies \text{test } (|R\rangle\rangle p)$
by (*simp add: edia d-test*)

lemma *abox-sp*:
assumes *test p*
and *test q*
shows $|R](p * q) = |R]p * |R]q$

proof –

have $|R](p * q) = a\text{Dom } (ne (R\downarrow) * (\imath p \cup \imath q))$
by (*metis (no-types, lifting) abox-1 ad-test-sp assms cl9-var d-ne-down-dp-complement-test sp-test test-double-complement test-oi-closed*)
also have ... = $a\text{Dom } (ne (R\downarrow) * \imath p) * a\text{Dom } (ne (R\downarrow) * \imath q)$
by (*smt ad-test-sp cl9-var d-complement-ad d-dist-ou d-test-sp semilattice-inf-class.inf-le2 split-sp-test-7*)
also have ... = $|R]p * |R]q$
by (*simp add: abox assms*)
finally show ?thesis

•
qed

lemma *adia-ou-below-ne-down*:
assumes *test p*
shows $|R\rangle(p \cup \imath q) \subseteq |R]p \cup |ne (R\downarrow)\rangle(\imath q)$
by (*metis adia assms d-dist-ou split-sp-test-6 test-complement-closed test-ou-closed*)

lemma *abox-adia-mp*:
assumes *test p*

and *test q*
shows $|R\rangle(p \rightarrow q) * |R]p \subseteq |R\rangle q$
by (*smt adia-ou-below-ne-down test-shunting abox adia assms d-complement-ad sup-commute test-complement-closed test-implication-closed*)

lemma *adia-abox-mp:*

assumes *test p*
and *test q*
shows $|R\rangle p * |R](p \rightarrow q) \subseteq |R\rangle q$
proof –
have $p \subseteq p \rightarrow q \rightarrow q$
using *assms(1)* **by** *blast*
hence $|R\rangle p \subseteq |R](p \rightarrow q) \rightarrow q$
by (*simp add: adia-right-isotone assms*)
thus *?thesis*
by (*smt abox-adia-mp abox-test adia-test assms(2) semilattice-inf-class.inf.orderE semilattice-inf-class.le-infI2 test-implication-closed test-shunting*)
qed

lemma *abox-implication-adia:*

assumes *test p*
and *test q*
shows $|R](p \rightarrow q) \subseteq |R]p \rightarrow |R\rangle q$
by (*metis adia-abox-mp test-shunting test-sp-commute Int-lower2 Un-commute abox-test adia-test assms test-ou-closed*)

lemma *abox-adia-implication:*

assumes *test p*
and *test q*
shows $|R]p \subseteq |R]q \rightarrow |R\rangle(p * q)$
proof –
have $p \subseteq q \rightarrow p * q$
by (*metis assms subset-refl test-galois-1 test-sp-commute*)
hence $|R]p \subseteq |R](q \rightarrow p * q)$
by (*simp add: abox-right-isotone assms test-galois-1*)
thus *?thesis*
by (*metis (no-types, lifting) Int-Un-eq(2) abox-implication-adia assms le-sup-iff subset-Un-eq test-galois-1*)
qed

lemma *abox-mp:*

assumes *test p*
and *test q*
shows $|R]p * |R](p \rightarrow q) \subseteq |R]q$
by (*metis (no-types, lifting) abox-right-isotone abox-sp assms semilattice-inf-class.inf.absorb-iff1 sp-test-dist-oi-left subset-refl sup-commute test-implication-closed test-shunting test-sp-commute*)

lemma *abox-implication*:

assumes *test p*

and *test q*

shows $|R|(p \rightarrow q) \subseteq |R]p \rightarrow |R]q$

by (*metis abox-mp test-shunting test-sp-commute abox-test assms sup-commute test-implication-closed*)

lemma *ebox-left-dist-ou*:

assumes *test p*

shows $|R \cup S]]p = |R]]p * |S]]p$

by (*auto simp: mr-simp*)

lemma *abox-left-dist-ou*:

assumes *test p*

shows $|R \cup S]p = |R]p * |S]p$

by (*simp add: abox-ebox assms ebox-left-dist-ou ii-right-dist-ou ne-dist-ou*)

lemma *adia-left-dist-ou*:

assumes *test p*

shows $|R \cup S\rangle p = |R\rangle p \cup |S\rangle p$

by (*auto simp: mr-simp*)

lemma *edia-left-dist-ou*:

assumes *test p*

shows $|R \cup S\rangle\rangle p = |R\rangle\rangle p \cup |S\rangle\rangle p$

by (*simp add: assms boolean-algebra.conj-disj-distrib2 d-dist-ou edia-1*)

lemma *abox-dist-iu-1*:

assumes *test p*

shows $|R \cup\cup S]p = |Dom R * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p$

proof

have *1*: $|R \cup\cup S]p \subseteq |Dom R * ne (S\downarrow)]p$

by (*metis abox-ebox assms d-sp-ne-down-below-ne-iu-down ebox-left-antitone*)

have $|R \cup\cup S]p \subseteq |Dom S * ne (R\downarrow)]p$

by (*metis abox-ebox assms d-sp-ne-down-below-ne-iu-down ebox-left-antitone iu-commute*)

thus $|R \cup\cup S]p \subseteq |Dom R * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p$

using *1* **by** (*simp add: assms ebox*)

next

have $|Dom R * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p \subseteq |Dom R * Dom S * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p$

apply (*clarsimp simp: mr-simp*)

by (*metis UN-I singletonI*)

also have $\dots \subseteq |Dom R * Dom S * ne (S\downarrow)]p * |Dom R * Dom S * ne (R\downarrow)]p$

by (*simp add: assms d-lb2 ebox-left-antitone s-prod-isol s-prod-isol*)

also have $\dots = |Dom R * Dom S * ne (S\downarrow) \cup Dom R * Dom S * ne (R\downarrow)]p$

using *assms ebox-left-dist-ou* **by** *blast*

also have $\dots = |ne (Dom R * Dom S * S\downarrow) \cup ne (Dom R * Dom S * R\downarrow)]p$

by (*metis d-dist-ii d-test test-sp-ne*)

also have ... = $|ne ((Dom R * Dom S * S)\downarrow) \cup ne ((Dom R * Dom S * R)\downarrow)]p$
by (*simp add: down-dist-sp*)
also have ... = $aDom ((ne ((Dom R * Dom S * S)\downarrow) \cup ne ((Dom R * Dom S * R)\downarrow)) * \wr p)$
using *assms ebox by blast*
also have ... $\subseteq aDom ((ne ((Dom R * Dom S * S \cup Dom R * Dom S * R)\downarrow)) * \wr p)$
using *d-ne-iu-down-sp-test-ou by blast*
also have ... = $|Dom R * Dom S * S \cup Dom R * Dom S * R]p$
using *abox assms by blast*
also have ... = $|Dom R * Dom S * (R \cup S)]p$
by (*metis d-assoc1 d-inter-r p-prod-comm*)
also have ... = $|R \cup S]p$
by (*metis c1 cl8-var d-dist-iu*)
finally show $|Dom R * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p \subseteq |R \cup S]p$
qed

lemma *abox-dist-iu-2:*

assumes *test p*
shows $|R \cup S]p = |Dom R * S]p * |Dom S * R]p$
proof –
have $|Dom R * ne (S\downarrow)]p * |Dom S * ne (R\downarrow)]p = |ne ((Dom R * S)\downarrow)]p * |ne ((Dom S * R)\downarrow)]p$
by (*simp add: d-test down-dist-sp test-sp-ne*)
also have ... = $|Dom R * S]p * |Dom S * R]p$
by (*simp add: abox-ebox assms*)
finally show *?thesis*
using *assms abox-dist-iu-1 by blast*
qed

lemma *abox-dist-iu-3:*

assumes *test p*
shows $|R \cup S]p = (|R]1 \rightarrow |S]p) * (|S]1 \rightarrow |R]p)$
by (*metis abox-dist-iu-2 adia assms abox-test-sp d-test s-prod-idr subset-refl*)

lemma *abox-adia-sp-one-set:*

$|R]|S]1 = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall b \in B . \exists D . (b, D) \in S) \}$
by (*auto simp: abox-def Dom-def adia*)

lemma *abox-abox-set:*

$|R]|S]p = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall C . (\exists b \in B . (b, C) \in S) \longrightarrow (\forall c \in C . (c, \{c\}) \in p)) \}$
by (*auto simp: abox-def*)

lemma *sp-abox-set:*

$|R * S]p = \{ (a, \{a\}) \mid a . \forall B . (a, B) \in R \longrightarrow (\forall C . (\exists f . (\forall b \in B . (b, f b) \in S) \wedge C = \bigcup \{ f b \mid b . b \in B \}) \longrightarrow (\forall c \in C . (c, \{c\}) \in p)) \}$
apply (*unfold abox-def s-prod-def*)

by *blast*

lemma *abox-sp-1*:

assumes *test p*

shows $|R||S\rangle 1 * |R * S\rangle p \subseteq |R||S\rangle p$

proof –

have $|R||S\rangle 1 * |R * S\rangle p = |R||S\rangle 1 \cap |R * S\rangle p$

by (*smt (verit, ccfv-SIG) abox-test adia-test assms convex-increasing inf.orderE inf-assoc sp-unit-convex test-s-prod-is-meet*)

also have $\dots \subseteq |R||S\rangle p$

proof

fix *x*

assume $x \in |R||S\rangle 1 \cap |R * S\rangle p$

from this obtain a where $1: x = (a, \{a\}) \wedge x \in |R||S\rangle 1 \wedge x \in |R * S\rangle p$

by (*metis Int-iff abox-test adia-test order-refl subid-aux2 subsetD surj-pair*)

hence $2: \forall B. (a, B) \in R \longrightarrow (\forall b \in B. \exists D. (b, D) \in S)$

by (*smt abox-adia-sp-one-set mem-Collect-eq old.prod.inject*)

have $3: \forall B. (a, B) \in R \longrightarrow (\forall C. (\exists f. (\forall b \in B. (b, f b) \in S) \wedge C = \bigcup \{ f b \mid b. b \in B \}) \longrightarrow (\forall c \in C. (c, \{c\}) \in p))$

using 1 **by** (*smt sp-abox-set mem-Collect-eq old.prod.inject*)

have $\forall B. (a, B) \in R \longrightarrow (\forall C. (\exists b \in B. (b, C) \in S) \longrightarrow (\forall c \in C. (c, \{c\}) \in p))$

proof (*rule allI, rule impI*)

fix *B*

assume $4: (a, B) \in R$

hence $\exists DD. \forall b \in B. (b, DD b) \in S$

using 2 **by** (*auto intro: bchoice*)

from this obtain DD where $5: \forall b \in B. (b, DD b) \in S$

by *auto*

show $\forall C. (\exists b \in B. (b, C) \in S) \longrightarrow (\forall c \in C. (c, \{c\}) \in p)$

proof (*rule allI, rule impI*)

fix *C*

assume $\exists b \in B. (b, C) \in S$

from this obtain b where $6: b \in B \wedge (b, C) \in S$

by *auto*

let $?f = \lambda x. \text{if } x = b \text{ then } C \text{ else } DD x$

let $?C = C \cup \bigcup \{ ?f x \mid x. x \in B \wedge x \neq b \}$

have $\exists f. (\forall b \in B. (b, f b) \in S) \wedge ?C = \bigcup \{ f b \mid b. b \in B \}$

apply (*rule exI[where ?x=?f]*)

using $5\ 6$ **by** *auto*

hence $\forall c \in ?C. (c, \{c\}) \in p$

using $3\ 4$ **by** *auto*

thus $\forall c \in C. (c, \{c\}) \in p$

by *blast*

qed

qed

thus $x \in |R||S\rangle p$

using 1 *abox-abox-set* **by** *blast*

qed

finally show *?thesis*

qed

lemma *abox-sp-2*:

assumes *test p*

shows $|R||S]p = |R\downarrow * S]p$

proof –

have $|R||S]p = aDom (ne (R\downarrow) * Dom (ne (S\downarrow) * \wr p))$

by (*metis abox abox-test assms d-complement-ad*)

also have $\dots = aDom (ne (R\downarrow) * ne (S\downarrow) * \wr p)$

by (*simp add: test-assoc3*)

also have $\dots = aDom (ne ((R\downarrow * S)\downarrow) * \wr p)$

by (*simp add: down-dist-sp ne-dist-down-sp*)

also have $\dots = |R\downarrow * S]p$

by (*simp add: abox assms*)

finally show *?thesis*

qed

lemma *abox-sp-3*:

assumes *test p*

shows $|R||S]p \subseteq |R * S]p$

by (*clarsimp simp: mr-simp*) auto

lemma *abox-sp-4*:

assumes *test p*

shows $|R * S]p \subseteq |R||S\rangle 1 \rightarrow |R||S]p$

proof –

have $|R||S\rangle 1 * |R * S]p \subseteq |R||S]p$

by (*auto simp: assms abox-sp-1*)

hence $|R||S\rangle 1 \cap |R * S]p \subseteq |R||S]p$

by (*smt (verit) abox-test adia-test assms convex-increasing inf.orderE sp-unit-convex test-oi-closed test-s-prod-is-meet*)

thus *?thesis*

using *abox-test assms* by *blast*

qed

lemma *abox-sp-5*:

assumes *test p*

shows $|R||S\rangle 1 * |R * S]p = |R||S\rangle 1 * |R||S]p$

proof (*rule antisym*)

have $|R * S]p \subseteq |R||S\rangle 1 \rightarrow |R||S]p$

by (*simp add: abox-sp-4 assms*)

hence $|R||S\rangle 1 \cap |R * S]p \subseteq |R||S]p$

by *blast*

hence $|R||S\rangle 1 \cap |R * S]p \subseteq |R||S\rangle 1 \cap |R||S]p$

by *blast*

thus $|R||S\rangle 1 * |R * S]p \subseteq |R||S\rangle 1 * |R||S]p$

by (*smt (verit, del-insts) abox-test adia-test assms convex-increasing inf.orderE sp-unit-convex test-oi-closed test-s-prod-is-meet*)
show $|R||S\rangle 1 * |R||S\rangle p \subseteq |R||S\rangle 1 * |R * S\rangle p$
by (*simp add: abox-sp-3 assms s-prod-isor*)
qed

lemma *abox-sp-6*:
assumes *test p*
shows $|R||S\rangle 1 \rightarrow |R * S\rangle p = |R||S\rangle 1 \rightarrow |R||S\rangle p$
by (*smt Int-commute abox-sp-3 abox-sp-4 assms inf-sup-distrib2 lattice-class.inf-sup-absorb semilattice-inf-class.inf.absorb-iff2 sup-commute*)

lemma *abox-sp-7*:
assumes *test p*
and *total S*
shows $|R * S\rangle p = |R||S\rangle p$
by (*metis (no-types, lifting) abox-ebox abox-sp-2 assms down-dist-sp total-down-dist-sp*)

lemma *adia-sp-associative*:
assumes *test p*
shows $|Q * (R * S)\rangle p = |(Q * R) * S\rangle p$
proof –
have $|Q * (R * S)\rangle p = |Q\rangle (|R\rangle (|S\rangle p))$
by (*metis (no-types, lifting) adia adia-test assms d-loc-ax inf.orderE test-assoc3*)
also have $\dots = |(Q * R) * S\rangle p$
by (*smt (verit, best) adia adia-test assms d-loc test-assoc3 test-double-complement*)
finally show $|Q * (R * S)\rangle p = |(Q * R) * S\rangle p$
qed

lemma *ebox-sp-associative*:
assumes *test p*
shows $|Q * (R * S)]\rangle p = |(Q * R) * S]]\rangle p$
by (*simp add: adia-sp-associative assms ebox-adia*)

lemma *edia-sp-associative*:
assumes *test p*
shows $|Q * (R * S)\rangle\rangle p = |(Q * R) * S\rangle\rangle p$
proof –
have $|fis (Q * (R * S))\rangle\rangle p = |fis (Q * Dom (R * S)) * (fis (R * Dom S) * fis S)\rangle\rangle p$
by (*metis fission-sp-dist*)
also have $\dots = |fis (Q * Dom (R * S)) * fis (R * Dom S) * fis S\rangle\rangle p$
by (*simp add: inner-deterministic-sp-assoc semilattice-inf-class.inf-commute semilattice-inf-class.le-infI1 fission-var*)
also have $\dots = |fis (Q * Dom (R * Dom S)) * fis (R * Dom S) * fis S\rangle\rangle p$

by *simp*
 also have ... = $|fis (Q * (R * Dom S)) * fis S\rangle\rangle p$
 by (*metis fission-sp-dist*)
 also have ... = $|fis ((Q * R) * Dom S) * fis S\rangle\rangle p$
 by (*metis d-complement-ad test-assoc3*)
 also have ... = $|fis ((Q * R) * S)\rangle\rangle p$
 by (*metis fission-sp-dist*)
 finally show *?thesis*
 using *assms edia-fission* by *blast*
 qed

lemma *abox-sp-associative*:
 assumes *test p*
 shows $|Q * (R * S)\rangle p = |(Q * R) * S\rangle p$
 by (*simp add: edia-sp-associative assms abox-edia*)

lemma *abox-oI*:
 assumes $X \neq \{\}$
 shows $|R\rangle \cap X = (\bigcap p \in X . |R\rangle p)$
 apply (*rule antisym*)
 apply (*clarsimp simp: mr-simp*)
 apply (*clarsimp simp: mr-simp*)
 using *assms* by *blast*

lemma *ebox-left-dist-oU*:
 assumes $X \neq \{\}$
 shows $|\bigcup X\rangle\rangle p = (\bigcap R \in X . |R\rangle\rangle p)$
 apply (*rule antisym*)
 apply (*clarsimp simp: mr-simp*)
 apply *blast*
 apply (*clarsimp simp: mr-simp*)
 using *assms* by *blast*

lemma *abox-left-dist-oU*:
 assumes $X \neq \{\}$
 shows $|\bigcup X\rangle p = (\bigcap R \in X . |R\rangle p)$
 apply (*rule antisym*)
 apply (*clarsimp simp: mr-simp*)
 apply *blast*
 apply (*clarsimp simp: mr-simp*)
 using *assms* by *blast*

lemma *adia-left-dist-oU*:
 $|\bigcup X\rangle p = (\bigcup R \in X . |R\rangle p)$
 apply (*clarsimp simp: mr-simp*)
 by *blast*

lemma *edia-left-dist-oU*:
 $|\bigcup X\rangle\rangle p = (\bigcup R \in X . |R\rangle\rangle p)$

apply (*clarsimp simp: mr-simp*)
by *blast*

8.6 Goldblatt's axioms with star

unbundle *no rtrancl-syntax*
notation *star* ($\langle \cdot^* \rangle$ [1000] 999)

lemma *star-induct-1*:
assumes $1 \subseteq X$
and $R * X \subseteq X$
shows $R^* \subseteq X$
apply (*unfold star-def*)
apply (*rule lfp-lowerbound*)
by (*simp add: assms*)

lemma *star-induct*:
assumes $S \subseteq 1 \cup 1_{\cup\cup}$
and $S \subseteq X$
and $R * X \subseteq X$
shows $R^* * S \subseteq X$
proof –
have $R^* \subseteq X \circ S$
proof (*rule star-induct-1*)
show $1 \subseteq X \circ S$
by (*metis (no-types, opaque-lifting) Int-subset-iff assms(2) dual-order.eq-iff sp-lres-galois test-sp*)
next
have $(X \circ S) * S \subseteq X$
by (*simp add: sp-lres-sp-below*)
hence $R * (X \circ S) * S \subseteq R * X$
by (*metis assms(1) s-prod-isor test-iu-test-sp-assoc-5*)
also have $\dots \subseteq X$
by (*simp add: assms(3)*)
finally show $R * (X \circ S) \subseteq X \circ S$
by (*simp add: sp-lres-galois*)
qed
thus *?thesis*
by (*simp add: sp-lres-galois*)
qed

lemma *star-total*:
total (R^*)
by (*metis s-prod-idl s-prod-isol star-refl total-4*)

lemma *star-down*:
 $R^* \downarrow = (R \downarrow)^* \cup 1_{\cup\cup}$
proof
have $R^* * (1 \cup 1_{\cup\cup}) \subseteq (R \downarrow)^* \cup 1_{\cup\cup}$

```

proof (rule star-induct)
  show  $1 \cup 1_{UU} \subseteq 1 \cup 1_{UU}$ 
    by simp
next
  show  $1 \cup 1_{UU} \subseteq (R\downarrow)^* \cup 1_{UU}$ 
    using star-refl by auto
next
  have  $ne (R * ((R\downarrow)^* \cup 1_{UU})) \subseteq ne (R\downarrow * ((R\downarrow)^* \cup 1_{UU}))$ 
    by (simp add: down-sp-sp sup-commute)
  also have  $\dots = ne (R\downarrow) * ne ((R\downarrow)^* \cup 1_{UU})$ 
    by (simp add: ne-dist-down-sp)
  also have  $\dots = ne (R\downarrow) * ne ((R\downarrow)^*)$ 
    by (metis down-idempotent down-sp-sp ne-dist-down-sp sup-commute)
  also have  $\dots \subseteq R\downarrow * (R\downarrow)^*$ 
    using sp-oi-subdist by blast
  also have  $\dots \subseteq (R\downarrow)^*$ 
    using star-unfold-eq by blast
  finally show  $R * ((R\downarrow)^* \cup 1_{UU}) \subseteq (R\downarrow)^* \cup 1_{UU}$ 
    by blast
qed
thus  $R^*\downarrow \subseteq (R\downarrow)^* \cup 1_{UU}$ 
  by (simp add: down-sp sup-commute)
next
  have  $(R\downarrow)^* \subseteq R^*\downarrow$ 
proof (rule star-induct-1)
  show  $1 \subseteq R^*\downarrow$ 
    by (simp add: star-refl subset-lower)
next
  show  $R\downarrow * R^*\downarrow \subseteq R^*\downarrow$ 
    by (metis total-dom Un-Int-eq(1) d-isotone d-test ii-right-dist-ou inf-le2
le-sup-iff s-subid-iff2 star-unfold-eq subset-antisym total-down-dist-sp)
qed
thus  $(R\downarrow)^* \cup 1_{UU} \subseteq R^*\downarrow$ 
  using star-total total-lower by blast
qed

```

lemma ne-star-down:
 $ne (R^*\downarrow) = ne ((R\downarrow)^*)$
by (simp add: ne-dist-ou star-down)

lemma ne-down-star:
 $ne ((R\downarrow)^*) = (ne (R\downarrow))^*$

```

proof
  have  $(R\downarrow)^* \subseteq (ne (R\downarrow))^* \cup 1_{UU}$ 
proof (rule star-induct-1)
  show  $1 \subseteq (ne (R\downarrow))^* \cup 1_{UU}$ 
    by (simp add: le-supI1 star-refl)
next
  have  $ne (R\downarrow * ((ne (R\downarrow))^* \cup 1_{UU})) = ne (R\downarrow) * ne ((ne (R\downarrow))^*)$ 

```

by (*metis down-idempotent down-sp-sp ne-dist-down-sp sup-commute*)
 also have $\dots \subseteq (ne (R\downarrow))^*$
 by (*metis (no-types, lifting) IntE UnCI inf.absorb-iff2 sp-oi-subdist*
star-unfold-eq subsetI)
 finally show $R\downarrow * ((ne (R\downarrow))^* \cup 1_{UU}) \subseteq ((ne (R\downarrow))^* \cup 1_{UU})$
 by *blast*
 qed
 thus $ne ((R\downarrow)^*) \subseteq (ne (R\downarrow))^*$
 by (*smt Compl-disjoint2 Int-commute Int-left-commute ne-dist-ou*
semilattice-inf-class.le-iff-inf sup-bot.right-neutral)
 next
 show $(ne (R\downarrow))^* \subseteq ne ((R\downarrow)^*)$
 proof (*rule star-induct-1*)
 show $1 \subseteq ne ((R\downarrow)^*)$
 using *star-refl test-ne by auto*
 next
 show $ne (R\downarrow) * ne ((R\downarrow)^*) \subseteq ne ((R\downarrow)^*)$
 by (*metis IntE IntI UnCI ne-dist-down-sp star-unfold-eq subsetI*)
 qed
 qed

lemma *abox-star-unfold*:
 $test\ p \implies |R^*]p = p * |R]|R^*]p$
 by (*metis abox-left-dist-ou abox-sp-7 sp-unit-abox star-total star-unfold-eq*)

lemma *star-sp-test-commute*:
 assumes $S \subseteq 1 \cup 1_{UU}$
 and $Q * S \subseteq S * R$
 shows $Q^* * S \subseteq S * R^*$
 proof (*rule star-induct*)
 show $S \subseteq 1 \cup 1_{UU}$
 by (*simp add: assms(1)*)
 next
 show $S \subseteq S * R^*$
 by (*metis s-prod-idr s-prod-isor star-refl*)
 next
 have $Q * (S * R^*) \subseteq S * R * R^*$
 by (*metis (no-types, lifting) assms s-prod-distr subset-Un-eq*
test-iu-test-sp-assoc-3)
 thus $Q * (S * R^*) \subseteq S * R^*$
 by (*metis (no-types, lifting) UnCI dual-order.trans s-prod-assoc1 s-prod-isor*
star-unfold subset-eq)
 qed

lemma *adia-star-induct*:
 assumes *test p*
 shows $|R\rangle p \subseteq p \longleftrightarrow |R^*\rangle p \subseteq p$
 proof
 assume $|R\rangle p \subseteq p$

hence $\lambda p * \text{Dom} (R * p) = \{\}$
by (*metis adia assms d-idem2 s-prod-isol subset-empty test-sp-shunting*)
hence $R * p \subseteq p * (R * p)$
by (*metis assms d-sp-strict subset-refl test-sp-shunting*)
hence $R * p \subseteq p * R$
by (*metis assms inf.absorb-iff2 inf-commute sp-test-dist-oi-right test-assoc3 test-sp-idempotent*)
hence $R^* * p \subseteq p * R^*$
by (*simp add: assms le-supI1 star-sp-test-commute*)
hence $R^* * p \subseteq p * (R^* * p)$
by (*metis assms inf.absorb-iff2 inf.orderE sp-oi-subdist test-assoc3 test-sp-idempotent*)
hence $\lambda p * (R^* * p) = \{\}$
by (*meson assms subset-empty test-sp-shunting*)
hence $\lambda p * \text{Dom} (R^* * p) = \{\}$
using *d-sp-strict* **by** *blast*
thus $|R^*\rangle p \subseteq p$
by (*metis adia assms d-test empty-subsetI semilattice-inf-class.le-inf-iff sp-test test-sp-shunting*)
next
assume $|R^*\rangle p \subseteq p$
thus $|R\rangle p \subseteq p$
by (*metis adia-left-isotone assms dual-order.trans s-prod-idr s-prod-isor star-refl star-unfold sup.coboundedI2*)
qed

lemma *ebox-star-induct*:

assumes *test p*
shows $p \subseteq |R\rangle p \longleftrightarrow p \subseteq |R^*\rangle p$
by (*smt (verit, best) adia adia-star-induct assms d-complement-ad ebox-adia test-complement-antitone test-double-complement*)

lemma *abox-star-induct*:

assumes *test p*
shows $p \subseteq |R\rangle p \longleftrightarrow p \subseteq |R^*\rangle p$
proof –
have $p \subseteq |ne (R\downarrow)\rangle p \longleftrightarrow p \subseteq |ne (R^*\downarrow)\rangle p$
by (*metis assms ebox-star-induct ne-down-star ne-star-down*)
thus *?thesis*
by (*metis abox-ebox assms*)

qed

lemma *edia-star-induct*:

assumes *test p*
shows $|R\rangle p \subseteq p \longleftrightarrow |R^*\rangle p \subseteq p$
by (*metis adia-star-induct assms edia-adia ne-down-star ne-star-down*)

lemma *abox-star-induct-1*:

assumes *test p*

and $test\ q$
and $q \subseteq p * |R]q$
shows $q \subseteq |R^*]p$
proof –
have $q \subseteq p \wedge q \subseteq |R^*]q$
by (*metis Int-subset-iff abox-star-induct abox-test assms test-sp test-sp-commute*)
thus *?thesis*
using *abox-right-isotone assms(1,2)* **by** *blast*
qed

lemma *adia-star-induct-1*:
assumes $test\ p$
and $test\ q$
and $p \cup |R]q \subseteq q$
shows $|R^*]p \subseteq q$
by (*meson adia-right-isotone adia-star-induct assms order.trans sup.bounded-iff*)

lemma *abox-segerberg*:
assumes $test\ p$
shows $|R^*](p \rightarrow |R]p) \subseteq p \rightarrow |R^*]p$
proof –
have $p * |R^*](p \rightarrow |R]p) \subseteq |R^*]p$
proof (*rule abox-star-induct-1*)
show $test\ p$
by (*simp add: assms*)
next
show $test\ (p * |R^*](p \rightarrow |R]p)$
by (*simp add: abox-test assms test-galois-1*)
next
have $p * |R^*](p \rightarrow |R]p) = p * (p \rightarrow |R]p) * |R]|R^*](p \rightarrow |R]p)$
by (*metis abox-star-unfold abox-test assms inf-le2 le-infE sp-unit-convex sp-unit-down test-iu-test-sp-assoc-1 test-ou-closed*)
also have $\dots = p * |R]p * |R]|R^*](p \rightarrow |R]p)$
by (*smt (verit, best) Un-Int-eq(4) abox-left-dist-ou abox-test assms equalityD1 le-infE s-prod-isol sp-unit-abox sp-unit-convex sp-unit-down subset-Un-eq subset-antisym test-galois-1 test-iu-test-sp-assoc-1 test-ou-closed test-sp-commute*)
also have $\dots = p * |R](p * |R^*](p \rightarrow |R]p))$
by (*metis (no-types, lifting) abox-sp abox-test abox-test-sp-3 assms test-assoc2 test-double-complement*)
finally show $p * |R^*](p \rightarrow |R]p) \subseteq p * |R](p * |R^*](p \rightarrow |R]p))$
by *simp*
qed
thus *?thesis*
by (*meson abox-test assms test-galois-1 test-implication-closed*)
qed

lemma *abox-segerberg-adia*:

```

assumes test p
shows  $|R^*|( |R\rangle p \rightarrow p ) \subseteq |R^*\rangle p \rightarrow p$ 
proof -
  let  $?q = |R^*|( |R\rangle p \rightarrow p )$ 
  have  $|R^*\rangle p \subseteq ?q \rightarrow p$ 
  proof (rule adia-star-induct-1)
    show test p
    by (simp add: assms)
  next
    show test ( ?q  $\rightarrow$  p )
    by (simp add: assms)
  next
    have  $|R\rangle(?q \rightarrow p) * |R\rangle ?q * ( |R\rangle p \rightarrow p ) \subseteq |R\rangle p * ( |R\rangle p \rightarrow p )$ 
    by (metis (no-types, lifting) abox-adia-mp abox-test assms inf.absorb-iff2
    sp-test-dist-oi test-implication-closed)
    also have  $\dots \subseteq p$ 
    by (meson adia-test assms equalityD2 test-galois-1 test-implication-closed)
    finally have  $|R\rangle(?q \rightarrow p) \subseteq ( |R\rangle p \rightarrow p ) * |R\rangle ?q \rightarrow p$ 
    by (smt (verit) abox-star-unfold abox-test adia assms d-complement-ad
    test-assoc3 test-double-complement test-galois-1 test-implication-closed
    test-sp-commute)
    also have  $\dots = ?q \rightarrow p$ 
    by (metis abox-star-unfold assms test-implication-closed)
    finally show  $p \cup |R\rangle(?q \rightarrow p) \subseteq ?q \rightarrow p$ 
    by (metis le-sup-iff order-refl)
  qed
thus ?thesis
  by (smt abox-test adia-test assms sup-commute test-galois-1
  test-implication-closed test-shunting)
qed

```

```

lemma s-p-id-sp:
   $(s\text{-id} \cup p\text{-id}) * R = R \cup p\text{-id}$ 
  by (simp add: s-prod-distr)

```

8.7 Propositional Hoare logic

```

abbreviation hoare :: ('a,'a) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  ('b,'b) mrel  $\Rightarrow$  bool (‹-|{-}›)
[50,60,50] 95)
  where  $p\{|R\rangle\}q \equiv p \subseteq |R\rangle q$ 

```

```

abbreviation if-then-else :: ('a,'a) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  ('a,'b) mrel  $\Rightarrow$  ('a,'b)
mrel
  where if-then-else p R S  $\equiv p * R \cup \imath p * S$ 

```

```

abbreviation while-do :: ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel  $\Rightarrow$  ('a,'a) mrel
  where while-do p R  $\equiv (p * R)^* * \imath p$ 

```

```

lemma hoare-skip:

```

assumes *test p*
shows $p \{I\} p$
by (*simp add: assms sp-unit-abox*)

lemma *hoare-cons*:
assumes *test s*
and $r \subseteq p$
and $q \subseteq s$
and $p \{R\} q$
shows $r \{R\} s$
by (*meson abox-right-isotone assms order-trans*)

lemma *hoare-seq*:
assumes *test q*
and *test r*
and $p \{R\} q$
and $q \{S\} r$
shows $p \{R*S\} r$
proof –
have $p \subseteq |R|q$
by (*simp add: assms(3)*)
also have $\dots \subseteq |R||S|r$
by (*simp add: abox-right-isotone abox-test assms(1,2,4)*)
also have $\dots \subseteq |R*S|r$
by (*simp add: abox-sp-3 assms(2)*)
finally show *?thesis*
by *simp*
qed

lemma *hoare-if*:
assumes *test p*
and *test q*
and *test r*
and $(p*q) \{R\} r$
and $((\lnot p)*q) \{S\} r$
shows $q \{if-then-else\} p R S \{r\}$
by (*smt (verit, ccfv-threshold) abox-left-dist-ou abox-test-sp assms inf.absorb-iff2 inf-le2 sp-oi-subdist test-galois-1 test-sp-idempotent*)

lemma *hoare-while*:
assumes *test p*
and *test q*
and $(p*q) \{R\} q$
shows $q \{while-do\} p R \{q*(\lnot p)\}$
proof –
have $q \subseteq p \rightarrow |R|q$
by (*meson assms test-galois-1*)
also have $\dots = |p||R|q$
by (*simp add: abox-test assms(1,2) test-abox*)

also have $\dots \subseteq |p * R|q$
by (*simp add: abox-sp-3 assms(2)*)
finally have $q \subseteq |(p * R)^*|q$
by (*meson abox-star-induct assms(2)*)
also have $\dots \subseteq |(p * R)^*|(q \cup p)$
by (*simp add: abox-right-isotone assms(1,2)*)
also have $\dots = |(p * R)^*|\lambda p \rightarrow (q * (\lambda p))$
by (*smt (verit, best) Un-assoc Un-upper1 assms(1,2) inf-commute*
sup commute sup.order-iff test-double-complement test-galois-1 test-s-prod-is-meet)
also have $\dots = |(p * R)^*|\lambda p|(q * (\lambda p))$
by (*simp add: assms(2) test-abox test-shunting*)
also have $\dots \subseteq |while\text{-}do\ p\ R|(q * (\lambda p))$
by (*simp add: abox-sp-3 assms(2) test-galois-1*)
finally show *?thesis*
by *simp*
qed

lemma *hoare-par:*

assumes *test q*
and $p \{R\} q$
and $p \{S\} q$
shows $p \{R \cup S\} q$

proof –

have $1: p \subseteq |R|1 \rightarrow |S|q$
by (*simp add: assms(3) le-supI2*)
have $p \subseteq |S|1 \rightarrow |R|q$
by (*simp add: assms(2) le-supI2*)
hence $p \subseteq (|R|1 \rightarrow |S|q) \cap (|S|1 \rightarrow |R|q)$
using 1 **by** *simp*
also have $\dots = |R \cup S|q$
by (*smt (verit) abox-dist-iu-3 abox-test assms(1) inf.orderE*
test-implication-closed test-oi-closed test-s-prod-is-meet)
finally show *?thesis*
by *simp*
qed

9 Counterexamples

locale *counterexamples*

begin

lemma *counter-01:*

$\neg ((U::('a, 'b)\ mrel) * \neg((U::('b, 'c)\ mrel) * (R::('c, 'd)\ mrel))) \subseteq \neg(U * R)$
by (*metis UNIV-I U-par-zero disjoint-eq-subset-Compl emptyE*
iu-unit-below-top-sp-test iu-unit-up le-inf-iff s-prod-zero subset-empty
top-upper-least)

abbreviation $a-1 \equiv finite-1.a_1$

lemma counter-02:

$\exists R::(Enum.finite-1, Enum.finite-1) \text{ mrel} . \exists p . \neg (test\ p \longrightarrow (R \cap -(U * p)) * U = R * -(p * U))$
apply (rule exI[**where** ?x={ (a-1, { }) }])
apply (rule exI[**where** ?x={ }])
apply (clarsimp simp: s-id-def)
by (smt (verit, ccfv-SIG) Compl-empty-eq Diff-eq Int-insert-left-if0 U-par-p-id cl8-var complement-test-sp-top d-U d-sp-strict dc empty-subsetI inf-le2 inf-top-left inner-total-2 insert-not-empty s-prod-zeroI x-split-var x-y-split zero-nc)

lemma counter-03:

$\exists R::(Enum.finite-1, Enum.finite-1) \text{ mrel} . \exists p . \neg (test\ p \longrightarrow (R \cap -(U * p)) * 1_{UU} = R * (-(p * U) \cap 1_{UU}))$
apply (rule exI[**where** ?x={ (a-1, { }) }])
apply (rule exI[**where** ?x={ }])
apply (clarsimp simp: s-id-def)
by (smt (z3) Int-Un-eq(3) Int-absorb2 U-c ad-sp-bot cd-iso dc-prop1 disjoint-eq-subset-Compl inf-compl-bot-right inner-total-2 insertI1 p-id-zero singleton-Un-iff sp-oi-subdist)

abbreviation b-1 \equiv finite-2.a₁

abbreviation b-2 \equiv finite-2.a₂

abbreviation b-1-0 \equiv (b-1, { })

abbreviation b-1-1 \equiv (b-1, { b-1 })

abbreviation b-1-2 \equiv (b-1, { b-2 })

abbreviation b-1-3 \equiv (b-1, { b-1, b-2 })

abbreviation b-2-0 \equiv (b-2, { })

abbreviation b-2-1 \equiv (b-2, { b-1 })

abbreviation b-2-2 \equiv (b-2, { b-2 })

abbreviation b-2-3 \equiv (b-2, { b-1, b-2 })

lemma counter-04:

$\exists R::(Enum.finite-2, Enum.finite-2) \text{ mrel} . \exists p\ q . \neg (test\ p \longrightarrow test\ q \longrightarrow |R * p\rangle q = |R\rangle |p\rangle q)$
apply (rule exI[**where** ?x={ b-1-3 }])
apply (rule exI[**where** ?x={ b-1-1 }])
apply (rule exI[**where** ?x={ }])
apply (subst sp-test)
apply (clarsimp simp: s-id-def)
apply (subst top-test)
apply (clarsimp simp: s-id-def)
apply (unfold abox-def)
apply (clarsimp simp: s-id-def)
by blast

lemma counter-05:

$\neg (\exists f . \forall R\ p . test\ p \longrightarrow |R\rangle p = |f\ R\rangle p)$
by (smt (verit, ccfv-threshold) Int-lower1 Int-lower2 abox-sp-unit adia-test-sp counter-01 iu-test-sp-left-zero s-prod-idl subset-refl)

lemma counter-06:

$\neg (\exists f . \forall R p . \text{test } p \longrightarrow |R|]p = |f R]p)$

by (*metis abox-adia-fusion abox-fusion abox-sp-unit adia counter-05 d-complement-ad disjoint-eq-subset-Compl ebox-adia empty-subsetI s-prod-idr order-refl*)

lemma counter-07:

$\neg (\exists f . \text{mono } f \wedge (\forall R . \text{fus } R = \text{lfp } (\lambda X . f R X)))$

proof

assume $\exists f :: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} . \text{mono } f \wedge (\forall R . \text{fus } R = \text{lfp } (\lambda X . f R X))$

from this obtain $f :: ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel} \Rightarrow ('a, 'b) \text{ mrel}$ **where** $\text{mono } f \wedge (\forall R . \text{fus } R = \text{lfp } (\lambda X . f R X))$

by auto

hence $\text{fus } \{\} \subseteq \text{fus } (U :: ('a, 'b) \text{ mrel})$

by (*simp add: le-fun-def mono-def lfp-mono*)

thus *False*

by (*auto simp: mr-simp*)

qed

abbreviation $c-1 \equiv \text{finite-3.a}_1$

abbreviation $c-2 \equiv \text{finite-3.a}_2$

abbreviation $c-3 \equiv \text{finite-3.a}_3$

lemma counter-08:

$\neg (\sim(1 :: (\text{Enum.finite-3}, \text{Enum.finite-3}) \text{ mrel}) * \sim 1 \in \{1, \sim 1\})$

proof –

let $?c = (c-1, \{c-1, c-2, c-3\})$

have $1: ?c \in \sim 1 * \sim 1$

apply (*clarsimp simp: mr-simp*)

apply (*rule exI[where ?x = \{c-2, c-3\}]*)

using *UNIV-finite-3* **by auto**

have $?c \notin 1 \wedge ?c \notin \sim 1$

by (*auto simp: mr-simp*)

thus *?thesis*

using *1* **by auto**

qed

lemma counter-09:

$\neg (\sim(1 :: (\text{Enum.finite-3}, \text{Enum.finite-3}) \text{ mrel}) \odot 1 \in \{1, \sim 1\})$

by (*metis counter-08 co-prod empty-iff ic-involutive insert-iff*)

lemma ex-2-cases:

$\exists b. b = b-1 \vee b = b-2$

by auto

lemma all-2-cases:

$(\forall b. b = b-2 \wedge b = b-1) = \text{False}$

by *auto*

lemma *impl-2-cases*:

$\bigcup \{ X . \exists b. (b = b-1 \longrightarrow X = Y) \wedge (b = b-2 \longrightarrow X = Z) \} = Y \cup Z$

by *auto*

lemma *ex-2-set-cases*:

$(\exists B::Enum.finite-2\ set . P\ B) \longleftrightarrow P\ \{\}\ \vee\ P\ \{b-1\} \vee P\ \{b-2\} \vee P\ \{b-1,b-2\}$

proof –

let $?U = UNIV::Enum.finite-2\ set\ set$

have $?U \subseteq \{\{\}, \{b-1\}, \{b-2\}, \{b-1,b-2\}\}$

proof

fix x

have $x \subseteq \{b-1,b-2\}$

by *auto*

thus $x \in \{\{\}, \{b-1\}, \{b-2\}, \{b-1,b-2\}\}$

by *auto*

qed

hence $?U = \{\{\}, \{b-1\}, \{b-2\}, \{b-1,b-2\}\}$

by *auto*

thus *?thesis*

by (*metis UNIV-I empty-iff insertE*)

qed

abbreviation $B-0 \equiv \{\}::Enum.finite-2\ set$

abbreviation $B-1 \equiv \{b-1\}$

abbreviation $B-2 \equiv \{b-2\}$

abbreviation $B-3 \equiv \{b-1,b-2\}$

abbreviation $mkf\ x\ y \equiv \lambda z . \text{if } z = b-1 \text{ then } x \text{ else } y$

lemma *mkf*:

$f = mkf\ (f\ b-1)\ (f\ b-2)$

by *auto*

lemma *mkf2*:

$f\ b-1 = X \wedge f\ b-2 = Y \implies f = mkf\ X\ Y$

by *auto*

lemma *ex-2-mrel-cases*:

$(\exists f::Enum.finite-2 \Rightarrow Enum.finite-2\ set . P\ f) \longleftrightarrow$

$P\ (mkf\ B-0\ B-0) \vee P\ (mkf\ B-0\ B-1) \vee P\ (mkf\ B-0\ B-2) \vee P\ (mkf\ B-0\ B-3) \vee$

$P\ (mkf\ B-1\ B-0) \vee P\ (mkf\ B-1\ B-1) \vee P\ (mkf\ B-1\ B-2) \vee P\ (mkf\ B-1\ B-3) \vee$

$P\ (mkf\ B-2\ B-0) \vee P\ (mkf\ B-2\ B-1) \vee P\ (mkf\ B-2\ B-2) \vee P\ (mkf\ B-2\ B-3) \vee$

$P\ (mkf\ B-3\ B-0) \vee P\ (mkf\ B-3\ B-1) \vee P\ (mkf\ B-3\ B-2) \vee P\ (mkf\ B-3\ B-3)$

proof

assume $\exists f::Enum.finite-2 \Rightarrow Enum.finite-2\ set . P\ f$

from *this* obtain f where $1: P\ f$

by *auto*

have $\bigwedge x . f\ x \subseteq B-3$

by auto
hence 2: $\bigwedge x . f x = B-0 \vee f x = B-1 \vee f x = B-2 \vee f x = B-3$
by auto
have $f = \text{mkf } B-0 \ B-0 \vee f = \text{mkf } B-0 \ B-1 \vee f = \text{mkf } B-0 \ B-2 \vee f = \text{mkf } B-0 \ B-3 \vee$
 $f = \text{mkf } B-1 \ B-0 \vee f = \text{mkf } B-1 \ B-1 \vee f = \text{mkf } B-1 \ B-2 \vee f = \text{mkf } B-1 \ B-3 \vee$
 $f = \text{mkf } B-2 \ B-0 \vee f = \text{mkf } B-2 \ B-1 \vee f = \text{mkf } B-2 \ B-2 \vee f = \text{mkf } B-2 \ B-3 \vee$
 $f = \text{mkf } B-3 \ B-0 \vee f = \text{mkf } B-3 \ B-1 \vee f = \text{mkf } B-3 \ B-2 \vee f = \text{mkf } B-3 \ B-3$
using 2[of b-1] 2[of b-2] mkf2[of f] **by blast**
thus $P (\text{mkf } B-0 \ B-0) \vee P (\text{mkf } B-0 \ B-1) \vee P (\text{mkf } B-0 \ B-2) \vee P (\text{mkf } B-0 \ B-3) \vee$
 $P (\text{mkf } B-1 \ B-0) \vee P (\text{mkf } B-1 \ B-1) \vee P (\text{mkf } B-1 \ B-2) \vee P (\text{mkf } B-1 \ B-3) \vee$
 $P (\text{mkf } B-2 \ B-0) \vee P (\text{mkf } B-2 \ B-1) \vee P (\text{mkf } B-2 \ B-2) \vee P (\text{mkf } B-2 \ B-3) \vee$
 $P (\text{mkf } B-3 \ B-0) \vee P (\text{mkf } B-3 \ B-1) \vee P (\text{mkf } B-3 \ B-2) \vee P (\text{mkf } B-3 \ B-3)$
using 1 by auto
next
assume $P (\text{mkf } B-0 \ B-0) \vee P (\text{mkf } B-0 \ B-1) \vee P (\text{mkf } B-0 \ B-2) \vee P (\text{mkf } B-0 \ B-3) \vee$
 $P (\text{mkf } B-1 \ B-0) \vee P (\text{mkf } B-1 \ B-1) \vee P (\text{mkf } B-1 \ B-2) \vee P (\text{mkf } B-1 \ B-3) \vee$
 $P (\text{mkf } B-2 \ B-0) \vee P (\text{mkf } B-2 \ B-1) \vee P (\text{mkf } B-2 \ B-2) \vee P (\text{mkf } B-2 \ B-3) \vee$
 $P (\text{mkf } B-3 \ B-0) \vee P (\text{mkf } B-3 \ B-1) \vee P (\text{mkf } B-3 \ B-2) \vee P (\text{mkf } B-3 \ B-3)$
thus $\exists f :: \text{Enum.finite-2} \Rightarrow \text{Enum.finite-2 set} . P f$
by auto
qed

lemma counter-10:

$\exists R :: (\text{Enum.finite-2}, \text{Enum.finite-2}) \text{ mrel} . \neg (U :: (\text{Enum.finite-2}, \text{Enum.finite-2})$
 $\text{mrel}) * (U * R) \subseteq U * R$
apply (rule exI[where ?x={b-1-1,b-1-2}])
apply (unfold s-prod-def)
apply (unfold ex-2-set-cases)
apply (unfold ex-2-mrel-cases)
apply (clarsimp simp: mr-simp ex-2-cases all-2-cases impl-2-cases)
by auto

lemma counter-11:

$\exists (R :: (\text{Enum.finite-2}, \text{Enum.finite-2}) \text{ mrel}) (s :: (\text{Enum.finite-2}, \text{Enum.finite-2})$
 $\text{mrel}) (t :: (\text{Enum.finite-2}, \text{Enum.finite-2}) \text{ mrel}) . \neg (\text{inner-univalent } s \wedge$
 $\text{inner-univalent } t \longrightarrow R * (s * t) = (R * s) * t)$
apply (rule exI[where ?x={b-1-3}])
apply (rule exI[where ?x={b-1-1,b-2-1}])
apply (rule exI[where ?x={b-1-1,b-1-2}])

apply (*unfold s-prod-def*)
apply (*unfold ex-2-set-cases*)
apply (*unfold ex-2-mrel-cases*)
apply (*clarsimp simp: mr-simp ex-2-cases all-2-cases impl-2-cases*)
by (*auto simp: times-eq-iff*)

lemma *counter-12*:

$\neg(\exists S . 1_{\cup\cup} \odot S = 1_{\cup\cup})$

by (*metis Int-absorb2 UNIV-I U-U cl9 co-prod cp-ii-unit-upper
disjoint-iff-not-equal ic-antidist-ii ic-iu-unit ic-top iu-unit-down p-prod-comm
p-prod-ild s-prod-idl s-prod-p-idl*)

lemma *counter-13*:

$\neg(\exists S . \forall R . R \odot S = R)$

by (*meson counter-12*)
end

end

References

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