# Morley's Theorem*

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## Contents



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**theory** *Complex-Angles*

**imports** *Complex-Geometry.Elementary-Complex-Geometry*

**begin**

# 1 Introduction

Morley's trisector theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle, called the first Morley triangle or simply the Morley triangle. This theorem is listed in the 100 theorem list [3].

In this theory, we define some basics elements on complex geometry such as axial symmetry, rotations. We also define basic property of congruent triangles in the complex field following the model already presented in the afp [1] In addition we demonstrate sines law in the complex context.

Finally we prove the Morley theorem using Alain Connes's proof [2].

# 2 Angles between three complex

**definition** *angle-c-def*:‹*angle-c a b c* ≡ ∠ (*a*−*b*) (*c*−*b*)›

**lemma** *angle-c-commute-pi*:
  **assumes** ‹*angle-c a b c* = *pi*›
  **shows** *angle-c a b c* = *angle-c c b a*
  **using** *angle-c-def assms ang-vec-sym-pi* **by** *presburger*

**lemma** *angle-c-commute*:
  **assumes** ‹*angle-c a b c* ≠ *pi*›
  **shows** *angle-c a b c* = −*angle-c c b a*
  **using** *angle-c-def assms ang-vec-sym* **by** *presburger*

**lemma** *angle-c-sum*:
  **assumes** *1*:‹*a* − *b*≠*0*› **and** *2*:‹*c* − *b*≠ *0*› **and** *3*:‹*b*−*d* ≠ *0*›
  **shows**‹⌊*angle-c a b c* + *angle-c c b d*⌋ = *angle-c a b d*›
  **using** *angle-c-def canon-ang-sum* **by** *force*

**lemma** *collinear-angle*:‹*collinear a b c* ⟹ *a*≠*b* ⟹ *b*≠*c* ⟹ *c*≠*a* ⟹ *angle-c a b c* = *0* ∨ *angle-c a b c* = *pi*›
  **unfolding** *collinear-def* **unfolding** *angle-c-def*
  **by** (*metis ang-vec-def arg-div collinear-def collinear-sym2′ is-real-arg2 right-minus-eq*)

**lemma** *cdist-commute*:‹*cdist a b* = *cdist b a*›
  **by** (*simp add*: *norm-minus-commute*)

**lemma** *angle-sum-triangle*:
  **assumes** $h{:}a \neq b \wedge b \neq c \wedge a \neq c$
  **shows** $\lfloor\angle\ (c-a)\ (b-a) + \angle\ (a-b)\ (c-b)\ \ + \angle\ (b-c)\ (a-c)\rfloor = pi$
**proof** −
  **have** *f0*:‹$b-a \neq 0$› ‹$c-a\neq 0$› ‹$b-c\neq 0$› **using** *h* **by** *auto*
  **then have** ‹$\angle c\ (c-a)\ (b-a) = abs\ (Arg\ ((b-a)/(c-a)))$›
    **unfolding** *ang-vec-def ang-vec-c-def*
    **apply**(*subst arg-div*[*OF f0*(*1*) *f0*(*2*), *symmetric*])
    **by**(*simp*)
  **have** ‹$(b-a)/(c-a) = rcis\ (cmod\ ((b-a)/(c-a)))\ (Arg\ ((b-a)/(c-a)))$›
    **by** (*simp add*: *rcis-cmod-Arg*)
  **have** ‹$(b-a)/(c-a) * (c-b)/(a-b) * (a-c)/(b-c) = -1$›
    **using** *h*
   **by** (*smt* (*z3*) *divide-eq-0-iff divide-eq-minus-1-iff f0*(*1*,*2*,*3*) *minus-diff-eq mult-minus-right*
      *nonzero-eq-divide-eq times-divide-eq-left*)
  **then have** ‹$Arg\ ((b-a)/(c-a) * (c-b)/(a-b) * (a-c)/(b-c)) = pi$›
    **by** (*metis arg-cis cis-pi pi-canonical*)
  **then have** ‹$Arg\ ((b-a)/(c-a) * (c-b)/(a-b) * (a-c)/(b-c))$
 $= \lfloor Arg\ ((b-a)/(c-a) * (c-b)/(a-b)) + Arg\ ((a-c)/(b-c))\rfloor$›
    **using** *arg-mult f0*
    **by** (*metis* (*no-types*, *lifting*) *Arg-zero arg-mult divide-divide-eq-left′ pi-neq-zero*
*times-divide-eq-left*
      *times-divide-eq-right*)
  **then have** ‹$\lfloor Arg\ ((b-a)/(c-a) * (c-b)/(a-b)) + Arg\ ((a-c)/(b-c))\rfloor$
$=\lfloor Arg\ ((b-a)/(c-a)) + Arg\ ((c-b)/(a-b)) + Arg\ ((a-c)/(b-c))\rfloor$›
   **by** (*metis* (*no-types*, *lifting*) *arg-mult canon-ang-arg canon-ang-sum divide-eq-0-iff*
*eq-iff-diff-eq-0 h*
      *no-zero-divisors times-divide-eq-right*)
  **then show** *?thesis*
    **using**
      ‹$Arg\ ((b - a) / (c - a) * (c - b) / (a - b) * (a - c) / (b - c)) = \lfloor Arg\ ((b$
$- a) / (c - a) * (c - b) / (a - b)) + Arg\ ((a - c) / (b - c))\rfloor$›
     ‹$Arg\ ((b - a) / (c - a) * (c - b) / (a - b) * (a - c) / (b - c)) = pi$› *arg-div*
*h* **by** *force*
**qed**

**lemma** *angle-sum-triangle-c*:
  **assumes** $h{:}a \neq b \wedge b \neq c \wedge a \neq c$
  **shows** $\lfloor angle\text{-}c\ c\ a\ b + angle\text{-}c\ a\ b\ c\ \ + angle\text{-}c\ b\ c\ a\rfloor = pi$
  **unfolding** *angle-c-def*
  **using** *h* **by** (*smt* (*z3*) *angle-sum-triangle h*)

**lemma** *angle-sum-triangle′*:
  **assumes** $h{:}a \neq 0 \wedge b \neq 0 \wedge c \neq 0$
  **shows** $\lfloor\angle\ (-a)\ b + \angle\ (-b)\ c\ \ + \angle\ (-c)\ a\rfloor = pi$
**proof** −
  **have** *f0*:‹$a \neq 0$› ‹$b\neq 0$› ‹$c\neq 0$› **using** *h* **by** *auto*
  **have** ‹$b/(-a) * a/(-c) * c/(-b) = -1$›

**using** *h* **by**(*auto*)
**then have** *f10*:‹*Arg* (*b*/(−*a*) ∗ *a*/(−*c*) ∗ *c*/(−*b*)) = *pi*›
**by** (*metis arg-cis cis-pi pi-canonical*)
**then have** *f1*:‹*Arg* (*b*/(−*a*) ∗ *a*/(−*c*) ∗ *c*/(−*b*))
= ⌊*Arg* (*b*/(−*a*) ∗ *a*/(−*c*)) + *Arg* (*c*/(−*b*))⌋›
**using** *arg-mult f0*
**by** (*metis* ‹*b* / − *a* ∗ *a* / − *c* ∗ *c* / − *b* = − *1*› *divide-eq-0-iff divide-eq-minus-1-iff times-divide-eq-right*)
**then have** *f2*:‹⌊*Arg* (*b*/(−*a*) ∗ *a*/(−*c*)) + *Arg* (*c*/(−*b*))⌋
=⌊*Arg* (*b*/(−*a*)) + *Arg* (*a*/(−*c*)) + *Arg* (*c*/(−*b*))⌋›
**by** (*metis* (*no-types*, *lifting*) *arg-mult canon-ang-arg canon-ang-sum divide-eq-0-iff f0*(*1*,*2*,*3*)
*neg-0-equal-iff-equal no-zero-divisors times-divide-eq-right*)
**then show** *?thesis*
**using** *f1 f2 arg-div h*
**by** (*smt* (*verit*) *f10 add.left-commute ang-vec-def neg-equal-0-iff-equal*)
**qed**

**lemma** *ang-c-in*:‹*angle-c a b c* ∈ {−*pi*<..*pi*}›
**unfolding** *angle-c-def* **by**(*auto simp*: *canon-ang*)

**lemma** *angle-c-aac*:‹*angle-c a a c* = ⌊*Arg* (*c*−*a*)⌋›
**by** (*simp add*: *Arg-zero angle-c-def*)

**lemma** *angle-c-caa*:‹*angle-c c a a* = ⌊−*Arg* (*c*−*a*)⌋›
**by** (*simp add*: *Arg-zero angle-c-def*)

**lemma** *angle-c-neq0*:‹*angle-c p q r* ≠ *0* ⟹ *p*≠*r* ›
**unfolding** *angle-c-def*
**by** *force*

**end**
**theory** *Complex-Triangles-Definitions*

**imports** *Complex-Angles*

**begin**

# 3 Complex triangles

In this section we define triangles and derive some useful lemmas on congruent triangles, following the model [1]

**locale** *congruent-ctriangle* =
**fixes** *a1 b1 c1* :: *complex* **and** *a2 b2 c2* :: *complex*
**assumes** *sides'*: *cdist a1 b1* = *cdist a2 b2 cdist a1 c1* = *cdist a2 c2 cdist b1 c1* = *cdist b2 c2*
**and** *angles'*: *angle-c b1 a1 c1* = *angle-c b2 a2 c2* ∨ *angle-c b1 a1 c1* = − *angle-c b2 a2 c2*

*angle-c a1 b1 c1* = *angle-c a2 b2 c2* ∨ *angle-c a1 b1 c1* = − *angle-c a2 b2 c2*
*angle-c a1 c1 b1* = *angle-c a2 c2 b2* ∨ *angle-c a1 c1 b1* = − *angle-c a2 c2 b2*
**begin**

**lemma** *sides*:
*cdist a1 b1* = *cdist a2 b2 cdist a1 c1* = *cdist a2 c2 cdist b1 c1* = *cdist b2 c2*
*cdist b1 a1* = *cdist a2 b2 cdist c1 a1* = *cdist a2 c2 cdist c1 b1* = *cdist b2 c2*
*cdist a1 b1* = *cdist b2 a2 cdist a1 c1* = *cdist c2 a2 cdist b1 c1* = *cdist c2 b2*
*cdist b1 a1* = *cdist b2 a2 cdist c1 a1* = *cdist c2 a2 cdist c1 b1* = *cdist c2 b2*
**using** *sides′*
**by** (*auto simp add*: *norm-minus-commute*)

**lemma** *angles*:
*angle-c b1 a1 c1* = *angle-c b2 a2 c2* ∨ *angle-c b1 a1 c1* = − *angle-c b2 a2 c2*
*angle-c a1 b1 c1* = *angle-c a2 b2 c2* ∨ *angle-c a1 b1 c1* = − *angle-c a2 b2 c2*
*angle-c a1 c1 b1* = *angle-c a2 c2 b2* ∨ *angle-c a1 c1 b1* = − *angle-c a2 c2 b2*
*angle-c c1 a1 b1* = *angle-c b2 a2 c2* ∨ *angle-c c1 a1 b1* = − *angle-c b2 a2 c2*
*angle-c c1 b1 a1* = *angle-c a2 b2 c2* ∨ *angle-c c1 b1 a1* = − *angle-c a2 b2 c2*
*angle-c b1 c1 a1* = *angle-c a2 c2 b2* ∨ *angle-c b1 c1 a1* = − *angle-c a2 c2 b2*
*angle-c b1 a1 c1* = *angle-c c2 a2 b2* ∨ *angle-c b1 a1 c1* = − *angle-c c2 a2 b2*
*angle-c a1 b1 c1* = *angle-c c2 b2 a2* ∨ *angle-c a1 b1 c1* = − *angle-c c2 b2 a2*
*angle-c a1 c1 b1* = *angle-c b2 c2 a2* ∨ *angle-c a1 c1 b1* = − *angle-c b2 c2 a2*
*angle-c c1 a1 b1* = *angle-c c2 a2 b2* ∨ *angle-c c1 a1 b1* = − *angle-c c2 a2 b2*
*angle-c c1 b1 a1* = *angle-c c2 b2 a2* ∨ *angle-c c1 b1 a1* = − *angle-c c2 b2 a2*
*angle-c b1 c1 a1* = *angle-c b2 c2 a2* ∨ *angle-c b1 c1 a1* = − *angle-c b2 c2 a2*
**using** *angles′ angle-c-commute*
**by**(*simp-all add*: *angle-c-commute*)(*metis add.inverse-inverse angle-c-commute*)+

**end**

**end**
**theory** *Complex-Trigonometry*

**imports** *HOL−Analysis.Convex Complex-Angles Complex-Triangles-Definitions*

**begin**

# 4 Complex trigonometry

In this section we add some trigonometric results, especially the law of sines

**lemma** *ang-sin*:
**shows**‹*Im* ((*b*−*a*)∗*cnj*(*c*−*a*)) = *cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*)›
**proof** −
**have** *f0*:‹*sin* (*Arg* (*cnj c* − *cnj a*)) = − *sin* (*Arg* ( *c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff neg-0-equal-iff-equal sin-minus sin-pi*)

**have** *f1*:‹*cos* (*Arg* (*cnj c* − *cnj a*)) = *cos* (*Arg* (*c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff cos-minus*)
**then have** ‹*b*−*a* = *cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∧ *cnj* (*c*−*a*) = *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))›
**using** *rcis-cmod-Arg rcis-def*
**by** (*metis complex-mod-cnj*)
**then have** ‹*Im*(*cis* (*Arg* (*b*−*a*)) ∗ *cis* (*Arg* (*cnj*(*c*−*a*)))) = *sin* (*angle-c c a b*)›
**unfolding** *ang-vec-def angle-c-def* **using** *f1 f0* **by**(*auto simp*:*cis.code sin-diff*)
**then have** ‹*Im* (*cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∗ *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))) =
*cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*) ›
**by** (*metis Im-complex-of-real*[*of cmod* (*c* − *a*)] *Im-complex-of-real*[*of cmod* (*b* − *a*)]
*Im-mult-real*[*of cor* (*cmod* (*c* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))]
*Im-mult-real*[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*)))]
*Re-complex-of-real*[*of cmod* (*c* − *a*)] *Re-complex-of-real*[*of cmod* (*b* − *a*)]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*)))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cis* (*Arg* (*cnj* (*c* − *a*)))
*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) *cor* (*cmod* (*c* − *a*))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cmod* (*c* − *a*) *cmod* (*b* − *a*)
*Im* (*cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*))))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) *cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))
*cor* (*cmod* (*c* − *a*))]
*mult.commute*[*of cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) ∗ *cor* (*cmod* (*c* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*)))]])
**then show** ‹*Im* ((*b* − *a*) ∗ *cnj* (*c* − *a*)) = *cmod* (*c* − *a*) ∗ *cmod* (*b* − *a*) ∗ *sin* (*angle-c c a b*)›
**by** (*metis* ‹*b* − *a* = *cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) ∧ *cnj* (*c* − *a*) = *cor* (*cmod* (*c* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*)))›
*ab-semigroup-mult-class.mult-ac*(*1*))
**qed**

**lemma** *ang-cos*:
**shows**‹*Re* ((*b*−*a*)∗*cnj*(*c*−*a*)) = *cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *cos* (*angle-c c a b*)›
**proof** −
**have** *f0*:‹*sin* (*Arg* (*cnj c* − *cnj a*)) = − *sin* (*Arg* ( *c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff neg-0-equal-iff-equal sin-minus sin-pi*)
**have** *f1*:‹*cos* (*Arg* (*cnj c* − *cnj a*)) = *cos* (*Arg* (*c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff cos-minus*)
**then have** *f2*:‹*b*−*a* = *cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∧ *cnj* (*c*−*a*) = *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))›

**using** *rcis-cmod-Arg rcis-def*
**by** (*metis complex-mod-cnj*)
**then have** ‹*Re*(*cis* (*Arg* (*b*−*a*)) ∗ *cis* (*Arg* (*cnj*(*c*−*a*)))) = *cos* (*angle-c c a b*)›
**unfolding** *ang-vec-def angle-c-def* **using** *f1 f0* **by**(*auto simp*:*cis.code cos-diff*)
**then have** ‹*Re* (*cmod* (*b*−*a*) ∗ *cmod* (*c*−*a*)∗ *cis* (*Arg* (*b*−*a*)) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))) =
*Re* (*cmod* (*c*−*a*) ∗*cmod* (*b*−*a*)) ∗ *cos* (*angle-c c a b*) ›
**proof** −
**have** *f1*: *cor* (*cmod* (*b* − *a*) ∗ *cmod* (*c* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*)))
= *cor* (*cmod* (*b* − *a*) ∗ *cmod* (*c* − *a*)) ∗ (*cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*))))
**using** *ab-semigroup-mult-class.mult-ac*(*1*) **by** *blast*
**have** *cmod* (*b* − *a*) ∗ *cmod* (*c* − *a*) = *cmod* (*c* − *a*) ∗ *cmod* (*b* − *a*)
**by** *argo*
**then show** *?thesis*
**using** *f1 Im-complex-of-real Re-mult-real* ‹*Re* (*cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*))))
= *cos* (*angle-c c a b*)› **by** *presburger*
**qed**
**then show** ‹*Re* ((*b*−*a*)∗*cnj*(*c*−*a*)) = *cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *cos* (*angle-c c a b*)›
**by** (*metis Re-complex-of-real f2 mult.assoc mult.commute of-real-mult*)

**qed**

**lemma** *law-of-cosines*′:
**assumes** *h*:‹*A*≠*C*› ‹*A*≠*B*›
**shows** ((*cdist B C*)$^2$ − (*cdist A C*)$^2$ − (*cdist A B*)$^2$ ) / (− *2*∗(*cdist A C*)∗(*cdist A B*)) = (*cos* (∠ (*C*−*A*) (*B*−*A*)))
**using** *law-of-cosines*[*of B C A*] *h* **by**(*auto simp*:*field-simps*)

**lemma** *law-of-cosines*′′:
**shows** (*cdist A C*)$^2$ = (*cdist B C*)$^2$ − (*cdist A B*)$^2$ + *2*∗(*cdist A C*)∗(*cdist A B*)∗(*cos* (∠ (*C*−*A*) (*B*−*A*)))
**using** *law-of-cosines*[*of B C A*] **by**(*auto simp*:*field-simps*)

**lemma** *law-of-cosines*′′′:
**shows** (*cdist A B*)$^2$ = (*cdist B C*)$^2$ − (*cdist A C*)$^2$ + *2*∗(*cdist A C*)∗(*cdist A B*)∗(*cos* (∠ (*C*−*A*) (*B*−*A*)))
**using** *law-of-cosines*[*of B C A*] **by**(*auto simp*:*field-simps*)

## 4.1 The law of sines

**theorem** *law-of-sines*:
**assumes** *h1*:‹*b* ≠ *a*› ‹*a* ≠ *c*› ‹*b* ≠ *c*›
**shows** *sin* (*angle-c a b c*) ∗ *cdist b c* = *sin* (*angle-c c a b*) ∗ *cdist a c* (**is** *?A* = *?B*)
**proof** −

{**fix** *a b c* ::*complex*
**have** *f0*:‹*sin* (*Arg* (*cnj c* − *cnj a*)) = − *sin* (*Arg* ( *c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff neg-0-equal-iff-equal sin-minus sin-pi*)
**have** *f1*:‹*cos* (*Arg* (*cnj c* − *cnj a*)) = *cos* (*Arg* (*c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff cos-minus*)
**assume** ‹*b*−*a* ≠ *0*› ‹*c*−*a* ≠ *0* ›
**then have** *f2*: ‹*b*−*a* = *cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∧ *cnj* (*c*−*a*) = *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))›
**using** *rcis-cmod-Arg rcis-def*
**by** (*metis complex-mod-cnj*)
**then have** ‹*Im*(*cis* (*Arg* (*b*−*a*)) ∗ *cis* (*Arg* (*cnj*(*c*−*a*)))) = *sin* (*angle-c c a b*)›
**unfolding** *ang-vec-def angle-c-def* **using** *f1 f0* **by**(*auto simp*:*cis.code sin-diff*)

**then have** ‹*Im* (*cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∗ *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))) =
*cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*) ›
**by** (*metis Im-complex-of-real*[*of cmod* (*c* − *a*)] *Im-complex-of-real*[*of cmod* (*b* − *a*)]
*Im-mult-real*[*of cor* (*cmod* (*c* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))]
*Im-mult-real*[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*)))]
*Re-complex-of-real*[*of cmod* (*c* − *a*)] *Re-complex-of-real*[*of cmod* (*b* − *a*)]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*)))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cis* (*Arg* (*cnj* (*c* − *a*)))
*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) *cor* (*cmod* (*c* − *a*))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cmod* (*c* − *a*) *cmod* (*b* − *a*)
*Im* (*cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*))))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) *cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))
*cor* (*cmod* (*c* − *a*))]
*mult.commute*[*of cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) ∗ *cor* (*cmod* (*c* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*)))]])
**then have** ‹*Im* ((*b*−*a*)∗*cnj*(*c*−*a*)) = *cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*)›
**using** *ang-sin* **by** *presburger*
}**note** *ang*=*this*
**have** *i2*:‹*sin* (*angle-c c a b*) = *Im*((*b*−*a*)∗*cnj*(*c*−*a*))/ (*cmod*(*b*−*a*)∗*cmod*(*c*−*a*))›
**using** *ang*[*of b a c*] *h1* **by**(*auto*)
**have** ‹*sin* (*angle-c a b c*) = *Im*((*c*−*b*)∗*cnj*(*a*−*b*)) / (*cmod* (*c*−*b*)∗*cmod* (*a*−*b*))›
**using** *ang h1* **by**(*auto*)
**then have** *imp1*:‹*cmod* (*c*−*b*) ∗ *sin* (*angle-c a b c*) = *Im*((*c*−*b*)∗*cnj*(*a*−*b*)) / (*cmod* (*a*−*b*)) ›

**by** *auto*
**from** *i2* **have** *imp2*:‹*cmod* (*c*−*a*) ∗ *sin* (*angle-c c a b*) = *Im*((*b*−*a*)∗*cnj*(*c*−*a*))/ (*cmod*(*b*−*a*)) ›
**by** *auto*
**show** *?thesis* **using** *imp1 imp2* **by**(*auto simp*:*field-simps norm-minus-commute*)

**qed**

**lemma** *law-of-sines'*: **assumes** *h1*:‹*b* ≠ *a*› ‹*a* ≠ *c*› ‹*b* ≠ *c*›
**shows** *sin* (*angle-c a b c*) ∗ *cdist b a* = *sin* (*angle-c b c a*) ∗ *cdist a c*
**proof** −
{**fix** *a b c* ::*complex*
**have** *f0*:‹*sin* (*Arg* (*cnj c* − *cnj a*)) = − *sin* (*Arg* ( *c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff neg-0-equal-iff-equal sin-minus sin-pi*)
**have** *f1*:‹*cos* (*Arg* (*cnj c* − *cnj a*)) = *cos* (*Arg* (*c* − *a*))›
**by** (*metis arg-cnj-not-pi arg-cnj-pi complex-cnj-diff cos-minus*)
**assume** ‹*b*−*a* ≠ *0*› ‹*c*−*a* ≠ *0* ›
**then have** *f2*:‹*b*−*a* = *cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∧ *cnj* (*c*−*a*) = *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))›
**using** *rcis-cmod-Arg rcis-def*
**by** (*metis complex-mod-cnj*)
**then have** ‹*Im*(*cis* (*Arg* (*b*−*a*)) ∗ *cis* (*Arg* (*cnj*(*c*−*a*)))) = *sin* (*angle-c c a b*)›
**unfolding** *ang-vec-def angle-c-def* **using** *f0 f1* **by**(*auto simp*:*cis.code sin-diff*)

**then have** ‹*Im* (*cmod* (*b*−*a*) ∗ *cis* (*Arg* (*b*−*a*)) ∗ *cmod* (*c*−*a*) ∗ *cis* (*Arg* (*cnj* (*c*−*a*)))) =
*cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*) ›
**by** (*metis Im-complex-of-real*[*of cmod* (*c* − *a*)] *Im-complex-of-real*[*of cmod* (*b* − *a*)]
*Im-mult-real*[*of cor* (*cmod* (*c* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))]
*Im-mult-real*[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*)))]
*Re-complex-of-real*[*of cmod* (*c* − *a*)] *Re-complex-of-real*[*of cmod* (*b* − *a*)]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cor* (*cmod* (*b* − *a*)) *cis* (*Arg* (*b* − *a*))
*cis* (*Arg* (*cnj* (*c* − *a*)))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cis* (*Arg* (*cnj* (*c* − *a*)))
*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) *cor* (*cmod* (*c* − *a*))]
*ab-semigroup-mult-class.mult-ac*(*1*)[*of cmod* (*c* − *a*) *cmod* (*b* − *a*)
*Im* (*cis* (*Arg* (*b* − *a*)) ∗ *cis* (*Arg* (*cnj* (*c* − *a*))))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) *cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*))]
*mult.commute*[*of cis* (*Arg* (*cnj* (*c* − *a*))) ∗ (*cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)))
*cor* (*cmod* (*c* − *a*))]
*mult.commute*[*of cor* (*cmod* (*b* − *a*)) ∗ *cis* (*Arg* (*b* − *a*)) ∗ *cor* (*cmod* (*c* − *a*))

*cis* (*Arg* (*cnj* (*c* − *a*)))])
**then have** ‹*Im* ((*b*−*a*)∗*cnj*(*c*−*a*)) = *cmod* (*c*−*a*) ∗*cmod* (*b*−*a*) ∗ *sin* (*angle-c c a b*)›
**using** *ang-sin* **by** *presburger*
**}note** *ang*=*this*
**have** *i2*:‹*sin* (*angle-c b c a*) = *Im*((*a*−*c*)∗*cnj*(*b* − *c*))/ (*cmod*(*b*−*c*)∗*cmod*(*a*−*c*))›
**using** *ang*[*of a c b*] *h1* **by**(*auto*)
**have** ‹*sin* (*angle-c a b c*) = *Im*((*c*−*b*)∗*cnj*(*a*−*b*)) / (*cmod* (*c*−*b*)∗*cmod* (*a*−*b*))›
**using** *ang h1* **by**(*auto*)
**then have** *imp1*:‹*cmod* (*a*−*b*) ∗ *sin* (*angle-c a b c*) = *Im*((*c*−*b*)∗*cnj*(*a*−*b*)) / (*cmod* (*c*−*b*)) ›
**by** *auto*
**from** *i2* **have** *imp2*:‹*cmod* (*a*−*c*) ∗ *sin* (*angle-c b c a*) = *Im*((*a*−*c*)∗*cnj*(*b*−*c*))/ (*cmod*(*b*−*c*)) ›
**by** *auto*
**show** *?thesis* **using** *imp1 imp2*
**by** (*metis cdist-commute h1*(*1*,*2*,*3*) *law-of-sines*)
**qed**

**lemma** *ang-pos-pos*:‹*q*≠*p* ⟹ *p*≠ *r* ⟹ *r* ≠ *q* ⟹ *angle-c q r p* ≥ *0* ⟹ *angle-c r p q* ≥*0*›
**using** *law-of-sines′*[*of r q p*]
**by** (*smt* (*verit*) *ang-vec-bounded angle-c-def cdist-def mult-neg-pos mult-nonneg-nonneg*

*right-minus-eq sin-ge-zero sin-gt-zero sin-minus zero-less-norm-iff*)

**lemma** *cmod-pos*:‹*cmod a* ≥ *0*›
**by** *simp*

**lemma** *ang-neg-neg*:‹*q*≠*p* ⟹ *p*≠ *r* ⟹ *r* ≠ *q* ⟹ *angle-c q r p* < *0* ⟹ *angle-c r p q* < *0*›
**proof** −
**assume** ‹*q*≠*p*› ‹*p*≠ *r*› ‹*r* ≠ *q*› ‹*angle-c q r p* < *0*›
**then have** ‹*sin* (*angle-c q r p*) < *0*›
**using** *ang-c-in*
**by** (*metis ang-vec-def angle-c-def canon-ang*(*1*) *minus-less-iff neg-0-less-iff-less sin-gt-zero sin-minus*)
**then have** ‹*sin* (*angle-c q r p*) ∗ *cdist r q* < *0*›
**using** ‹*r* ≠ *q*› *mult-neg-pos* **by** *fastforce*
**from** *law-of-sines′*[*of r q p*] **have** ‹*sin* (*angle-c r p q*)<*0*›
**by** (*metis* ‹*p* ≠ *r*› ‹*q* ≠ *p*› ‹*r* ≠ *q*› ‹*sin* (*angle-c q r p*) ∗ *cdist r q* < *0*›
*cdist-def linorder-not-less mult-less-0-iff norm-ge-zero*)
**then show** *?thesis*
**using** *ang-c-in*[*of r p q*]
**by** (*metis ang-vec-def angle-c-def canon-ang*(*2*) *linorder-not-less sin-ge-zero*)
**qed**

**lemma** *collinear-sin-neq-0*:

‹¬*collinear a2 b2 c2* ⟹ *sin* (*angle-c a2 c2 b2*) ≠ *0*›
**unfolding** *collinear-def angle-c-def*
**by** (*metis Im-complex-div-eq-0 ang-sin angle-c-def collinear-def collinear-sym1 collinear-sym2′ mult-eq-0-iff*)

**lemma** *collinear-sin-neq-pi*:
‹¬*collinear a2 b2 c2* ⟹ *sin* (*angle-c a2 c2 b2*) ≠ *pi*›
**unfolding** *collinear-def angle-c-def*
**by** (*metis add-cancel-right-left order.antisym order.trans le-add-same-cancel1 zero-le-one*

*one-add-one one-neq-zero pi-ge-two sin-le-one*)

**lemma** *collinear-iff*:
**assumes** ‹*a*≠*b* ∧ *b*≠*c* ∧ *c*≠*a*›
**shows** ‹*collinear a b c* ⟷ (*angle-c a b c* = *pi* ∨ *angle-c a b c* = *0*)›
**using** *assms collinear-angle collinear-sin-neq-0 collinear-sym1 sin-minus-pi sin-pi-minus*
**by** *fastforce*

**definition** ‹*innerprod a b* ≡ *cnj a* ∗ *b*›

**lemma** *left-lin-innerprod*:‹*innerprod* (*x* + *y*) *z* = *innerprod x z* + *innerprod y z*›
**unfolding** *innerprod-def*
**by** (*simp add*: *mult.commute ring-class.ring-distribs*(*1*))

**lemma** *right-lin-innerprod*:‹*innerprod x* (*y*+*z*) = *innerprod x y* + *innerprod x z*›
**unfolding** *innerprod-def*
**by** (*simp add*: *ring-class.ring-distribs*(*1*))

**lemma** *leftlin-innerprod*:‹*innerprod x* (*t*∗*y*) = *t* ∗ *innerprod x y*›
**unfolding** *innerprod-def* **by**(*auto*)

**lemma** *rightsesqlin-innerprod*:‹*innerprod* (*t*∗*x*) (*y*) = *cnj t* ∗ *innerprod x y*›
**unfolding** *innerprod-def* **by**(*auto*)

**lemma** *norm-eq-csqrt-inner*:‹*norm x* = *csqrt* (*innerprod x x*)›
**using** *complex-mod-sqrt-Re-mult-cnj innerprod-def* **by** *force*

**lemma** *abs2-eq-inner*:‹*abs* (*innerprod x y*)^*2* = *innerprod x y* ∗ *cnj* (*innerprod x y*)›
**unfolding** *innerprod-def abs-complex-def* **apply**(*rule complex-eqI*)
**by** (*metis comp-apply complex-mult-cnj-cmod of-real-power*)+

**lemma** *complex-add-inner-cnj*:‹*t*∗*innerprod x y* + *cnj* (*t*∗*innerprod x y*) = *2*∗ *Re* (*t*∗*innerprod x y*)›
**using** *complex-add-cnj* **by** *blast*

**lemma** *Re-innerprod-inner*:‹*Re* (*innerprod* (*a*−*b*) (*c*−*b*)) = (*a*−*b*)·(*c*−*b*)›
  **unfolding** *innerprod-def inner-complex-def* **by**(*auto simp*:*field-simps*)

**lemma** *angle-c-arccos-pos*:
  **assumes** *h*:‹*a*≠*b* ∧ *b*≠ *c*› ‹*angle-c a b c* ≥ *0*›
  **shows** ‹*angle-c a b c* = *arccos* ((*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))/(*cmod*(*a*−*b*)∗*cmod*(*c*−*b*)))›
**proof** −
  **have** ‹*Re* ((*a*−*b*)∗*cnj*(*c*−*b*)) = (*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))›
    **unfolding** *innerprod-def* **by**(*auto simp*:*field-simps*)
  **then have** ‹((*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))/(*cmod*(*a*−*b*)∗*cmod*(*c*−*b*))) = *cos* (*angle-c a b c*)›
    **by** (*metis* (*no-types*, *lifting*) *ang-cos angle-c-commute cos-minus divisors-zero h*(*1*) *mult.commute nonzero-mult-div-cancel-left norm-eq-zero right-minus-eq*)
  **then show** *?thesis*
    **using** *ang-vec-bounded angle-c-def arccos-cos h*(*2*) **by** *presburger*
**qed**

**lemma** *angle-c-arccos-neg*:
  **assumes** *h*:‹*a*≠*b* ∧ *b*≠ *c*› ‹*angle-c a b c* ≤ *0* ›
  **shows** ‹− *angle-c a b c* = *arccos* ((*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))/(*cmod*(*a*−*b*)∗*cmod*(*c*−*b*)))›
**proof** −
  **have** ‹*Re* ((*a*−*b*)∗*cnj*(*c*−*b*)) = (*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))›
    **unfolding** *innerprod-def* **by**(*auto simp*:*field-simps*)
  **then have** ‹((*Re* (*innerprod* (*a*−*b*) (*c* − *b*)))/(*cmod*(*a*−*b*)∗*cmod*(*c*−*b*))) = *cos* (*angle-c a b c*)›
    **by** (*metis* (*no-types*, *lifting*) *ang-cos angle-c-commute cos-minus divisors-zero h*(*1*) *mult.commute nonzero-mult-div-cancel-left norm-eq-zero right-minus-eq*)
  **then show** *?thesis*
    **using** *ang-vec-bounded angle-c-def arccos-cos h*(*2*)
    **by** (*metis arccos-cos2 less-le-not-le*)
**qed**

**end**
**theory** *Third-Unity-Root*

**imports** *Complex-Angles*

**begin**

# 5 Third unity root

In this section we prove some basic properties of the third unity root j.

**lemma** *root-unity-3*: ‹(*z*::*complex*)^*3* − *1* = *0* ⟷ (*z* = *cis* (*2*∗*pi*/*3*) ∨ *z*=*1* ∨ *z* = *cis* (*4*∗*pi*/*3*))›
**proof**(*rule iffI*)
  **have** ‹*cis* (*2* ∗ *pi* ∗ *real xa* / *3*) ≠ *1* ⟹

```
        cis (2 * pi * real xa / 3) ≠ cis (2 * pi / 3)
  ⟹ xa < 3 ⟹ cis (2 * pi * real xa / 3) = cis (4 * pi / 3) ›
    for xa
  proof(rule ccontr)
    assume h:‹cis (2 * pi * real xa / 3) ≠ 1›
      ‹cis (2 * pi * real xa / 3) ≠ cis (2 * pi / 3)›
      ‹xa < 3› ‹cis (2 * pi * real xa / 3) ≠ cis (4 * pi / 3)›
    then have ‹xa = 1 ∨ xa = 0›
      using less-Suc-eq numeral-3-eq-3 by fastforce
    then show False
      using h(1) h(2) by auto
  qed
  then have f0:‹(λk. cis (2 * pi * real k / real 3)) ‘ {..<3} = {1, cis(2*pi/3),
cis (4*pi/3)}›
    unfolding image-def  by(auto simp: image-def intro:bexI[where x=0]
        bexI[where x=1] bexI[where x=2])
  have ‹{z. z^3 = 1} = {1, cis(2*pi/3), cis (4*pi/3)}›
    apply(insert bij-betw-roots-unity[of 3, simplified])
    apply(frule bij-betw-imp-surj-on)
    using f0 by auto
  then show ‹z ^ 3 − 1 = 0 ⟹ z = cis (2 * pi / 3) ∨ z = 1 ∨ z = cis (4 * pi
/ 3)›
    unfolding bij-betw-def inj-on-def by(auto)
  show ‹ z = cis (2 * pi / 3) ∨ z = 1 ∨ z = cis (4 * pi / 3) ⟹ z ^ 3 − 1 = 0
›
    using cis-2pi cis-multiple-2pi[of 2] by(auto simp:DeMoivre field-simps)
qed

definition root3 where ‹root3≡cis(2*pi/3)›

lemma root3-eq-0:‹1 + root3 + root3^2 = 0›
proof −
  have ‹(z::complex)^3 − 1 = (z − 1)*(1 + z + z^2)› for z
    by(auto simp:field-simps power2-eq-square power3-eq-cube)
  moreover have ‹root3^3 − 1 = 0›
    using root-unity-3 unfolding root3-def by blast
  moreover have ‹root3−1 ≠ 0›
    by(simp add:cis.code Complex-eq-1 cos-120 root3-def)
  ultimately show ?thesis
    by simp
qed

lemma root-unity-carac:
  assumes h1:‹a1*a2*a3 = j› ‹1−a1*a2 ≠ 0 ∧ 1−a2*a3≠0 ∧ 1−a1*a3 ≠ 0›
‹j=root3›
    and  R : ‹R=(a1 * b2 + b1) / (1 − a1 * a2)› (is ‹R=?R›)
    and P :‹P=(a2*b3 + b2)/(1−a2*a3)› (is ‹P=?P›)
    and Q:‹Q=(a3*b1 + b3)/(1−a3*a1)› (is ‹Q=?Q›)
 shows ‹(a1^2+a1+1)*b1 + a1^3*(a2^2+a2+1)*b2 +a1^3*a2^3*(a3^2+a3+1)*b3
```

=
 −j∗a1^2∗a2∗(a1−j)∗(a2−j)∗(a3−j)∗(?R +j∗?P +j^2∗?Q)›
**proof** −
 **have** simp-rules-for-eq:‹1−a1∗a2 ≠ 0 ∧ 1−a2∗a3≠0 ∧ 1−a1∗a3 ≠ 0› ‹a1∗a2∗a3 = j›
 ‹1 + j +j^2 = 0› ‹j∗j∗j = 1› ‹(1−a1∗a2)∗a3 = (a3−j)› ‹(1−a2∗a3)∗a1 = (a1−j)› ‹(1−a1∗a3)∗a2 = (a2−j)›
 **using** h1 root3-eq-0 root-unity-3 **unfolding** root3-def
 **by**(auto simp:power3-eq-cube mult.left-commute mult.commute right-diff-distrib)

 **have** y:‹ (− j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗
 (a1 ∗ b2) + − j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ b1) /
 (1 − a1 ∗ a2) = (− j ∗ a1^2 ∗ a2 ∗ ((1 − a3 ∗ a2) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ( a3) ∗ (a1 ∗ b2)
 + − j ∗ a1^2 ∗ a2 ∗ ((1 − a3 ∗ a2) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ (a3) ∗ b1)›
 **apply**(subst add-divide-distrib)
 **using** simp-rules-for-eq(1)
 **by** (simp add: mult.commute)
 **have** y2:‹(− j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j ∗ (a2 ∗ b3)) +
 − j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j ∗ b2)) /
 (1 − a2 ∗ a3) = (− j ∗ a1^2 ∗ a2 ∗ (a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j ∗ (a2 ∗ b3)) +
 − j ∗ a1^2 ∗ a2 ∗ (a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j ∗ b2))›
 **apply**(subst add-divide-distrib)
 **using** simp-rules-for-eq(1) **by** auto
 **have** y3:‹(− j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j^2 ∗ (a3 ∗ b1)) +
 − j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ ((1 − a1 ∗ a3) ∗ a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j^2 ∗ b3)) /
 (1 − a3 ∗ a1) =
(− j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ (a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j^2 ∗ (a3 ∗ b1)) +
 − j ∗ a1^2 ∗ a2 ∗ ((1 − a2 ∗ a3) ∗ a1) ∗ (a2) ∗ ((1 − a1 ∗ a2) ∗ a3) ∗ (j^2 ∗ b3))›
 **apply**(subst add-divide-distrib)
 **by** (smt (verit, best) mult.commute nonzero-mult-div-cancel-left times-divide-eq-left simp-rules-for-eq(1))
 **have** ko:‹a∗(b∗(c)) = a∗b∗c› **for** a b c::complex
 **by** auto
 **have** ko′:‹a +(b +(c)) = a+b+c› **for** a b c::complex
 **by** auto
 **have** ∗:‹a1 ∗ a1 ∗ a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a2 ∗ a2 ∗ a3 ∗ a3 ∗ a3 ∗ a3 ∗ b1 = a1∗a1∗a2∗a3∗b1›

```
    using simp-rules-for-eq by (auto simp add: mult.assoc mult.left-commute)
  have **:‹a1 ∗ a1 ∗ a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a2 ∗ a2 ∗ a2 ∗ a3 ∗ a3 ∗ a3 ∗
b3 = a1∗a1∗a2∗a2∗b3›
    using simp-rules-for-eq by (auto simp add: mult.assoc mult.left-commute)
  have ***:‹a1 ∗ a1 ∗ a1 ∗ a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a2 ∗ a2 ∗ a2 ∗ a3 ∗ a3 ∗
a3 ∗ a3 ∗ b3
              = a1∗a1∗a1∗a2∗a2∗a3∗b3›
    using simp-rules-for-eq by (auto simp add: mult.assoc mult.left-commute)
  have fin:‹a1 ∗ b1 + a1 ∗ a1 ∗ a1 ∗ a2 ∗ b2 + a1 ∗ a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗
a3 ∗ b2 +
    a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a2 ∗ a3 ∗ b3 + a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a3 ∗ a3
∗ b1 +
    a1 ∗ a1 ∗ a1 ∗ a1 ∗ a1 ∗ a2 ∗ a2 ∗ a2 ∗ a3 ∗ a3 ∗ b2 +a1 ∗ a1 ∗ a1 ∗ a1 ∗
a2 ∗ a2 ∗ a2 ∗ a2 ∗
    a3 ∗ a3 ∗ b3 + a1 ∗ a1 ∗ a2 ∗ a2 ∗ b3 + a1 ∗ a1 ∗ a2 ∗ a3 ∗ b1 = a1∗b1
+ j∗a1∗b1 + j^2∗a1∗b1 +
   a1∗a1∗a1∗a2∗b2 + a1∗a1∗a1∗a2∗b2∗j + a1∗a1∗a1∗a2∗b2∗j^2 + a1∗a1∗a2∗a2∗b3
+ a1∗a1∗a2∗a2∗b3∗j +
    a1∗a1∗a2∗a2∗b3∗j^2›
   using simp-rules-for-eq by(auto simp:algebra-simps power2-eq-square power3-eq-cube)
  then have fin′:‹. . . = (a1∗b1 + a1∗a1∗a1∗a2∗b2 + a1∗a1∗a2∗a2∗b3)∗(1+j+j^2)›
    by(auto simp:algebra-simps power2-eq-square power3-eq-cube)
  then show graal:‹(a1^2+a1+1)∗b1 + a1^3∗(a2^2+a2+1)∗b2 +a1^3∗a2^3∗(a3^2+a3+1)∗b3
=
      −j∗a1^2∗a2∗(a1−j)∗(a2−j)∗(a3−j)∗(?R +j∗?P +j^2∗?Q)›
    apply(simp only: simp-rules-for-eq(5−7)[symmetric])
   apply(simp only:y y2 y3 distrib-left distrib-right times-divide-eq-left times-divide-eq-right)

    apply(simp only:simp-rules-for-eq algebra-simps del:mult.assoc mult.commute)

    apply(simp only:ko ko′)
   using simp-rules-for-eq apply(auto simp add: power2-eq-square power3-eq-cube
algebra-simps)
     apply(simp only:ko ko′) apply(subst ∗) apply(subst ∗∗)  apply(subst ∗∗∗)
apply(simp)
    apply(simp only:ko ko′) apply(subst fin) apply(subst fin′)
    using simp-rules-for-eq by fastforce
qed

end
theory Complex-Triangles

imports Complex-Trigonometry Third-Unity-Root

begin


lemma similar-triangles′:
  assumes h:a ≠ 0 ∧ b ≠ 0 ∧ 0 ≠ c ∧ a′ ≠ 0 ∧ b′≠0 ∧ c′≠0
    and h1:‹∠ (−a) b = ∠ (−a′) b′› ‹∠ (−b′) c′  = ∠ (−b) c›
```

```
  shows ‹∠ (−c) a = ∠ (−c′) a′›
proof −
  have ‹⌊∠ (−a) b + ∠ (−b) c  + ∠ (−c) a⌋ = pi›
    using angle-sum-triangle′ h by auto
  also have ‹⌊∠ (−a′) b′ + ∠ (−b′) c′  + ∠ (−c′) a′⌋ = pi›
    using angle-sum-triangle′ h by auto
  also have ‹⌊∠ (−a′) b′ + ∠ (−b′) c′  + ∠ (−c) a⌋ = pi›
    using ‹⌊∠ (− a) b + ∠ (− b) c + ∠ (− c) a⌋ = pi› h1(1,2) by auto
  then have ‹⌊∠ (− c) a⌋ = ⌊∠ (− c′) a′⌋›
   by (smt (verit) ‹⌊∠ (− a′) b′ + ∠ (− b′) c′ + ∠ (− c′) a′⌋ = pi› add.inverse-inverse
ang-vec-bounded
      ang-vec-opposite1 ang-vec-opposite2 ang-vec-opposite-opposite ang-vec-plus-pi1
ang-vec-plus-pi2
        canon-ang-diff canon-ang-id h neg-0-equal-iff-equal)
  ultimately show ?thesis
    by (metis ang-vec-bounded canon-ang-id)
qed


lemma similar-triangles:
  assumes h:a ≠ b ∧ b ≠ c ∧ a ≠ c ∧ a′ ≠ b′ ∧ b′≠ c′ ∧ c′≠a′
    and h1:‹∠ (c−a) (b−a) = ∠ (c′−a′) (b′−a′)› ‹∠ (a−b) (c−b)  =  ∠ (a′−b′)
(c′−b′)›
  shows    ‹ ∠ (b−c) (a−c) =  ∠ (b′−c′) (a′−c′)›
proof −
  have ‹a−b≠0› ‹b−c≠0› ‹a−c≠0› ‹a′−b′≠0› ‹b′−c′≠0› ‹c′−a′≠0›
    using h by auto
  then show ?thesis
    by (smt (z3) diff-numeral-special(12) h1(1,2) minus-diff-eq similar-triangles′)
qed

lemma similar-triangles-c:
  assumes h:a ≠ b ∧ b ≠ c ∧ a ≠ c ∧ a′ ≠ b′ ∧ b′≠ c′ ∧ c′≠a′
    and h1:‹angle-c c a b = angle-c c′ a′ b′› ‹angle-c a b c  =  angle-c a′ b′ c′›
  shows    ‹angle-c b c a =  angle-c b′ c′ a′›
proof −
  have ‹a−b≠0› ‹b−c≠0› ‹a−c≠0› ‹a′−b′≠0› ‹b′−c′≠0› ‹c′−a′≠0›
    using h by auto
  then show ?thesis
    unfolding angle-c-def
    by (metis angle-c-def h h1(1,2) similar-triangles)
qed


lemmas congruent-ctriangleD = congruent-ctriangle.sides congruent-ctriangle.angles


lemma congruent-ctriangles-sss:
```

**assumes** *h*:$a \neq b \wedge b \neq c \wedge a \neq c$
**and** *h1*:‹*cmod* $(b - a)$= *cmod* $(b'-a')$› ‹*cmod* $(b - c)$ = *cmod* $(b' - c')$› ‹*cmod* $(c - a)$ = *cmod* $(c' - a')$›
**shows** ‹*congruent-ctriangle* $a\ b\ c\ a'\ b'\ c'$›
**proof** –
{**fix** $c\ b\ a\ c'\ b'\ a'$
**assume** *h*:$a \neq b \wedge b \neq c \wedge a \neq c$
**and** *h1*:‹*cmod* $(b - a)$= *cmod* $(b'-a')$› ‹*cmod* $(b - c)$ = *cmod* $(b' - c')$› ‹*cmod* $(c - a)$ = *cmod* $(c' - a')$›
**then have** *h'*:‹$a'\neq b' \wedge b'\neq c' \wedge c'\neq a'$›
**by** *auto*
**have** ‹*cos* $(\angle\ (c-a)\ (b-a))$ = $((cdist\ b\ c)^2 - (cdist\ a\ c)^2 - (cdist\ a\ b)^2\ )\ /\ (-\ 2*(cdist\ a\ c)*(cdist\ a\ b))$ ›
**using** *law-of-cosines'* *h* **by** *auto*
**moreover have** ‹*cos* $(\angle\ (c'-a')\ (b'-a'))$ = $((cdist\ b'\ c')^2 - (cdist\ a'\ c')^2 - (cdist\ a'\ b')^2\ )\ /\ (-\ 2*(cdist\ a'\ c')*(cdist\ a'\ b'))$ ›
**using** *law-of-cosines'* *h h'* **by** *auto*
**ultimately have** *f0*:‹*cos* $(\angle\ (c-a)\ (b-a))$ = *cos* $(\angle\ (c'-a')\ (b'-a'))$›
**by** (*metis cdist-def h1*(*1*,*2*,*3*) *norm-minus-commute*)
**have** ‹$\angle\ (c-a)\ (b-a) \in \{-pi\ ..\ pi\}$› ‹$\angle\ (c'-a')\ (b'-a') \in \{-pi..pi\}$›
**unfolding** *ang-vec-def*
**using** *canon-ang*(*1*,*2*) *less-eq-real-def* **by** *auto*
**with** *f0* **have** ‹$\angle\ (c-a)\ (b-a) = \angle\ (c'-a')\ (b'-a') \vee \angle\ (c-a)\ (b-a) = -\ \angle\ (c'-a')\ (b'-a')$›
**unfolding** *ang-vec-def*
**by** (*smt* (*verit*) *arccos-cos2 arccos-minus-1 arccos-unique canon-ang*(*1*,*2*))}**note** *dem*=*this*
**then show** *?thesis*
**by** (*metis* (*mono-tags*, *lifting*) *angle-c-def cdist-def congruent-ctriangle-def h h1*(*1*) *h1*(*2*) *h1*(*3*) *norm-minus-commute*)
**qed**

**lemma** *congruent-ctriangleI-sss*:
**assumes** *h*:$a \neq b \wedge b \neq c \wedge a \neq c$
**and** *h1*:‹*cdist* $a\ b$ = *cdist* $a'\ b'$› ‹*cdist* $b\ c$ = *cdist* $b'\ c'$› ‹*cdist* $a\ c$ = *cdist* $a'\ c'$›
**shows** ‹*congruent-ctriangle* $a\ b\ c\ a'\ b'\ c'$›
**using** *cdist-def congruent-ctriangles-sss* *h1*(*1*,*2*,*3*) *h*
**by** (*metis dist-commute dist-complex-def*)

**lemmas** *congruent-ctriangle-sss* = *congruent-ctriangleD*[*OF congruent-ctriangleI-sss*]

**lemma** *isosceles-triangles*:
**assumes** ‹*cdist* $a\ b$ = *cdist* $b\ c$›
**shows** ‹*angle-c* $b\ c\ a$ = *angle-c* $b\ a\ c$ ∨ *angle-c* $b\ c\ a$ = – *angle-c* $b\ a\ c$ ›
**by** (*metis assms cdist-commute cdist-def congruent-ctriangle-sss*(*14*) *norm-eq-zero right-minus-eq*)

**lemma** *non-collinear-independant*:¬ *collinear* $a\ b\ c \Longrightarrow a \neq b \wedge b \neq c \wedge a \neq c$

```
  using collinear-ex-real by force


lemma congruent-ctriangleI-sas:
  assumes ‹a1 ≠ b1 ∧ b1 ≠ c1 ∧ a1 ≠ c1›
  assumes h1:cdist a1 b1 = cdist a2 b2
  assumes h2:cdist b1 c1 = cdist b2 c2
  assumes h3:angle-c a1 b1 c1 = angle-c a2 b2 c2 ∨ angle-c a1 b1 c1 = − angle-c
a2 b2 c2
  shows   congruent-ctriangle a1 b1 c1 a2 b2 c2
proof (rule congruent-ctriangleI-sss)
  have h1':cdist a1 b1 = cdist b2 a2
    using cdist-commute h1 by auto
  show cdist a1 c1 = cdist a2 c2
  proof (rule power2-eq-imp-eq)
    show (cdist a1 c1)^2 = (cdist a2 c2)^2
      apply(insert law-of-cosines[of a1 c1 b1] law-of-cosines[of a2 c2 b2])
      apply(subst (asm) (1 2) h2) apply(subst (asm) (1 2) h1'[symmetric])
      using assms unfolding angle-c-def
      by (smt (verit) congruent-ctriangle-sss(7) angle-c-commute angle-c-def
          cos-minus)
  qed simp-all
  show a1 ≠ b1 ∧ b1 ≠ c1 ∧ a1 ≠ c1
    cdist a1 b1 = cdist a2 b2
    cdist b1 c1 = cdist b2 c2 using assms cdist-commute by auto
qed

lemmas congruent-ctriangle-sas = congruent-ctriangleD[OF congruent-ctriangleI-sas]


lemma congruent-ctriangleI-aas:
  assumes h1:angle-c a1 b1 c1 = angle-c a2 b2 c2
  assumes h2:angle-c b1 c1 a1 = angle-c b2 c2 a2
  assumes h3:cdist a1 b1 = cdist a2 b2
  assumes h4:¬collinear a1 b1 c1 ¬collinear a2 b2 c2
  shows   congruent-ctriangle a1 b1 c1 a2 b2 c2
proof (rule congruent-ctriangleI-sas)
  from h4 have neq: a1 ≠ b1 unfolding collinear-def by auto
  with assms(3) have neq': a2 ≠ b2 by auto
  have h0:‹a1 ≠ b1› ‹b1 ≠ c1› ‹a1 ≠ c1›
    using assms(4) non-collinear-independant by auto
  have h0':‹a2 ≠ b2› ‹b2 ≠ c2› ‹a2 ≠ c2›
    using assms(5) non-collinear-independant by auto
  have A: angle-c c1 a1 b1 = angle-c c2 a2 b2 using neq neq' assms
     apply(insert angle-sum-triangle-c[of a1 b1 c1] angle-sum-triangle-c[of a2 b2
c2])
    by (metis h0 assms(5) h1 h2 non-collinear-independant similar-triangles-c)
  have ‹−∠ (b1 − c1) (a1 − c1) = − ∠ (b2 − c2) (a2 − c2)›
```

**using** *h2* **unfolding** *angle-c-def* **by** *auto*
**then have** *A1*:‹*angle-c a1 c1 b1* = *angle-c a2 c2 b2*›
**using** *h2* **unfolding** *angle-c-def*
**apply**(*cases* ‹*angle-c a1 c1 b1* = *pi*›)
**apply** (*metis ang-vec-def angle-c-def canon-ang-uminus-pi minus-diff-eq*)
**by** (*metis ang-vec-sym ang-vec-sym-pi*)
**have** ‹∠ (*b1* − *a1*) (*c1* − *a1*) = ∠ (*b2* − *a2*) (*c2* − *a2*)›
**by** (*metis ang-vec-sym ang-vec-sym-pi angle-c-def A*)
**have** *sine1*:‹ *sin* (*angle-c a1 c1 b1*) ∗ *cdist c1 b1* = *sin* (*angle-c b1 a1 c1*) ∗ *cdist a1 b1*›
**by** (*metis h0 law-of-sines*)
**have** *sine2*:‹*sin* (*angle-c a2 c2 b2*) ∗ *cdist c2 b2* = *sin* (*angle-c b2 a2 c2*) ∗ *cdist a2 b2*›
**by** (*metis assms*(*5*) *law-of-sines non-collinear-independant*)
**have** ‹¬*collinear a2 b2 c2* ⟹ *sin* (*angle-c a2 c2 b2*) ≠ *0*›
**unfolding** *collinear-def angle-c-def*
**using** *angle-c-def collinear-sin-neq-0 h4*(*2*) **by** *presburger*
**have** *h3'*:‹*cdist b1 a1* = *cdist b2 a2*›
**using** *h3 cdist-commute* **by** *auto*
**have** *A2*:‹*angle-c b1 a1 c1* = *angle-c b2 a2 c2*›
**using** ‹∠ (*b1* − *a1*) (*c1* − *a1*) = ∠ (*b2* − *a2*) (*c2* − *a2*)› *angle-c-def* **by** *auto*
**have** ‹ *cdist c1 b1* = *sin* (*angle-c b2 a2 c2*) ∗ *cdist a2 b2* / *sin* (*angle-c a2 c2 b2*) ∧
*cdist c2 b2* = *sin* (*angle-c b2 a2 c2*) ∗ *cdist a2 b2* / *sin* (*angle-c a2 c2 b2*) ›
**apply**(*cases* ‹ *sin* (*angle-c a2 c2 b2*) = *0*›)
**apply** (*simp add*:
‹¬ *Elementary-Complex-Geometry.collinear a2 b2 c2* ⟹ *sin* (*angle-c a2 c2 b2*) ≠ *0*›
‹¬ *Elementary-Complex-Geometry.collinear a2 b2 c2*›)
**by** (*metis A1 A2 h3 nonzero-mult-div-cancel-left sine1 sine2*)
**then show** *cdist b1 c1* = *cdist b2 c2*
**using** *cdist-commute* **by** *auto*
**show** *a1* ≠ *b1* ∧ *b1* ≠ *c1* ∧ *a1* ≠ *c1*
*cdist a1 b1* = *cdist a2 b2*
*angle-c a1 b1 c1* = *angle-c a2 b2 c2* ∨ *angle-c a1 b1 c1* = − *angle-c a2 b2 c2*
**using** *assms h0 neq* **by** *auto*
**qed**

**lemmas** *congruent-ctriangle-aas* = *congruent-ctriangleD*[*OF congruent-ctriangleI-aas*]

**lemma** *congruent-ctriangleI-asa*:
**assumes** *angle-c a1 b1 c1* = *angle-c a2 b2 c2*
**assumes** *cdist a1 b1* = *cdist a2 b2*
**assumes** *h0*:*angle-c b1 a1 c1* = *angle-c b2 a2 c2*
**assumes** *h4*:¬*collinear a1 b1 c1* ¬*collinear a2 b2 c2*
**shows** *congruent-ctriangle a1 b1 c1 a2 b2 c2*
**proof** (*rule congruent-ctriangleI-aas*)
**from** *assms* **have** *neq*: *a1* ≠ *b1 a2* ≠ *b2* **unfolding** *collinear-def* **by** *auto*

```
  show angle-c b1 c1 a1 = angle-c b2 c2 a2
    apply (rule similar-triangles-c)
    using assms(4,5) non-collinear-independant apply auto[1]
    using h0 unfolding angle-c-def
     apply (metis ang-vec-sym ang-vec-sym-pi)
    using angle-c-def assms(1) by auto
qed fact+


lemmas congruent-ctriangle-asa = congruent-ctriangleD[OF congruent-ctriangleI-asa]


lemma orientation-respect-letter-order:
  assumes angle-c a b c = angle-c b a c ¬ collinear a b c
  shows   False
  by (smt (verit, ccfv-threshold) angle-c-commute assms(1)
      assms(2) collinear-sin-neq-0 collinear-sym1 similar-triangles-c sin-pi sin-zero)


lemma isoscele-iff-pr-cis-qr:
  assumes h':‹q≠r›
  shows ‹(cdist q r = cdist p r) ⟷ (p−r) = cis(angle-c q r p)∗(q−r)›
proof (rule iffI)
  assume h:‹(cdist q r = cdist p r)›
  then have f0:‹cmod(p−r) = cmod(q−r)›
    by (simp add: norm-minus-commute)
  have ‹(p−r)/(q−r) = (p−r)∗cnj(q−r)/cmod(q−r)^2›
    using complex-div-cnj by blast
  have f1:‹Re ((p−r)∗cnj(q−r)) = cmod(p−r)∗cmod(q−r)∗cos(angle-c q r p)›
    by (metis (no-types, lifting)  ang-cos cdist-commute cdist-def h(1))
  have f2: ‹Im ((p−r)∗cnj(q−r)) = cmod(p−r)∗cmod(q−r)∗ sin(angle-c q r p)›
    using ang-sin h(1) by auto
 then have ‹ (p−r)∗cnj(q−r)/cmod(q−r)^2 = cmod(p−r)∗cmod(q−r)∗cis(angle-c
q r p)/cmod(q−r)^2›
    apply(intro complex-eqI)
    using f1 f2 by auto
 then have f3:‹(p−r)/(q−r) = cmod(p−r)∗cmod(q−r)∗cis(angle-c q r p)/cmod(q−r)^2›
    by (simp add: ‹(p − r) / (q − r) = (p − r) ∗ cnj (q − r) / cor ((cmod (q −
r))²)›)
  have ‹(p−r)/(q−r) = cis (angle-c q r p)›
    apply(subst f3) apply(subst f0) by(simp add:h h' power2-eq-square)
  then show ‹p − r = cis (angle-c q r p) ∗ (q − r)›
    using divide-eq-eq h h' by force
next
  assume ‹p − r = cis (angle-c q r p) ∗ (q − r) ›
  have ‹(p−r)/(q−r) = (p−r)∗cnj(q−r)/cmod(q−r)^2›
    using complex-div-cnj by blast
  have f1:‹Re ((p−r)∗cnj(q−r)) = cmod(p−r)∗cmod(q−r)∗cos(angle-c q r p)›
    using ang-cos by fastforce
```

```
  have f2: ‹Im ((p−r)∗cnj(q−r)) = cmod(p−r)∗cmod(q−r)∗ sin(angle-c q r p)›
    using ang-sin by fastforce
  then have f4:‹ (p−r)∗cnj(q−r)/cmod(q−r)^2 = cmod(p−r)∗cmod(q−r)∗cis(angle-c
q r p)/cmod(q−r)^2›
    apply(intro complex-eqI)
    using f1 f2 by auto
  then have f3:‹(p−r)/(q−r) = cmod(p−r)∗cmod(q−r)∗cis(angle-c q r p)/cmod(q−r)^2›
    by (simp add: ‹(p − r) / (q − r) = (p − r) ∗ cnj (q − r) / cor ((cmod (q −
r))²)›)
  have ‹cmod(p−r)∗cmod(q−r)∗cis(angle-c q r p)/cmod(q−r)^2 = cis (angle-c q
r p)›
    using ‹p − r = cis (angle-c q r p) ∗ (q − r)› ‹q ≠ r› f3 by auto
  then have ‹cmod(p−r)∗cmod(q−r)/cmod(q−r)^2 = 1›
     by (metis f4 ‹q ≠ r› cmod-power2 complex-mod-cnj complex-mod-mult-cnj
complex-mult-cnj
      eq-iff-diff-eq-0 nonzero-norm-divide norm-cis norm-eq-zero norm-mult power-not-zero)
  then show ‹cdist q r = cdist p r›
   by (metis cdist-commute cdist-def eq-divide-eq-1 mult.commute power2-eq-square
real-divide-square-eq)
qed


lemma equilateral-imp-pi3:
  assumes ‹q≠r› cdist q r = cdist p r cdist p r = cdist p q
  shows ⌊(angle-c q r p)⌋ = pi/3 ∨ ⌊(angle-c q r p)⌋ = −pi/3
    (angle-c q r p) = (angle-c p q r) ∧ (angle-c q r p) = (angle-c r p q)
proof −
  have f0:‹q≠r ∧ r≠p ∧ p≠q› using assms by(auto)
  have ‹angle-c q r p ≠ 0 ∧ angle-c q r p ≠ pi›
     by (smt (verit, best) Re-complex-of-real angle-c-commute angle-c-commute-pi
assms(1) assms(2)
        assms(3) cdist-commute cdist-def congruent-ctriangle-sss(20) cor-cmod-real
isoscele-iff-pr-cis-qr
      minus-complex.simps(1) minus-complex.simps(2) minus-diff-eq mult-minus-right
norm-eq-zero right-minus-eq)
  then have f1:‹¬collinear q r p›
    using collinear-angle f0 by blast
  then have f1':‹¬collinear p q r›
    by (simp add: collinear-sym1 collinear-sym2)
  have f12:‹(angle-c q r p) = angle-c r p q ∨ (angle-c q r p) = − angle-c r p q›
    apply(rule congruent-ctriangle-sss)
    using f0 assms by(auto simp:norm-minus-commute)
  have f4:‹(angle-c q r p) = angle-c p q r ∨ (angle-c q r p) = − angle-c p q r›
    apply(rule congruent-ctriangle-sss)
    using f0 assms by(auto simp:norm-minus-commute)
  have f5:‹(angle-c q r p) ≠ pi›
    using f1
   by (metis angle-c-commute canon-ang-sin canon-ang-uminus-pi collinear-sin-neq-0
f1' pi-canonical
```

```
      sin-pi)
  from f4 have ‹ ⌊(angle-c q r p) +  (angle-c r p q) + (angle-c p q r)⌋ = pi›
    using angle-sum-triangle-c[of p q r] f0 by auto
  have ‹∀x∈{−pi..0}. 3∗x ≠ pi›
   by (metis atLeastAtMost-iff dual-order.trans mult-eq-0-iff mult-less-cancel-right2
numeral-le-one-iff
      pi-ge-zero pi-neq-zero semiring-norm(70) verit-comp-simplify1(3) verit-la-disequality)
  also have the-x:‹∃!x∈{−pi<..pi}. 3∗x = pi›
    by(auto intro:exI[where x=‹pi/3›])
  moreover have ‹(angle-c x y z)  ∈{−pi<..pi}› for x y z
    using ang-c-in by auto
  have ‹x∈{−3∗pi<..3∗pi} ∧ x−2∗pi = pi ⟹ ( x = 3∗pi)› for x k::real
    by(auto simp:)
  have ‹x∈{−3∗pi<..3∗pi} ∧ x−2∗k∗pi = pi ∧ ( k≥2 ∨ k≤ − 2) ⟹ False› for
x and k::int
  proof −
    assume h:‹x∈{−3∗pi<..3∗pi} ∧ x−2∗k∗pi = pi ∧ ( k≥2 ∨ k≤ −2)›
    then have f0:‹ r = 2 ⟹ x − 2∗ r∗pi ≤ −pi› for r by auto
    also have f1:‹k > 2 ⟹ x − 2∗ k∗pi < x−2∗2∗pi›
      by(auto)
    have f3:‹x−2∗2∗pi ≤ −pi›
      using f0[of 2] by(auto simp:abs-real-def)
    then have f2:‹ k > 2 ⟹ x − 2∗ k∗pi < −pi›
      using f1 by argo
    ultimately have f3:‹ k ≥2 ⟹ x−2∗ k∗pi ≤ −pi›
      apply(cases ‹abs k=2›) using f0 by(auto)
    have f0':‹r=−2 ⟹ x − 2∗r∗pi > pi› for r ::int
      using h by(auto)
    have ‹x+2∗2∗pi > pi› using h by auto
    also have ‹k< −2 ⟹ x −2∗k∗pi > x − 2∗(−2)∗pi›
      using f0'[of 2]
    proof −
      assume a1: k < − 2
      have f2: pi < x − real-of-int (− 4) ∗ pi
        using f0' by force
      have k + k ≤ − 4
        using a1 by force
      then show ?thesis
         using f2 by (metis (no-types) diff-left-mono h mult.commute mult-2-right
of-int-le-iff ordered-comm-semiring-class.comm-mult-left-mono pi-ge-zero verit-comp-simplify1(3))
    qed
    ultimately have f4:‹k ≤ −2 ⟹ x−2∗ k∗pi > pi›
      apply(cases ‹k=−2›) using h
        ‹k < − 2 ⟹ x − 2 ∗ − 2 ∗ pi < x − real-of-int (2 ∗ k) ∗ pi› by auto
    show False
      using f3 f4 h by force
  qed
  then have possible-k:‹x∈{−3∗pi<..3∗pi} ∧ x−2∗k∗pi = pi ⟹ k=−1 ∨ k=1
∨ k=0› for k::int and x
```

**by** *force*
**have** *ang-3-in*:‹$y \in \{-3*pi<..3*pi\} \wedge \lfloor y\rfloor = pi \longleftrightarrow y=pi \vee y = 3*pi \vee y=-pi$›
**for** *y*
**proof**(*rule iffI*)
**assume** *hyp*:‹$y \in \{- 3 * pi<..3 * pi\} \wedge \lfloor y\rfloor = pi$›
**have** ‹$\exists k.\ y - 2*k*pi = pi$›
**by** (*metis add-diff-cancel-left′ diff-add-cancel divide-self-if field-sum-of-halves mult.commute mult-1 mult-2*
*pi-neq-zero times-divide-eq-right*)
**with** *hyp* **show** ‹$y = pi \vee y=3*pi \vee y= -pi$›
**using** *hyp possible-k canon-ang-pi-3pi*[*of y*] *canon-ang-id canon-ang-uminus-pi*

**by** (*smt* (*verit*, *del-insts*) *canon-ang-minus-3pi-minus-pi greaterThanAtMost-iff*)
**next**
**assume** ‹$y = pi \vee y = 3 * pi \vee y = - pi$›
**then show** ‹$y \in \{- 3 * pi<..3 * pi\} \wedge \lfloor y\rfloor = pi$›
**by**(*auto simp*:*canon-ang-uminus-pi canon-ang-pi-3pi*)
**qed**
**ultimately have** ‹$\lfloor 3*($*angle-c q r p*$)\rfloor = pi \Longrightarrow$ *angle-c q r p* $= -pi/3 \vee$ *angle-c q r p* $= pi/3$›
**proof** −
**assume** ‹$\lfloor 3*($*angle-c q r p*$)\rfloor = pi$›
**have** ‹$3*($*angle-c q r p*$) \in \{-3*pi<..3*pi\}$›
**using** *ang-c-in* **by** *auto*
**then have** *f0*:‹$3*($*angle-c q r p*$) = -pi \vee 3*($*angle-c q r p*$) = pi \vee 3*($*angle-c q r p*$) = 3*pi$›
**using** *ang-3-in* ‹$\lfloor 3*($*angle-c q r p*$)\rfloor = pi$› **by** *blast*
**then have** ‹$3*($*angle-c q r p*$) = 3*pi \Longrightarrow$ *False*›
**using** *collinear-sin-neq-pi* **by** (*simp add*: *f5*)
**then show** ‹*angle-c q r p* $= -pi/3 \vee$ *angle-c q r p* $= pi/3$›
**using** *f0* **by** *auto*
**qed**
**have** *f2*:‹$\lfloor$*angle-c x y z* $\rfloor$ = *angle-c x y z*› **for** *x y z*::*complex*
**using** *ang-vec-bounded angle-c-def canon-ang-id* **by** *fastforce*
**then have** ‹$\lfloor 3*($*angle-c q r p*$)\rfloor = pi$›
**apply** (*cases* ‹(*angle-c q r p*) = − *angle-c r p q*›)
**apply**(*simp add*:*f1 f2*)
**apply** (*metis add.commute add.right-inverse add.right-neutral angle-sum-triangle-c canon-ang-uminus-pi f0 f2 f4 f5*)
**by** (*smt* (*verit*) ‹$\lfloor$*angle-c q r p* + *angle-c r p q* + *angle-c p q r*$\rfloor = pi$›
‹*angle-c q r p* = *angle-c p q r* ∨ *angle-c q r p* = − *angle-c p q r*›
‹*angle-c q r p* = *angle-c r p q* ∨ *angle-c q r p* = − *angle-c r p q*› *canon-ang-uminus-pi*
*f2*)
**then have** ‹*angle-c q r p* $= -pi/3 \vee$ *angle-c q r p* $= pi/3$›
**using** ‹$\lfloor 3 *$ *angle-c q r p*$\rfloor = pi \Longrightarrow$ *angle-c q r p* $= - pi / 3 \vee$ *angle-c q r p* $= pi / 3$› **by** *auto*
**then show** ‹$\lfloor$*angle-c q r p*$\rfloor = pi / 3 \vee \lfloor$*angle-c q r p*$\rfloor = - pi / 3$›
**using** *f2* **by** *auto*

**then show** ‹*angle-c q r p* = *angle-c p q r* ∧ *angle-c q r p* = *angle-c r p q*›
**by** (*smt* (*verit*, *del-insts*) ‹⌊*angle-c q r p* + *angle-c r p q* + *angle-c p q r*⌋ = *pi*›

*f12 collinear-sin-neq-0 collinear-sym1 f1' f2 f4 f5 sin-pi*)
**qed**

**lemma** *isosceles-triangle-converse*:
**assumes** *angle-c a b c* = *angle-c c a b* ¬*collinear a b c*
**shows** *dist a c* = *dist b c*
**by** (*metis assms*(*1*) *assms*(*2*) *cdist-def collinear-sin-neq-0 collinear-sym1 congruent-ctriangle-asa*(*8*)
*congruent-ctriangle-asa*(*9*) *dist-complex-def law-of-sines mult-cancel-left non-collinear-independant*)

**lemma** *pi3-imp-equilateral*:
**assumes** ‹*q*≠*r*› ‹*p*≠*q*› ‹*r*≠*p*›
**and** ‹(*angle-c q p r*) = *pi*/*3* ∨ (*angle-c q p r*) = −*pi*/*3*›
**and** ‹(*angle-c q p r*) = (*angle-c r q p*)›
**and** ‹(*angle-c q p r*) = (*angle-c p r q*)›
**shows** ‹*cdist p r* = *cdist q r* ∧ *cdist p r* = *cdist p q*›
**proof**(*safe*)
**have** *f0*:‹¬ *collinear q p r*›
**using** *assms*(*1*−*3*) **unfolding** *collinear-def*
**by** (*smt* (*verit*, *ccfv-SIG*) *arccos-one-half arcsin-one-half arcsin-plus-arccos assms*(*4*)
*collinear-angle collinear-def divide-le-eq-1-pos divide-pos-pos minus-divide-left pi-def pi-gt-zero pi-half*)
**then show** ‹*cdist p r* = *cdist q r*›
**using** *isosceles-triangle-converse*[*of q p r*] *assms*
**by** (*simp add*: *dist-complex-def norm-minus-commute*)
**show** ‹*cdist p r* = *cdist p q* ›
**using** *f0 isosceles-triangle-converse*[*of p r q*] *assms*
**by** (*metis* ‹*cdist p r* = *cdist q r*› *cdist-def collinear-iff dist-commute dist-norm*)
**qed**

**lemma** *pi3-isoscele-imp-equilateral*:
**assumes** ‹*q*≠*r*› ‹*p*≠*q*› *cdist q r* = *cdist p r*
**and** ⌊(*angle-c q p r*)⌋ = *pi*/*3* ∨ ⌊(*angle-c q p r*)⌋ = −*pi*/*3*
**shows** ‹*cdist p q* = *cdist r q*›
**proof**( − )
**have** *f*:‹*angle-c p q r* = (*angle-c r p q*)›
**by** (*smt* (*verit*, *best*) *angle-c-commute angle-c-commute-pi assms*(*2*) *assms*(*3*) *cdist-commute*
*collinear-angle isosceles-triangles orientation-respect-letter-order*)
**have** *f0*:‹*r*≠*p*›
**using** *assms*(*1*) *assms*(*3*) **by** *force*
**then have** ‹*cdist q p* = *cdist r q*›
**by** (*smt* (*verit*) *congruent-ctriangle-sss*(*19*) *add-divide-distrib ang-vec-bounded*

*angle-c-commute angle-c-def angle-sum-triangle-c arccos-minus-one-half arccos-one-half arcsin-minus-one-half*
*arcsin-one-half arcsin-plus-arccos assms(2) assms(3) assms(4) canon-ang-id canon-ang-minus-3pi-minus-pi*
*cdist-commute field-sum-of-halves law-of-sines mult-cancel-right pi3-imp-equilateral sin-gt-zero sin-periodic-pi*)
**then show** ‹*cdist p q = cdist r q*›
**using** *cdist-commute* **by** *force*
**qed**

**lemma** *pi3-isoscele-imp-equilateral'*:
**assumes** ‹*q≠r*› ‹*p≠q*› *cdist q r = cdist p r*
**and** ⌊(*angle-c q p r*)⌋ = *pi/3* ∨ ⌊(*angle-c q p r*)⌋ = −*pi/3*
**shows** ‹*cdist p r = cdist p q*›
**by** (*metis assms(2) assms(3) assms(4) cdist-commute pi3-isoscele-imp-equilateral*)

**lemma** *equilateral-caracterization*:‹*q≠r* ⟹ (*cdist q r = cdist p r* ∧ *cdist p r = cdist p q*)
⟷ ((*p−r*) = *cis*(*pi/3*)∗(*q−r*) ∨ (*p−r*) = *cis*(−*pi/3*)∗(*q−r*))›
**proof**(*rule iffI*)
**assume** *h*: ‹*q ≠ r*› ‹*cdist q r = cdist p r* ∧ *cdist p r = cdist p q* ›
**then have** *f1*:‹*p − r = cis* (*angle-c q r p*) ∗ (*q − r*) ∨ *p − r = cis* (− *angle-c q r p*) ∗ (*q − r*)›
**using** *isoscele-iff-pr-cis-qr* **by** *blast*
**have** *f0*:‹*q≠r* ∧ *r ≠ p* ∧ *p ≠ q*› **using** *h* **by** *auto*
**have** ‹*angle-c q r p = pi/3* ∨ *angle-c q r p* = −*pi/3*›
**using** *equilateral-imp-pi3(1)[of q r p]*
**by** (*metis ang-vec-bounded angle-c-def canon-ang-id h(1) h(2)*)
**then show** ‹*p − r = cis* (*pi / 3*) ∗ (*q − r*) ∨ *p − r = cis* (− *pi / 3*) ∗ (*q − r*)›
**using** *f1*
**by** (*metis add.inverse-inverse minus-divide-left*)
**next**
**assume** *h*:‹*q ≠ r*› ‹*p − r = cis* (*pi / 3*) ∗ (*q − r*) ∨ *p − r = cis* (− *pi / 3*) ∗ (*q − r*)›
**have** ‹(*p−r*)/(*q−r*) = (*p−r*)∗*cnj*(*q−r*)/*cmod*(*q−r*)^*2*›
**using** *complex-div-cnj* **by** *blast*
**have** *f1*:‹*Re* ((*p−r*)∗*cnj*(*q−r*)) = *cmod*(*p−r*)∗*cmod*(*q−r*)∗*cos*(*angle-c q r p*)›
**using** *ang-cos* **by** *force*
**have** *f2*: ‹*Im* ((*p−r*)∗*cnj*(*q−r*)) = *cmod*(*p−r*)∗*cmod*(*q−r*)∗ *sin*(*angle-c q r p*)›
**using** *ang-sin h(1)* **by** *auto*
**then have** ‹ (*p−r*)∗*cnj*(*q−r*)/*cmod*(*q−r*)^*2* = *cmod*(*p−r*)∗*cmod*(*q−r*)∗*cis*(*angle-c q r p*)/*cmod*(*q−r*)^*2*›
**apply**(*intro complex-eqI*)
**using** *f1 f2* **by** *auto*
**then have** *f3*:‹(*p−r*)/(*q−r*) = *cmod*(*p−r*)∗*cmod*(*q−r*)∗*cis*(*angle-c q r p*)/*cmod*(*q−r*)^*2*›
**by** (*simp add*: ‹(*p − r*) / (*q − r*) = (*p − r*) ∗ *cnj* (*q − r*) / *cor* ((*cmod* (*q −

*r*))$^2$)›)
  **have** ‹(*p*−*r*)/(*q*−*r*) = *cis* (*angle-c q r p*)›
    **apply**(*subst f3*)
    **by** (*smt* (*z3*) ‹(*p* − *r*) ∗ *cnj* (*q* − *r*) / *cor* ((*cmod* (*q* − *r*))$^2$) = *cor* (*cmod* (*p* − *r*) ∗ *cmod* (*q* − *r*))
∗ *cis* (*angle-c q r p*) / *cor* ((*cmod* (*q* − *r*))$^2$)› *cis-divide cis-mult cis-neq-zero cis-zero*
      *cmod-cor-divide complex-mod-cnj eq-iff-diff-eq-0 f3 h*(*1*) *h*(*2*) *nonzero-mult-divide-mult-cancel-left*

        *nonzero-mult-divide-mult-cancel-right norm-cis norm-mult numeral-One of-real-1 of-real-divide*
        *power2-less-0 times-divide-eq-left*)
  **then have** *fpi*:‹*angle-c q r p* = *pi*/*3* ∨ *angle-c q r p* = −*pi*/*3*›
    **using** *h ang-c-in*
   **by** (*smt* (*verit*, *best*) *add-divide-distrib ang-vec-bounded angle-c-def arccos-one-half*

      *arcsin-one-half arcsin-plus-arccos cis-inj diff-eq-diff-eq diff-numeral-special*(*9*)

        *divide-nonneg-pos field-sum-of-halves nonzero-divide-eq-eq*)
  **then have** *ff*:‹*cdist p r* = *cdist q r*›
    **by** (*metis* ‹(*p* − *r*) / (*q* − *r*) = *cis* (*angle-c q r p*)› *h*(*1*) *isoscele-iff-pr-cis-qr nonzero-divide-eq-eq right-minus-eq*)
  **then have** ‹*angle-c p q r* = *angle-c q p r* ∨ *angle-c p q r* = −*angle-c q p r* ›
      **by** (*metis congruent-ctriangle-sss*(*13*) *cdist-commute cdist-def norm-eq-zero right-minus-eq*)
  **then have** *f10*:‹⌊*angle-c p q r* + *angle-c q r p* + *angle-c r p q*⌋ = *pi*›
     **by** (*metis* ‹*angle-c q r p* = *pi* / *3* ∨ *angle-c q r p* = − *pi* / *3*› *angle-c-neq0 angle-sum-triangle-c*
        *divide-eq-0-iff h*(*1*) *neg-equal-0-iff-equal pi-neq-zero zero-neq-numeral*)
  **then have** ‹⌊*angle-c q p r*⌋ = *pi* / *3* ∨ ⌊*angle-c q p r*⌋ = − *pi* / *3*›
    **by** (*smt* (*verit*) ‹*angle-c p q r* = *angle-c q p r* ∨ *angle-c p q r* = − *angle-c q p r*›
        *add-divide-distrib ang-pos-pos ang-vec-bounded ang-vec-sym angle-c-def arccos-minus-one-half*
          *arccos-one-half arcsin-minus-one-half arcsin-one-half arcsin-plus-arccos canon-ang-id*
        *canon-ang-minus-3pi-minus-pi canon-ang-pi-3pi field-sum-of-halves fpi*)
  **show** ‹*cdist q r* = *cdist p r* ∧ *cdist p r* = *cdist p q*›
    **apply**(*rule conjI*)
    **using** *ff* **apply**(*auto simp*: *norm-minus-commute*)[*1*]
    **apply**(*rule pi3-isoscele-imp-equilateral′*)
    **using** *h*(*1*) **apply** *auto*[*1*]
      **apply** (*metis* ‹*angle-c q r p* = *pi* / *3* ∨ *angle-c q r p* = − *pi* / *3*›
      *angle-c-neq0 divide-eq-0-iff neg-equal-0-iff-equal pi-neq-zero zero-neq-numeral*)
    **using** *ff* **apply** *presburger*
    **using** ‹⌊*angle-c q p r*⌋ = *pi* / *3* ∨ ⌊*angle-c q p r*⌋ = − *pi* / *3*› **by** *blast*
**qed**

**lemmas** *equilateral-imp-prcispi3* = *equilateral-caracterization*[*THEN iffD1*]

**lemmas** *prcispi3-imp-equilateral* = *equilateral-caracterization*[*THEN iffD2*]

**lemma** *equilateral-caracterization-neg*:
  **fixes** *p q r*::*complex*
  **assumes** *h1*:‹*p*≠*r*›
  **shows** ‹(*cdist p r* = *cdist p q* ∧ *cdist p q* = *cdist q r* ∧ *angle-c q r p* = −*pi*/*3*)
⟷ *p* + *root3* ∗ *q* + *root3* ^ *2* ∗ *r* = *0*›
**proof**(*rule iffI*)
  **assume** *h*:‹*cdist p r* = *cdist p q* ∧ *cdist p q* = *cdist q r* ∧ *angle-c q r p* = −*pi*/*3*›
  **then have** ‹*p*−*r* = *cis*(−*pi*/*3*)∗(*q*−*r*)›
    **using** *h1 isoscele-iff-pr-cis-qr*[*of q r p*]
    **by** *force*
  **with** *h* **have** ‹*p*−*r*+*cis*(*2*∗*pi*/*3*)∗(*q*−*r*) = *0*›
    **apply**(*simp add*:*cis.code*)
    **apply**(*subst sin-120*)
    **apply**(*intro complex-eqI*)
    **using** *cos-120 cos-60 sin-120 sin-60*
    **by**(*auto simp add*:*field-simps*)
  **then have** ‹*p* + *root3*∗*q* − (*root3*+*1*)∗*r* = *0*› **by**(*auto simp*:*field-simps root3-def*)
  **then show** ‹*p* + *root3* ∗ *q* + *root3*^*2* ∗ *r* = *0*›
    **by** (*metis* (*no-types*, *lifting*) *add.commute eq-iff-diff-eq-0 left-diff-distrib*
        *ring-class.ring-distribs*(*2*) *root3-eq-0*)
**next**
  **assume** *h*:‹*p* + *root3* ∗ *q* + *root3*^*2* ∗ *r* = *0*›
  **then have** ‹*p* + *root3*∗*q* − (*root3*+*1*)∗*r* = *0*›
   **by** (*metis add.commute add-diff-cancel-left diff-numeral-special*(*9*) *left-diff-distrib*
*ring-class.ring-distribs*(*2*)
       *root3-eq-0*)
  **then have** ‹*p*−*r*+*root3*∗(*q*−*r*) = *0*›
    **by**(*auto simp*:*field-simps*)
  **have** ‹−*root3* = *cis*(−*pi*/*3*)›
    **using** *cos-120 cos-60 sin-120 sin-60*
    **by** (*auto intro*:*complex-eqI simp*: *root3-def*)
  **then have** *f1'*:‹*p* − *r* = *cis*(−*pi*/*3*)∗(*q*−*r*)›
    **by** (*metis* ‹*p* − *r* + *root3* ∗ (*q* − *r*) = *0*› *eq-neg-iff-add-eq-0 mult-minus-left*)
  **then have** *f1*:‹*p* − *r* = *cis*(*pi*/*3*)∗(*q*−*r*) ∨ *p* − *r* = *cis*(−*pi*/*3*)∗(*q*−*r*)›
    **by** *auto*
  **then have** *f2*:‹*cmod* (*p*−*r*) = *cmod* (*q*−*r*)›
    **by**(*auto simp*:*norm-mult*)
  **then have** *f3*:‹*dist p r* = *dist q r*›
    **by** (*simp add*: *dist-complex-def*)
  **have** *ang-dem*:‹*angle-c q r p* = −*pi*/*3*›
  **proof** −
    **have** ∗:‹*q*≠*r*›
      **using** *f2 h1* **by** *auto*
    **then have** ∗∗:‹(*p* − *r*) / (*q*−*r*) = *cis*(−*pi*/*3*)›
      **using** *f1'* **by** *auto*
    **have** ∗∗∗:‹ *angle-c q r p* = *Arg* ((*p* − *r*) /(*q*−*r*))›

**by** (*metis ang-vec-def angle-c-def arg-div f1' h1 mult-zero-right right-minus-eq*)
**then have** ‹*Arg* ((*p* − *r*) /(*q*−*r*)) = ⌊−*pi*/*3*⌋›
**by** (*simp add*:*arg-cis* ∗∗)
**then show** *?thesis* **using** ∗ ∗∗ ∗∗∗
**using** *canon-ang-id pi-ge-two* **by** *force*
**qed**
**have** ‹*cmod* (*q* − *p*) = *cmod* (*r* − *q*)›
**using** *f2*
**by** (*metis cdist-def dist-eq-0-iff equilateral-caracterization f1' f3*)
**then show** ‹*cdist p r* = *cdist p q* ∧ *cdist p q* = *cdist q r* ∧ *angle-c q r p* = −*pi*/*3*›

**using** *equilateral-caracterization*[*THEN iffD2*, *of q r p*, *OF - f1*]
**using** *f2 ang-dem* **by**(*auto simp*: *f2 field-simps intro*!:)
**qed**

**end**
**theory** *Complex-Axial-Symmetry*

**imports** *Complex-Angles Complex-Triangles*

**begin**

# 6 Axial symmetry in complex field

In the following we define the axial symmetry and prove basics properties.

**context**
**fixes** *z1 z2* :: *complex* **and** α β :: *complex*
**assumes** *neq0*:‹*z1*≠*z2*›
**defines** ‹α ≡ (*z1*−*z2*)/(*cnj z1* − *cnj z2*)›
**defines** ‹β ≡ (*z2*∗*cnj z1* − *z1*∗*cnj z2*)/(*cnj z1* − *cnj z2*)›
**begin**

**definition** *axial-symmetry*::‹*complex* ⇒ *complex*› **where**
‹*axial-symmetry z* ≡ *cnj z*∗(*z1* − *z2*)/(*cnj z1* − *cnj z2*) + (*z2*∗*cnj z1* − *z1*∗*cnj z2*)/(*cnj z1* − *cnj z2*)›

**lemma** *norm-α-eq-1*:‹*cmod* (α) = *1*›
**by** (*auto simp*: *neq0 α-def β-def*)

**lemma** *z1-inv*:‹*axial-symmetry z1* = *z1*›
**unfolding** *axial-symmetry-def*
**by** (*metis* (*mono-tags*, *lifting*) *add-diff-eq add-divide-distrib complex-cnj-cancel-iff*

*diff-add-cancel eq-iff-diff-eq-0 mult.commute neq0 nonzero-eq-divide-eq right-diff-distrib'*)

**lemma** *z2-inv*:‹*axial-symmetry z2* = *z2*›

**unfolding** *axial-symmetry-def*
**by** (*smt* (*z3*) *add-diff-cancel-left add-diff-cancel-left′ add-divide-distrib axial-symmetry-def*

*complex-cnj-cancel-iff diff-add-cancel divide-divide-eq-right eq-divide-imp mult-commute-abs right-minus-eq z1-inv*)

**lemma** *cmod-axial*:‹*cmod* (*axial-symmetry z* − *axial-symmetry z′*) = *cmod* (α∗(*cnj z* − *cnj z′*))›
**unfolding** *axial-symmetry-def*
**by** (*auto simp*: α*-def* β*-def*) (*simp add*: *diff-divide-distrib mult.commute right-diff-distrib′*)

**lemma** *cmod-axial-inv*:‹*cmod* (*axial-symmetry z* − *axial-symmetry z′*) = *cmod* (*z* − *z′*)›
**by** (*metis cmod-axial complex-cnj-diff complex-mod-cnj mult.commute mult.right-neutral norm-*α*-eq-1 norm-mult*)

**lemma** *axial-symmetry-dist1*:‹*cdist z1 z* = *cdist z1* (*axial-symmetry z*)›
**by** (*metis cdist-def cmod-axial-inv z1-inv*)

**lemma** *axial-symmetry-dist2*:‹*cdist z2 z* = *dist z2* (*axial-symmetry z*)›
**by** (*metis cdist-def cmod-axial-inv dist-commute dist-norm z2-inv*)

**lemma** αβ: ‹α∗*cnj* β + β = *0*›
**by** (*simp add*: *add-divide-eq-iff* α*-def* β*-def*)

**lemma** *involution-symmetry*:‹*axial-symmetry* (*axial-symmetry z*) = *z*›
**proof** −
**have** ‹α∗*cnj*(α∗*cnj z* + β) + β = α∗*cnj* α∗*z* + α∗*cnj* β +β›
**by** (*simp add*: *ring-class.ring-distribs*(*1*))
**then show** *?thesis* **unfolding** *axial-symmetry-def*
**by** (*smt* (*verit*, *best*) αβ *add.right-neutral cmod-eq-one group-cancel.add1 mult.commute mult-cancel-right2 norm-*α*-eq-1 times-divide-eq-left* α*-def* β*-def*)
**qed**

**lemma** *arg-*α:‹*Arg* α = ↓*2*∗*Arg* (*z1*−*z2*)↓›
**unfolding** α*-def* β*-def*
**by** (*smt* (*verit*, *del-insts*) *arg-cnj-not-pi arg-div arg-pi-iff canon-ang-id canon-ang-pi-3pi*

*complex-cnj-cancel-iff complex-cnj-diff eq-cnj-iff-real eq-iff-diff-eq-0 neq0 pi-ge-two*)

**lemma** *Arg-invol*:‹*Arg* (*axial-symmetry* (*axial-symmetry z*)) = *Arg z*›
**by** (*simp add*: *involution-symmetry*)

**lemma** *angle-sum-symmetry*:‹*z*≠*z1* ⟹ ↓*angle-c z z1 z2* + *angle-c z2 z1* (*axial-symmetry z*)↓ = *angle-c z z1* (*axial-symmetry z*)›
**proof** −
**assume** ‹*z*≠*z1*›

```
  have ‹z2−z1 ≠ 0› using angle-c-sum neq0 by auto
  have ‹z1 − (axial-symmetry z) ≠ 0›
    by (metis ‹z ≠ z1› eq-iff-diff-eq-0 involution-symmetry z1-inv)
  then show ?thesis
    using angle-c-sum
    by (metis ‹z2 − z1 ≠ 0› right-minus-eq z1-inv)
qed

lemma angle-symmetry-eq-imp:
  assumes h:‹z1≠z› ‹z2≠z›
  shows‹angle-c z z1 z2 = − angle-c (axial-symmetry z) z1 z2 ∨ angle-c z z1 z2
= angle-c (axial-symmetry z) z1 z2›
  by (metis (mono-tags, lifting) axial-symmetry-dist1 cdist-def cmod-axial-inv
          congruent-ctriangle-sss(22) h(1) neq0 z2-inv)

lemma angle-symmetry:
  assumes h:‹z1≠z› ‹z2≠z›
    and ‹angle-c z z1 z2 = angle-c (axial-symmetry z) z1 z2›
  shows ‹z = axial-symmetry z›
proof −
  have ‹cmod (z − z1) = cmod (axial-symmetry z − z1)›
    using axial-symmetry-dist1 by auto
  have ‹cmod (z − z2) = cmod (axial-symmetry z − z2)›
    by (metis axial-symmetry-dist2 cdist-def dist-commute dist-complex-def)
  have ‹z−z1 = axial-symmetry z−z1›
     by (metis ‹cmod (z − z1) = cmod (local.axial-symmetry z − z1)› ang-cos
ang-sin assms(3)
        complex-cnj-cnj complex-eq-iff eq-iff-diff-eq-0 mult-cancel-left neq0)
  then show ?thesis
    by simp
qed

lemma line-is-inv:‹z∈ line z1 z2 ∧ z≠z2 ∧ z≠z1 ⟹ z = axial-symmetry z›
proof −
  assume ‹z∈ line z1 z2 ∧ z≠z2 ∧ z≠z1›
  then have ‹angle-c z z1 z2 = 0 ∨ angle-c z z1 z2 = pi›
    unfolding line-def using neq0 collinear-angle
    using collinear-sym1 collinear-sym2' by blast
  then show ?thesis
    by (smt (verit) ‹z ∈ line z1 z2 ∧ z ≠ z2 ∧ z ≠ z1›
      angle-c-commute angle-c-commute-pi angle-symmetry angle-symmetry-eq-imp)
qed

lemma dist-inv:‹cdist a b = cdist (axial-symmetry a) (axial-symmetry b)›
  by (simp add: cmod-axial-inv)

lemma collinear-inv: assumes ‹collinear a b c› and ‹a ≠ b∧ b ≠ c ∧ c≠a›
  shows ‹collinear (axial-symmetry a) (axial-symmetry b) (axial-symmetry c)›
proof −
```

**have** ‹*angle-c a b c = pi* ∨ *angle-c a b c = 0*›
**using** *assms(1) assms(2) collinear-angle* **by** *blast*
**then have** ‹*angle-c (axial-symmetry a) (axial-symmetry b) (axial-symmetry c) = angle-c a b c*
∨ *angle-c (axial-symmetry a) (axial-symmetry b) (axial-symmetry c) = − angle-c a b c*›
**by** (*metis congruent-ctriangle-sss(24) assms(2) dist-inv minus-equation-iff*)
**then show** *?thesis*
**using** ‹*angle-c a b c = pi* ∨ *angle-c a b c = 0*› *collinear-sin-neq-0 collinear-sym1* **by** *fastforce*
**qed**

**lemma** *axial-symmetry-eq-line*:‹*z≠z1* ∧ *z≠z2* ⟹ *z = axial-symmetry z* ⟹ *z* ∈ *line z1 z2*›
**proof** −
**assume** ‹*z≠z1* ∧ *z≠z2*› ‹*z = axial-symmetry z*›
**then have** *g0*:‹*z = α∗cnj z + β*› **unfolding** *axial-symmetry-def*
**by** (*simp add: mult.commute α-def β-def*)
**then have** *g1*:‹*cnj z = (z−β)∗cnj α* ›
**by** (*smt (verit, ccfv-threshold) add-diff-cancel complex-cnj-cnj complex-cnj-diff α-def β-def*
*complex-cnj-divide mult.commute neq0 nonzero-eq-divide-eq right-minus-eq times-divide-eq-left*)
**also have** *g2*:‹*z = (cnj z − cnj β)∗α*›
**by** (*metis calculation complex-cnj-cnj complex-cnj-diff complex-cnj-mult*)
**have** ‹*α/(z1−z2) = 1 /(cnj z1 − cnj z2)*›
**by** (*auto simp: α-def β-def*)
**have** ‹*Im w = (w−cnj w)/(2∗*i)› **for** *w*
**using** *Im-express-cnj* **by** *blast*
**then have** ‹*Im (z2∗cnj z1) = (z2∗cnj z1 − cnj (z2∗cnj z1))/(2∗*i)›
**by** *presburger*
**then have** ‹*Im (z2∗cnj z1)∗2∗*i = (*z2∗cnj z1 − cnj z2∗ z1*)›
**by** (*metis complex-cnj-cnj complex-cnj-mult complex-diff-cnj mult.commute*)
**have** *f0*:‹*(cnj z1 − cnj z2)∗(z1−z2) = cmod z1^2 + cmod z2^2 − cnj z1 ∗ z2 − cnj z2 ∗ z1*›
**by** (*smt (verit) cancel-ab-semigroup-add-class.diff-right-commute complex-mult-cnj-cmod*

*diff-diff-eq2 left-diff-distrib mult.assoc mult.commute norm-minus-commute norm-mult of-real-add of-real-eq-iff*)
**have** *f1*:‹*cmod z1^2 + cmod z2^2 − cnj z1 ∗ z2 − cnj z2 ∗ z1 = cmod z1^2 + cmod z2^2 − 2∗Re z1∗Re z2 − 2∗Im z1∗Im z2*›
**by**(*auto simp:field-simps intro:complex-eqI*)
**have** ‹*β = 2∗*i∗*Im (z2∗cnj z1) / (cnj z1−cnj z2)*›
**using** ‹*cor (Im (z2 ∗ cnj z1)) = (z2 ∗ cnj z1 − cnj (z2 ∗ cnj z1)) / (2 ∗* i)›
**unfolding** *α-def β-def* **by** *auto*
**then have** *f2*:‹*β/(z1−z2) = 2∗*i∗*Im (z2∗cnj z1) / ((cnj z1−cnj z2)∗(z1−z2))*›
**by** *simp*
**then have** ‹*β/(z1−z2) = 2∗*i∗*Im (z2∗cnj z1) /(cmod z1^2 + cmod z2^2 − 2∗Re z1∗Re z2 − 2∗Im z1∗Im z2)*›

**using** *f0 f1* **by** *presburger*
**then have** ‹*is-real* $((z - z1)/(z1-z2))$›
**unfolding** *axial-symmetry-def* **using** *g2*
**by** (*smt* (*verit*, *del-insts*) *add-diff-cancel-right axial-symmetry-def complex-cnj-cnj complex-cnj-diff complex-cnj-divide diff-divide-distrib divide-divide-eq-right eq-cnj-iff-real*
*g0 g2 mult.commute times-divide-eq-right z1-inv z2-inv* $\alpha$*-def* $\beta$*-def*)
**then show** ‹$z \in$ *line z1 z2*›
**unfolding** *line-def collinear-def*
**by** (*metis* (*mono-tags*, *lifting*) *Im-i-times Re-i-times cnj.simps*(*2*) *complex-i-mult-minus*

*eq-cnj-iff-real mem-Collect-eq minus-diff-eq minus-divide-right*)
**qed**

**lemma** *angle-symmetry-eq*:
**assumes** *h*:‹$z1 \neq z$› ‹$z2 \neq z$› ‹$z \notin$ *line z1 z2*›
**shows**‹*angle-c z z1 z2* = − *angle-c* (*axial-symmetry z*) *z1 z2*›
**proof** −
**have** *f0*:‹*angle-c z z1 z2* = − *angle-c* (*axial-symmetry z*) *z1 z2* ∨
*angle-c z z1 z2* = *angle-c* (*axial-symmetry z*) *z1 z2*›
**using** *angle-symmetry-eq-imp h*(*1*) *h*(*2*) **by** *blast*
**have** *f1*:‹*angle-c z z2 z1* = − *angle-c* (*axial-symmetry z*) *z2 z1*
∨ *angle-c z z2 z1* = *angle-c* (*axial-symmetry z*) *z2 z1*›
**by** (*metis axial-symmetry-dist1 congruent-ctriangle-sss*(*24*) *dist-inv h*(*1*) *neq0 z2-inv*)
**show** *?thesis*
**using** *angle-symmetry axial-symmetry-eq-line f0 h*(*1*) *h*(*2*) *h*(*3*) **by** *presburger*
**qed**

**end**
**end**
**theory** *Morley*

**imports** *Complex-Axial-Symmetry*

**begin**

# 7 Rotations

**locale** *complex-rotation* =
**fixes** *A*::*complex* **and** $\vartheta$::*real*
**begin**

**definition** ‹$r\ z = A + (z-A)*cis(\vartheta)$›

**lemma** *cmod-inv-rotation*:‹*cmod* $(z-A)$ = *cmod* $(r\ z - A)$›
**unfolding** *r-def*
**by** (*simp add*: *norm-mult*)

**lemma** *inner-ang*:‹*cos* (∠ *z1 z2*)∗(*cmod z1* ∗*cmod z2*) = *Re* (*innerprod z1 z2*)›
**proof** −
  **have** ‹*Re* (*innerprod z1 z2*) = *Re* (*scalprod z1 z2*)›
    **unfolding** *innerprod-def* **by**(*auto*)
  **then show** *?thesis*
    **by** (*metis cos-cmod-scalprod mult.commute*)
**qed**

**lemma** *ang-eq-cos-theta*:‹*z*≠*A* ⟹ *cos* (*angle-c z A* (*r z*)) = *cos* (ϑ)›
**proof** −
  **assume** ‹*z*≠*A*›
  **then have** ‹*innerprod* (*z*−*A*) ((*r z* − *A*)) = (*z*−*A*)∗*cis*(ϑ)∗*cnj*(*z* − *A*)›
    **unfolding** *innerprod-def r-def* **by** *auto*
  **then have** *f0*:‹*innerprod* (*z*−*A*) ((*r z* − *A*)) = *cmod* (*z*−*A*)^*2*∗*cis*(ϑ)›
   **by** (*metis ab-semigroup-mult-class.mult-ac*(*1*) *complex-mult-cnj-cmod mult.commute*)
  **then have** ‹*cos* (*angle-c z A* (*r z*))∗*cmod* (*z*−*A*)∗*cmod*(*z* − *A*) = *Re* (*innerprod* (*z*−*A*) ((*r z* − *A*)))›
    **unfolding** *angle-c-def*
    **by** (*metis inner-ang mult.assoc cmod-inv-rotation*)
  **then have** ‹*cos* (*angle-c z A* (*r z*)) = *Re* (*cis* ϑ)›
    **by** (*simp add*: ‹*z* ≠ *A*› *f0 power2-eq-square*)
  **then show** *?thesis*
    **by**(*auto simp*:*cis.code*)
**qed**

**lemma** *cdist-dist*:‹*cdist* = *dist*›
  **using** *cdist-commute dist-complex-def* **by** *fastforce*

**lemma** *ang-eq-theta*:**assumes** *h*:‹*z*≠*A*› **shows** ‹*angle-c z A* (*r z*) = ⌊ϑ⌋›
**proof**(*cases* ‹*angle-c z A* (*r z*) = ⌊− ϑ⌋›)
  **case** *True*
  **then have** ‹*r z* = *A* + (*z*−*A*)∗*cis*(−ϑ)›
   **by** (*metis add-diff-cancel-left′ arg-cis cdist-def cis-cmod cis-neq-zero cmod-inv-rotation*
         *isoscele-iff-pr-cis-qr mult.left-commute mult.right-neutral mult-cancel-left norm-cis*
       *norm-minus-commute of-real-1 r-def right-minus-eq*)
  **then show** *?thesis*
   **by** (*smt* (*verit*, *del-insts*) *True add-diff-cancel-left′ angle-c-neq0 arg-cis arg-mult-eq*
         *canon-ang-cos canon-ang-sin cis.code cis-cnj diff-add-cancel divisors-zero r-def*)
**next**
  **case** *False*
  **have** ‹*cos* (*angle-c z A* (*r z*)) = *cos* (ϑ)›
    **using** *ang-eq-cos-theta h* **by** *auto*
  **then show** *?thesis*
    **by** (*smt* (*verit*, *ccfv-threshold*) *cmod-inv-rotation r-def add-diff-cancel-left′ angle-c-neq0*

*angle-c-sum arg-cis canon-ang-sin cdist-def cis.code cis-neq-zero divisors-zero eq-iff-diff-eq-0 h*
*isoscele-iff-pr-cis-qr nonzero-mult-div-cancel-left nonzero-mult-div-cancel-right norm-minus-commute*)
**qed**

**lemma** *inj-r*:‹*inj r*›
**unfolding** *inj-on-def* **by**(*auto simp*:*r-def*)

**lemma** *img-eqI*:‹*cdist A z1* = *cdist A z2* ∧ *angle-c z1 A z2* = ϑ ⟹ *z2* = *r z1*›
**apply**(*cases* ‹*z1* = *A* ∨ *z2* = *A*›)
**using** *r-def add-minus-cancel isoscele-iff-pr-cis-qr* **apply** *force*
**unfolding** *r-def*
**by** (*metis add.commute cdist-commute diff-add-cancel isoscele-iff-pr-cis-qr mult.commute*)

**lemma** *r-id-iff*:‹⌊ϑ⌋ = *0* ⟷ *r* = *id*›
**proof** −
**obtain** *cc* :: (*complex* ⇒ *complex*) ⇒ *complex*
**and** *cca* :: (*complex* ⇒ *complex*) ⇒ *complex* **where**
*f1*: ∀ *f*. (*id* = *f* ∨ *f* (*cc f*) ≠ *cc f*) ∧ ((∀ *c*. *f c* = *c*) ∨ *id* ≠ *f*)
**by** (*metis* (*no-types*) *eq-id-iff*)
**have** *f2*: ∀ *c ca ra*. *ca* + (*c* − *ca*) ∗ *Complex* (*cos ra*) (*sin ra*) = *complex-rotation.r ca ra c*
**by** (*simp add*: *complex-rotation.r-def cis.code*)
**then have** ∀ *c ca*. *complex-rotation.r c 0 ca* = *ca*
**by** (*metis* (*no-types*) *add-diff-cancel-left′ cos-zero diff-diff-eq2*
*lambda-one mult.commute one-complex.code sin-zero*)
**then show** *?thesis*
**using** *f2 f1*
**by** (*metis* (*lifting*, *full-types*) *ang-eq-theta angle-c-neq0*
*canon-ang-cos canon-ang-sin complex-i-not-zero*)
**qed**

**end**

**lemma** *axial-symmetry-eq*:‹*axial-symmetry B C P* = *axial-symmetry C B P*› **if** ‹*C*≠*B*› **for** *C B P*
**unfolding** *axial-symmetry-def*[*OF that*] *axial-symmetry-def*[*OF that*[*symmetric*]]

**by** (*metis* (*no-types*, *lifting*) *complex-cnj-cancel-iff eq-iff-diff-eq-0*
*minus-diff-eq nonzero-minus-divide-divide times-divide-eq-right*)

**lemma** *img-r-sym*:
**assumes** *h*:‹*z1* ≠ *z2*› ‹*z* ∉ *line z1 z2*›
**shows** ‹*axial-symmetry z1 z2 z* = *complex-rotation.r z1* (⌊*2*∗*angle-c z z1 z2*⌋) *z*›
**proof** −
**interpret** *complex-rotation z1* ⌊*2*∗*angle-c z z1 z2*⌋ **.**
**let** *?z* = ‹*axial-symmetry z1 z2 z*›

**from** *h* **have** ‹*z1*≠*z*› ‹*z2* ≠ *z*›
  **unfolding** *line-def Elementary-Complex-Geometry.collinear-def* **by**(*auto*)
**then have** ‹*angle-c ?z z1 z2* = −*angle-c z z1 z2*›
  **using** *angle-symmetry-eq h*(*1*) *h*(*2*) **by** *force*
**then have** ‹*angle-c z z1 ?z* = ⌊*2*∗*angle-c z z1 z2*⌋›
  **by** (*metis* ‹*z1* ≠ *z*› *add.inverse-inverse angle-c-commute angle-c-commute-pi angle-sum-symmetry h*(*1*) *mult-2-right mult-commute-abs*)
**have** ‹ *cdist z1 z* = *cdist z1* (*axial-symmetry z1 z2 z*)›
  **using** *axial-symmetry-dist1 h*(*1*) **by** *blast*
**then show** *?thesis*
  **apply**(*intro img-eqI*)
  **by** (*metis* ‹*angle-c z z1* (*axial-symmetry z1 z2 z*) = ⌊*2* ∗ *angle-c z z1 z2*⌋›)
**qed**

**lemma** *img-r-sym′*:
  **assumes** *h*:‹*z1* ≠ *z2*› ‹*z*∉*line z1 z2*›
  **shows** ‹*axial-symmetry z1 z2 z* = *complex-rotation.r z1* (⌊−*2*∗*angle-c z2 z1 z*⌋) *z*›
  **by** (*metis angle-c-commute angle-c-neq0 axial-symmetry-eq-line cancel-comm-monoid-add-class.diff-cancel complex-rotation.img-eqI complex-rotation.r-def h*(*1*,*2*) *img-r-sym mult-eq-0-iff mult-minus-left mult-minus-right pi-neq-zero two-pi-canonical*)

**lemma** *equality-for-pqr*:
  **assumes** *1*:‹(*a2*::*complex*)∗*a3*≠*1*› **and** *2*:‹⋀*z*. *h z* = *a3*∗*z* + *b3*› **and** *3*:‹⋀*z*. *g z* = *a2*∗*z* + *b2*› **and** *4*:‹*g* (*h z*) = *z*›
  **shows** ‹*z* = (*a2*∗*b3* + *b2*)/(*1*−*a2*∗*a3*)›
**proof** −
  **have** *f21*:‹*g* (*h z*) = *a2*∗*a3*∗*z* + *a2*∗*b3* +*b2*›
    **using** *assms* **by**(*auto simp*:*2 3field-simps*)
  **then have** ‹ *g* (*h z*) = *a2*∗*a3*∗*z* + *a2*∗*b3* +*b2* ⟷ *z*∗(*1*−*a2*∗*a3*) = *a2*∗*b3* + *b2*›
    **by**(*auto simp*:*field-simps 4*)
  **then have** ‹*a2*∗*a3* ≠*1* ⟹ *z* = (*a2*∗*b3* + *b2*)/(*1*−*a2*∗*a3*)›
    **using** *f21* **by**(*auto simp*:*field-simps*)
  **then show** *?thesis* **using** *1* **by** *auto*
**qed**

**lemma** *equality-for-comp*:
  **assumes** *2*:‹⋀*z*. *h z* = (*a3*::*complex*)∗*z* + *b3*› **and** *3*:‹⋀*z*. *g z* = *a2*∗*z* + *b2*›
    **and** *4*:‹⋀*z*. *f z* = *a1*∗*z* +*b1*›
  **shows** ‹((*f*∘*f*∘*f*)∘(*g*∘*g*∘*g*)∘(*h*∘*h*∘*h*)) *z* = (*a1*∗*a2*∗*a3*)^*3*∗*z* +(*a1*^*2*+*a1*+*1*)∗*b1* +*a1*^*3*∗(*a2*^*2*+*a2*+*1*)∗*b2* +*a1*^*3*∗*a2*^*3*∗(*a3*^*2*+*a3*+*1*)∗*b3* ›
  **using** *assms* **unfolding** *comp-def* **by**(*auto simp*:*fun-eq-iff power2-eq-square power3-eq-cube field-simps*)

**lemma** *eq-translation-id*:
  **assumes** ‹*h* = *complex-rotation.r A 0*› ‹*h B* = *B*›

**shows** ‹$h = id$›
**using** *assms*(*1*) *complex-rotation.r-id-iff* **by** *auto*

**lemma** *r-eqI*:
**assumes** ‹$A = B$› ‹$\vartheta 1 = \vartheta 2$›
**shows** ‹$r\ A\ \vartheta 1 = r\ B\ \vartheta 2$›
**using** *assms*(*1*) *assms*(*2*) **by** *force*

**lemma** *r-eqI′*:
**assumes** ‹$A = B$› ‹$\vartheta 1 = \vartheta 2$›
**shows** ‹$r\ A\ \vartheta 1\ z = r\ B\ \vartheta 2\ z$›
**using** *assms*(*1*) *assms*(*2*) **by** *force*

**lemma** *composed-rotations-same-center*:
**shows** ‹(*complex-rotation.r* $A\ \vartheta 1$ ∘ *complex-rotation.r* $A\ \vartheta 2$) = *complex-rotation.r* $A\ (\vartheta 1 + \vartheta 2)$›
**unfolding** *complex-rotation.r-def* **by** (*auto simp*: *fun-eq-iff cis-mult add-ac*)

**lemma** *composed-rotations*:
**assumes** *h*:‹$\lfloor \vartheta 1 + \vartheta 2 \rfloor \neq 0$›
**shows** ‹(*complex-rotation.r* $A\ \vartheta 1$ ∘ *complex-rotation.r* $B\ \vartheta 2$) =
*complex-rotation.r* $((A*(1-cis\ \vartheta 1) + B*cis\ \vartheta 1*(1-cis\ \vartheta 2))/(1-cis\ (\vartheta 1+\vartheta 2)))\ (\vartheta 1 + \vartheta 2)$›
**proof** −
**have** ‹$cis\ (\vartheta 1 + \vartheta 2) \neq 1$›
**by** (*metis arg-cis assms cis-zero zero-canonical*)
**with** *h* **have** ‹(*complex-rotation.r* $A\ \vartheta 1$ ∘ *complex-rotation.r* $B\ \vartheta 2$)
$((A*(1-cis\ \vartheta 1) + B*cis\ \vartheta 1*(1-cis\ \vartheta 2))/(1-cis\ (\vartheta 1+\vartheta 2)))$
$= (A*(1-cis\ \vartheta 1) + B*cis\ \vartheta 1*(1-cis\ \vartheta 2))/(1-cis\ (\vartheta 1+\vartheta 2))$ ›
**unfolding** *complex-rotation.r-def* **using** *assms*
**by**(*auto simp*:*cis-mult field-simps intro*!:)
**with** *h* **show** *?thesis*
**apply**(*cases* ‹$\vartheta 1 = 0 \vee \vartheta 2 = 0$›)
**unfolding** *complex-rotation.r-def* **using** *assms*
**by**(*auto simp*:*cis-mult field-simps fun-eq-iff diff-divide-distrib add-divide-distrib intro*!:)
**qed**

**lemma** *composed-rotation-is-trans*:
**assumes** ‹$\lfloor \vartheta 1 + \vartheta 2 \rfloor = 0$›
**shows** ‹(*complex-rotation.r* $A\ \vartheta 1$ ∘ *complex-rotation.r* $B\ \vartheta 2$) $z = z + (B - A)*(cis(\vartheta 1) - 1)$›
**using** *assms*
**by** (*auto simp*:*complex-rotation.r-def add-divide-distrib diff-divide-distrib field-simps*)

(*metis add.commute canon-ang-cos canon-ang-sin cis.code cis-mult cis-zero lambda-one mult.commute*)

# 8 Morley's theorem

We begin by proving the Morley's theorem in the case where angles are positives then using the congruence between two triangles with the same angles only not of the same sign we prove Morley's theorem when angles are negatives.

We then proceed to conclude because in a triangle either angles are all negatives or all the angles are positives depending on orientation.

```
theorem Morley-pos:
  assumes‹¬collinear A B C›
    ‹angle-c A B R = angle-c A B C / 3› (is ‹?abr = ?abc›)
    angle-c B A R = angle-c B A C / 3 (is ‹?bar = ?α›)
    angle-c B C P = angle-c B C A / 3 (is ‹?bcp = ?bca›)
    angle-c C B P = angle-c C B A / 3 (is ‹?cbp = ?β›)
    angle-c C A Q = angle-c C A B / 3 (is ‹?caq = ?cab›)
    angle-c A C Q = angle-c A C B / 3 (is ‹?acq = ?γ›)
   and hhh:‹⌊angle-c B A C / 3+angle-c C B A / 3+angle-c A C B / 3⌋ = pi/3›
  shows  ‹cdist R P = cdist P Q ∧ cdist Q R = cdist P Q›
proof −
  have bundle-line:‹A∉ line B C› ‹B∉ line A C› ‹C ∉ line A B› ‹A≠B› ‹B≠C›
‹C≠A›
   using assms(1) non-collinear-independant by (auto simp:collinear-sym1 collinear-sym2
line-def)
  {fix A B C γ
    assume ABC-nline:‹A ∉ line B C›
      and eq-3c:‹angle-c A C B = 3∗γ›
    then have neq-PI:‹abs γ < pi/3›
    proof −
      have ‹angle-c A C B ≠ pi›
        using ABC-nline(1) unfolding line-def
      by (metis angle-c-commute-pi collinear-iff mem-Collect-eq non-collinear-independant)
      then have ‹angle-c A C B ∈ {−pi<..<pi}›
        using ang-c-in less-eq-real-def by auto
      then show ‹abs γ < pi/3›
        using eq-3c by force
    qed}note ang-inf-pi3=this
  have ‹⌊angle-c B A C + angle-c C B A  + angle-c A C B⌋ = pi›
    by (metis collinear-def add.commute
        angle-sum-triangle-c assms(1) collinear-sym1 collinear-sym2')
  have eq-pi:‹⌊3∗?α + 3∗?β + 3∗?γ⌋ = pi›
    using ‹⌊angle-c B A C + angle-c C B A + angle-c A C B⌋ = pi› by force
  then have neq-pi:‹⌊?β⌋ ≠ pi ∧ ⌊?γ⌋ ≠ pi ∧ ⌊?α⌋ ≠ pi›
    by (smt (verit) ang-neg-neg angle-c-commute angle-c-neq0 assms
      canon-ang-sin collinear-angle collinear-sin-neq-0 divide-eq-0-iff divide-nonneg-pos

        minus-divide-left pi-neq-zero sin-pi sin-pi-minus zero-canonical)
  moreover have ‹3∗?α ≠ 0∧3∗?β ≠ 0∧3∗?γ ≠ 0›
    using bundle-line collinear-sin-neq-0 line-def angle-c-commute assms(1) bun-
```

*dle-line*(*4*)
*bundle-line*(*5*) *bundle-line*(*6*) *collinear-iff assms*(*1*) *collinear-sin-neq-0*
**by** (*metis collinear-sym2′ divide-eq-eq-numeral1* (*1*) *mem-Collect-eq mult.commute zero-neq-numeral*)
**ultimately have** *neq-0*:‹$\lfloor ?\beta \rfloor \neq 0 \wedge \lfloor ?\gamma \rfloor \neq 0 \wedge \lfloor ?\alpha \rfloor \neq 0$›
**by** (*metis ang-vec-def angle-c-def arg-cis assms*(*3*) *assms*(*5*) *assms*(*7*) *canon-ang-arg mult-zero-right*)

**interpret** *rot1*: *complex-rotation A 2∗?α* **.**
**interpret** *rot2*: *complex-rotation B 2∗?β* **.**
**interpret** *rot3*: *complex-rotation C 2∗?γ* **.**

**let** *?f*=‹*rot1.r*›
**have** *f0*:‹*rot1.r A* = *A*›
**unfolding** *rot1.r-def* **by** *auto*
**have** *f1*:‹*rot2.r B* = *B*›
**unfolding** *rot2.r-def* **by** *auto*
**have** *f2*:‹*rot3.r C* = *C*›
**unfolding** *rot3.r-def* **by** *auto*

**have** ‹*cmod* (*rot3.r z* − *C*) = *cmod* (*z*−*C*)› **for** *z*
**using** *rot3.cmod-inv-rotation* **by** *presburger*
**have** *f2*:‹*B*≠*C*›
**by** (*metis collinear-def assms*(*1*) *collinear-sym1 collinear-sym2*)
**have** *f5*:‹*angle-c C B P* = *?β*› ‹*angle-c P B C* = −*?β*›
**by** (*auto simp add*: *assms*(*5*) *angle-c-commute*)
(*metis angle-c-commute angle-c-commute-pi assms*(*5*) *neq-pi pi-canonical*)
**then have** *f3*:‹*P* ∉ *line C B*›
**by** (*smt* (*verit*, *ccfv-SIG*) *neq-0 neq-pi angle-c-commute angle-c-neq0 assms*(*4*)
*collinear-angle divide-eq-0-iff line-def mem-Collect-eq pi-canonical zero-canonical*)
**then have** *f3′*:‹*P* ∉ *line B C*›
**using** *collinear-sym2′ line-def* **by** *blast*
**then have** *f4*:‹*P*≠*C* ∧ *P*≠*B*›
**by** (*metis collinear-def collinear-sym1 collinear-sym2 line-def mem-Collect-eq*)
**then have** ‹*angle-c P B C* = − *angle-c* (*axial-symmetry B C P*) *B C*›
**using** *angle-symmetry-eq*[*OF f2 - - f3′*] **by** *auto*
**then have** ‹*angle-c* (*axial-symmetry B C P*) *B C* = *?β*›
**using** *f5*(*2*) **by** *fastforce*

**have** *f13*:‹*angle-c P C B* = *?γ*›
**by** (*smt* (*verit*, *ccfv-threshold*) *angle-c-commute angle-c-commute-pi assms*(*1*) *assms*(*4*) *assms*(*7*)
*collinear-sin-neq-0 nonzero-minus-divide-divide nonzero-minus-divide-right sin-pi*)
**let** *?P′*= ‹(*axial-symmetry B C P*)›

**have** ‹*angle-c* (*axial-symmetry B C P*) *B P* = ⌊*2∗?β*⌋›
**by** (*metis* ‹*P* ≠ *C* ∧ *P* ≠ *B*› ‹*angle-c* (*axial-symmetry B C P*) *B C* = *angle-c C B A* / *3*›

*angle-sum-symmetry f2 f5(1) involution-symmetry mult.commute mult-2-right z1-inv*)
**have** ‹*cdist B* (*P*) = *cdist B ?P′*›
**by**(*auto*)(*metis cmod-axial-inv f2 z1-inv*)
**then have** ‹*cdist B* (*rot2.r P*) = *cdist B ?P′*›
**unfolding** *cdist-def*
**using** *rot2.cmod-inv-rotation* **by** *presburger*
**have** *f16*:‹*rot2.r ?P′* = *P*›
**by** (*metis* ‹*angle-c* (*axial-symmetry B C P*) *B P* = ⌊*2* ∗ (*angle-c C B A* / *3*)⌋›
‹*cdist B P* = *cdist B* (*axial-symmetry B C P*)› *canon-ang-cos*
*canon-ang-sin cis.code complex-rotation.img-eqI complex-rotation.r-def*)
**have** *f10*:‹*angle-c P C B* = ⌊*?γ*⌋›
**by** (*metis* ‹*angle-c P C B* = *angle-c A C B* / *3*› *ang-vec-def angle-c-def arg-cis canon-ang-arg*)

**from** *f10* **have** *f11*:‹*angle-c B C ?P′* = ⌊*?γ*⌋›
**by** (*metis axial-symmetry-eq* ‹*P* ≠ *C* ∧ *P* ≠ *B*› ‹*angle-c P C B* = *angle-c A C B* / *3*› *angle-c-commute*
*angle-symmetry angle-symmetry-eq-imp axial-symmetry-eq-line canon-ang-uminus-pi f2 f3 pi-canonical*)
**then have** *f12*:‹*angle-c P C ?P′* = ⌊*2*∗*?γ*⌋›
**by** (*metis axial-symmetry-eq f13 angle-sum-symmetry f10 f2 f4 mult.commute mult-2-right*)
**then have** *f15*:‹*rot3.r P* = *?P′*›
**by** (*metis axial-symmetry-eq canon-ang-cos canon-ang-sin cis.code f13 f2 f3 img-r-sym complex-rotation.r-def*)
**have** *P-inv*:‹*rot2.r* (*rot3.r P*) = *P*›
**using** *f16 f15* **by** *presburger*
**let** *?Q′*=‹*axial-symmetry A C Q*›
**have** ‹*angle-c C A Q* = −*?α*›
**by** (*metis angle-c-commute assms*(*1*) *assms*(*6*) *collinear-sin-neq-0*
*collinear-sym1 collinear-sym2′ minus-divide-left sin-pi*)
**then have** ‹*angle-c Q A C* = *?α*›
**by** (*metis add.inverse-inverse angle-c-commute canon-ang-uminus-pi neq-pi pi-canonical*)
**have** ‹*A*≠*C* ∧ *Q*∉*line A C*›
**by** (*metis* ‹*angle-c Q A C* = *angle-c B A C* / *3*› *angle-c-neq0 assms*(*7*) *collinear-angle div-0*
*f10 f13 line-def mem-Collect-eq neq-0 neq-pi zero-canonical*)
**then have** *f17*:‹*rot1.r Q* = *?Q′*›
**using** *img-r-sym*[*of A C Q*]
**using** ‹*angle-c Q A C* = *angle-c B A C* / *3*› *canon-ang-cos canon-ang-sin cis.code complex-rotation.r-def*
**by** *presburger*
**then have** ‹*angle-c ?Q′ C A* = *?γ*›
**by** (*smt* (*verit*, *ccfv-SIG*) ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› ‹*angle-c Q A C* = *angle-c B A C* / *3*›
*complex-rotation.ang-eq-theta angle-c-commute angle-symmetry angle-symmetry-eq-imp assms*(*7*)

*axial-symmetry-eq axial-symmetry-eq-line f10 f13 complex-rotation.img-eqI neq-0 neq-pi*)
**then have** ‹*angle-c A C Q* = *?γ*›
**using** *assms*(*7*) **by** *blast*
**then have** ‹*angle-c ?Q′ C Q* =⌊*2*∗*?γ*⌋›
**by** (*metis* ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› ‹*angle-c* (*axial-symmetry A C Q*) *C A* = *angle-c A C B* / *3*›
‹*angle-c Q A C* = *angle-c B A C* / *3*› *angle-c-neq0 angle-sum-symmetry angle-symmetry-eq*
*axial-symmetry-eq involution-symmetry mult-2-right mult-commute-abs neg-equal-0-iff-equal*
*neq-0 zero-canonical*)
**then have** *f18*:‹*rot3*.*r ?Q′* = *Q*›
**using** *img-r-sym*[*of C A ?Q′*]
**by** (*metis* ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› *axial-symmetry-dist1*
*axial-symmetry-eq canon-ang-cos canon-ang-sin cis.code*
*complex-rotation.img-eqI complex-rotation.r-def*)
**have** *Q-inv*:‹*rot3*.*r* (*rot1*.*r Q*) = *Q*›
**using** *f17 f18* **by** *presburger*
**let** *?R′*=‹*axial-symmetry A B R*›
**have** ‹*angle-c B A R* = *?α*›
**by** (*simp add*: *assms*(*3*))
**then have** ‹*angle-c R A B* = −*?α*›
**by** (*metis angle-c-commute canon-ang-uminus-pi neq-pi pi-canonical*)
**have** ‹*angle-c R B A* = *?β*›
**by** (*metis* ‹*angle-c* (*axial-symmetry B C P*) *B C* = *angle-c C B A* / *3*› *angle-c-commute*
*angle-c-commute-pi assms*(*1*) *assms*(*2*) *collinear-sin-neq-0 collinear-sym1 minus-divide-left sin-pi*)
**have** ‹*A*≠*B* ∧ *R*∉*line A B*›
**by** (*metis* (*no-types*, *opaque-lifting*) ‹*angle-c Q A C* = *angle-c B A C* / *3*› ‹*angle-c R A B* = − (*angle-c B A C* / *3*)› *angle-c-commute angle-c-commute-pi angle-c-neq0 assms*(*2*) *collinear-angle*
*collinear-sym2′ diff-zero divide-eq-0-iff line-def mem-Collect-eq minus-diff-eq neg-equal-zero*
*neq-0 zero-canonical*)
**then have** *f17*:‹*rot2*.*r R* = *?R′*›
**using** *img-r-sym*[*of B A R*]
**using** ‹*angle-c R B A* = *angle-c C B A* / *3*› *axial-symmetry-eq*
*cis.code collinear-sym2 line-def complex-rotation.r-def* **by** *auto*
**then have** ‹*angle-c ?R′ A B* = *?α*›
**by** (*metis* ‹*A* ≠ *B* ∧ *R* ∉ *line A B*› ‹*angle-c R A B* = − (*angle-c B A C* / *3*)›
‹*angle-c R B A* = *angle-c C B A* / *3*› *add.inverse-inverse add.inverse-neutral angle-c-neq0*
*angle-symmetry-eq neq-0 zero-canonical*)
**then have** *f18*:‹*rot1*.*r ?R′* = *R*›
**using** *img-r-sym*[*of A B ?R′*]
**by** (*metis* ‹*A* ≠ *B* ∧ *R* ∉ *line A B*› *axial-symmetry-eq canon-ang-cos canon-ang-sin cis.code*

*involution-symmetry line-is-inv complex-rotation.r-def z2-inv*)
**have** *R-inv*:‹*rot1.r* (*rot2.r R*) = *R*›
**using** *f17 f18* **by** *presburger*
**let** *?a1* = ‹*cis*(*2*∗*?α*)› **let** *?b1*=‹*A*∗(*1*−*?a1*)› **let** *?a2*=‹*cis*(*2*∗*?β*)› **let** *?b2*=‹*B*∗(*1*−*?a2*)›
**let** *?a3*=‹*cis*(*2*∗*?γ*)› **let** *?b3*=‹*C*∗(*1*−*?a3*)›
**have** *possi-abc*:‹⌊*?α*+*?β*+*?γ*⌋ =*pi*/*3* ∨ ⌊*?α*+*?β*+*?γ*⌋ = −*pi*/*3* ∨ ⌊*?α*+*?β*+*?γ*⌋
= *pi*›
**proof** −
**have** ‹⌊*?α*+*?β*+*?γ*⌋∈{−*pi*<..*pi*}›
**by** (*simp add*: *canon-ang*(*1*) *canon-ang*(*2*))
**then have** ‹*3*∗⌊*?α*+*?β*+*?γ*⌋ ∈ {−*3*∗*pi*<..*3*∗*pi*}›
**by**(*auto*)
**then have** ‹⌊⌊*?α*+*?β*+*?γ*⌋+⌊*?α*+*?β*+*?γ*⌋+⌊*?α*+*?β*+*?γ*⌋⌋ = ⌊*?α*+*?β*+*?γ*+*?α*+*?β*+*?γ*+*?α*+*?β*+*?γ*⌋›
**by** (*smt* (*verit*, *ccfv-SIG*) *add.commute arg-cis canon-ang-arg*
*canon-ang-sum distrib-left is-num-normalize*(*1*) *mult-2*)
**then have** ‹⌊*3*∗⌊*?α*+*?β*+*?γ*⌋⌋=*pi*›
**using** *eq-pi* **by** *argo*
**then have** ‹*3*∗⌊*?α*+*?β*+*?γ*⌋ = *3*∗*pi* ∨ *3*∗⌊*?α*+*?β*+*?γ*⌋ = −*pi* ∨ *3*∗⌊*?α*+*?β*+*?γ*⌋=*pi*›
**by** (*smt* (*verit*, *del-insts*) *canon-ang*(*1*) *canon-ang*(*2*) *canon-ang-id*
*canon-ang-minus-3pi-minus-pi canon-ang-pi-3pi*)
**then show** *?thesis*
**by** *force*
**qed**
**have** *jj*:‹*B* ∉ *line C A*› ‹*Q* ∉ *line A C*› ‹*R* ∉ *line A B*›
**using** *assms f3'* ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› ‹*A* ≠ *B* ∧ *R* ∉ *line A B*›
**by** (*simp-all add*: *collinear-sym1 collinear-sym2 line-def*)
**then have** *inf-pi3*:‹*abs ?α* < *pi*/*3*› ‹*abs ?β* < *pi*/*3*› ‹*abs ?γ* < *pi*/*3*›
**using** *ang-inf-pi3*[*of B C A ?α*] *ang-inf-pi3*[*of C A B ?β*] *ang-inf-pi3*[*of A B C*
*?γ*]
**by** (*auto simp add*:*collinear-sym2 collinear-sym1 assms line-def*)
**then have** ‹*abs* (*?α*+*?β*+*?γ*) < *pi*›
**by** *argo*
**have** *j1*:‹⌊*?α*+*?β*+*?γ*⌋ = *pi*/*3* ⟹ *?a1*∗*?a2*∗*?a3* = *root3*›
**proof** −
**assume** *hh*:‹⌊*?α*+*?β*+*?γ*⌋ = *pi*/*3*›
**have** *f20*:‹*cis* (*2* ∗ (*angle-c B A C* / *3*)) ∗ *cis* (*2* ∗ (*angle-c C B A* / *3*)) ∗ *cis*
(*2* ∗ (*angle-c A C B* / *3*))
= *cis* (*2*∗(*?α*+*?β*+*?γ*))›
**by** (*simp add*: *cis-mult*)
**then have** *f21*:‹*cis* (*2*∗(⌊*?α*+*?β*+*?γ*⌋)) = *cis* (*2*∗(*?α*+*?β*+*?γ*))›
**by** (*smt* (*verit*, *ccfv-threshold*) *canon-ang-cos canon-ang-sin canon-ang-sum*
*cis.code*)
**with** *hh* **show** *?thesis*
**by** (*metis* (*no-types*, *opaque-lifting*) *f21 f20 times-divide-eq-right root3-def*)
**qed**
**have** ‹*rot1.r*∘*rot1.r* = *complex-rotation.r A* (*4*∗*?α*)›
**using** *composed-rotations-same-center* **by** *auto*
**have** *g10*:‹((*rot1.r*∘*rot1.r*∘*rot1.r*)∘(*rot2.r*∘*rot2.r*∘*rot2.r*)∘(*rot3.r* ∘*rot3.r*∘*rot3.r*))
=

complex-rotation.r A (6∗?α) ∘ complex-rotation.r B (6∗?β) ∘ complex-rotation.r C (6∗?γ)›
using composed-rotations-same-center **by** auto
**have** f26:‹⌊6∗?α + 6∗?β + 6∗?γ⌋ = 0›
**by** (smt (verit, ccfv-SIG) canon-ang-sum eq-pi two-pi-canonical)
**also have** ‹⌊6∗?γ⌋ ≠ 0›
**proof** (rule ccontr)
**assume** ‹¬ ⌊6 ∗ (angle-c A C B / 3)⌋ ≠ 0›
**then obtain** k::int **where** ‹6 ∗ ⌊(angle-c A C B / 3)⌋ = 2∗k∗pi›
**by** (metis add.commute add.inverse-neutral add-0 canon-ang(3)
diff-conv-add-uminus f10 f13 of-int-mult of-int-numeral)
**then have** ‹3∗⌊(angle-c A C B / 3)⌋ = k∗pi›
**by** force
**then show** False
**by** (metis (no-types, opaque-lifting) assms(1) collinear-sin-neq-0 divide-eq-eq-numeral1(1)
f10 f13 mult-1 sin-kpi times-divide-eq-left zero-neq-numeral)
**qed**
**then have** f24:‹⌊6∗?α + 6∗?β⌋ ≠ 0›
**by** (metis add.commute add.right-neutral arg-cis calculation
canon-ang-cos canon-ang-sin canon-ang-sum cis.code)
**let** ?A1=‹(A∗(1−cis (6∗?α)) + B∗cis (6∗?α)∗(1−cis (6∗?β)))/(1−cis (6∗?β+6∗?α))›
**have** g11:‹complex-rotation.r A (6∗?α) ∘ complex-rotation.r B (6∗?β) ∘ complex-rotation.r C (6∗?γ) =
complex-rotation.r ?A1 (6∗?α + 6∗?β)∘ complex-rotation.r C (6∗?γ) ›
**using** composed-rotations[of ‹6∗?α› ‹6∗?β› A B, OF f24]
**by** argo
**then have** f27:‹complex-rotation.r ?A1 (6∗?α + 6∗?β)∘ complex-rotation.r C (6∗?γ) = (λz. z + (C −
(A ∗ (1 − cis (6 ∗ (angle-c B A C / 3))) + B ∗ cis (6 ∗ (angle-c B A C / 3)) ∗ (1 − cis (6 ∗ (angle-c C B A / 3)))) /
(1 − cis (6 ∗ (angle-c C B A / 3) + 6 ∗ (angle-c B A C / 3)))) ∗
(cis (6 ∗ (angle-c B A C / 3) + 6 ∗ (angle-c C B A / 3)) − 1))› (**is** ‹?l = (λz. z + ?A2)›)
**using** composed-rotation-is-trans[of ‹6∗?α + 6∗?β› ‹6∗?γ› ?A1 C, OF f26]
**by** auto
**have** f30:‹2∗angle-c A C B = 6∗?γ›
**by** auto
**have** ‹axial-symmetry C B A = complex-rotation.r C (⌊2∗angle-c A C B⌋) A›
**using** img-r-sym collinear-sym1 f2 jj(1) line-def **by** auto
**then have** first:‹complex-rotation.r C (6∗?γ) A = axial-symmetry C B A› (**is** ‹?rr = ?axA›)
**using** canon-ang-cos canon-ang-sin cis.code f30 complex-rotation.r-def **by** presburger
**have** f31:‹axial-symmetry B C ?axA = complex-rotation.r B (⌊2∗angle-c ?axA B C⌋) ?axA›
**by** (metis ‹A ≠ C ∧ Q ∉ line A C› ‹⌊6 ∗ (angle-c A C B / 3)⌋ ≠ 0›
‹complex-rotation.r C (6 ∗ (angle-c A C B / 3)) A = axial-symmetry C B A›
complex-rotation.ang-eq-theta angle-c-neq0

*axial-symmetry-eq f2 img-r-sym involution-symmetry line-is-inv z1-inv*)
**have** *f32*:‹*angle-c C B A* = *3*∗*?β*›
**by** *simp*
**then have** *f33*:‹*angle-c ?axA B C* = *3*∗*?β*›
**by** (*smt* (*verit*) ‹*A* ≠ *B* ∧ *R* ∉ *line A B*› ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› ‹⌊*6* ∗ (*angle-c A C B* / *3*)⌋ ≠ *0*›
‹*complex-rotation.r C* (*6* ∗ (*angle-c A C B* / *3*)) *A* = *axial-symmetry C B A*›
*complex-rotation.ang-eq-theta angle-c-commute angle-c-neq0 angle-symmetry-eq assms*(*1*)
*axial-symmetry-eq collinear-sin-neq-0 collinear-sym1 f2 line-is-inv sin-pi*)
**then have** *snd*:‹*complex-rotation.r B* (*6*∗*?β*) ((*axial-symmetry B C A*)) = *A*›
**by** (*smt* (*verit*, *ccfv-SIG*) ‹*A* ≠ *C* ∧ *Q* ∉ *line A C*› ‹⌊*6* ∗ (*angle-c A C B* / *3*)⌋ ≠ *0*›
‹*complex-rotation.r C* (*6* ∗ (*angle-c A C B* / *3*)) *A* = *axial-symmetry C B A*›
*complex-rotation.ang-eq-theta angle-c-neq0 axial-symmetry-eq canon-ang-cos canon-ang-sin*
*cis.code f2 img-r-sym involution-symmetry line-is-inv complex-rotation.r-def z1-inv*)
**then have** *thrd*:‹*complex-rotation.r A* (*6*∗*?α*) *A* = *A*›
**unfolding** *complex-rotation.r-def* **by** *auto*
**then have** ‹(*complex-rotation.r A* (*6*∗*?α*) ∘ *complex-rotation.r B* (*6*∗*?β*) ∘ *complex-rotation.r C* (*6*∗*?γ*)) *A* = *A*›
**apply**(*simp*)
**by** (*smt* (*verit*, *best*) *axial-symmetry-eq f30 f32 first snd*)
**then have** *g21*:‹((*rot1.r*∘*rot1.r*∘*rot1.r*)∘(*rot2.r*∘*rot2.r*∘*rot2.r*)∘(*rot3.r* ∘*rot3.r*∘*rot3.r*)) = (λ*z*. *z* + *?A2*)›
**apply**(*subst g10*) **apply**(*subst g11*) **apply**(*subst f27*) **by**(*simp add*:*fun-eq-iff*)
**then have** ‹*?A2* = *0*›
**by** (*metis* ‹(*complex-rotation.r A* (*6* ∗ (*angle-c B A C* / *3*)) ∘
*complex-rotation.r B* (*6* ∗ (*angle-c C B A* / *3*)) ∘
*complex-rotation.r C* (*6* ∗ (*angle-c A C B* / *3*))) *A* = *A*›
*add-diff-cancel-left′ cancel-comm-monoid-add-class.diff-cancel g10*)
**then have** *g22*:‹((*rot1.r*∘*rot1.r*∘*rot1.r*)∘(*rot2.r*∘*rot2.r*∘*rot2.r*)∘(*rot3.r* ∘*rot3.r*∘*rot3.r*)) = *id*›
**using** *g21* **by**(*auto simp*:*fun-eq-iff*)
**have** *g20*:‹*rot1.r z* = *?a1*∗*z* +*?b1*› ‹*rot2.r z* =*?a2*∗*z* + *?b2*› ‹*rot3.r z* = *?a3* ∗ *z* + *?b3*› **for** *z*
**by**(*auto simp*:*field-simps complex-rotation.r-def*)
**then have** ‹((*rot1.r*∘*rot1.r*∘*rot1.r*)∘(*rot2.r*∘*rot2.r*∘*rot2.r*)∘(*rot3.r* ∘*rot3.r*∘*rot3.r*)) *z* = (*cis* (*2* ∗ (*angle-c B A C* / *3*)) ∗ *cis* (*2* ∗ (*angle-c C B A* / *3*))
∗ *cis* (*2* ∗ (*angle-c A C B* / *3*))) ^ *3* ∗ *z* +
((*cis* (*2* ∗ (*angle-c B A C* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c B A C* / *3*)) + *1*) ∗ (*A* ∗ (*1* − *cis* (*2* ∗ (*angle-c B A C* / *3*)))) +
*cis* (*2* ∗ (*angle-c B A C* / *3*)) ^ *3* ∗ ((*cis* (*2* ∗ (*angle-c C B A* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c C B A* / *3*)) + *1*) ∗
(*B* ∗ (*1* − *cis* (*2* ∗ (*angle-c C B A* / *3*)))) +
*cis* (*2* ∗ (*angle-c B A C* / *3*)) ^ *3* ∗ *cis* (*2* ∗ (*angle-c C B A* / *3*)) ^ *3* ∗

((*cis* (*2* ∗ (*angle-c A C B* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c A C B* / *3*)) + *1*) ∗
(*C* ∗ (*1* − *cis* (*2* ∗ (*angle-c A C B* / *3*))))› (**is** ‹*?l z* = *?A3 z*›) **for** *z*
**using** *equality-for-comp*[*of rot3.r ?a3 ?b3 rot2.r ?a2 ?b2 rot1.r ?a1 ?b1*, *OF g20*(*3*) *g20*(*2*) *g20*(*1*)]
**by** *blast*
**then have** ‹*z* = *?A3 z*› **for** *z*
**by** (*metis g22 id-apply*)
**then have** *very-imp*:‹((*cis* (*2* ∗ (*angle-c B A C* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c B A C* / *3*)) + *1*) ∗ (*A* ∗ (*1* − *cis* (*2* ∗ (*angle-c B A C* / *3*)))) +
*cis* (*2* ∗ (*angle-c B A C* / *3*)) ^ *3* ∗ ((*cis* (*2* ∗ (*angle-c C B A* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c C B A* / *3*)) + *1*) ∗
(*B* ∗ (*1* − *cis* (*2* ∗ (*angle-c C B A* / *3*)))) +
*cis* (*2* ∗ (*angle-c B A C* / *3*)) ^ *3* ∗ *cis* (*2* ∗ (*angle-c C B A* / *3*)) ^ *3* ∗
((*cis* (*2* ∗ (*angle-c A C B* / *3*)))$^2$ + *cis* (*2* ∗ (*angle-c A C B* / *3*)) + *1*) ∗
(*C* ∗ (*1* − *cis* (*2* ∗ (*angle-c A C B* / *3*)))) = *0*›
**by** (*metis comm-monoid-add-class.add-0 mult-zero-class.mult-zero-right*)
{**assume** *hhh*:‹⌊*?α*+*?β*+*?γ*⌋ = *pi*/*3*›
**have** *j5*:‹*?a1*∗*?a2* ≠*1*›
**proof**(*rule ccontr*)
**assume** ‹¬ *cis* (*2* ∗ (*angle-c B A C* / *3*)) ∗ *cis* (*2* ∗ (*angle-c C B A* / *3*)) ≠ *1*›
**then have** ‹*?a1*∗*?a2* = *cis 0*›
**using** *cis-zero* **by** *presburger*
**then have** ‹⌊*2*∗(*?α*+*?β*)⌋ = *0*›
**by** (*metis arg-cis cis-mult distrib-left zero-canonical*)
**then have** *t*:‹⌊*?α*+*?β*⌋ = *pi* ∨ ⌊*?α*+*?β*⌋ = *0*›
**by** (*smt* (*verit*, *del-insts*) *canon-ang*(*1*) *canon-ang-id canon-ang-sum canon-ang-uminus*)
**then have** ‹⌊*?α*+*?β*⌋ = *0* ⟹ *False*›
**by** (*metis add-0 assms*(*1*) *canon-ang-sum collinear-sin-neq-0*
*divide-cancel-right f10 f13 hhh sin-pi zero-neq-numeral*)
**then have** ‹⌊*1*/*3*∗(*3*∗*?α*+*3*∗*?β*)⌋=*pi*› **using** *t* **by**(*auto simp*:*algebra-simps*)
**then have** ‹⌊(*pi*−⌊*3*∗*?γ*⌋)⌋ = ⌊(*3*∗*?α*+*3*∗*?β*)⌋›
**by** (*smt* (*verit*, *ccfv-SIG*) *canon-ang-diff eq-pi*)
**then have** ‹⌊(*pi*/*3*−⌊*?γ*⌋)⌋ = *pi*›
**by** (*smt* (*verit*, *best*) ‹⌊*angle-c B A C* / *3* + *angle-c C B A* / *3*⌋ = *0* ⟹ *False*›
*canon-ang-diff hhh t*)
**then have** ‹⌊*?γ*⌋ ∈ {*0*<..<*pi*/*3*}›
**by** (*smt* (*verit*, *ccfv-SIG*) *add-divide-distrib ang-vec-bounded angle-c-def*
*arccos-minus-one-half arccos-one-half arcsin-minus-one-half arcsin-one-half*

*arcsin-plus-arccos canon-ang-id canon-ang-pi-3pi divide-cancel-right f13 field-sum-of-halves*)
**then show** *False*
**using** ‹⌊*pi* / *3* − ⌊*angle-c A C B* / *3*⌋⌋ = *pi*› *add-divide-distrib canon-ang-id* **by** *auto*
**qed**
**have** *r-eq-all*:‹*rot1.r z* = *?a1*∗*z* + *?b1*› ‹*rot2.r z* = *?a2*∗*z* + *?b2*› ‹*rot3.r z* = *?a3*∗*z*+*?b3*› **for** *z*

**by**(*auto simp:field-simps complex-rotation.r-def cis.code*)
**have** *f21*:‹*rot2.r* (*rot3.r P*) = *?a2*∗*?a3*∗*P* + *?a2*∗*?b3* +*?b2*›
**apply**(*subst r-eq-all*(*2*)) **apply**(*subst r-eq-all*(*3*)) **by**(*auto simp:field-simps*)

**then have** ‹ *rot2.r* (*rot3.r P*) = *?a2*∗*?a3*∗*P* + *?a2*∗*?b3* +*?b2* ⟷ *P*∗(*1*−*?a2*∗*?a3*)
= *?a2*∗*?b3* + *?b2*›
**by** (*smt* (*verit*) *add.commute add-diff-cancel-left′ add-diff-eq f15*
*f16 mult.commute mult.right-neutral right-diff-distrib′*)
**then have** *j2*:‹*?a2*∗*?a3* ≠*1* ⟹ *P* = (*?a2*∗*?b3* + *?b2*)/(*1*−*?a2*∗*?a3*)›
**using** *f21* **by**(*auto simp:field-simps*)
**have** *j4*:‹*?a1*∗*?a3* ≠ *1*›
**proof**(*rule ccontr*)
**assume** ‹¬ *cis* (*2* ∗ (*angle-c B A C* / *3*)) ∗ *cis* (*2* ∗ (*angle-c A C B* / *3*)) ≠
*1*›
**then have** ‹*?a1*∗*?a3* = *cis 0*›
**using** *cis-zero* **by** *presburger*
**then have** ‹⌊*2*∗(*?α*+*?γ*)⌋ = *0*›
**by** (*metis arg-cis cis-mult distrib-left zero-canonical*)
**then have** *t*:‹⌊*?α*+*?γ*⌋ = *pi* ∨ ⌊*?α*+*?γ*⌋ = *0*›
**by** (*smt* (*verit*, *del-insts*) *canon-ang*(*1*) *canon-ang-id canon-ang-sum canon-ang-uminus*)
**then have** *k0*:‹⌊*?α*+*?γ*⌋ = *0* ⟹ *False*›
**by** (*smt* (*verit*, *ccfv-SIG*) ‹*A* ≠ *B* ∧ *R* ∉ *line A B*› ‹*A* ≠ *C* ∧ *Q* ∉ *line A*
*C*›
‹|*angle-c B A C* / *3* + *angle-c C B A* / *3* + *angle-c A C B* / *3*| < *pi*›
‹*angle-c* (*axial-symmetry A B R*) *A B* = *angle-c B A C* / *3*› ‹*angle-c R*
*A B* = − (*angle-c B A C* / *3*)›
‹*angle-c R B A* = *angle-c C B A* / *3*› *ang-pos-pos assms*(*3*) *canon-ang-id*
*f2 f30 f32 neq-0 z1-inv*)

**then have** ‹⌊*1*/*3*∗(*3*∗*?α*+*3*∗*?γ*)⌋=*pi*› **using** *t* **by**(*auto simp:algebra-simps*)
**then have** ‹⌊(*pi*−⌊*3*∗*?β*⌋)⌋ = ⌊(*3*∗*?α*+*3*∗*?γ*)⌋›
**by** (*smt* (*verit*, *ccfv-SIG*) *canon-ang-diff eq-pi*)
**then have** ‹⌊(*pi*/*3*−⌊*?β*⌋)⌋ = *pi*›
**by** (*smt* (*verit*, *best*) *k0*
*canon-ang-diff hhh t*)
**then have** *k1*:‹⌊*?β*⌋ ∈ {*0*<..<*pi*/*3*}›
**by** (*smt* (*verit*, *del-insts*) *arccos-one-half arcsin-one-half arcsin-plus-arccos*
*canon-ang-id divide-nonneg-pos field-sum-of-halves inf-pi3*(*2*) *pi-ge-zero*)
**then show** *False*
**using** *k1 add-divide-distrib canon-ang-id* ‹⌊*pi* / *3* − ⌊*angle-c C B A* / *3*⌋⌋
= *pi*› **by** *auto*
**qed**
**have** *j3*:‹*?a2*∗*?a3* ≠ *1*›
**proof**(*rule ccontr*)
**assume** ‹¬ *cis* (*2* ∗ (*angle-c C B A* / *3*)) ∗ *cis* (*2* ∗ (*angle-c A C B* / *3*)) ≠
*1*›
**then have** ‹*?a2*∗*?a3* = *cis 0*›
**using** *cis-zero* **by** *presburger*
**then have** ‹⌊*2*∗(*?β*+*?γ*)⌋ = *0*›

```
      by (metis arg-cis cis-mult distrib-left zero-canonical)
    then have t:‹⌊?β+?γ⌋ = pi ∨ ⌊?β+?γ⌋ = 0›
    by (smt (verit, del-insts) canon-ang(1) canon-ang-id canon-ang-sum canon-ang-uminus)
    then have k0:‹⌊?β+?γ⌋ = 0 ⟹ False›
      by (smt (verit, ccfv-SIG) ‹A ≠ B ∧ R ∉ line A B› ‹A ≠ C ∧ Q ∉ line A
C›
          ‹|angle-c B A C / 3 + angle-c C B A / 3 + angle-c A C B / 3| < pi›
          ‹angle-c (axial-symmetry A B R) A B = angle-c B A C / 3› ‹angle-c R
A B = − (angle-c B A C / 3)›
          ‹angle-c R B A = angle-c C B A / 3› ang-pos-pos assms(3) canon-ang-id
f2 f30 f32 neq-0 z1-inv)

    then have ‹⌊1/3*(3*?β+3*?γ)⌋=pi› using t by(auto simp:algebra-simps)
    then have ‹⌊(pi−⌊3*?α⌋)⌋ = ⌊(3*?β+3*?γ)⌋›
      by (smt (verit, ccfv-SIG) canon-ang-diff eq-pi)
    then have ‹⌊(pi/3−⌊?α⌋)⌋ = pi›
      by (smt (verit, best) k0
          canon-ang-diff hhh t)
    then have k1:‹⌊?α⌋ ∈ {0<..<pi/3}›
      by (smt (verit, del-insts) arccos-one-half arcsin-one-half arcsin-plus-arccos
          canon-ang-id divide-nonneg-pos field-sum-of-halves inf-pi3(1) pi-ge-zero)
    then show False
      using k1 add-divide-distrib canon-ang-id ‹⌊(pi/3−⌊?α⌋)⌋ = pi› by auto
  qed
  have f21':‹rot3.r (rot1.r Q) = ?a3*?a1*Q + ?a3*?b1 +?b3›
    apply(subst r-eq-all(1)) apply(subst r-eq-all(3)) by(auto simp:field-simps)
  have f21'':‹rot1.r (rot2.r R) = ?a1*?a2*R + ?a1*?b2 +?b1›
    apply(subst r-eq-all(1)) apply(subst r-eq-all(2)) by(auto simp:field-simps)
  have jjj:‹?a1 ≠ 0 ∧ ?a2 ≠ 0 ∧ ?a3 ≠ 0›
    using cis-neq-zero by blast
  then have eq-j:‹?a1*?a2*?a3 = root3›
    using j1 hhh by blast
  then have P-def:‹P = (?a2*?b3 + ?b2)/(1−?a2*?a3)›(is ‹-=?P›)
    using j2 j3 by blast
  have n1: ‹1−?a2*?a3 = (?a1 − root3)/?a1›
    by (smt (verit, ccfv-threshold) cis-divide cis-times-cis-opposite cis-zero
        diff-divide-distrib eq-j minus-real-def mult-1 times-divide-eq-left)
  then have P-last:‹P = ?a1*(?a2*?b3 + ?b2)/(?a1−root3)›
    using P-def jjj by (simp add: )
  then have Q-def:‹Q = (?a3*?b1 + ?b3)/(1−?a3*?a1)› (is ‹-=?Q›)
    using f21' j4 Q-inv by(auto simp:field-simps)
  have n2: ‹1−?a3*?a1 = (?a2 − root3)/?a2›
    by (smt (verit, ccfv-threshold) cis-divide cis-times-cis-opposite cis-zero
        diff-divide-distrib eq-j minus-real-def mult-1 times-divide-eq-left)
  then have Q-last:‹Q = ?a2*(?a3*?b1 + ?b3)/(?a2−root3)›
    using Q-def jjj by simp
  then have R-def:‹R = (?a1*?b2 + ?b1)/(1−?a1*?a2)› (is ‹-=?R›)
    using f21'' j5 R-inv by(auto simp:field-simps)
  have n3:‹1−?a1*?a2 = (?a3 − root3)/?a3›
```

**by** (*smt* (*verit*, *ccfv-threshold*) *cis-divide cis-times-cis-opposite cis-zero diff-divide-distrib eq-j minus-real-def mult-1 times-divide-eq-left*)
**then have** *R-last*:‹$R = ?a3*(?a1*?b2 + ?b1)/(?a3-root3)$›
**using** *R-def jjj* **by** *simp*
**have** ‹$?a1 - root3 \neq 0 \wedge ?a2-root3\neq 0 \wedge ?a3-root3 \neq 0$›
**using** *eq-j j3 j4 j5* **by** *auto*
**have** *rule-s*:‹$c \neq 0 \Longrightarrow a/c + b/c = (a+b)/c$› **for** *a b c*::*real*
**by** *argo*
**define** $j'$ **where**
*defs*: ‹$j'\equiv root3$›
**have** *simp-rules-for-eq*:‹$P = (?a2*?b3 + ?b2)/(1-?a2*?a3)$› ‹$R = (?a1*?b2 + ?b1)/(1-?a1*?a2)$›
‹$Q = (?a3*?b1 + ?b3)/(1-?a1*?a3)$› ‹$1-?a1*?a2 \neq 0 \wedge 1-?a2*?a3\neq 0 \wedge 1-?a1*?a3 \neq 0$›
‹$?a1*?a2*?a3 = j'$ ›
‹$1 + j' +j'\hat{\ }2 = 0$› ‹$((1 - ?a1 * ?a3) * ((1 - ?a1 * ?a2) * (1 - ?a2 * ?a3)))\neq 0$›
‹$j'\hat{\ }3 = 1$› ‹$j'*j'*j' = 1$› ‹$?a1\neq 0$› ‹$?a2\neq 0$› ‹$?a3\neq 0$›
‹$(1-?a1*?a2)*?a3 = (?a3-j')$› ‹$(1-?a2*?a3)*?a1 = (?a1-j')$› ‹$(1-?a1*?a3)*?a2 = (?a2-j')$›
**using** *Q-def P-def R-def eq-j j3 jjj j4 j5 root3-eq-0 root-unity-3 n1 n2 n3*
**by**(*auto simp add*:*mult.commute power3-eq-cube defs root3-def*)
**have** *graal*:‹$(?a1\hat{\ }2+?a1+1)*?b1 + ?a1\hat{\ }3*(?a2\hat{\ }2+?a2+1)*?b2 +?a1\hat{\ }3*?a2\hat{\ }3*(?a3\hat{\ }2+?a3+1)*?b3 =$
$-j'*?a1\hat{\ }2*?a2*(?a1-j')*(?a2-j')*(?a3-j')*(?R +j'*?P +j'\hat{\ }2*?Q)$›
**using** *root-unity-carac*[*of ?a1 ?a2 ?a3 j' ?R ?b2 ?b1 ?P ?b3 ?Q*]
**using** *simp-rules-for-eq defs* **by**(*auto simp*:)
**then have** ‹$(?a1\hat{\ }2+?a1+1)*?b1 + ?a1\hat{\ }3*(?a2\hat{\ }2+?a2+1)*?b2 +?a1\hat{\ }3*?a2\hat{\ }3*(?a3\hat{\ }2+?a3+1)*?b3 = 0$›
**unfolding** *defs* **using** *very-imp* **by** *auto*
**then have** ‹$(?R +j'*?P +j'\hat{\ }2*?Q) = 0$›
**using** *graal simp-rules-for-eq*(*13*) *simp-rules-for-eq*(*14*)
*simp-rules-for-eq*(*15*) *simp-rules-for-eq*(*11*) *simp-rules-for-eq*(*12*) *simp-rules-for-eq*(*4*)
**by** *force*
**then have** *impp*:‹$R +j'*P +j'\hat{\ }2*Q = 0$›
**using** *R-def P-def Q-def*
**by** (*metis simp-rules-for-eq*(*1*) *simp-rules-for-eq*(*2*) )
**have** *neq-all*:‹$A\neq B \wedge B\neq C \wedge C\neq A$›
**using** ‹$A \neq B \wedge R \notin line\ A\ B$› ‹$A \neq C \wedge Q \notin line\ A\ C$› *f2* **by** *argo*
**have** ‹$R \neq Q$›
**proof** (*rule ccontr*)
**assume** ‹$\neg R \neq Q$›
**then have** *q1*:‹*angle-c A B R* = *angle-c A B Q*›
‹*angle-c B A R* = *angle-c B A Q*› ‹*angle-c C A Q* = *angle-c C A R*›
‹*angle-c A C Q* = *angle-c A C R*›
**using** *assms* **by** *auto*
**then have** ‹*angle-c B A R* = $?\alpha$› **using** *assms* **by** *auto*
**then have** ‹*angle-c B A Q* = $?\alpha$›
**using** *q1* **by** *auto*

**then have** ‹*angle-c A B Q = angle-c A B R + angle-c R B Q*›
**using** ‹¬ $R \neq Q$› *angle-c-neq0* **by** *auto*
**have** ‹*angle-c C A B* = − *3*?α*› **using** *assms*
**using** ‹*angle-c C A Q* = − (*angle-c B A C / 3*)› **by** *argo*
**then have** ‹*angle-c* (*axial-symmetry B A R*) *A B* = *?α*›
**by** (*metis* ‹*angle-c* (*axial-symmetry A B R*) *A B* = *angle-c B A C / 3*› *axial-symmetry-eq*)
**have** ‹$C - A \neq 0 \wedge Q - A \neq 0 \wedge A - R \neq 0 \wedge A - B \neq 0$›
**using** ‹$A \neq C \wedge Q \notin$ *line A C*› ‹*angle-c A B Q = angle-c A B R + angle-c R B Q*›
‹*angle-c R B A = angle-c C B A / 3*› *neq-0 q1*(*1*) *neq-all* **by** *fastforce*
**then have** ‹*angle-c C A B = angle-c C A Q + angle-c Q A R + angle-c R A B*›
**using** *angle-c-sum*[*of C A Q R*] *angle-c-sum*[*of C A R B*]
**by** (*smt* (*verit*) ‹¬ $R \neq Q$› ‹*angle-c C A B* = − *3* ∗ (*angle-c B A C / 3*)›
‹*angle-c C A Q* = − (*angle-c B A C / 3*)› ‹*angle-c R A B* = − (*angle-c B A C / 3*)›
*angle-c-neq0 canon-ang*(*1*) *canon-ang*(*2*) *canon-ang-id*)
**then show** *False*
**using** ‹¬ $R \neq Q$›
**by** (*smt* (*verit*, *best*) ‹*angle-c C A B* = − *3* ∗ (*angle-c B A C / 3*)›
‹*angle-c C A Q* = − (*angle-c B A C / 3*)› ‹*angle-c R A B* = − (*angle-c B A C / 3*)›
*angle-c-neq0 neq-0 zero-canonical*)
**qed**
**then have** ‹*cdist R P = cdist R Q* ∧ *cdist R Q = cdist Q P*›
**using** *equilateral-caracterization-neg*[*of R Q P*] *impp* **unfolding** *defs*
**by** (*metis cdist-commute*)
**then have** *?thesis*
**using** *cdist-commute* **by** *auto*
**}note** *case-pi3=this*
**then show** *?thesis* **using** *hhh* **by** *auto*
**qed**

**theorem** *Morley-neg*:
**assumes**‹¬*collinear A B C*›
‹*angle-c A B R = angle-c A B C / 3*› (**is** ‹*?abr = ?abc*›)
*angle-c B A R = angle-c B A C / 3* (**is** ‹*?bar = ?α*›)
*angle-c B C P = angle-c B C A / 3* (**is** ‹*?bcp = ?bca*›)
*angle-c C B P = angle-c C B A / 3* (**is** ‹*?cbp = ?β*›)
*angle-c C A Q = angle-c C A B / 3* (**is** ‹*?caq = ?cab*›)
*angle-c A C Q = angle-c A C B / 3* (**is** ‹*?acq = ?γ*›)
**and** *hhh*:‹⌊*angle-c B A C / 3+angle-c C B A / 3+angle-c A C B / 3*⌋ = −*pi/3*›
**shows** ‹*cdist R P = cdist P Q* ∧ *cdist Q R = cdist P Q*›
**proof** −
**have** ‹⌊−*angle-c B A C / 3*+−*angle-c C B A / 3*+−*angle-c A C B / 3*⌋ = *pi/3*›
**using** *hhh*
**by** (*metis* (*no-types*, *opaque-lifting*) *canon-ang*(*1*) *canon-ang-uminus less-eq-real-def*

*linorder-not-le minus-add-distrib minus-divide-left mult-le-0-iff*
*nonzero-mult-div-cancel-left not-numeral-le-zero pi-gt-zero times-divide-eq-right*
*verit-minus-simplify*(*4*) *zero-canonical*)
**then show** *?thesis* **using** *Morley-pos*[*of C B A P Q R*]
**by** (*smt* (*verit*, *best*) *Morley-pos angle-c-commute assms*(*1*) *assms*(*2*) *assms*(*3*)
*assms*(*4*) *assms*(*5*)
*assms*(*6*) *assms*(*7*) *cdist-commute collinear-sin-neq-0 collinear-sym1 collinear-sym2*
*sin-pi*)
**qed**

**theorem** *Morley*:
**assumes**‹¬*collinear A B C*›
‹*angle-c A B R* = *angle-c A B C* / *3*› (**is** ‹*?abr* = *?abc*›)
*angle-c B A R* = *angle-c B A C* / *3* (**is** ‹*?bar* = *?α*›)
*angle-c B C P* = *angle-c B C A* / *3* (**is** ‹*?bcp* = *?bca*›)
*angle-c C B P* = *angle-c C B A* / *3* (**is** ‹*?cbp* = *?β*›)
*angle-c C A Q* = *angle-c C A B* / *3* (**is** ‹*?caq* = *?cab*›)
*angle-c A C Q* = *angle-c A C B* / *3* (**is** ‹*?acq* = *?γ*›)
**shows** ‹*cdist R P* = *cdist P Q* ∧ *cdist Q R* = *cdist P Q*›
**proof** −
{**fix** *A B C γ*
**assume** *ABC-nline*:‹*A* ∉ *line B C*›
**and** *eq-3c*:‹*angle-c A C B* = *3*∗*γ*›
**then have** *neq-PI*:‹*abs γ* < *pi*/*3*›
**proof** −
**have** ‹*angle-c A C B* ≠ *pi*› **using** *ABC-nline*(*1*) **unfolding** *line-def*
**by** (*metis angle-c-commute-pi collinear-iff mem-Collect-eq non-collinear-independant*)
**then have** ‹*angle-c A C B* ∈ {−*pi*<..<*pi*}›
**using** *ang-c-in less-eq-real-def* **by** *auto*
**then show** ‹*abs γ* < *pi*/*3*›
**using** *eq-3c* **by** *force*
**qed**}**note** *ang-inf-pi3*=*this*
**have** ‹⌊*angle-c B A C* + *angle-c C B A* + *angle-c A C B*⌋ = *pi*›
**by** (*metis Elementary-Complex-Geometry.collinear-def add.commute*
*angle-sum-triangle-c assms*(*1*) *collinear-sym1 collinear-sym2′*)
**have** *eq-pi*:‹⌊*3*∗*?α* + *3*∗*?β* + *3*∗*?γ*⌋ = *pi*›
**using** ‹⌊*angle-c B A C* + *angle-c C B A* + *angle-c A C B*⌋ = *pi*› **by** *force*
**have** ‹*A*∉*line B C* ∧ *B*∉ *line A C* ∧ *C* ∉ *line A B*›
**using** *assms*(*1*) **unfolding** *line-def* **using** *collinear-sym1 collinear-sym2′* **by**(*auto*)

**then have** *f2*:‹*abs ?α* < *pi*/*3* ∧ *abs ?β* < *pi*/*3* ∧ *abs ?γ* < *pi*/*3*›
**using** *ang-inf-pi3*
**by** (*smt* (*verit*, *ccfv-SIG*) *ang-vec-bounded angle-c-def assms*(*1*) *collinear-sin-neq-0*

*collinear-sym1 collinear-sym2 divide-less-cancel minus-divide-left sin-pi*)
**have** *possi-abc*:‹⌊*?α*+*?β*+*?γ*⌋ =*pi*/*3* ∨ ⌊*?α*+*?β*+*?γ*⌋ = −*pi*/*3*›
**proof** −
**have** ‹*?α*+*?β*+*?γ* ≤ *pi*›

**using** *f2* **by**(*auto simp*:*field-simps*)
**then have** ‹⌊?α+?β+?γ⌋ = ?α+?β+?γ›
**using** *canon-ang-id f2* **by** *fastforce*
**then have** ‹⌊3*⌊?α+?β+?γ⌋⌋ = pi›
**using** *eq-pi* **by** *force*
**then show** *?thesis*
**by** (*smt* (*verit*) *add-divide-distrib arccos-minus-one-half arccos-one-half arcsin-minus-one-half*
*arcsin-one-half arcsin-plus-arccos canon-ang-id canon-ang-minus-3pi-minus-pi canon-ang-pi-3pi*

*eq-pi f2 field-sum-of-halves*)
**qed**
**then show** *?thesis*
**using** *Morley-neg Morley-pos assms*(*1*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*) *assms*(*6*) *assms*(*7*)
**by** *meson*
**qed**

**end**

# References


[1] Manuel Eberl *Basic Geometric Properties of Triangles*, Archive of Formal Proofs, December 2015 https://isa-afp.org/entries/Triangle.html

[2] Morley theorem http://denisevellachemla.eu/Alain-Connes-Theoreme-Morley.pdf.

[3] Wiedijk's catalogue "Formalizing 100 Theorems" https://www.cs.ru.nl/~freek/100/,Itappearsatposition121....