

Mereology

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Abstract

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

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1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.¹ We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].²

We will use the following notation throughout.³

```

typedecl i
consts part :: i ⇒ i ⇒ bool (P)
consts overlap :: i ⇒ i ⇒ bool (O)
consts proper-part :: i ⇒ i ⇒ bool (PP)
consts sum :: i ⇒ i ⇒ i (infix ⊕ 52)
consts product :: i ⇒ i ⇒ i (infix ⊗ 53)
consts difference :: i ⇒ i ⇒ i (infix ⊖ 51)
consts complement:: i ⇒ i (−)
consts universe :: i (u)
consts general-sum :: (i ⇒ bool) ⇒ i (binder σ 9)
consts general-product :: (i ⇒ bool) ⇒ i (binder π [8] 9)

```

2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.⁴ In other words, parthood is assumed to be a partial ordering relation.⁵ Overlap is defined as common parthood.⁶

```

locale PM =
  assumes part-reflexivity: P x x
  assumes part-transitivity : P x y ⇒ P y z ⇒ P x z

```

¹For similar developments see [Sen, 2017] and [Bittner, 2018].

²For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.

³See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.

⁴For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].

⁵Hence the name *premereology*, from [Parsons, 2014] p. 6.

⁶See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
begin

2.1 Parthood

lemma *identity-implies-part* : $x = y \implies P\ x\ y$
<proof>

2.2 Overlap

lemma *overlap-intro*: $P\ z\ x \implies P\ z\ y \implies O\ x\ y$
<proof>

lemma *part-implies-overlap*: $P\ x\ y \implies O\ x\ y$
<proof>

lemma *overlap-reflexivity*: $O\ x\ x$
<proof>

lemma *overlap-symmetry*: $O\ x\ y \implies O\ y\ x$
<proof>

lemma *overlap-monotonicity*: $P\ x\ y \implies O\ z\ x \implies O\ z\ y$
<proof>

The next lemma is from [Hovda, 2009] p. 66.

lemma *overlap-lemma*: $\exists\ x.\ (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$
<proof>

2.3 Disjointness

lemma *disjoint-implies-distinct*: $\neg\ O\ x\ y \implies x \neq y$
<proof>

lemma *disjoint-implies-not-part*: $\neg\ O\ x\ y \implies \neg\ P\ x\ y$
<proof>

lemma *disjoint-symmetry*: $\neg\ O\ x\ y \implies \neg\ O\ y\ x$
<proof>

lemma *disjoint-demonotonicity*: $P\ x\ y \implies \neg\ O\ z\ y \implies \neg\ O\ z\ x$
<proof>

end

3 Ground Mereology

The theory of *ground mereology* adds to premereology the anti-symmetry of parthood, and defines proper parthood as nonidentical parthood.⁷ In other words, ground mereology assumes that parthood is a partial order.

locale $M = PM +$
 assumes *part-antisymmetry*: $P x y \implies P y x \implies x = y$
 assumes *nip-eq*: $PP x y \longleftrightarrow P x y \wedge x \neq y$
begin

3.1 Proper Parthood

lemma *proper-implies-part*: $PP x y \implies P x y$
<proof>

lemma *proper-implies-distinct*: $PP x y \implies x \neq y$
<proof>

lemma *proper-implies-not-part*: $PP x y \implies \neg P y x$
<proof>

lemma *proper-part-asymmetry*: $PP x y \implies \neg PP y x$
<proof>

lemma *proper-implies-overlap*: $PP x y \implies O x y$
<proof>

end

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.⁸ In the presence of antisymmetry, the two definitions of proper parthood are equivalent.⁹

locale $M1 = PM +$
 assumes *nmp-eq*: $PP x y \longleftrightarrow P x y \wedge \neg P y x$
 assumes *part-antisymmetry*: $P x y \implies P y x \implies x = y$

⁷For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

⁸See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

⁹See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

sublocale $M \subseteq M1$

<proof>

sublocale $M1 \subseteq M$

<proof>

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.¹⁰

locale $M2 = PM +$

assumes *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$

assumes *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg P\ y\ x$

sublocale $M \subseteq M2$

<proof>

sublocale $M2 \subseteq M$

<proof>

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.¹¹

locale $M3 = PM +$

assumes *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$

assumes *part-extensionality*: $x = y \longleftrightarrow (\forall z. P\ z\ x \longleftrightarrow P\ z\ y)$

sublocale $M \subseteq M3$

<proof>

sublocale $M3 \subseteq M$

<proof>

The fourth axiomatization adopts proper parthood as primitive.¹² Improper parthood is defined as proper parthood or identity.

locale $M4 =$

assumes *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$

assumes *proper-part-asymmetry*: $PP\ x\ y \implies \neg PP\ y\ x$

assumes *proper-part-transitivity*: $PP\ x\ y \implies PP\ y\ z \implies PP\ x\ z$

begin

¹⁰For this point see especially [Parsons, 2014] pp. 9-10.

¹¹For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.

¹²See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

lemma *proper-part-irreflexivity*: $\neg PP\ x\ x$
<proof>

end

sublocale $M \subseteq M4$
<proof>

sublocale $M4 \subseteq M$
<proof>

4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.¹³

locale $MM = M +$
assumes *weak-supplementation*: $PP\ y\ x \implies (\exists z. P\ z\ x \wedge \neg O\ z\ y)$

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.¹⁴

locale $MM1 = M +$
assumes *proper-weak-supplementation*:
 $PP\ y\ x \implies (\exists z. PP\ z\ x \wedge \neg O\ z\ y)$

sublocale $MM \subseteq MM1$
<proof>

sublocale $MM1 \subseteq MM$
<proof>

The following two corollaries are sometimes found in the literature.¹⁵

context MM
begin

corollary *weak-company*: $PP\ y\ x \implies (\exists z. PP\ z\ x \wedge z \neq y)$
<proof>

¹³See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

¹⁴See [Simons, 1987] p. 28.

¹⁵See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].

corollary *strong-companly*: $PP\ y\ x \implies (\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y)$
 ⟨*proof*⟩

end

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.¹⁶

locale $MM2 = PM +$
assumes *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$
assumes *weak-supplementation*: $PP\ y\ x \implies (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

sublocale $MM2 \subseteq MM$
 ⟨*proof*⟩

sublocale $MM \subseteq MM2$
 ⟨*proof*⟩

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.¹⁷

locale $MM3 =$
assumes *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$
assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
assumes *proper-part-transitivity*: $PP\ x\ y \implies PP\ y\ z \implies PP\ x\ z$
assumes *weak-supplementation*: $PP\ y\ x \implies (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

lemma *part-reflexivity*: $P\ x\ x$
 ⟨*proof*⟩

lemma *proper-part-irreflexivity*: $\neg\ PP\ x\ x$
 ⟨*proof*⟩

end

sublocale $MM3 \subseteq M4$
 ⟨*proof*⟩

sublocale $MM3 \subseteq MM$
 ⟨*proof*⟩

sublocale $MM \subseteq MM3$

¹⁶See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.

¹⁷See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

<proof>

5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.¹⁸

locale $EM = M +$

assumes *strong-supplementation*:

$$\neg P x y \implies (\exists z. P z x \wedge \neg O z y)$$

begin

Strong supplementation entails weak supplementation.¹⁹

lemma *weak-supplementation*: $PP x y \implies (\exists z. P z y \wedge \neg O z x)$

<proof>

end

So minimal mereology is a subtheory of extensional mereology.²⁰

sublocale $EM \subseteq MM$

<proof>

Strong supplementation also entails the proper parts principle.²¹

context EM

begin

lemma *proper-parts-principle*:

$$(\exists z. PP z x) \implies (\forall z. PP z x \longrightarrow P z y) \implies P x y$$

<proof>

Which with antisymmetry entails the extensionality of parthood.²²

theorem *proper-part-extensionality*:

$$(\exists z. PP z x \vee PP z y) \implies x = y \iff (\forall z. PP z x \iff PP z y)$$

<proof>

It also follows from strong supplementation that parthood is definable in terms of overlap.²³

lemma *part-overlap-eq*: $P x y \iff (\forall z. O z x \longrightarrow O z y)$

<proof>

Which entails the extensionality of overlap.

¹⁸See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.

¹⁹See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.

²⁰[Casati and Varzi, 1999] p. 40.

²¹See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.

²²See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.

²³See [Parsons, 2014] p. 4.

theorem *overlap-extensionality*: $x = y \longleftrightarrow (\forall z. O z x \longleftrightarrow O z y)$
 ⟨proof⟩

end

6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.²⁴

locale $CM = M +$

assumes *sum-eq*: $x \oplus y = (THE z. \forall v. O v z \longleftrightarrow O v x \vee O v y)$

assumes *sum-closure*: $\exists z. \forall v. O v z \longleftrightarrow O v x \vee O v y$

assumes *product-eq*:

$x \otimes y = (THE z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$

assumes *product-closure*:

$O x y \implies \exists z. \forall v. P v z \longleftrightarrow P v x \wedge P v y$

begin

6.1 Products

lemma *product-intro*:

$(\forall w. P w z \longleftrightarrow (P w x \wedge P w y)) \implies x \otimes y = z$

⟨proof⟩

lemma *product-idempotence*: $x \otimes x = x$

⟨proof⟩

lemma *product-character*:

$O x y \implies (\forall w. P w (x \otimes y) \longleftrightarrow (P w x \wedge P w y))$

⟨proof⟩

lemma *product-commutativity*: $O x y \implies x \otimes y = y \otimes x$

⟨proof⟩

lemma *product-in-factors*: $O x y \implies P (x \otimes y) x \wedge P (x \otimes y) y$

⟨proof⟩

lemma *product-in-first-factor*: $O x y \implies P (x \otimes y) x$

⟨proof⟩

lemma *product-in-second-factor*: $O x y \implies P (x \otimes y) y$

⟨proof⟩

lemma *nonpart-implies-proper-product*:

²⁴See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

$\neg P x y \wedge O x y \implies PP (x \otimes y) x$
 $\langle \text{proof} \rangle$

lemma *common-part-in-product*: $P z x \wedge P z y \implies P z (x \otimes y)$
 $\langle \text{proof} \rangle$

lemma *product-part-in-factors*:
 $O x y \implies P z (x \otimes y) \implies P z x \wedge P z y$
 $\langle \text{proof} \rangle$

corollary *product-part-in-first-factor*:
 $O x y \implies P z (x \otimes y) \implies P z x$
 $\langle \text{proof} \rangle$

corollary *product-part-in-second-factor*:
 $O x y \implies P z (x \otimes y) \implies P z y$
 $\langle \text{proof} \rangle$

lemma *part-product-identity*: $P x y \implies x \otimes y = x$
 $\langle \text{proof} \rangle$

lemma *product-overlap*: $P z x \implies O z y \implies O z (x \otimes y)$
 $\langle \text{proof} \rangle$

lemma *disjoint-from-second-factor*:
 $P x y \wedge \neg O x (y \otimes z) \implies \neg O x z$
 $\langle \text{proof} \rangle$

lemma *converse-product-overlap*:
 $O x y \implies O z (x \otimes y) \implies O z y$
 $\langle \text{proof} \rangle$

lemma *part-product-in-whole-product*:
 $O x y \implies P x v \wedge P y z \implies P (x \otimes y) (v \otimes z)$
 $\langle \text{proof} \rangle$

lemma *right-associated-product*: $(\exists w. P w x \wedge P w y \wedge P w z) \implies$
 $(\forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge (P w y \wedge P w z))$
 $\langle \text{proof} \rangle$

lemma *left-associated-product*: $(\exists w. P w x \wedge P w y \wedge P w z) \implies$
 $(\forall w. P w ((x \otimes y) \otimes z) \longleftrightarrow (P w x \wedge P w y) \wedge P w z)$
 $\langle \text{proof} \rangle$

theorem *product-associativity*:
 $(\exists w. P w x \wedge P w y \wedge P w z) \implies x \otimes (y \otimes z) = (x \otimes y) \otimes z$
 $\langle \text{proof} \rangle$

end

6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.²⁵

locale $CMD = CM +$

assumes *difference-eq*:

$$x \ominus y = (\text{THE } z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$$

assumes *difference-closure*:

$$(\exists w. P w x \wedge \neg O w y) \implies (\exists z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$$

begin

lemma *difference-intro*:

$$(\forall w. P w z \longleftrightarrow P w x \wedge \neg O w y) \implies x \ominus y = z$$

<proof>

lemma *difference-idempotence*: $\neg O x y \implies (x \ominus y) = x$

<proof>

lemma *difference-character*: $(\exists w. P w x \wedge \neg O w y) \implies$

$$(\forall w. P w (x \ominus y) \longleftrightarrow P w x \wedge \neg O w y)$$

<proof>

lemma *difference-disjointness*:

$$(\exists z. P z x \wedge \neg O z y) \implies \neg O y (x \ominus y)$$

<proof>

end

6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.²⁶

locale $CMU = CM +$

assumes *universe-eq*: $u = (\text{THE } z. \forall w. P w z)$

assumes *universe-closure*: $\exists y. \forall x. P x y$

begin

lemma *universe-intro*: $(\forall w. P w z) \implies u = z$

<proof>

lemma *universe-character*: $P x u$

<proof>

lemma $\neg PP u x$

<proof>

²⁵See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.

²⁶See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

lemma *product-universe-implies-factor-universe*:

$O x y \implies x \otimes y = u \implies x = u$
 $\langle \text{proof} \rangle$

end

6.4 Complements

As is a condition ensuring the existence of complements.²⁷

locale *CMC* = *CM* +

assumes *complement-eq*: $-x = (\text{THE } z. \forall w. P w z \longleftrightarrow \neg O w x)$

assumes *complement-closure*:

$(\exists w. \neg O w x) \implies (\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$

assumes *difference-eq*:

$x \ominus y = (\text{THE } z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$

begin

lemma *complement-intro*:

$(\forall w. P w z \longleftrightarrow \neg O w x) \implies -x = z$
 $\langle \text{proof} \rangle$

lemma *complement-character*:

$(\exists w. \neg O w x) \implies (\forall w. P w (-x) \longleftrightarrow \neg O w x)$
 $\langle \text{proof} \rangle$

lemma *not-complement-part*: $\exists w. \neg O w x \implies \neg P x (-x)$

$\langle \text{proof} \rangle$

lemma *complement-part*: $\neg O x y \implies P x (-y)$

$\langle \text{proof} \rangle$

lemma *complement-overlap*: $\neg O x y \implies O x (-y)$

$\langle \text{proof} \rangle$

lemma *or-complement-overlap*: $\forall y. O y x \vee O y (-x)$

$\langle \text{proof} \rangle$

lemma *complement-disjointness*: $\exists v. \neg O v x \implies \neg O x (-x)$

$\langle \text{proof} \rangle$

lemma *part-disjoint-from-complement*:

$\exists v. \neg O v x \implies P y x \implies \neg O y (-x)$
 $\langle \text{proof} \rangle$

lemma *product-complement-character*: $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\forall w. P w (x \otimes (-y)) \longleftrightarrow (P w x \wedge (\neg O w y)))$

²⁷See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

<proof>

theorem *difference-closure*: $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\exists z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$

<proof>

end

sublocale $CMC \subseteq CMD$

<proof>

corollary (in CMC) *difference-is-product-of-complement*:

$(\exists w. P w x \wedge \neg O w y) \implies (x \ominus y) = x \otimes (-y)$

<proof>

Universe and difference closure entail complement closure, since the difference of an individual and the universe is the individual's complement.

locale $CMUD = CMU + CMD +$

assumes *complement-eq*: $-x = (THE z. \forall w. P w z \longleftrightarrow \neg O w x)$

begin

lemma *universe-difference*:

$(\exists w. \neg O w x) \implies (\forall w. P w (u \ominus x) \longleftrightarrow \neg O w x)$

<proof>

theorem *complement-closure*:

$(\exists w. \neg O w x) \implies (\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$

<proof>

end

sublocale $CMUD \subseteq CMC$

<proof>

corollary (in $CMUD$) *complement-universe-difference*:

$(\exists y. \neg O y x) \implies -x = (u \ominus x)$

<proof>

7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.²⁸

locale $CEM = CM + EM$

Likewise, closed minimal mereology combines closed mereology

²⁸See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.

with minimal mereology.²⁹

locale $CMM = CM + MM$

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.³⁰

sublocale $CMM \subseteq CEM$

<proof>

sublocale $CEM \subseteq CMM$ *<proof>*

7.1 Sums

context CEM

begin

lemma *sum-intro*:

$(\forall w. O w z \longleftrightarrow (O w x \vee O w y)) \implies x \oplus y = z$
<proof>

lemma *sum-idempotence*: $x \oplus x = x$

<proof>

lemma *part-sum-identity*: $P y x \implies x \oplus y = x$

<proof>

lemma *sum-character*: $\forall w. O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$

<proof>

lemma *sum-overlap*: $O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$

<proof>

lemma *sum-part-character*:

$P w (x \oplus y) \longleftrightarrow (\forall v. O v w \longrightarrow O v x \vee O v y)$
<proof>

lemma *sum-commutativity*: $x \oplus y = y \oplus x$

<proof>

lemma *first-summand-overlap*: $O z x \implies O z (x \oplus y)$

<proof>

lemma *first-summand-disjointness*: $\neg O z (x \oplus y) \implies \neg O z x$

<proof>

²⁹See [Casati and Varzi, 1999] p. 43.

³⁰See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

lemma *first-summand-in-sum*: $P x (x \oplus y)$
<proof>

lemma *common-first-summand*: $P x (x \oplus y) \wedge P x (x \oplus z)$
<proof>

lemma *common-first-summand-overlap*: $O (x \oplus y) (x \oplus z)$
<proof>

lemma *second-summand-overlap*: $O z y \implies O z (x \oplus y)$
<proof>

lemma *second-summand-disjointness*: $\neg O z (x \oplus y) \implies \neg O z y$
<proof>

lemma *second-summand-in-sum*: $P y (x \oplus y)$
<proof>

lemma *second-summands-in-sums*: $P y (x \oplus y) \wedge P v (z \oplus v)$
<proof>

lemma *disjoint-from-sum*: $\neg O z (x \oplus y) \longleftrightarrow \neg O z x \wedge \neg O z y$
<proof>

lemma *summands-part-implies-sum-part*:
 $P x z \wedge P y z \implies P (x \oplus y) z$
<proof>

lemma *sum-part-implies-summands-part*:
 $P (x \oplus y) z \implies P x z \wedge P y z$
<proof>

lemma *in-second-summand*: $P z (x \oplus y) \wedge \neg O z x \implies P z y$
<proof>

lemma *disjoint-second-summands*:
 $P v (x \oplus y) \wedge P v (x \oplus z) \implies \neg O y z \implies P v x$
<proof>

lemma *right-associated-sum*:
 $O w (x \oplus (y \oplus z)) \longleftrightarrow O w x \vee (O w y \vee O w z)$
<proof>

lemma *left-associated-sum*:
 $O w ((x \oplus y) \oplus z) \longleftrightarrow (O w x \vee O w y) \vee O w z$
<proof>

theorem *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
<proof>

7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

lemma *common-summand-in-product*: $P x ((x \oplus y) \otimes (x \oplus z))$
 $\langle proof \rangle$

lemma *product-in-first-summand*:
 $\neg O y z \implies P ((x \oplus y) \otimes (x \oplus z)) x$
 $\langle proof \rangle$

lemma *product-is-first-summand*:
 $\neg O y z \implies (x \oplus y) \otimes (x \oplus z) = x$
 $\langle proof \rangle$

lemma *sum-over-product-left*: $O y z \implies P (x \oplus (y \otimes z)) ((x \oplus y) \otimes (x \oplus z))$
 $\langle proof \rangle$

lemma *sum-over-product-right*:
 $O y z \implies P ((x \oplus y) \otimes (x \oplus z)) (x \oplus (y \otimes z))$
 $\langle proof \rangle$

Sums distribute over products.

theorem *sum-over-product*:
 $O y z \implies x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
 $\langle proof \rangle$

lemma *product-in-factor-by-sum*:
 $O x y \implies P (x \otimes y) (x \otimes (y \oplus z))$
 $\langle proof \rangle$

lemma *product-of-first-summand*:
 $O x y \implies \neg O x z \implies P (x \otimes (y \oplus z)) (x \otimes y)$
 $\langle proof \rangle$

theorem *disjoint-product-over-sum*:
 $O x y \implies \neg O x z \implies x \otimes (y \oplus z) = x \otimes y$
 $\langle proof \rangle$

lemma *product-over-sum-left*:
 $O x y \wedge O x z \implies P (x \otimes (y \oplus z)) ((x \otimes y) \oplus (x \otimes z))$
 $\langle proof \rangle$

lemma *product-over-sum-right*:
 $O x y \wedge O x z \implies P ((x \otimes y) \oplus (x \otimes z)) (x \otimes (y \oplus z))$
 $\langle proof \rangle$

theorem *product-over-sum*:

$O x y \wedge O x z \implies x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
 $\langle proof \rangle$

lemma *joint-identical-sums*:

$v \oplus w = x \oplus y \implies O x v \wedge O x w \implies ((x \otimes v) \oplus (x \otimes w)) = x$
 $\langle proof \rangle$

lemma *disjoint-identical-sums*:

$v \oplus w = x \oplus y \implies \neg O y v \wedge \neg O w x \implies x = v \wedge y = w$
 $\langle proof \rangle$

end

7.3 Differences

locale $CEMD = CEM + CMD$

begin

lemma *plus-minus*: $PP y x \implies y \oplus (x \ominus y) = x$
 $\langle proof \rangle$

end

7.4 The Universe

locale $CEMU = CEM + CMU$

begin

lemma *something-disjoint*: $x \neq u \implies (\exists v. \neg O v x)$
 $\langle proof \rangle$

lemma *overlaps-universe*: $O x u$
 $\langle proof \rangle$

lemma *universe-absorbing*: $x \oplus u = u$
 $\langle proof \rangle$

lemma *second-summand-not-universe*: $x \oplus y \neq u \implies y \neq u$
 $\langle proof \rangle$

lemma *first-summand-not-universe*: $x \oplus y \neq u \implies x \neq u$
 $\langle proof \rangle$

end

7.5 Complements

locale $CEMC = CEM + CMC +$

assumes *universe-eq*: $u = (THE x. \forall y. P y x)$

begin

lemma *complement-sum-character*: $\forall y. P y (x \oplus (-x))$

\langle *proof* \rangle

lemma *universe-closure*: $\exists x. \forall y. P y x$

\langle *proof* \rangle

end

sublocale $CEMC \subseteq CEMU$

\langle *proof* \rangle

sublocale $CEMC \subseteq CEMD$

\langle *proof* \rangle

context $CEMC$

begin

corollary *universe-is-complement-sum*: $u = x \oplus (-x)$

\langle *proof* \rangle

lemma *strong-complement-character*:

$x \neq u \implies (\forall v. P v (-x) \longleftrightarrow \neg O v x)$

\langle *proof* \rangle

lemma *complement-part-not-part*: $x \neq u \implies P y (-x) \implies \neg P y x$

\langle *proof* \rangle

lemma *complement-involution*: $x \neq u \implies x = -(-x)$

\langle *proof* \rangle

lemma *part-complement-reversal*: $y \neq u \implies P x y \implies P (-y) (-x)$

\langle *proof* \rangle

lemma *complements-overlap*: $x \oplus y \neq u \implies O(-x)(-y)$

\langle *proof* \rangle

lemma *sum-complement-in-complement-product*:

$x \oplus y \neq u \implies P(-(x \oplus y))(-x \otimes -y)$

\langle *proof* \rangle

lemma *complement-product-in-sum-complement*:

$x \oplus y \neq u \implies P(-x \otimes -y)(-(x \oplus y))$

\langle *proof* \rangle

theorem *sum-complement-is-complements-product*:

$x \oplus y \neq u \implies -(x \oplus y) = (-x \otimes -y)$

\langle *proof* \rangle

lemma *complement-sum-in-product-complement:*

$O x y \implies x \neq u \implies y \neq u \implies P((-x) \oplus (-y))(-(x \otimes y))$
(proof)

lemma *product-complement-in-complements-sum:*

$x \neq u \implies y \neq u \implies P(-(x \otimes y))((-x) \oplus (-y))$
(proof)

theorem *complement-of-product-is-sum-of-complements:*

$O x y \implies x \oplus y \neq u \implies -(x \otimes y) = (-x) \oplus (-y)$
(proof)

end

8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.³¹

locale $GM = M +$

assumes *fusion:*

$\exists x. \varphi x \implies \exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x)$

begin

Fusion entails sum closure.

theorem *sum-closure:* $\exists z. \forall w. O w z \longleftrightarrow (O w a \vee O w b)$

(proof)

end

9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.³²

locale $GMM = GM + MM$

begin

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general minimal mereology and general extensional mereology.³³ But

³¹See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

³²See [Casati and Varzi, 1999] p. 46.

³³For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

lemma *product-closure*:

$$O x y \implies (\exists z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$$

nitpick [*expect = genuine*] *<proof>*

lemma *strong-supplementation*: $\neg P x y \implies (\exists z. P z x \wedge \neg O z y)$

nitpick [*expect = genuine*] *<proof>*

end

10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.³⁴

locale *GEM = GM + EM +*

assumes *sum-eq*: $x \oplus y = (\text{THE } z. \forall v. O v z \longleftrightarrow O v x \vee O v y)$

assumes *product-eq*:

$$x \otimes y = (\text{THE } z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$$

assumes *difference-eq*:

$$x \ominus y = (\text{THE } z. \forall w. P w z = (P w x \wedge \neg O w y))$$

assumes *complement-eq*: $\neg x = (\text{THE } z. \forall w. P w z \longleftrightarrow \neg O w x)$

assumes *universe-eq*: $u = (\text{THE } x. \forall y. P y x)$

assumes *fusion-eq*: $\exists x. F x \implies$

$$(\sigma x. F x) = (\text{THE } x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z))$$

assumes *general-product-eq*: $(\pi x. F x) = (\sigma x. \forall y. F y \longrightarrow P x y)$

sublocale *GEM* \subseteq *GMM*

<proof>

10.1 General Sums

context *GEM*

begin

lemma *fusion-intro*:

$$(\forall y. O y z \longleftrightarrow (\exists x. F x \wedge O y x)) \implies (\sigma x. F x) = z$$

<proof>

lemma *fusion-idempotence*: $(\sigma x. z = x) = z$

³⁴For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

$\langle proof \rangle$

The whole is the sum of its parts.

lemma *fusion-absorption*: $(\sigma x. P x z) = z$
 $\langle proof \rangle$

lemma *part-fusion*: $P w (\sigma v. P v x) \implies P w x$
 $\langle proof \rangle$

lemma *fusion-character*:
 $\exists x. F x \implies (\forall y. O y (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O y x))$
 $\langle proof \rangle$

The next lemma characterises fusions in terms of parthood.³⁵

lemma *fusion-part-character*: $\exists x. F x \implies$
 $(\forall y. P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)))$
 $\langle proof \rangle$

lemma *fusion-part*: $F x \implies P x (\sigma x. F x)$
 $\langle proof \rangle$

lemma *common-part-fusion*:
 $O x y \implies (\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y))$
 $\langle proof \rangle$

theorem *product-closure*:
 $O x y \implies (\exists z. \forall w. P w z \longleftrightarrow (P w x \wedge P w y))$
 $\langle proof \rangle$

end

sublocale $GEM \subseteq CEM$
 $\langle proof \rangle$

context GEM
begin

corollary $O x y \implies x \otimes y = (\sigma v. P v x \wedge P v y)$
 $\langle proof \rangle$

lemma *disjoint-fusion*:
 $\exists w. \neg O w x \implies (\forall w. P w (\sigma z. \neg O z x) \longleftrightarrow \neg O w x)$
 $\langle proof \rangle$

theorem *complement-closure*:
 $\exists w. \neg O w x \implies (\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$
 $\langle proof \rangle$

³⁵See [Pontow, 2004] pp. 202-9.

end

sublocale $GEM \subseteq CEMC$

\langle *proof* \rangle

context GEM

begin

corollary *complement-is-disjoint-fusion*:

$$\exists w. \neg O w x \implies \neg x = (\sigma z. \neg O z x)$$

\langle *proof* \rangle

theorem *strong-fusion*: $\exists x. F x \implies$

$$\exists x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$$

\langle *proof* \rangle

theorem *strong-fusion-eq*: $\exists x. F x \implies (\sigma x. F x) =$

$$(THE x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$$

\langle *proof* \rangle

lemma *strong-sum-eq*: $x \oplus y = (THE z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y))$

\langle *proof* \rangle

10.2 General Products

lemma *general-product-intro*: $(\forall y. O y x \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)) \implies (\pi x. F x) = x$

\langle *proof* \rangle

lemma *general-product-idempotence*: $(\pi z. z = x) = x$

\langle *proof* \rangle

lemma *general-product-absorption*: $(\pi z. P x z) = x$

\langle *proof* \rangle

lemma *general-product-character*: $\exists z. \forall y. F y \longrightarrow P z y \implies$

$$\forall y. O y (\pi x. F x) \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)$$

\langle *proof* \rangle

corollary $\neg (\exists x. F x) \implies u = (\pi x. F x)$

\langle *proof* \rangle

end

10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.³⁶

locale $GEM1 = MM +$

assumes *strong-fusion*: $\exists x. F x \implies \exists x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$

assumes *strong-sum-eq*: $x \oplus y = (THE z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y))$

assumes *product-eq*:

$x \otimes y = (THE z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$

assumes *difference-eq*:

$x \ominus y = (THE z. \forall w. P w z = (P w x \wedge \neg O w y))$

assumes *complement-eq*: $\neg x = (THE z. \forall w. P w z \longleftrightarrow \neg O w x)$

assumes *universe-eq*: $u = (THE x. \forall y. P y x)$

assumes *strong-fusion-eq*: $\exists x. F x \implies (\sigma x. F x) = (THE x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$

assumes *general-product-eq*: $(\pi x. F x) = (\sigma x. \forall y. F y \longrightarrow P x y)$

begin

theorem *fusion*:

$\exists x. \varphi x \implies (\exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x))$

<proof>

lemma *pair*: $\exists v. (\forall w. (w = x \vee w = y) \longrightarrow P w v) \wedge (\forall w. P w v \longrightarrow (\exists z. (z = x \vee z = y) \wedge O w z))$

<proof>

lemma *or-id*: $(v = x \vee v = y) \wedge O w v \implies O w x \vee O w y$

<proof>

lemma *strong-sum-closure*:

$\exists z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$

<proof>

end

sublocale $GEM1 \subseteq GMM$

<proof>

context $GEM1$

begin

lemma *least-upper-bound*:

assumes *sf*:

³⁶See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

$((\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$
shows lub:
 $(\forall y. F y \longrightarrow P y x) \wedge (\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P x z)$
 <proof>

corollary strong-fusion-intro: $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)) \implies (\sigma x. F x) = x$
 <proof>

lemma strong-fusion-character: $\exists x. F x \implies ((\forall y. F y \longrightarrow P y (\sigma x. F x)) \wedge (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)))$
 <proof>

lemma F-in: $\exists x. F x \implies (\forall y. F y \longrightarrow P y (\sigma x. F x))$
 <proof>

lemma parts-overlap-Fs:
 $\exists x. F x \implies (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z))$
 <proof>

lemma in-strong-fusion: $P z (\sigma x. z = x)$
 <proof>

lemma strong-fusion-in: $P (\sigma x. z = x) z$
 <proof>

lemma strong-fusion-idempotence: $(\sigma x. z = x) = z$
 <proof>

10.4 Strong Sums

lemma pair-fusion: $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y) \longrightarrow (\sigma z. z = x \vee z = y) = z$
 <proof>

corollary strong-sum-fusion: $x \oplus y = (\sigma z. z = x \vee z = y)$
 <proof>

corollary strong-sum-intro:
 $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y) \longrightarrow x \oplus y = z$
 <proof>

corollary strong-sum-character: $(P x (x \oplus y) \wedge P y (x \oplus y)) \wedge (\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y)$
 <proof>

corollary summands-in: $(P x (x \oplus y) \wedge P y (x \oplus y))$
 <proof>

corollary *first-summand-in*: $P x (x \oplus y) \langle \text{proof} \rangle$

corollary *second-summand-in*: $P y (x \oplus y) \langle \text{proof} \rangle$

corollary *sum-part-overlap*: $(\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y) \langle \text{proof} \rangle$

lemma *strong-sum-absorption*: $y = (x \oplus y) \implies P x y \langle \text{proof} \rangle$

theorem *strong-supplementation*: $\neg P x y \implies (\exists z. P z x \wedge \neg O z y) \langle \text{proof} \rangle$

lemma *sum-character*: $\forall v. O v (x \oplus y) \longleftrightarrow (O v x \vee O v y) \langle \text{proof} \rangle$

lemma *sum-eq*: $x \oplus y = (\text{THE } z. \forall v. O v z = (O v x \vee O v y)) \langle \text{proof} \rangle$

theorem *fusion-eq*: $\exists x. F x \implies (\sigma x. F x) = (\text{THE } x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z)) \langle \text{proof} \rangle$

end

sublocale $GEM1 \subseteq GEM \langle \text{proof} \rangle$

sublocale $GEM \subseteq GEM1 \langle \text{proof} \rangle$

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