Mereology

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Abstract

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

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1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.[1] We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].[2]

We will use the following notation throughout.[3]

typedecl i
consts *part* :: $i \Rightarrow i \Rightarrow$ *bool* (P)
consts *overlap* :: $i \Rightarrow i \Rightarrow$ *bool* (O)
consts *proper-part* :: $i \Rightarrow i \Rightarrow$ *bool* (PP)
consts *sum* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\oplus$ *52*)
consts *product* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\otimes$ *53*)
consts *difference* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\ominus$ *51*)
consts *complement*:: $i \Rightarrow i$ $(-)$
consts *universe* :: i (u)
consts *general-sum* :: $(i \Rightarrow$ *bool*$) \Rightarrow i$ (**binder** σ *9*)
consts *general-product* :: $(i \Rightarrow$ *bool*$) \Rightarrow i$ (**binder** π [*8*] *9*)

2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.[4] In other words, parthood is assumed to be a partial ordering relation.[5] Overlap is defined as common parthood.[6]

locale PM =
  assumes *part-reflexivity*: $P\ x\ x$
  assumes *part-transitivity* : $P\ x\ y \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$

[1] For similar developments see [Sen, 2017] and [Bittner, 2018].

[2] For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.

[3] See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.

[4] For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].

[5] Hence the name *premereology*, from [Parsons, 2014] p. 6.

[6] See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
begin

2.1 Parthood

lemma *identity-implies-part* : $x = y \Longrightarrow P\ x\ y$
proof −
assume $x = y$
moreover have $P\ x\ x$ **by** (*rule part-reflexivity*)
ultimately show $P\ x\ y$ **by** (*rule subst*)
qed

2.2 Overlap

lemma *overlap-intro*: $P\ z\ x \Longrightarrow P\ z\ y \Longrightarrow O\ x\ y$
proof−
assume $P\ z\ x$
moreover assume $P\ z\ y$
ultimately have $P\ z\ x \wedge P\ z\ y$**..**
hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
with *overlap-eq* **show** $O\ x\ y$**..**
qed

lemma *part-implies-overlap*: $P\ x\ y \Longrightarrow O\ x\ y$
proof −
assume $P\ x\ y$
with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$**..**
hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
with *overlap-eq* **show** $O\ x\ y$**..**
qed

lemma *overlap-reflexivity*: $O\ x\ x$
proof −
have $P\ x\ x \wedge P\ x\ x$ **using** *part-reflexivity part-reflexivity***..**
hence $\exists\ z.\ P\ z\ x \wedge P\ z\ x$**..**
with *overlap-eq* **show** $O\ x\ x$**..**
qed

lemma *overlap-symmetry*: $O\ x\ y \Longrightarrow O\ y\ x$
proof−
assume $O\ x\ y$
with *overlap-eq* **have** $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
hence $\exists\ z.\ P\ z\ y \wedge P\ z\ x$ **by** *auto*
with *overlap-eq* **show** $O\ y\ x$**..**
qed

lemma *overlap-monotonicity*: $P\ x\ y \Longrightarrow O\ z\ x \Longrightarrow O\ z\ y$
proof −
assume $P\ x\ y$
assume $O\ z\ x$

```
  with overlap-eq have ∃ v. P v z ∧ P v x..
  then obtain v where v: P v z ∧ P v x..
  hence P v z..
  moreover from v have P v x..
  hence P v y using ‹P x y› by (rule part-transitivity)
  ultimately have P v z ∧ P v y..
  hence ∃ v. P v z ∧ P v y..
  with overlap-eq show O z y..
qed
```

The next lemma is from [Hovda, 2009] p. 66.

```
lemma overlap-lemma: ∃x. (P x y ∧ O z x) ⟶ O y z
proof −
  fix x
  have P x y ∧ O z x ⟶ O y z
  proof
    assume antecedent: P x y ∧ O z x
    hence O z x..
    with overlap-eq have ∃v. P v z ∧ P v x..
    then obtain v where v: P v z ∧ P v x..
    hence P v x..
    moreover from antecedent have P x y..
    ultimately have P v y by (rule part-transitivity)
    moreover from v have P v z..
    ultimately have P v y ∧ P v z..
    hence ∃v. P v y ∧ P v z..
    with overlap-eq show O y z..
  qed
  thus ∃x. (P x y ∧ O z x) ⟶ O y z..
qed
```

2.3 Disjointness

```
lemma disjoint-implies-distinct: ¬ O x y ⟹ x ≠ y
proof −
  assume ¬ O x y
  show x ≠ y
  proof
    assume x = y
    hence ¬ O y y using ‹¬ O x y› by (rule subst)
    thus False using overlap-reflexivity..
  qed
qed

lemma disjoint-implies-not-part: ¬ O x y ⟹ ¬ P x y
proof −
  assume ¬ O x y
  show ¬ P x y
  proof
```

```
    assume P x y
    hence O x y by (rule part-implies-overlap)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-symmetry: ¬ O x y ⟹ ¬ O y x
proof −
  assume ¬ O x y
  show ¬ O y x
  proof
    assume O y x
    hence O x y by (rule overlap-symmetry)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-demonotonicity: P x y ⟹ ¬ O z y ⟹ ¬ O z x
proof −
  assume P x y
  assume ¬ O z y
  show ¬ O z x
  proof
    assume O z x
    with ‹P x y› have O z y
      by (rule overlap-monotonicity)
    with ‹¬ O z y› show False..
  qed
qed

end
```

3 Ground Mereology

The theory of *ground mereology* adds to premereology the antisymmetry of parthood, and defines proper parthood as nonidentical parthood.[7] In other words, ground mereology assumes that parthood is a partial order.

```
locale M = PM +
  assumes part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
begin
```

[7] For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

3.1 Proper Parthood

lemma *proper-implies-part*: $PP\ x\ y \Longrightarrow P\ x\ y$
proof −
  assume $PP\ x\ y$
  with *nip-eq* **have** $P\ x\ y \wedge x \neq y$**..**
  thus $P\ x\ y$**..**
qed

lemma *proper-implies-distinct*: $PP\ x\ y \Longrightarrow x \neq y$
proof −
  assume $PP\ x\ y$
  with *nip-eq* **have** $P\ x\ y \wedge x \neq y$**..**
  thus $x \neq y$**..**
qed

lemma *proper-implies-not-part*: $PP\ x\ y \Longrightarrow \neg\ P\ y\ x$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  show $\neg\ P\ y\ x$
  proof
    from ‹$PP\ x\ y$› **have** $x \neq y$ **by** (*rule proper-implies-distinct*)
    moreover assume $P\ y\ x$
    with ‹$P\ x\ y$› **have** $x = y$ **by** (*rule part-antisymmetry*)
    ultimately show *False***..**
  qed
qed

lemma *proper-part-asymmetry*: $PP\ x\ y \Longrightarrow \neg\ PP\ y\ x$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  from ‹$PP\ x\ y$› **have** $x \neq y$ **by** (*rule proper-implies-distinct*)
  show $\neg\ PP\ y\ x$
  proof
    assume $PP\ y\ x$
    hence $P\ y\ x$ **by** (*rule proper-implies-part*)
    with ‹$P\ x\ y$› **have** $x = y$ **by** (*rule part-antisymmetry*)
    with ‹$x \neq y$› **show** *False***..**
  qed
qed

lemma *proper-implies-overlap*: $PP\ x\ y \Longrightarrow O\ x\ y$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  thus $O\ x\ y$ **by** (*rule part-implies-overlap*)
qed

end

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.[8] In the presence of antisymmetry, the two definitions of proper parthood are equivalent.[9]

```
locale M1 = PM +
  assumes nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x
  assumes part-antisymmetry: P x y ⟹ P y x ⟹ x = y

sublocale M ⊆ M1
proof
  fix x y
  show nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x
  proof
    assume PP x y
    with nip-eq have nip: P x y ∧ x ≠ y..
    hence x ≠ y..
    from nip have P x y..
    moreover have ¬ P y x
    proof
      assume P y x
      with ‹P x y› have x = y by (rule part-antisymmetry)
      with ‹x ≠ y› show False..
    qed
    ultimately show P x y ∧ ¬ P y x..
  next
    assume nmp: P x y ∧ ¬ P y x
    hence ¬ P y x..
    from nmp have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence ¬ P y y using ‹¬ P y x› by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately have P x y ∧ x ≠ y..
    with nip-eq show PP x y..
  qed
  show  P x y ⟹ P y x ⟹ x = y using part-antisymmetry.
qed
```

[8] See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

[9] See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

```
sublocale M1 ⊆ M
proof
  fix x y
  show nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    with nmp-eq have nmp: P x y ∧ ¬ P y x..
    hence ¬ P y x..
    from nmp have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence ¬ P y y using ‹¬ P y x› by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume nip: P x y ∧ x ≠ y
    hence x ≠ y..
    from nip have P x y..
    moreover have ¬ P y x
    proof
      assume P y x
      with ‹P x y› have x = y by (rule part-antisymmetry)
      with ‹x ≠ y› show False..
    qed
    ultimately have P x y ∧ ¬ P y x..
    with nmp-eq show PP x y..
  qed
  show  P x y ⟹ P y x ⟹ x = y using part-antisymmetry.
qed
```

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.[10]

```
locale M2 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x

sublocale M ⊆ M2
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show PP x y ⟷ P x y ∧ ¬ P y x using nmp-eq.
qed
```

[10]For this point see especially [Parsons, 2014] pp. 9-10.

```
sublocale M2 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    show x = y
    proof (rule ccontr)
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      with nmp-eq have P x y ∧ ¬ P y x..
      hence ¬ P y x..
      thus False using ‹P y x›..
    qed
  qed
qed
```

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.[11]

```
locale M3 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)

sublocale M ⊆ M3
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)
  proof
    assume x = y
    moreover have ∀ z. P z x ⟷ P z x by simp
    ultimately show ∀ z. P z x ⟷ P z y by (rule subst)
  next
    assume z: ∀ z. P z x ⟷ P z y
    show x = y
    proof (rule part-antisymmetry)
      from z have P y x ⟷ P y y..
      moreover have P y y by (rule part-reflexivity)
      ultimately show P y x..
    next
      from z have P x x ⟷ P x y..
      moreover have P x x by (rule part-reflexivity)
      ultimately show P x y..
```

[11] For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.

```
    qed
  qed
qed

sublocale M3 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    have ∀ z. P z x ⟷ P z y
    proof
      fix z
      show P z x ⟷ P z y
      proof
        assume P z x
        thus P z y using ‹P x y› by (rule part-transitivity)
      next
        assume P z y
        thus P z x using ‹P y x› by (rule part-transitivity)
      qed
    qed
    with part-extensionality show x = y..
  qed
qed
```

The fourth axiomatization adopts proper parthood as primitive.[12] Improper parthood is defined as proper parthood or identity.

```
locale M4 =
  assumes part-eq: P x y ⟷ PP x y ∨ x = y
  assumes  overlap-eq: O x y ⟷ (∃ z. P z x ∧ P z y)
  assumes proper-part-asymmetry: PP x y ⟹ ¬ PP y x
  assumes proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
begin

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ¬ PP x x by (rule proper-part-asymmetry)
  thus False using ‹PP x x›..
qed

end
```

[12]See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

```
sublocale M ⊆ M4
proof
  fix x y z
  show part-eq: P x y ⟷ (PP x y ∨ x = y)
  proof
    assume P x y
    show PP x y ∨ x = y
    proof cases
      assume x = y
      thus PP x y ∨ x = y..
    next
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      thus PP x y ∨ x = y..
    qed
  next
    assume PP x y ∨ x = y
    thus P x y
    proof
      assume PP x y
      thus P x y by (rule proper-implies-part)
    next
      assume x = y
      thus P x y by (rule identity-implies-part)
    qed
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show PP x y ⟹ ¬ PP y x using proper-part-asymmetry.
  show proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
  proof −
    assume PP x y
    assume PP y z
    have P x z ∧ x ≠ z
    proof
      from ‹PP x y› have P x y by (rule proper-implies-part)
     moreover from ‹PP y z› have P y z by (rule proper-implies-part)
      ultimately show P x z by (rule part-transitivity)
    next
      show x ≠ z
      proof
        assume x = z
        hence PP y x using ‹PP y z› by (rule ssubst)
        hence ¬ PP x y by (rule proper-part-asymmetry)
        thus False using ‹PP x y›..
      qed
    qed
    with nip-eq show PP x z..
  qed
```

```
qed

sublocale M4 ⊆ M
proof
  fix x y z
  show proper-part-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    hence PP x y ∨ x = y..
    with part-eq have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence PP y y using ‹PP x y› by (rule subst)
      with proper-part-irreflexivity show False..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume rhs: P x y ∧ x ≠ y
    hence x ≠ y..
    from rhs have P x y..
    with part-eq have PP x y ∨ x = y..
    thus PP x y
    proof
      assume PP x y
      thus PP x y.
    next
      assume x = y
      with ‹x ≠ y› show PP x y..
    qed
  qed
  show P x x
  proof −
    have x = x by (rule refl)
    hence PP x x ∨ x = x..
    with part-eq show P x x..
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    from part-eq have PP x y ∨ x = y using ‹P x y›..
    thus x = y
    proof
      assume PP x y
      hence ¬ PP y x by (rule proper-part-asymmetry)
      from part-eq have PP y x ∨ y = x using ‹P y x›..
      thus x = y
```

```
      proof
        assume PP y x
        with ‹¬ PP y x› show x = y..
      next
        assume y = x
        thus x = y..
      qed
    qed
  qed
  show P x y ⟹ P y z ⟹ P x z
  proof −
    assume P x y
    assume P y z
    with part-eq have PP y z ∨ y = z..
    hence PP x z ∨ x = z
    proof
      assume PP y z
      from part-eq have PP x y ∨ x = y using ‹P x y›..
      hence PP x z
      proof
        assume PP x y
        thus PP x z using ‹PP y z› by (rule proper-part-transitivity)
      next
        assume x = y
        thus PP x z using ‹PP y z› by (rule ssubst)
      qed
      thus PP x z ∨ x = z..
    next
      assume y = z
      moreover from part-eq have PP x y ∨ x = y using ‹P x y›..
      ultimately show PP x z ∨ x = z by (rule subst)
    qed
    with part-eq show P x z..
  qed
qed
```

4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.[13]

```
locale MM = M +
  assumes weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
```

[13] See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.[14]

```
locale MM1 = M +
  assumes proper-weak-supplementation:
    PP y x ⟹ (∃ z. PP z x ∧ ¬ O z y)

sublocale MM ⊆ MM1
proof
  fix x y
  show PP y x ⟹ (∃ z. PP z x ∧ ¬ O z y)
  proof −
    assume PP y x
    hence ∃ z. P z x ∧ ¬ O z y by (rule weak-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence ¬ O z y..
    from z have P z x..
    hence P z x ∧ z ≠ x
    proof
      show z ≠ x
      proof
        assume z = x
        hence PP y z
          using ‹PP y x› by (rule ssubst)
        hence O y z by (rule proper-implies-overlap)
        hence O z y by (rule overlap-symmetry)
        with ‹¬ O z y› show False..
      qed
    qed
    with nip-eq have PP z x..
    hence PP z x ∧ ¬ O z y
      using ‹¬ O z y›..
    thus ∃ z. PP z x ∧ ¬ O z y..
  qed
qed

sublocale MM1 ⊆ MM
proof
  fix x y
  show weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
  proof −
    assume PP y x
   hence ∃ z. PP z x ∧ ¬ O z y by (rule proper-weak-supplementation)
    then obtain z where z: PP z x ∧ ¬ O z y..
    hence PP z x..
    hence P z x by (rule proper-implies-part)
```

[14]See [Simons, 1987] p. 28.

moreover from z **have** $\neg\ O\ z\ y$**..**
ultimately have $P\ z\ x \wedge \neg\ O\ z\ y$**..**
thus $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$**..**
qed
qed

The following two corollaries are sometimes found in the literature.[15]

context *MM*
begin

corollary *weak-company*: $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge z \neq y)$
proof −
assume $PP\ y\ x$
hence $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)
then obtain z **where** z: $PP\ z\ x \wedge \neg\ O\ z\ y$**..**
hence $PP\ z\ x$**..**
from z **have** $\neg\ O\ z\ y$**..**
hence $z \neq y$ **by** (*rule disjoint-implies-distinct*)
with ‹$PP\ z\ x$› **have** $PP\ z\ x \wedge z \neq y$**..**
thus $\exists\ z.\ PP\ z\ x \wedge z \neq y$**..**
qed

corollary *strong-company*: $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y)$
proof −
assume $PP\ y\ x$
hence $\exists z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)
then obtain z **where** z: $PP\ z\ x \wedge \neg\ O\ z\ y$**..**
hence $PP\ z\ x$**..**
from z **have** $\neg\ O\ z\ y$**..**
hence $\neg\ P\ z\ y$ **by** (*rule disjoint-implies-not-part*)
with ‹$PP\ z\ x$› **have** $PP\ z\ x \wedge \neg\ P\ z\ y$**..**
thus $\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y$**..**
qed

end

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.[16]

locale *MM2* = *PM* +
assumes *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$
assumes *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

[15] See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].

[16] See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.

```
sublocale MM2 ⊆ MM
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    show x = y
    proof (rule ccontr)
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      hence ∃ z. P z y ∧ ¬ O z x by (rule weak-supplementation)
      then obtain z where z: P z y ∧ ¬ O z x..
      hence ¬ O z x..
      hence ¬ P z x by (rule disjoint-implies-not-part)
      from z have P z y..
      hence P z x using ‹P y x› by (rule part-transitivity)
      with ‹¬ P z x› show False..
    qed
  qed
  show PP y x ⟹ ∃ z. P z x ∧ ¬ O z y using weak-supplementation.
qed

sublocale MM ⊆ MM2
proof
  fix x y
  show PP x y ⟷ (P x y ∧ x ≠ y) using nip-eq.
  show PP y x ⟹ ∃ z. P z x ∧ ¬ O z y using weak-supplementation.
qed
```

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.[17]

```
locale MM3 =
  assumes part-eq: P x y ⟷ PP x y ∨ x = y
  assumes overlap-eq: O x y ⟷ (∃ z. P z x ∧ P z y)
  assumes proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
  assumes weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
begin

lemma part-reflexivity: P x x
proof −
  have x = x..
```

[17] See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

```
  hence PP x x ∨ x = x..
  with part-eq show P x x..
qed

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ∃ z. P z x ∧ ¬ O z x by (rule weak-supplementation)
  then obtain z where z: P z x ∧ ¬ O z x..
  hence ¬ O z x..
  from z have P z x..
  with part-reflexivity have P z z ∧ P z x..
  hence ∃ v. P v z ∧ P v x..
  with overlap-eq have O z x..
  with ‹¬ O z x› show False..
qed

end

sublocale MM3 ⊆ M4
proof
  fix x y z
  show P x y ⟷ PP x y ∨ x = y using part-eq.
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show proper-part-irreflexivity: PP x y ⟹ ¬ PP y x
  proof −
    assume PP x y
    show ¬ PP y x
    proof
      assume PP y x
      hence PP y y using ‹PP x y› by (rule proper-part-transitivity)
      with proper-part-irreflexivity show False..
    qed
  qed
  show PP x y ⟹ PP y z ⟹ PP x z using proper-part-transitivity.
qed

sublocale MM3 ⊆ MM
proof
  fix x y
 show PP y x ⟹ (∃ z. P z x ∧ ¬ O z y) using weak-supplementation.
qed

sublocale MM ⊆ MM3
proof
  fix x y z
  show P x y ⟷ (PP x y ∨ x = y) using part-eq.
  show O x y ⟷ (∃z. P z x ∧ P z y) using overlap-eq.
  show PP x y ⟹ PP y z ⟹ PP x z using proper-part-transitivity.
```

show $PP\ y\ x \Longrightarrow \exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.[18]

locale $EM = M\ +$
assumes *strong-supplementation*:
$\neg\ P\ x\ y \Longrightarrow (\exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

Strong supplementation entails weak supplementation.[19]

lemma *weak-supplementation*: $PP\ x\ y \Longrightarrow (\exists\, z.\ P\ z\ y \wedge \neg\ O\ z\ x)$
proof −
assume $PP\ x\ y$
hence $\neg\ P\ y\ x$ **by** (*rule proper-implies-not-part*)
thus $\exists\, z.\ P\ z\ y \wedge \neg\ O\ z\ x$ **by** (*rule strong-supplementation*)
qed

end

So minimal mereology is a subtheory of extensional mereology.[20]

sublocale $EM \subseteq MM$
proof
fix $y\ x$
show $PP\ y\ x \Longrightarrow \exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

Strong supplementation also entails the proper parts principle.[21]

context EM
begin

lemma *proper-parts-principle*:
$(\exists\, z.\ PP\ z\ x) \Longrightarrow (\forall\, z.\ PP\ z\ x \longrightarrow P\ z\ y) \Longrightarrow P\ x\ y$
proof −
assume $\exists\, z.\ PP\ z\ x$
then obtain v **where** v: $PP\ v\ x$..
hence $P\ v\ x$ **by** (*rule proper-implies-part*)
assume *antecedent*: $\forall\, z.\ PP\ z\ x \longrightarrow P\ z\ y$
hence $PP\ v\ x \longrightarrow P\ v\ y$..
hence $P\ v\ y$ **using** ‹$PP\ v\ x$›..

[18] See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.
[19] See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.
[20] [Casati and Varzi, 1999] p. 40.
[21] See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.

```
  with ‹P v x› have P v x ∧ P v y..
  hence ∃ v. P v x ∧ P v y..
  with overlap-eq have O x y..
  show P x y
  proof (rule ccontr)
    assume ¬ P x y
    hence ∃ z. P z x ∧ ¬ O z y
      by (rule strong-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence P z x..
    moreover have z ≠ x
    proof
      assume z = x
      moreover from z have ¬ O z y..
      ultimately have ¬ O x y by (rule subst)
      thus False  using ‹O x y›..
    qed
    ultimately have P z x ∧ z ≠ x..
    with nip-eq have PP z x..
    from antecedent have PP z x ⟶ P z y..
    hence P z y using ‹PP z x›..
    hence O z y by (rule part-implies-overlap)
    from z have ¬ O z y..
    thus False using ‹O z y›..
  qed
qed
```

Which with antisymmetry entails the extensionality of proper parthood.[22]

```
theorem proper-part-extensionality:
(∃ z. PP z x ∨ PP z y) ⟹ x = y ⟷ (∀ z. PP z x ⟷ PP z y)
proof −
  assume antecedent: ∃ z. PP z x ∨ PP z y
  show x = y ⟷ (∀ z. PP z x ⟷ PP z y)
  proof
    assume x = y
    moreover have ∀ z. PP z x ⟷ PP z x by simp
    ultimately show ∀ z. PP z x ⟷ PP z y by (rule subst)
  next
    assume right: ∀ z. PP z x ⟷ PP z y
    have ∀ z. PP z x ⟶ P z y
    proof
      fix z
      show PP z x ⟶ P z y
      proof
        assume PP z x
        from right have PP z x ⟷ PP z y..
```

[22]See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.

```
      hence PP z y using ‹PP z x›..
      thus P z y by (rule proper-implies-part)
    qed
  qed
  have ∀ z. PP z y ⟶ P z x
  proof
    fix z
    show PP z y ⟶ P z x
    proof
      assume PP z y
      from right have PP z x ⟷ PP z y..
      hence PP z x using ‹PP z y›..
      thus P z x by (rule proper-implies-part)
    qed
  qed
  from antecedent obtain z where z: PP z x ∨ PP z y..
  thus x = y
  proof (rule disjE)
    assume PP z x
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z y using ‹PP z x›..
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    with ‹P x y› show x = y
      by (rule part-antisymmetry)
  next
    assume PP z y
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z x using ‹PP z y›..
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
        by (rule proper-parts-principle)
    thus x = y
      using ‹P y x› by (rule part-antisymmetry)
  qed
 qed
qed
```

It also follows from strong supplementation that parthood is definable in terms of overlap.[23]

```
lemma part-overlap-eq: P x y ⟷ (∀ z. O z x ⟶ O z y)
```

[23]See [Parsons, 2014] p. 4.

```
proof
  assume P x y
  show (∀ z. O z x ⟶ O z y)
  proof
    fix z
    show O z x ⟶ O z y
    proof
      assume O z x
      with ‹P x y› show O z y
        by (rule overlap-monotonicity)
    qed
  qed
next
  assume right: ∀ z. O z x ⟶ O z y
  show P x y
  proof (rule ccontr)
    assume ¬ P x y
    hence ∃ z. P z x ∧ ¬ O z y
      by (rule strong-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence ¬ O z y..
    from right have O z x ⟶ O z y..
    moreover from z have P z x..
    hence O z x by (rule part-implies-overlap)
    ultimately have O z y..
    with ‹¬ O z y› show False..
  qed
qed
```

Which entails the extensionality of overlap.

```
theorem overlap-extensionality: x = y ⟷ (∀ z. O z x ⟷ O z y)
proof
  assume x = y
  moreover have ∀ z. O z x ⟷ O z x
  proof
    fix z
    show O z x ⟷ O z x..
  qed
  ultimately show ∀ z. O z x ⟷ O z y
    by (rule subst)
next
  assume right: ∀ z. O z x ⟷ O z y
  have ∀ z. O z y ⟶ O z x
  proof
    fix z
    from right have O z x ⟷ O z y..
    thus O z y ⟶ O z x..
  qed
  with part-overlap-eq have P y x..
```

have $\forall z.\ O\ z\ x \longrightarrow O\ z\ y$
proof
fix z
from *right* **have** $O\ z\ x \longleftrightarrow O\ z\ y$**..**
thus $O\ z\ x \longrightarrow O\ z\ y$**..**
qed
with *part-overlap-eq* **have** $P\ x\ y$**..**
thus $x = y$
using ‹$P\ y\ x$› **by** (*rule part-antisymmetry*)
qed

end

6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.[24]

locale $CM = M\ +$
assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
assumes *sum-closure*: $\exists z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
assumes *product-closure*:
$O\ x\ y \Longrightarrow \exists z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y$
begin

6.1 Products

lemma *product-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)) \Longrightarrow x \otimes y = z$
proof −
assume z: $\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
hence $(THE\ v.\ \forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \wedge P\ w\ y) = z$
proof (*rule the-equality*)
fix v
assume v: $\forall w.\ P\ w\ v \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w
from z **have** $P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**

[24] See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

```
  qed
  thus x ⊗ y = z
    using product-eq by (rule subst)
qed

lemma product-idempotence: x ⊗ x = x
proof −
  have ∀ w. P w x ⟷ P w x ∧ P w x
  proof
    fix w
    show P w x ⟷ P w x ∧ P w x
    proof
      assume P w x
      thus P w x ∧ P w x using ‹P w x›..
    next
      assume P w x ∧ P w x
      thus P w x..
    qed
  qed
  thus x ⊗ x = x by (rule product-intro)
qed

lemma product-character:
  O x y ⟹ (∀ w. P w (x ⊗ y) ⟷ (P w x ∧ P w y))
proof −
  assume O x y
  hence ∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y) by (rule product-closure)
  then obtain z where z: ∀ w. P w z ⟷ (P w x ∧ P w y)..
  hence x ⊗ y = z by (rule product-intro)
  thus ∀ w. P w (x ⊗ y) ⟷ P w x ∧ P w y
    using z by (rule ssubst)
qed

lemma product-commutativity: O x y ⟹ x ⊗ y = y ⊗ x
proof −
  assume O x y
  hence O y x by (rule overlap-symmetry)
   hence ∀ w. P w (y ⊗ x) ⟷ (P w y ∧ P w x) by (rule prod-
uct-character)
  hence ∀ w. P w (y ⊗ x) ⟷ (P w x ∧ P w y) by auto
  thus x ⊗ y = y ⊗ x by (rule product-intro)
qed

lemma product-in-factors: O x y ⟹ P (x ⊗ y) x ∧ P (x ⊗ y) y
proof −
  assume O x y
   hence ∀ w. P w (x ⊗ y) ⟷ P w x ∧ P w y by (rule prod-
uct-character)
  hence P (x ⊗ y) (x ⊗ y) ⟷ P (x ⊗ y) x ∧ P (x ⊗ y) y..
```

```
  moreover have P (x ⊗ y) (x ⊗ y) by (rule part-reflexivity)
  ultimately show P (x ⊗ y) x ∧ P (x ⊗ y) y..
qed

lemma product-in-first-factor: O x y ⟹ P (x ⊗ y) x
proof −
  assume O x y
  hence P (x ⊗ y) x ∧ P (x ⊗ y) y by (rule product-in-factors)
  thus P (x ⊗ y) x..
qed

lemma product-in-second-factor: O x y ⟹ P (x ⊗ y) y
proof −
  assume O x y
  hence P (x ⊗ y) x ∧ P (x ⊗ y) y by (rule product-in-factors)
  thus P (x ⊗ y) y..
qed

lemma nonpart-implies-proper-product:
  ¬ P x y ∧ O x y ⟹ PP (x ⊗ y) x
proof −
  assume antecedent: ¬ P x y ∧ O x y
  hence ¬ P x y..
  from antecedent have O x y..
  hence P (x ⊗ y) x by (rule product-in-first-factor)
  moreover have (x ⊗ y) ≠ x
  proof
    assume (x ⊗ y) = x
    hence ¬ P (x ⊗ y) y
      using ‹¬ P x y› by (rule ssubst)
    moreover have P (x ⊗ y) y
      using ‹O x y› by (rule product-in-second-factor)
    ultimately show False..
  qed
  ultimately have P (x ⊗ y) x ∧ x ⊗ y ≠ x..
  with nip-eq show PP (x ⊗ y) x..
qed

lemma common-part-in-product: P z x ∧ P z y ⟹ P z (x ⊗ y)
proof −
  assume antecedent: P z x ∧ P z y
  hence ∃ z. P z x ∧ P z y..
  with overlap-eq have O x y..
  hence ∀ w. P w (x ⊗ y) ⟷ (P w x ∧ P w y)
    by (rule product-character)
  hence P z (x ⊗ y) ⟷ (P z x ∧ P z y)..
  thus P z (x ⊗ y)
    using ‹P z x ∧ P z y›..
qed
```

lemma *product-part-in-factors*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x \wedge P\ z\ y$
proof −
assume $O\ x\ y$
hence $\forall\, w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
by (*rule product-character*)
hence $P\ z\ (x \otimes y) \longleftrightarrow (P\ z\ x \wedge P\ z\ y)$**..**
moreover assume $P\ z\ (x \otimes y)$
ultimately show $P\ z\ x \wedge P\ z\ y$**..**
qed

corollary *product-part-in-first-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x$
proof −
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by (*rule product-part-in-factors*)
thus $P\ z\ x$**..**
qed

corollary *product-part-in-second-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ y$
proof −
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by (*rule product-part-in-factors*)
thus $P\ z\ y$**..**
qed

lemma *part-product-identity*: $P\ x\ y \Longrightarrow x \otimes y = x$
proof −
assume $P\ x\ y$
with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$**..**
hence $P\ x\ (x \otimes y)$ **by** (*rule common-part-in-product*)
have $O\ x\ y$ **using** ‹$P\ x\ y$› **by** (*rule part-implies-overlap*)
hence $P\ (x \otimes y)\ x$ **by** (*rule product-in-first-factor*)
thus $x \otimes y = x$ **using** ‹$P\ x\ (x \otimes y)$› **by** (*rule part-antisymmetry*)
qed

lemma *product-overlap*: $P\ z\ x \Longrightarrow O\ z\ y \Longrightarrow O\ z\ (x \otimes y)$
proof −
assume $P\ z\ x$
assume $O\ z\ y$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ z \wedge P\ v\ y$**..**
then obtain v **where** v: $P\ v\ z \wedge P\ v\ y$**..**
hence $P\ v\ y$**..**

```
  from v have P v z..
  hence P v x using ‹P z x› by (rule part-transitivity)
  hence P v x ∧ P v y using ‹P v y›..
  hence P v (x ⊗ y) by (rule common-part-in-product)
  with ‹P v z› have P v z ∧ P v (x ⊗ y)..
  hence ∃ v. P v z ∧ P v (x ⊗ y)..
  with overlap-eq show O z (x ⊗ y)..
qed

lemma disjoint-from-second-factor:
  P x y ∧ ¬ O x (y ⊗ z) ⟹ ¬ O x z
proof −
  assume antecedent: P x y ∧ ¬ O x (y ⊗ z)
  hence ¬ O x (y ⊗ z)..
  show ¬ O x z
  proof
    from antecedent have P x y..
    moreover assume O x z
    ultimately have O x (y ⊗ z)
      by (rule product-overlap)
    with ‹¬ O x (y ⊗ z)› show False..
  qed
qed

lemma converse-product-overlap:
  O x y ⟹ O z (x ⊗ y) ⟹ O z y
proof −
  assume O x y
  hence P (x ⊗ y) y by (rule product-in-second-factor)
  moreover assume O z (x ⊗ y)
  ultimately show O z y
    by (rule overlap-monotonicity)
qed

lemma part-product-in-whole-product:
  O x y ⟹ P x v ∧ P y z ⟹ P (x ⊗ y) (v ⊗ z)
proof −
  assume O x y
  assume P x v ∧ P y z
  have ∀ w. P w (x ⊗ y) ⟶ P w (v ⊗ z)
  proof
    fix w
    show P w (x ⊗ y) ⟶ P w (v ⊗ z)
    proof
      assume P w (x ⊗ y)
      with ‹O x y› have P w x ∧ P w y
        by (rule product-part-in-factors)
      have P w v ∧ P w z
      proof
```

from ‹$P\ w\ x \wedge P\ w\ y$› **have** $P\ w\ x$**..**
moreover from ‹$P\ x\ v \wedge P\ y\ z$› **have** $P\ x\ v$**..**
ultimately show $P\ w\ v$ **by** (*rule part-transitivity*)
next
from ‹$P\ w\ x \wedge P\ w\ y$› **have** $P\ w\ y$**..**
moreover from ‹$P\ x\ v \wedge P\ y\ z$› **have** $P\ y\ z$**..**
ultimately show $P\ w\ z$ **by** (*rule part-transitivity*)
qed
thus $P\ w\ (v \otimes z)$ **by** (*rule common-part-in-product*)
qed
qed
hence $P\ (x \otimes y)\ (x \otimes y) \longrightarrow P\ (x \otimes y)\ (v \otimes z)$**..**
moreover have $P\ (x \otimes y)\ (x \otimes y)$ **by** (*rule part-reflexivity*)
ultimately show $P\ (x \otimes y)\ (v \otimes z)$**..**
qed

lemma *right-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
proof −
assume *antecedent*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
then obtain w **where** w: $P\ w\ x \wedge P\ w\ y \wedge P\ w\ z$**..**
hence $P\ w\ x$**..**
from w **have** $P\ w\ y \wedge P\ w\ z$**..**
hence $\exists\, w.\ P\ w\ y \wedge P\ w\ z$**..**
with *overlap-eq* **have** $O\ y\ z$**..**
hence *yz*: $\forall\, w.\ P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$
by (*rule product-character*)
hence $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$**..**
hence $P\ w\ (y \otimes z)$
using ‹$P\ w\ y \wedge P\ w\ z$›**..**
with ‹$P\ w\ x$› **have** $P\ w\ x \wedge P\ w\ (y \otimes z)$**..**
hence $\exists\, w.\ P\ w\ x \wedge P\ w\ (y \otimes z)$**..**
with *overlap-eq* **have** $O\ x\ (y \otimes z)$**..**
hence *xyz*: $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$
by (*rule product-character*)
show $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
proof
fix w
from *yz* **have** *wyz*: $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$**..**
moreover from *xyz* **have**
$P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$**..**
ultimately show
$P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
by (*rule subst*)
qed
qed

lemma *left-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$

proof −
  assume *antecedent*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
  then obtain w **where** w: $P\ w\ x \wedge P\ w\ y \wedge P\ w\ z$**..**
  hence $P\ w\ y \wedge P\ w\ z$**..**
  hence $P\ w\ y$**..**
  have $P\ w\ z$
    using ‹$P\ w\ y \wedge P\ w\ z$›**..**
  from w **have** $P\ w\ x$**..**
  hence $P\ w\ x \wedge P\ w\ y$
    using ‹$P\ w\ y$›**..**
  hence $\exists\, z.\ P\ z\ x \wedge P\ z\ y$**..**
  with *overlap-eq* **have** $O\ x\ y$**..**
  hence *xy*: $\forall\, w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
    by (*rule product-character*)
  hence $P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
  hence $P\ w\ (x \otimes y)$
    using ‹$P\ w\ x \wedge P\ w\ y$›**..**
  hence $P\ w\ (x \otimes y) \wedge P\ w\ z$
    using ‹$P\ w\ z$›**..**
  hence $\exists\, w.\ P\ w\ (x \otimes y) \wedge P\ w\ z$**..**
  with *overlap-eq* **have** $O\ (x \otimes y)\ z$**..**
  hence *xyz*: $\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow P\ w\ (x \otimes y) \wedge P\ w\ z$
    by (*rule product-character*)
  show $\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z$
  proof
    fix v
    from *xy* **have** *vxy*: $P\ v\ (x \otimes y) \longleftrightarrow (P\ v\ x \wedge P\ v\ y)$**..**
    moreover from *xyz* **have**
      $P\ v\ ((x \otimes y) \otimes z) \longleftrightarrow P\ v\ (x \otimes y) \wedge P\ v\ z$**..**
    ultimately show $P\ v\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ v\ x \wedge P\ v\ y) \wedge P\ v\ z$
      by (*rule subst*)
  qed
qed

theorem *product-associativity*:
  $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow x \otimes (y \otimes z) = (x \otimes y) \otimes z$
proof −
  assume *ante*:$(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
  hence $(\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
    by (*rule right-associated-product*)
  moreover from *ante* **have**
    $(\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$
    by (*rule left-associated-product*)
  ultimately have $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ ((x \otimes y) \otimes z)$
    by *simp*
  with *part-extensionality* **show** $x \otimes (y \otimes z) = (x \otimes y) \otimes z$**..**
qed

end

6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.[25]

locale *CMD* = *CM* +
  assumes *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
  assumes *difference-closure*:
    $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
begin

lemma *difference-intro*:
  $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow x \ominus y = z$
proof −
  assume *antecedent*: $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
  hence $(THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y) = z$
  proof (*rule the-equality*)
    fix v
    assume v: $(\forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
    have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
    proof
      fix w
      from *antecedent* **have** $P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$**..**
      moreover from v **have** $P\ w\ v \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$**..**
      ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
    qed
    with *part-extensionality* **show** $v = z$**..**
  qed
  with *difference-eq* **show** $x \ominus y = z$ **by** (*rule ssubst*)
qed

lemma *difference-idempotence*: $\neg\ O\ x\ y \Longrightarrow (x \ominus y) = x$
proof −
  assume $\neg\ O\ x\ y$
  hence $\neg\ O\ y\ x$ **by** (*rule disjoint-symmetry*)
  have $\forall w.\ P\ w\ x \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
  proof
    fix w
    show $P\ w\ x \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
    proof
      assume $P\ w\ x$
      hence $\neg\ O\ y\ w$ **using** ‹$\neg\ O\ y\ x$›
        by (*rule disjoint-demonotonicity*)
      hence $\neg\ O\ w\ y$ **by** (*rule disjoint-symmetry*)
      with ‹$P\ w\ x$› **show** $P\ w\ x \wedge \neg\ O\ w\ y$**..**
    next

[25] See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.

```
      assume P w x ∧ ¬ O w y
      thus P w x..
    qed
  qed
  thus (x ⊖ y) = x by (rule difference-intro)
qed

lemma difference-character: (∃ w. P w x ∧ ¬ O w y) ⟹
  (∀ w. P w (x ⊖ y) ⟷ P w x ∧ ¬ O w y)
proof −
  assume ∃ w. P w x ∧ ¬ O w y
   hence ∃ z. ∀ w. P w z ⟷ P w x ∧ ¬ O w y by (rule differ-
ence-closure)
  then obtain z where z: ∀ w. P w z ⟷ P w x ∧ ¬ O w y..
  hence (x ⊖ y) = z by (rule difference-intro)
   thus ∀ w. P w (x ⊖ y) ⟷ P w x ∧ ¬ O w y using z by (rule
ssubst)
qed

lemma difference-disjointness:
  (∃ z. P z x ∧ ¬ O z y) ⟹ ¬ O y (x ⊖ y)
proof −
  assume ∃ z. P z x ∧ ¬ O z y
  hence xmy: ∀ w. P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)
    by (rule difference-character)
  show ¬ O y (x ⊖ y)
  proof
    assume O y (x ⊖ y)
    with overlap-eq have ∃ v. P v y ∧ P v (x ⊖ y)..
    then obtain v where v: P v y ∧ P v (x ⊖ y)..
    from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
    moreover from v have P v (x ⊖ y)..
    ultimately have P v x ∧ ¬ O v y..
    hence ¬ O v y..
    moreover from v have P v y..
    hence O v y by (rule part-implies-overlap)
    ultimately show False..
  qed
qed

end
```

6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.[26]

```
locale CMU = CM +
```

[26] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

```
  assumes universe-eq: u = (THE z. ∀ w. P w z)
  assumes universe-closure: ∃ y. ∀ x. P x y
begin

lemma universe-intro: (∀ w. P w z) ⟹ u = z
proof −
  assume z: ∀ w. P w z
  hence (THE z. ∀ w. P w z) = z
  proof (rule the-equality)
    fix v
    assume v: ∀ w. P w v
    have ∀ w. P w v ⟷ P w z
    proof
      fix w
      show P w v ⟷ P w z
      proof
        assume P w v
        from z show P w z..
      next
        assume P w z
        from v show P w v..
      qed
    qed
    with part-extensionality show v = z..
  qed
  thus u = z using universe-eq by (rule subst)
qed

lemma universe-character: P x u
proof −
  from universe-closure obtain y where y: ∀ x. P x y..
  hence u = y by (rule universe-intro)
  hence ∀ x. P x u using y by (rule ssubst)
  thus P x u..
qed

lemma ¬ PP u x
proof
  assume PP u x
  hence ¬ P x u by (rule proper-implies-not-part)
  thus False using universe-character..
qed

lemma product-universe-implies-factor-universe:
  O x y ⟹ x ⊗ y = u ⟹ x = u
proof −
  assume x ⊗ y = u
  moreover assume O x y
  hence P (x ⊗ y) x
```

by (*rule product-in-first-factor*)
ultimately have $P\ u\ x$
by (*rule subst*)
with *universe-character* **show** $x = u$
by (*rule part-antisymmetry*)
qed

end

6.4 Complements

As is a condition ensuring the existence of complements.[27]

locale $CMC = CM\ +$
assumes *complement-eq*: $-x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *complement-closure*:
$(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
begin

lemma *complement-intro*:
$(\forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) \Longrightarrow -x = z$
proof −
assume *antecedent*: $\forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$
hence $(THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) = z$
proof (*rule the-equality*)
fix v
assume v: $\forall\, w.\ P\ w\ v \longleftrightarrow \neg\ O\ w\ x$
have $\forall\, w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w
from *antecedent* **have** $P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow \neg\ O\ w\ x$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**
qed
with *complement-eq* **show** $-x = z$ **by** (*rule ssubst*)
qed

lemma *complement-character*:
$(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x)$
proof −
assume $\exists\, w.\ \neg\ O\ w\ x$
hence $(\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$ **by** (*rule complement-closure*)
then obtain z **where** z: $\forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
hence $-x = z$ **by** (*rule complement-intro*)

[27] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

```
  thus ∀ w. P w (−x) ⟷ ¬ O w x
    using z by (rule ssubst)
qed

lemma not-complement-part: ∃ w. ¬ O w x ⟹ ¬ P x (−x)
proof −
  assume ∃ w. ¬ O w x
  hence ∀ w. P w (−x) ⟷ ¬ O w x
    by (rule complement-character)
  hence P x (−x) ⟷ ¬ O x x..
  show ¬ P x (−x)
  proof
    assume P x (−x)
    with ‹P x (−x) ⟷ ¬ O x x› have ¬ O x x..
    thus False using overlap-reflexivity..
  qed
qed

lemma complement-part: ¬ O x y ⟹ P x (−y)
proof −
  assume ¬ O x y
  hence ∃ z. ¬ O z y..
  hence ∀ w. P w (−y) ⟷ ¬ O w y
    by (rule complement-character)
  hence P x (−y) ⟷ ¬ O x y..
  thus P x (−y) using ‹¬ O x y›..
qed

lemma complement-overlap: ¬ O x y ⟹ O x (−y)
proof −
  assume ¬ O x y
  hence P x (−y)
    by (rule complement-part)
  thus O x (−y)
    by (rule part-implies-overlap)
qed

lemma or-complement-overlap: ∀ y. O y x ∨ O y (−x)
proof
  fix y
  show O y x ∨ O y (−x)
  proof cases
    assume O y x
    thus O y x ∨ O y (−x)..
  next
    assume ¬ O y x
    hence O y (−x)
      by (rule complement-overlap)
    thus O y x ∨ O y (−x)..
```

qed
qed

lemma *complement-disjointness*: $\exists\, v.\ \neg\ O\ v\ x \Longrightarrow \neg\ O\ x\ (-x)$
proof −
assume $\exists\, v.\ \neg\ O\ v\ x$
hence w: $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule complement-character*)
show $\neg\ O\ x\ (-x)$
proof
assume $O\ x\ (-x)$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ x \wedge P\ v\ (-x)$**..**
then obtain v **where** v: $P\ v\ x \wedge P\ v\ (-x)$**..**
from w **have** $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$**..**
moreover from v **have** $P\ v\ (-x)$**..**
ultimately have $\neg\ O\ v\ x$**..**
moreover from v **have** $P\ v\ x$**..**
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
ultimately show *False***..**
qed
qed

lemma *part-disjoint-from-complement*:
$\exists\, v.\ \neg\ O\ v\ x \Longrightarrow P\ y\ x \Longrightarrow \neg\ O\ y\ (-x)$
proof
assume $\exists\, v.\ \neg\ O\ v\ x$
hence $\neg\ O\ x\ (-x)$ **by** (*rule complement-disjointness*)
assume $P\ y\ x$
assume $O\ y\ (-x)$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ y \wedge P\ v\ (-x)$**..**
then obtain v **where** v: $P\ v\ y \wedge P\ v\ (-x)$**..**
hence $P\ v\ y$**..**
hence $P\ v\ x$ **using** ‹$P\ y\ x$› **by** (*rule part-transitivity*)
moreover from v **have** $P\ v\ (-x)$**..**
ultimately have $P\ v\ x \wedge P\ v\ (-x)$**..**
hence $\exists\, v.\ P\ v\ x \wedge P\ v\ (-x)$**..**
with *overlap-eq* **have** $O\ x\ (-x)$**..**
with ‹$\neg\ O\ x\ (-x)$› **show** *False***..**
qed

lemma *product-complement-character*: $(\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y)))$
proof −
assume *antecedent*: $\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y$
then obtain w **where** w: $P\ w\ x \wedge \neg\ O\ w\ y$**..**
hence $P\ w\ x$**..**
moreover from w **have** $\neg\ O\ w\ y$**..**
hence $P\ w\ (-y)$ **by** (*rule complement-part*)
ultimately have $P\ w\ x \wedge P\ w\ (-y)$**..**

hence $\exists\, w.\ P\ w\ x \wedge P\ w\ (-y)$**..**
with *overlap-eq* **have** $O\ x\ (-y)$**..**
hence *prod*: $(\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ (-y)))$
by (*rule product-character*)
show $\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y))$
proof
fix v
from w **have** $\neg\ O\ w\ y$**..**
hence $\exists\, w.\ \neg\ O\ w\ y$**..**
hence $\forall\, w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule complement-character*)
hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$**..**
moreover have $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge P\ v\ (-y))$
using *prod***..**
ultimately show $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge (\neg\ O\ v\ y))$
by (*rule subst*)
qed
qed

theorem *difference-closure*: $(\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
proof −
assume $\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
by (*rule product-complement-character*)
thus $(\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$ **by** (*rule exI*)
qed

end

sublocale *CMC* $\subseteq$ *CMD*
proof
fix $x\ y$
show $x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq***.**
show $(\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists\, z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-closure***.**
qed

corollary (**in** *CMC*) *difference-is-product-of-complement*:
$(\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow (x \ominus y) = x \otimes (-y)$
proof −
assume *antecedent*: $\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
by (*rule product-complement-character*)
thus $(x \ominus y) = x \otimes (-y)$ **by** (*rule difference-intro*)
qed

Universe and difference closure entail complement closure, since

the difference of an individual and the universe is the individual's complement.

locale *CMUD* = *CMU* + *CMD* +
  assumes *complement-eq*: $-x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
begin

lemma *universe-difference*:
  $(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x)$
proof −
  assume $\exists\, w.\ \neg\ O\ w\ x$
  then obtain w **where** w: $\neg\ O\ w\ x$**..**
  from *universe-character* **have** $P\ w\ u$**.**
  hence $P\ w\ u \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›**..**
  hence $\exists\, z.\ P\ z\ u \wedge \neg\ O\ z\ x$**..**
  hence *ux*: $\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg\ O\ w\ x)$
    by (*rule difference-character*)
  show $\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
  proof
    fix w
    from *ux* **have** *wux*: $P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg\ O\ w\ x)$**..**
    show $P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
    proof
      assume $P\ w\ (u \ominus x)$
      with *wux* **have** $P\ w\ u \wedge \neg\ O\ w\ x$**..**
      thus $\neg\ O\ w\ x$**..**
    next
      assume $\neg\ O\ w\ x$
      from *universe-character* **have** $P\ w\ u$**.**
      hence $P\ w\ u \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›**..**
      with *wux* **show** $P\ w\ (u \ominus x)$**..**
    qed
  qed
qed

theorem *complement-closure*:
  $(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
proof −
  assume $\exists\, w.\ \neg\ O\ w\ x$
  hence $\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule universe-difference*)
  thus $\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
qed

end

sublocale *CMUD* $\subseteq$ *CMC*
proof
  fix $x\ y$
  show $-x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow (\neg\ O\ w\ x))$

using *complement-eq*.
show $\exists\, w.\ \neg\ O\ w\ x \Longrightarrow \exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow (\neg\ O\ w\ x)$
using *complement-closure*.
show $x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
qed

corollary (**in** *CMUD*) *complement-universe-difference*:
$(\exists\, y.\ \neg\ O\ y\ x) \Longrightarrow -x = (u \ominus x)$
proof −
assume $\exists\, w.\ \neg\ O\ w\ x$
hence $\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule universe-difference*)
thus $-x = (u \ominus x)$
by (*rule complement-intro*)
qed

7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.[28]

locale *CEM* = *CM* + *EM*

Likewise, closed minimal mereology combines closed mereology with minimal mereology.[29]

locale *CMM* = *CM* + *MM*

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.[30]

sublocale $CMM \subseteq CEM$
proof
fix $x\ y$
show *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
proof −
assume $\neg\ P\ x\ y$
show $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$
proof *cases*
assume $O\ x\ y$
with ‹$\neg\ P\ x\ y$› **have** $\neg\ P\ x\ y \wedge O\ x\ y$..
hence $PP\ (x \otimes y)\ x$ **by** (*rule nonpart-implies-proper-product*)

[28] See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.
[29] See [Casati and Varzi, 1999] p. 43.
[30] See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

```
    hence ∃ z. P z x ∧ ¬ O z (x ⊗ y) by (rule weak-supplementation)
     then obtain z where z: P z x ∧ ¬ O z (x ⊗ y)..
     hence ¬ O z y by (rule disjoint-from-second-factor)
     moreover from z have P z x..
     hence  P z x ∧ ¬ O z y
       using ‹¬ O z y›..
     thus ∃ z. P z x ∧ ¬ O z y..
   next
     assume ¬ O x y
     with part-reflexivity have P x x ∧ ¬ O x y..
     thus (∃ z. P z x ∧ ¬ O z y)..
   qed
  qed
qed

sublocale CEM ⊆ CMM..
```

7.1 Sums

```
context CEM
begin

lemma sum-intro:
   (∀ w. O w z ⟷ (O w x ∨ O w y)) ⟹ x ⊕ y = z
proof −
  assume sum: ∀ w. O w z ⟷ (O w x ∨ O w y)
  hence (THE v. ∀ w. O w v ⟷ (O w x ∨ O w y)) = z
  proof (rule the-equality)
    fix a
    assume a: ∀ w. O w a ⟷ (O w x ∨ O w y)
    have ∀ w. O w a ⟷ O w z
    proof
      fix w
      from sum have O w z ⟷ (O w x ∨ O w y)..
      moreover from a have O w a ⟷ (O w x ∨ O w y)..
      ultimately show O w a ⟷ O w z by (rule ssubst)
      qed
      with overlap-extensionality show a = z..
  qed
  thus x ⊕ y = z
    using sum-eq by (rule subst)
qed

lemma sum-idempotence: x ⊕ x = x
proof −
  have ∀ w. O w x ⟷ (O w x ∨ O w x)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w x)
```

```
    proof (rule iffI)
      assume O w x
      thus O w x ∨ O w x..
    next
      assume O w x ∨ O w x
      thus O w x by (rule disjE)
    qed
  qed
  thus x ⊕ x = x by (rule sum-intro)
qed

lemma part-sum-identity: P y x ⟹ x ⊕ y = x
proof −
  assume P y x
  have ∀ w. O w x ⟷ (O w x ∨ O w y)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w y)
    proof
      assume O w x
      thus O w x ∨ O w y..
    next
      assume O w x ∨ O w y
      thus O w x
      proof
        assume O w x
        thus O w x.
      next
        assume O w y
        with ‹P y x› show O w x
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  thus x ⊕ y = x by (rule sum-intro)
qed

lemma sum-character: ∀ w. O w (x ⊕ y) ⟷ (O w x ∨ O w y)
proof −
  from sum-closure have (∃ z. ∀ w. O w z ⟷ (O w x ∨ O w y)).
  then obtain a where a: ∀ w. O w a ⟷ (O w x ∨ O w y)..
  hence x ⊕ y = a by (rule sum-intro)
  thus ∀ w. O w (x ⊕ y) ⟷ (O w x ∨ O w y)
    using a by (rule ssubst)
qed

lemma sum-overlap: O w (x ⊕ y) ⟷ (O w x ∨ O w y)
  using sum-character..
```

lemma *sum-part-character*:
  $P\ w\ (x \oplus y) \longleftrightarrow (\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y)$
proof
  assume $P\ w\ (x \oplus y)$
  show $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
  proof
    fix v
    show $O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
    proof
      assume $O\ v\ w$
      with ‹$P\ w\ (x \oplus y)$› **have** $O\ v\ (x \oplus y)$
        by (*rule overlap-monotonicity*)
      with *sum-overlap* **show** $O\ v\ x \vee O\ v\ y$**..**
    qed
  qed
next
  assume *right*: $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
  have $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
  proof
    fix v
    from *right* **have** $O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$**..**
    with *sum-overlap* **show** $O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
      by (*rule ssubst*)
  qed
  with *part-overlap-eq* **show** $P\ w\ (x \oplus y)$**..**
qed

lemma *sum-commutativity*: $x \oplus y = y \oplus x$
proof −
  from *sum-character* **have** $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ y \vee O\ w\ x$**.**
  hence $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ x \vee O\ w\ y$ **by** *metis*
  thus $x \oplus y = y \oplus x$ **by** (*rule sum-intro*)
qed

lemma *first-summand-overlap*: $O\ z\ x \Longrightarrow O\ z\ (x \oplus y)$
proof −
  assume $O\ z\ x$
  hence $O\ z\ x \vee O\ z\ y$**..**
  with *sum-overlap* **show** $O\ z\ (x \oplus y)$**..**
qed

lemma *first-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ x$
proof −
  assume $\neg\ O\ z\ (x \oplus y)$
  show $\neg\ O\ z\ x$
  proof
    assume $O\ z\ x$
    hence $O\ z\ (x \oplus y)$ **by** (*rule first-summand-overlap*)
    with ‹$\neg\ O\ z\ (x \oplus y)$› **show** *False***..**

```
  qed
qed

lemma first-summand-in-sum: P x (x ⊕ y)
proof −
  have ∀ w. O w x ⟶ O w (x ⊕ y)
  proof
    fix w
    show O w x ⟶ O w (x ⊕ y)
    proof
      assume O w x
      thus O w (x ⊕ y)
        by (rule first-summand-overlap)
    qed
  qed
  with part-overlap-eq show P x (x ⊕ y)..
qed

lemma common-first-summand: P x (x ⊕ y) ∧ P x (x ⊕ z)
proof
  from first-summand-in-sum show P x (x ⊕ y).
  from first-summand-in-sum show P x (x ⊕ z).
qed

lemma common-first-summand-overlap: O (x ⊕ y) (x ⊕ z)
proof −
  from first-summand-in-sum have P x (x ⊕ y).
  moreover from first-summand-in-sum have P x (x ⊕ z).
  ultimately have P x (x ⊕ y) ∧ P x (x ⊕ z)..
  hence ∃ v. P v (x ⊕ y) ∧ P v (x ⊕ z)..
  with overlap-eq show ?thesis..
qed

lemma second-summand-overlap: O z y ⟹ O z (x ⊕ y)
proof −
  assume O z y
  from sum-character have O z (x ⊕ y) ⟷ (O z x ∨ O z y)..
  moreover from ‹O z y› have O z x ∨ O z y..
  ultimately show O z (x ⊕ y)..
qed

lemma second-summand-disjointness: ¬ O z (x ⊕ y) ⟹ ¬ O z y
proof −
  assume ¬ O z (x ⊕ y)
  show ¬ O z y
  proof
    assume  O z y
    hence O z (x ⊕ y)
      by (rule second-summand-overlap)
```

```
    with ‹¬ O z (x ⊕ y)› show False..
  qed
qed

lemma second-summand-in-sum: P y (x ⊕ y)
proof −
  have ∀ w. O w y ⟶ O w (x ⊕ y)
  proof
    fix w
    show O w y ⟶ O w (x ⊕ y)
    proof
      assume O w y
      thus O w (x ⊕ y)
        by (rule second-summand-overlap)
    qed
  qed
  with part-overlap-eq show P y (x ⊕ y)..
qed

lemma second-summands-in-sums: P y (x ⊕ y) ∧ P v (z ⊕ v)
proof
  show P y (x ⊕ y) using second-summand-in-sum.
  show P v (z ⊕ v) using second-summand-in-sum.
qed

lemma disjoint-from-sum: ¬ O z (x ⊕ y) ⟷ ¬ O z x ∧ ¬ O z y
proof −
  from sum-character have O z (x ⊕ y) ⟷ (O z x ∨ O z y)..
  thus ?thesis by simp
qed

lemma summands-part-implies-sum-part:
  P x z ∧ P y z ⟹ P (x ⊕ y) z
proof −
  assume antecedent: P x z ∧ P y z
  have ∀ w. O w (x ⊕ y) ⟶ O w z
  proof
    fix w
    have w: O w (x ⊕ y) ⟷ (O w x ∨ O w y)
      using sum-character..
    show O w (x ⊕ y) ⟶ O w z
    proof
      assume O w (x ⊕ y)
      with w have O w x ∨ O w y..
      thus O w z
      proof
        from antecedent have P x z..
        moreover assume O w x
        ultimately show O w z
```

```
        by (rule overlap-monotonicity)
      next
        from antecedent have P y z..
        moreover assume O w y
        ultimately show O w z
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  with part-overlap-eq show P (x ⊕ y) z..
qed

lemma sum-part-implies-summands-part:
  P (x ⊕ y) z ⟹ P x z ∧ P y z
proof −
  assume antecedent: P (x ⊕ y) z
  show P x z ∧ P y z
  proof
    from first-summand-in-sum show P x z
      using antecedent by (rule part-transitivity)
  next
    from second-summand-in-sum show P y z
      using antecedent by (rule part-transitivity)
  qed
qed

lemma in-second-summand: P z (x ⊕ y) ∧ ¬ O z x ⟹ P z y
proof −
  assume antecedent: P z (x ⊕ y) ∧ ¬ O z x
  hence P z (x ⊕ y)..
  show P z y
  proof (rule ccontr)
    assume ¬ P z y
    hence ∃ v. P v z ∧ ¬ O v y
      by (rule strong-supplementation)
    then obtain v where v: P v z ∧ ¬ O v y..
    hence ¬ O v y..
    from v have P v z..
    hence P v (x ⊕ y)
      using ‹P z (x ⊕ y)› by (rule part-transitivity)
    hence O v (x ⊕ y) by (rule part-implies-overlap)
    from sum-character have O v (x ⊕ y) ⟷ O v x ∨ O v y..
    hence O v x ∨ O v y using ‹O v (x ⊕ y)›..
    thus False
    proof (rule disjE)
      from antecedent have ¬ O z x..
      moreover assume O v x
      hence O x v by (rule overlap-symmetry)
      with ‹P v z› have O x z
```

```
      by (rule overlap-monotonicity)
    hence O z x by (rule overlap-symmetry)
    ultimately show False..
  next
    assume O v y
    with ‹¬ O v y› show False..
  qed
 qed
qed

lemma disjoint-second-summands:
 P v (x ⊕ y) ∧ P v (x ⊕ z) ⟹ ¬ O y z ⟹ P v x
proof −
 assume antecedent: P v (x ⊕ y) ∧ P v (x ⊕ z)
 hence P v (x ⊕ z)..
 assume ¬ O y z
 show P v x
 proof (rule ccontr)
  assume ¬ P v x
  hence ∃ w. P w v ∧ ¬ O w x by (rule strong-supplementation)
  then obtain w where w: P w v ∧ ¬ O w x..
  hence ¬ O w x..
  from w have P w v..
  moreover from antecedent have P v (x ⊕ z)..
  ultimately have P w (x ⊕ z) by (rule part-transitivity)
  hence P w (x ⊕ z) ∧ ¬ O w x using ‹¬ O w x›..
  hence P w z by (rule in-second-summand)
  from antecedent have P v (x ⊕ y)..
  with ‹P w v› have P w (x ⊕ y) by (rule part-transitivity)
  hence P w (x ⊕ y) ∧ ¬ O w x using ‹¬ O w x›..
  hence P w y by (rule in-second-summand)
  hence P w y ∧ P w z using ‹P w z›..
  hence ∃ w. P w y ∧ P w z..
  with overlap-eq have O y z..
  with ‹¬ O y z› show False..
 qed
qed

lemma right-associated-sum:
 O w (x ⊕ (y ⊕ z)) ⟷ O w x ∨ (O w y ∨ O w z)
proof −
 from sum-character have O w (y ⊕ z) ⟷ O w y ∨ O w z..
 moreover from sum-character have
  O w (x ⊕ (y ⊕ z)) ⟷ (O w x ∨ O w (y ⊕ z))..
 ultimately show ?thesis
  by (rule subst)
qed

lemma left-associated-sum:
```

$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow (O\ w\ x \vee O\ w\ y) \vee O\ w\ z$
proof −
from *sum-character* **have** $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \vee O\ w\ y)$**..**
moreover from *sum-character* **have**
$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow O\ w\ (x \oplus y) \vee O\ w\ z$**..**
ultimately show *?thesis*
by (*rule subst*)
qed

theorem *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
proof −
have $\forall\ w.\ O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$
proof
fix w
have $O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow (O\ w\ x \vee O\ w\ y) \vee O\ w\ z$
using *right-associated-sum* **by** *simp*
with *left-associated-sum* **show**
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$ **by** (*rule ssubst*)
qed
with *overlap-extensionality* **show** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$**..**
qed

7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

lemma *common-summand-in-product*: $P\ x\ ((x \oplus y) \otimes (x \oplus z))$
using *common-first-summand* **by** (*rule common-part-in-product*)

lemma *product-in-first-summand*:
$\neg\ O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ x$
proof −
assume $\neg\ O\ y\ z$
have $\forall\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
proof
fix v
show $P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
proof
assume $P\ v\ ((x \oplus y) \otimes (x \oplus z))$
with *common-first-summand-overlap* **have**
$P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)
thus $P\ v\ x$ **using** ‹$\neg\ O\ y\ z$› **by** (*rule disjoint-second-summands*)
qed
qed
hence $P\ ((x \oplus y) \otimes (x \oplus z))\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow$
$P\ ((x \oplus y) \otimes (x \oplus z))\ x$**..**
thus $P\ ((x \oplus y) \otimes (x \oplus z))\ x$ **using** *part-reflexivity***..**
qed

lemma *product-is-first-summand*:
$\neg\ O\ y\ z \Longrightarrow (x \oplus y) \otimes (x \oplus z) = x$
proof −
assume $\neg\ O\ y\ z$
hence $P\ ((x \oplus y) \otimes (x \oplus z))\ x$
by (*rule product-in-first-summand*)
thus $(x \oplus y) \otimes (x \oplus z) = x$
using *common-summand-in-product*
by (*rule part-antisymmetry*)
qed

lemma *sum-over-product-left*: $O\ y\ z \Longrightarrow P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
proof −
assume $O\ y\ z$
hence $P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$ **using** *second-summands-in-sums*
by (*rule part-product-in-whole-product*)
with *common-summand-in-product* **have**
$P\ x\ ((x \oplus y) \otimes (x \oplus z)) \wedge P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$**..**
thus $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
by (*rule summands-part-implies-sum-part*)
qed

lemma *sum-over-product-right*:
$O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
proof −
assume $O\ y\ z$
show $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
proof (*rule ccontr*)
assume $\neg\ P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
hence $\exists\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v:
$P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$**..**
hence $\neg\ O\ v\ (x \oplus (y \otimes z))$**..**
with *disjoint-from-sum* **have** *vd*: $\neg\ O\ v\ x \wedge \neg\ O\ v\ (y \otimes z)$**..**
hence $\neg\ O\ v\ (y \otimes z)$**..**
from *vd* **have** $\neg\ O\ v\ x$**..**
from v **have** $P\ v\ ((x \oplus y) \otimes (x \oplus z))$**..**
with *common-first-summand-overlap* **have**
vs: $P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)
hence $P\ v\ (x \oplus y)$**..**
hence $P\ v\ (x \oplus y) \wedge \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›**..**
hence $P\ v\ y$ **by** (*rule in-second-summand*)
moreover from *vs* **have** $P\ v\ (x \oplus z)$**..**
hence $P\ v\ (x \oplus z) \wedge \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›**..**
hence $P\ v\ z$ **by** (*rule in-second-summand*)
ultimately have $P\ v\ y \wedge P\ v\ z$**..**
hence $P\ v\ (y \otimes z)$ **by** (*rule common-part-in-product*)

hence $O\ v\ (y \otimes z)$ **by** (*rule part-implies-overlap*)
with ‹$\neg\ O\ v\ (y \otimes z)$› **show** *False*..
qed
qed

Sums distribute over products.

theorem *sum-over-product*:
$O\ y\ z \Longrightarrow x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
proof −
assume $O\ y\ z$
hence $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
by (*rule sum-over-product-left*)
moreover have $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
using ‹$O\ y\ z$› **by** (*rule sum-over-product-right*)
ultimately show $x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
by (*rule part-antisymmetry*)
qed

lemma *product-in-factor-by-sum*:
$O\ x\ y \Longrightarrow P\ (x \otimes y)\ (x \otimes (y \oplus z))$
proof −
assume $O\ x\ y$
hence $P\ (x \otimes y)\ x$
by (*rule product-in-first-factor*)
moreover have $P\ (x \otimes y)\ y$
using ‹$O\ x\ y$› **by** (*rule product-in-second-factor*)
hence $P\ (x \otimes y)\ (y \oplus z)$
using *first-summand-in-sum* **by** (*rule part-transitivity*)
with ‹$P\ (x \otimes y)\ x$› **have** $P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ (y \oplus z)$..
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule common-part-in-product*)
qed

lemma *product-of-first-summand*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))\ (x \otimes y)$
proof −
assume $O\ x\ y$
hence $O\ x\ (y \oplus z)$
by (*rule first-summand-overlap*)
assume $\neg\ O\ x\ z$
show $P\ (x \otimes (y \oplus z))\ (x \otimes y)$
proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))\ (x \otimes y)$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$..
hence $P\ v\ (x \otimes (y \oplus z))$..
with ‹$O\ x\ (y \oplus z)$› **have** $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)

hence $P\ v\ x$**..**
moreover from v **have** $\neg\ O\ v\ (x \otimes y)$**..**
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$**..**
hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from ‹$P\ v\ x \wedge P\ v\ (y \oplus z)$› **have** $P\ v\ (y \oplus z)$**..**
hence $P\ v\ (y \oplus z) \wedge \neg\ O\ v\ y$ **using** ‹$\neg\ O\ v\ y$›**..**
hence $P\ v\ z$ **by** (*rule in-second-summand*)
with ‹$P\ v\ x$› **have** $P\ v\ x \wedge P\ v\ z$**..**
hence $\exists\ v.\ P\ v\ x \wedge P\ v\ z$**..**
with *overlap-eq* **have** $O\ x\ z$**..**
with ‹$\neg\ O\ x\ z$› **show** *False***..**
qed
qed

theorem *disjoint-product-over-sum*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow x \otimes (y \oplus z) = x \otimes y$
proof −
assume $O\ x\ y$
moreover assume $\neg\ O\ x\ z$
ultimately have $P\ (x \otimes (y \oplus z))\ (x \otimes y)$
by (*rule product-of-first-summand*)
moreover have $P\ (x \otimes y)(x \otimes (y \oplus z))$
using ‹$O\ x\ y$› **by** (*rule product-in-factor-by-sum*)
ultimately show $x \otimes (y \oplus z) = x \otimes y$
by (*rule part-antisymmetry*)
qed

lemma *product-over-sum-left*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof −
assume $O\ x\ y \wedge O\ x\ z$
hence $O\ x\ y$**..**
hence $O\ x\ (y \oplus z)$ **by** (*rule first-summand-overlap*)
show $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v:
$P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$**..**
hence $\neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$**..**
with *disjoint-from-sum* **have** *oxyz*:
$\neg\ O\ v\ (x \otimes y) \wedge \neg\ O\ v\ (x \otimes z)$**..**
from v **have** $P\ v\ (x \otimes (y \oplus z))$**..**
with ‹$O\ x\ (y \oplus z)$› **have** *pxyz*: $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)
hence $P\ v\ x$**..**
moreover from *oxyz* **have** $\neg\ O\ v\ (x \otimes y)$**..**
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$**..**

hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from *oxyz* **have** $\neg\ O\ v\ (x \otimes z)$**..**
with ‹$P\ v\ x$› **have** $P\ v\ x \wedge \neg\ O\ v\ (x \otimes z)$**..**
hence $\neg\ O\ v\ z$ **by** (*rule disjoint-from-second-factor*)
with ‹$\neg\ O\ v\ y$› **have** $\neg\ O\ v\ y \wedge \neg\ O\ v\ z$**..**
with *disjoint-from-sum* **have** $\neg\ O\ v\ (y \oplus z)$**..**
from *pxyz* **have** $P\ v\ (y \oplus z)$**..**
hence $O\ v\ (y \oplus z)$ **by** (*rule part-implies-overlap*)
with ‹$\neg\ O\ v\ (y \oplus z)$› **show** *False***..**
qed
qed

lemma *product-over-sum-right*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
proof −
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
have $P\ (x \otimes y)\ (x \otimes (y \oplus z)) \wedge P\ (x \otimes z)\ (x \otimes (y \oplus z))$
proof
from *antecedent* **have** $O\ x\ y$**..**
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule product-in-factor-by-sum*)
next
from *antecedent* **have** $O\ x\ z$**..**
hence $P\ (x \otimes z)\ (x \otimes (z \oplus y))$
by (*rule product-in-factor-by-sum*)
with *sum-commutativity* **show** $P\ (x \otimes z)\ (x \otimes (y \oplus z))$
by (*rule subst*)
qed
thus $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
by (*rule summands-part-implies-sum-part*)
qed

theorem *product-over-sum*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
proof −
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
hence $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
by (*rule product-over-sum-left*)
moreover have $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
using *antecedent* **by** (*rule product-over-sum-right*)
ultimately show $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
by (*rule part-antisymmetry*)
qed

lemma *joint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow O\ x\ v \wedge O\ x\ w \Longrightarrow ((x \otimes v) \oplus (x \otimes w)) = x$
proof −
assume $v \oplus w = x \oplus y$
moreover assume $O\ x\ v \wedge O\ x\ w$

```
  hence x ⊗ (v ⊕ w) = x ⊗ v ⊕ x ⊗ w
    by (rule product-over-sum)
  ultimately have x ⊗ (x ⊕ y) = x ⊗ v ⊕ x ⊗ w by (rule subst)
  moreover have (x ⊗ (x ⊕ y)) = x using first-summand-in-sum
    by (rule part-product-identity)
  ultimately show ((x ⊗ v) ⊕ (x ⊗ w)) = x by (rule subst)
qed

lemma disjoint-identical-sums:
  v ⊕ w = x ⊕ y ⟹ ¬ O y v ∧ ¬ O w x ⟹ x = v ∧ y = w
proof −
  assume identical: v ⊕ w = x ⊕ y
  assume disjoint: ¬ O y v ∧ ¬ O w x
  show x = v ∧ y = w
  proof
    from disjoint have ¬ O y v..
    hence (x ⊕ y) ⊗ (x ⊕ v) = x
      by (rule product-is-first-summand)
    with identical have (v ⊕ w) ⊗ (x ⊕ v) = x
      by (rule ssubst)
    moreover from disjoint have ¬ O w x..
    hence (v ⊕ w) ⊗ (v ⊕ x) = v
      by (rule product-is-first-summand)
    with sum-commutativity have (v ⊕ w) ⊗ (x ⊕ v) = v
      by (rule subst)
    ultimately show x = v by (rule subst)
  next
    from disjoint have ¬ O w x..
    hence (y ⊕ w) ⊗ (y ⊕ x) = y
      by (rule product-is-first-summand)
    moreover from disjoint have ¬ O y v..
    hence (w ⊕ y) ⊗ (w ⊕ v) = w
      by (rule product-is-first-summand)
    with sum-commutativity have (w ⊕ y) ⊗ (v ⊕ w) = w
      by (rule subst)
    with identical have (w ⊕ y) ⊗ (x ⊕ y) = w
      by (rule subst)
    with sum-commutativity have (w ⊕ y) ⊗ (y ⊕ x) = w
      by (rule subst)
    with sum-commutativity have (y ⊕ w) ⊗ (y ⊕ x) = w
      by (rule subst)
    ultimately show y = w
      by (rule subst)
  qed
qed

end
```

7.3 Differences

```
locale CEMD = CEM + CMD
begin

lemma plus-minus: PP y x ⟹ y ⊕ (x ⊖ y) = x
proof −
  assume PP y x
  hence ∃ z. P z x ∧ ¬ O z y by (rule weak-supplementation)
  hence xmy:∀ w. P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)
    by (rule difference-character)
  have ∀ w. O w x ⟷ (O w y ∨ O w (x ⊖ y))
  proof
    fix w
    from xmy have w: P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)..
    show O w x ⟷ (O w y ∨ O w (x ⊖ y))
    proof
      assume O w x
      with overlap-eq have ∃ v. P v w ∧ P v x..
      then obtain v where v: P v w ∧ P v x..
      hence P v w..
      from v have P v x..
      show O w y ∨ O w (x ⊖ y)
      proof cases
        assume O v y
        hence O y v by (rule overlap-symmetry)
        with ‹P v w› have O y w by (rule overlap-monotonicity)
        hence O w y by (rule overlap-symmetry)
        thus O w y ∨ O w (x ⊖ y)..
      next
        from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
        moreover assume ¬ O v y
        with ‹P v x› have  P v x ∧ ¬ O v y..
        ultimately have P v (x ⊖ y)..
        with ‹P v w› have P v w ∧ P v (x ⊖ y)..
        hence ∃ v. P v w ∧ P v (x ⊖ y)..
        with overlap-eq have O w (x ⊖ y)..
        thus O w y ∨ O w (x ⊖ y)..
      qed
    next
      assume O w y ∨ O w (x ⊖ y)
      thus O w x
      proof
        from ‹PP y x› have P y x
          by (rule proper-implies-part)
        moreover assume O w y
        ultimately show O w x
          by (rule overlap-monotonicity)
      next
        assume O w (x ⊖ y)
```

```
      with overlap-eq have ∃ v. P v w ∧ P v (x ⊖ y)..
      then obtain v where v: P v w ∧ P v (x ⊖ y)..
      hence P v w..
      from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
      moreover from v have P v (x ⊖ y)..
      ultimately have P v x ∧ ¬ O v y..
      hence P v x..
      with ‹P v w› have P v w ∧ P v x..
      hence ∃ v. P v w ∧ P v x..
      with overlap-eq show O w x..
    qed
  qed
 qed
 thus y ⊕ (x ⊖ y) = x
  by (rule sum-intro)
qed

end
```

7.4 The Universe

```
locale CEMU = CEM + CMU
begin

lemma something-disjoint: x ≠ u ⟹ (∃ v. ¬ O v x)
proof −
 assume x ≠ u
 with universe-character have P x u ∧ x ≠ u..
 with nip-eq have PP x u..
 hence ∃ v. P v u ∧ ¬ O v x
  by (rule weak-supplementation)
 then obtain v where P v u ∧ ¬ O v x..
 hence ¬ O v x..
 thus ∃ v. ¬ O v x..
qed

lemma overlaps-universe: O x u
proof −
 from universe-character have P x u.
 thus O x u by (rule part-implies-overlap)
qed

lemma universe-absorbing: x ⊕ u = u
proof −
 from universe-character have P (x ⊕ u) u.
 thus x ⊕ u = u using second-summand-in-sum
  by (rule part-antisymmetry)
qed
```

lemma *second-summand-not-universe*: $x \oplus y \neq u \Longrightarrow y \neq u$
proof −
  assume *antecedent*: $x \oplus y \neq u$
  show $y \neq u$
  proof
    assume $y = u$
    hence $x \oplus u \neq u$ **using** *antecedent* **by** (*rule subst*)
    thus *False* **using** *universe-absorbing*..
  qed
qed

lemma *first-summand-not-universe*: $x \oplus y \neq u \Longrightarrow x \neq u$
proof −
  assume $x \oplus y \neq u$
  with *sum-commutativity* **have** $y \oplus x \neq u$ **by** (*rule subst*)
  thus $x \neq u$ **by** (*rule second-summand-not-universe*)
qed

end

7.5 Complements

locale *CEMC* = *CEM* + *CMC* +
  assumes *universe-eq*: $u = (THE\ x.\ \forall\ y.\ P\ y\ x)$
begin

lemma *complement-sum-character*: $\forall\ y.\ P\ y\ (x \oplus (-x))$
proof
  fix y
  have $\forall\ v.\ O\ v\ y \longrightarrow O\ v\ x \vee O\ v\ (-x)$
  proof
    fix v
    show $O\ v\ y \longrightarrow O\ v\ x \vee O\ v\ (-x)$
    proof
      assume $O\ v\ y$
      show $O\ v\ x \vee O\ v\ (-x)$
        using *or-complement-overlap*..
    qed
  qed
  with *sum-part-character* **show** $P\ y\ (x \oplus (-x))$..
qed

lemma *universe-closure*: $\exists\ x.\ \forall\ y.\ P\ y\ x$
  using *complement-sum-character* **by** (*rule exI*)

end

sublocale *CEMC* $\subseteq$ *CEMU*
proof

show $u = (THE\ z.\ \forall\, w.\ P\ w\ z)$ **using** *universe-eq*.
show $\exists\ x.\ \forall\ y.\ P\ y\ x$ **using** *universe-closure*.
qed

sublocale $CEMC \subseteq CEMD$
proof
qed

context *CEMC*
begin

corollary *universe-is-complement-sum*: $u = x \oplus (-x)$
using *complement-sum-character* **by** (*rule universe-intro*)

lemma *strong-complement-character*:
$x \neq u \Longrightarrow (\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x)$
proof −
assume $x \neq u$
hence $\exists\ v.\ \neg\ O\ v\ x$ **by** (*rule something-disjoint*)
thus $\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$ **by** (*rule complement-character*)
qed

lemma *complement-part-not-part*: $x \neq u \Longrightarrow P\ y\ (-x) \Longrightarrow \neg\ P\ y\ x$
proof −
assume $x \neq u$
hence $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule strong-complement-character*)
hence y: $P\ y\ (-x) \longleftrightarrow \neg\ O\ y\ x$..
moreover assume $P\ y\ (-x)$
ultimately have $\neg\ O\ y\ x$..
thus $\neg\ P\ y\ x$
by (*rule disjoint-implies-not-part*)
qed

lemma *complement-involution*: $x \neq u \Longrightarrow x = -(-x)$
proof −
assume $x \neq u$
have $\neg\ P\ u\ x$
proof
assume $P\ u\ x$
with *universe-character* **have** $x = u$
by (*rule part-antisymmetry*)
with ‹$x \neq u$› **show** *False*..
qed
hence $\exists\ v.\ P\ v\ u \wedge \neg\ O\ v\ x$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ u \wedge \neg\ O\ v\ x$..
hence $\neg\ O\ v\ x$..
hence $\exists\ v.\ \neg\ O\ v\ x$..

hence *notx*: $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
  by (*rule complement-character*)
have $-x \neq u$
proof
  assume $-x = u$
  hence $\forall\ w.\ P\ w\ u \longleftrightarrow \neg\ O\ w\ x$ **using** *notx* **by** (*rule subst*)
  hence $P\ x\ u \longleftrightarrow \neg\ O\ x\ x$**..**
  hence $\neg\ O\ x\ x$ **using** *universe-character***..**
  thus *False* **using** *overlap-reflexivity***..**
qed
have $\neg\ P\ u\ (-x)$
proof
  assume $P\ u\ (-x)$
  with *universe-character* **have** $-x = u$
    by (*rule part-antisymmetry*)
  with ‹$-x \neq u$› **show** *False***..**
qed
hence $\exists\ v.\ P\ v\ u \wedge \neg\ O\ v\ (-x)$
  by (*rule strong-supplementation*)
then obtain w **where** w: $P\ w\ u \wedge \neg\ O\ w\ (-x)$**..**
hence $\neg\ O\ w\ (-x)$**..**
hence $\exists\ v.\ \neg\ O\ v\ (-x)$**..**
hence *notnotx*: $\forall\ w.\ P\ w\ (-(-x)) \longleftrightarrow \neg\ O\ w\ (-x)$
  by (*rule complement-character*)
hence $P\ x\ (-(-x)) \longleftrightarrow \neg\ O\ x\ (-x)$**..**
moreover have $\neg\ O\ x\ (-x)$
proof
  assume $O\ x\ (-x)$
  with *overlap-eq* **have** $\exists\ s.\ P\ s\ x \wedge P\ s\ (-x)$**..**
  then obtain s **where** s: $P\ s\ x \wedge P\ s\ (-x)$**..**
  hence $P\ s\ x$**..**
  hence $O\ s\ x$ **by** (*rule part-implies-overlap*)
  from *notx* **have** $P\ s\ (-x) \longleftrightarrow \neg\ O\ s\ x$**..**
  moreover from s **have** $P\ s\ (-x)$**..**
  ultimately have $\neg\ O\ s\ x$**..**
  thus *False* **using** ‹$O\ s\ x$›**..**
qed
ultimately have $P\ x\ (-(-x))$**..**
moreover have $P\ (-(-x))\ x$
proof (*rule ccontr*)
  assume $\neg\ P\ (-(-x))\ x$
  hence $\exists\ s.\ P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$
    by (*rule strong-supplementation*)
  then obtain s **where** s: $P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$**..**
  hence $\neg\ O\ s\ x$**..**
  from *notnotx* **have** $P\ s\ (-(-x)) \longleftrightarrow (\neg\ O\ s\ (-x))$**..**
  moreover from s **have** $P\ s\ (-(-x))$**..**
  ultimately have $\neg\ O\ s\ (-x)$**..**
  from *or-complement-overlap* **have** $O\ s\ x \vee O\ s\ (-x)$**..**

```
    thus False
    proof
      assume O s x
      with ‹¬ O s x› show False..
    next
      assume O s (−x)
      with ‹¬ O s (−x )› show False..
    qed
  qed
  ultimately show x = −(−x)
    by (rule part-antisymmetry)
qed

lemma part-complement-reversal: y ≠ u ⟹ P x y ⟹ P (−y) (−x)
proof −
  assume y ≠ u
  hence ny: ∀ w. P w (−y) ⟷ ¬ O w y
    by (rule strong-complement-character)
  assume P x y
  have x ≠ u
  proof
    assume x = u
    hence P u y using ‹P x y› by (rule subst)
    with universe-character have y = u
      by (rule part-antisymmetry)
    with ‹y ≠ u› show False..
  qed
  hence ∀ w. P w (−x) ⟷ ¬ O w x
    by (rule strong-complement-character)
  hence P (−y) (−x) ⟷ ¬ O (−y) x..
  moreover have ¬ O (−y) x
  proof
    assume O (−y) x
    with overlap-eq have ∃ v. P v (−y) ∧ P v x..
    then obtain v where v: P v (−y) ∧ P v x..
    hence P v (−y)..
    from ny have P v (−y) ⟷ ¬ O v y..
    hence ¬ O v y using ‹P v (−y)›..
    moreover from v have P v x..
    hence P v y using ‹P x y›
      by (rule part-transitivity)
    hence O v y
      by (rule part-implies-overlap)
    ultimately show False..
  qed
  ultimately show P (−y) (−x)..
qed

lemma complements-overlap: x ⊕ y ≠ u ⟹ O(−x)(−y)
```

proof −
  assume $x \oplus y \neq u$
  hence $\exists\ z.\ \neg\ O\ z\ (x \oplus y)$
    by (*rule something-disjoint*)
  then obtain z **where** z:$\neg\ O\ z\ (x \oplus y)$**..**
  hence $\neg\ O\ z\ x$ **by** (*rule first-summand-disjointness*)
  hence $P\ z\ (-x)$ **by** (*rule complement-part*)
  moreover from z **have** $\neg\ O\ z\ y$
    by (*rule second-summand-disjointness*)
  hence $P\ z\ (-y)$ **by** (*rule complement-part*)
  ultimately show $O(-x)(-y)$
    by (*rule overlap-intro*)
qed

lemma *sum-complement-in-complement-product*:
  $x \oplus y \neq u \Longrightarrow P(-(x \oplus y))(-x \otimes -y)$
proof −
  assume $x \oplus y \neq u$
  hence $O\ (-x)\ (-y)$
    by (*rule complements-overlap*)
  hence $\forall\ w.\ P\ w\ (-x \otimes -y) \longleftrightarrow (P\ w\ (-x) \wedge P\ w\ (-y))$
    by (*rule product-character*)
  hence $P(-(x \oplus y))(-x \otimes -y) \longleftrightarrow (P(-(x \oplus y))(-x) \wedge P(-(x \oplus y))(-y))$**..**
  moreover have $P\ (-(x \oplus y))(-x) \wedge P\ (-(x \oplus y))(-y)$
  proof
    show $P\ (-(x \oplus y))(-x)$ **using** ‹$x \oplus y \neq u$› *first-summand-in-sum*
      by (*rule part-complement-reversal*)
  next
    show $P\ (-(x \oplus y))(-y)$ **using** ‹$x \oplus y \neq u$› *second-summand-in-sum*
      by (*rule part-complement-reversal*)
  qed
  ultimately show $P\ (-(x \oplus y))(-x \otimes -y)$**..**
qed

lemma *complement-product-in-sum-complement*:
  $x \oplus y \neq u \Longrightarrow P(-x \otimes -y)(-(x \oplus y))$
proof −
  assume $x \oplus y \neq u$
  hence $\forall\, w.\ P\ w\ (-(x \oplus y)) \longleftrightarrow \neg\ O\ w\ (x \oplus y)$
    by (*rule strong-complement-character*)
  hence $P\ (-x \otimes -y)\ (-(x \oplus y)) \longleftrightarrow (\neg\ O\ (-x \otimes -y)\ (x \oplus y))$**..**
  moreover have $\neg\ O\ (-x \otimes -y)\ (x \oplus y)$
  proof
    have $O(-x)(-y)$ **using** ‹$x \oplus y \neq u$› **by** (*rule complements-overlap*)
    hence p: $\forall\ v.\ P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$
      by (*rule product-character*)
    have $O(-x \otimes -y)(x \oplus y) \longleftrightarrow (O(-x \otimes -y)\ x \vee O(-x \otimes -y)y)$
      using *sum-character***..**

moreover assume $O\ (-x \otimes -y)(x \oplus y)$
ultimately have $O\ (-x \otimes -y)\ x \vee O\ (-x \otimes -y)\ y$**..**
thus *False*
proof
assume $O\ (-x \otimes -y)\ x$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ x$**..**
then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ x$**..**
hence $P\ v\ (-x \otimes -y)$**..**
from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$**..**
hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›**..**
hence $P\ v\ (-x)$**..**
have $x \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule first-summand-not-universe*)
hence $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule strong-complement-character*)
hence $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$**..**
hence $\neg\ O\ v\ x$ **using** ‹$P\ v\ (-x)$›**..**
moreover from v **have** $P\ v\ x$**..**
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
ultimately show *False***..**
next
assume $O\ (-x \otimes -y)\ y$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ y$**..**
then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ y$**..**
hence $P\ v\ (-x \otimes -y)$**..**
from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$**..**
hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›**..**
hence $P\ v\ (-y)$**..**
have $y \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule second-summand-not-universe*)
hence $\forall\, w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule strong-complement-character*)
hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$**..**
hence $\neg\ O\ v\ y$ **using** ‹$P\ v\ (-y)$›**..**
moreover from v **have** $P\ v\ y$**..**
hence $O\ v\ y$ **by** (*rule part-implies-overlap*)
ultimately show *False***..**
qed
qed
ultimately show $P\ (-x \otimes -y)\ (-(x \oplus y))$**..**
qed

theorem *sum-complement-is-complements-product*:
$x \oplus y \neq u \Longrightarrow -(x \oplus y) = (-x \otimes -y)$
proof −
assume $x \oplus y \neq u$
show $-(x \oplus y) = (-x \otimes -y)$
proof (*rule part-antisymmetry*)
show $P\ (-\ (x \oplus y))\ (-\ x \otimes -\ y)$ **using** ‹$x \oplus y \neq u$›

by (*rule sum-complement-in-complement-product*)
show $P\ (-\ x \otimes -\ y)\ (-\ (x \oplus y))$ **using** ‹$x \oplus y \neq u$›
by (*rule complement-product-in-sum-complement*)
qed
qed

lemma *complement-sum-in-product-complement*:
$O\ x\ y \Longrightarrow x \neq u \Longrightarrow y \neq u \Longrightarrow P\ ((-x) \oplus (-y))(-(x \otimes y))$
proof −
assume $O\ x\ y$
assume $x \neq u$
assume $y \neq u$
have $x \otimes y \neq u$
proof
assume $x \otimes y = u$
with ‹$O\ x\ y$› **have** $x = u$
by (*rule product-universe-implies-factor-universe*)
with ‹$x \neq u$› **show** *False*..
qed
hence *notxty*: $\forall\ w.\ P\ w\ (-(x \otimes y)) \longleftrightarrow \neg\ O\ w\ (x \otimes y)$
by (*rule strong-complement-character*)
hence $P((-x)\oplus(-y))(-(x \otimes y)) \longleftrightarrow \neg O((-x)\oplus(-y))(x \otimes y)$..
moreover have $\neg\ O\ ((-x) \oplus (-y))\ (x \otimes y)$
proof
from *sum-character* **have**
$\forall\ w.\ O\ w\ ((-x) \oplus (-y)) \longleftrightarrow (O\ w\ (-x) \vee O\ w\ (-y))$.
hence $O(x \otimes y)((-x)\oplus(-y)) \longleftrightarrow (O(x \otimes y)(-x) \vee O(x \otimes y)(-y))$..
moreover assume $O\ ((-x) \oplus (-y))\ (x \otimes y)$
hence $O\ (x \otimes y)\ ((-x) \oplus (-y))$ **by** (*rule overlap-symmetry*)
ultimately have $O\ (x \otimes y)\ (-x) \vee O\ (x \otimes y)\ (-y)$..
thus *False*
proof
assume $O\ (x \otimes y)(-x)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \wedge P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \wedge P\ v\ (-x)$..
hence $P\ v\ (-x)$..
with ‹$x \neq u$› **have** $\neg\ P\ v\ x$
by (*rule complement-part-not-part*)
moreover from v **have** $P\ v\ (x \otimes y)$..
with ‹$O\ x\ y$› **have** $P\ v\ x$ **by** (*rule product-part-in-first-factor*)
ultimately show *False*..
next
assume $O\ (x \otimes y)\ (-y)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \wedge P\ v\ (-y)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \wedge P\ v\ (-y)$..
hence $P\ v\ (-y)$..
with ‹$y \neq u$› **have** $\neg\ P\ v\ y$
by (*rule complement-part-not-part*)

moreover from v **have** $P\ v\ (x \otimes y)$**..**
with ‹$O\ x\ y$› **have** $P\ v\ y$ **by** (*rule product-part-in-second-factor*)
ultimately show *False***..**
qed
qed
ultimately show $P\ ((-x) \oplus (-y))(-(x \otimes y))$**..**
qed

lemma *product-complement-in-complements-sum*:
$x \neq u \Longrightarrow y \neq u \Longrightarrow P(-(x \otimes y))((-x) \oplus (-y))$
proof −
assume $x \neq u$
hence $x = -(-x)$
by (*rule complement-involution*)
assume $y \neq u$
hence $y = -(-y)$
by (*rule complement-involution*)
show $P\ (-(x \otimes y))((-x) \oplus (-y))$
proof *cases*
assume $-x \oplus -y = u$
thus $P\ (-(x \otimes y))((-x) \oplus (-y))$
using *universe-character* **by** (*rule ssubst*)
next
assume $-x \oplus -y \neq u$
hence $-x \oplus -y = -(-(-x \oplus -\ y))$
by (*rule complement-involution*)
moreover have $-(-x \oplus -y) = -(-x) \otimes -(-y)$
using ‹$-x \oplus -y \neq u$›
by (*rule sum-complement-is-complements-product*)
with ‹$x = -(-x)$› **have** $-(-x \oplus -y) = x \otimes -(-y)$
by (*rule ssubst*)
with ‹$y = -(-y)$› **have** $-(-x \oplus -y) = x \otimes y$
by (*rule ssubst*)
hence $P\ (-(x \otimes y))(-(-(-x \oplus -y)))$
using *part-reflexivity* **by** (*rule subst*)
ultimately show $P\ (-(x \otimes y))(-x \oplus -y)$
by (*rule ssubst*)
qed
qed

theorem *complement-of-product-is-sum-of-complements*:
$O\ x\ y \Longrightarrow x \oplus y \neq u \Longrightarrow -(x \otimes y) = (-x) \oplus (-y)$
proof −
assume $O\ x\ y$
assume $x \oplus y \neq u$
show $-(x \otimes y) = (-x) \oplus (-y)$
proof (*rule part-antisymmetry*)
have $x \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule first-summand-not-universe*)

```
    have y ≠ u using ‹x ⊕ y ≠ u›
      by (rule second-summand-not-universe)
    show P (− (x ⊗ y)) (− x ⊕ − y)
     using ‹x ≠ u› ‹y ≠ u› by (rule product-complement-in-complements-sum)
    show  P (− x ⊕ − y) (− (x ⊗ y))
     using ‹O x y› ‹x ≠ u› ‹y ≠ u› by (rule complement-sum-in-product-complement)
  qed
qed

end
```

8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.[31]

```
locale GM = M +
  assumes fusion:
    ∃ x. φ x ⟹ ∃ z. ∀ y. O y z ⟷ (∃ x. φ x ∧ O y x)
begin
```

Fusion entails sum closure.

```
theorem sum-closure: ∃ z. ∀ w. O w z ⟷ (O w a ∨ O w b)
proof −
  have a = a..
  hence a = a ∨ a = b..
  hence ∃ x. x = a ∨ x = b..
  hence (∃ z. ∀ y. O y z ⟷ (∃ x. (x = a ∨ x = b) ∧ O y x))
    by (rule fusion)
  then obtain z where z:
    ∀ y. O y z ⟷ (∃ x. (x = a ∨ x = b) ∧ O y x)..
  have ∀ w. O w z ⟷ (O w a ∨ O w b)
  proof
    fix w
    from z have w: O w z ⟷ (∃ x. (x = a ∨ x = b) ∧ O w x)..
    show O w z ⟷ (O w a ∨ O w b)
    proof
      assume O w z
      with w have ∃ x. (x = a ∨ x = b) ∧ O w x..
      then obtain x where x: (x = a ∨ x = b) ∧ O w x..
      hence O w x..
      from x have x = a ∨ x = b..
      thus O w a ∨ O w b
      proof (rule disjE)
        assume x = a
        hence O w a using ‹O w x› by (rule subst)
```

[31] See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

```
      thus O w a ∨ O w b..
    next
      assume x = b
      hence O w b using ‹O w x› by (rule subst)
      thus O w a ∨ O w b..
    qed
  next
    assume O w a ∨ O w b
    hence ∃ x. (x = a ∨ x = b) ∧ O w x
    proof (rule disjE)
      assume O w a
      with ‹a = a ∨ a = b› have (a = a ∨ a = b) ∧ O w a..
      thus ∃ x. (x = a ∨ x = b) ∧ O w x..
    next
      have b = b..
      hence b = a ∨ b = b..
      moreover assume O w b
      ultimately have (b = a ∨ b = b) ∧ O w b..
      thus ∃ x. (x = a ∨ x = b) ∧ O w x..
    qed
    with w show O w z..
  qed
 qed
 thus ∃ z. ∀ w. O w z ⟷ (O w a ∨ O w b)..
qed

end
```

9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.[32]

locale *GMM* = *GM* + *MM*
begin

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general minimal mereology and general extensional mereology.[33] But this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does

[32] See [Casati and Varzi, 1999] p. 46.

[33] For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

lemma *product-closure*:
$O\ x\ y \Longrightarrow (\exists\ z.\ \forall\ v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
nitpick [*expect* = *genuine*] **oops**

lemma *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
nitpick [*expect* = *genuine*] **oops**

end

10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.[34]

locale *GEM* = *GM* + *EM* +
assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
assumes *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
$(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$

sublocale $GEM \subseteq GMM$
proof
qed

10.1 General Sums

context *GEM*
begin

lemma *fusion-intro*:
$(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)) \Longrightarrow (\sigma\ x.\ F\ x) = z$
proof −
assume *antecedent*: $(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
hence $(THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) = z$
proof (*rule the-equality*)
fix a
assume a: $(\forall y.\ O\ y\ a \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$

[34] For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

```
    have ∀x. O x a ⟷ O x z
    proof
      fix b
      from antecedent have O b z ⟷ (∃x. F x ∧ O b x)..
      moreover from a have O b a ⟷ (∃x. F x ∧ O b x)..
      ultimately show O b a ⟷ O b z by (rule ssubst)
    qed
    with overlap-extensionality show a = z..
  qed
  moreover from antecedent have O z z ⟷ (∃x. F x ∧ O z x)..
  hence ∃x. F x ∧ O z x using overlap-reflexivity..
  hence ∃x. F x by auto
  hence (σ x. F x) = (THE x. ∀y. O y x ⟷ (∃z. F z ∧ O y z))
    by (rule fusion-eq)
  ultimately show (σ v. F v) = z by (rule subst)
qed

lemma fusion-idempotence: (σ x. z = x) = z
proof −
  have ∀y. O y z ⟷ (∃x. z = x ∧ O y x)
  proof
    fix y
    show O y z ⟷ (∃x. z = x ∧ O y x)
    proof
      assume O y z
      with refl have z = z ∧ O y z..
      thus ∃x. z = x ∧ O y x..
    next
      assume ∃x. z = x ∧ O y x
      then obtain x where x: z = x ∧ O y x..
      hence z = x..
      moreover from x have O y x..
      ultimately show O y z by (rule ssubst)
    qed
  qed
  thus (σ x. z = x) = z
    by (rule fusion-intro)
qed
```

The whole is the sum of its parts.

```
lemma fusion-absorption: (σ x. P x z) = z
proof −
  have (∀y. O y z ⟷ (∃x. P x z ∧ O y x))
  proof
    fix y
    show O y z ⟷ (∃x. P x z ∧ O y x)
    proof
      assume O y z
      with part-reflexivity have P z z ∧ O y z..
```

thus $\exists x.\ P\ x\ z \wedge O\ y\ x$**..**
next
assume $\exists x.\ P\ x\ z \wedge O\ y\ x$
then obtain x **where** x: $P\ x\ z \wedge O\ y\ x$**..**
hence $P\ x\ z$**..**
moreover from x **have** $O\ y\ x$**..**
ultimately show $O\ y\ z$ **by** (*rule overlap-monotonicity*)
qed
qed
thus $(\sigma\ x.\ P\ x\ z) = z$
by (*rule fusion-intro*)
qed

lemma *part-fusion*: $P\ w\ (\sigma\ v.\ P\ v\ x) \Longrightarrow P\ w\ x$
proof −
assume $P\ w\ (\sigma\ v.\ P\ v\ x)$
with *fusion-absorption* **show** $P\ w\ x$ **by** (*rule subst*)
qed

lemma *fusion-character*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
proof −
assume $\exists x.\ F\ x$
hence $\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion*)
then obtain z **where** z: $\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$**..**
hence $(\sigma\ v.\ F\ v) = z$ **by** (*rule fusion-intro*)
thus $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$ **using** z **by** (*rule ssubst*)
qed

The next lemma characterises fusions in terms of parthood.[35]

lemma *fusion-part-character*: $\exists x.\ F\ x \Longrightarrow$
$(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)))$
proof −
assume $(\exists x.\ F\ x)$
hence F: $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion-character*)
show $\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
proof
fix y
show $P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
proof
assume $P\ y\ (\sigma\ v.\ F\ v)$
show $\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)$
proof
fix w
from F **have** w: $O\ w\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ w\ x)$**..**

[35] See [Pontow, 2004] pp. 202-9.

```
      show P w y ⟶ (∃ v. F v ∧ O w v)
      proof
        assume P w y
        hence P w (σ v. F v) using ‹P y (σ v. F v)›
          by (rule part-transitivity)
        hence O w (σ v. F v) by (rule part-implies-overlap)
        with w show ∃ x. F x ∧ O w x..
      qed
    qed
  next
    assume right: ∀ w. P w y ⟶ (∃ v. F v ∧ O w v)
    show P y (σ v. F v)
    proof (rule ccontr)
      assume ¬ P y (σ v. F v)
      hence ∃ v. P v y ∧ ¬ O v (σ v. F v)
        by (rule strong-supplementation)
      then obtain v where v: P v y ∧ ¬ O v (σ v. F v)..
      hence ¬ O v (σ v. F v)..
      from right have P v y ⟶ (∃ w. F w ∧ O v w)..
      moreover from v have P v y..
      ultimately have ∃ w. F w ∧ O v w..
      from F have O v (σ v. F v) ⟷ (∃ x. F x ∧ O v x)..
      hence O v (σ v. F v) using ‹∃ w. F w ∧ O v w›..
      with ‹¬ O v (σ v. F v)› show False..
    qed
  qed
 qed
qed

lemma fusion-part: F x ⟹ P x (σ x. F x)
proof −
 assume F x
 hence ∃ x. F x..
 hence ∀ y. P y (σ v. F v) ⟷ (∀ w. P w y ⟶ (∃ v. F v ∧ O w v))
   by (rule fusion-part-character)
 hence P x (σ v. F v) ⟷ (∀ w. P w x ⟶ (∃ v. F v ∧ O w v))..
 moreover have ∀ w. P w x ⟶ (∃ v. F v ∧ O w v)
 proof
   fix w
   show P w x ⟶ (∃ v. F v ∧ O w v)
   proof
     assume P w x
     hence O w x by (rule part-implies-overlap)
     with ‹F x› have F x ∧ O w x..
     thus ∃ v. F v ∧ O w v..
   qed
 qed
 ultimately show P x (σ v. F v)..
qed
```

lemma *common-part-fusion*:
  $O\ x\ y \Longrightarrow (\forall w.\ P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ y))$
proof −
  assume $O\ x\ y$
  with *overlap-eq* **have** $\exists z.\ (P\ z\ x \wedge P\ z\ y)$**..**
  hence *sum*: $(\forall w.\ P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y)) \longleftrightarrow$
    $(\forall z.\ P\ z\ w \longrightarrow (\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v)))$
    by (*rule fusion-part-character*)
  show $\forall w.\ P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
  proof
    fix w
    from *sum* **have** w: $P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y))$
      $\longleftrightarrow (\forall z.\ P\ z\ w \longrightarrow (\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v))$**..**
    show $P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
    proof
      assume $P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y))$
      with w **have** *bla*:
        $(\forall z.\ P\ z\ w \longrightarrow (\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v))$**..**
      show $P\ w\ x \wedge P\ w\ y$
      proof
        show $P\ w\ x$
        proof (*rule ccontr*)
          assume $\neg\ P\ w\ x$
          hence $\exists z.\ P\ z\ w \wedge \neg\ O\ z\ x$
            by (*rule strong-supplementation*)
          then obtain z **where** z: $P\ z\ w \wedge \neg\ O\ z\ x$**..**
          hence $\neg\ O\ z\ x$**..**
          from *bla* **have** $P\ z\ w \longrightarrow (\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v)$**..**
          moreover from z **have** $P\ z\ w$**..**
          ultimately have $\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v$**..**
          then obtain v **where** v: $(P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v$**..**
          hence $P\ v\ x \wedge P\ v\ y$**..**
          hence $P\ v\ x$**..**
          moreover from v **have** $O\ z\ v$**..**
          ultimately have $O\ z\ x$
            by (*rule overlap-monotonicity*)
          with ‹$\neg\ O\ z\ x$› **show** *False***..**
        qed
        show $P\ w\ y$
        proof (*rule ccontr*)
          assume $\neg\ P\ w\ y$
          hence $\exists z.\ P\ z\ w \wedge \neg\ O\ z\ y$
            by (*rule strong-supplementation*)
          then obtain z **where** z: $P\ z\ w \wedge \neg\ O\ z\ y$**..**
          hence $\neg\ O\ z\ y$**..**
          from *bla* **have** $P\ z\ w \longrightarrow (\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v)$**..**
          moreover from z **have** $P\ z\ w$**..**
          ultimately have $\exists v.\ (P\ v\ x \wedge P\ v\ y) \wedge O\ z\ v$**..**

```
        then obtain v where v: (P v x ∧ P v y) ∧ O z v..
        hence P v x ∧ P v y..
        hence P v y..
        moreover from v have O z v..
        ultimately have O z y
          by (rule overlap-monotonicity)
        with ‹¬ O z y› show False..
      qed
    qed
  next
    assume P w x ∧ P w y
    thus P w (σ v. (P v x ∧ P v y))
      by (rule fusion-part)
  qed
 qed
qed

theorem product-closure:
 O x y ⟹ (∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y))
proof −
 assume O x y
 hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
   by (rule common-part-fusion)
 thus ∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y)..
qed

end

sublocale GEM ⊆ CEM
proof
 fix x y
 show ∃ z. ∀ w. O w z = (O w x ∨ O w y)
   using sum-closure.
 show x ⊕ y = (THE z. ∀ v. O v z ⟷ O v x ∨ O v y)
   using sum-eq.
 show x ⊗ y = (THE z. ∀ v. P v z ⟷ P v x ∧ P v y)
   using product-eq.
 show O x y ⟹ (∃ z. ∀ w. P w z = (P w x ∧ P w y))
   using product-closure.
qed

context GEM
begin

corollary O x y ⟹ x ⊗ y = (σ v. P v x ∧ P v y)
proof −
 assume O x y
 hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
   by (rule common-part-fusion)
```

```
  thus x ⊗ y = (σ v. P v x ∧ P v y) by (rule product-intro)
qed

lemma disjoint-fusion:
  ∃ w. ¬ O w x ⟹ (∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x)
proof −
  assume antecedent: ∃ w. ¬ O w x
  hence ∀ y. O y (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O y v)
    by (rule fusion-character)
  hence x: O x (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O x v)..
  show ∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x
  proof
    fix y
    show P y (σ z. ¬ O z x) ⟷ ¬ O y x
    proof
      assume P y (σ z. ¬ O z x)
      moreover have ¬ O x (σ z. ¬ O z x)
      proof
        assume O x (σ z. ¬ O z x)
        with x have (∃ v. ¬ O v x ∧ O x v)..
        then obtain v where v: ¬ O v x ∧ O x v..
        hence ¬ O v x..
        from v have O x v..
        hence O v x by (rule overlap-symmetry)
        with ‹¬ O v x› show False..
      qed
      ultimately have ¬ O x y
        by (rule disjoint-demonotonicity)
      thus ¬ O y x by (rule disjoint-symmetry)
    next
      assume ¬ O y x
      thus P y (σ v. ¬ O v x)
        by (rule fusion-part)
    qed
  qed
qed

theorem complement-closure:
  ∃ w. ¬ O w x ⟹ (∃ z. ∀ w. P w z ⟷ ¬ O w x)
proof −
  assume (∃ w. ¬ O w x)
  hence ∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x
    by (rule disjoint-fusion)
  thus ∃ z. ∀ w. P w z ⟷ ¬ O w x..
qed

end

sublocale GEM ⊆ CEMC
```

proof
  fix $x\ y$
  show $-\ x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
    using *complement-eq*.
  show $(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\exists\, z.\ \forall\, w.\ P\ w\ z = (\neg\ O\ w\ x))$
    using *complement-closure*.
  show $x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
    using *difference-eq*.
  show $u = (THE\ x.\ \forall\, y.\ P\ y\ x)$
    using *universe-eq*.
qed

context *GEM*
begin

corollary *complement-is-disjoint-fusion*:
  $\exists\, w.\ \neg\ O\ w\ x \Longrightarrow -\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
proof −
  assume $\exists\, w.\ \neg\ O\ w\ x$
  hence $\forall\, w.\ P\ w\ (\sigma\ z.\ \neg\ O\ z\ x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule disjoint-fusion*)
  thus $-\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
    by (*rule complement-intro*)
qed

theorem *strong-fusion*: $\exists\, x.\ F\ x \Longrightarrow$
  $\exists\, x.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z))$
proof −
  assume $\exists\, x.\ F\ x$
  have $(\forall\, y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)) \wedge$
    $(\forall\, y.\ P\ y\ (\sigma\ v.\ F\ v) \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z))$
  proof
    show $\forall\, y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
    proof
      fix y
      show $F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
      proof
        assume $F\ y$
        thus $P\ y\ (\sigma\ v.\ F\ v)$
          by (*rule fusion-part*)
      qed
    qed
  next
    have $(\forall\, y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow$
      $(\forall\, w.\ P\ w\ y \longrightarrow (\exists\, v.\ F\ v \wedge O\ w\ v)))$
      using ‹$\exists\, x.\ F\ x$› **by** (*rule fusion-part-character*)
    hence $P\ (\sigma\ v.\ F\ v)\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall\, w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow$
      $(\exists\, v.\ F\ v \wedge O\ w\ v))$..
  thus $\forall\, w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow (\exists\, v.\ F\ v \wedge O\ w\ v)$ **using** *part-reflexivity*..

```
  qed
  thus ?thesis..
qed

theorem strong-fusion-eq: ∃ x. F x ⟹ (σ x. F x) =
  (THE x. (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z)))
proof −
  assume ∃ x. F x
  have (THE x. (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O
y z))) = (σ x. F x)
  proof (rule the-equality)
    show (∀ y. F y ⟶ P y (σ x. F x)) ∧ (∀ y. P y (σ x. F x) ⟶
(∃ z. F z ∧ O y z))
    proof
      show ∀ y. F y ⟶ P y (σ x. F x)
      proof
        fix y
        show F y ⟶ P y (σ x. F x)
        proof
          assume F y
          thus P y (σ x. F x)
            by (rule fusion-part)
        qed
      qed
    next
      show (∀ y. P y (σ x. F x) ⟶ (∃ z. F z ∧ O y z))
      proof
        fix y
        show P y (σ x. F x) ⟶ (∃ z. F z ∧ O y z)
        proof
          have  ∀ y. P y (σ v. F v) ⟷ (∀ w. P w y ⟶ (∃ v. F v ∧ O
w v))
            using ‹∃ x. F x› by (rule fusion-part-character)
          hence P y (σ v. F v) ⟷ (∀ w. P w y ⟶ (∃ v. F v ∧ O w
v))..
          moreover assume P y (σ x. F x)
          ultimately have ∀ w. P w y ⟶ (∃ v. F v ∧ O w v)..
          hence P y y ⟶ (∃ v. F v ∧ O y v)..
          thus ∃ v. F v ∧ O y v using part-reflexivity..
        qed
      qed
    qed
  next
    fix x
    assume x: (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y
z))
    have ∀ y. O y x ⟷ (∃ z. F z ∧ O y z)
    proof
      fix y
```

```
      show O y x ⟷ (∃ z. F z ∧ O y z)
      proof
        assume O y x
        with overlap-eq have ∃ v. P v y ∧ P v x..
        then obtain v where v: P v y ∧ P v x..
        from x have ∀ y. P y x ⟶ (∃ z. F z ∧ O y z)..
        hence P v x ⟶ (∃ z. F z ∧ O v z)..
        moreover from v have P v x..
        ultimately have ∃ z. F z ∧ O v z..
        then obtain z where z: F z ∧ O v z..
        hence F z..
        from v have P v y..
        moreover from z have O v z..
        hence O z v by (rule overlap-symmetry)
        ultimately have O z y by (rule overlap-monotonicity)
        hence O y z by (rule overlap-symmetry)
        with ‹F z› have F z ∧ O y z..
        thus ∃ z. F z ∧ O y z..
      next
        assume ∃ z. F z ∧ O y z
        then obtain z where z: F z ∧ O y z..
        from x have ∀ y. F y ⟶ P y x..
        hence F z ⟶ P z x..
        moreover from z have F z..
        ultimately have P z x..
        moreover from z have O y z..
        ultimately show O y x
          by (rule overlap-monotonicity)
      qed
    qed
    hence (σ x. F x) = x
      by (rule fusion-intro)
    thus x = (σ x. F x)..
  qed
  thus ?thesis..
qed

lemma strong-sum-eq: x ⊕ y = (THE z. (P x z ∧ P y z) ∧ (∀ w. P w
z ⟶ O w x ∨ O w y))
proof −
  have (THE z. (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y))
= x ⊕ y
  proof (rule the-equality)
    show (P x (x ⊕ y) ∧ P y (x ⊕ y)) ∧ (∀ w. P w (x ⊕ y) ⟶ O w
x ∨ O w y)
    proof
      show P x (x ⊕ y) ∧ P y (x ⊕ y)
        proof
          show P x (x ⊕ y) using first-summand-in-sum.
```

```
      show P y (x ⊕ y) using second-summand-in-sum.
    qed
  show ∀ w. P w (x ⊕ y) ⟶ O w x ∨ O w y
  proof
    fix w
    show P w (x ⊕ y) ⟶ O w x ∨ O w y
    proof
      assume P w (x ⊕ y)
      hence O w (x ⊕ y) by (rule part-implies-overlap)
      with sum-overlap show O w x ∨ O w y..
    qed
  qed
qed
fix z
assume z: (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
hence P x z ∧ P y z..
have ∀ w. O w z ⟷ (O w x ∨ O w y)
proof
  fix w
  show O w z ⟷ (O w x ∨ O w y)
  proof
    assume O w z
    with overlap-eq have ∃ v. P v w ∧ P v z..
    then obtain v where v: P v w ∧ P v z..
    hence P v w..
    from z have ∀ w. P w z ⟶ O w x ∨ O w y..
    hence P v z ⟶ O v x ∨ O v y..
    moreover from v have P v z..
    ultimately have O v x ∨ O v y..
    thus O w x ∨ O w y
    proof
      assume O v x
      hence O x v by (rule overlap-symmetry)
      with ‹P v w› have O x w by (rule overlap-monotonicity)
      hence O w x by (rule overlap-symmetry)
      thus O w x ∨ O w y..
    next
      assume O v y
      hence O y v by (rule overlap-symmetry)
      with ‹P v w› have O y w by (rule overlap-monotonicity)
      hence O w y by (rule overlap-symmetry)
      thus O w x ∨ O w y..
    qed
  next
    assume O w x ∨ O w y
    thus O w z
    proof
      from ‹P x z ∧ P y z› have P x z..
      moreover assume O w x
```

ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
next
from ‹$P\ x\ z \wedge P\ y\ z$› **have** $P\ y\ z$**..**
moreover assume $O\ w\ y$
ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
qed
qed
qed
hence $x \oplus y = z$ **by** (*rule sum-intro*)
thus $z = x \oplus y$**..**
qed
thus *?thesis***..**
qed

10.2 General Products

lemma *general-product-intro*: $(\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)) \Longrightarrow (\pi\ x.\ F\ x) = x$
proof −
assume $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
hence $(\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y) = x$ **by** (*rule fusion-intro*)
with *general-product-eq* **show** $(\pi\ x.\ F\ x) = x$ **by** (*rule ssubst*)
qed

lemma *general-product-idempotence*: $(\pi\ z.\ z = x) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ y = x \longrightarrow P\ z\ y) \wedge O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ z = x) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-absorption*: $(\pi\ z.\ P\ x\ z) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ P\ x\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ P\ x\ z) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-character*: $\exists z.\ \forall y.\ F\ y \longrightarrow P\ z\ y \Longrightarrow$
$\forall y.\ O\ y\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
proof −
assume $(\exists z.\ \forall y.\ F\ y \longrightarrow P\ z\ y)$
hence $(\exists x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$
by (*rule fusion*)
then obtain x **where** x:
$\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$**..**
hence $(\pi\ x.\ F\ x) = x$ **by** (*rule general-product-intro*)

thus $(\forall y.\ O\ y\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$
using x **by** (*rule ssubst*)
qed

corollary $\neg\ (\exists x.\ F\ x) \Longrightarrow u = (\pi\ x.\ F\ x)$
proof −
assume *antecedent*: $\neg\ (\exists x.\ F\ x)$
have $\forall y.\ P\ y\ (\pi\ x.\ F\ x)$
proof
fix y
show $P\ y\ (\pi\ x.\ F\ x)$
proof (*rule ccontr*)
assume $\neg\ P\ y\ (\pi\ x.\ F\ x)$
hence $\exists z.\ P\ z\ y \wedge \neg\ O\ z\ (\pi\ x.\ F\ x)$ **by** (*rule strong-supplementation*)
then obtain z **where** z: $P\ z\ y \wedge \neg\ O\ z\ (\pi\ x.\ F\ x)$**..**
hence $\neg\ O\ z\ (\pi\ x.\ F\ x)$**..**
from *antecedent* **have** *bla*: $\forall\ y.\ F\ y \longrightarrow P\ z\ y$ **by** *simp*
hence $\exists\ v.\ \forall\ y.\ F\ y \longrightarrow P\ v\ y$**..**
hence $(\forall y.\ O\ y\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$ **by** (*rule general-product-character*)
hence $O\ z\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists v.\ (\forall y.\ F\ y \longrightarrow P\ v\ y) \wedge O\ z\ v)$**..**
moreover from *bla* **have** $(\forall\ y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ z\ z$
using *overlap-reflexivity***..**
hence $\exists\ v.\ (\forall\ y.\ F\ y \longrightarrow P\ v\ y) \wedge O\ z\ v$**..**
ultimately have $O\ z\ (\pi\ x.\ F\ x)$**..**
with ‹$\neg\ O\ z\ (\pi\ x.\ F\ x)$› **show** *False***..**
qed
qed
thus $u = (\pi\ x.\ F\ x)$
by (*rule universe-intro*)
qed

end

10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.[36]

locale *GEM1* = *MM* +
assumes *strong-fusion*: $\exists x.\ F\ x \Longrightarrow \exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
assumes *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$

[36] See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
assumes *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$
begin

theorem *fusion*:
$\exists x.\ \varphi\ x \Longrightarrow (\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$
proof −
assume $\exists x.\ \varphi\ x$
hence $\exists x.\ (\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$ **by** (*rule strong-fusion*)
then obtain x **where** x:
$(\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$**..**
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
fix y
show $O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
assume $O\ y\ x$
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ x$**..**
then obtain z **where** z: $P\ z\ y \wedge P\ z\ x$**..**
hence $P\ z\ x$**..**
from x **have** $\forall y.\ P\ y\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$**..**
hence $P\ z\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ z\ v)$**..**
hence $\exists v.\ \varphi\ v \wedge O\ z\ v$ **using** ‹$P\ z\ x$›**..**
then obtain v **where** v: $\varphi\ v \wedge O\ z\ v$**..**
hence $O\ z\ v$**..**
with *overlap-eq* **have** $\exists w.\ P\ w\ z \wedge P\ w\ v$**..**
then obtain w **where** w: $P\ w\ z \wedge P\ w\ v$**..**
hence $P\ w\ z$**..**
moreover from z **have** $P\ z\ y$**..**
ultimately have $P\ w\ y$
by (*rule part-transitivity*)
moreover from w **have** $P\ w\ v$**..**
ultimately have $P\ w\ y \wedge P\ w\ v$**..**
hence $\exists w.\ P\ w\ y \wedge P\ w\ v$**..**
with *overlap-eq* **have** $O\ y\ v$**..**
from v **have** $\varphi\ v$**..**
hence $\varphi\ v \wedge O\ y\ v$ **using** ‹$O\ y\ v$›**..**
thus $\exists v.\ \varphi\ v \wedge O\ y\ v$**..**
next
assume $\exists v.\ \varphi\ v \wedge O\ y\ v$
then obtain v **where** v: $\varphi\ v \wedge O\ y\ v$**..**
hence $O\ y\ v$**..**
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ v$**..**

then obtain z **where** z: $P\ z\ y \wedge P\ z\ v$**..**
hence $P\ z\ v$**..**
from x **have** $\forall y.\ \varphi\ y \longrightarrow P\ y\ x$**..**
hence $\varphi\ v \longrightarrow P\ v\ x$**..**
moreover from v **have** $\varphi\ v$**..**
ultimately have $P\ v\ x$**..**
with ‹$P\ z\ v$› **have** $P\ z\ x$
by (*rule part-transitivity*)
from z **have** $P\ z\ y$**..**
thus $O\ y\ x$ **using** ‹$P\ z\ x$›
by (*rule overlap-intro*)
qed
qed
thus $(\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$**..**
qed

lemma *pair*: $\exists v.\ (\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ v) \wedge (\forall w.\ P\ w\ v \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ w\ z))$
proof −
have $x = x$**..**
hence $x = x \vee x = y$**..**
hence $\exists v.\ v = x \vee v = y$**..**
thus *?thesis*
by (*rule strong-fusion*)
qed

lemma *or-id*: $(v = x \vee v = y) \wedge O\ w\ v \Longrightarrow O\ w\ x \vee O\ w\ y$
proof −
assume v: $(v = x \vee v = y) \wedge O\ w\ v$
hence $O\ w\ v$**..**
from v **have** $v = x \vee v = y$**..**
thus $O\ w\ x \vee O\ w\ y$
proof
assume $v = x$
hence $O\ w\ x$ **using** ‹$O\ w\ v$› **by** (*rule subst*)
thus $O\ w\ x \vee O\ w\ y$**..**
next
assume $v = y$
hence $O\ w\ y$ **using** ‹$O\ w\ v$› **by** (*rule subst*)
thus $O\ w\ x \vee O\ w\ y$**..**
qed
qed

lemma *strong-sum-closure*:
$\exists z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
proof −
from *pair* **obtain** z **where** z: $(\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow (\exists v.\ (v = x \vee v = y) \wedge O\ w\ v))$**..**
have $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$

```
  proof
    from z have allw: ∀ w. (w = x ∨ w = y) ⟶ P w z..
    hence x = x ∨ x = y ⟶ P x z..
    moreover have x = x ∨ x = y using refl..
    ultimately have P x z..
    from allw have y = x ∨ y = y ⟶ P y z..
    moreover have y = x ∨ y = y using refl..
    ultimately have P y z..
    with ‹P x z› show P x z ∧ P y z..
  next
    show ∀ w. P w z ⟶ O w x ∨ O w y
    proof
      fix w
      show P w z ⟶ O w x ∨ O w y
      proof
        assume P w z
        from z have ∀ w. P w z ⟶ (∃ v. (v = x ∨ v = y) ∧ O w v)..
        hence P w z ⟶ (∃ v. (v = x ∨ v = y) ∧ O w v)..
        hence ∃ v. (v = x ∨ v = y) ∧ O w v using ‹P w z›..
        then obtain v where v: (v = x ∨ v = y) ∧ O w v..
        thus O w x ∨ O w y by (rule or-id)
      qed
    qed
  qed
  thus ?thesis..
qed

end

sublocale GEM1 ⊆ GMM
proof
  fix x y φ
  show (∃ x. φ x) ⟹ (∃ z. ∀ y. O y z ⟷ (∃ x. φ x ∧ O y x)) using
fusion.
qed

context GEM1
begin

lemma least-upper-bound:
  assumes sf:
    ((∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z)))
  shows lub:
      (∀ y. F y ⟶ P y x) ∧ (∀ z. (∀ y. F y ⟶ P y z) ⟶ P x z)
proof
 from sf show ∀ y. F y ⟶ P y x..
next
 show (∀ z. (∀ y. F y ⟶ P y z) ⟶ P x z)
 proof
```

fix z
show $(\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z$
proof
assume z: $\forall y.\ F\ y \longrightarrow P\ y\ z$
from *pair* **obtain** v **where** v: $(\forall w.\ (w = x \lor w = z) \longrightarrow P\ w\ v)$
$\land\ (\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y))$**..**
hence *left*: $(\forall w.\ (w = x \lor w = z) \longrightarrow P\ w\ v)$**..**
hence $(x = x \lor x = z) \longrightarrow P\ x\ v$**..**
moreover have $x = x \lor x = z$ **using** *refl***..**
ultimately have $P\ x\ v$**..**
have $z = v$
proof (*rule ccontr*)
assume $z \neq v$
from *left* **have** $z = x \lor z = z \longrightarrow P\ z\ v$**..**
moreover have $z = x \lor z = z$ **using** *refl***..**
ultimately have $P\ z\ v$**..**
hence $P\ z\ v \land z \neq v$ **using** ‹$z \neq v$›**..**
with *nip-eq* **have** $PP\ z\ v$**..**
hence $\exists w.\ P\ w\ v \land \neg\ O\ w\ z$ **by** (*rule weak-supplementation*)
then obtain w **where** w: $P\ w\ v \land \neg\ O\ w\ z$**..**
hence $P\ w\ v$**..**
from v **have** *right*:
$\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y)$**..**
hence $P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y)$**..**
hence $\exists y.\ (y = x \lor y = z) \land O\ w\ y$ **using** ‹$P\ w\ v$›**..**
then obtain s **where** s: $(s = x \lor s = z) \land O\ w\ s$**..**
hence $s = x \lor s = z$**..**
thus *False*
proof
assume $s = x$
moreover from s **have** $O\ w\ s$**..**
ultimately have $O\ w\ x$ **by** (*rule subst*)
with *overlap-eq* **have** $\exists t.\ P\ t\ w \land P\ t\ x$**..**
then obtain t **where** t: $P\ t\ w \land P\ t\ x$**..**
hence $P\ t\ x$**..**
from *sf* **have** $(\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \land O\ y\ z))$**..**
hence $P\ t\ x \longrightarrow (\exists z.\ F\ z \land O\ t\ z)$**..**
hence $\exists z.\ F\ z \land O\ t\ z$ **using** ‹$P\ t\ x$›**..**
then obtain a **where** a: $F\ a \land O\ t\ a$**..**
hence $F\ a$**..**
from *sf* **have** *ub*: $\forall y.\ F\ y \longrightarrow P\ y\ x$**..**
hence $F\ a \longrightarrow P\ a\ x$**..**
hence $P\ a\ x$ **using** ‹$F\ a$›**..**
moreover from a **have** $O\ t\ a$**..**
ultimately have $O\ t\ x$
by (*rule overlap-monotonicity*)
from t **have** $P\ t\ w$**..**
moreover have $O\ z\ t$
proof −

from z **have** $F\ a \longrightarrow P\ a\ z$**..**
moreover from a **have** $F\ a$**..**
ultimately have $P\ a\ z$**..**
moreover from a **have** $O\ t\ a$**..**
ultimately have $O\ t\ z$
by (*rule overlap-monotonicity*)
thus $O\ z\ t$ **by** (*rule overlap-symmetry*)
qed
ultimately have $O\ z\ w$
by (*rule overlap-monotonicity*)
hence $O\ w\ z$ **by** (*rule overlap-symmetry*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** ‹$O\ w\ z$›**..**
next
assume $s = z$
moreover from s **have** $O\ w\ s$**..**
ultimately have $O\ w\ z$ **by** (*rule subst*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** ‹$O\ w\ z$›**..**
qed
qed
thus $P\ x\ z$ **using** ‹$P\ x\ v$› **by** (*rule ssubst*)
qed
qed
qed

corollary *strong-fusion-intro*: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) \Longrightarrow (\sigma\ x.\ F\ x) = x$
proof −
assume *antecedent*: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
with *least-upper-bound* **have** *lubx*:
$(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$**.**
from *antecedent* **have** $\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$**..**
hence $P\ x\ x \longrightarrow (\exists z.\ F\ z \wedge O\ x\ z)$**..**
hence $\exists z.\ F\ z \wedge O\ x\ z$ **using** *part-reflexivity***..**
then obtain z **where** z: $F\ z \wedge O\ x\ z$**..**
hence $F\ z$**..**
hence $\exists z.\ F\ z$**..**
hence $(\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion-eq*)
moreover have $(THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))) = x$
proof (*rule the-equality*)
show $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
using *antecedent***.**
next
fix w
assume w:

$(\forall y.\ F\ y \longrightarrow P\ y\ w) \wedge (\forall y.\ P\ y\ w \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
with *least-upper-bound* **have** *lubw*:
$(\forall y.\ F\ y \longrightarrow P\ y\ w) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ w\ z)$**.**
hence $(\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ w\ z)$**..**
hence $(\forall y.\ F\ y \longrightarrow P\ y\ x) \longrightarrow P\ w\ x$**..**
moreover from *antecedent* **have** $\forall y.\ F\ y \longrightarrow P\ y\ x$**..**
ultimately have $P\ w\ x$**..**
from *lubx* **have** $(\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$**..**
hence $(\forall y.\ F\ y \longrightarrow P\ y\ w) \longrightarrow P\ x\ w$**..**
moreover from *lubw* **have** $(\forall y.\ F\ y \longrightarrow P\ y\ w)$**..**
ultimately have $P\ x\ w$**..**
with ‹$P\ w\ x$› **show** $w = x$
by (*rule part-antisymmetry*)
qed
ultimately show $(\sigma\ x.\ F\ x) = x$ **by** (*rule ssubst*)
qed

lemma *strong-fusion-character*: $\exists x.\ F\ x \Longrightarrow ((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
proof −
assume $\exists x.\ F\ x$
hence $(\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion*)
then obtain x **where** x:
$(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$**..**
hence $(\sigma\ x.\ F\ x) = x$ **by** (*rule strong-fusion-intro*)
thus *?thesis* **using** x **by** (*rule ssubst*)
qed

lemma *F-in*: $\exists x.\ F\ x \Longrightarrow (\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x))$
proof −
assume $\exists x.\ F\ x$
hence $((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion-character*)
thus $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$**..**
qed

lemma *parts-overlap-Fs*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof −
assume $\exists x.\ F\ x$
hence $((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion-character*)
thus $(\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$**..**
qed

lemma *in-strong-fusion*: $P\ z\ (\sigma\ x.\ z = x)$
proof −
have $\exists y.\ z = y$ **using** *refl***..**

hence $\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)$
by (*rule F-in*)
hence $z = z \longrightarrow P\ z\ (\sigma\ x.\ z = x)$**..**
thus $P\ z\ (\sigma\ x.\ z = x)$ **using** *refl***..**
qed

lemma *strong-fusion-in*: $P\ (\sigma\ x.\ z = x)\ z$
proof −
have $\exists y.\ z = y$ **using** *refl***..**
hence *sf*:
$(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ z = x) \longrightarrow (\exists v.\ z = v \wedge O\ y\ v))$
by (*rule strong-fusion-character*)
with *least-upper-bound* **have** *lub*: $(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$**.**
hence $(\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$**..**
hence $(\forall y.\ z = y \longrightarrow P\ y\ z) \longrightarrow P\ (\sigma\ x.\ z = x)\ z$**..**
moreover have $(\forall y.\ z = y \longrightarrow P\ y\ z)$
proof
fix y
show $z = y \longrightarrow P\ y\ z$
proof
assume $z = y$
thus $P\ y\ z$ **using** *part-reflexivity* **by** (*rule subst*)
qed
qed
ultimately show $P\ (\sigma\ x.\ z = x)\ z$**..**
qed

lemma *strong-fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
using *strong-fusion-in in-strong-fusion* **by** (*rule part-antisymmetry*)

10.4 Strong Sums

lemma *pair-fusion*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow (\sigma\ z.\ z = x \vee z = y) = z$
proof
assume z: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
have $(\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z) \wedge (\forall v.\ P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z))$
proof
show $\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z$
proof
fix w
from z **have** $P\ x\ z \wedge P\ y\ z$**..**
show $w = x \vee w = y \longrightarrow P\ w\ z$
proof
assume $w = x \vee w = y$
thus $P\ w\ z$

```
    proof
     assume w = x
     moreover from ‹P x z ∧ P y z› have P x z..
     ultimately show P w z by (rule ssubst)
    next
     assume w = y
     moreover from ‹P x z ∧ P y z› have P y z..
     ultimately show P w z by (rule ssubst)
    qed
   qed
  qed
  show ∀ v. P v z ⟶ (∃ z. (z = x ∨ z = y) ∧ O v z)
  proof
   fix v
   show P v z ⟶ (∃ z. (z = x ∨ z = y) ∧ O v z)
   proof
    assume P v z
    from z have ∀ w. P w z ⟶ O w x ∨ O w y..
    hence P v z ⟶ O v x ∨ O v y..
    hence O v x ∨ O v y using ‹P v z›..
    thus ∃ z. (z = x ∨ z = y) ∧ O v z
    proof
     assume O v x
     have x = x ∨ x = y using refl..
     hence (x = x ∨ x = y) ∧ O v x using ‹O v x›..
     thus ∃ z. (z = x ∨ z = y) ∧ O v z..
    next
     assume O v y
     have y = x ∨ y = y using refl..
     hence (y = x ∨ y = y) ∧ O v y using ‹O v y›..
     thus ∃ z. (z = x ∨ z = y) ∧ O v z..
    qed
   qed
  qed
 qed
  thus (σ z. z = x ∨ z = y) = z
    by (rule strong-fusion-intro)
qed

corollary strong-sum-fusion: x ⊕ y = (σ z. z = x ∨ z = y)
proof −
  have (THE z. (P x z ∧ P y z) ∧
    (∀ w. P w z ⟶ O w x ∨ O w y)) = (σ z. z = x ∨ z = y)
  proof (rule the-equality)
    have x = x ∨ x = y using refl..
    hence exz: ∃ z. z = x ∨ z = y..
    hence allw: (∀ w. w = x ∨ w = y ⟶ P w (σ z. z = x ∨ z = y))
      by (rule F-in)
    show (P x (σ z. z = x ∨ z = y) ∧ P y (σ z. z = x ∨ z = y)) ∧
```

```
      (∀ w. P w (σ z. z = x ∨ z = y) ⟶ O w x ∨ O w y)
    proof
      show (P x (σ z. z = x ∨ z = y) ∧ P y (σ z. z = x ∨ z = y))
      proof
        from allw have x = x ∨ x = y ⟶ P x (σ z. z = x ∨ z = y)..
        thus P x (σ z. z = x ∨ z = y)
          using ‹x = x ∨ x = y›..
      next
        from allw have y = x ∨ y = y ⟶ P y (σ z. z = x ∨ z = y)..
        moreover have y = x ∨ y = y
          using refl..
        ultimately show P y (σ z. z = x ∨ z = y)..
      qed
    next
      show ∀ w. P w (σ z. z = x ∨ z = y) ⟶ O w x ∨ O w y
      proof
        fix w
        show P w (σ z. z = x ∨ z = y) ⟶ O w x ∨ O w y
        proof
          have ∀ v. P v (σ z. z = x ∨ z = y) ⟶ (∃ z. (z = x ∨ z =
y) ∧ O v z) using exz by (rule parts-overlap-Fs)
          hence P w (σ z. z = x ∨ z = y) ⟶ (∃ z. (z = x ∨ z = y) ∧
O w z)..
          moreover assume P w (σ z. z = x ∨ z = y)
          ultimately have (∃ z. (z = x ∨ z = y) ∧ O w z)..
          then obtain z where z: (z = x ∨ z = y) ∧ O w z..
          thus O w x ∨ O w y by (rule or-id)
        qed
      qed
    qed
  next
    fix z
    assume z: (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
    with pair-fusion have (σ z. z = x ∨ z = y) = z..
    thus z = (σ z. z = x ∨ z = y)..
  qed
  with strong-sum-eq show x ⊕ y = (σ z. z = x ∨ z = y)
    by (rule ssubst)
qed

corollary strong-sum-intro:
  (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y) ⟶ x ⊕ y = z
proof
  assume z: (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
  with pair-fusion have (σ z. z = x ∨ z = y) = z..
  with strong-sum-fusion show (x ⊕ y) = z
    by (rule ssubst)
qed
```

corollary *strong-sum-character*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
proof −
  from *strong-sum-closure* **obtain** z **where** z:
    $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$**..**
 with *strong-sum-intro* **have** $x \oplus y = z$**..**
 thus *?thesis* **using** z **by** (*rule ssubst*)
qed

corollary *summands-in*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y))$
  using *strong-sum-character***..**

corollary *first-summand-in*: $P\ x\ (x \oplus y)$ **using** *summands-in***..**

corollary *second-summand-in*: $P\ y\ (x \oplus y)$ **using** *summands-in***..**

corollary *sum-part-overlap*: $(\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
using *strong-sum-character***..**

lemma *strong-sum-absorption*: $y = (x \oplus y) \Longrightarrow P\ x\ y$
proof −
 assume $y = (x \oplus y)$
 thus $P\ x\ y$ **using** *first-summand-in* **by** (*rule ssubst*)
qed

theorem *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
proof −
 assume $\neg\ P\ x\ y$
 have $\neg\ (\forall z.\ P\ z\ x \longrightarrow O\ z\ y)$
 proof
  assume z: $\forall z.\ P\ z\ x \longrightarrow O\ z\ y$
  have $(\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)) \wedge$
    $(\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z))$
  proof
   show $\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)$
   proof
    fix v
    show $y = v \longrightarrow P\ v\ (x \oplus y)$
    proof
     assume $y = v$
     thus $P\ v\ (x \oplus y)$
       using *second-summand-in* **by** (*rule subst*)
    qed
   qed
   show $\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z)$
   proof
    fix v
    show $P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z)$
    proof

```
      assume P v (x ⊕ y)
      moreover from sum-part-overlap have
        P v (x ⊕ y) ⟶ O v x ∨ O v y..
      ultimately have O v x ∨ O v y by (rule rev-mp)
      hence O v y
      proof
       assume O v x
       with overlap-eq have ∃ w. P w v ∧ P w x..
       then obtain w where w: P w v ∧ P w x..
       from z have P w x ⟶ O w y..
       moreover from w have P w x..
       ultimately have O w y..
       with overlap-eq have ∃ t. P t w ∧ P t y..
       then obtain t where t: P t w ∧ P t y..
       hence P t w..
       moreover from w have P w v..
       ultimately have P t v
         by (rule part-transitivity)
       moreover from t have P t y..
       ultimately show O v y
         by (rule overlap-intro)
      next
       assume O v y
       thus O v y.
      qed
      with refl have y = y ∧ O v y..
      thus ∃ z. y = z ∧ O v z..
     qed
    qed
   qed
   hence  (σ z. y = z) = (x ⊕ y) by (rule strong-fusion-intro)
   with strong-fusion-idempotence have y = x ⊕ y by (rule subst)
   hence P x y by (rule strong-sum-absorption)
   with ‹¬ P x y› show False..
  qed
  thus ∃ z. P z x ∧ ¬ O z y by simp
qed

lemma sum-character: ∀ v. O v (x ⊕ y) ⟷ (O v x ∨ O v y)
proof
  fix v
  show O v (x ⊕ y) ⟷ (O v x ∨ O v y)
  proof
    assume O v (x ⊕ y)
    with overlap-eq have ∃ w. P w v ∧ P w (x ⊕ y)..
    then obtain w where w: P w v ∧ P w (x ⊕ y)..
    hence P w v..
    have P w (x ⊕ y) ⟶ O w x ∨ O w y using sum-part-overlap..
    moreover from w have P w (x ⊕ y)..
```

```
    ultimately have O w x ∨ O w y..
    thus O v x ∨ O v y
    proof
      assume O w x
      hence O x w
        by (rule overlap-symmetry)
      with ‹P w v› have O x v
        by (rule overlap-monotonicity)
      hence O v x
        by (rule overlap-symmetry)
      thus O v x ∨ O v y..
    next
      assume O w y
      hence O y w
        by (rule overlap-symmetry)
      with ‹P w v› have O y v
        by (rule overlap-monotonicity)
      hence O v y by (rule overlap-symmetry)
      thus O v x ∨ O v y..
    qed
  next
    assume O v x ∨ O v y
    thus O v (x ⊕ y)
    proof
      assume O v x
      with overlap-eq have ∃ w. P w v ∧ P w x..
      then obtain w where w: P w v ∧ P w x..
      hence P w v..
      moreover from w have P w x..
      hence P w (x ⊕ y) using first-summand-in
        by (rule part-transitivity)
      ultimately show O v (x ⊕ y)
        by (rule overlap-intro)
    next
      assume O v y
      with overlap-eq have ∃ w. P w v ∧ P w y..
      then obtain w where w: P w v ∧ P w y..
      hence P w v..
      moreover from w have P w y..
      hence P w (x ⊕ y) using second-summand-in
        by (rule part-transitivity)
      ultimately show O v (x ⊕ y)
        by (rule overlap-intro)
    qed
  qed
qed

lemma sum-eq: x ⊕ y = (THE z. ∀ v. O v z = (O v x ∨ O v y))
proof −
```

```
have (THE z. ∀ v. O v z ⟷ (O v x ∨ O v y)) = x ⊕ y
proof (rule the-equality)
  show ∀ v. O v (x ⊕ y) ⟷ (O v x ∨ O v y) using sum-character.
next
  fix z
  assume z: ∀ v. O v z ⟷ (O v x ∨ O v y)
  have (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
  proof
    show P x z ∧ P y z
    proof
      show P x z
      proof (rule ccontr)
        assume ¬ P x z
        hence ∃ v. P v x ∧ ¬ O v z
          by (rule strong-supplementation)
        then obtain v where v: P v x ∧ ¬ O v z..
        hence ¬ O v z..
        from z have O v z ⟷ (O v x ∨ O v y)..
        moreover from v have P v x..
        hence O v x by (rule part-implies-overlap)
        hence O v x ∨ O v y..
        ultimately have O v z..
        with ‹¬ O v z› show False..
      qed
    next
      show P y z
      proof (rule ccontr)
        assume ¬ P y z
        hence ∃ v. P v y ∧ ¬ O v z
          by (rule strong-supplementation)
        then obtain v where v: P v y ∧ ¬ O v z..
        hence ¬ O v z..
        from z have O v z ⟷ (O v x ∨ O v y)..
        moreover from v have P v y..
        hence O v y by (rule part-implies-overlap)
        hence O v x ∨ O v y..
        ultimately have O v z..
        with ‹¬ O v z› show False..
      qed
    qed
    show ∀ w. P w z ⟶ (O w x ∨ O w y)
    proof
      fix w
      show P w z ⟶ (O w x ∨ O w y)
      proof
        from z have O w z ⟷ O w x ∨ O w y..
        moreover assume P w z
        hence O w z by (rule part-implies-overlap)
        ultimately show O w x ∨ O w y..
```

```
      qed
    qed
  qed
  with strong-sum-intro have x ⊕ y = z..
  thus z = x ⊕ y..
 qed
 thus ?thesis..
qed

theorem fusion-eq: ∃ x. F x ⟹
 (σ x. F x) = (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z))
proof −
 assume ∃ x. F x
 hence bla: ∀ y. P y (σ x. F x) ⟶ (∃ z. F z ∧ O y z)
  by (rule parts-overlap-Fs)
 have (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z)) = (σ x. F x)
 proof (rule the-equality)
  show ∀ y. O y (σ x. F x) ⟷ (∃ z. F z ∧ O y z)
  proof
   fix y
   show O y (σ x. F x) ⟷ (∃ z. F z ∧ O y z)
   proof
    assume O y (σ x. F x)
    with overlap-eq have ∃ v. P v y ∧ P v (σ x. F x)..
    then obtain v where v: P v y ∧ P v (σ x. F x)..
    hence P v y..
    from bla have P v (σ x. F x) ⟶ (∃ z. F z ∧ O v z)..
    moreover from v have P v (σ x. F x)..
    ultimately have (∃ z. F z ∧ O v z)..
    then obtain z where z: F z ∧ O v z..
    hence F z..
    moreover from z have O v z..
    hence O z v by (rule overlap-symmetry)
    with ‹P v y› have O z y by (rule overlap-monotonicity)
    hence O y z by (rule overlap-symmetry)
    ultimately have F z ∧ O y z..
    thus (∃ z. F z ∧ O y z)..
   next
    assume ∃ z. F z ∧ O y z
    then obtain z where z: F z ∧ O y z..
    from‹∃ x. F x› have (∀ y. F y ⟶ P y (σ x. F x))
     by (rule F-in)
    hence F z ⟶  P z (σ x. F x)..
    moreover from z have F z..
    ultimately have P z (σ x. F x)..
    moreover from z have O y z..
    ultimately show O y (σ x. F x)
     by (rule overlap-monotonicity)
   qed
```

```
    qed
  next
    fix x
    assume x: ∀ y. O y x ⟷ (∃ v. F v ∧ O y v)
    have (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z))
    proof
      show ∀ y. F y ⟶ P y x
      proof
        fix y
        show F y ⟶ P y x
        proof
          assume F y
          show P y x
          proof (rule ccontr)
            assume ¬ P y x
            hence ∃ z. P z y ∧ ¬ O z x
              by (rule strong-supplementation)
            then obtain z where z: P z y ∧ ¬ O z x..
            hence ¬ O z x..
            from x have O z x ⟷ (∃ v. F v ∧ O z v)..
            moreover from z have P z y..
            hence O z y by (rule part-implies-overlap)
            with ‹F y› have F y ∧ O z y..
            hence ∃ y. F y ∧ O z y..
            ultimately have O z x..
            with ‹¬ O z x› show False..
          qed
        qed
      qed
      show ∀ y. P y x ⟶ (∃ z. F z ∧ O y z)
      proof
        fix y
        show P y x ⟶ (∃ z. F z ∧ O y z)
        proof
          from x have O y x ⟷ (∃ z. F z ∧ O y z)..
          moreover assume P y x
          hence O y x by (rule part-implies-overlap)
          ultimately show ∃ z. F z ∧ O y z..
        qed
      qed
    qed
    hence (σ x. F x) = x
      by (rule strong-fusion-intro)
    thus x = (σ x. F x)..
  qed
  thus (σ x. F x) = (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z))..
qed

end
```

sublocale *GEM1* $\subseteq$ *GEM*
proof
fix $x\ y\ F$
show $\neg\ P\ x\ y \Longrightarrow \exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$
using *strong-supplementation*.
show $x \oplus y = (THE\ z.\ \forall\, v.\ O\ v\ z \longleftrightarrow (O\ v\ x \vee O\ v\ y))$
using *sum-eq*.
show $x \otimes y = (THE\ z.\ \forall\, v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
using *product-eq*.
show $x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
show $-\ x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
using *complement-eq*.
show $u = (THE\ x.\ \forall\, y.\ P\ y\ x)$
using *universe-eq*.
show $\exists\, x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ \forall\, y.\ O\ y\ x \longleftrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z))$ **using** *fusion-eq*.
show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall\, y.\ F\ y \longrightarrow P\ x\ y)$
using *general-product-eq*.
qed

sublocale *GEM* $\subseteq$ *GEM1*
proof
fix $x\ y\ F$
show $\exists\, x.\ F\ x \Longrightarrow (\exists\, x.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)))$ **using** *strong-fusion*.
show $\exists\, x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)))$ **using** *strong-fusion-eq*.
show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall\, y.\ F\ y \longrightarrow P\ x\ y)$ **using** *general-product-eq*.
show $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall\, w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$ **using** *strong-sum-eq*.
show $x \otimes y = (THE\ z.\ \forall\, v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
using *product-eq*.
show $x \ominus y = (THE\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
show $-\ x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$ **using** *complement-eq*.
show $u = (THE\ x.\ \forall\, y.\ P\ y\ x)$ **using** *universe-eq*.
qed

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