Mereology

Ben Blumson

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Abstract

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

Contents

1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.[1] We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].[2]

We will use the following notation throughout.[3]

typedecl i
consts *part* :: $i \Rightarrow i \Rightarrow$ *bool* (P)
consts *overlap* :: $i \Rightarrow i \Rightarrow$ *bool* (O)
consts *proper-part* :: $i \Rightarrow i \Rightarrow$ *bool* (PP)
consts *sum* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\oplus$ *52*)
consts *product* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\otimes$ *53*)
consts *difference* :: $i \Rightarrow i \Rightarrow i$ (**infix** $\ominus$ *51*)
consts *complement*:: $i \Rightarrow i$ ($-$)
consts *universe* :: i (u)
consts *general-sum* :: ($i \Rightarrow$ *bool*) $\Rightarrow i$ (**binder** σ *9*)
consts *general-product* :: ($i \Rightarrow$ *bool*) $\Rightarrow i$ (**binder** π [*8*] *9*)

2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.[4] In other words, parthood is assumed to be a partial ordering relation.[5] Overlap is defined as common parthood.[6]

locale *PM* =
  assumes *part-reflexivity*: $P\ x\ x$
  assumes *part-transitivity* : $P\ x\ y \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$

[1] For similar developments see [Sen, 2017] and [Bittner, 2018].

[2] For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.

[3] See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.

[4] For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].

[5] Hence the name *premereology*, from [Parsons, 2014] p. 6.

[6] See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
begin

2.1 Parthood

lemma *identity-implies-part* : $x = y \Longrightarrow P\ x\ y$
proof −
  assume $x = y$
  moreover have $P\ x\ x$ **by** (*rule part-reflexivity*)
  ultimately show $P\ x\ y$ **by** (*rule subst*)
qed

2.2 Overlap

lemma *overlap-intro*: $P\ z\ x \Longrightarrow P\ z\ y \Longrightarrow O\ x\ y$
proof−
  assume $P\ z\ x$
  moreover assume $P\ z\ y$
  ultimately have $P\ z\ x \wedge P\ z\ y$**..**
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
  with *overlap-eq* **show** $O\ x\ y$**..**
qed

lemma *part-implies-overlap*: $P\ x\ y \Longrightarrow O\ x\ y$
proof −
  assume $P\ x\ y$
  with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$**..**
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
  with *overlap-eq* **show** $O\ x\ y$**..**
qed

lemma *overlap-reflexivity*: $O\ x\ x$
proof −
  have $P\ x\ x \wedge P\ x\ x$ **using** *part-reflexivity part-reflexivity*..
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ x$**..**
  with *overlap-eq* **show** $O\ x\ x$**..**
qed

lemma *overlap-symmetry*: $O\ x\ y \Longrightarrow O\ y\ x$
proof−
  assume $O\ x\ y$
  with *overlap-eq* **have** $\exists\ z.\ P\ z\ x \wedge P\ z\ y$**..**
  hence $\exists\ z.\ P\ z\ y \wedge P\ z\ x$ **by** *auto*
  with *overlap-eq* **show** $O\ y\ x$**..**
qed

lemma *overlap-monotonicity*: $P\ x\ y \Longrightarrow O\ z\ x \Longrightarrow O\ z\ y$
proof −
  assume $P\ x\ y$
  assume $O\ z\ x$

with *overlap-eq* **have** $\exists\ v.\ P\ v\ z \wedge P\ v\ x$..
then obtain v **where** v: $P\ v\ z \wedge P\ v\ x$..
hence $P\ v\ z$..
moreover from v **have** $P\ v\ x$..
hence $P\ v\ y$ **using** ‹$P\ x\ y$› **by** (*rule part-transitivity*)
ultimately have $P\ v\ z \wedge P\ v\ y$..
hence $\exists\ v.\ P\ v\ z \wedge P\ v\ y$..
with *overlap-eq* **show** $O\ z\ y$..
qed

The next lemma is from [Hovda, 2009] p. 66.

lemma *overlap-lemma*: $\exists x.\ (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$
proof –
  fix x
  have $P\ x\ y \wedge O\ z\ x \longrightarrow O\ y\ z$
  proof
    assume *antecedent*: $P\ x\ y \wedge O\ z\ x$
    hence $O\ z\ x$..
    with *overlap-eq* **have** $\exists v.\ P\ v\ z \wedge P\ v\ x$..
    then obtain v **where** v: $P\ v\ z \wedge P\ v\ x$..
    hence $P\ v\ x$..
    moreover from *antecedent* **have** $P\ x\ y$..
    ultimately have $P\ v\ y$ **by** (*rule part-transitivity*)
    moreover from v **have** $P\ v\ z$..
    ultimately have $P\ v\ y \wedge P\ v\ z$..
    hence $\exists v.\ P\ v\ y \wedge P\ v\ z$..
    with *overlap-eq* **show** $O\ y\ z$..
  qed
  thus $\exists x.\ (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$..
qed

2.3 Disjointness

lemma *disjoint-implies-distinct*: $\neg\ O\ x\ y \Longrightarrow x \neq y$
proof –
  assume $\neg\ O\ x\ y$
  show $x \neq y$
  proof
    assume $x = y$
    hence $\neg\ O\ y\ y$ **using** ‹$\neg\ O\ x\ y$› **by** (*rule subst*)
    thus *False* **using** *overlap-reflexivity*..
  qed
qed

lemma *disjoint-implies-not-part*: $\neg\ O\ x\ y \Longrightarrow \neg\ P\ x\ y$
proof –
  assume $\neg\ O\ x\ y$
  show $\neg\ P\ x\ y$
  proof

```
    assume P x y
    hence O x y by (rule part-implies-overlap)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-symmetry: ¬ O x y ⟹ ¬ O y x
proof −
  assume ¬ O x y
  show ¬ O y x
  proof
    assume O y x
    hence O x y by (rule overlap-symmetry)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-demonotonicity: P x y ⟹ ¬ O z y ⟹ ¬ O z x
proof −
  assume P x y
  assume ¬ O z y
  show ¬ O z x
  proof
    assume O z x
    with ‹P x y› have O z y
      by (rule overlap-monotonicity)
    with ‹¬ O z y› show False..
  qed
qed

end
```

3 Ground Mereology

The theory of *ground mereology* adds to premereology the antisymmetry of parthood, and defines proper parthood as nonidentical parthood.[7] In other words, ground mereology assumes that parthood is a partial order.

```
locale M = PM +
  assumes part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
begin
```

[7]For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

3.1 Proper Parthood

```
lemma proper-implies-part: PP x y ⟹ P x y
proof −
  assume PP x y
  with nip-eq have P x y ∧ x ≠ y..
  thus P x y..
qed

lemma proper-implies-distinct: PP x y ⟹ x ≠ y
proof −
  assume PP x y
  with nip-eq have P x y ∧ x ≠ y..
  thus x ≠ y..
qed

lemma proper-implies-not-part: PP x y ⟹ ¬ P y x
proof −
  assume PP x y
  hence P x y by (rule proper-implies-part)
  show ¬ P y x
  proof
    from ‹PP x y› have x ≠ y by (rule proper-implies-distinct)
    moreover assume P y x
    with ‹P x y› have x = y by (rule part-antisymmetry)
    ultimately show False..
  qed
qed

lemma proper-part-asymmetry: PP x y ⟹ ¬ PP y x
proof −
  assume PP x y
  hence P x y by (rule proper-implies-part)
  from ‹PP x y› have x ≠ y by (rule proper-implies-distinct)
  show ¬ PP y x
  proof
    assume PP y x
    hence P y x by (rule proper-implies-part)
    with ‹P x y› have x = y by (rule part-antisymmetry)
    with ‹x ≠ y› show False..
  qed
qed

lemma proper-implies-overlap: PP x y ⟹ O x y
proof −
  assume PP x y
  hence P x y by (rule proper-implies-part)
  thus O x y by (rule part-implies-overlap)
qed
```

```
end
```

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.[8] In the presence of antisymmetry, the two definitions of proper parthood are equivalent.[9]

```
locale M1 = PM +
  assumes nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x
  assumes part-antisymmetry: P x y ⟹ P y x ⟹ x = y

sublocale M ⊆ M1
proof
  fix x y
  show nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x
  proof
    assume PP x y
    with nip-eq have nip: P x y ∧ x ≠ y..
    hence x ≠ y..
    from nip have P x y..
    moreover have ¬ P y x
    proof
      assume P y x
      with ‹P x y› have x = y by (rule part-antisymmetry)
      with ‹x ≠ y› show False..
    qed
    ultimately show P x y ∧ ¬ P y x..
  next
    assume nmp: P x y ∧ ¬ P y x
    hence ¬ P y x..
    from nmp have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence ¬ P y y using ‹¬ P y x› by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately have P x y ∧ x ≠ y..
    with nip-eq show PP x y..
  qed
  show  P x y ⟹ P y x ⟹ x = y using part-antisymmetry.
qed
```

[8]See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

[9]See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

```
sublocale M1 ⊆ M
proof
  fix x y
  show nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    with nmp-eq have nmp: P x y ∧ ¬ P y x..
    hence ¬ P y x..
    from nmp have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence ¬ P y y using ‹¬ P y x› by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume nip: P x y ∧ x ≠ y
    hence x ≠ y..
    from nip have P x y..
    moreover have ¬ P y x
    proof
      assume P y x
      with ‹P x y› have x = y by (rule part-antisymmetry)
      with ‹x ≠ y› show False..
    qed
    ultimately have P x y ∧ ¬ P y x..
    with nmp-eq show PP x y..
  qed
  show  P x y ⟹ P y x ⟹ x = y using part-antisymmetry.
qed
```

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.[10]

```
locale M2 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x

sublocale M ⊆ M2
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show PP x y ⟷ P x y ∧ ¬ P y x using nmp-eq.
qed
```

[10]For this point see especially [Parsons, 2014] pp. 9-10.

```
sublocale M2 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof -
    assume P x y
    assume P y x
    show x = y
    proof (rule ccontr)
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      with nmp-eq have P x y ∧ ¬ P y x..
      hence ¬ P y x..
      thus False using ‹P y x›..
    qed
  qed
qed
```

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.[11]

```
locale M3 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)

sublocale M ⊆ M3
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)
  proof
    assume x = y
    moreover have ∀ z. P z x ⟷ P z x by simp
    ultimately show ∀ z. P z x ⟷ P z y by (rule subst)
  next
    assume z: ∀ z. P z x ⟷ P z y
    show x = y
    proof (rule part-antisymmetry)
      from z have P y x ⟷ P y y..
      moreover have P y y by (rule part-reflexivity)
      ultimately show P y x..
    next
      from z have P x x ⟷ P x y..
      moreover have P x x by (rule part-reflexivity)
      ultimately show P x y..
```

[11]For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.

```
    qed
  qed
qed

sublocale M3 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    have ∀ z. P z x ⟷ P z y
    proof
      fix z
      show P z x ⟷ P z y
      proof
        assume P z x
        thus P z y using ‹P x y› by (rule part-transitivity)
      next
        assume P z y
        thus P z x using ‹P y x› by (rule part-transitivity)
      qed
    qed
    with part-extensionality show x = y..
  qed
qed
```

The fourth axiomatization adopts proper parthood as primitive.[12] Improper parthood is defined as proper parthood or identity.

```
locale M4 =
  assumes part-eq: P x y ⟷ PP x y ∨ x = y
  assumes  overlap-eq: O x y ⟷ (∃ z. P z x ∧ P z y)
  assumes proper-part-asymmetry: PP x y ⟹ ¬ PP y x
  assumes proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
begin

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ¬ PP x x by (rule proper-part-asymmetry)
  thus False using ‹PP x x›..
qed

end
```

[12]See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

```
sublocale M ⊆ M4
proof
  fix x y z
  show part-eq: P x y ⟷ (PP x y ∨ x = y)
  proof
    assume P x y
    show PP x y ∨ x = y
    proof cases
      assume x = y
      thus PP x y ∨ x = y..
    next
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      thus PP x y ∨ x = y..
    qed
  next
    assume PP x y ∨ x = y
    thus P x y
    proof
      assume PP x y
      thus P x y by (rule proper-implies-part)
    next
      assume x = y
      thus P x y by (rule identity-implies-part)
    qed
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show PP x y ⟹ ¬ PP y x using proper-part-asymmetry.
  show proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
  proof −
    assume PP x y
    assume PP y z
    have P x z ∧ x ≠ z
    proof
      from ‹PP x y› have P x y by (rule proper-implies-part)
     moreover from ‹PP y z› have P y z by (rule proper-implies-part)
      ultimately show P x z by (rule part-transitivity)
    next
      show x ≠ z
      proof
        assume x = z
        hence PP y x using ‹PP y z› by (rule ssubst)
        hence ¬ PP x y by (rule proper-part-asymmetry)
        thus False using ‹PP x y›..
      qed
    qed
    with nip-eq show PP x z..
  qed
```

```
qed

sublocale M4 ⊆ M
proof
  fix x y z
  show proper-part-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    hence PP x y ∨ x = y..
    with part-eq have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence PP y y using ‹PP x y› by (rule subst)
      with proper-part-irreflexivity show False..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume rhs: P x y ∧ x ≠ y
    hence x ≠ y..
    from rhs have P x y..
    with part-eq have PP x y ∨ x = y..
    thus PP x y
    proof
      assume PP x y
      thus PP x y.
    next
      assume x = y
      with ‹x ≠ y› show PP x y..
    qed
  qed
  show P x x
  proof -
    have x = x by (rule refl)
    hence PP x x ∨ x = x..
    with part-eq show P x x..
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof -
    assume P x y
    assume P y x
    from part-eq have PP x y ∨ x = y using ‹P x y›..
    thus x = y
    proof
      assume PP x y
      hence ¬ PP y x by (rule proper-part-asymmetry)
      from part-eq have PP y x ∨ y = x using ‹P y x›..
      thus x = y
```

```
      proof
        assume PP y x
        with ‹¬ PP y x› show x = y..
      next
        assume y = x
        thus x = y..
      qed
    qed
  qed
  show P x y ⟹ P y z ⟹ P x z
  proof –
    assume P x y
    assume P y z
    with part-eq have PP y z ∨ y = z..
    hence PP x z ∨ x = z
    proof
      assume PP y z
      from part-eq have PP x y ∨ x = y using ‹P x y›..
      hence PP x z
      proof
        assume PP x y
        thus PP x z using ‹PP y z› by (rule proper-part-transitivity)
      next
        assume x = y
        thus PP x z using ‹PP y z› by (rule ssubst)
      qed
      thus PP x z ∨ x = z..
    next
      assume y = z
      moreover from part-eq have PP x y ∨ x = y using ‹P x y›..
      ultimately show PP x z ∨ x = z by (rule subst)
    qed
    with part-eq show P x z..
  qed
qed
```

4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.[13]

```
locale MM = M +
  assumes weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
```

[13] See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.[14]

```
locale MM1 = M +
  assumes proper-weak-supplementation:
    PP y x ⟹ (∃ z. PP z x ∧ ¬ O z y)

sublocale MM ⊆ MM1
proof
  fix x y
  show PP y x ⟹ (∃ z. PP z x ∧ ¬ O z y)
  proof −
    assume PP y x
    hence ∃ z. P z x ∧ ¬ O z y by (rule weak-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence ¬ O z y..
    from z have P z x..
    hence P z x ∧ z ≠ x
    proof
      show z ≠ x
      proof
        assume z = x
        hence PP y z
          using ‹PP y x› by (rule ssubst)
        hence O y z by (rule proper-implies-overlap)
        hence O z y by (rule overlap-symmetry)
        with ‹¬ O z y› show False..
      qed
    qed
    with nip-eq have PP z x..
    hence PP z x ∧ ¬ O z y
      using ‹¬ O z y›..
    thus ∃ z. PP z x ∧ ¬ O z y..
  qed
qed

sublocale MM1 ⊆ MM
proof
  fix x y
  show weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
  proof −
    assume PP y x
   hence ∃ z. PP z x ∧ ¬ O z y by (rule proper-weak-supplementation)
    then obtain z where z: PP z x ∧ ¬ O z y..
    hence PP z x..
    hence P z x by (rule proper-implies-part)
```

[14]See [Simons, 1987] p. 28.

```
    moreover from z have ¬ O z y..
    ultimately have P z x ∧ ¬ O z y..
    thus ∃ z. P z x ∧ ¬ O z y..
  qed
qed
```

The following two corollaries are sometimes found in the literature.[15]

```
context MM
begin

corollary weak-company: PP y x ⟹ (∃ z. PP z x ∧ z ≠ y)
proof −
  assume PP y x
  hence ∃ z. PP z x ∧ ¬ O z y by (rule proper-weak-supplementation)
  then obtain z where z: PP z x ∧ ¬ O z y..
  hence PP z x..
  from z have ¬ O z y..
  hence z ≠ y by (rule disjoint-implies-distinct)
  with ‹PP z x› have PP z x ∧ z ≠ y..
  thus ∃ z. PP z x ∧ z ≠ y..
qed

corollary strong-company: PP y x ⟹ (∃ z. PP z x ∧ ¬ P z y)
proof −
  assume PP y x
  hence ∃z. PP z x ∧ ¬ O z y by (rule proper-weak-supplementation)
  then obtain z where z: PP z x ∧ ¬ O z y..
  hence PP z x..
  from z have ¬ O z y..
  hence ¬ P z y by (rule disjoint-implies-not-part)
  with ‹PP z x› have  PP z x ∧ ¬ P z y..
  thus ∃ z. PP z x ∧ ¬ P z y..
qed

end
```

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.[16]

```
locale MM2 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes weak-supplementation: PP y x ⟹ (∃ z. P z x ∧ ¬ O z y)
```

[15] See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].

[16] See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.

sublocale *MM2* $\subseteq$ *MM*
proof
  fix $x\ y$
  show $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$ **using** *nip-eq*.
  show *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$
  proof –
    assume $P\ x\ y$
    assume $P\ y\ x$
    show $x = y$
    proof (*rule ccontr*)
      assume $x \neq y$
      with ‹$P\ x\ y$› **have** $P\ x\ y \wedge x \neq y$..
      with *nip-eq* **have** $PP\ x\ y$..
      hence $\exists\ z.\ P\ z\ y \wedge \neg\ O\ z\ x$ **by** (*rule weak-supplementation*)
      then obtain z **where** z: $P\ z\ y \wedge \neg\ O\ z\ x$..
      hence $\neg\ O\ z\ x$..
      hence $\neg\ P\ z\ x$ **by** (*rule disjoint-implies-not-part*)
      from z **have** $P\ z\ y$..
      hence $P\ z\ x$ **using** ‹$P\ y\ x$› **by** (*rule part-transitivity*)
      with ‹$\neg\ P\ z\ x$› **show** *False*..
    qed
  qed
  show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

sublocale *MM* $\subseteq$ *MM2*
proof
  fix $x\ y$
  show $PP\ x\ y \longleftrightarrow (P\ x\ y \wedge x \neq y)$ **using** *nip-eq*.
  show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.[17]

locale *MM3* =
  assumes *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$
  assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
  assumes *proper-part-transitivity*: $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$
  assumes *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

lemma *part-reflexivity*: $P\ x\ x$
proof –
  have $x = x$..

[17] See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

```
  hence PP x x ∨ x = x..
  with part-eq show P x x..
qed

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ∃ z. P z x ∧ ¬ O z x by (rule weak-supplementation)
  then obtain z where z: P z x ∧ ¬ O z x..
  hence ¬ O z x..
  from z have P z x..
  with part-reflexivity have P z z ∧ P z x..
  hence ∃ v. P v z ∧ P v x..
  with overlap-eq have O z x..
  with ‹¬ O z x› show False..
qed

end

sublocale MM3 ⊆ M4
proof
  fix x y z
  show P x y ⟷ PP x y ∨ x = y using part-eq.
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show proper-part-irreflexivity: PP x y ⟹ ¬ PP y x
  proof −
    assume PP x y
    show ¬ PP y x
    proof
      assume PP y x
      hence PP y y using ‹PP x y› by (rule proper-part-transitivity)
      with proper-part-irreflexivity show False..
    qed
  qed
  show PP x y ⟹ PP y z ⟹ PP x z using proper-part-transitivity.
qed

sublocale MM3 ⊆ MM
proof
  fix x y
  show PP y x ⟹ (∃ z. P z x ∧ ¬ O z y) using weak-supplementation.
qed

sublocale MM ⊆ MM3
proof
  fix x y z
  show P x y ⟷ (PP x y ∨ x = y) using part-eq.
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show PP x y ⟹ PP y z ⟹ PP x z using proper-part-transitivity.
```

show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.[18]

locale *EM* = *M* +
assumes *strong-supplementation*:
$\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

Strong supplementation entails weak supplementation.[19]

lemma *weak-supplementation*: $PP\ x\ y \Longrightarrow (\exists z.\ P\ z\ y \wedge \neg\ O\ z\ x)$
proof −
assume $PP\ x\ y$
hence $\neg\ P\ y\ x$ **by** (*rule proper-implies-not-part*)
thus $\exists z.\ P\ z\ y \wedge \neg\ O\ z\ x$ **by** (*rule strong-supplementation*)
qed

end

So minimal mereology is a subtheory of extensional mereology.[20]

sublocale $EM \subseteq MM$
proof
fix $y\ x$
show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

Strong supplementation also entails the proper parts principle.[21]

context *EM*
begin

lemma *proper-parts-principle*:
$(\exists z.\ PP\ z\ x) \Longrightarrow (\forall z.\ PP\ z\ x \longrightarrow P\ z\ y) \Longrightarrow P\ x\ y$
proof −
assume $\exists z.\ PP\ z\ x$
then obtain v **where** v: $PP\ v\ x$..
hence $P\ v\ x$ **by** (*rule proper-implies-part*)
assume *antecedent*: $\forall z.\ PP\ z\ x \longrightarrow P\ z\ y$
hence $PP\ v\ x \longrightarrow P\ v\ y$..
hence $P\ v\ y$ **using** ‹$PP\ v\ x$›..

[18] See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.
[19] See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.
[20] [Casati and Varzi, 1999] p. 40.
[21] See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.

```
  with ‹P v x› have P v x ∧ P v y..
  hence ∃ v. P v x ∧ P v y..
  with overlap-eq have O x y..
  show P x y
  proof (rule ccontr)
    assume ¬ P x y
    hence ∃ z. P z x ∧ ¬ O z y
      by (rule strong-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence P z x..
    moreover have z ≠ x
    proof
      assume z = x
      moreover from z have ¬ O z y..
      ultimately have ¬ O x y by (rule subst)
      thus False  using ‹O x y›..
    qed
    ultimately have P z x ∧ z ≠ x..
    with nip-eq have PP z x..
    from antecedent have PP z x ⟶ P z y..
    hence P z y using ‹PP z x›..
    hence O z y by (rule part-implies-overlap)
    from z have ¬ O z y..
    thus False using ‹O z y›..
  qed
qed
```

Which with antisymmetry entails the extensionality of proper parthood.[22]

```
theorem proper-part-extensionality:
(∃ z. PP z x ∨ PP z y) ⟹ x = y ⟷ (∀ z. PP z x ⟷ PP z y)
proof −
  assume antecedent: ∃ z. PP z x ∨ PP z y
  show x = y ⟷ (∀ z. PP z x ⟷ PP z y)
  proof
    assume x = y
    moreover have ∀ z. PP z x ⟷ PP z x by simp
    ultimately show ∀ z. PP z x ⟷ PP z y by (rule subst)
  next
    assume right: ∀ z. PP z x ⟷ PP z y
    have ∀ z. PP z x ⟶ P z y
    proof
      fix z
      show PP z x ⟶ P z y
      proof
        assume PP z x
        from right have PP z x ⟷ PP z y..
```

[22]See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.

```
      hence PP z y using ‹PP z x›..
      thus P z y by (rule proper-implies-part)
    qed
  qed
  have ∀ z. PP z y ⟶ P z x
  proof
    fix z
    show PP z y ⟶ P z x
    proof
      assume PP z y
      from right have PP z x ⟷ PP z y..
      hence PP z x using ‹PP z y›..
      thus P z x by (rule proper-implies-part)
    qed
  qed
  from antecedent obtain z where z: PP z x ∨ PP z y..
  thus x = y
  proof (rule disjE)
    assume PP z x
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z y using ‹PP z x›..
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    with ‹P x y› show x = y
      by (rule part-antisymmetry)
  next
    assume PP z y
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z x using ‹PP z y›..
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
        by (rule proper-parts-principle)
    thus x = y
      using ‹P y x› by (rule part-antisymmetry)
  qed
 qed
qed
```

It also follows from strong supplementation that parthood is definable in terms of overlap.[23]

lemma *part-overlap-eq*: $P\ x\ y \longleftrightarrow (\forall z.\ O\ z\ x \longrightarrow O\ z\ y)$

[23] See [Parsons, 2014] p. 4.

```
proof
  assume P x y
  show (∀ z. O z x ⟶ O z y)
  proof
    fix z
    show O z x ⟶ O z y
    proof
      assume O z x
      with ‹P x y› show O z y
        by (rule overlap-monotonicity)
    qed
  qed
next
  assume right: ∀ z. O z x ⟶ O z y
  show P x y
  proof (rule ccontr)
    assume ¬ P x y
    hence ∃ z. P z x ∧ ¬ O z y
      by (rule strong-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence ¬ O z y..
    from right have O z x ⟶ O z y..
    moreover from z have P z x..
    hence O z x by (rule part-implies-overlap)
    ultimately have O z y..
    with ‹¬ O z y› show False..
  qed
qed
```

Which entails the extensionality of overlap.

```
theorem overlap-extensionality: x = y ⟷ (∀ z. O z x ⟷ O z y)
proof
  assume x = y
  moreover have ∀ z. O z x ⟷ O z x
  proof
    fix z
    show O z x ⟷ O z x..
  qed
  ultimately show ∀ z. O z x ⟷ O z y
    by (rule subst)
next
  assume right: ∀ z. O z x ⟷ O z y
  have ∀ z. O z y ⟶ O z x
  proof
    fix z
    from right have O z x ⟷ O z y..
    thus O z y ⟶ O z x..
  qed
  with part-overlap-eq have P y x..
```

have $\forall z.\ O\ z\ x \longrightarrow O\ z\ y$
proof
fix z
from *right* **have** $O\ z\ x \longleftrightarrow O\ z\ y$**..**
thus $O\ z\ x \longrightarrow O\ z\ y$**..**
qed
with *part-overlap-eq* **have** $P\ x\ y$**..**
thus $x = y$
using ‹$P\ y\ x$› **by** (*rule part-antisymmetry*)
qed

end

6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.[24]

locale $CM = M +$
assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
assumes *sum-closure*: $\exists z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
assumes *product-closure*:
$O\ x\ y \Longrightarrow \exists z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y$
begin

6.1 Products

lemma *product-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)) \Longrightarrow x \otimes y = z$
proof −
assume z: $\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
hence $(THE\ v.\ \forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \wedge P\ w\ y) = z$
proof (*rule the-equality*)
fix v
assume v: $\forall w.\ P\ w\ v \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w
from z **have** $P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**

[24] See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

  qed
  thus $x \otimes y = z$
    using *product-eq* **by** (*rule subst*)
qed

lemma *product-idempotence*: $x \otimes x = x$
proof −
  have $\forall w.\ P\ w\ x \longleftrightarrow P\ w\ x \wedge P\ w\ x$
  proof
    fix w
    show $P\ w\ x \longleftrightarrow P\ w\ x \wedge P\ w\ x$
    proof
      assume $P\ w\ x$
      thus $P\ w\ x \wedge P\ w\ x$ **using** ‹$P\ w\ x$›**..**
    next
      assume $P\ w\ x \wedge P\ w\ x$
      thus $P\ w\ x$**..**
    qed
  qed
  thus $x \otimes x = x$ **by** (*rule product-intro*)
qed

lemma *product-character*:
  $O\ x\ y \Longrightarrow (\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y))$
proof −
  assume $O\ x\ y$
  hence $\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$ **by** (*rule product-closure*)
  then obtain z **where** z: $\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
  hence $x \otimes y = z$ **by** (*rule product-intro*)
  thus $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow P\ w\ x \wedge P\ w\ y$
    using z **by** (*rule ssubst*)
qed

lemma *product-commutativity*: $O\ x\ y \Longrightarrow x \otimes y = y \otimes x$
proof −
  assume $O\ x\ y$
  hence $O\ y\ x$ **by** (*rule overlap-symmetry*)
  hence $\forall w.\ P\ w\ (y \otimes x) \longleftrightarrow (P\ w\ y \wedge P\ w\ x)$ **by** (*rule product-character*)
  hence $\forall w.\ P\ w\ (y \otimes x) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$ **by** *auto*
  thus $x \otimes y = y \otimes x$ **by** (*rule product-intro*)
qed

lemma *product-in-factors*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$
proof −
  assume $O\ x\ y$
  hence $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow P\ w\ x \wedge P\ w\ y$ **by** (*rule product-character*)
  hence $P\ (x \otimes y)\ (x \otimes y) \longleftrightarrow P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$**..**

```
  moreover have P (x ⊗ y) (x ⊗ y) by (rule part-reflexivity)
  ultimately show P (x ⊗ y) x ∧ P (x ⊗ y) y..
qed

lemma product-in-first-factor: O x y ⟹ P (x ⊗ y) x
proof −
  assume O x y
  hence P (x ⊗ y) x ∧ P (x ⊗ y) y by (rule product-in-factors)
  thus P (x ⊗ y) x..
qed

lemma product-in-second-factor: O x y ⟹ P (x ⊗ y) y
proof −
  assume O x y
  hence P (x ⊗ y) x ∧ P (x ⊗ y) y by (rule product-in-factors)
  thus P (x ⊗ y) y..
qed

lemma nonpart-implies-proper-product:
  ¬ P x y ∧ O x y ⟹ PP (x ⊗ y) x
proof −
  assume antecedent: ¬ P x y ∧ O x y
  hence ¬ P x y..
  from antecedent have O x y..
  hence P (x ⊗ y) x by (rule product-in-first-factor)
  moreover have (x ⊗ y) ≠ x
  proof
    assume (x ⊗ y) = x
    hence ¬ P (x ⊗ y) y
      using ‹¬ P x y› by (rule ssubst)
    moreover have P (x ⊗ y) y
      using ‹O x y› by (rule product-in-second-factor)
    ultimately show False..
  qed
  ultimately have P (x ⊗ y) x ∧ x ⊗ y ≠ x..
  with nip-eq show PP (x ⊗ y) x..
qed

lemma common-part-in-product: P z x ∧ P z y ⟹ P z (x ⊗ y)
proof −
  assume antecedent: P z x ∧ P z y
  hence ∃ z. P z x ∧ P z y..
  with overlap-eq have O x y..
  hence ∀ w. P w (x ⊗ y) ⟷ (P w x ∧ P w y)
    by (rule product-character)
  hence P z (x ⊗ y) ⟷ (P z x ∧ P z y)..
  thus P z (x ⊗ y)
    using ‹P z x ∧ P z y›..
qed
```

lemma *product-part-in-factors*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x \wedge P\ z\ y$
proof –
assume $O\ x\ y$
hence $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
by (*rule product-character*)
hence $P\ z\ (x \otimes y) \longleftrightarrow (P\ z\ x \wedge P\ z\ y)$**..**
moreover assume $P\ z\ (x \otimes y)$
ultimately show $P\ z\ x \wedge P\ z\ y$**..**
qed

corollary *product-part-in-first-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x$
proof –
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by (*rule product-part-in-factors*)
thus $P\ z\ x$**..**
qed

corollary *product-part-in-second-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ y$
proof –
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by (*rule product-part-in-factors*)
thus $P\ z\ y$**..**
qed

lemma *part-product-identity*: $P\ x\ y \Longrightarrow x \otimes y = x$
proof –
assume $P\ x\ y$
with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$**..**
hence $P\ x\ (x \otimes y)$ **by** (*rule common-part-in-product*)
have $O\ x\ y$ **using** ‹$P\ x\ y$› **by** (*rule part-implies-overlap*)
hence $P\ (x \otimes y)\ x$ **by** (*rule product-in-first-factor*)
thus $x \otimes y = x$ **using** ‹$P\ x\ (x \otimes y)$› **by** (*rule part-antisymmetry*)
qed

lemma *product-overlap*: $P\ z\ x \Longrightarrow O\ z\ y \Longrightarrow O\ z\ (x \otimes y)$
proof –
assume $P\ z\ x$
assume $O\ z\ y$
with *overlap-eq* **have** $\exists v.\ P\ v\ z \wedge P\ v\ y$**..**
then obtain v **where** v: $P\ v\ z \wedge P\ v\ y$**..**
hence $P\ v\ y$**..**

```
  from v have P v z..
  hence P v x using ‹P z x› by (rule part-transitivity)
  hence P v x ∧ P v y using ‹P v y›..
  hence P v (x ⊗ y) by (rule common-part-in-product)
  with ‹P v z› have P v z ∧ P v (x ⊗ y)..
  hence ∃ v. P v z ∧ P v (x ⊗ y)..
  with overlap-eq show O z (x ⊗ y)..
qed

lemma disjoint-from-second-factor:
  P x y ∧ ¬ O x (y ⊗ z) ⟹ ¬ O x z
proof –
  assume antecedent: P x y ∧ ¬ O x (y ⊗ z)
  hence ¬ O x (y ⊗ z)..
  show ¬ O x z
  proof
    from antecedent have P x y..
    moreover assume O x z
    ultimately have O x (y ⊗ z)
      by (rule product-overlap)
    with ‹¬ O x (y ⊗ z)› show False..
  qed
qed

lemma converse-product-overlap:
  O x y ⟹ O z (x ⊗ y) ⟹ O z y
proof –
  assume O x y
  hence P (x ⊗ y) y by (rule product-in-second-factor)
  moreover assume O z (x ⊗ y)
  ultimately show O z y
    by (rule overlap-monotonicity)
qed

lemma part-product-in-whole-product:
  O x y ⟹ P x v ∧ P y z ⟹ P (x ⊗ y) (v ⊗ z)
proof –
  assume O x y
  assume P x v ∧ P y z
  have ∀ w. P w (x ⊗ y) ⟶ P w (v ⊗ z)
  proof
    fix w
    show P w (x ⊗ y) ⟶ P w (v ⊗ z)
    proof
      assume P w (x ⊗ y)
      with ‹O x y› have P w x ∧ P w y
        by (rule product-part-in-factors)
      have P w v ∧ P w z
      proof
```

from ‹$P\ w\ x \wedge P\ w\ y$› **have** $P\ w\ x$..
moreover from ‹$P\ x\ v \wedge P\ y\ z$› **have** $P\ x\ v$..
ultimately show $P\ w\ v$ **by** (*rule part-transitivity*)
next
from ‹$P\ w\ x \wedge P\ w\ y$› **have** $P\ w\ y$..
moreover from ‹$P\ x\ v \wedge P\ y\ z$› **have** $P\ y\ z$..
ultimately show $P\ w\ z$ **by** (*rule part-transitivity*)
qed
thus $P\ w\ (v \otimes z)$ **by** (*rule common-part-in-product*)
qed
qed
hence $P\ (x \otimes y)\ (x \otimes y) \longrightarrow P\ (x \otimes y)\ (v \otimes z)$..
moreover have $P\ (x \otimes y)\ (x \otimes y)$ **by** (*rule part-reflexivity*)
ultimately show $P\ (x \otimes y)\ (v \otimes z)$..
qed

lemma *right-associated-product*: $(\exists w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
proof −
assume *antecedent*: $(\exists w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
then obtain w **where** w: $P\ w\ x \wedge P\ w\ y \wedge P\ w\ z$..
hence $P\ w\ x$..
from w **have** $P\ w\ y \wedge P\ w\ z$..
hence $\exists w.\ P\ w\ y \wedge P\ w\ z$..
with *overlap-eq* **have** $O\ y\ z$..
hence *yz*: $\forall w.\ P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$
by (*rule product-character*)
hence $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$..
hence $P\ w\ (y \otimes z)$
using ‹$P\ w\ y \wedge P\ w\ z$›..
with ‹$P\ w\ x$› **have** $P\ w\ x \wedge P\ w\ (y \otimes z)$..
hence $\exists w.\ P\ w\ x \wedge P\ w\ (y \otimes z)$..
with *overlap-eq* **have** $O\ x\ (y \otimes z)$..
hence *xyz*: $\forall w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$
by (*rule product-character*)
show $\forall w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
proof
fix w
from *yz* **have** *wyz*: $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$..
moreover from *xyz* **have**
$P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$..
ultimately show
$P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
by (*rule subst*)
qed
qed

lemma *left-associated-product*: $(\exists w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$

```
proof −
  assume antecedent: (∃ w. P w x ∧ P w y ∧ P w z)
  then obtain w where w: P w x ∧ P w y ∧ P w z..
  hence P w y ∧ P w z..
  hence P w y..
  have P w z
    using ‹P w y ∧ P w z›..
  from w have P w x..
  hence P w x ∧ P w y
    using ‹P w y›..
  hence ∃ z. P z x ∧ P z y..
  with overlap-eq have O x y..
  hence xy: ∀ w. P w (x ⊗ y) ⟷ (P w x ∧ P w y)
    by (rule product-character)
  hence P w (x ⊗ y) ⟷ (P w x ∧ P w y)..
  hence P w (x ⊗ y)
    using ‹P w x ∧ P w y›..
  hence P w (x ⊗ y) ∧ P w z
    using ‹P w z›..
  hence ∃ w. P w (x ⊗ y) ∧ P w z..
  with overlap-eq have O (x ⊗ y) z..
  hence xyz: ∀ w. P w ((x ⊗ y) ⊗ z) ⟷ P w (x ⊗ y) ∧ P w z
    by (rule product-character)
  show ∀ w. P w ((x ⊗ y) ⊗ z) ⟷ (P w x ∧ P w y) ∧ P w z
  proof
    fix v
    from xy have vxy: P v (x ⊗ y) ⟷ (P v x ∧ P v y)..
    moreover from xyz have
      P v ((x ⊗ y) ⊗ z) ⟷ P v (x ⊗ y) ∧ P v z..
    ultimately show P v ((x ⊗ y) ⊗ z) ⟷ (P v x ∧ P v y) ∧ P v z
      by (rule subst)
  qed
qed

theorem product-associativity:
  (∃ w. P w x ∧ P w y ∧ P w z) ⟹ x ⊗ (y ⊗ z) = (x ⊗ y) ⊗ z
proof −
  assume ante:(∃ w. P w x ∧ P w y ∧ P w z)
  hence (∀ w. P w (x ⊗ (y ⊗ z)) ⟷ P w x ∧ (P w y ∧ P w z))
    by (rule right-associated-product)
  moreover from ante have
    (∀ w. P w ((x ⊗ y) ⊗ z) ⟷ (P w x ∧ P w y) ∧ P w z)
    by (rule left-associated-product)
  ultimately have ∀ w. P w (x ⊗ (y ⊗ z)) ⟷ P w ((x ⊗ y) ⊗ z)
    by simp
  with part-extensionality show x ⊗ (y ⊗ z) = (x ⊗ y) ⊗ z..
qed

end
```

6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.[25]

locale *CMD* = *CM* +
  assumes *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
  assumes *difference-closure*:
    $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
begin

lemma *difference-intro*:
  $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow x \ominus y = z$
proof −
  assume *antecedent*: $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
  hence $(THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y) = z$
  proof (*rule the-equality*)
    fix v
    assume v: $(\forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
    have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
    proof
      fix w
      from *antecedent* **have** $P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$..
      moreover from v **have** $P\ w\ v \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$..
      ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
    qed
    with *part-extensionality* **show** $v = z$..
  qed
  with *difference-eq* **show** $x \ominus y = z$ **by** (*rule ssubst*)
qed

lemma *difference-idempotence*: $\neg\ O\ x\ y \Longrightarrow (x \ominus y) = x$
proof −
  assume $\neg\ O\ x\ y$
  hence $\neg\ O\ y\ x$ **by** (*rule disjoint-symmetry*)
  have $\forall w.\ P\ w\ x \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
  proof
    fix w
    show $P\ w\ x \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
    proof
      assume $P\ w\ x$
      hence $\neg\ O\ y\ w$ **using** ‹$\neg\ O\ y\ x$›
        by (*rule disjoint-demonotonicity*)
      hence $\neg\ O\ w\ y$ **by** (*rule disjoint-symmetry*)
      with ‹$P\ w\ x$› **show** $P\ w\ x \wedge \neg\ O\ w\ y$..
    next

[25] See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.

```
      assume P w x ∧ ¬ O w y
      thus P w x..
    qed
  qed
  thus (x ⊖ y) = x by (rule difference-intro)
qed

lemma difference-character: (∃ w. P w x ∧ ¬ O w y) ⟹
  (∀ w. P w (x ⊖ y) ⟷ P w x ∧ ¬ O w y)
proof –
  assume ∃ w. P w x ∧ ¬ O w y
  hence ∃ z. ∀ w. P w z ⟷ P w x ∧ ¬ O w y by (rule difference-closure)
  then obtain z where z: ∀ w. P w z ⟷ P w x ∧ ¬ O w y..
  hence (x ⊖ y) = z by (rule difference-intro)
  thus ∀ w. P w (x ⊖ y) ⟷ P w x ∧ ¬ O w y using z by (rule ssubst)
qed

lemma difference-disjointness:
  (∃ z. P z x ∧ ¬ O z y) ⟹ ¬ O y (x ⊖ y)
proof –
  assume ∃ z. P z x ∧ ¬ O z y
  hence xmy: ∀ w. P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)
    by (rule difference-character)
  show ¬ O y (x ⊖ y)
  proof
    assume O y (x ⊖ y)
    with overlap-eq have ∃ v. P v y ∧ P v (x ⊖ y)..
    then obtain v where v: P v y ∧ P v (x ⊖ y)..
    from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
    moreover from v have P v (x ⊖ y)..
    ultimately have P v x ∧ ¬ O v y..
    hence ¬ O v y..
    moreover from v have P v y..
    hence O v y by (rule part-implies-overlap)
    ultimately show False..
  qed
qed

end
```

6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.[26]

```
locale CMU = CM +
```

[26]See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

```
  assumes universe-eq: u = (THE z. ∀ w. P w z)
  assumes universe-closure: ∃ y. ∀ x. P x y
begin

lemma universe-intro: (∀ w. P w z) ⟹ u = z
proof −
  assume z: ∀ w. P w z
  hence (THE z. ∀ w. P w z) = z
  proof (rule the-equality)
    fix v
    assume v: ∀ w. P w v
    have ∀ w. P w v ⟷ P w z
    proof
      fix w
      show P w v ⟷ P w z
      proof
        assume P w v
        from z show P w z..
      next
        assume P w z
        from v show P w v..
      qed
    qed
    with part-extensionality show v = z..
  qed
  thus u = z using universe-eq by (rule subst)
qed

lemma universe-character: P x u
proof −
  from universe-closure obtain y where y: ∀ x. P x y..
  hence u = y by (rule universe-intro)
  hence ∀ x. P x u using y by (rule ssubst)
  thus P x u..
qed

lemma ¬ PP u x
proof
  assume PP u x
  hence ¬ P x u by (rule proper-implies-not-part)
  thus False using universe-character..
qed

lemma product-universe-implies-factor-universe:
  O x y ⟹ x ⊗ y = u ⟹ x = u
proof −
  assume x ⊗ y = u
  moreover assume O x y
  hence P (x ⊗ y) x
```

by (*rule product-in-first-factor*)
ultimately have $P\ u\ x$
by (*rule subst*)
with *universe-character* **show** $x = u$
by (*rule part-antisymmetry*)
qed

end

6.4 Complements

As is a condition ensuring the existence of complements.[27]

locale *CMC* = *CM* +
assumes *complement-eq*: $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *complement-closure*:
$(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
begin

lemma *complement-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) \Longrightarrow -x = z$
proof −
assume *antecedent*: $\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$
hence $(THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) = z$
proof (*rule the-equality*)
fix v
assume v: $\forall w.\ P\ w\ v \longleftrightarrow \neg\ O\ w\ x$
have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w
from *antecedent* **have** $P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow \neg\ O\ w\ x$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**
qed
with *complement-eq* **show** $-x = z$ **by** (*rule ssubst*)
qed

lemma *complement-character*:
$(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x)$
proof −
assume $\exists w.\ \neg\ O\ w\ x$
hence $(\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$ **by** (*rule complement-closure*)
then obtain z **where** z: $\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
hence $-x = z$ **by** (*rule complement-intro*)

[27] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

thus $\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
  using z **by** (*rule ssubst*)
qed

lemma *not-complement-part*: $\exists w.\ \neg\ O\ w\ x \Longrightarrow \neg\ P\ x\ (-x)$
proof −
  assume $\exists w.\ \neg\ O\ w\ x$
  hence $\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule complement-character*)
  hence $P\ x\ (-x) \longleftrightarrow \neg\ O\ x\ x$..
  show $\neg\ P\ x\ (-x)$
  proof
    assume $P\ x\ (-x)$
    with ‹$P\ x\ (-x) \longleftrightarrow \neg\ O\ x\ x$› **have** $\neg\ O\ x\ x$..
    thus *False* **using** *overlap-reflexivity*..
  qed
qed

lemma *complement-part*: $\neg\ O\ x\ y \Longrightarrow P\ x\ (-y)$
proof −
  assume $\neg\ O\ x\ y$
  hence $\exists z.\ \neg\ O\ z\ y$..
  hence $\forall w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
    by (*rule complement-character*)
  hence $P\ x\ (-y) \longleftrightarrow \neg\ O\ x\ y$..
  thus $P\ x\ (-y)$ **using** ‹$\neg\ O\ x\ y$›..
qed

lemma *complement-overlap*: $\neg\ O\ x\ y \Longrightarrow O\ x\ (-y)$
proof −
  assume $\neg\ O\ x\ y$
  hence $P\ x\ (-y)$
    by (*rule complement-part*)
  thus $O\ x\ (-y)$
    by (*rule part-implies-overlap*)
qed

lemma *or-complement-overlap*: $\forall y.\ O\ y\ x \lor O\ y\ (-x)$
proof
  fix y
  show $O\ y\ x \lor O\ y\ (-x)$
  proof *cases*
    assume $O\ y\ x$
    thus $O\ y\ x \lor O\ y\ (-x)$..
  next
    assume $\neg\ O\ y\ x$
    hence $O\ y\ (-x)$
      by (*rule complement-overlap*)
    thus $O\ y\ x \lor O\ y\ (-x)$..

qed
qed

lemma *complement-disjointness*: $\exists v.\ \neg\ O\ v\ x \Longrightarrow \neg\ O\ x\ (-x)$
proof −
assume $\exists v.\ \neg\ O\ v\ x$
hence *w*: $\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule complement-character*)
show $\neg\ O\ x\ (-x)$
proof
assume $O\ x\ (-x)$
with *overlap-eq* **have** $\exists v.\ P\ v\ x \wedge P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ x \wedge P\ v\ (-x)$..
from w **have** $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$..
moreover from v **have** $P\ v\ (-x)$..
ultimately have $\neg\ O\ v\ x$..
moreover from v **have** $P\ v\ x$..
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
ultimately show *False*..
qed
qed

lemma *part-disjoint-from-complement*:
$\exists v.\ \neg\ O\ v\ x \Longrightarrow P\ y\ x \Longrightarrow \neg\ O\ y\ (-x)$
proof
assume $\exists v.\ \neg\ O\ v\ x$
hence $\neg\ O\ x\ (-x)$ **by** (*rule complement-disjointness*)
assume $P\ y\ x$
assume $O\ y\ (-x)$
with *overlap-eq* **have** $\exists v.\ P\ v\ y \wedge P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ y \wedge P\ v\ (-x)$..
hence $P\ v\ y$..
hence $P\ v\ x$ **using** ‹$P\ y\ x$› **by** (*rule part-transitivity*)
moreover from v **have** $P\ v\ (-x)$..
ultimately have $P\ v\ x \wedge P\ v\ (-x)$..
hence $\exists v.\ P\ v\ x \wedge P\ v\ (-x)$..
with *overlap-eq* **have** $O\ x\ (-x)$..
with ‹$\neg\ O\ x\ (-x)$› **show** *False*..
qed

lemma *product-complement-character*: $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y)))$
proof −
assume *antecedent*: $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
then obtain w **where** w: $P\ w\ x \wedge \neg\ O\ w\ y$..
hence $P\ w\ x$..
moreover from w **have** $\neg\ O\ w\ y$..
hence $P\ w\ (-y)$ **by** (*rule complement-part*)
ultimately have $P\ w\ x \wedge P\ w\ (-y)$..

hence $\exists w.\ P\ w\ x \wedge P\ w\ (-y)$..
with *overlap-eq* **have** $O\ x\ (-y)$..
hence *prod*: $(\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ (-y)))$
by (*rule product-character*)
show $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y))$
proof
fix v
from w **have** $\neg\ O\ w\ y$..
hence $\exists w.\ \neg\ O\ w\ y$..
hence $\forall w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule complement-character*)
hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$..
moreover have $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge P\ v\ (-y))$
using *prod*..
ultimately show $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge (\neg\ O\ v\ y))$
by (*rule subst*)
qed
qed

theorem *difference-closure*: $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
proof –
assume $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
by (*rule product-complement-character*)
thus $(\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$ **by** (*rule exI*)
qed

end

sublocale $CMC \subseteq CMD$
proof
fix $x\ y$
show $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
show $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-closure*.
qed

corollary (**in** *CMC*) *difference-is-product-of-complement*:
$(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow (x \ominus y) = x \otimes (-y)$
proof –
assume *antecedent*: $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
by (*rule product-complement-character*)
thus $(x \ominus y) = x \otimes (-y)$ **by** (*rule difference-intro*)
qed

Universe and difference closure entail complement closure, since the difference of an individual and the universe is the individual's complement.

```
locale CMUD = CMU + CMD +
  assumes complement-eq: −x = (THE z. ∀w. P w z ⟷ ¬ O w x)
begin

lemma universe-difference:
  (∃w. ¬ O w x) ⟹ (∀w. P w (u ⊖ x) ⟷ ¬ O w x)
proof −
  assume ∃w. ¬ O w x
  then obtain w where w: ¬ O w x..
  from universe-character have P w u.
  hence P w u ∧ ¬ O w x using ‹¬ O w x›..
  hence ∃z. P z u ∧ ¬ O z x..
  hence ux: ∀w. P w (u ⊖ x) ⟷ (P w u ∧ ¬ O w x)
    by (rule difference-character)
  show ∀w. P w (u ⊖ x) ⟷ ¬ O w x
  proof
    fix w
    from ux have wux: P w (u ⊖ x) ⟷ (P w u ∧ ¬ O w x)..
    show P w (u ⊖ x) ⟷ ¬ O w x
    proof
      assume P w (u ⊖ x)
      with wux have P w u ∧ ¬ O w x..
      thus ¬ O w x..
    next
      assume ¬ O w x
      from universe-character have P w u.
      hence P w u ∧ ¬ O w x using ‹¬ O w x›..
      with wux show P w (u ⊖ x)..
    qed
  qed
qed

theorem complement-closure:
  (∃w. ¬ O w x) ⟹ (∃z. ∀w. P w z ⟷ ¬ O w x)
proof −
  assume ∃w. ¬ O w x
  hence ∀w. P w (u ⊖ x) ⟷ ¬ O w x
    by (rule universe-difference)
  thus ∃z. ∀w. P w z ⟷ ¬ O w x..
qed

end

sublocale CMUD ⊆ CMC
proof
  fix x y
  show −x = (THE z. ∀w. P w z ⟷ (¬ O w x))
```

using *complement-eq*.
show $\exists\, w.\ \neg\ O\ w\ x \Longrightarrow \exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow (\neg\ O\ w\ x)$
using *complement-closure*.
show $x \ominus y = (\mathit{THE}\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
qed

corollary (**in** *CMUD*) *complement-universe-difference*:
$(\exists\, y.\ \neg\ O\ y\ x) \Longrightarrow -x = (u \ominus x)$
proof −
assume $\exists\, w.\ \neg\ O\ w\ x$
hence $\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule universe-difference*)
thus $-x = (u \ominus x)$
by (*rule complement-intro*)
qed

7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.[28]

locale *CEM* = *CM* + *EM*

Likewise, closed minimal mereology combines closed mereology with minimal mereology.[29]

locale *CMM* = *CM* + *MM*

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.[30]

sublocale *CMM* ⊆ *CEM*
proof
fix $x\ y$
show *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
proof −
assume $\neg\ P\ x\ y$
show $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$
proof *cases*
assume $O\ x\ y$
with ‹$\neg\ P\ x\ y$› **have** $\neg\ P\ x\ y \wedge O\ x\ y$..
hence $\mathit{PP}\ (x \otimes y)\ x$ **by** (*rule nonpart-implies-proper-product*)

[28]See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.
[29]See [Casati and Varzi, 1999] p. 43.
[30]See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

```
     hence ∃ z. P z x ∧ ¬ O z (x ⊗ y) by (rule weak-supplementation)
      then obtain z where z: P z x ∧ ¬ O z (x ⊗ y)..
      hence ¬ O z y by (rule disjoint-from-second-factor)
      moreover from z have P z x..
      hence  P z x ∧ ¬ O z y
        using ‹¬ O z y›..
      thus ∃ z. P z x ∧ ¬ O z y..
    next
      assume ¬ O x y
      with part-reflexivity have P x x ∧ ¬ O x y..
      thus (∃ z. P z x ∧ ¬ O z y)..
    qed
  qed
qed

sublocale CEM ⊆ CMM..
```

7.1 Sums

```
context CEM
begin

lemma sum-intro:
   (∀ w. O w z ⟷ (O w x ∨ O w y)) ⟹ x ⊕ y = z
proof −
  assume sum: ∀ w. O w z ⟷ (O w x ∨ O w y)
  hence (THE v. ∀ w. O w v ⟷ (O w x ∨ O w y)) = z
  proof (rule the-equality)
    fix a
    assume a: ∀ w. O w a ⟷ (O w x ∨ O w y)
    have ∀ w. O w a ⟷ O w z
    proof
      fix w
      from sum have O w z ⟷ (O w x ∨ O w y)..
      moreover from a have O w a ⟷ (O w x ∨ O w y)..
      ultimately show O w a ⟷ O w z by (rule ssubst)
      qed
      with overlap-extensionality show a = z..
  qed
  thus x ⊕ y = z
    using sum-eq by (rule subst)
qed

lemma sum-idempotence: x ⊕ x = x
proof −
  have ∀ w. O w x ⟷ (O w x ∨ O w x)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w x)
```

```
    proof (rule iffI)
      assume O w x
      thus O w x ∨ O w x..
    next
      assume O w x ∨ O w x
      thus O w x by (rule disjE)
    qed
  qed
  thus x ⊕ x = x by (rule sum-intro)
qed

lemma part-sum-identity: P y x ⟹ x ⊕ y = x
proof −
  assume P y x
  have ∀ w. O w x ⟷ (O w x ∨ O w y)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w y)
    proof
      assume O w x
      thus O w x ∨ O w y..
    next
      assume O w x ∨ O w y
      thus O w x
      proof
        assume O w x
        thus O w x.
      next
        assume O w y
        with ‹P y x› show O w x
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  thus x ⊕ y = x by (rule sum-intro)
qed

lemma sum-character: ∀ w. O w (x ⊕ y) ⟷ (O w x ∨ O w y)
proof −
  from sum-closure have (∃ z. ∀ w. O w z ⟷ (O w x ∨ O w y)).
  then obtain a where a: ∀ w. O w a ⟷ (O w x ∨ O w y)..
  hence x ⊕ y = a by (rule sum-intro)
  thus ∀ w. O w (x ⊕ y) ⟷ (O w x ∨ O w y)
    using a by (rule ssubst)
qed

lemma sum-overlap: O w (x ⊕ y) ⟷ (O w x ∨ O w y)
  using sum-character..
```

lemma *sum-part-character*:
  $P\ w\ (x \oplus y) \longleftrightarrow (\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y)$
proof
  assume $P\ w\ (x \oplus y)$
  show $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
  proof
    fix v
    show $O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
    proof
      assume $O\ v\ w$
      with ‹$P\ w\ (x \oplus y)$› **have** $O\ v\ (x \oplus y)$
        by (*rule overlap-monotonicity*)
      with *sum-overlap* **show** $O\ v\ x \vee O\ v\ y$**..**
    qed
  qed
next
  assume *right*: $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$
  have $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
  proof
    fix v
    from *right* **have** $O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y$**..**
    with *sum-overlap* **show** $O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
      by (*rule ssubst*)
  qed
  with *part-overlap-eq* **show** $P\ w\ (x \oplus y)$**..**
qed

lemma *sum-commutativity*: $x \oplus y = y \oplus x$
proof −
  from *sum-character* **have** $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ y \vee O\ w\ x$**.**
  hence $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ x \vee O\ w\ y$ **by** *metis*
  thus $x \oplus y = y \oplus x$ **by** (*rule sum-intro*)
qed

lemma *first-summand-overlap*: $O\ z\ x \Longrightarrow O\ z\ (x \oplus y)$
proof −
  assume $O\ z\ x$
  hence $O\ z\ x \vee O\ z\ y$**..**
  with *sum-overlap* **show** $O\ z\ (x \oplus y)$**..**
qed

lemma *first-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ x$
proof −
  assume $\neg\ O\ z\ (x \oplus y)$
  show $\neg\ O\ z\ x$
  proof
    assume $O\ z\ x$
    hence $O\ z\ (x \oplus y)$ **by** (*rule first-summand-overlap*)
    with ‹$\neg\ O\ z\ (x \oplus y)$› **show** *False***..**

qed
qed

lemma *first-summand-in-sum*: $P\ x\ (x \oplus y)$
proof $-$
have $\forall\ w.\ O\ w\ x \longrightarrow O\ w\ (x \oplus y)$
proof
fix w
show $O\ w\ x \longrightarrow O\ w\ (x \oplus y)$
proof
assume $O\ w\ x$
thus $O\ w\ (x \oplus y)$
by (*rule first-summand-overlap*)
qed
qed
with *part-overlap-eq* **show** $P\ x\ (x \oplus y)$..
qed

lemma *common-first-summand*: $P\ x\ (x \oplus y) \wedge P\ x\ (x \oplus z)$
proof
from *first-summand-in-sum* **show** $P\ x\ (x \oplus y)$.
from *first-summand-in-sum* **show** $P\ x\ (x \oplus z)$.
qed

lemma *common-first-summand-overlap*: $O\ (x \oplus y)\ (x \oplus z)$
proof $-$
from *first-summand-in-sum* **have** $P\ x\ (x \oplus y)$.
moreover from *first-summand-in-sum* **have** $P\ x\ (x \oplus z)$.
ultimately have $P\ x\ (x \oplus y) \wedge P\ x\ (x \oplus z)$..
hence $\exists\ v.\ P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$..
with *overlap-eq* **show** *?thesis*..
qed

lemma *second-summand-overlap*: $O\ z\ y \Longrightarrow O\ z\ (x \oplus y)$
proof $-$
assume $O\ z\ y$
from *sum-character* **have** $O\ z\ (x \oplus y) \longleftrightarrow (O\ z\ x \vee O\ z\ y)$..
moreover from ‹$O\ z\ y$› **have** $O\ z\ x \vee O\ z\ y$..
ultimately show $O\ z\ (x \oplus y)$..
qed

lemma *second-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ y$
proof $-$
assume $\neg\ O\ z\ (x \oplus y)$
show $\neg\ O\ z\ y$
proof
assume $O\ z\ y$
hence $O\ z\ (x \oplus y)$
by (*rule second-summand-overlap*)

```
    with ‹¬ O z (x ⊕ y)› show False..
  qed
qed

lemma second-summand-in-sum: P y (x ⊕ y)
proof −
  have ∀ w. O w y ⟶ O w (x ⊕ y)
  proof
    fix w
    show O w y ⟶ O w (x ⊕ y)
    proof
      assume O w y
      thus O w (x ⊕ y)
        by (rule second-summand-overlap)
    qed
  qed
  with part-overlap-eq show P y (x ⊕ y)..
qed

lemma second-summands-in-sums: P y (x ⊕ y) ∧ P v (z ⊕ v)
proof
  show P y (x ⊕ y) using second-summand-in-sum.
  show P v (z ⊕ v) using second-summand-in-sum.
qed

lemma disjoint-from-sum: ¬ O z (x ⊕ y) ⟷ ¬ O z x ∧ ¬ O z y
proof −
  from sum-character have O z (x ⊕ y) ⟷ (O z x ∨ O z y)..
  thus ?thesis by simp
qed

lemma summands-part-implies-sum-part:
  P x z ∧ P y z ⟹ P (x ⊕ y) z
proof −
  assume antecedent: P x z ∧ P y z
  have ∀ w. O w (x ⊕ y) ⟶ O w z
  proof
    fix w
    have w: O w (x ⊕ y) ⟷ (O w x ∨ O w y)
      using sum-character..
    show O w (x ⊕ y) ⟶ O w z
    proof
      assume O w (x ⊕ y)
      with w have O w x ∨ O w y..
      thus O w z
      proof
        from antecedent have P x z..
        moreover assume O w x
        ultimately show O w z
```

```
        by (rule overlap-monotonicity)
      next
        from antecedent have P y z..
        moreover assume O w y
        ultimately show O w z
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  with part-overlap-eq show P (x ⊕ y) z..
qed

lemma sum-part-implies-summands-part:
  P (x ⊕ y) z ⟹ P x z ∧ P y z
proof −
  assume antecedent: P (x ⊕ y) z
  show P x z ∧ P y z
  proof
    from first-summand-in-sum show P x z
      using antecedent by (rule part-transitivity)
  next
    from second-summand-in-sum show P y z
      using antecedent by (rule part-transitivity)
  qed
qed

lemma in-second-summand: P z (x ⊕ y) ∧ ¬ O z x ⟹ P z y
proof −
  assume antecedent: P z (x ⊕ y) ∧ ¬ O z x
  hence P z (x ⊕ y)..
  show P z y
  proof (rule ccontr)
    assume ¬ P z y
    hence ∃ v. P v z ∧ ¬ O v y
      by (rule strong-supplementation)
    then obtain v where v: P v z ∧ ¬ O v y..
    hence ¬ O v y..
    from v have P v z..
    hence P v (x ⊕ y)
      using ‹P z (x ⊕ y)› by (rule part-transitivity)
    hence O v (x ⊕ y) by (rule part-implies-overlap)
    from sum-character have O v (x ⊕ y) ⟷ O v x ∨ O v y..
    hence O v x ∨ O v y using ‹O v (x ⊕ y)›..
    thus False
    proof (rule disjE)
      from antecedent have ¬ O z x..
      moreover assume O v x
      hence O x v by (rule overlap-symmetry)
      with ‹P v z› have O x z
```

by (*rule overlap-monotonicity*)
hence $O\ z\ x$ **by** (*rule overlap-symmetry*)
ultimately show *False*..
next
assume $O\ v\ y$
with ‹$\neg\ O\ v\ y$› **show** *False*..
qed
qed
qed

lemma *disjoint-second-summands*:
$P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z) \Longrightarrow \neg\ O\ y\ z \Longrightarrow P\ v\ x$
proof −
assume *antecedent*: $P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$
hence $P\ v\ (x \oplus z)$..
assume $\neg\ O\ y\ z$
show $P\ v\ x$
proof (*rule ccontr*)
assume $\neg\ P\ v\ x$
hence $\exists\ w.\ P\ w\ v \wedge \neg\ O\ w\ x$ **by** (*rule strong-supplementation*)
then obtain w **where** w: $P\ w\ v \wedge \neg\ O\ w\ x$..
hence $\neg\ O\ w\ x$..
from w **have** $P\ w\ v$..
moreover from *antecedent* **have** $P\ v\ (x \oplus z)$..
ultimately have $P\ w\ (x \oplus z)$ **by** (*rule part-transitivity*)
hence $P\ w\ (x \oplus z) \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›..
hence $P\ w\ z$ **by** (*rule in-second-summand*)
from *antecedent* **have** $P\ v\ (x \oplus y)$..
with ‹$P\ w\ v$› **have** $P\ w\ (x \oplus y)$ **by** (*rule part-transitivity*)
hence $P\ w\ (x \oplus y) \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›..
hence $P\ w\ y$ **by** (*rule in-second-summand*)
hence $P\ w\ y \wedge P\ w\ z$ **using** ‹$P\ w\ z$›..
hence $\exists\ w.\ P\ w\ y \wedge P\ w\ z$..
with *overlap-eq* **have** $O\ y\ z$..
with ‹$\neg\ O\ y\ z$› **show** *False*..
qed
qed

lemma *right-associated-sum*:
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ x \vee (O\ w\ y \vee O\ w\ z)$
proof −
from *sum-character* **have** $O\ w\ (y \oplus z) \longleftrightarrow O\ w\ y \vee O\ w\ z$..
moreover from *sum-character* **have**
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow (O\ w\ x \vee O\ w\ (y \oplus z))$..
ultimately show *?thesis*
by (*rule subst*)
qed

lemma *left-associated-sum*:

$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow (O\ w\ x \vee O\ w\ y) \vee O\ w\ z$
proof −
 from *sum-character* **have** $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \vee O\ w\ y)$**..**
 moreover from *sum-character* **have**
  $O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow O\ w\ (x \oplus y) \vee O\ w\ z$**..**
 ultimately show *?thesis*
  by (*rule subst*)
qed

theorem *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
proof −
 have $\forall\ w.\ O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$
 proof
  fix w
  have $O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow (O\ w\ x \vee O\ w\ y) \vee O\ w\ z$
   using *right-associated-sum* **by** *simp*
  with *left-associated-sum* **show**
   $O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$ **by** (*rule ssubst*)
 qed
 with *overlap-extensionality* **show** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$**..**
qed

7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

lemma *common-summand-in-product*: $P\ x\ ((x \oplus y) \otimes (x \oplus z))$
  using *common-first-summand* **by** (*rule common-part-in-product*)

lemma *product-in-first-summand*:
 $\neg\ O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ x$
proof −
 assume $\neg\ O\ y\ z$
 have $\forall\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
 proof
  fix v
  show $P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
  proof
   assume $P\ v\ ((x \oplus y) \otimes (x \oplus z))$
   with *common-first-summand-overlap* **have**
    $P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)
   thus $P\ v\ x$ **using** ‹$\neg\ O\ y\ z$› **by** (*rule disjoint-second-summands*)
  qed
 qed
 hence $P\ ((x \oplus y) \otimes (x \oplus z))\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow$
  $P\ ((x \oplus y) \otimes (x \oplus z))\ x$**..**
 thus $P\ ((x \oplus y) \otimes (x \oplus z))\ x$ **using** *part-reflexivity***..**
qed

lemma *product-is-first-summand*:
  $\neg$ $O\ y\ z \Longrightarrow (x \oplus y) \otimes (x \oplus z) = x$
proof –
  assume $\neg$ $O\ y\ z$
  hence $P\ ((x \oplus y) \otimes (x \oplus z))\ x$
    by (*rule product-in-first-summand*)
  thus $(x \oplus y) \otimes (x \oplus z) = x$
    using *common-summand-in-product*
    by (*rule part-antisymmetry*)
qed

lemma *sum-over-product-left*: $O\ y\ z \Longrightarrow P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
proof –
  assume $O\ y\ z$
  hence $P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$ **using** *second-summands-in-sums*
    by (*rule part-product-in-whole-product*)
  with *common-summand-in-product* **have**
    $P\ x\ ((x \oplus y) \otimes (x \oplus z)) \wedge P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$..
  thus $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
    by (*rule summands-part-implies-sum-part*)
qed

lemma *sum-over-product-right*:
  $O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
proof –
  assume $O\ y\ z$
  show $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
  proof (*rule ccontr*)
    assume $\neg$ $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
    hence $\exists\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$
      by (*rule strong-supplementation*)
    then obtain v **where** v:
      $P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$..
    hence $\neg\ O\ v\ (x \oplus (y \otimes z))$..
    with *disjoint-from-sum* **have** *vd*: $\neg\ O\ v\ x \wedge \neg\ O\ v\ (y \otimes z)$..
    hence $\neg\ O\ v\ (y \otimes z)$..
    from *vd* **have** $\neg\ O\ v\ x$..
    from v **have** $P\ v\ ((x \oplus y) \otimes (x \oplus z))$..
    with *common-first-summand-overlap* **have**
      vs: $P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)
    hence $P\ v\ (x \oplus y)$..
    hence $P\ v\ (x \oplus y) \wedge \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›..
    hence $P\ v\ y$ **by** (*rule in-second-summand*)
    moreover from *vs* **have** $P\ v\ (x \oplus z)$..
    hence $P\ v\ (x \oplus z) \wedge \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›..
    hence $P\ v\ z$ **by** (*rule in-second-summand*)
    ultimately have $P\ v\ y \wedge P\ v\ z$..
    hence $P\ v\ (y \otimes z)$ **by** (*rule common-part-in-product*)

hence $O\ v\ (y \otimes z)$ **by** (*rule part-implies-overlap*)
with ‹$\neg\ O\ v\ (y \otimes z)$› **show** *False*..
qed
qed

Sums distribute over products.

theorem *sum-over-product*:
$O\ y\ z \Longrightarrow x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
proof −
assume $O\ y\ z$
hence $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
by (*rule sum-over-product-left*)
moreover have $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
using ‹$O\ y\ z$› **by** (*rule sum-over-product-right*)
ultimately show $x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
by (*rule part-antisymmetry*)
qed

lemma *product-in-factor-by-sum*:
$O\ x\ y \Longrightarrow P\ (x \otimes y)\ (x \otimes (y \oplus z))$
proof −
assume $O\ x\ y$
hence $P\ (x \otimes y)\ x$
by (*rule product-in-first-factor*)
moreover have $P\ (x \otimes y)\ y$
using ‹$O\ x\ y$› **by** (*rule product-in-second-factor*)
hence $P\ (x \otimes y)\ (y \oplus z)$
using *first-summand-in-sum* **by** (*rule part-transitivity*)
with ‹$P\ (x \otimes y)\ x$› **have** $P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ (y \oplus z)$..
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule common-part-in-product*)
qed

lemma *product-of-first-summand*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))\ (x \otimes y)$
proof −
assume $O\ x\ y$
hence $O\ x\ (y \oplus z)$
by (*rule first-summand-overlap*)
assume $\neg\ O\ x\ z$
show $P\ (x \otimes (y \oplus z))\ (x \otimes y)$
proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))\ (x \otimes y)$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$..
hence $P\ v\ (x \otimes (y \oplus z))$..
with ‹$O\ x\ (y \oplus z)$› **have** $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)

hence $P\ v\ x$**..**
moreover from v **have** $\neg\ O\ v\ (x \otimes y)$**..**
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$**..**
hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from ‹$P\ v\ x \wedge P\ v\ (y \oplus z)$› **have** $P\ v\ (y \oplus z)$**..**
hence $P\ v\ (y \oplus z) \wedge \neg\ O\ v\ y$ **using** ‹$\neg\ O\ v\ y$›**..**
hence $P\ v\ z$ **by** (*rule in-second-summand*)
with ‹$P\ v\ x$› **have** $P\ v\ x \wedge P\ v\ z$**..**
hence $\exists\ v.\ P\ v\ x \wedge P\ v\ z$**..**
with *overlap-eq* **have** $O\ x\ z$**..**
with ‹$\neg\ O\ x\ z$› **show** *False***..**
qed
qed

theorem *disjoint-product-over-sum*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow x \otimes (y \oplus z) = x \otimes y$
proof −
assume $O\ x\ y$
moreover assume $\neg\ O\ x\ z$
ultimately have $P\ (x \otimes (y \oplus z))\ (x \otimes y)$
by (*rule product-of-first-summand*)
moreover have $P\ (x \otimes y)(x \otimes (y \oplus z))$
using ‹$O\ x\ y$› **by** (*rule product-in-factor-by-sum*)
ultimately show $x \otimes (y \oplus z) = x \otimes y$
by (*rule part-antisymmetry*)
qed

lemma *product-over-sum-left*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof −
assume $O\ x\ y \wedge O\ x\ z$
hence $O\ x\ y$**..**
hence $O\ x\ (y \oplus z)$ **by** (*rule first-summand-overlap*)
show $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v:
$P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$**..**
hence $\neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$**..**
with *disjoint-from-sum* **have** *oxyz*:
$\neg\ O\ v\ (x \otimes y) \wedge \neg\ O\ v\ (x \otimes z)$**..**
from v **have** $P\ v\ (x \otimes (y \oplus z))$**..**
with ‹$O\ x\ (y \oplus z)$› **have** *pxyz*: $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)
hence $P\ v\ x$**..**
moreover from *oxyz* **have** $\neg\ O\ v\ (x \otimes y)$**..**
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$**..**

hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from *oxyz* **have** $\neg\ O\ v\ (x \otimes z)$**..**
with $\langle P\ v\ x \rangle$ **have** $P\ v\ x \wedge \neg\ O\ v\ (x \otimes z)$**..**
hence $\neg\ O\ v\ z$ **by** (*rule disjoint-from-second-factor*)
with $\langle \neg\ O\ v\ y \rangle$ **have** $\neg\ O\ v\ y \wedge \neg\ O\ v\ z$**..**
with *disjoint-from-sum* **have** $\neg\ O\ v\ (y \oplus z)$**..**
from *pxyz* **have** $P\ v\ (y \oplus z)$**..**
hence $O\ v\ (y \oplus z)$ **by** (*rule part-implies-overlap*)
with $\langle \neg\ O\ v\ (y \oplus z) \rangle$ **show** *False***..**
qed
qed

lemma *product-over-sum-right*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
proof −
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
have $P\ (x \otimes y)\ (x \otimes (y \oplus z)) \wedge P\ (x \otimes z)\ (x \otimes (y \oplus z))$
proof
from *antecedent* **have** $O\ x\ y$**..**
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule product-in-factor-by-sum*)
next
from *antecedent* **have** $O\ x\ z$**..**
hence $P\ (x \otimes z)\ (x \otimes (z \oplus y))$
by (*rule product-in-factor-by-sum*)
with *sum-commutativity* **show** $P\ (x \otimes z)\ (x \otimes (y \oplus z))$
by (*rule subst*)
qed
thus $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
by (*rule summands-part-implies-sum-part*)
qed

theorem *product-over-sum*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
proof −
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
hence $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
by (*rule product-over-sum-left*)
moreover have $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
using *antecedent* **by** (*rule product-over-sum-right*)
ultimately show $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
by (*rule part-antisymmetry*)
qed

lemma *joint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow O\ x\ v \wedge O\ x\ w \Longrightarrow ((x \otimes v) \oplus (x \otimes w)) = x$
proof −
assume $v \oplus w = x \oplus y$
moreover assume $O\ x\ v \wedge O\ x\ w$

hence $x \otimes (v \oplus w) = x \otimes v \oplus x \otimes w$
by (*rule product-over-sum*)
ultimately have $x \otimes (x \oplus y) = x \otimes v \oplus x \otimes w$ **by** (*rule subst*)
moreover have $(x \otimes (x \oplus y)) = x$ **using** *first-summand-in-sum*
by (*rule part-product-identity*)
ultimately show $((x \otimes v) \oplus (x \otimes w)) = x$ **by** (*rule subst*)
qed

lemma *disjoint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow \neg\ O\ y\ v \wedge \neg\ O\ w\ x \Longrightarrow x = v \wedge y = w$
proof −
assume *identical*: $v \oplus w = x \oplus y$
assume *disjoint*: $\neg\ O\ y\ v \wedge \neg\ O\ w\ x$
show $x = v \wedge y = w$
proof
from *disjoint* **have** $\neg\ O\ y\ v$..
hence $(x \oplus y) \otimes (x \oplus v) = x$
by (*rule product-is-first-summand*)
with *identical* **have** $(v \oplus w) \otimes (x \oplus v) = x$
by (*rule ssubst*)
moreover from *disjoint* **have** $\neg\ O\ w\ x$..
hence $(v \oplus w) \otimes (v \oplus x) = v$
by (*rule product-is-first-summand*)
with *sum-commutativity* **have** $(v \oplus w) \otimes (x \oplus v) = v$
by (*rule subst*)
ultimately show $x = v$ **by** (*rule subst*)
next
from *disjoint* **have** $\neg\ O\ w\ x$..
hence $(y \oplus w) \otimes (y \oplus x) = y$
by (*rule product-is-first-summand*)
moreover from *disjoint* **have** $\neg\ O\ y\ v$..
hence $(w \oplus y) \otimes (w \oplus v) = w$
by (*rule product-is-first-summand*)
with *sum-commutativity* **have** $(w \oplus y) \otimes (v \oplus w) = w$
by (*rule subst*)
with *identical* **have** $(w \oplus y) \otimes (x \oplus y) = w$
by (*rule subst*)
with *sum-commutativity* **have** $(w \oplus y) \otimes (y \oplus x) = w$
by (*rule subst*)
with *sum-commutativity* **have** $(y \oplus w) \otimes (y \oplus x) = w$
by (*rule subst*)
ultimately show $y = w$
by (*rule subst*)
qed
qed

end

7.3 Differences

```
locale CEMD = CEM + CMD
begin

lemma plus-minus: PP y x ⟹ y ⊕ (x ⊖ y) = x
proof –
  assume PP y x
  hence ∃ z. P z x ∧ ¬ O z y by (rule weak-supplementation)
  hence xmy:∀ w. P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)
    by (rule difference-character)
  have ∀ w. O w x ⟷ (O w y ∨ O w (x ⊖ y))
  proof
    fix w
    from xmy have w: P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)..
    show O w x ⟷ (O w y ∨ O w (x ⊖ y))
    proof
      assume O w x
      with overlap-eq have ∃ v. P v w ∧ P v x..
      then obtain v where v: P v w ∧ P v x..
      hence P v w..
      from v have P v x..
      show O w y ∨ O w (x ⊖ y)
      proof cases
        assume O v y
        hence O y v by (rule overlap-symmetry)
        with ‹P v w› have O y w by (rule overlap-monotonicity)
        hence O w y by (rule overlap-symmetry)
        thus O w y ∨ O w (x ⊖ y)..
      next
        from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
        moreover assume ¬ O v y
        with ‹P v x› have  P v x ∧ ¬ O v y..
        ultimately have P v (x ⊖ y)..
        with ‹P v w› have P v w ∧ P v (x ⊖ y)..
        hence ∃ v. P v w ∧ P v (x ⊖ y)..
        with overlap-eq have O w (x ⊖ y)..
        thus O w y ∨ O w (x ⊖ y)..
      qed
    next
      assume O w y ∨ O w (x ⊖ y)
      thus O w x
      proof
        from ‹PP y x› have P y x
          by (rule proper-implies-part)
        moreover assume O w y
        ultimately show O w x
          by (rule overlap-monotonicity)
      next
        assume O w (x ⊖ y)
```

```
      with overlap-eq have ∃ v. P v w ∧ P v (x ⊖ y)..
      then obtain v where v: P v w ∧ P v (x ⊖ y)..
      hence P v w..
      from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
      moreover from v have P v (x ⊖ y)..
      ultimately have P v x ∧ ¬ O v y..
      hence P v x..
      with ‹P v w› have P v w ∧ P v x..
      hence ∃ v. P v w ∧ P v x..
      with overlap-eq show O w x..
    qed
   qed
  qed
  thus y ⊕ (x ⊖ y) = x
   by (rule sum-intro)
qed

end
```

7.4 The Universe

```
locale CEMU = CEM + CMU
begin

lemma something-disjoint: x ≠ u ⟹ (∃ v. ¬ O v x)
proof −
  assume x ≠ u
  with universe-character have P x u ∧ x ≠ u..
  with nip-eq have PP x u..
  hence ∃ v. P v u ∧ ¬ O v x
   by (rule weak-supplementation)
  then obtain v where P v u ∧ ¬ O v x..
  hence ¬ O v x..
  thus ∃ v. ¬ O v x..
qed

lemma overlaps-universe: O x u
proof −
  from universe-character have P x u.
  thus O x u by (rule part-implies-overlap)
qed

lemma universe-absorbing: x ⊕ u = u
proof −
  from universe-character have P (x ⊕ u) u.
  thus x ⊕ u = u using second-summand-in-sum
   by (rule part-antisymmetry)
qed
```

lemma *second-summand-not-universe*: $x \oplus y \neq u \Longrightarrow y \neq u$
proof −
  assume *antecedent*: $x \oplus y \neq u$
  show $y \neq u$
  proof
    assume $y = u$
    hence $x \oplus u \neq u$ **using** *antecedent* **by** (*rule subst*)
    thus *False* **using** *universe-absorbing*..
  qed
qed

lemma *first-summand-not-universe*: $x \oplus y \neq u \Longrightarrow x \neq u$
proof −
  assume $x \oplus y \neq u$
  with *sum-commutativity* **have** $y \oplus x \neq u$ **by** (*rule subst*)
  thus $x \neq u$ **by** (*rule second-summand-not-universe*)
qed

end

7.5 Complements

locale *CEMC* = *CEM* + *CMC* +
  assumes *universe-eq*: $u = (THE\ x.\ \forall\ y.\ P\ y\ x)$
begin

lemma *complement-sum-character*: $\forall\ y.\ P\ y\ (x \oplus (-x))$
proof
  fix y
  have $\forall\ v.\ O\ v\ y \longrightarrow O\ v\ x \vee O\ v\ (-x)$
  proof
    fix v
    show $O\ v\ y \longrightarrow O\ v\ x \vee O\ v\ (-x)$
    proof
      assume $O\ v\ y$
      show $O\ v\ x \vee O\ v\ (-x)$
        using *or-complement-overlap*..
    qed
  qed
  with *sum-part-character* **show** $P\ y\ (x \oplus (-x))$..
qed

lemma *universe-closure*: $\exists\ x.\ \forall\ y.\ P\ y\ x$
  using *complement-sum-character* **by** (*rule exI*)

end

sublocale *CEMC* $\subseteq$ *CEMU*
proof

show $u = (THE\ z.\ \forall w.\ P\ w\ z)$ **using** *universe-eq*.
show $\exists\ x.\ \forall\ y.\ P\ y\ x$ **using** *universe-closure*.
qed

sublocale *CEMC* $\subseteq$ *CEMD*
proof
qed

context *CEMC*
begin

corollary *universe-is-complement-sum*: $u = x \oplus (-x)$
using *complement-sum-character* **by** (*rule universe-intro*)

lemma *strong-complement-character*:
$x \neq u \Longrightarrow (\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x)$
proof −
assume $x \neq u$
hence $\exists\ v.\ \neg\ O\ v\ x$ **by** (*rule something-disjoint*)
thus $\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$ **by** (*rule complement-character*)
qed

lemma *complement-part-not-part*: $x \neq u \Longrightarrow P\ y\ (-x) \Longrightarrow \neg\ P\ y\ x$
proof −
assume $x \neq u$
hence $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule strong-complement-character*)
hence y: $P\ y\ (-x) \longleftrightarrow \neg\ O\ y\ x$..
moreover assume $P\ y\ (-x)$
ultimately have $\neg\ O\ y\ x$..
thus $\neg\ P\ y\ x$
by (*rule disjoint-implies-not-part*)
qed

lemma *complement-involution*: $x \neq u \Longrightarrow x = -(-x)$
proof −
assume $x \neq u$
have $\neg\ P\ u\ x$
proof
assume $P\ u\ x$
with *universe-character* **have** $x = u$
by (*rule part-antisymmetry*)
with ‹$x \neq u$› **show** *False*..
qed
hence $\exists\ v.\ P\ v\ u \land \neg\ O\ v\ x$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ u \land \neg\ O\ v\ x$..
hence $\neg\ O\ v\ x$..
hence $\exists\ v.\ \neg\ O\ v\ x$..

hence *notx*: $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
  by (*rule complement-character*)
have $-x \neq u$
proof
  assume $-x = u$
  hence $\forall\ w.\ P\ w\ u \longleftrightarrow \neg\ O\ w\ x$ **using** *notx* **by** (*rule subst*)
  hence $P\ x\ u \longleftrightarrow \neg\ O\ x\ x$**..**
  hence $\neg\ O\ x\ x$ **using** *universe-character***..**
  thus *False* **using** *overlap-reflexivity***..**
qed
have $\neg\ P\ u\ (-x)$
proof
  assume $P\ u\ (-x)$
  with *universe-character* **have** $-x = u$
    by (*rule part-antisymmetry*)
  with ‹$-x \neq u$› **show** *False***..**
qed
hence $\exists\ v.\ P\ v\ u \wedge \neg\ O\ v\ (-x)$
  by (*rule strong-supplementation*)
then obtain w **where** w: $P\ w\ u \wedge \neg\ O\ w\ (-x)$**..**
hence $\neg\ O\ w\ (-x)$**..**
hence $\exists\ v.\ \neg\ O\ v\ (-x)$**..**
hence *notnotx*: $\forall\ w.\ P\ w\ (-(-x)) \longleftrightarrow \neg\ O\ w\ (-x)$
  by (*rule complement-character*)
hence $P\ x\ (-(-x)) \longleftrightarrow \neg\ O\ x\ (-x)$**..**
moreover have $\neg\ O\ x\ (-x)$
proof
  assume $O\ x\ (-x)$
  with *overlap-eq* **have** $\exists\ s.\ P\ s\ x \wedge P\ s\ (-x)$**..**
  then obtain s **where** s: $P\ s\ x \wedge P\ s\ (-x)$**..**
  hence $P\ s\ x$**..**
  hence $O\ s\ x$ **by** (*rule part-implies-overlap*)
  from *notx* **have** $P\ s\ (-x) \longleftrightarrow \neg\ O\ s\ x$**..**
  moreover from s **have** $P\ s\ (-x)$**..**
  ultimately have $\neg\ O\ s\ x$**..**
  thus *False* **using** ‹$O\ s\ x$›**..**
qed
ultimately have $P\ x\ (-(-x))$**..**
moreover have $P\ (-(-x))\ x$
proof (*rule ccontr*)
  assume $\neg\ P\ (-(-x))\ x$
  hence $\exists\ s.\ P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$
    by (*rule strong-supplementation*)
  then obtain s **where** s: $P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$**..**
  hence $\neg\ O\ s\ x$**..**
  from *notnotx* **have** $P\ s\ (-(-x)) \longleftrightarrow (\neg\ O\ s\ (-x))$**..**
  moreover from s **have** $P\ s\ (-(-x))$**..**
  ultimately have $\neg\ O\ s\ (-x)$**..**
  from *or-complement-overlap* **have** $O\ s\ x \vee O\ s\ (-x)$**..**

```
    thus False
    proof
      assume O s x
      with ‹¬ O s x› show False..
    next
      assume O s (−x)
      with ‹¬ O s (−x )› show False..
    qed
  qed
  ultimately show x = −(−x)
    by (rule part-antisymmetry)
qed

lemma part-complement-reversal: y ≠ u ⟹ P x y ⟹ P (−y) (−x)
proof −
  assume y ≠ u
  hence ny: ∀ w. P w (−y) ⟷ ¬ O w y
    by (rule strong-complement-character)
  assume P x y
  have x ≠ u
  proof
    assume x = u
    hence P u y using ‹P x y› by (rule subst)
    with universe-character have y = u
      by (rule part-antisymmetry)
    with ‹y ≠ u› show False..
  qed
  hence ∀ w. P w (−x) ⟷ ¬ O w x
    by (rule strong-complement-character)
  hence P (−y) (−x) ⟷ ¬ O (−y) x..
  moreover have ¬ O (−y) x
  proof
    assume O (−y) x
    with overlap-eq have ∃ v. P v (−y) ∧ P v x..
    then obtain v where v: P v (−y) ∧ P v x..
    hence P v (−y)..
    from ny have P v (−y) ⟷ ¬ O v y..
    hence ¬ O v y using ‹P v (−y)›..
    moreover from v have P v x..
    hence P v y using ‹P x y›
      by (rule part-transitivity)
    hence O v y
      by (rule part-implies-overlap)
    ultimately show False..
  qed
  ultimately show P (−y) (−x)..
qed

lemma complements-overlap: x ⊕ y ≠ u ⟹ O(−x)(−y)
```

proof −
  assume $x \oplus y \neq u$
  hence $\exists\ z.\ \neg\ O\ z\ (x \oplus y)$
    by (*rule something-disjoint*)
  then obtain z **where** z:$\neg\ O\ z\ (x \oplus y)$**..**
  hence $\neg\ O\ z\ x$ **by** (*rule first-summand-disjointness*)
  hence $P\ z\ (-x)$ **by** (*rule complement-part*)
  moreover from z **have** $\neg\ O\ z\ y$
    by (*rule second-summand-disjointness*)
  hence $P\ z\ (-y)$ **by** (*rule complement-part*)
  ultimately show $O(-x)(-y)$
    by (*rule overlap-intro*)
qed

lemma *sum-complement-in-complement-product*:
  $x \oplus y \neq u \Longrightarrow P(-(x \oplus y))(-x \otimes -y)$
proof −
  assume $x \oplus y \neq u$
  hence $O\ (-x)\ (-y)$
    by (*rule complements-overlap*)
  hence $\forall\ w.\ P\ w\ (-x \otimes -y) \longleftrightarrow (P\ w\ (-x) \wedge P\ w\ (-y))$
    by (*rule product-character*)
  hence $P(-(x \oplus y))(-x \otimes -y) \longleftrightarrow (P(-(x \oplus y))(-x) \wedge P(-(x \oplus y))(-y))$**..**
  moreover have $P\ (-(x \oplus y))(-x) \wedge P\ (-(x \oplus y))(-y)$
  proof
   show $P\ (-(x \oplus y))(-x)$ **using** ‹$x \oplus y \neq u$› *first-summand-in-sum*
     by (*rule part-complement-reversal*)
  next
   show $P\ (-(x \oplus y))(-y)$ **using** ‹$x \oplus y \neq u$› *second-summand-in-sum*
     by (*rule part-complement-reversal*)
  qed
  ultimately show $P\ (-(x \oplus y))(-x \otimes -y)$**..**
qed

lemma *complement-product-in-sum-complement*:
  $x \oplus y \neq u \Longrightarrow P(-x \otimes -y)(-(x \oplus y))$
proof −
  assume $x \oplus y \neq u$
  hence $\forall w.\ P\ w\ (-(x \oplus y)) \longleftrightarrow \neg\ O\ w\ (x \oplus y)$
    by (*rule strong-complement-character*)
  hence $P\ (-x \otimes -y)\ (-(x \oplus y)) \longleftrightarrow (\neg\ O\ (-x \otimes -y)\ (x \oplus y))$**..**
  moreover have $\neg\ O\ (-x \otimes -y)\ (x \oplus y)$
  proof
   have $O(-x)(-y)$ **using** ‹$x \oplus y \neq u$› **by** (*rule complements-overlap*)
    hence p: $\forall\ v.\ P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$
     by (*rule product-character*)
    have $O(-x \otimes -y)(x \oplus y) \longleftrightarrow (O(-x \otimes -y)\ x \vee O(-x \otimes -y)y)$
     using *sum-character***..**

moreover assume $O\ (-x \otimes -y)(x \oplus y)$
ultimately have $O\ (-x \otimes -y)\ x \vee O\ (-x \otimes -y)\ y$..
thus *False*
proof
assume $O\ (-x \otimes -y)\ x$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ x$..
then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ x$..
hence $P\ v\ (-x \otimes -y)$..
from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$..
hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›..
hence $P\ v\ (-x)$..
have $x \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule first-summand-not-universe*)
hence $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule strong-complement-character*)
hence $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$..
hence $\neg\ O\ v\ x$ **using** ‹$P\ v\ (-x)$›..
moreover from v **have** $P\ v\ x$..
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
ultimately show *False*..
next
assume $O\ (-x \otimes -y)\ y$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ y$..
then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ y$..
hence $P\ v\ (-x \otimes -y)$..
from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$..
hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›..
hence $P\ v\ (-y)$..
have $y \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule second-summand-not-universe*)
hence $\forall\, w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule strong-complement-character*)
hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$..
hence $\neg\ O\ v\ y$ **using** ‹$P\ v\ (-y)$›..
moreover from v **have** $P\ v\ y$..
hence $O\ v\ y$ **by** (*rule part-implies-overlap*)
ultimately show *False*..
qed
qed
ultimately show $P\ (-x \otimes -y)\ (-(x \oplus y))$..
qed

theorem *sum-complement-is-complements-product*:
$x \oplus y \neq u \Longrightarrow -(x \oplus y) = (-x \otimes -y)$
proof −
assume $x \oplus y \neq u$
show $-(x \oplus y) = (-x \otimes -y)$
proof (*rule part-antisymmetry*)
show $P\ (-\ (x \oplus y))\ (-\ x \otimes -\ y)$ **using** ‹$x \oplus y \neq u$›

by (*rule sum-complement-in-complement-product*)
show $P\ (-\ x \otimes -\ y)\ (-\ (x \oplus y))$ **using** ‹$x \oplus y \neq u$›
by (*rule complement-product-in-sum-complement*)
qed
qed

lemma *complement-sum-in-product-complement*:
$O\ x\ y \Longrightarrow x \neq u \Longrightarrow y \neq u \Longrightarrow P\ ((-x) \oplus (-y))(-(x \otimes y))$
proof −
assume $O\ x\ y$
assume $x \neq u$
assume $y \neq u$
have $x \otimes y \neq u$
proof
assume $x \otimes y = u$
with ‹$O\ x\ y$› **have** $x = u$
by (*rule product-universe-implies-factor-universe*)
with ‹$x \neq u$› **show** *False*..
qed
hence *notxty*: $\forall\ w.\ P\ w\ (-(x \otimes y)) \longleftrightarrow \neg\ O\ w\ (x \otimes y)$
by (*rule strong-complement-character*)
hence $P((-x)\oplus(-y))(-(x \otimes y)) \longleftrightarrow \neg O((-x)\oplus(-y))(x \otimes y)$..
moreover have $\neg\ O\ ((-x) \oplus (-y))\ (x \otimes y)$
proof
from *sum-character* **have**
$\forall\ w.\ O\ w\ ((-x) \oplus (-y)) \longleftrightarrow (O\ w\ (-x) \lor O\ w\ (-y))$.
hence $O(x \otimes y)((-x)\oplus(-y)) \longleftrightarrow (O(x \otimes y)(-x) \lor O(x \otimes y)(-y))$..
moreover assume $O\ ((-x) \oplus (-y))\ (x \otimes y)$
hence $O\ (x \otimes y)\ ((-x) \oplus (-y))$ **by** (*rule overlap-symmetry*)
ultimately have $O\ (x \otimes y)\ (-x) \lor O\ (x \otimes y)\ (-y)$..
thus *False*
proof
assume $O\ (x \otimes y)(-x)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \land P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \land P\ v\ (-x)$..
hence $P\ v\ (-x)$..
with ‹$x \neq u$› **have** $\neg\ P\ v\ x$
by (*rule complement-part-not-part*)
moreover from v **have** $P\ v\ (x \otimes y)$..
with ‹$O\ x\ y$› **have** $P\ v\ x$ **by** (*rule product-part-in-first-factor*)
ultimately show *False*..
next
assume $O\ (x \otimes y)\ (-y)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \land P\ v\ (-y)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \land P\ v\ (-y)$..
hence $P\ v\ (-y)$..
with ‹$y \neq u$› **have** $\neg\ P\ v\ y$
by (*rule complement-part-not-part*)

moreover from v **have** $P\ v\ (x \otimes y)$**..**
with ‹$O\ x\ y$› **have** $P\ v\ y$ **by** (*rule product-part-in-second-factor*)
ultimately show *False***..**
qed
qed
ultimately show $P\ ((-x) \oplus (-y))(-(x \otimes y))$**..**
qed

lemma *product-complement-in-complements-sum*:
$x \neq u \Longrightarrow y \neq u \Longrightarrow P(-(x \otimes y))((-x) \oplus (-y))$
proof −
assume $x \neq u$
hence $x = -(-x)$
by (*rule complement-involution*)
assume $y \neq u$
hence $y = -(-y)$
by (*rule complement-involution*)
show $P\ (-(x \otimes y))((-x) \oplus (-y))$
proof *cases*
assume $-x \oplus -y = u$
thus $P\ (-(x \otimes y))((-x) \oplus (-y))$
using *universe-character* **by** (*rule ssubst*)
next
assume $-x \oplus -y \neq u$
hence $-x \oplus -y = -(-(-x \oplus - y))$
by (*rule complement-involution*)
moreover have $-(-x \oplus -y) = -(-x) \otimes -(-y)$
using ‹$-x \oplus -y \neq u$›
by (*rule sum-complement-is-complements-product*)
with ‹$x = -(-x)$› **have** $-(-x \oplus -y) = x \otimes -(-y)$
by (*rule ssubst*)
with ‹$y = -(-y)$› **have** $-(-x \oplus -y) = x \otimes y$
by (*rule ssubst*)
hence $P\ (-(x \otimes y))(-(-(-x \oplus -y)))$
using *part-reflexivity* **by** (*rule subst*)
ultimately show $P\ (-(x \otimes y))(-x \oplus -y)$
by (*rule ssubst*)
qed
qed

theorem *complement-of-product-is-sum-of-complements*:
$O\ x\ y \Longrightarrow x \oplus y \neq u \Longrightarrow -(x \otimes y) = (-x) \oplus (-y)$
proof −
assume $O\ x\ y$
assume $x \oplus y \neq u$
show $-(x \otimes y) = (-x) \oplus (-y)$
proof (*rule part-antisymmetry*)
have $x \neq u$ **using** ‹$x \oplus y \neq u$›
by (*rule first-summand-not-universe*)

```
      have y ≠ u using ‹x ⊕ y ≠ u›
        by (rule second-summand-not-universe)
      show P (− (x ⊗ y)) (− x ⊕ − y)
       using ‹x ≠ u› ‹y ≠ u› by (rule product-complement-in-complements-sum)
      show  P (− x ⊕ − y) (− (x ⊗ y))
       using ‹O x y› ‹x ≠ u› ‹y ≠ u› by (rule complement-sum-in-product-complement)
    qed
  qed

end
```

8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.[31]

```
locale GM = M +
  assumes fusion:
    ∃ x. φ x ⟹ ∃ z. ∀ y. O y z ⟷ (∃ x. φ x ∧ O y x)
begin
```

Fusion entails sum closure.

```
theorem sum-closure: ∃ z. ∀ w. O w z ⟷ (O w a ∨ O w b)
proof −
  have a = a..
  hence a = a ∨ a = b..
  hence ∃ x. x = a ∨ x = b..
  hence (∃ z. ∀ y. O y z ⟷ (∃ x. (x = a ∨ x = b) ∧ O y x))
    by (rule fusion)
  then obtain z where z:
    ∀ y. O y z ⟷ (∃ x. (x = a ∨ x = b) ∧ O y x)..
  have ∀ w. O w z ⟷ (O w a ∨ O w b)
  proof
    fix w
    from z have w: O w z ⟷ (∃ x. (x = a ∨ x = b) ∧ O w x)..
    show O w z ⟷ (O w a ∨ O w b)
    proof
      assume O w z
      with w have ∃ x. (x = a ∨ x = b) ∧ O w x..
      then obtain x where x: (x = a ∨ x = b) ∧ O w x..
      hence O w x..
      from x have x = a ∨ x = b..
      thus O w a ∨ O w b
      proof (rule disjE)
        assume x = a
        hence O w a using ‹O w x› by (rule subst)
```

[31]See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

```
      thus O w a ∨ O w b..
    next
      assume x = b
      hence O w b using ‹O w x› by (rule subst)
      thus O w a ∨ O w b..
    qed
  next
    assume O w a ∨ O w b
    hence ∃ x. (x = a ∨ x = b) ∧ O w x
    proof (rule disjE)
      assume O w a
      with ‹a = a ∨ a = b› have (a = a ∨ a = b) ∧ O w a..
      thus ∃ x. (x = a ∨ x = b) ∧ O w x..
    next
      have b = b..
      hence b = a ∨ b = b..
      moreover assume O w b
      ultimately have (b = a ∨ b = b) ∧ O w b..
      thus ∃ x. (x = a ∨ x = b) ∧ O w x..
    qed
    with w show O w z..
  qed
 qed
 thus ∃ z. ∀ w. O w z ⟷ (O w a ∨ O w b)..
qed

end
```

9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.[32]

locale *GMM* = *GM* + *MM*
begin

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general minimal mereology and general extensional mereology.[33] But this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does

[32] See [Casati and Varzi, 1999] p. 46.

[33] For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

lemma *product-closure*:
  $O\ x\ y \Longrightarrow (\exists\ z.\ \forall\ v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  nitpick [*expect* = *genuine*] **oops**

lemma *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
  nitpick [*expect* = *genuine*] **oops**

end

10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.[34]

locale *GEM* = *GM* + *EM* +
  assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
  assumes *product-eq*:
    $x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  assumes *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
  assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
  assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
  assumes *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
    $(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
  assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$

sublocale *GEM* ⊆ *GMM*
proof
qed

10.1 General Sums

context *GEM*
begin

lemma *fusion-intro*:
$(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)) \Longrightarrow (\sigma\ x.\ F\ x) = z$
proof −
  assume *antecedent*: $(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
  hence $(THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) = z$
  proof (*rule the-equality*)
    fix *a*
    assume *a*: $(\forall y.\ O\ y\ a \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$

[34] For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

have $\forall x.\ O\ x\ a \longleftrightarrow O\ x\ z$
proof
fix b
from *antecedent* **have** $O\ b\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ b\ x)$**..**
moreover from a **have** $O\ b\ a \longleftrightarrow (\exists x.\ F\ x \wedge O\ b\ x)$**..**
ultimately show $O\ b\ a \longleftrightarrow O\ b\ z$ **by** (*rule ssubst*)
qed
with *overlap-extensionality* **show** $a = z$**..**
qed
moreover from *antecedent* **have** $O\ z\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ z\ x)$**..**
hence $\exists x.\ F\ x \wedge O\ z\ x$ **using** *overlap-reflexivity***..**
hence $\exists x.\ F\ x$ **by** *auto*
hence $(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
by (*rule fusion-eq*)
ultimately show $(\sigma\ v.\ F\ v) = z$ **by** (*rule subst*)
qed

lemma *fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
proof −
have $\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ z = x \wedge O\ y\ x)$
proof
fix y
show $O\ y\ z \longleftrightarrow (\exists x.\ z = x \wedge O\ y\ x)$
proof
assume $O\ y\ z$
with *refl* **have** $z = z \wedge O\ y\ z$**..**
thus $\exists x.\ z = x \wedge O\ y\ x$**..**
next
assume $\exists x.\ z = x \wedge O\ y\ x$
then obtain x **where** x: $z = x \wedge O\ y\ x$**..**
hence $z = x$**..**
moreover from x **have** $O\ y\ x$**..**
ultimately show $O\ y\ z$ **by** (*rule ssubst*)
qed
qed
thus $(\sigma\ x.\ z = x) = z$
by (*rule fusion-intro*)
qed

The whole is the sum of its parts.

lemma *fusion-absorption*: $(\sigma\ x.\ P\ x\ z) = z$
proof −
have $(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ P\ x\ z \wedge O\ y\ x))$
proof
fix y
show $O\ y\ z \longleftrightarrow (\exists x.\ P\ x\ z \wedge O\ y\ x)$
proof
assume $O\ y\ z$
with *part-reflexivity* **have** $P\ z\ z \wedge O\ y\ z$**..**

thus $\exists x.\ P\ x\ z \wedge O\ y\ x$..
next
assume $\exists x.\ P\ x\ z \wedge O\ y\ x$
then obtain x **where** x: $P\ x\ z \wedge O\ y\ x$..
hence $P\ x\ z$..
moreover from x **have** $O\ y\ x$..
ultimately show $O\ y\ z$ **by** (*rule overlap-monotonicity*)
qed
qed
thus $(\sigma\ x.\ P\ x\ z) = z$
by (*rule fusion-intro*)
qed

lemma *part-fusion*: $P\ w\ (\sigma\ v.\ P\ v\ x) \Longrightarrow P\ w\ x$
proof –
assume $P\ w\ (\sigma\ v.\ P\ v\ x)$
with *fusion-absorption* **show** $P\ w\ x$ **by** (*rule subst*)
qed

lemma *fusion-character*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
proof –
assume $\exists x.\ F\ x$
hence $\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion*)
then obtain z **where** z: $\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$..
hence $(\sigma\ v.\ F\ v) = z$ **by** (*rule fusion-intro*)
thus $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$ **using** z **by** (*rule ssubst*)
qed

The next lemma characterises fusions in terms of parthood.[35]

lemma *fusion-part-character*: $\exists x.\ F\ x \Longrightarrow$
$(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)))$
proof –
assume $(\exists x.\ F\ x)$
hence F: $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion-character*)
show $\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
proof
fix y
show $P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
proof
assume $P\ y\ (\sigma\ v.\ F\ v)$
show $\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)$
proof
fix w
from F **have** w: $O\ w\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ w\ x)$..

[35] See [Pontow, 2004] pp. 202-9.

```
      show P w y ⟶ (∃v. F v ∧ O w v)
      proof
        assume P w y
        hence P w (σ v. F v) using ‹P y (σ v. F v)›
          by (rule part-transitivity)
        hence O w (σ v. F v) by (rule part-implies-overlap)
        with w show ∃x. F x ∧ O w x..
      qed
    qed
  next
    assume right: ∀w. P w y ⟶ (∃v. F v ∧ O w v)
    show P y (σ v. F v)
    proof (rule ccontr)
      assume ¬ P y (σ v. F v)
      hence ∃v. P v y ∧ ¬ O v (σ v. F v)
        by (rule strong-supplementation)
      then obtain v where v: P v y ∧ ¬ O v (σ v. F v)..
      hence ¬ O v (σ v. F v)..
      from right have P v y ⟶ (∃w. F w ∧ O v w)..
      moreover from v have P v y..
      ultimately have ∃w. F w ∧ O v w..
      from F have O v (σ v. F v) ⟷ (∃x. F x ∧ O v x)..
      hence O v (σ v. F v) using ‹∃w. F w ∧ O v w›..
      with ‹¬ O v (σ v. F v)› show False..
    qed
  qed
 qed
qed

lemma fusion-part: F x ⟹ P x (σ x. F x)
proof −
 assume F x
 hence ∃x. F x..
 hence ∀y. P y (σ v. F v) ⟷ (∀w. P w y ⟶ (∃v. F v ∧ O w v))
  by (rule fusion-part-character)
 hence P x (σ v. F v) ⟷ (∀w. P w x ⟶ (∃v. F v ∧ O w v))..
 moreover have ∀w. P w x ⟶ (∃v. F v ∧ O w v)
 proof
  fix w
  show P w x ⟶ (∃v. F v ∧ O w v)
  proof
   assume P w x
   hence O w x by (rule part-implies-overlap)
   with ‹F x› have F x ∧ O w x..
   thus ∃v. F v ∧ O w v..
  qed
 qed
 ultimately show P x (σ v. F v)..
qed
```

```
lemma common-part-fusion:
  O x y ⟹ (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
proof −
  assume O x y
  with overlap-eq have ∃ z. (P z x ∧ P z y)..
  hence sum: (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷
    (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)))
    by (rule fusion-part-character)
  show ∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y)
  proof
    fix w
    from sum have w: P w (σ v. (P v x ∧ P v y))
      ⟷ (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v))..
    show P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y)
    proof
      assume P w (σ v. (P v x ∧ P v y))
      with w have bla:
        (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v))..
      show P w x ∧ P w y
      proof
        show P w x
        proof (rule ccontr)
          assume ¬ P w x
          hence ∃ z. P z w ∧ ¬ O z x
            by (rule strong-supplementation)
          then obtain z where z: P z w ∧ ¬ O z x..
          hence ¬ O z x..
          from bla have P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)..
          moreover from z have P z w..
          ultimately have ∃ v. (P v x ∧ P v y) ∧ O z v..
          then obtain v where v: (P v x ∧ P v y) ∧ O z v..
          hence P v x ∧ P v y..
          hence P v x..
          moreover from v have O z v..
          ultimately have O z x
            by (rule overlap-monotonicity)
          with ‹¬ O z x› show False..
        qed
        show P w y
        proof (rule ccontr)
          assume ¬ P w y
          hence ∃ z. P z w ∧ ¬ O z y
            by (rule strong-supplementation)
          then obtain z where z: P z w ∧ ¬ O z y..
          hence ¬ O z y..
          from bla have P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)..
          moreover from z have P z w..
          ultimately have ∃ v. (P v x ∧ P v y) ∧ O z v..
```

```
        then obtain v where v: (P v x ∧ P v y) ∧ O z v..
        hence P v x ∧ P v y..
        hence P v y..
        moreover from v have O z v..
        ultimately have O z y
          by (rule overlap-monotonicity)
        with ‹¬ O z y› show False..
      qed
    qed
  next
    assume P w x ∧ P w y
    thus P w (σ v. (P v x ∧ P v y))
      by (rule fusion-part)
  qed
 qed
qed

theorem product-closure:
  O x y ⟹ (∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y))
proof −
  assume O x y
  hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
    by (rule common-part-fusion)
  thus ∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y)..
qed

end

sublocale GEM ⊆ CEM
proof
  fix x y
  show ∃ z. ∀ w. O w z = (O w x ∨ O w y)
    using sum-closure.
  show x ⊕ y = (THE z. ∀ v. O v z ⟷ O v x ∨ O v y)
    using sum-eq.
  show x ⊗ y = (THE z. ∀ v. P v z ⟷ P v x ∧ P v y)
    using product-eq.
  show O x y ⟹ (∃ z. ∀ w. P w z = (P w x ∧ P w y))
    using product-closure.
qed

context GEM
begin

corollary O x y ⟹ x ⊗ y = (σ v. P v x ∧ P v y)
proof −
  assume O x y
  hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
    by (rule common-part-fusion)
```

```
  thus x ⊗ y = (σ v. P v x ∧ P v y) by (rule product-intro)
qed

lemma disjoint-fusion:
  ∃ w. ¬ O w x ⟹ (∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x)
proof −
  assume antecedent: ∃ w. ¬ O w x
  hence ∀ y. O y (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O y v)
    by (rule fusion-character)
  hence x: O x (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O x v)..
  show ∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x
  proof
    fix y
    show P y (σ z. ¬ O z x) ⟷ ¬ O y x
    proof
      assume P y (σ z. ¬ O z x)
      moreover have ¬ O x (σ z. ¬ O z x)
      proof
        assume O x (σ z. ¬ O z x)
        with x have (∃ v. ¬ O v x ∧ O x v)..
        then obtain v where v: ¬ O v x ∧ O x v..
        hence ¬ O v x..
        from v have O x v..
        hence O v x by (rule overlap-symmetry)
        with ‹¬ O v x› show False..
      qed
      ultimately have ¬ O x y
        by (rule disjoint-demonotonicity)
      thus ¬ O y x by (rule disjoint-symmetry)
    next
      assume ¬ O y x
      thus P y (σ v. ¬ O v x)
        by (rule fusion-part)
    qed
  qed
qed

theorem complement-closure:
  ∃ w. ¬ O w x ⟹ (∃ z. ∀ w. P w z ⟷ ¬ O w x)
proof −
  assume (∃ w. ¬ O w x)
  hence ∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x
    by (rule disjoint-fusion)
  thus ∃ z. ∀ w. P w z ⟷ ¬ O w x..
qed

end

sublocale GEM ⊆ CEMC
```

proof
  fix $x\ y$
  show $-\ x = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
    using *complement-eq*.
  show $(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z = (\neg\ O\ w\ x))$
    using *complement-closure*.
  show $x \ominus y = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
    using *difference-eq*.
  show $u = (\mathit{THE}\ x.\ \forall y.\ P\ y\ x)$
    using *universe-eq*.
qed

context *GEM*
begin

corollary *complement-is-disjoint-fusion*:
  $\exists w.\ \neg\ O\ w\ x \Longrightarrow -\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
proof −
  assume $\exists w.\ \neg\ O\ w\ x$
  hence $\forall w.\ P\ w\ (\sigma\ z.\ \neg\ O\ z\ x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule disjoint-fusion*)
  thus $-\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
    by (*rule complement-intro*)
qed

theorem *strong-fusion*: $\exists x.\ F\ x \Longrightarrow$
  $\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof −
  assume $\exists x.\ F\ x$
  have $(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)) \wedge$
     $(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
  proof
    show $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
    proof
      fix y
      show $F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
      proof
        assume $F\ y$
        thus $P\ y\ (\sigma\ v.\ F\ v)$
          by (*rule fusion-part*)
      qed
    qed
  next
    have $(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow$
      $(\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)))$
      using $\langle \exists x.\ F\ x \rangle$ **by** (*rule fusion-part-character*)
    hence $P\ (\sigma\ v.\ F\ v)\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow$
      $(\exists v.\ F\ v \wedge O\ w\ v))$..
   thus $\forall w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)$ **using** *part-reflexivity*..

qed
thus *?thesis*..
qed

theorem *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
proof –
assume $\exists x.\ F\ x$
have $(THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))) = (\sigma\ x.\ F\ x)$
proof (*rule the-equality*)
show $(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof
show $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$
proof
fix y
show $F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$
proof
assume $F\ y$
thus $P\ y\ (\sigma\ x.\ F\ x)$
by (*rule fusion-part*)
qed
qed
next
show $(\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof
fix y
show $P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
proof
have $\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
using ‹$\exists x.\ F\ x$› **by** (*rule fusion-part-character*)
hence $P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$..
moreover assume $P\ y\ (\sigma\ x.\ F\ x)$
ultimately have $\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)$..
hence $P\ y\ y \longrightarrow (\exists v.\ F\ v \wedge O\ y\ v)$..
thus $\exists v.\ F\ v \wedge O\ y\ v$ **using** *part-reflexivity*..
qed
qed
qed
next
fix x
assume x: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
proof
fix y

show $O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
proof
assume $O\ y\ x$
with *overlap-eq* **have** $\exists v.\ P\ v\ y \wedge P\ v\ x$**..**
then obtain v **where** v: $P\ v\ y \wedge P\ v\ x$**..**
from x **have** $\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$**..**
hence $P\ v\ x \longrightarrow (\exists z.\ F\ z \wedge O\ v\ z)$**..**
moreover from v **have** $P\ v\ x$**..**
ultimately have $\exists z.\ F\ z \wedge O\ v\ z$**..**
then obtain z **where** z: $F\ z \wedge O\ v\ z$**..**
hence $F\ z$**..**
from v **have** $P\ v\ y$**..**
moreover from z **have** $O\ v\ z$**..**
hence $O\ z\ v$ **by** (*rule overlap-symmetry*)
ultimately have $O\ z\ y$ **by** (*rule overlap-monotonicity*)
hence $O\ y\ z$ **by** (*rule overlap-symmetry*)
with ‹$F\ z$› **have** $F\ z \wedge O\ y\ z$**..**
thus $\exists z.\ F\ z \wedge O\ y\ z$**..**
next
assume $\exists z.\ F\ z \wedge O\ y\ z$
then obtain z **where** z: $F\ z \wedge O\ y\ z$**..**
from x **have** $\forall y.\ F\ y \longrightarrow P\ y\ x$**..**
hence $F\ z \longrightarrow P\ z\ x$**..**
moreover from z **have** $F\ z$**..**
ultimately have $P\ z\ x$**..**
moreover from z **have** $O\ y\ z$**..**
ultimately show $O\ y\ x$
by (*rule overlap-monotonicity*)
qed
qed
hence $(\sigma\ x.\ F\ x) = x$
by (*rule fusion-intro*)
thus $x = (\sigma\ x.\ F\ x)$**..**
qed
thus *?thesis***..**
qed

lemma *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$
proof −
have $(THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)) = x \oplus y$
proof (*rule the-equality*)
show $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
proof
show $P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)$
proof
show $P\ x\ (x \oplus y)$ **using** *first-summand-in-sum***.**

```
      show P y (x ⊕ y) using second-summand-in-sum.
    qed
  show ∀ w. P w (x ⊕ y) ⟶ O w x ∨ O w y
  proof
    fix w
    show P w (x ⊕ y) ⟶ O w x ∨ O w y
    proof
      assume P w (x ⊕ y)
      hence O w (x ⊕ y) by (rule part-implies-overlap)
      with sum-overlap show O w x ∨ O w y..
    qed
  qed
qed
fix z
assume z: (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
hence P x z ∧ P y z..
have ∀ w. O w z ⟷ (O w x ∨ O w y)
proof
  fix w
  show O w z ⟷ (O w x ∨ O w y)
  proof
    assume O w z
    with overlap-eq have ∃ v. P v w ∧ P v z..
    then obtain v where v: P v w ∧ P v z..
    hence P v w..
    from z have ∀ w. P w z ⟶ O w x ∨ O w y..
    hence P v z ⟶ O v x ∨ O v y..
    moreover from v have P v z..
    ultimately have O v x ∨ O v y..
    thus O w x ∨ O w y
    proof
      assume O v x
      hence O x v by (rule overlap-symmetry)
      with ‹P v w› have O x w by (rule overlap-monotonicity)
      hence O w x by (rule overlap-symmetry)
      thus O w x ∨ O w y..
    next
      assume O v y
      hence O y v by (rule overlap-symmetry)
      with ‹P v w› have O y w by (rule overlap-monotonicity)
      hence O w y by (rule overlap-symmetry)
      thus O w x ∨ O w y..
    qed
  next
    assume O w x ∨ O w y
    thus O w z
    proof
      from ‹P x z ∧ P y z› have P x z..
      moreover assume O w x
```

ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
next
from ‹$P\ x\ z \wedge P\ y\ z$› **have** $P\ y\ z$..
moreover assume $O\ w\ y$
ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
qed
qed
qed
hence $x \oplus y = z$ **by** (*rule sum-intro*)
thus $z = x \oplus y$..
qed
thus *?thesis*..
qed

10.2 General Products

lemma *general-product-intro*: $(\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)) \Longrightarrow (\pi\ x.\ F\ x) = x$
proof −
assume $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
hence $(\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y) = x$ **by** (*rule fusion-intro*)
with *general-product-eq* **show** $(\pi\ x.\ F\ x) = x$ **by** (*rule ssubst*)
qed

lemma *general-product-idempotence*: $(\pi\ z.\ z = x) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ y = x \longrightarrow P\ z\ y) \wedge O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ z = x) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-absorption*: $(\pi\ z.\ P\ x\ z) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ P\ x\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ P\ x\ z) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-character*: $\exists z.\ \forall y.\ F\ y \longrightarrow P\ z\ y \Longrightarrow$
$\forall y.\ O\ y\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
proof −
assume $(\exists z.\ \forall y.\ F\ y \longrightarrow P\ z\ y)$
hence $(\exists x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$
by (*rule fusion*)
then obtain x **where** x:
$\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$..
hence $(\pi\ x.\ F\ x) = x$ **by** (*rule general-product-intro*)

```
  thus (∀ y. O y (π x. F x) ⟷ (∃ z. (∀ y. F y ⟶ P z y) ∧ O y z))
    using x by (rule ssubst)
qed

corollary ¬ (∃ x. F x) ⟹ u = (π x. F x)
proof −
  assume antecedent: ¬ (∃ x. F x)
  have ∀ y. P y (π x. F x)
  proof
    fix y
    show P y (π x. F x)
    proof (rule ccontr)
      assume ¬ P y (π x. F x)
     hence ∃ z. P z y ∧ ¬ O z (π x. F x) by (rule strong-supplementation)
      then obtain z where z: P z y ∧ ¬ O z (π x. F x)..
      hence ¬ O z (π x. F x)..
      from antecedent have bla: ∀ y. F y ⟶ P z y by simp
      hence ∃ v. ∀ y. F y ⟶ P v y..
      hence (∀ y. O y (π x. F x) ⟷ (∃ z. (∀ y. F y ⟶ P z y) ∧ O y
z)) by (rule general-product-character)
      hence O z (π x. F x) ⟷ (∃ v. (∀ y. F y ⟶ P v y) ∧ O z v)..
      moreover from bla have (∀ y. F y ⟶ P z y) ∧ O z z
        using overlap-reflexivity..
      hence ∃ v. (∀ y. F y ⟶ P v y) ∧ O z v..
      ultimately have O z (π x. F x)..
      with ‹¬ O z (π x. F x)› show False..
    qed
  qed
  thus u = (π x. F x)
    by (rule universe-intro)
qed

end
```

10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.[36]

```
locale GEM1 = MM +
 assumes strong-fusion: ∃ x. F x ⟹ ∃ x. (∀ y. F y ⟶ P y x) ∧ (∀ y.
P y x ⟶ (∃ z. F z ∧ O y z))
  assumes strong-sum-eq: x ⊕ y = (THE z. (P x z ∧ P y z) ∧ (∀ w.
P w z ⟶ O w x ∨ O w y))
  assumes product-eq:
    x ⊗ y = (THE z. ∀ v. P v z ⟷ P v x ∧ P v y)
```

[36] See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
assumes *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$
begin

theorem *fusion*:
$\exists x.\ \varphi\ x \Longrightarrow (\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$
proof $-$
assume $\exists x.\ \varphi\ x$
hence $\exists x.\ (\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$ **by** (*rule strong-fusion*)
then obtain x **where** x:
$(\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$**..**
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
fix y
show $O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
assume $O\ y\ x$
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ x$**..**
then obtain z **where** z: $P\ z\ y \wedge P\ z\ x$**..**
hence $P\ z\ x$**..**
from x **have** $\forall y.\ P\ y\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$**..**
hence $P\ z\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ z\ v)$**..**
hence $\exists v.\ \varphi\ v \wedge O\ z\ v$ **using** ‹$P\ z\ x$›**..**
then obtain v **where** v: $\varphi\ v \wedge O\ z\ v$**..**
hence $O\ z\ v$**..**
with *overlap-eq* **have** $\exists w.\ P\ w\ z \wedge P\ w\ v$**..**
then obtain w **where** w: $P\ w\ z \wedge P\ w\ v$**..**
hence $P\ w\ z$**..**
moreover from z **have** $P\ z\ y$**..**
ultimately have $P\ w\ y$
by (*rule part-transitivity*)
moreover from w **have** $P\ w\ v$**..**
ultimately have $P\ w\ y \wedge P\ w\ v$**..**
hence $\exists w.\ P\ w\ y \wedge P\ w\ v$**..**
with *overlap-eq* **have** $O\ y\ v$**..**
from v **have** $\varphi\ v$**..**
hence $\varphi\ v \wedge O\ y\ v$ **using** ‹$O\ y\ v$›**..**
thus $\exists v.\ \varphi\ v \wedge O\ y\ v$**..**
next
assume $\exists v.\ \varphi\ v \wedge O\ y\ v$
then obtain v **where** v: $\varphi\ v \wedge O\ y\ v$**..**
hence $O\ y\ v$**..**
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ v$**..**

then obtain z **where** z: $P\ z\ y \wedge P\ z\ v$..
hence $P\ z\ v$..
from x **have** $\forall y.\ \varphi\ y \longrightarrow P\ y\ x$..
hence $\varphi\ v \longrightarrow P\ v\ x$..
moreover from v **have** $\varphi\ v$..
ultimately have $P\ v\ x$..
with ‹$P\ z\ v$› **have** $P\ z\ x$
by (*rule part-transitivity*)
from z **have** $P\ z\ y$..
thus $O\ y\ x$ **using** ‹$P\ z\ x$›
by (*rule overlap-intro*)
qed
qed
thus $(\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$..
qed

lemma *pair*: $\exists v.\ (\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ v) \wedge (\forall w.\ P\ w\ v \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ w\ z))$
proof –
have $x = x$..
hence $x = x \vee x = y$..
hence $\exists v.\ v = x \vee v = y$..
thus *?thesis*
by (*rule strong-fusion*)
qed

lemma *or-id*: $(v = x \vee v = y) \wedge O\ w\ v \Longrightarrow O\ w\ x \vee O\ w\ y$
proof –
assume v: $(v = x \vee v = y) \wedge O\ w\ v$
hence $O\ w\ v$..
from v **have** $v = x \vee v = y$..
thus $O\ w\ x \vee O\ w\ y$
proof
assume $v = x$
hence $O\ w\ x$ **using** ‹$O\ w\ v$› **by** (*rule subst*)
thus $O\ w\ x \vee O\ w\ y$..
next
assume $v = y$
hence $O\ w\ y$ **using** ‹$O\ w\ v$› **by** (*rule subst*)
thus $O\ w\ x \vee O\ w\ y$..
qed
qed

lemma *strong-sum-closure*:
$\exists z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
proof –
from *pair* **obtain** z **where** z: $(\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow (\exists v.\ (v = x \vee v = y) \wedge O\ w\ v))$..
have $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$

```
  proof
    from z have allw: ∀ w. (w = x ∨ w = y) ⟶ P w z..
    hence x = x ∨ x = y ⟶ P x z..
    moreover have x = x ∨ x = y using refl..
    ultimately have P x z..
    from allw have y = x ∨ y = y ⟶ P y z..
    moreover have y = x ∨ y = y using refl..
    ultimately have P y z..
    with ‹P x z› show P x z ∧ P y z..
  next
    show ∀ w. P w z ⟶ O w x ∨ O w y
    proof
      fix w
      show P w z ⟶ O w x ∨ O w y
      proof
        assume P w z
        from z have ∀ w. P w z ⟶ (∃ v. (v = x ∨ v = y) ∧ O w v)..
        hence P w z ⟶ (∃ v. (v = x ∨ v = y) ∧ O w v)..
        hence ∃ v. (v = x ∨ v = y) ∧ O w v using ‹P w z›..
        then obtain v where v: (v = x ∨ v = y) ∧ O w v..
        thus O w x ∨ O w y by (rule or-id)
      qed
    qed
  qed
  thus ?thesis..
qed

end

sublocale GEM1 ⊆ GMM
proof
  fix x y φ
  show (∃ x. φ x) ⟹ (∃ z. ∀ y. O y z ⟷ (∃ x. φ x ∧ O y x)) using
fusion.
qed

context GEM1
begin

lemma least-upper-bound:
  assumes sf:
    ((∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z)))
  shows lub:
      (∀ y. F y ⟶ P y x) ∧ (∀ z. (∀ y. F y ⟶ P y z) ⟶ P x z)
proof
 from sf show ∀ y. F y ⟶ P y x..
next
 show (∀ z. (∀ y. F y ⟶ P y z) ⟶ P x z)
 proof
```

fix z
show $(\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z$
proof
assume z: $\forall y.\ F\ y \longrightarrow P\ y\ z$
from *pair* **obtain** v **where** v: $(\forall w.\ (w = x \lor w = z) \longrightarrow P\ w\ v)$
$\land (\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y))$..
hence *left*: $(\forall w.\ (w = x \lor w = z) \longrightarrow P\ w\ v)$..
hence $(x = x \lor x = z) \longrightarrow P\ x\ v$..
moreover have $x = x \lor x = z$ **using** *refl*..
ultimately have $P\ x\ v$..
have $z = v$
proof (*rule ccontr*)
assume $z \neq v$
from *left* **have** $z = x \lor z = z \longrightarrow P\ z\ v$..
moreover have $z = x \lor z = z$ **using** *refl*..
ultimately have $P\ z\ v$..
hence $P\ z\ v \land z \neq v$ **using** ‹$z \neq v$›..
with *nip-eq* **have** $PP\ z\ v$..
hence $\exists w.\ P\ w\ v \land \neg\ O\ w\ z$ **by** (*rule weak-supplementation*)
then obtain w **where** w: $P\ w\ v \land \neg\ O\ w\ z$..
hence $P\ w\ v$..
from v **have** *right*:
$\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y)$..
hence $P\ w\ v \longrightarrow (\exists y.\ (y = x \lor y = z) \land O\ w\ y)$..
hence $\exists y.\ (y = x \lor y = z) \land O\ w\ y$ **using** ‹$P\ w\ v$›..
then obtain s **where** s: $(s = x \lor s = z) \land O\ w\ s$..
hence $s = x \lor s = z$..
thus *False*
proof
assume $s = x$
moreover from s **have** $O\ w\ s$..
ultimately have $O\ w\ x$ **by** (*rule subst*)
with *overlap-eq* **have** $\exists t.\ P\ t\ w \land P\ t\ x$..
then obtain t **where** t: $P\ t\ w \land P\ t\ x$..
hence $P\ t\ x$..
from *sf* **have** $(\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \land O\ y\ z))$..
hence $P\ t\ x \longrightarrow (\exists z.\ F\ z \land O\ t\ z)$..
hence $\exists z.\ F\ z \land O\ t\ z$ **using** ‹$P\ t\ x$›..
then obtain a **where** a: $F\ a \land O\ t\ a$..
hence $F\ a$..
from *sf* **have** *ub*: $\forall y.\ F\ y \longrightarrow P\ y\ x$..
hence $F\ a \longrightarrow P\ a\ x$..
hence $P\ a\ x$ **using** ‹$F\ a$›..
moreover from a **have** $O\ t\ a$..
ultimately have $O\ t\ x$
by (*rule overlap-monotonicity*)
from t **have** $P\ t\ w$..
moreover have $O\ z\ t$
proof –

from z **have** $F\ a \longrightarrow P\ a\ z$**..**
moreover from a **have** $F\ a$**..**
ultimately have $P\ a\ z$**..**
moreover from a **have** $O\ t\ a$**..**
ultimately have $O\ t\ z$
by (*rule overlap-monotonicity*)
thus $O\ z\ t$ **by** (*rule overlap-symmetry*)
qed
ultimately have $O\ z\ w$
by (*rule overlap-monotonicity*)
hence $O\ w\ z$ **by** (*rule overlap-symmetry*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** $\langle O\ w\ z \rangle$**..**
next
assume $s = z$
moreover from s **have** $O\ w\ s$**..**
ultimately have $O\ w\ z$ **by** (*rule subst*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** $\langle O\ w\ z \rangle$**..**
qed
qed
thus $P\ x\ z$ **using** $\langle P\ x\ v \rangle$ **by** (*rule ssubst*)
qed
qed
qed

corollary *strong-fusion-intro*: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) \Longrightarrow (\sigma\ x.\ F\ x) = x$
proof −
assume *antecedent*: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
with *least-upper-bound* **have** *lubx*:
$(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$**.**
from *antecedent* **have** $\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$**..**
hence $P\ x\ x \longrightarrow (\exists z.\ F\ z \wedge O\ x\ z)$**..**
hence $\exists z.\ F\ z \wedge O\ x\ z$ **using** *part-reflexivity***..**
then obtain z **where** z: $F\ z \wedge O\ x\ z$**..**
hence $F\ z$**..**
hence $\exists z.\ F\ z$**..**
hence $(\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion-eq*)
moreover have $(THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))) = x$
proof (*rule the-equality*)
show $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
using *antecedent***.**
next
fix w
assume w:

($\forall y.\ F\ y \longrightarrow P\ y\ w) \wedge (\forall y.\ P\ y\ w \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
with *least-upper-bound* **have** *lubw*:
($\forall y.\ F\ y \longrightarrow P\ y\ w) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ w\ z)$.
hence ($\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ w\ z$)..
hence ($\forall y.\ F\ y \longrightarrow P\ y\ x) \longrightarrow P\ w\ x$..
moreover from *antecedent* **have** $\forall y.\ F\ y \longrightarrow P\ y\ x$..
ultimately have $P\ w\ x$..
from *lubx* **have** ($\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z$)..
hence ($\forall y.\ F\ y \longrightarrow P\ y\ w) \longrightarrow P\ x\ w$..
moreover from *lubw* **have** ($\forall y.\ F\ y \longrightarrow P\ y\ w$)..
ultimately have $P\ x\ w$..
with ‹$P\ w\ x$› **show** $w = x$
by (*rule part-antisymmetry*)
qed
ultimately show ($\sigma\ x.\ F\ x) = x$ **by** (*rule ssubst*)
qed

lemma *strong-fusion-character*: $\exists x.\ F\ x \Longrightarrow ((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
proof −
assume $\exists x.\ F\ x$
hence ($\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$) **by** (*rule strong-fusion*)
then obtain x **where** x:
($\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$)..
hence ($\sigma\ x.\ F\ x) = x$ **by** (*rule strong-fusion-intro*)
thus *?thesis* **using** x **by** (*rule ssubst*)
qed

lemma *F-in*: $\exists x.\ F\ x \Longrightarrow (\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x))$
proof −
assume $\exists x.\ F\ x$
hence ($(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$) **by** (*rule strong-fusion-character*)
thus $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$..
qed

lemma *parts-overlap-Fs*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof −
assume $\exists x.\ F\ x$
hence ($(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$) **by** (*rule strong-fusion-character*)
thus ($\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$)..
qed

lemma *in-strong-fusion*: $P\ z\ (\sigma\ x.\ z = x)$
proof −
have $\exists y.\ z = y$ **using** *refl*..

hence $\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)$
by (*rule F-in*)
hence $z = z \longrightarrow P\ z\ (\sigma\ x.\ z = x)$**..**
thus $P\ z\ (\sigma\ x.\ z = x)$ **using** *refl***..**
qed

lemma *strong-fusion-in*: $P\ (\sigma\ x.\ z = x)\ z$
proof −
have $\exists y.\ z = y$ **using** *refl***..**
hence *sf*:
$(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ z = x) \longrightarrow (\exists v.\ z = v \wedge O\ y\ v))$
by (*rule strong-fusion-character*)
with *least-upper-bound* **have** *lub*: $(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$**.**
hence $(\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$**..**
hence $(\forall y.\ z = y \longrightarrow P\ y\ z) \longrightarrow P\ (\sigma\ x.\ z = x)\ z$**..**
moreover have $(\forall y.\ z = y \longrightarrow P\ y\ z)$
proof
fix y
show $z = y \longrightarrow P\ y\ z$
proof
assume $z = y$
thus $P\ y\ z$ **using** *part-reflexivity* **by** (*rule subst*)
qed
qed
ultimately show $P\ (\sigma\ x.\ z = x)\ z$**..**
qed

lemma *strong-fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
using *strong-fusion-in in-strong-fusion* **by** (*rule part-antisymmetry*)

10.4 Strong Sums

lemma *pair-fusion*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow (\sigma\ z.\ z = x \vee z = y) = z$
proof
assume *z*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
have $(\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z) \wedge (\forall v.\ P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z))$
proof
show $\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z$
proof
fix w
from *z* **have** $P\ x\ z \wedge P\ y\ z$**..**
show $w = x \vee w = y \longrightarrow P\ w\ z$
proof
assume $w = x \vee w = y$
thus $P\ w\ z$

proof
assume $w = x$
moreover from ‹$P\ x\ z \wedge P\ y\ z$› **have** $P\ x\ z$**..**
ultimately show $P\ w\ z$ **by** (*rule ssubst*)
next
assume $w = y$
moreover from ‹$P\ x\ z \wedge P\ y\ z$› **have** $P\ y\ z$**..**
ultimately show $P\ w\ z$ **by** (*rule ssubst*)
qed
qed
qed
show $\forall v.\ P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z)$
proof
fix v
show $P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z)$
proof
assume $P\ v\ z$
from z **have** $\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y$**..**
hence $P\ v\ z \longrightarrow O\ v\ x \vee O\ v\ y$**..**
hence $O\ v\ x \vee O\ v\ y$ **using** ‹$P\ v\ z$›**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$
proof
assume $O\ v\ x$
have $x = x \vee x = y$ **using** *refl***..**
hence $(x = x \vee x = y) \wedge O\ v\ x$ **using** ‹$O\ v\ x$›**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$**..**
next
assume $O\ v\ y$
have $y = x \vee y = y$ **using** *refl***..**
hence $(y = x \vee y = y) \wedge O\ v\ y$ **using** ‹$O\ v\ y$›**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$**..**
qed
qed
qed
qed
thus $(\sigma\ z.\ z = x \vee z = y) = z$
by (*rule strong-fusion-intro*)
qed

corollary *strong-sum-fusion*: $x \oplus y = (\sigma\ z.\ z = x \vee z = y)$
proof –
have $(THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge$
$(\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)) = (\sigma\ z.\ z = x \vee z = y)$
proof (*rule the-equality*)
have $x = x \vee x = y$ **using** *refl***..**
hence *exz*: $\exists z.\ z = x \vee z = y$**..**
hence *allw*: $(\forall w.\ w = x \vee w = y \longrightarrow P\ w\ (\sigma\ z.\ z = x \vee z = y))$
by (*rule F-in*)
show $(P\ x\ (\sigma\ z.\ z = x \vee z = y) \wedge P\ y\ (\sigma\ z.\ z = x \vee z = y)) \wedge$

$(\forall w.\ P\ w\ (\sigma\ z.\ z = x \lor z = y) \longrightarrow O\ w\ x \lor O\ w\ y)$
proof
show $(P\ x\ (\sigma\ z.\ z = x \lor z = y) \land P\ y\ (\sigma\ z.\ z = x \lor z = y))$
proof
from *allw* **have** $x = x \lor x = y \longrightarrow P\ x\ (\sigma\ z.\ z = x \lor z = y)$**..**
thus $P\ x\ (\sigma\ z.\ z = x \lor z = y)$
using ‹$x = x \lor x = y$›**..**
next
from *allw* **have** $y = x \lor y = y \longrightarrow P\ y\ (\sigma\ z.\ z = x \lor z = y)$**..**
moreover have $y = x \lor y = y$
using *refl***..**
ultimately show $P\ y\ (\sigma\ z.\ z = x \lor z = y)$**..**
qed
next
show $\forall w.\ P\ w\ (\sigma\ z.\ z = x \lor z = y) \longrightarrow O\ w\ x \lor O\ w\ y$
proof
fix w
show $P\ w\ (\sigma\ z.\ z = x \lor z = y) \longrightarrow O\ w\ x \lor O\ w\ y$
proof
have $\forall v.\ P\ v\ (\sigma\ z.\ z = x \lor z = y) \longrightarrow (\exists z.\ (z = x \lor z = y) \land O\ v\ z)$ **using** *exz* **by** (*rule parts-overlap-Fs*)
hence $P\ w\ (\sigma\ z.\ z = x \lor z = y) \longrightarrow (\exists z.\ (z = x \lor z = y) \land O\ w\ z)$**..**
moreover assume $P\ w\ (\sigma\ z.\ z = x \lor z = y)$
ultimately have $(\exists z.\ (z = x \lor z = y) \land O\ w\ z)$**..**
then obtain z **where** z: $(z = x \lor z = y) \land O\ w\ z$**..**
thus $O\ w\ x \lor O\ w\ y$ **by** (*rule or-id*)
qed
qed
qed
next
fix z
assume z: $(P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y)$
with *pair-fusion* **have** $(\sigma\ z.\ z = x \lor z = y) = z$**..**
thus $z = (\sigma\ z.\ z = x \lor z = y)$**..**
qed
with *strong-sum-eq* **show** $x \oplus y = (\sigma\ z.\ z = x \lor z = y)$
by (*rule ssubst*)
qed

corollary *strong-sum-intro*:
$(P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y) \longrightarrow x \oplus y = z$
proof
assume z: $(P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y)$
with *pair-fusion* **have** $(\sigma\ z.\ z = x \lor z = y) = z$**..**
with *strong-sum-fusion* **show** $(x \oplus y) = z$
by (*rule ssubst*)
qed

corollary *strong-sum-character*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.$ $P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
proof –
  from *strong-sum-closure* **obtain** z **where** z:
    $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$**..**
 with *strong-sum-intro* **have** $x \oplus y = z$**..**
 thus *?thesis* **using** z **by** (*rule ssubst*)
qed

corollary *summands-in*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y))$
  using *strong-sum-character***..**

corollary *first-summand-in*: $P\ x\ (x \oplus y)$ **using** *summands-in***..**

corollary *second-summand-in*: $P\ y\ (x \oplus y)$ **using** *summands-in***..**

corollary *sum-part-overlap*: $(\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
using *strong-sum-character***..**

lemma *strong-sum-absorption*: $y = (x \oplus y) \Longrightarrow P\ x\ y$
proof –
 assume $y = (x \oplus y)$
 thus $P\ x\ y$ **using** *first-summand-in* **by** (*rule ssubst*)
qed

theorem *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
proof –
 assume $\neg\ P\ x\ y$
 have $\neg\ (\forall z.\ P\ z\ x \longrightarrow O\ z\ y)$
 proof
  assume z: $\forall z.\ P\ z\ x \longrightarrow O\ z\ y$
  have $(\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)) \wedge$
    $(\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z))$
  proof
   show $\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)$
   proof
    fix v
    show $y = v \longrightarrow P\ v\ (x \oplus y)$
    proof
     assume $y = v$
     thus $P\ v\ (x \oplus y)$
       using *second-summand-in* **by** (*rule subst*)
    qed
   qed
   show $\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z)$
   proof
    fix v
    show $P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z)$
    proof

```
      assume P v (x ⊕ y)
      moreover from sum-part-overlap have
        P v (x ⊕ y) ⟶ O v x ∨ O v y..
      ultimately have O v x ∨ O v y by (rule rev-mp)
      hence O v y
      proof
       assume O v x
       with overlap-eq have ∃ w. P w v ∧ P w x..
       then obtain w where w: P w v ∧ P w x..
       from z have P w x ⟶ O w y..
       moreover from w have P w x..
       ultimately have O w y..
       with overlap-eq have ∃ t. P t w ∧ P t y..
       then obtain t where t: P t w ∧ P t y..
       hence P t w..
       moreover from w have P w v..
       ultimately have P t v
         by (rule part-transitivity)
       moreover from t have P t y..
       ultimately show O v y
         by (rule overlap-intro)
      next
       assume O v y
       thus O v y.
      qed
      with refl have y = y ∧ O v y..
      thus ∃ z. y = z ∧ O v z..
     qed
    qed
   qed
   hence (σ z. y = z) = (x ⊕ y) by (rule strong-fusion-intro)
   with strong-fusion-idempotence have y = x ⊕ y by (rule subst)
   hence P x y by (rule strong-sum-absorption)
   with ‹¬ P x y› show False..
  qed
  thus ∃ z. P z x ∧ ¬ O z y by simp
qed

lemma sum-character: ∀ v. O v (x ⊕ y) ⟷ (O v x ∨ O v y)
proof
  fix v
  show O v (x ⊕ y) ⟷ (O v x ∨ O v y)
  proof
    assume O v (x ⊕ y)
    with overlap-eq have ∃ w. P w v ∧ P w (x ⊕ y)..
    then obtain w where w: P w v ∧ P w (x ⊕ y)..
    hence P w v..
    have P w (x ⊕ y) ⟶ O w x ∨ O w y using sum-part-overlap..
    moreover from w have P w (x ⊕ y)..
```

ultimately have *O w x* ∨ *O w y*..
thus *O v x* ∨ *O v y*
proof
assume *O w x*
hence *O x w*
by (*rule overlap-symmetry*)
with ‹*P w v*› **have** *O x v*
by (*rule overlap-monotonicity*)
hence *O v x*
by (*rule overlap-symmetry*)
thus *O v x* ∨ *O v y*..
next
assume *O w y*
hence *O y w*
by (*rule overlap-symmetry*)
with ‹*P w v*› **have** *O y v*
by (*rule overlap-monotonicity*)
hence *O v y* **by** (*rule overlap-symmetry*)
thus *O v x* ∨ *O v y*..
qed
next
assume *O v x* ∨ *O v y*
thus *O v* (*x* ⊕ *y*)
proof
assume *O v x*
with *overlap-eq* **have** ∃ *w*. *P w v* ∧ *P w x*..
then obtain *w* **where** *w*: *P w v* ∧ *P w x*..
hence *P w v*..
moreover from *w* **have** *P w x*..
hence *P w* (*x* ⊕ *y*) **using** *first-summand-in*
by (*rule part-transitivity*)
ultimately show *O v* (*x* ⊕ *y*)
by (*rule overlap-intro*)
next
assume *O v y*
with *overlap-eq* **have** ∃ *w*. *P w v* ∧ *P w y*..
then obtain *w* **where** *w*: *P w v* ∧ *P w y*..
hence *P w v*..
moreover from *w* **have** *P w y*..
hence *P w* (*x* ⊕ *y*) **using** *second-summand-in*
by (*rule part-transitivity*)
ultimately show *O v* (*x* ⊕ *y*)
by (*rule overlap-intro*)
qed
qed
qed

lemma *sum-eq*: *x* ⊕ *y* = (*THE z*. ∀ *v*. *O v z* = (*O v x* ∨ *O v y*))
proof −

```
have (THE z. ∀ v. O v z ⟷ (O v x ∨ O v y)) = x ⊕ y
proof (rule the-equality)
 show ∀ v. O v (x ⊕ y) ⟷ (O v x ∨ O v y) using sum-character.
next
  fix z
  assume z: ∀ v. O v z ⟷ (O v x ∨ O v y)
  have (P x z ∧ P y z) ∧ (∀ w. P w z ⟶ O w x ∨ O w y)
  proof
    show P x z ∧ P y z
    proof
      show P x z
      proof (rule ccontr)
        assume ¬ P x z
        hence ∃ v. P v x ∧ ¬ O v z
          by (rule strong-supplementation)
        then obtain v where v: P v x ∧ ¬ O v z..
        hence ¬ O v z..
        from z have O v z ⟷ (O v x ∨ O v y)..
        moreover from v have P v x..
        hence O v x by (rule part-implies-overlap)
        hence O v x ∨ O v y..
        ultimately have O v z..
        with ‹¬ O v z› show False..
      qed
    next
      show P y z
      proof (rule ccontr)
        assume ¬ P y z
        hence ∃ v. P v y ∧ ¬ O v z
          by (rule strong-supplementation)
        then obtain v where v: P v y ∧ ¬ O v z..
        hence ¬ O v z..
        from z have O v z ⟷ (O v x ∨ O v y)..
        moreover from v have P v y..
        hence O v y by (rule part-implies-overlap)
        hence O v x ∨ O v y..
        ultimately have O v z..
        with ‹¬ O v z› show False..
      qed
    qed
    show ∀ w. P w z ⟶ (O w x ∨ O w y)
    proof
      fix w
      show P w z ⟶ (O w x ∨ O w y)
      proof
        from z have O w z ⟷ O w x ∨ O w y..
        moreover assume P w z
        hence O w z by (rule part-implies-overlap)
        ultimately show O w x ∨ O w y..
```

```
      qed
    qed
  qed
  with strong-sum-intro have x ⊕ y = z..
  thus z = x ⊕ y..
 qed
 thus ?thesis..
qed

theorem fusion-eq: ∃ x. F x ⟹
 (σ x. F x) = (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z))
proof −
 assume ∃ x. F x
 hence bla: ∀ y. P y (σ x. F x) ⟶ (∃ z. F z ∧ O y z)
  by (rule parts-overlap-Fs)
 have (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z)) = (σ x. F x)
 proof (rule the-equality)
  show ∀ y. O y (σ x. F x) ⟷ (∃ z. F z ∧ O y z)
  proof
   fix y
   show O y (σ x. F x) ⟷ (∃ z. F z ∧ O y z)
   proof
    assume O y (σ x. F x)
    with overlap-eq have ∃ v. P v y ∧ P v (σ x. F x)..
    then obtain v where v: P v y ∧ P v (σ x. F x)..
    hence P v y..
    from bla have P v (σ x. F x) ⟶ (∃ z. F z ∧ O v z)..
    moreover from v have P v (σ x. F x)..
    ultimately have (∃ z. F z ∧ O v z)..
    then obtain z where z: F z ∧ O v z..
    hence F z..
    moreover from z have O v z..
    hence O z v by (rule overlap-symmetry)
    with ‹P v y› have O z y by (rule overlap-monotonicity)
    hence O y z by (rule overlap-symmetry)
    ultimately have F z ∧ O y z..
    thus (∃ z. F z ∧ O y z)..
   next
    assume ∃ z. F z ∧ O y z
    then obtain z where z: F z ∧ O y z..
    from‹∃ x. F x› have (∀ y. F y ⟶ P y (σ x. F x))
     by (rule F-in)
    hence F z ⟶  P z (σ x. F x)..
    moreover from z have F z..
    ultimately have P z (σ x. F x)..
    moreover from z have O y z..
    ultimately show O y (σ x. F x)
     by (rule overlap-monotonicity)
   qed
```

```
    qed
  next
    fix x
    assume x: ∀ y. O y x ⟷ (∃ v. F v ∧ O y v)
    have (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z))
    proof
      show ∀ y. F y ⟶ P y x
      proof
        fix y
        show F y ⟶ P y x
        proof
          assume F y
          show P y x
          proof (rule ccontr)
            assume ¬ P y x
            hence ∃ z. P z y ∧ ¬ O z x
              by (rule strong-supplementation)
            then obtain z where z: P z y ∧ ¬ O z x..
            hence ¬ O z x..
            from x have O z x ⟷ (∃ v. F v ∧ O z v)..
            moreover from z have P z y..
            hence O z y by (rule part-implies-overlap)
            with ‹F y› have F y ∧ O z y..
            hence ∃ y. F y ∧ O z y..
            ultimately have O z x..
            with ‹¬ O z x› show False..
          qed
        qed
      qed
      show ∀ y. P y x ⟶ (∃ z. F z ∧ O y z)
      proof
        fix y
        show P y x ⟶ (∃ z. F z ∧ O y z)
        proof
          from x have O y x ⟷ (∃ z. F z ∧ O y z)..
          moreover assume P y x
          hence O y x by (rule part-implies-overlap)
          ultimately show ∃ z. F z ∧ O y z..
        qed
      qed
    qed
    hence (σ x. F x) = x
      by (rule strong-fusion-intro)
    thus x = (σ x. F x)..
  qed
  thus (σ x. F x) = (THE x. ∀ y. O y x ⟷ (∃ z. F z ∧ O y z))..
qed

end
```

sublocale *GEM1* $\subseteq$ *GEM*
proof
  fix x y F
  show $\neg\, P\ x\ y \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\, O\ z\ y$
    using *strong-supplementation*.
  show $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow (O\ v\ x \vee O\ v\ y))$
    using *sum-eq*.
  show $x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
    using *product-eq*.
  show $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\, O\ w\ y))$
    using *difference-eq*.
  show $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\, O\ w\ x)$
    using *complement-eq*.
  show $u = (THE\ x.\ \forall y.\ P\ y\ x)$
    using *universe-eq*.
  show $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$ **using** *fusion-eq*.
  show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$
    using *general-product-eq*.
qed

sublocale *GEM* $\subseteq$ *GEM1*
proof
  fix x y F
  show $\exists x.\ F\ x \Longrightarrow (\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **using** *strong-fusion*.
  show $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **using** *strong-fusion-eq*.
  show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$ **using** *general-product-eq*.
  show $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$ **using** *strong-sum-eq*.
  show $x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
    using *product-eq*.
  show $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\, O\ w\ y))$
    using *difference-eq*.
  show $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\, O\ w\ x)$ **using** *complement-eq*.
  show $u = (THE\ x.\ \forall y.\ P\ y\ x)$ **using** *universe-eq*.
qed

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