Hall's Marriage Theorem

Dongchen Jiang and Tobias Nipkow

December 7, 2022

Abstract

A proof of Hall's Marriage Theorem due to Halmos and Vaughan [1].

theory *Marriage*
imports *Main*
begin

theorem *marriage-necessary*:
  fixes $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  assumes $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  and $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$ (**is** $\exists R.\ ?R\ R\ A\ \&\ ?inj\ R\ A$)
  shows $\forall J{\subseteq}I.\ card\ J \le card\ (\bigcup (A\ `\ J))$
⟨*proof*⟩

The proof by Halmos and Vaughan:

theorem *marriage-HV*:
  fixes $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  assumes $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  and $\forall J{\subseteq}I.\ card\ J \le card\ (\bigcup (A\ `\ J))$ (**is** $?M\ A\ I$)
  shows $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$
    (**is** $?SDR\ A\ I$ **is** $\exists R.\ ?R\ R\ A\ I\ \&\ ?inj\ R\ A\ I$)
⟨*proof*⟩

The proof by Rado:

theorem *marriage-Rado*:
  fixes $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  assumes $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  and $\forall J{\subseteq}I.\ card\ J \le card\ (\bigcup (A\ `\ J))$ (**is** $?M\ A$)
  shows $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$
    (**is** $?SDR\ A$ **is** $\exists R.\ ?R\ R\ A\ \&\ ?inj\ R\ A$)
⟨*proof*⟩

end

References

[1] P. R. Halmos and H. E. Vaughan. The marriage problem. *American Journal of Mathematics*, 72:214–215, 1950.